DIRICHLET L-FUNCTIONS OF QUADRATIC CHARACTERS OF PRIME CONDUCTOR AT THE CENTRAL POINT

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ABSTRACT. We prove that more than nine percent of the central values $L(\frac{1}{2}, \chi_p)$ are non-zero, where $p \equiv 1 \pmod 8$ ranges over primes and χ_p is the real primitive Dirichlet character of conductor p. Previously, it was not known whether a positive proportion of these central values are non-zero. As a by-product, we obtain the order of magnitude of the second moment of $L(\frac{1}{2}, \chi_p)$, and conditionally we obtain the order of magnitude of the third moment. Assuming the Generalized Riemann Hypothesis, we show that our lower bound for the second moment is asymptotically sharp.

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1. Introduction and results

The values of L-functions at special points on the complex plane are of great interest. At the fixed point of the functional equation, called the central point, the question of non-vanishing is particularly important. For instance, the well-known Birch and Swinnerton-Dyer conjecture [43] relates the order of vanishing of certain L-functions at the central point to the arithmetic of elliptic curves. Katz and Sarnak [22] discuss several examples of families of L-functions and describe how the zeros close to $s = \frac{1}{2}$ give evidence of some underlying symmetry group for each of these families. They suggest that understanding these symmetries may in turn lead to finding a natural spectral interpretation of the zeros of the L-functions. The analysis of each family they discuss leads to a *Density Conjecture* that, if true, would imply that almost all L-functions in the family do not vanish at the central point. Iwaniec and Sarnak [19] show that the non-vanishing of L-functions associated with holomorphic cusp forms is closely related to the Landau-Siegel zero problem. Thus the question of non-vanishing at the central point is connected to many deep arithmetical problems.

A considerable amount of research has been done towards answering this question for families of Dirichlet L-functions. Chowla conjectured that $L(\frac{1}{2}, \chi) \neq 0$ for χ a primitive quadratic Dirichlet character [7, p. 82, problem 3]. It has since become a sort of folklore conjecture that $L(\frac{1}{2}, \chi) \neq 0$ for all primitive Dirichlet characters χ. One family that has attracted a lot of attention is the family of $L(s, \chi)$ with χ varying over primitive characters modulo a fixed conductor. This family is widely believed to have a unitary symmetry type, as in the philosophy of Katz and Sarnak. Balasubramanian and Murty [3] were the first to prove that a (small) positive proportion of this family does not vanish at the central point. They used the celebrated technique of mollified moments, a method that has been highly useful in other contexts (see, for example, [4, 9, 38]). Iwaniec and Sarnak [18] developed a simpler, stronger version of the method and improved this proportion to $\frac{1}{3}$. The approach of Iwaniec and Sarnak has since become standard in the study of non-vanishing of L-functions at the central point. Bui [5] and Khan and Ngo [26] introduced new ideas and further improved the lower bound $\frac{1}{3}$. The second author [35] has shown that more than fifty percent of the central values are non-vanishing when one additionally averages over the conductors. For further interesting research on this and other families of L-functions, see [6, 10, 23, 24, 25, 27, 28, 29, 30, 31].

The family of $L(s, \chi)$ with χ varying over all real primitive characters has also been extensively studied. This family is of particular significance because it seems to be of symplectic rather than unitary symmetry. Thus we encounter new phenomena not seen in the unitary case. For d a fundamental discriminant, set $\chi_d(\cdot) = \left(\frac{d}{\cdot}\right)$, the Kronecker symbol. Then

χ_d is a real primitive character with conductor $|d|$. The hypothetical positivity of central values $L(\frac{1}{2}, \chi_d)$ has implications for the class number of imaginary quadratic fields [17, p. 514]. Jutila [21] initiated the study of non-vanishing at the central point for this family and proved that $L(\frac{1}{2}, \chi_d) \neq 0$ for infinitely many fundamental discriminants d. His methods show that $\gg X/\log X$ of the quadratic characters χ_d with $|d| \leq X$ have $L(\frac{1}{2}, \chi_d) \neq 0$. Özlük and Snyder [32] examined the low-lying zeros of this family, and found the first evidence of its symplectic behavior. Assuming the Generalized Riemann Hypothesis (GRH), they showed that at least $\frac{15}{16}$ of the central values $L(\frac{1}{2}, \chi_d)$ are non-zero [33]. Katz and Sarnak independently obtained the same result in unpublished work (see [22, 39]).

Soundararajan [39] made a breakthrough when he proved unconditionally that at least $\frac{7}{8}$ of the central values $L(\frac{1}{2}, \chi_d)$ with $d \equiv 0 \pmod 8$ are non-zero. The biggest difficulty lies in analyzing the contribution of the "off-diagonal" terms in the evaluation of a mollified second moment. Soundararajan discovered that there is, in fact, a main contribution arising from these off-diagonal terms. (See Section 3 for more discussion.)

The case of real primitive characters with prime conductor is more difficult still. Jutila [21] initiated the study of $L(\frac{1}{2}, \chi_p)$, where p is a prime. His methods yield that $\gg X/(\log X)^3$ of the primes $p \leq X$ satisfy $L(\frac{1}{2}, \chi_p) \neq 0$. The difficulty in studying this family is that its moments involve sums over primes, and thus are more complicated to investigate. In fact, Jutila only evaluated the first moment of this family. As far as the authors are aware, no asymptotic evaluation of the second moment has appeared in the literature. However, Andrade and Keating [2] asymptotically evaluated the second moment of an analogous family over function fields. Andrade and the first author [1] have continued the study of the family of $L(\frac{1}{2}, \chi_p)$, showing that it is likely governed by a symplectic law. Conditionally on GRH, they prove that at least 75% of primes $p \leq X$ satisfy $L(\frac{1}{2}, \chi_p) \neq 0$.

We prove an unconditional positive proportion result for the central values $L(\frac{1}{2}, \chi_p)$. In fact, we prove that more than nine percent of these central values are non-zero.

Theorem 1.1. *There exists an absolute, effective constant X_0 such that if $X \geq X_0$ then*

$$\sum_{\substack{p \leq X \\ p \equiv 1 \,(\mathrm{mod}\ 8) \\ L(\frac{1}{2},\chi_p) \neq 0}} 1 \geq .0964 \sum_{\substack{p \leq X \\ p \equiv 1 \,(\mathrm{mod}\ 8)}} 1.$$

The proof of Theorem 1.1 proceeds via the mollification method, which we discuss briefly in Section 3 below. Our methods build on those of Jutila [21] and Soundararajan [39]. As in the work of Soundararajan, the main difficulty lies in evaluating the contribution of certain off-diagonal terms. The difference now is that we are summing over primes instead of over square-free integers, and so we cannot directly use his approach. A key idea in the proof of Theorem 1.1 is the use of upper bound sieves to turn intractable sums over primes into manageable sums over integers. The use of sieves in studying central values of L-functions has also appeared in some other contexts (see [16], also [36, p. 1035]).

The tools developed for the proof of Theorem 1.1 allow us to obtain the order of magnitude of the second moment of $L(\frac{1}{2}, \chi_p)$.

Theorem 1.2. *Let $\mathfrak{c}$ be the positive constant*

$$\mathfrak{c} := \left(144\zeta(2) \left(1 - \frac{1}{\sqrt{2}} \right)^2 \right)^{-1} = .0492\ldots .$$

For large X we have

$$(\mathfrak{c} - o(1))\frac{X}{4}(\log X)^3 \leq \sum_{\substack{p \leq X \\ p \equiv 1 \,(\mathrm{mod}\ 8)}} (\log p) L\left(\tfrac{1}{2}, \chi_p\right)^2 \leq (4\mathfrak{c} + o(1))\frac{X}{4}(\log X)^3.$$

One would rather have an upper bound in Theorem 1.2 that asymptotically matches the lower bound, but this seems difficult to prove unconditionally. By adapting a method of Soundararajan and Young [41] we are able, however, to prove such an asymptotic formula on GRH.

Theorem 1.3. *Let $\mathfrak{c}$ be as in Theorem 1.2. Assume the Riemann Hypothesis for $\zeta(s)$ and for all Dirichlet L-functions $L(s, \chi_p)$ with $p \equiv 1$ (mod 8). Then*

$$\sum_{\substack{p \leq X \\ p \equiv 1 \,(\mathrm{mod}\ 8)}} (\log p) L\left(\tfrac{1}{2}, \chi_p\right)^2 = \mathfrak{c}\frac{X}{4}(\log X)^3 + O(X(\log X)^{11/4}).$$

After we completed this paper, Maksym Radziwiłł informed us about work in progress with Julio Andrade, Roger Heath-Brown, Xiannan Li, and K. Soundararajan in which they derive an unconditional asymptotic formula for the second moment of $L(\tfrac{1}{2}, \chi_p)$. Their approach similarly introduces sieve weights, and they also observed that this idea could lead to a non-vanishing result.

Our methods further yield the order of magnitude of the third moment of $L(\tfrac{1}{2}, \chi_p)$, assuming that the central values $L(\tfrac{1}{2}, \chi_n)$ are non-negative for certain fundamental discriminants n. This non-negativity hypothesis follows, of course, from GRH.

Theorem 1.4. *Assume that for all positive square-free integers n with $n \equiv 1$ (mod 8) it holds that $L(\tfrac{1}{2}, \chi_n) \geq 0$. Then for large X*

$$\sum_{\substack{p \leq X \\ p \equiv 1 \,(\mathrm{mod}\ 8)}} (\log p) L\left(\tfrac{1}{2}, \chi_p\right)^3 \asymp X(\log X)^6.$$

Throughout this paper, we work exclusively with $p \equiv 1$ (mod 8) for convenience, but our methods are not specific to this residue class. With some modifications one could state similar results for other residue classes modulo 8. See the end of Section 3 for more details.

Our work indicates that Soundararajan's lower bound [39] for the proportion of non-vanishing for fundamental discrimimants $d \equiv 0$ (mod 8) also holds for the case of fundamental discriminants $d \equiv 1$ (mod 8). Proving this involves re-doing the calculations in Section 7, but without applying an upper bound sieve. To complete the proof, one would also need a first moment calculation. We omit the details and instead refer the reader to [39, Section 4].

It is natural to ask about the limitations of our method, and how much we can increase the lower bound in Theorem 1.1. If we assume that we can use arbitrarily long mollifiers [12], then we obtain a higher percentage of non-vanishing. However, in view of the parity problem of sieve theory [13], we could not reach a proportion greater than $\frac{1}{2}$ via our method. On the other hand, by a different method [1], the Density Conjecture of Katz and Sarnak would imply that 100% of the central values $L(\tfrac{1}{2}, \chi_p)$ are nonzero.

The outline of the rest of the paper is as follows. In Section 2 we establish some notation and conventions that hold throughout this work. Section 3 outlines the basic strategy for

the proof of Theorem 1.1. In Sections 4 and 5 we state a number of important technical results which are used in the proofs of our theorems. The proof of Theorem 1.1 is spread across Sections 6, 7, and 8. In Section 6 and its subsections we study the mollified first moment problem. The very long Section 7 and its subsections handle the mollified second moment. We choose our mollifier and finish the proof of Theorem 1.1 in Section 8. We prove Theorems 1.2 and 1.3 in Section 9, and we prove Theorem 1.4 in Section 10.

2. NOTATION AND CONVENTIONS

We define $\chi_n(\cdot) = \left(\frac{n}{\cdot}\right)$, the Kronecker symbol, for all nonzero integers n, even if n is not a fundamental discriminant. Note that this means χ_n has conductor $|n|$ only when n is a fundamental discriminant. We write $S(Q)$ for the set of all real primitive characters χ with conductor $\leq Q$. For an integer n, we write $n = \square$ or $n \neq \square$ according to whether or not n is a perfect square.

We let $\varepsilon > 0$ denote an arbitrarily small constant whose value may vary from one line to the next. When ε is present, in some fashion, in an inequality or error term, we allow implied constants to depend on ε without necessarily indicating this in the notation. At times we indicate the dependence of implied constants on other quantities by use of subscripts: for example, $Y \ll_A Z$.

Throughout this paper, we denote by $\Phi(x)$ a smooth function, compactly supported in $[\frac{1}{2}, 1]$, which satisfies $\Phi(x) = 1$ for $x \in [\frac{1}{2} + \frac{1}{\log X}, 1 - \frac{1}{\log X}]$ and $\Phi^{(j)}(x) \ll_j (\log X)^j$ for all $j \geq 0$. We could state our results for arbitrary smooth functions supported in $[\frac{1}{2}, 1]$, but we avoid this in an attempt to achieve some simplicity.

We write $e(x) = e^{2\pi i x}$. For g a compactly supported smooth function, we define the Fourier transform $\hat{g}(y)$ of g by

$$\hat{g}(y) = \int_{\mathbb{R}} g(x)e(-xy)dx.$$

At times, however, we find it convenient to use a slightly different normalization of the Fourier transform (see Lemma 5.2).

We define the Mellin transform $g^{\dagger}(s)$ of g by

$$g^{\dagger}(s) = \int_0^{\infty} g(x)x^{s-1}dx.$$

It is also helpful to define a modified Mellin transform $\check{g}(w)$ by

$$\check{g}(w) = \int_0^{\infty} g(x)x^w dx.$$

Observe that $\check{g}(w) = g^{\dagger}(1 + w)$. Lastly, for a complex number s, we define

$$g_s(t) = g(t)t^{s/2}.$$

Note that

$$\hat{\Phi}(0) = \Phi^{\dagger}(1) = \check{\Phi}(0) = \frac{1}{2} + O\left(\frac{1}{\log X}\right).$$

The letter p always denotes a prime number. We write φ for the Euler phi function, and d_k for the k-fold divisor function. If a and b are integers we write $[a, b]$ for their least common multiple and (a, b) for their greatest common divisor. It will always be clear from context whether $[a, b]$, say, denotes a least common multiple or a real interval.

Given coprime integers a and q, we write $\bar{a} \pmod{q}$ for the multiplicative inverse of a modulo q.

3. Outline of the proof of Theorem 1.1

The proof of Theorem 1.1 proceeds through the mollification method. The method was introduced by Bohr and Landau [4], but later greatly refined in the hands of Selberg [38]. The idea is to introduce a Dirichlet polynomial $M(p)$, known as a mollifier, which dampens the occasional wild behavior of the central values $L(\frac{1}{2}, \chi_p)$. We study the first and second moments

$$S_1 := \sum_{p \equiv 1 \,(\mathrm{mod}\, 8)} (\log p)\Phi\left(\frac{p}{X}\right) L\left(\tfrac{1}{2}, \chi_p\right) M(p),$$

(3.1)

$$S_2 := \sum_{p \equiv 1 \,(\mathrm{mod}\, 8)} (\log p)\Phi\left(\frac{p}{X}\right) L\left(\tfrac{1}{2}, \chi_p\right)^2 M(p)^2.$$

If the mollifier is chosen well then $S_1 \gg X$ and $S_2 \ll X$. By the Cauchy-Schwarz inequality we have

(3.2)
$$\sum_{\substack{p \equiv 1 \,(\mathrm{mod}\, 8) \\ L(\frac{1}{2}, \chi_p) \neq 0}} (\log p)\Phi\left(\frac{p}{X}\right) \geq \frac{S_1^2}{S_2},$$

and this implies that a positive proportion of $L(\frac{1}{2}, \chi_p)$ are non-zero.

Our mollifier takes the form

(3.3)
$$M(p) := \sum_{\substack{m \leq M \\ m \text{ odd}}} \frac{b_m}{\sqrt{m}} \chi_p(m),$$

for some coefficients b_m we describe shortly. Here we set

(3.4)
$$M = X^\theta, \qquad \theta \in \left(0, \tfrac{1}{2}\right) \quad \text{fixed.}$$

The larger one can take θ, the better proportion of non-vanishing one can achieve.

The coefficients b_m are a smoothed version of the Möbius function $\mu(m)$. Specifically, we choose

(3.5)
$$b_m = \mu(m) H\left(\frac{\log m}{\log M}\right),$$

where $H(t)$ is smooth function compactly supported in $[-1, 1]$ which we choose in Section 8. It will be convenient in a number of places that b_m is supported on square-free integers.

We outline our strategy for estimating S_1 and S_2. We simplify the presentation here in comparison to the actual proofs. The sum S_1 is by far the simpler of the two, so we start here (see Section 6). Using an approximate functional equation for the central value $L(\frac{1}{2}, \chi_p)$ (Lemma 4.2), we write S_1 as

$$S_1 \approx \sum_{m \leq M} \frac{b_m}{\sqrt{m}} \sum_{k \leq X^{1/2+\varepsilon}} \frac{1}{\sqrt{k}} \sum_{p \equiv 1 \,(\mathrm{mod}\, 8)} (\log p)\Phi\left(\frac{p}{X}\right) \chi_p(mk).$$

The main term arises from the "diagonal" terms $mk = \square$. The character values $\chi_p(mk)$ are then all equal to one, and we simply use the prime number theorem for arithmetic progressions modulo eight to handle the sum on p. The sum over k contributes a logarithmic factor,

but this logarithmic loss is canceled out by a logarithmic gain coming from a cancellation in the mollifier coefficients. This yields the main term for S_1, which is of size $\asymp X$ (Proposition 6.1).

The "off-diagonal" terms $mk \neq \square$ contribute only to the error term. After some manipulations the off-diagonal terms are essentially of the form

$$\mathcal{E} := \sum_{\substack{q \leq MX^{1/2+\varepsilon} \\ q \neq \square}} \frac{\alpha(q)}{q^{\frac{1}{2}}} \sum_p (\log p) \Phi \left(\frac{p}{X} \right) \chi_q(p),$$

where $\alpha(q)$ is some function satisfying $|\alpha(q)| \ll_\varepsilon q^\varepsilon$. We assume here for simplicity that all of the characters χ_q are primitive characters. We bound the character sum over primes in $\mathcal{E}$ in three different ways, depending on the size of q. These three regimes correspond to small, medium, and large values of q. Some of the arguments are similar to those of Jutila [21].

In the regime of small q we appeal to the prime number theorem for arithmetic progressions with error term. The sum over primes p is small, except in the case where one of the characters χ_{q^*} is exceptional: that is, the associated L-function $L(s, \chi_{q^*})$ has a real zero β_* very close to $s = 1$. Siegel's theorem gives $q^* \geq c(B)(\log X)^B$ with $B > 0$ arbitrarily large. This would immediately dispatch any exceptional characters, but unfortunately the constant $c(B)$ is not effectively computable. To get an effective estimate we use Page's theorem, which states that at most one such exceptional character χ_{q^*} exists. We then study carefully the contribution of this one exceptional character and show it is acceptably small.

In regimes of medium and large q, we take advantage of the averaging over q present in $\mathcal{E}$. We bound $\mathcal{E}$ in terms of instances of

$$\mathcal{E}(Q) := Q^{-\frac{1}{2}+\varepsilon} \sum_{\substack{Q/2 < q \leq Q \\ q \neq \square}} \left| \sum_p (\log p) \Phi \left(\frac{p}{X} \right) \chi_q(p) \right|,$$

where Q is of moderate size, or is large.

When Q is medium-sized, we use the explicit formula to bound $\mathcal{E}(Q)$ by sums over zeros of the L-functions $L(s, \chi_q)$. We then use zero-density estimates.

We are left with the task of bounding $\mathcal{E}(Q)$ when Q is large, which means Q is larger than X^δ for some small, fixed $\delta > 0$. Rather than treating the sum on primes analytically, as we did when Q was small or medium-sized, we treat the sum on primes combinatorially. We use Vaughan's identity to write the character sum over the primes as a linear combination of linear and bilinear sums. The linear sums are handled easily with the Pólya-Vinogradov inequality. We bound the bilinear sums by appealing to a large sieve inequality for real characters due to Heath-Brown (Lemma 4.4).

We now describe our plan of attack for S_2 (see Section 7). Recall that

$$S_2 = \sum_{p \equiv 1 \,(\mathrm{mod}\, 8)} (\log p) \Phi \left(\frac{p}{X} \right) L \left(\tfrac{1}{2}, \chi_p \right)^2 M(p)^2.$$

As we see from Theorem 1.3, we only barely obtain an asymptotic formula for the second moment

$$\sum_{\substack{p \leq X \\ p \equiv 1 \,(\mathrm{mod}\, 8)}} (\log p) L \left(\tfrac{1}{2}, \chi_p \right)^2$$

under the assumption of the Generalized Riemann Hypothesis. Thus, it might seem doubtful that one can say anything useful about S_2, since the central value $L(\frac{1}{2}, \chi_p)^2$ is further twisted by the square of a Dirichlet polynomial. The key idea is that we do not need an asymptotic formula for S_2, but only an upper bound of the right order of magnitude (with a good constant). We therefore avail ourselves of sieve methods (see Section 5). By positivity we have

$$S_2 \leq (\log X) \sum_{n \equiv 1 \,(\mathrm{mod}\ 8)} \mu^2(n) \Phi\left(\frac{n}{X}\right) \left(\sum_{d|n} \lambda_d\right) L\left(\tfrac{1}{2}, \chi_n\right)^2 M(n)^2,$$

where

$$\sum_{d|n} \lambda_d$$

is an upper bound sieve supported on coefficients with $d \leq D$. Since we are now working with ordinary integers instead of prime numbers, the analysis for S_2 becomes similar to the second moment problem considered in [39] (see [39, Section 5]).

We begin by writing

(3.6) $$\mu^2(n) = N_Y(n) + R_Y(n),$$

where

(3.7) $$N_Y(n) := \sum_{\substack{\ell^2|n \\ \ell \leq Y}} \mu(\ell), \qquad R_Y(n) := \sum_{\substack{\ell^2|n \\ \ell > Y}} \mu(\ell),$$

and Y is a small power of X. The sum

$$\sum_{n \equiv 1 \,(\mathrm{mod}\ 8)} \Phi\left(\frac{n}{X}\right) R_Y(n) \left(\sum_{d|n} \lambda_d\right) L\left(\tfrac{1}{2}, \chi_n\right)^2 M(n)^2$$

is an error term, and is shown to be small in a straightforward fashion by applying moment estimates for $L(\frac{1}{2}, \chi_n)$ due to Heath-Brown (Lemma 4.5).

The main task is therefore to asymptotically evaluate the sum

$$\sum_{n \equiv 1 \,(\mathrm{mod}\ 8)} \Phi\left(\frac{n}{X}\right) N_Y(n) \left(\sum_{d|n} \lambda_d\right) L\left(\tfrac{1}{2}, \chi_n\right)^2 M(n)^2.$$

We use an approximate functional equation to represent the central values $L\left(\frac{1}{2}, \chi_n\right)^2$ and arrive at expressions of the form

$$\sum_{\ell \leq Y} \mu(\ell) \sum_{d \leq D} \lambda_d \sum\sum_{m_1, m_2 \leq M} \frac{b_{m_1} b_{m_2}}{\sqrt{m_1 m_2}} \sum_{\nu=1}^{\infty} \frac{d(\nu)}{\sqrt{\nu}} \sum_{\substack{n \equiv 1 \,(\mathrm{mod}\ 8) \\ d|n \\ \ell^2|n}} \left(\frac{m_1 m_2 \nu}{n}\right) \Phi\left(\frac{n}{X}\right) \omega\left(\frac{\nu}{n}\right),$$

where $\omega(x)$ is some rapidly decaying smooth function that satisfies $\omega(x) \approx 1$ for small x. We then make the change of variables $n = m[d, \ell^2]$.

We use Poisson summation to transform the sum over m into a sum basically of the form

$$\sum_{k \in \mathbb{Z}} \left(\frac{[d, \ell^2]k}{m_1 m_2 \nu} \right) e \left(\frac{k\overline{[d, \ell^2]m_1 m_2 \nu}}{8} \right) \hat{F}_\nu \left(\frac{kX}{[d, \ell^2]m_1 m_2 \nu} \right),$$

for some smooth function F_ν. The zero frequency $k = 0$ gives rise to a main term. Since $\left(\frac{0}{h} \right) = 1$ or 0 depending on whether h is a square, the $k = 0$ contribution represents the expected "diagonal" contribution from $m_1 m_2 \nu = \square$. There is an additional, off-diagonal, main term which arises, essentially, from the terms with $[d, \ell^2]k = \square$. We adapt here the delicate off-diagonal analysis of [39]. The situation is complicated by the presence of the additive character $e(\cdot)$, which is not present in [39]. The additive character necessitates a division of the integers k into residue classes modulo 8. We then use Fourier expansion to write the additive character as a linear combination of multiplicative characters. After many calculations the off-diagonal main term arises as a sum of complex line integrals. When we combine the various pieces the integrand becomes an even function, exhibiting a symmetry which none of the pieces separately possessed. This fact proves to be very convenient in the final steps of the main term analysis.

One intriguing feature of the main term in S_2 is a kind of "double mollification". We must account for the savings coming from the mollifier $M(n)$, but must also account for the savings coming from the sieve weights λ_d, which act as a sort of mollifier on the natural numbers. It is crucial that we get savings in both places, and therefore our sieve process must be very precise. We find that a variation on the ideas of Selberg (see e.g. [17, Section 6.5]) is sufficient.

At length we arrive at an upper bound $S_{2,U}$, say, for S_2 of size $S_{2,U} \ll X$. We make an optimal choice of the function $H(x)$ in Section 8 to maximize the ratio $S_1^2/S_{2,U}$. The resulting mollifier is not the optimal mollifier, but it gives results that are asymptotically equivalent to those attained with the optimal mollifier. This yields Theorem 1.1.

To treat other residue classes of p (mod 8), we make the following changes. First, we change the definition of $\chi_p(\cdot)$ to $\left(\frac{(-1)^{\mathfrak{a}}p}{\cdot} \right)$, where $\mathfrak{a} = 0$ if $p \equiv 1$ (mod 4) and $\mathfrak{a} = 1$ if $p \equiv 3$ (mod 4). Thus χ_p is still a primitive character of conductor p. Second, we use a variant of the approximate functional equation (Lemma 4.2) with ω_j, defined in (4.1), replaced by

$$\frac{1}{2\pi i} \int_{(c)} \frac{\Gamma \left(\frac{s}{2} + \frac{1+2\mathfrak{a}}{4} \right)^j}{\Gamma \left(\frac{1+2\mathfrak{a}}{4} \right)^j} \left(1 - \frac{\chi_p(2)}{2^{\frac{1}{2}-s}} \right)^j \xi^{-s} W(s) \frac{ds}{s}.$$

The function $W(s)$ here is $16 \left(s^2 - \frac{1}{4} \right)^2$. Its purpose is to cancel potential poles at $s = \frac{1}{2}$ in the analysis.

4. LEMMATA

We represent the central values of L-functions by using an approximate functional equation. We first investigate some properties of the smooth functions which appear in our approximate functional equations. For $j = 1, 2$ and $c > 0$, define

(4.1) $$\omega_j(\xi) = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma \left(\frac{s}{2} + \frac{1}{4} \right)^j}{\Gamma \left(\frac{1}{4} \right)^j} \left(1 - \frac{1}{2^{\frac{1}{2}-s}} \right)^j \xi^{-s} \frac{ds}{s}.$$

Lemma 4.1. *Let* $j = 1, 2$. *The function* $\omega_j(\xi)$ *is real-valued and smooth on* $(0, \infty)$. *If* $\xi > 0$ *we have*

$$\omega_j(\xi) = \left(1 - \frac{1}{\sqrt{2}}\right)^j + O_\varepsilon(\xi^{\frac{1}{2}-\varepsilon}).$$

For any fixed integer $\nu \geq 0$ *and* $\xi \geq 4\nu + 10$, *we have*

$$\omega_j^{(\nu)}(\xi) \ll (\xi/2)^{\nu+3} \exp\left(-\frac{1}{4}\xi^{\frac{2}{j}}\right) \ll_\nu \exp\left(-\frac{1}{8}\xi^{\frac{2}{j}}\right).$$

Proof. The proof is similar to [39, Lemma 2.1], but we give details for completeness. The function $\omega_j(s)$ is real-valued because the change of variable $\mathrm{Im}(s) \to -\mathrm{Im}(s)$ shows that ω_j is equal to its complex conjugate. Moreover, uniform convergence for ξ in compact subintervals of $(0, \infty)$ shows that ω_j is smooth.

To prove the first estimate of the lemma, move the line of integration in the definition of $\omega_j(\xi)$ to $c = -\frac{1}{2} + \varepsilon$. The pole at $s = 0$ contributes $\left(1 - \frac{1}{\sqrt{2}}\right)^j$, and the new integral is $O_\varepsilon(\xi^{\frac{1}{2}-\varepsilon})$.

Let us turn to the last estimate of the lemma. We may suppose $\xi^{\frac{2}{j}} \geq 4\nu + 10$. By differentiation under the integral sign we find

$$\omega_j^{(\nu)}(\xi) = \frac{(-1)^\nu}{2\pi i} \int_{(c)} \frac{\Gamma\left(\frac{s}{2} + \frac{1}{4}\right)^j}{\Gamma\left(\frac{1}{4}\right)^j} \left(1 - \frac{1}{2^{\frac{1}{2}-s}}\right)^j s(s+1)\cdots(s+\nu-1)\xi^{-s-\nu}\frac{ds}{s}.$$

Recall that $|\Gamma(x + iy)| \leq \Gamma(x)$ for $x \geq 1$ and $z\Gamma(z) = \Gamma(z+1)$. Thus, for $c \geq 2$ we obtain

$$|\omega_j^{(\nu)}(\xi)| \ll \Gamma\left(\frac{c}{2} + \frac{5}{4} + \nu\right)^j \left(1 + \frac{2^c}{\sqrt{2}}\right)^j \xi^{-c-v} \int_{(c)} \frac{1}{|s||\frac{s}{2} + \frac{1}{4} + \nu|} \prod_{k=0}^{\nu-1} \frac{|s+k|}{|\frac{s}{2} + \frac{1}{4} + k|} |ds|$$

$$\ll \Gamma\left(\frac{c}{2} + \frac{5}{4} + \nu\right)^j \left(\frac{2^j}{\xi}\right)^c \left(\frac{2}{\xi}\right)^\nu c^{-1},$$

where the implied constants are absolute. By Stirling's formula this is

$$\ll \left(\frac{c + 2\nu + 3}{2e}\right)^{\frac{j}{2}(c+2\nu+3)} \left(\frac{2^j}{\xi}\right)^c \left(\frac{2}{\xi}\right)^\nu.$$

We choose $c = \frac{1}{2}\xi^{\frac{2}{j}} - 2\nu - 3$, which we note is > 2. Thus, the quantity in question is

$$\ll \left(\frac{\xi}{2}\right)^{\nu+3} \exp\left(-\frac{1}{4}\xi^{\frac{2}{j}}\right),$$

as desired. $\qquad\square$

We will find it technically convenient to use an approximate functional equation in which the variable of summation is restricted to odd integers.

Lemma 4.2. *Let* $n \equiv 1 \pmod{8}$ *be square-free and satisfy* $n > 1$. *Let* $\chi_n(\cdot) = \left(\frac{n}{\cdot}\right)$ *denote the real primitive character of conductor* n. *Then for* $j = 1, 2$ *we have*

$$L\left(\tfrac{1}{2}, \chi_n\right)^j = \frac{2}{\left(1 - \frac{1}{\sqrt{2}}\right)^{2j}} \sum_{\substack{\nu=1 \\ \nu \text{ odd}}}^{\infty} \frac{\chi_n(\nu)d_j(\nu)}{\sqrt{\nu}} \omega_j\left(\nu\left(\frac{\pi}{n}\right)^{j/2}\right) =: \mathcal{D}_j(n).$$

Proof. The proof follows along standard lines (e.g. [17, Theorem 5.3]), but we give a proof since our situation is slightly different.

Let $\Lambda(z, \chi_n) = \left(\frac{n}{\pi}\right)^{z/2} \Gamma\left(\frac{z}{2}\right) L(z, \chi_n)$. Since $n \equiv 1 \pmod 4$ we have $\chi_n(-1) = 1$, and therefore we have the functional equation (see [8, Proposition 2.2.24], [11, Chapter 9])

$$\Lambda(z, \chi_n) = \Lambda(1 - z, \chi_n).$$

Recall also that $\Lambda(z, \chi_n)$ is entire because χ_n is primitive.

Now consider the sum

$$I := \sum_{\nu \text{ odd}} \frac{\chi_n(\nu) d_j(\nu)}{\sqrt{\nu}} \omega_j \left(\nu \left(\frac{\pi}{n}\right)^{j/2}\right).$$

We use the definition of ω_j and interchange the order of summation and integration. Since $\chi_n(2) = 1$ we have

$$I = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma\left(\frac{s}{2} + \frac{1}{4}\right)^j}{\Gamma\left(\frac{1}{4}\right)^j} \left(1 - \frac{1}{2^{\frac{1}{2}-s}}\right)^j \left(1 - \frac{1}{2^{\frac{1}{2}+s}}\right)^j \left(\frac{n}{\pi}\right)^{js/2} L\left(\frac{1}{2} + s, \chi_n\right)^j \frac{ds}{s}$$

$$= \frac{1}{2\pi i} \int_{(c)} \frac{\left(\frac{n}{\pi}\right)^{-j/4}}{\Gamma\left(\frac{1}{4}\right)^j} \left(1 - \frac{1}{2^{\frac{1}{2}-s}}\right)^j \left(1 - \frac{1}{2^{\frac{1}{2}+s}}\right)^j \Lambda\left(\frac{1}{2} + s, \chi_n\right)^j \frac{ds}{s}.$$

We move the line of integration to $\text{Re}(s) = -c$, picking up a contribution from the simple pole at $s = 0$:

$$I = \frac{\left(\frac{n}{\pi}\right)^{-j/4}}{\Gamma\left(\frac{1}{4}\right)^j} \left(1 - \frac{1}{\sqrt 2}\right)^{2j} \Lambda\left(\frac{1}{2}, \chi_n\right)^j$$

$$+ \frac{1}{2\pi i} \int_{(-c)} \frac{\left(\frac{n}{\pi}\right)^{-j/4}}{\Gamma\left(\frac{1}{4}\right)^j} \left(1 - \frac{1}{2^{\frac{1}{2}-s}}\right)^j \left(1 - \frac{1}{2^{\frac{1}{2}+s}}\right)^j \Lambda\left(\frac{1}{2} + s, \chi_n\right)^j \frac{ds}{s}.$$

In this latter integral we change variables $s \to -s$ and then apply the functional equation $\Lambda\left(\frac{1}{2} - s, \chi_n\right) = \Lambda\left(\frac{1}{2} + s, \chi_n\right)$ to obtain

$$\frac{\left(\frac{n}{\pi}\right)^{-j/4}}{\Gamma\left(\frac{1}{4}\right)^j} \left(1 - \frac{1}{\sqrt 2}\right)^{2j} \Lambda\left(\frac{1}{2}, \chi_n\right)^j = 2I = 2 \sum_{\nu \text{ odd}} \frac{\chi_n(\nu) d_j(\nu)}{\sqrt{\nu}} \omega_j \left(\nu \left(\frac{\pi}{n}\right)^{j/2}\right).$$

We then rearrange to obtain the desired conclusion. $\qquad\square$

We frequently encounter exponential sums which are analogous to Gauss sums. Given an odd integer n, we define for all integers k

$$(4.2) \qquad G_k(n) = \left(\frac{1 - i}{2} + \left(\frac{-1}{n}\right) \frac{1 + i}{2}\right) \sum_{a(\text{mod } n)} \left(\frac{a}{n}\right) e\left(\frac{ak}{n}\right)$$

and

$$(4.3) \qquad \tau_k(n) = \sum_{a(\text{mod } n)} \left(\frac{a}{n}\right) e\left(\frac{ak}{n}\right) = \left(\frac{1 + i}{2} + \left(\frac{-1}{n}\right) \frac{1 - i}{2}\right) G_k(n).$$

We require knowledge of $G_k(n)$ for all n.

Lemma 4.3. *(i) (Multiplicativity) Suppose m and n are coprime odd integers. Then $G_k(mn) = G_k(m)G_k(n)$.*
(ii) Suppose p^α is the largest power of p dividing k. (If $k = 0$ set $\alpha = \infty$.) Then for $\beta \geq 1$

$$G_k(p^\beta) = \begin{cases} 0 & \text{if } \beta \leq \alpha \text{ is odd,} \\ \varphi(p^\beta) & \text{if } \beta \leq \alpha \text{ is even,} \\ -p^\alpha & \text{if } \beta = \alpha + 1 \text{ is even,} \\ (\frac{kp^{-\alpha}}{p})p^\alpha\sqrt{p} & \text{if } \beta = \alpha + 1 \text{ is odd,} \\ 0 & \text{if } \beta \geq \alpha + 2. \end{cases}$$

Proof. This is [39, Lemma 2.3]. ☐

The following two results are useful for bounding various character sums that arise. Both results are corollaries of a large sieve inequality for quadratic characters developed by Heath-Brown [15].

Lemma 4.4. *Let N and Q be positive integers, and let $a_1, \ldots, a_N$ be arbitrary complex numbers. Then*

$$\sum_{\chi \in S(Q)} \left| \sum_{n \leq N} a_n \chi(n) \right|^2 \ll_\varepsilon (QN)^\varepsilon (Q + N) \sum_{n_1 n_2 = \square} |a_{n_1} a_{n_2}|,$$

for any $\varepsilon > 0$. Let M be a positive integer, and for each $|m| \leq M$ write $4m = m_1 m_2^2$, where m_1 is a fundamental discriminant, and m_2 is positive. Suppose the sequence a_n satisfies $|a_n| \ll n^\varepsilon$. Then

$$\sum_{|m| \leq M} \frac{1}{m_2} \left| \sum_{n \leq N} a_n \left(\frac{m}{n} \right) \right|^2 \ll (MN)^\varepsilon N(M + N).$$

Proof. This is [39, Lemma 2.4]. ☐

Lemma 4.5. *Suppose $\sigma + it$ is a complex number with $\sigma \geq \frac{1}{2}$. Then*

$$\sum_{\chi \in S(Q)} |L(\sigma + it, \chi)|^4 \ll Q^{1+\varepsilon}(1 + |t|)^{1+\varepsilon}$$

and

$$\sum_{\chi \in S(Q)} |L(\sigma + it, \chi)|^2 \ll Q^{1+\varepsilon}(1 + |t|)^{\frac{1}{2}+\varepsilon}.$$

Proof. This is [39, Lemma 2.5]. ☐

5. SIEVE ESTIMATES

Our main sieve will be a variant of the Selberg sieve (see [14, Chapter 7]). To lessen the volume of calculations, we also use Brun's pure sieve [14, Chapter 6] as a preliminary sieve to handle small prime factors. We set

$$(5.1) \qquad\qquad z_0 := \exp((\log X)^{1/3})$$

and

$$(5.2) \qquad\qquad R := X^\vartheta, \qquad \vartheta \in \left(0, \tfrac{1}{2}\right) \quad \text{fixed.}$$

Given a set $\mathcal{A}$ of integers we write $\mathbf{1}_{\mathcal{A}}(n)$ for the indicator function of this set. For $y > 2$ we define

$$P(y) = \prod_{p \leq y} p.$$

Then, for $n \asymp X$, our basic sieve inequality is

(5.3)
$$\mathbf{1}_{\{n:n \text{ prime}\}} \leq \mathbf{1}_{\{n:(n,P(z_0))=1\}} \mathbf{1}_{\{n:(n,P(R)/P(z_0))=1\}},$$

We write $\omega(n)$ for the number of distinct prime factors of n. To bound the first factor on the right-hand side of (5.3), we use Brun's upper bound sieve condition (see [14, (6.1)])

(5.4)
$$\mathbf{1}_{\{n:(n,P(z_0))=1\}}(n) \leq \sum_{\substack{b|(n,P(z_0)) \\ \omega(b) \leq 2r_0}} \mu(b),$$

where

$$r_0 := \lfloor (\log X)^{1/3} \rfloor.$$

We use an "analytic" Selberg sieve (e.g. [34]) for the second factor of (5.3). We introduce a smooth, non-negative function $G(t)$ which is supported on the interval $[-1, 1]$. We further require $G(t)$ to satisfy $|G(t)| \ll 1, |G^{(j)}(t)| \ll_j (\log \log X)^{j-1}$ for j a positive integer, and on the interval $[0, 1]$ we require $G(t) = 1 - t$ for $t \leq 1 - (\log \log X)^{-1}$. Then

(5.5)
$$\mathbf{1}_{\{n:(n,P(R)/P(z_0))=1\}}(n) \leq \left(\sum_{\substack{d|n \\ (d,P(z_0))=1}} \mu(d) G\left(\frac{\log d}{\log R} \right) \right)^2$$

$$= \sum_{\substack{j,k \leq R \\ [j,k]|n \\ (jk,P(z_0))=1}} \mu(j)\mu(k) G\left(\frac{\log j}{\log R} \right) G\left(\frac{\log k}{\log R} \right).$$

We mention also that the properties of G imply

(5.6)
$$\int_0^\infty G'(t)^2 dt = 1 + O\left(\frac{1}{\log \log X} \right) = 1 + o(1).$$

Note that the fundamental theorem of calculus and Cauchy-Schwarz yield the lower bound

$$\int_0^\infty G'(t)^2 dt \geq 1.$$

From (5.3), (5.4), and (5.5), we arrive at the upper bound sieve condition

(5.7)
$$\mathbf{1}_{\{n:n \text{ prime}\}}(n) \leq \sum_{d|n} \lambda_d,$$

where the coefficients λ_d are defined by

(5.8)
$$\lambda_d = \sum_{\substack{b|P(z_0) \\ \omega(b) \leq 2r_0}} \sum_{\substack{m,n \leq R \\ b[m,n]=d \\ (mn,P(z_0))=1}} \mu(b)\mu(m)\mu(n) G\left(\frac{\log m}{\log R} \right) G\left(\frac{\log n}{\log R} \right).$$

If $b|P(z_0)$ and $\omega(b) \leq 2r_0$, then $b \leq z_0^{2r_0} = \exp(2(\log X)^{2/3})$. Hence $\lambda_d \neq 0$ only for $d \leq D$, where

$$(5.9) \qquad D = R^2 \exp(2(\log X)^{2/3}) \ll_\varepsilon R^2 X^\varepsilon.$$

In our evaluation of sums involving the sieve coefficients (5.8) we use the following version of the fundamental lemma of sieve theory (see also [14, Section 6.5]).

Lemma 5.1. *Let $0 < \delta < 1$ be a fixed constant, r a positive integer with $r \asymp (\log X)^\delta$, and z_0 as in (5.1). Suppose that g is a multiplicative function such that $|g(p)| \ll 1$ uniformly for all primes p. Then*

$$\sum_{\substack{b|P(z_0) \\ \omega(b) \leq r \\ (b,\ell)=1}} \frac{\mu(b)}{b} g(b) = \prod_{\substack{p \leq z_0 \\ p \nmid \ell}} \left(1 - \frac{g(p)}{p}\right) + O\left(\exp(-r \log \log r)\right)$$

uniformly for all positive integers ℓ.

Proof. The proof is standard. Complete the sum on the left-hand side by adding to it all the terms with $\omega(b) > r$, dropping by positivity the condition $(b, \ell) = 1$. The error introduced in doing so is $\ll \exp(-(1 + o(1))r \log r) \ll \exp(-r \log \log r)$ (e.g. [17, §6.3]). The completed sum is equal to the Euler product on the right-hand side. □

The basic tool in our application of the Selberg sieve is the following lemma.

Lemma 5.2. *Let $z_0 = \exp((\log X)^{1/3})$. Let G be as above. Suppose h is a function such that $|h(p)| \ll_\varepsilon p^{-\varepsilon}$ uniformly for all primes p. Let $A > 0$ be a fixed real number. Then there exists a function $E_0(X)$, which depends only on X, G, and ϑ (see (5.2)) with $E_0(X) \to 0$ as $X \to \infty$, such that*

$$\sum_{\substack{m,n \leq R \\ (mn,\ell P(z_0))=1}} \frac{\mu(m)\mu(n)}{[m,n]} G\left(\frac{\log m}{\log R}\right) G\left(\frac{\log n}{\log R}\right) \prod_{p|mn} \left(1 + h(p)\right)$$

$$(5.10) \qquad = \frac{1 + E_0(X)}{\log R} \prod_{p \leq z_0} \left(1 - \frac{1}{p}\right)^{-1} + O_{\varepsilon,A}\left(\frac{1}{(\log R)^A}\right),$$

uniformly for $\ell \ll X^{O(1)}$.

Proof. Let $\mathcal{S}$ denote the left-hand side of (5.10). If $m, n \leq R$ and $(mn, P(z_0)) = 1$, then $\omega(mn) \ll \log R$, and each prime dividing mn is larger than z_0. Thus

$$\prod_{p|mn} \left(1 + h(p)\right) = 1 + O_\varepsilon\left(\frac{\log R}{z_0^\varepsilon}\right),$$

and so

$$(5.11) \qquad \mathcal{S} = \sum_{\substack{m,n \leq R \\ (mn,\ell P(z_0))=1}} \frac{\mu(m)\mu(n)}{[m,n]} G\left(\frac{\log m}{\log R}\right) G\left(\frac{\log n}{\log R}\right) + O\left(\frac{(\log R)^4}{z_0^\varepsilon}\right).$$

We may ignore the condition $(mn, \ell) = 1$ in (5.11) because

$$\sum_{\substack{m,n \leq R \\ (mn,P(z_0))=1 \\ (mn,\ell)>1}} \frac{1}{[m,n]} \leq \sum_{\substack{m,n \leq R \\ (mn,P(z_0))=1}} \frac{1}{[m,n]} \sum_{\substack{p|\ell \\ p|mn}} 1 \ll (\log R)^3 \sum_{\substack{p|\ell \\ p>z_0}} \frac{1}{p} \ll \frac{(\log \ell)(\log R)^3}{z_0}.$$

We next insert the Fourier inversion formula

$$(5.12) \qquad\qquad G(t) = \int_{-\infty}^{\infty} g(z) e^{-t(1+iz)} \, dz$$

into (5.11), where

$$(5.13) \qquad\qquad g(z) = \int_{-\infty}^{\infty} e^t G(t) e^{izt} \, dt.$$

We then interchange the order of summation and integration and write the sum as an Euler product to deduce that
(5.14)

$$\mathcal{S} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(z_1) g(z_2) \prod_{p>z_0} \left(1 - \frac{1}{p^{1+\frac{1+iz_1}{\log R}}} - \frac{1}{p^{1+\frac{1+iz_2}{\log R}}} + \frac{1}{p^{1+\frac{2+iz_1+iz_2}{\log R}}} \right) dz_1 dz_2 + O\left(\frac{(\log R)^4}{z_0^\varepsilon} \right).$$

By integrating (5.13) by parts repeatedly we see

$$g(z) \ll_A \left(\frac{\log \log X}{1+|z|} \right)^A,$$

and we have the trivial bound

$$\prod_{p>z_0} \left(1 - \frac{1}{p^{1+\frac{1+iz_1}{\log R}}} - \frac{1}{p^{1+\frac{1+iz_2}{\log R}}} + \frac{1}{p^{1+\frac{2+iz_1+iz_2}{\log R}}} \right) \ll (\log R)^{O(1)}.$$

Therefore, we may truncate the double integral in (5.14) to the region $|z_1|, |z_2| \leq \sqrt{\log R}$, with an error of size $O_A((\log R)^{-A})$. After doing so, we multiply and divide the integrand by Euler products of zeta-functions to arrive at
(5.15)

$$\mathcal{S} = \iint_{|z_i| \leq \sqrt{\log R}} g(z_1) g(z_2) \frac{\zeta\left(1 + \frac{2+iz_1+iz_2}{\log R}\right)}{\zeta\left(1 + \frac{1+iz_1}{\log R}\right) \zeta\left(1 + \frac{1+iz_2}{\log R}\right)}$$

$$\times \prod_{p \leq z_0} \frac{\left(1 - \frac{1}{p^{1+\frac{2+iz_1+iz_2}{\log R}}}\right)}{\left(1 - \frac{1}{p^{1+\frac{1+iz_1}{\log R}}}\right) \left(1 - \frac{1}{p^{1+\frac{1+iz_2}{\log R}}}\right)} \prod_{p>z_0} \left(1 + O\left(\frac{1}{p^2}\right)\right) dz_1 dz_2 + O\left(\frac{1}{(\log R)^A}\right).$$

The product over primes $p > z_0$ in (5.15) is $1 + O(1/z_0)$. To estimate the product over $p \leq z_0$, observe that if $|s| \ll \sqrt{\log R}$, then

$$\sum_{p \leq z_0} \frac{1}{p-1} \left(1 - p^{-s}\right) \ll \sum_{p \leq z_0} \frac{|s| \log p}{p} \ll |s| \log z_0 \ll \frac{(\log X)^{1/3}}{(\log R)^{1/2}},$$

which implies that

$$\prod_{p \leq z_0} \left(1 - \frac{1}{p^{1+s}} \right) = \exp \left(\sum_{p \leq z_0} \log \left(1 + \frac{1}{p-1} \left(1 - p^{-s} \right) \right) \right) \prod_{p \leq z_0} \left(1 - \frac{1}{p} \right)$$

$$= \left(1 + O \left(\frac{(\log X)^{1/3}}{(\log R)^{1/2}} \right) \right) \prod_{p \leq z_0} \left(1 - \frac{1}{p} \right).$$

We may also expand each zeta-function in (5.15) into its Laurent series. With these approximations, we deduce from (5.15) that

$$\mathcal{S} = \frac{1}{\log R} \prod_{p \leq z_0} \left(1 - \frac{1}{p} \right)^{-1} \int\int_{|z_i| \leq \sqrt{\log R}} g(z_1) g(z_2) \frac{(1 + iz_1)(1 + iz_2)}{2 + iz_1 + iz_2} \left(1 + E(X, \vartheta, z_1, z_2) \right) dz_1 dz_2$$

$$+ O \Big((\log R)^{-A} \Big),$$

uniformly for $\log \ell \ll \log X$. Here $E(X, \vartheta, z_1, z_2)$ tends to zero as $X \to \infty$. By the rapid decay of $g(z)$, we may extend the range of integration to $\mathbb{R}^2$ without affecting our bound for the error term. By differentiating (5.12) under the integral sign and Fubini's theorem, we find

$$(5.16) \qquad \int\int_{\mathbb{R}^2} g(z_1) g(z_2) \frac{(1 + iz_1)(1 + iz_2)}{2 + iz_1 + iz_2} dz_2 dz_1 = \int_0^\infty G'(t)^2 dt.$$

The lemma now follows from (5.16) and (5.6). □

Lemma 5.3. *Let λ_d and D be as defined in (5.8) and (5.9), respectively. Suppose that g is a multiplicative function such that $g(p) = 1 + O(p^{-\varepsilon})$ for all primes p. Then with $E_0(X)$ as in Lemma 5.2 we have*

$$\sum_{\substack{d \leq D \\ (d, \ell) = 1}} \frac{\lambda_d}{d} g(d) = \frac{1 + E_0(X)}{\log R} \prod_{\substack{p \leq z_0 \\ p \nmid \ell}} \left(1 - \frac{g(p)}{p} \right) \prod_{p \leq z_0} \left(1 - \frac{1}{p} \right)^{-1} + O_\varepsilon \left(\frac{1}{(\log R)^{2018}} \right),$$

uniformly in $\ell \ll X^{O(1)}$.

Proof. The definitions (5.8) and (5.9) of λ_d and D imply

$$\sum_{\substack{d \leq D \\ (d, \ell) = 1}} \frac{\lambda_d}{d} g(d) = \sum_{\substack{b | P(z_0) \\ \omega(b) \leq 2r_0 \\ (b, \ell) = 1}} \sum_{\substack{m, n \leq R \\ (mn, \ell P(z_0)) = 1}} \frac{\mu(b) \mu(m) \mu(n)}{b[m, n]} G \left(\frac{\log m}{\log R} \right) G \left(\frac{\log n}{\log R} \right) g(b[m, n]).$$

In the sum on the right-hand side, $g(b[m, n]) = g(b)g([m, n])$ because b and mn are coprime. Thus we may apply Lemma 5.2 and then Lemma 5.1 to arrive at Lemma 5.3. □

Lemma 5.4. *Let λ_d, D, g be as in Lemma 5.3. Suppose that h is a function such that $|h(p)| \ll_\varepsilon p^{-1+\varepsilon}$ for all primes p. Then with $E_0(X)$ as in Lemma 5.2 we have*

$$
\sum_{\substack{d \leq D \\ (d,\ell)=1}} \frac{\lambda_d}{d} g(d) \sum_{p|d} h(p) = - \frac{1 + E_0(X)}{\log R} \prod_{p \leq z_0} \left(1 - \frac{1}{p}\right)^{-1}
$$

$$
\times \sum_{\substack{p \leq z_0 \\ p \nmid \ell}} \frac{g(p)h(p)}{p} \prod_{\substack{q \leq z_0 \\ q \nmid p\ell}} \left(1 - \frac{g(q)}{q}\right) + O_\varepsilon\left(\frac{1}{(\log R)^{2018}}\right),
$$

uniformly for all integers ℓ such that $\log \ell \ll \log X$. (Here, the index q runs over primes q.)

Proof. The definitions (5.8) and (5.9) of λ_d and D imply

$$
\sum_{\substack{d \leq D \\ (d,\ell)=1}} \frac{\lambda_d}{d} g(d) \sum_{p|d} h(p) = \sum_{\substack{b|P(z_0) \\ \omega(b) \leq 2r_0 \\ (b,\ell)=1}} \sum_{\substack{m,n \leq R \\ (mn,\ell P(z_0))=1}} \frac{\mu(b)\mu(m)\mu(n)}{b[m,n]} G\left(\frac{\log m}{\log R}\right) G\left(\frac{\log n}{\log R}\right)
$$

$$
\times g(b[m,n]) \sum_{p|bmn} h(p).
$$

Since b and mn are coprime, $g(b[m,n]) = g(b)g([m,n])$ and

$$
\sum_{p|bmn} h(p) = \sum_{p|b} h(p) + \sum_{p|mn} h(p).
$$

We may ignore the sum over the $p|mn$ because the conditions $(mn, P(z_0)) = 1$ and $mn \leq R^2$ imply

$$
\sum_{p|mn} h(p) \ll \sum_{p|mn} p^{-1+\varepsilon} \ll \frac{\log R}{z_0^{1-\varepsilon}}.
$$

We factor out $g(b)$ and $\sum_{p|b} h(p)$ from the sum over m, n and then apply Lemma 5.2 to deduce that

(5.17)
$$
\sum_{\substack{d \leq D \\ (d,\ell)=1}} \frac{\lambda_d}{d} g(d) \sum_{p|d} h(p) = \frac{1 + E_0(X)}{\log R} \prod_{p \leq z_0} \left(1 - \frac{1}{p}\right)^{-1}
$$

$$
\times \sum_{\substack{b|P(z_0) \\ \omega(b) \leq 2r_0 \\ (b,\ell)=1}} \frac{\mu(b)}{b} g(b) \sum_{p|b} h(p) + O\left(\frac{1}{(\log R)^{2018}}\right).
$$

To estimate the b-sum, we interchange the order of summation and then relabel b as bp to write

$$
\sum_{\substack{b|P(z_0) \\ \omega(b) \leq 2r_0 \\ (b,\ell)=1}} \frac{\mu(b)}{b} g(b) \sum_{p|b} h(p) = \sum_{\substack{p \leq z_0 \\ p \nmid \ell}} h(p) \sum_{\substack{b|P(z_0) \\ \omega(b) \leq 2r_0 \\ (b,\ell)=1 \\ p|b}} \frac{\mu(b)}{b} g(b) = - \sum_{\substack{p \leq z_0 \\ p \nmid \ell}} \frac{g(p)h(p)}{p} \sum_{\substack{b|P(z_0) \\ \omega(b) \leq 2r_0-1 \\ (b,p\ell)=1}} \frac{\mu(b)}{b} g(b).
$$

Lemma 5.4 now follows from Lemma 5.1 and (5.17). $\square$

6. The mollified first moment

Our goal in this section is to asymptotically evaluate S_1. Recall from (3.1) that

$$S_1 = \sum_{p \equiv 1 \,(\mathrm{mod}\, 8)} (\log p) \Phi\left(\frac{p}{X}\right) L\left(\tfrac{1}{2}, \chi_p\right) M(p).$$

Recall the definition of $M(p)$ from (3.3), and the choice (3.5) we made for the mollifier coefficients b_m. We shall prove the following result.

Proposition 6.1. *Let $0 < \theta < \frac{1}{2}$ be fixed. If $X \geq X_0(\theta)$, then*

$$S_1 = \frac{1}{2(1 - \frac{1}{\sqrt{2}})} \left(H(0) - \frac{1}{2\theta} H'(0) \right) \frac{X}{4} + O\left(\frac{X}{(\log X)^{1-\varepsilon}} \right).$$

The implied constant in the error term is effectively computable.

Let us begin in earnest, following the outline in Section 3. We apply Lemma 4.2 to write $L(\tfrac{1}{2}, \chi_p)$ as a Dirichlet series. We insert the definition of $M(p)$ and obtain

$$S_1 = \frac{2}{\left(1 - \frac{1}{\sqrt{2}}\right)^2} \sum_{\substack{m \leq M \\ m \text{ odd}}} \frac{b_m}{\sqrt{m}} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{\sqrt{n}} \sum_{p \equiv 1 \,(\mathrm{mod}\, 8)} (\log p) \Phi\left(\frac{p}{X}\right) \omega_1\left(n\sqrt{\frac{\pi}{p}}\right)\left(\frac{mn}{p}\right).$$

The main term arises from the terms with $mn = \square$. Let us denote this portion of S_1 by $S_1^{\square}$. We denote the complementary portion with $mn \neq \square$ by $S_1^{\neq}$. Therefore

$$S_1 = S_1^{\square} + S_1^{\neq},$$

where

$$S_1^{\square} = \frac{2}{\left(1 - \frac{1}{\sqrt{2}}\right)^2} \sum_{\substack{m \leq M \\ m \text{ odd}}} \sum_{\substack{n=1 \\ n \text{ odd} \\ mn = \square}}^{\infty} \frac{b_m}{\sqrt{m}} \frac{1}{\sqrt{n}} \sum_{p \equiv 1 \,(\mathrm{mod}\, 8)} (\log p) \Phi\left(\frac{p}{X}\right) \omega_1\left(n\sqrt{\frac{\pi}{p}}\right)\left(\frac{mn}{p}\right),$$

(6.1)

$$S_1^{\neq} = \frac{2}{\left(1 - \frac{1}{\sqrt{2}}\right)^2} \sum_{\substack{m \leq M \\ m \text{ odd}}} \sum_{\substack{n=1 \\ n \text{ odd} \\ mn \neq \square}}^{\infty} \frac{b_m}{\sqrt{m}} \frac{1}{\sqrt{n}} \sum_{p \equiv 1 \,(\mathrm{mod}\, 8)} (\log p) \Phi\left(\frac{p}{X}\right) \omega_1\left(n\sqrt{\frac{\pi}{p}}\right)\left(\frac{mn}{p}\right).$$

We treat first the main term $S_1^{\square}$, and later we will bound the error term $S_1^{\neq}$.

6.1. **Main term.** Recall that b_m is supported on square-free integers m. Therefore, $mn = \square$ if and only if $n = mk^2$, where k is a positive integer. We make this change of variables and then interchange orders of summation to obtain

$$S_1^{\square} = \frac{2}{\left(1 - \frac{1}{\sqrt{2}}\right)^2} \sum_{p \equiv 1 \,(\mathrm{mod}\, 8)} (\log p) \Phi\left(\frac{p}{X}\right) \sum_{\substack{m \leq M \\ (m, 2p)=1}} \frac{b_m}{m} \sum_{\substack{k=1 \\ (k, 2p)=1}}^{\infty} \frac{1}{k} \omega_1\left(mk^2 \sqrt{\frac{\pi}{p}}\right).$$

By the rapid decay of ω_1 (Lemma 4.1) we see that the contribution from those k with $(k, p) > 1$ is $O_A(X^{-A})$, so we may safely ignore this condition. We may also ignore the

condition $(m, p) = 1$, since $m \leq M < p$. We insert the definition (4.1) of $\omega_1(\xi)$ and interchange to deduce that for any $c > 0$ we have

$$\sum_{\substack{k=1 \\ (k,2)=1}}^{\infty} \frac{1}{k} \omega_1\left(mk^2\sqrt{\frac{\pi}{p}}\right)$$

$$= \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(\frac{s}{2} + \frac{1}{4})}{\Gamma(\frac{1}{4})} \left(1 - \frac{1}{2^{\frac{1}{2}-s}}\right)\left(1 - \frac{1}{2^{1+2s}}\right) \zeta(1 + 2s) \left(\frac{p}{\pi}\right)^{s/2} m^{-s} \frac{ds}{s}.$$

We move the line of integration to $\operatorname{Re} s = -\frac{1}{2} + \varepsilon$, leaving a residue at $s = 0$. The new integral is $O_\varepsilon\left(p^{-\frac{1}{4}+\varepsilon}m^{\frac{1}{2}-\varepsilon}\right)$. Using $b_m \ll 1$, we see that the total contribution of this error term is $\ll X^{\frac{3}{4}+\varepsilon}M^{\frac{1}{2}}$. This is $O(X^{1-\varepsilon})$ by (3.4). Writing the residue at $s = 0$ as an integral along a small circle around 0, we deduce that

(6.1.1)
$$S_1^\square = O(X^{1-\varepsilon}) + \frac{2}{\left(1 - \frac{1}{\sqrt{2}}\right)^2} \sum_{p \equiv 1 \,(\mathrm{mod}\ 8)} (\log p)\Phi\left(\frac{p}{X}\right) \sum_{\substack{m \leq M \\ (m,2)=1}} \frac{b_m}{m}$$

$$\times \frac{1}{2\pi i} \oint_{|s|=\frac{1}{2\log X}} \frac{\Gamma(\frac{s}{2} + \frac{1}{4})}{\Gamma(\frac{1}{4})} \left(1 - \frac{1}{2^{\frac{1}{2}-s}}\right)\left(1 - \frac{1}{2^{1+2s}}\right) \zeta(1 + 2s) \left(\frac{p}{\pi}\right)^{s/2} m^{-s} \frac{ds}{s}.$$

We next use the definition $b_m = \mu(m)H\left(\frac{\log m}{\log M}\right)$ and the Fourier inversion formula (compare with (5.12),(5.13))

(6.1.2)
$$H(t) = \int_{-\infty}^{\infty} h(z)e^{-t(1+iz)}\, dz,$$

where

(6.1.3)
$$h(z) = \int_{-\infty}^{\infty} e^t H(t)e^{izt}\, dt,$$

to write

$$\sum_{\substack{m \leq M \\ (m,2)=1}} \frac{b_m}{m} m^{-s} = \int_{-\infty}^{\infty} h(z) \sum_{\substack{m=1 \\ (m,2)=1}}^{\infty} \frac{\mu(m)}{m^{1+s+\frac{1+iz}{\log M}}}\, dz$$

$$= \int_{-\infty}^{\infty} h(z) \left(1 - \frac{1}{2^{1+s+\frac{1+iz}{\log M}}}\right)^{-1} \zeta^{-1}\left(1 + s + \frac{1 + iz}{\log M}\right) dz.$$

From repeated integration by parts we obtain

(6.1.4)
$$h(z) \ll_j \frac{1}{(1 + |z|)^j},$$

and therefore we may truncate this integral to the range $|z| \leq \sqrt{\log M}$. Thus,

$$\sum_{\substack{m \leq M \\ (m,2)=1}} \frac{b_m}{m} m^{-s} = \int_{|z| \leq \sqrt{\log M}} h(z) \left(1 - \frac{1}{2^{1+s+\frac{1+iz}{\log M}}}\right)^{-1} \zeta^{-1}\left(1 + s + \frac{1+iz}{\log M}\right) dz$$

$$+ O_A\left(\frac{1}{(\log X)^A}\right).$$

For $|s| = \frac{1}{2 \log X}$ and $|z| \leq \sqrt{\log M}$, we may write $\left(1 - \frac{1}{2^{1+s+\frac{1+iz}{\log M}}}\right)^{-1} \zeta^{-1}\left(1 + s + \frac{1+iz}{\log M}\right)$ as a power series and arrive at

$$\sum_{\substack{m \leq M \\ (m,2)=1}} \frac{b_m}{m} m^{-s} = 2\int_{|z| \leq \sqrt{\log M}} h(z)\left(s + \frac{1+iz}{\log M}\right) dz + O\left(\frac{1}{(\log X)^2}\right).$$

We may extend the range of integration to the entire real line, with negligible error, because of (6.1.4). The definition of $H(t)$ implies that

$$H'(t) = -(1+iz)\int_{-\infty}^{\infty} h(z)e^{-t(1+iz)} dz.$$

Therefore

$$\int_{-\infty}^{\infty} h(z)\left(s + \frac{1+iz}{\log M}\right) dz = sH(0) - \frac{1}{\log M}H'(0),$$

and hence

(6.1.5) $$\sum_{\substack{m \leq M \\ (m,2)=1}} \frac{b_m}{m} m^{-s} = 2sH(0) - \frac{2}{\log M}H'(0) + O\left(\frac{1}{(\log X)^2}\right).$$

We insert (6.1.5) into (6.1.1) to obtain

$$S_1^{\square} = \frac{4}{\left(1 - \frac{1}{\sqrt{2}}\right)^2} \sum_{p \equiv 1 \,(\mathrm{mod}\, 8)} (\log p)\Phi\left(\frac{p}{X}\right) \frac{1}{2\pi i} \oint_{|s|=\frac{1}{2 \log X}} \frac{\Gamma(\frac{s}{2} + \frac{1}{4})}{\Gamma(\frac{1}{4})}\left(1 - \frac{1}{2^{\frac{1}{2}-s}}\right)$$

$$\times \left(1 - \frac{1}{2^{1+2s}}\right)\zeta(1+2s)\left(\frac{p}{\pi}\right)^{s/2}\left(sH(0) - \frac{1}{\log M}H'(0)\right)\frac{ds}{s} + O\left(\frac{X}{\log X}\right).$$

We evaluate the integral using the formula

(6.1.6) $$\operatorname*{Res}_{s=0} g(s) = \frac{1}{(n-1)!}\frac{d^{n-1}}{ds^{n-1}}s^n g(s)\bigg|_{s=0}$$

for a pole of a function $g(s)$ at $s = 0$ of order at most n. This yields

$$S_1^{\square} = \frac{1}{\left(1 - \frac{1}{\sqrt{2}}\right)} \sum_{p \equiv 1 \,(\mathrm{mod}\, 8)} (\log p)\Phi\left(\frac{p}{X}\right)\left(H(0) - \frac{\log p}{2 \log M}H'(0)\right) + O\left(\frac{X}{\log X}\right).$$

By the support of Φ we have $\log p = \log X + O(1)$. We then use the prime number theorem in arithmetic progressions and partial summation to obtain

$$(6.1.7) \qquad S_1^\square = \frac{1}{\left(1 - \frac{1}{\sqrt{2}}\right)} \left(H(0) - \frac{\log X}{2 \log M} H'(0)\right) \frac{X}{4} \widehat{\Phi}(0) + O\left(\frac{X}{\log X}\right).$$

Now (6.1.7) gives the main term for Proposition 6.1.

6.2. Preparation of the off-diagonal.

We turn to bounding $S_1^{\neq}$. In order to complete the proof of Proposition 6.1, we prove

$$(6.2.1) \qquad S_1^{\neq} \ll \frac{X}{(\log X)^{1-\varepsilon}}.$$

We need to perform some technical massaging before $S_1^{\neq}$ is in a suitable form. Recall from (6.1) that

$$S_1^{\neq} = \frac{2}{\left(1 - \frac{1}{\sqrt{2}}\right)^2} \sum_{\substack{m \leq M \\ m \text{ odd}}} \sum_{\substack{n=1 \\ n \text{ odd} \\ mn \neq \square}}^{\infty} \frac{b_m}{\sqrt{mn}} \sum_{p \equiv 1 \,(\mathrm{mod}\ 8)} (\log p)\Phi\left(\frac{p}{X}\right) \omega_1\left(n\sqrt{\frac{\pi}{p}}\right) \left(\frac{mn}{p}\right).$$

We begin by uniquely writing $n = rk^2$, where r is square-free and k is an integer (this variable k is unrelated to the variable k appearing in the analysis for $S_1^\square$). The condition $mn \neq \square$ is equivalent to $m \neq r$, since both m and r are square-free. It follows that

$$S_1^{\neq} = \frac{2}{\left(1 - \frac{1}{\sqrt{2}}\right)^2} \sum_{\substack{m \leq M \\ m \text{ odd}}} \frac{b_m}{\sqrt{m}} \sum_{\substack{r=1 \\ r \text{ odd} \\ r \neq m}}^{\infty} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{\mu^2(r)}{k\sqrt{r}} \sum_{p \equiv 1 \,(\mathrm{mod}\ 8)} (\log p)\Phi\left(\frac{p}{X}\right) \omega_1\left(rk^2\sqrt{\frac{\pi}{p}}\right) \left(\frac{mrk^2}{p}\right)$$

We next factor out the greatest common divisor, say g, of m and r. We change variables $m \to gm, r \to gr$ and obtain

$$S_1^{\neq} = \frac{2}{\left(1 - \frac{1}{\sqrt{2}}\right)^2} \sum_{g \text{ odd}} \frac{\mu^2(g)}{g} \sum_{\substack{m \leq M/g \\ (m,2g)=1}} \frac{b_{mg}}{\sqrt{m}} \sum_{\substack{r=1 \\ (r,2g)=1 \\ (m,r)=1 \\ mr>1}}^{\infty} \frac{\mu^2(r)}{\sqrt{r}} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{k}$$

$$\times \sum_{p \equiv 1 \,(\mathrm{mod}\ 8)} (\log p)\Phi\left(\frac{p}{X}\right) \omega_1\left(grk^2\sqrt{\frac{\pi}{p}}\right) \left(\frac{mrg^2k^2}{p}\right).$$

Observe that the support of b_{gm} forces $g \leq M < X^{\frac{1}{2}}$, but we prefer not to indicate this explicitly.

Clearly we have $\left(\frac{g^2k^2}{p}\right) = 1$ for $p \nmid gk$ and $= 0$ otherwise. Since $g \leq M < p$ the condition $p \nmid g$ is automatically satisfied. By Lemma 4.1 we may truncate the sum over k to $k \leq X^{\frac{1}{4}+\varepsilon}$ at the cost of an error $O(X^{-1})$, say. We may similarly truncate the sum on r to $r \leq X^{\frac{1}{2}+\varepsilon}$. With k suitably reduced we may drop the condition $p \nmid k$, and then we use the rapid decay

of ω_1 again to extend the sum on k to infinity. It follows that

$$
S_1^{\neq} = \frac{2}{\left(1 - \frac{1}{\sqrt{2}}\right)^2} \sum_{g \text{ odd}} \frac{\mu^2(g)}{g} \sum_{\substack{m \leq M/g \\ (m,2g)=1}} \frac{b_{mg}}{\sqrt{m}} \sum_{\substack{r \leq X^{1/2+\varepsilon} \\ (r,2g)=1 \\ (m,r)=1 \\ mr>1}} \frac{\mu^2(r)}{\sqrt{r}} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{k}
$$

(6.2.2)

$$
\times \sum_{p \equiv 1 \,(\mathrm{mod}\ 8)} (\log p) \Phi\left(\frac{p}{X}\right) \omega_1\left(grk^2\sqrt{\frac{\pi}{p}}\right)\left(\frac{mr}{p}\right) + O(X^{-1}).
$$

We next detect the congruence condition $p \equiv 1 \pmod 8$ with multiplicative characters modulo 8. Therefore

$$
\sum_{p \equiv 1 \,(\mathrm{mod}\ 8)} (\log p) \Phi\left(\frac{p}{X}\right) \omega_1\left(grk^2\sqrt{\frac{\pi}{p}}\right)\left(\frac{mr}{p}\right)
$$

(6.2.3)

$$
= \frac{1}{4} \sum_{\gamma \in \{\pm 1, \pm 2\}} \sum_p (\log p) \Phi\left(\frac{p}{X}\right) \omega_1\left(grk^2\sqrt{\frac{\pi}{p}}\right)\left(\frac{\gamma mr}{p}\right).
$$

Since m and r are odd and square-free and $(m,r)=1$, it follows that mr is odd and square-free. Hence, for each $\gamma \in \{1, -1, 2, -2\}$, the integer γmr is square-free. Therefore $\gamma mr \equiv 1$, 2, or 3 $\pmod 4$. If $\gamma mr \equiv 1 \pmod 4$, then $\left(\frac{\gamma mr}{\cdot}\right)$ is a real primitive character modulo $|\gamma mr|$, while if $\gamma mr \equiv 2$ or 3 $\pmod 4$, then $\left(\frac{4\gamma mr}{\cdot}\right)$ is a real primitive character modulo $|4\gamma mr|$ (see [8, Theorem 2.2.15]). Moreover, for p odd, $\left(\frac{4\gamma mr}{p}\right) = \left(\frac{\gamma mr}{p}\right)$. Therefore the sum in (6.2.3) is equal to

(6.2.4)
$$
\frac{1}{4} \sum_{\gamma \in \{\pm 1, \pm 2\}} \sum_p (\log p) \Phi\left(\frac{p}{X}\right) \omega_1\left(grk^2\sqrt{\frac{\pi}{p}}\right) \chi_{\gamma mr}(p),
$$

where $\chi_{\gamma mr}(\cdot) = \left(\frac{\gamma mr}{\cdot}\right)$ if $\gamma mr \equiv 1 \pmod 4$, and $\chi_{\gamma mr}(\cdot) = \left(\frac{4\gamma mr}{\cdot}\right)$ if $\gamma mr \equiv 2$ or 3 $\pmod 4$, so that $\chi_{\gamma mr}(\cdot)$ is a real primitive character for all the relevant γ, m, r. Also, since $mr > 1$, we see that γmr is never 1, so each $\chi_{\gamma mr}$ is nonprincipal.

We insert the definition of ω_1 into (6.2.4) in order to facilitate a separation of variables. Recalling (6.2.2) and (6.2.3), we interchange the order of summation and integration to obtain

$$
S_1^{\neq} = O(1) + \frac{2}{\left(1 - \frac{1}{\sqrt{2}}\right)^2} \sum_{g \text{ odd}} \frac{\mu^2(g)}{g} \sum_{\substack{m \leq M/g \\ (m,2g)=1}} \frac{b_{mg}}{\sqrt{m}} \sum_{\substack{r \leq X^{1/2+\varepsilon} \\ (r,2g)=1 \\ (m,r)=1 \\ mr>1}} \frac{\mu^2(r)}{\sqrt{r}} \frac{1}{4} \sum_{\gamma \in \{\pm 1, \pm 2\}} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{k}
$$

$$
\times \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma\left(\frac{s}{2} + \frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \left(1 - \frac{1}{2^{\frac{1}{2}-s}}\right) \pi^{-s/2} \left(grk^2\right)^{-s} \sum_p (\log p) \Phi\left(\frac{p}{X}\right) \chi_{\gamma mr}(p) p^{s/2} \frac{ds}{s}.
$$

We choose $c = \frac{1}{\log X}$, so that $p^{s/2}$ is bounded in absolute value. We can put the summation on k inside of the integral, where it becomes a zeta factor, and we obtain

$$S_1^{\neq} = O(1) + \frac{2}{\left(1 - \frac{1}{\sqrt{2}}\right)^2} \sum_{g \text{ odd}} \frac{\mu^2(g)}{g} \sum_{\substack{m \leq M/g \\ (m,2g)=1}} \frac{b_{mg}}{\sqrt{m}} \sum_{\substack{r \leq X^{1/2+\varepsilon} \\ (r,2g)=1 \\ (m,r)=1 \\ mr>1}} \frac{\mu^2(r)}{\sqrt{r}} \frac{1}{4} \sum_{\gamma \in \{\pm 1, \pm 2\}} \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(\frac{s}{2} + \frac{1}{4})}{\Gamma(\frac{1}{4})}$$

$$\times \left(1 - \frac{1}{2^{\frac{1}{2}-s}}\right) \left(1 - \frac{1}{2^{1+2s}}\right) \zeta(1+2s) \pi^{-s/2} (gr)^{-s} \sum_p (\log p) \Phi\left(\frac{p}{X}\right) \chi_{\gamma mr}(p) p^{s/2} \frac{ds}{s}.$$

It is more convenient to replace the $\log p$ factor with the von Mangoldt function $\Lambda(n)$. By trivial estimation we have

$$\sum_p (\log p) \Phi\left(\frac{p}{X}\right) \chi_{\gamma mr}(p) p^{s/2} = \sum_n \Lambda(n) \Phi\left(\frac{n}{X}\right) \chi_{\gamma mr}(n) n^{s/2} + O(X^{1/2}).$$

When we sum the error term over m, g, r and integrate over s, the total contribution is $O(X^{1-\varepsilon})$, provided $\varepsilon = \varepsilon(\theta) > 0$ is sufficiently small. By the rapid decay of the Γ function in vertical strips we can truncate the integral to $|\text{Im}(s)| \leq (\log X)^2$, at the cost of a negligible error. We therefore obtain

$$S_1^{\neq} = O(X^{1-\varepsilon}) + \frac{2}{\left(1 - \frac{1}{\sqrt{2}}\right)^2} \sum_{g \text{ odd}} \frac{\mu^2(g)}{g} \sum_{\substack{m \leq M/g \\ (m,2g)=1}} \frac{b_{mg}}{\sqrt{m}} \sum_{\substack{r \leq X^{1/2+\varepsilon} \\ (r,2g)=1 \\ (m,r)=1 \\ mr>1}} \frac{\mu^2(r)}{\sqrt{r}} \frac{1}{4} \sum_{\gamma \in \{\pm 1, \pm 2\}}$$

(6.2.5)

$$\times \frac{1}{2\pi i} \int_{\frac{1}{\log X} - i(\log X)^2}^{\frac{1}{\log X} + i(\log X)^2} \frac{\Gamma(\frac{s}{2} + \frac{1}{4})}{\Gamma(\frac{1}{4})} \left(1 - \frac{1}{2^{\frac{1}{2}-s}}\right) \left(1 - \frac{1}{2^{1+2s}}\right) \zeta(1+2s)$$

$$\times \left(\frac{X}{\pi}\right)^{s/2} (gr)^{-s} \sum_n \Lambda(n) \Phi_s\left(\frac{n}{X}\right) \chi_{\gamma mr}(n) \frac{ds}{s}.$$

Having arrived at (6.2.5), we are finished with the preparatory technical manipulations. We proceed to show that $S_1^{\neq}$ is small. As discussed in Section 3, we apply three different arguments, depending on the size of mr. We call these ranges Regimes I, II, and III, which correspond to small, medium, and large values of mr. In Regime I we have $1 < mr \ll \exp(\varpi\sqrt{\log x})$, where $\varpi > 0$ is a sufficiently small, fixed constant. Regime II corresponds to $\exp(\varpi\sqrt{\log x}) \ll mr \ll X^{\frac{1}{10}}$, and Regime III corresponds to $X^{\frac{1}{10}} \ll mr \ll MX^{\frac{1}{2}+\varepsilon}$. We then write

(6.2.6) $$S_1^{\neq} = E_1 + E_2,$$

where E_1 contains those terms with $mr \ll \exp(\varpi\sqrt{\log x})$, and E_2 contains those terms with $mr \gg \exp(\varpi\sqrt{\log x})$. We claim the bounds

(6.2.7)
$$E_1 \ll \frac{X}{(\log X)^{1-\varepsilon}},$$

$$E_2 \ll X \exp(-c\varpi\sqrt{\log x}),$$

where $c > 0$ is some absolute constant. Taking together (6.2.6) and (6.2.7) clearly gives (6.2.1), and this yields Proposition 6.1. It therefore suffices to show (6.2.7).

6.3. **Regime I.** We first bound E_1, which is precisely the contribution of Regime I. By definition, we have

$$E_1 := \frac{2}{\left(1 - \frac{1}{\sqrt{2}}\right)^2} \sum_{g \text{ odd}} \frac{\mu^2(g)}{g} \sum_{\substack{m \leq M/g \\ (m,2g)=1}} \frac{b_{mg}}{\sqrt{m}} \sum_{\substack{r \leq X^{1/2+\varepsilon} \\ (r,2g)=1 \\ (m,r)=1 \\ 1 < mr \ll \exp(\varpi\sqrt{\log x})}} \frac{\mu^2(r)}{\sqrt{r}} \frac{1}{4} \sum_{\gamma \in \{\pm 1, \pm 2\}}$$

(6.3.1)

$$\times \frac{1}{2\pi i} \int_{\frac{1}{\log X} - i(\log X)^2}^{\frac{1}{\log X} + i(\log X)^2} \frac{\Gamma(\frac{s}{2} + \frac{1}{4})}{\Gamma(\frac{1}{4})} \left(1 - \frac{1}{2^{\frac{1}{2}-s}}\right) \left(1 - \frac{1}{2^{1+2s}}\right) \zeta(1 + 2s)$$

$$\times \left(\frac{X}{\pi}\right)^{s/2} (gr)^{-s} \sum_n \Lambda(n) \Phi_s\left(\frac{n}{X}\right) \chi_{\gamma mr}(n) \frac{ds}{s}.$$

We transform the sum on n with partial summation to obtain

(6.3.2)
$$\sum_n \Lambda(n) \Phi_s\left(\frac{n}{X}\right) \chi_{\gamma mr}(n) = -\int_0^\infty \frac{1}{X} \Phi_s'\left(\frac{w}{X}\right) \left(\sum_{n \leq w} \Lambda(n) \chi_{bmr}(n)\right) dw.$$

By [11, equation (8) of Chapter 20], we have

(6.3.3)
$$\sum_{n \leq w} \Lambda(n) \chi_{\gamma mr}(n) = -\frac{w^{\beta_1}}{\beta_1} + O\left(w \exp(-c_1\sqrt{\log w})\right),$$

where $c_1 > 0$ is some absolute constant, and the term $-w^{\beta_1}/\beta_1$ only appears if $L(s, \chi_{\gamma mr})$ has a real zero β_1 which satisfies $\beta_1 > 1 - \frac{c_2}{\log|\gamma mr|}$ for some sufficiently small constant $c_2 > 0$. All the constants in (6.3.3), implied or otherwise, are effective.

The contribution from the error term in (6.3.3) is easy to control. Observe that

(6.3.4)
$$\int_0^\infty \frac{1}{X} \left|\Phi_s'\left(\frac{w}{X}\right)\right| dw = \int_0^\infty |\Phi_s'(u)| \, du \ll |s| + 1,$$

uniformly in s with $\mathrm{Re}(s)$ bounded. Taking (6.3.1),(6.3.2) and (6.3.4) together, we see the error term of (6.3.3) contributes

(6.3.5)
$$\ll X \exp(c_3(\varpi - c_1)\sqrt{\log X})$$

to E_1, where $c_3 > 0$ is some absolute constant. The bound (6.3.5) is more than adequate for (6.2.7) provided we choose $\varpi > 0$ sufficiently small in terms of c_1.

The conductor of the primitive character $\chi_{\gamma mr}$ is $\ll \exp(\varpi\sqrt{\log X}) \leq \exp(2\varpi\sqrt{\log X})$. We apply Page's theorem [11, equation (9) of Chapter 14], which implies that, for some fixed absolute constant $c_4 > 0$, there is at most one real primitive character $\chi_{\gamma mr}$ with modulus $\leq \exp(2\varpi\sqrt{\log X})$ for which the L-function $L(s, \chi_{\gamma mr})$ has a real zero satisfying

(6.3.6)
$$\beta_1 > 1 - \frac{c_4}{2\varpi\sqrt{\log X}}.$$

To estimate the contribution of the possible term $-\frac{w^{\beta_1}}{\beta_1}$, we evaluate the integral

$$\int_0^\infty \frac{w^{\beta_1}}{\beta_1} \frac{1}{X} \Phi_s'\left(\frac{w}{X}\right) dw$$

arising from (6.3.2) and (6.3.3). We make the change of variable $\frac{w}{X} \mapsto u$ and integrate by parts to see that this integral equals

$$X^{\beta_1} \int_0^\infty \frac{u^{\beta_1}}{\beta_1} \Phi'_s(u)\, du = -X^{\beta_1} \int_0^\infty \Phi_s(u) u^{\beta_1 - 1}\, du = -X^{\beta_1} \Phi^\dagger \left(\frac{s}{2} + \beta_1 \right).$$

We assume that a real zero satisfying (6.3.6) does exist, for otherwise we already have an acceptable bound for E_1. Let q^* denote the conductor of the exceptional character $\chi_{\gamma mr}$ for which the real zero β_1 satisfying (6.3.6) exists. Then we have

(6.3.7)

$$E_1 = -\frac{1}{2\pi i} \frac{\sqrt{\gamma^*}}{2 \left(1 - \frac{1}{\sqrt{2}} \right)^2} \frac{X^{\beta_1}}{\sqrt{|q^*|}} \int_{\frac{1}{\log X} - i(\log X)^2}^{\frac{1}{\log X} + i(\log X)^2} \frac{\Gamma(\frac{s}{2} + \frac{1}{4})}{\Gamma(\frac{1}{4})} \left(1 - \frac{1}{2^{\frac{1}{2} - s}} \right) \left(1 - \frac{1}{2^{1+2s}} \right)$$

$$\times \left(\frac{X}{\pi} \right)^{s/2} \zeta(1+2s) \Phi^\dagger \left(\frac{s}{2} + \beta_1 \right) \sum_{\substack{1 < mr \ll \exp(\varpi \sqrt{\log X}) \\ (mr,2)=1 \\ (m,r)=1 \\ \gamma mr = q^*}} \frac{\mu^2(r)}{r^s} \sum_{(g,2mr)=1} \frac{\mu^2(g) b_{gm}}{g^{1+s}} \frac{ds}{s}$$

$$+ O\left(X \exp(-c_5 \sqrt{\log X}) \right),$$

where $c_5 > 0$ is some constant, and γ^* is some bounded power of two.

We next write $b_{gm} = \mu(gm) H(\frac{\log gm}{\log M})$ and apply Fourier inversion as in (6.1.2),(6.1.3) to obtain

(6.3.8)

$$\sum_{(g,2mr)=1} \frac{b_{mg}}{g^{1+s}} = \mu(m) \int_{-\infty}^\infty \frac{1}{m^{\frac{1+iz}{\log M}}} h(z) \prod_{p|2mr} \left(1 - \frac{1}{p^{1+s+\frac{1+iz}{\log M}}} \right)^{-1} \zeta^{-1} \left(1 + s + \frac{1+iz}{\log M} \right) dz.$$

By (6.1.4) we can truncate the integral in (6.3.8) to $|z| \leq \sqrt{\log M}$ at the cost of an error of size $O_B(d_2(mr)(\log X)^{-B})$. This error contributes to (6.3.7)

$$\ll_B \frac{X}{(\log X)^{B+O(1)}},$$

which is acceptable. We therefore have

(6.3.9)

$$E_1 = -\frac{X^{\beta_1}}{2 \left(1 - \frac{1}{\sqrt{2}} \right)^2} \frac{\sqrt{\gamma^*}}{\sqrt{|q^*|}} \sum_{\substack{m \leq M, r \leq X^{\frac{1}{2}+\varepsilon} \\ (mr,2)=1 \\ (m,r)=1 \\ \gamma mr = q^*}} \mu(m)\mu^2(r) \frac{1}{2\pi i} \int_{\frac{1}{\log X} - i(\log X)^2}^{\frac{1}{\log X} + i(\log X)^2} \frac{\Gamma(\frac{s}{2} + \frac{1}{4})}{\Gamma(\frac{1}{4})} \left(1 - \frac{1}{2^{\frac{1}{2}-s}} \right) r^{-s}$$

$$\times \left(1 - \frac{1}{2^{1+2s}} \right) \zeta(1+2s) \left(\frac{X}{\pi} \right)^{s/2} \Phi^\dagger \left(\frac{s}{2} + \beta_1 \right) \int_{|z| \leq \sqrt{\log M}} \frac{1}{m^{\frac{1+iz}{\log M}}} h(z)$$

$$\times \prod_{p|2mr} \left(1 - \frac{1}{p^{1+s+\frac{1+iz}{\log M}}} \right)^{-1} \zeta^{-1} \left(1 + s + \frac{1+iz}{\log M} \right) dz \frac{ds}{s} + O\left(\frac{X}{\log X} \right).$$

We handle the s-integral in (6.3.9) by moving the line of integration to $\mathrm{Re}(s) = -\frac{c_6}{\log\log X}$, where $c_6 > 0$ is small enough that $\zeta(1 + s + \frac{1+iz}{\log M})$ has no zeros in the region $\mathrm{Re}(s) \geq -\frac{c_6}{\log\log X}, \mathrm{Im}(s) \leq (\log X)^2$. By moving the line of integration we pick up a contribution from the pole at $s = 0$. We write this residue as an integral around a circle of small radius centered at the origin, and thereby deduce

(6.3.10)

$$
E_1 = -\frac{X^{\beta_1}}{2\left(1 - \frac{1}{\sqrt{2}}\right)^2} \frac{\sqrt{\gamma^*}}{\sqrt{|q^*|}} \sum_{\substack{m \leq M, r \leq X^{1/2+\varepsilon} \\ (mr,2)=1 \\ (m,r)=1 \\ \gamma mr = q^*}} \mu(m)\mu^2(r) \frac{1}{2\pi i} \oint_{|s|=\frac{1}{\log X}} \frac{\Gamma(\frac{s}{2} + \frac{1}{4})}{\Gamma(\frac{1}{4})} \left(1 - \frac{1}{2^{\frac{1}{2}-s}}\right) r^{-s}
$$

$$
\times \left(1 - \frac{1}{2^{1+2s}}\right) \zeta(1 + 2s) \left(\frac{X}{\pi}\right)^{s/2} \Phi^\dagger\left(\frac{s}{2} + \beta_1\right) \int_{|z| \leq \sqrt{\log M}} \frac{1}{m^{\frac{1+iz}{\log M}}} h(z)
$$

$$
\times \prod_{p \mid 2mr} \left(1 - \frac{1}{p^{1+s+\frac{1+iz}{\log M}}}\right)^{-1} \zeta^{-1}\left(1 + s + \frac{1+iz}{\log M}\right) dz \frac{ds}{s} + O\left(\frac{X}{\log X}\right).
$$

We have the bound

(6.3.11)
$$
\beta_1 < 1 - \frac{c_7}{\sqrt{|q^*|}(\log|q^*|)^2},
$$

where $c_7 > 0$ is a fixed absolute constant (see [11, equation (12) of Chapter 14]). If q^* satisfies $|q^*| \leq (\log X)^{2-\varepsilon}$ then by (6.3.11) we derive

$$
X^{\beta_1} \ll X \exp(-c_7 (\log X)^{\varepsilon/3}).
$$

By estimating (6.3.10) trivially we then obtain

$$
E_1 \ll X \exp(-c_7 (\log X)^{\varepsilon/4}),
$$

which is an acceptable bound. We may therefore assume that q^* satisfies

(6.3.12)
$$
|q^*| > (\log X)^{2-\varepsilon}.
$$

For $|s| = \frac{1}{\log X}$ we have the bounds

$$
\zeta(1 + 2s) \ll \log X, \qquad \zeta^{-1}\left(1 + s + \frac{1+iz}{\log M}\right) \ll \frac{1+|z|}{\log X}.
$$

Using these bounds and (6.3.12) we deduce by trivial estimation that

$$
(6.3.10) \ll \frac{X}{|q^*|^{1/2-o(1)}} \ll \frac{X}{(\log X)^{1-\varepsilon}}.
$$

This completes the proof of the bound for E_1 in (6.2.7).

6.4. **Regime II.** It remains to prove the bound for E_2 in (6.2.7). From (6.2.5) and (6.2.6) we see that E_2 is the contribution from those m and r in Regimes II and III. The estimates in regimes II and III are less delicate than those in regime I, and consequently the arguments are easier.

In (6.2.5) we write $q = \gamma m r$. After breaking q into dyadic segments we find

$$E_2 \ll (\log X)^{O(1)} \sum_{\substack{Q=2^j \\ Q \gg \exp(\varpi \sqrt{\log X}) \\ Q \ll MX^{1/2+\varepsilon}}} \mathcal{E}(Q),$$

where

$$\mathcal{E}(Q) := Q^{-\frac{1}{2}+\varepsilon} \sum_{\chi \in S(Q)} \left| \sum_n \Lambda(n) \Phi_{s_0}\left(\frac{n}{X}\right) \chi(n) \right|.$$

Here s_0 is some complex number with $\operatorname{Re}(s_0) = \frac{1}{\log X}$ and $|\operatorname{Im}(s_0)| \le (\log X)^2$. In order to prove (6.2.7) it therefore suffices to show that

(6.4.1) $$\mathcal{E}(Q) \ll X \exp(-c_8 \varpi \sqrt{\log X})$$

for each Q satisfying $\exp(\varpi\sqrt{\log X}) \ll Q \ll MX^{\frac{1}{2}+\varepsilon}$. In this subsection we treat the Q belonging to Regime II, that is, those Q which satisfy $Q \ll X^{\frac{1}{10}}$. In the next subsection we treat the Q in Regime III, which satisfy $Q \gg X^{\frac{1}{10}}$.

In Regime II we employ zero-density estimates. We begin by writing Φ_{s_0} as the integral of its Mellin transform, yielding

$$\sum_n \Lambda(n) \, \Phi_{s_0}\left(\frac{n}{X}\right) \chi(n) = \frac{1}{2\pi i} \int_{(2)} X^w \Phi^\dagger\left(w + \frac{s_0}{2}\right)\left(-\frac{L'}{L}(w, \chi)\right) dw.$$

Observe that from repeated integration by parts we have

(6.4.2) $$\left| \Phi^\dagger(\sigma + it + \frac{s_0}{2}) \right| \ll_{\sigma,j} (\log X)^j \left(1 + \left| t - \frac{\operatorname{Im}(s_0)}{2} \right| \right)^{-j}$$

for every non-negative integer j.

We shift the line of integration to $\operatorname{Re}(w) = -\frac{1}{2}$, picking up residues from all of the zeros in the critical strip. On the line $\operatorname{Re}(w) = -\frac{1}{2}$ we have the bound

$$\left| \frac{L'}{L}(w, \chi) \right| \ll \log(q|w|),$$

and this yields

$$\sum_n \Lambda(n) \, \Phi_{s_0}\left(\frac{n}{X}\right) \chi(n) = \sum_{\substack{L(\rho,\chi)=0 \\ 0 \le \beta \le 1}} X^\rho \Phi^\dagger\left(\rho + \frac{s_0}{2}\right) + O\left(\frac{(\log X)^{O(1)}}{X^{1/2}}\right).$$

We have written here $\rho = \beta + i\gamma$. The error term is, of course, completely acceptable for (6.4.1) when summed over $q \ll Q$.

By (6.4.2), the contribution to $\mathcal{E}(Q)$ from those ρ with $|\gamma| > Q^{1/2}$ is

$$\ll XQ^{-100},$$

say, and this gives an acceptable bound. We have therefore obtained

(6.4.3) $$\mathcal{E}(Q) \ll X \exp(-\varpi\sqrt{\log X}) + Q^{-\frac{1}{2}+\varepsilon} \sum_{\chi \in S(Q)} \sum_{\substack{L(\rho,\chi)=0 \\ 0 \le \beta \le 1 \\ |\gamma| \le Q^{1/2}}} X^\beta.$$

In order to bound the right side of (6.4.3), we first need to introduce some notation. For a primitive Dirichlet character χ modulo q, let $N(T, \chi)$ denote the number of zeros of $L(s, \chi)$ in the rectangle

$$0 \leq \beta \leq 1, \qquad |\gamma| \leq T.$$

For $T \geq 2$, say, we have [11, Chapter 16]

$$(6.4.4) \qquad\qquad N(T, \chi) \ll T \log(qT).$$

For $\frac{1}{2} \leq \alpha \leq 1$, define $N(\alpha, T, \chi)$ to be the number of zeros $\rho = \beta + i\gamma$ of $L(s, \chi)$ in the rectangle

$$\alpha \leq \beta \leq 1, \qquad |\gamma| \leq T,$$

and define

$$N(\alpha, Q, T) = \sum_{q \leq Q} \sideset{}{^*}\sum_{\chi (\mathrm{mod}\ q)} N(\alpha, T, \chi).$$

In $N(\alpha, Q, T)$ the summation on χ is over primitive characters. We employ Jutila's zero-density estimate [20, (1.7)]

$$(6.4.5) \qquad\qquad N(\alpha, Q, T) \ll (QT)^{4(1-\alpha)+\varepsilon},$$

which holds for $\alpha \geq \frac{4}{5}$.

In (6.4.3), we separate the zeros ρ according to whether $\beta < \frac{4}{5}$ or $\beta \geq \frac{4}{5}$. Using (6.4.4) we deduce

$$(6.4.6) \qquad\qquad Q^{-\frac{1}{2}+\varepsilon} \sum_{\chi \in S(Q)} \sum_{\substack{L(\rho,\chi)=0 \\ 0 \leq \beta < 4/5 \\ |\gamma| \leq Q^{1/2}}} X^\beta \ll X^{\frac{4}{5}} Q^{1+\varepsilon}.$$

For those zeros with $\beta \geq \frac{4}{5}$ we write

$$X^\beta = X^{4/5} + (\log X) \int_{4/5}^\beta X^\alpha d\alpha.$$

We then embed $S(Q)$ into the set of all primitive characters with conductors $\leq Q$. Applying (6.4.6) and (6.4.5), we obtain

$$\sum_{\chi \in S(Q)} \sum_{\substack{L(\rho,\chi_q)=0 \\ 4/5 \leq \beta \leq 1 \\ |\gamma| \leq Q^{\frac{1}{2}}}} X^\beta \ll X^{\frac{4}{5}} Q^{1+\varepsilon} + (\log X) \int_{4/5}^1 X^\alpha N(\alpha, Q, Q^{\frac{1}{2}}) d\alpha$$

$$\ll X^{\frac{4}{5}} Q^{1+\varepsilon} + Q^\varepsilon \int_{4/5}^1 X^\alpha Q^{6(1-\alpha)} d\alpha.$$

Since $Q \ll X^{\frac{1}{10}}$ the integrand of this latter integral is maximized when $\alpha = 1$. It follows that

$$(6.4.7) \qquad\qquad Q^{-\frac{1}{2}+\varepsilon} \sum_{\chi \in S(Q)} \sum_{\substack{L(\rho,\chi_q)=0 \\ 4/5 \leq \beta \leq 1 \\ |\gamma| \leq Q^{\frac{1}{2}}}} X^\beta \ll X^{\frac{4}{5}} Q^{1+\varepsilon} + XQ^{-\frac{1}{2}+\varepsilon} \ll XQ^{-\frac{1}{2}+\varepsilon}.$$

Combining (6.4.7) and (6.4.6) yields

$$\mathcal{E}(Q) \ll XQ^{-\frac{1}{2}+\varepsilon},$$

and this suffices for (6.4.1).

6.5. **Regime III.** In Regime III we have $X^{\frac{1}{10}} \ll Q \ll MX^{\frac{1}{2}+\varepsilon} = X^{\frac{1}{2}+\theta+\varepsilon}$ (recall (3.4)). Here we depart from the philosophy of the previous two regimes, in that we do not bound $\mathcal{E}(Q)$ by considerations of zeros of L-functions. Rather, we exploit the combinatorial structure of the von Mangoldt function and Lemma 4.4.

We observe that in Regime III one may still proceed with zero-density estimates by appealing to Heath-Brown's zero-density estimate for L-functions of quadratic characters [15, Theorem 3]. We present our method for the sake of variety, and because it might prove useful in other contexts.

Let us move to our treatment of $\mathcal{E}(Q)$ for these large Q. Given an arithmetic function $f : \mathbb{N} \to \mathbb{C}$ and a real number $W > 1$, let $f_{\leq W}(n)$ denote the arithmetic function

$$f_{\leq W}(n) = \begin{cases} f(n), & n \leq W, \\ 0, & n > W. \end{cases}$$

We write $f_{>W}(n) = f(n) - f_{\leq W}(n)$.

We write $\star$ for Dirichlet convolution. Our starting place is Vaughan's identity [17, Proposition 13.4]. Given a parameter $V > 1$, we have

$$(6.5.1) \qquad \Lambda(n) = \Lambda_{\leq V}(n) + (\mu_{\leq V} \star \log)(n) - (\mu_{\leq V} \star \Lambda_{\leq V} \star 1)(n) + (\mu_{>V} \star \Lambda_{>V} \star 1)(n).$$

We apply (6.5.1) for $n \asymp X$, and we set $V := X^{\frac{1}{3}(\frac{1}{2}-\theta)}$. This reduces the estimation of $\mathcal{E}(Q)$ to the estimation of three different sums, say $\mathcal{E}_i(Q)$, for $i \in \{1, 2, 3\}$. Observe that there are four terms on the right side of (6.5.1), but $\Lambda_{\leq V}(n)$ is identically zero for $n \asymp X$.

We have

$$\mathcal{E}_1(Q) := Q^{-\frac{1}{2}+\varepsilon} \sum_{\chi \in S(Q)} \left| \sum_n (\mu_{\leq V} \star \log)(n) \Phi_{s_0}\left(\frac{n}{X}\right) \chi(n) \right|$$

$$\ll Q^{-\frac{1}{2}+\varepsilon} \sum_{\chi \in S(Q)} \sum_{v \leq V} \mu^2(v) \left| \sum_m (\log m) \Phi_{s_0}\left(\frac{mv}{X}\right) \chi(m) \right|.$$

By partial summation and the Pólya-Vinogradov inequality, we find that

$$(6.5.2) \qquad \mathcal{E}_1(Q) \ll Q^{1+\varepsilon}V \ll X^{\frac{1}{2}+\theta+\frac{1}{3}(\frac{1}{2}-\theta)+\varepsilon} \ll X^{1-\varepsilon},$$

the last inequality holding for $\varepsilon = \varepsilon(\theta) > 0$ sufficiently small.

The estimation of $\mathcal{E}_2(Q)$ is entirely similar, and we obtain

$$(6.5.3) \qquad \mathcal{E}_2(Q) := Q^{-\frac{1}{2}+\varepsilon} \sum_{\chi \in S(Q)} \left| \sum_n (\mu_{\leq V} \star \Lambda_{\leq V} \star 1)(n) \Phi_{s_0}\left(\frac{n}{X}\right) \chi(n) \right|$$

$$\ll Q^{1+\varepsilon}V^2 \ll X^{\frac{1}{2}+\theta+\frac{2}{3}(\frac{1}{2}-\theta)+\varepsilon} \ll X^{1-\varepsilon}.$$

The last sum to estimate is $\mathcal{E}_3(Q)$:

$$\mathcal{E}_3(Q) := Q^{-\frac{1}{2}+\varepsilon} \sum_{\chi \in S(Q)} \left| \sum_n (\mu_{>V} \star \Lambda_{>V} \star 1)(n) \Phi_{s_0}\left(\frac{n}{X}\right) \chi(n) \right|$$

$$= Q^{-\frac{1}{2}+\varepsilon} \sum_{\chi \in S(Q)} \left| \sum_k \sum_\ell \alpha(k)\beta(\ell) \Phi_{s_0}\left(\frac{k\ell}{X}\right) \chi(k\ell) \right|,$$

where $\alpha(k) = \mu_{>V}(k)$ and $\beta(\ell) = (\Lambda_{>V} \star 1)(\ell)$. Observe that both $\alpha(\cdot)$ and $\beta(\cdot)$ are supported on integers m satisfying

$$V \ll m \ll XV^{-1}.$$

We further observe that $|\alpha(k)| \le 1, |\beta(\ell)| \le \log(\ell)$. We perform dyadic decompositions on the ranges of k and ℓ, so that $k \asymp K, \ell \asymp L$, with

$$V \ll K \ll XV^{-1}, \qquad V \ll L \ll XV^{-1},$$

and $KL \asymp X$.

We next separate the variables by Mellin inversion on Φ_{s_0}:

$$\mathcal{E}_1(Q) \ll (\log X)^{O(1)} \sup_{K,L} \int_{(0)} \left| \Phi_{s_0}^\dagger \left(w + \frac{s_0}{2}\right) \right| Q^{-\frac{1}{2}+\varepsilon} \sum_{\chi \in S(Q)} \left| \sum_{\substack{k \asymp K \\ \ell \asymp L}} \sum \alpha(k)\beta(\ell)(k\ell)^{-w} \chi(k\ell) \right| \, dw.$$

The integral of $|\Phi_{s_0}^\dagger|$ has size $\ll (\log X)^{O(1)}$, so we obtain

$$\mathcal{E}_3(Q) \ll \sup_{K,L} Q^{-\frac{1}{2}+\varepsilon} \sum_{\chi \in S(Q)} \left| \sum_{\substack{k \asymp K \\ \ell \asymp L}} \sum \tilde\alpha(k)\tilde\beta(\ell)\chi(k\ell) \right|,$$

where $\tilde\alpha, \tilde\beta$ are complex sequences with $|\tilde\alpha(k)| = |\alpha(k)|, |\tilde\beta(\ell)| = |\beta(\ell)|$ for all k, ℓ.

By multiplicativity and Cauchy-Schwarz we obtain

$$\mathcal{E}_3(Q) \ll \sup_{K,L} Q^{-\frac{1}{2}+\varepsilon} \left(\sum_{\chi \in S(Q)} \left| \sum_{k \asymp K} \tilde\alpha(k)\chi(k) \right|^2 \right)^{\frac{1}{2}} \left(\sum_{\chi \in S(Q)} \left| \sum_{\ell \asymp L} \tilde\beta(\ell)\chi(\ell) \right|^2 \right)^{\frac{1}{2}}.$$

Applying Lemma 4.4 yields

$$\mathcal{E}_3(Q) \ll \sup_{K,L} \frac{X^\varepsilon}{Q^{\frac{1}{2}}} ((Q+K)K)^{\frac{1}{2}} ((Q+L)L)^{\frac{1}{2}}$$

(6.5.4)
$$\ll \sup_{K,L} X^\varepsilon \left(X^{\frac{1}{2}} Q^{\frac{1}{2}} + \frac{KL}{K^{\frac{1}{2}}} + \frac{KL}{L^{\frac{1}{2}}} + \frac{KL}{Q^{\frac{1}{2}}} \right)$$

$$\ll X^\varepsilon \left(X^{\frac{3}{4}+\frac{\theta}{2}} + \frac{X}{V^{\frac{1}{2}}} + \frac{X}{Q^{\frac{1}{2}}} \right) \ll X^{1-\varepsilon}.$$

The last inequality follows since $V = X^{\frac{1}{3}(\frac{1}{2}-\theta)}$ and $Q \gg X^{\frac{1}{10}}$. Then (6.5.2), (6.5.3), and (6.5.4) imply

$$\mathcal{E}(Q) \ll X^{1-\varepsilon},$$

and this suffices for (6.4.1).

6.6. Dénouement. We can extract from our proof of Proposition 6.1 the following result on character sums over primes, which we shall have occasion to use later.

Lemma 6.1. *Let X be a large real number, and let $\delta > 0$ be small and fixed. Let s_0 be a complex number with $|\mathrm{Re}(s_0)| \leq \frac{A_1}{\log X}$ and $|\mathrm{Im}(s_0)| \leq (\log X)^{A_2}$, for some positive real numbers A_1 and A_2. Given any positive real numbers A_3, A_4, and B, we have*

$$\sum_{\substack{q \leq X^{1-\delta} \\ q \text{ odd} \\ q \neq \square}} \frac{\tau(q)^{A_1}(\log q)^{A_2}}{\sqrt{q}} \left| \sum_{p \equiv 1 \,(\mathrm{mod}\, 8)} (\log p)\Phi_{s_0}\left(\frac{p}{X}\right)\left(\frac{q}{p}\right) \right| \ll_{A_1,A_2,A_3,A_4,B,\delta} \frac{X}{(\log X)^B}.$$

The implied constant is ineffective.

Proof. Follow the proof of (6.2.7), but instead use the lower bound $q^* > c(D)(\log X)^D$, which holds for arbitrary $D > 0$. The constant $c(D)$ is ineffective if $D \geq 2$. $\qquad\square$

Lemma 6.1 is quite strong since it corresponds, roughly, to square root cancellation on average in the sums over p. Thus, one would not expect to be able to prove an analogue of Lemma 6.1 with the upper bound for q replaced by $X^{1+\varepsilon}$ for any $\varepsilon > 0$.

7. The mollified second moment

In this section we derive an upper bound of the correct order of magnitude for the sum S_2 defined in (3.1). Our main result for this section is the following (recall (3.4) and (5.2)).

Proposition 7.1. *Let $\delta > 0$ be small and fixed, and let θ, ϑ satisfy $\theta + 2\vartheta < \frac{1}{2}$. If $X \geq X_0(\delta, \theta, \vartheta)$, then*

$$S_2 \leq \frac{1+\delta}{2(1-\frac{1}{\sqrt{2}})^2}\frac{\mathfrak{I}}{\vartheta}\frac{X}{4},$$

where

$$\mathfrak{I} = -2\int_0^1 H(x)H'(x)dx + \frac{1}{\theta}\int_0^1 H(x)H''(x)dx + \frac{1}{\theta}\int_0^1 H'(x)^2dx$$

$$- \frac{1}{2\theta^2}\int_0^1 H'(x)H''(x)dx + \frac{1}{24\theta^3}\int_0^1 H''(x)^2dx.$$

The proof of Proposition 7.1 follows the ideas outlined in Section 3. First, we note that $\log p \leq \log X$ in (3.1) because Φ is supported on $[\frac{1}{2}, 1]$. By positivity we may apply the upper bound sieve condition (5.7) to write

$$S_2 \leq (\log X)S^+,$$

where S^+ is defined by

$$(7.1) \qquad S^+ = \sum_{n \equiv 1 \,(\mathrm{mod}\, 8)} \mu^2(n)\left(\sum_{\substack{d|n \\ d \leq D}}\lambda_d\right)\Phi\left(\frac{n}{X}\right)L(\tfrac{1}{2}, \chi_n)^2 M(n)^2.$$

Note that d is odd since $d \mid n$ and $n \equiv 1 \pmod 8$. Also, $\lambda_d \neq 0$ only for square-free d by the definition (5.8), and so $\lambda_d = \mu^2(d)\lambda_d$. We use Lemma 4.2 to write $L(\tfrac{1}{2}, \chi_n)^2 = \mathcal{D}_2(n)$, then insert (3.6) into (7.1) to write

$$(7.2) \qquad S^+ = S_N^+ + S_R^+,$$

where

$$(7.3) \qquad S_N^+ = \sum_{n \equiv 1 \,(\mathrm{mod}\ 8)} N_Y(n) \left(\sum_{\substack{d|n \\ d \leq D}} \mu^2(d) \lambda_d \right) \Phi\left(\frac{n}{X}\right) \mathcal{D}_2(n) M(n)^2$$

and

$$S_R^+ = \sum_{n \equiv 1 \,(\mathrm{mod}\ 8)} R_Y(n) \left(\sum_{\substack{d|n \\ d \leq D}} \mu^2(d) \lambda_d \right) \Phi\left(\frac{n}{X}\right) \mathcal{D}_2(n) M(n)^2$$

We first obtain a bound on S_R^+. The remainder of this section will then be devoted to an analysis of S_N^+.

7.1. **The contribution of S_R^+.** In this subsection we show

$$(7.1.1) \qquad S_R^+ \ll X^\varepsilon \left(\frac{X}{Y} + X^{1/2} M \right).$$

The arguments here are almost identical to those in [39, Section 3]. Observe that $R_Y(n) = 0$ unless $n = \ell^2 h$ with $\ell > Y$ and h square-free. If $n \equiv 1 \,(\mathrm{mod}\ 8)$ then ℓ and h are odd and $h \equiv 1 \,(\mathrm{mod}\ 8)$. By the divisor bound we have

$$|R_Y(n)| \ll n^\varepsilon, \qquad \left| \sum_{\substack{d|n \\ d \leq D}} \mu^2(d) \lambda_d \right| \ll n^\varepsilon,$$

and therefore

$$S_R^+ \ll X^\varepsilon \sum_{\substack{Y < \ell \leq \sqrt{X} \\ 2 \nmid \ell}} \sum_{\substack{X/2\ell^2 < h \leq X/\ell^2 \\ h \equiv 1 \,(\mathrm{mod}\ 8)}} \mu^2(h) |M(\ell^2 h)^2 \mathcal{D}_2(\ell^2 h)|.$$

There is a mild complication compared to [39] in that it is possible to have $h = 1$, in which case the character χ_h is principal.

We apply Cauchy-Schwarz and obtain

$$(7.1.2)$$

$$S_R^+ \ll X^\varepsilon \sum_{\substack{Y < \ell \leq \sqrt{X} \\ 2 \nmid \ell}} \left(\sum_{\substack{X/2\ell^2 < h \leq X/\ell^2 \\ h \equiv 1 \,(\mathrm{mod}\ 8)}} \mu^2(h) |M(\ell^2 h)^2|^2 \right)^{1/2} \left(\sum_{\substack{X/2\ell^2 < h \leq X/\ell^2 \\ h \equiv 1 \,(\mathrm{mod}\ 8)}} \mu^2(h) |\mathcal{D}_2(\ell^2 h)|^2 \right)^{1/2}.$$

We have

$$M(\ell^2 h)^2 = \sum_{\substack{m \leq M^2 \\ (m,2\ell)=1}} \frac{\alpha(m)}{\sqrt{m}} \left(\frac{h}{m}\right)$$

for some coefficients $\alpha(m)$ satisfying $|\alpha(m)| \ll m^\varepsilon$. For $h = 1$ we use the trivial bound $M(\ell^2)^4 \ll M^2 X^\varepsilon$. For $h > 1$ we use Lemma 4.4. We therefore have

$$(7.1.3) \qquad \sum_{\substack{X/2\ell^2 < h \leq X/\ell^2 \\ h \equiv 1 \,(\mathrm{mod}\ 8)}} \mu^2(h) |M(\ell^2 h)^2|^2 \ll X^\varepsilon \left(\frac{X}{\ell^2} + M^2 \right).$$

Now observe that, for any $c > \frac{1}{2}$,

$$\mathcal{D}_2(\ell^2 h) = \frac{2}{(1 - \frac{1}{\sqrt{2}})^4} \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma^2\left(\frac{s}{2} + \frac{1}{4}\right)}{\Gamma^2\left(\frac{1}{4}\right)} \left(1 - \frac{1}{2^{1/2-s}}\right)^2 \left(\frac{\ell^2 h}{\pi}\right)^s L^2\left(\frac{1}{2} + s, \chi_h\right) \mathcal{E}(s, 2\ell) \frac{ds}{s},$$

where

$$\mathcal{E}(s, k) = \prod_{p | k} \left(1 - \frac{\chi_h(p)}{p^{1/2+s}}\right)^s.$$

If $h = 1$ then $L^2\left(\frac{1}{2} + s, \chi_h\right) = \zeta^2(\frac{1}{2} + s)$. In any case, we move the line of integration to $c = \frac{1}{\log X}$, and we do not pick up contributions from any poles. When $h > 1$ this is obvious, and when $h = 1$ the double pole of $\zeta^2(\frac{1}{2} + s)$ is canceled out by the double zero of $(1 - 2^{-(1/2-s)})^2$. By trivial estimation we have then $|\mathcal{D}_2(\ell^2)| \ll X^\varepsilon$. For $h > 1$ we apply Cauchy-Schwarz to obtain

$$|\mathcal{D}_2(\ell^2 h)|^2 \ll X^\varepsilon \int_{(\frac{1}{\log X})} \left|\Gamma\left(\frac{s}{2} + \frac{1}{4}\right)\right|^2 \left|L\left(\frac{1}{2} + s, \chi_h\right)\right|^4 |ds|.$$

Summing over h and using Lemma 4.5, we obtain

$$(7.1.4) \qquad \sum_{\substack{X/2\ell^2 < h \le X/\ell^2 \\ h \equiv 1 \,(\mathrm{mod}\, 8)}} \mu^2(h) |\mathcal{D}_2(\ell^2 h)|^2 \ll \frac{X^{1+\varepsilon}}{\ell^2}.$$

Combining (7.1.2), (7.1.3), and (7.1.4) yields (7.1.1).

7.2. **Poisson summation.**

We begin our evaluation of S_N^+ by inserting into (7.3) the definition (3.3) of the mollifier $M(n)$. We then use the definition of $\mathcal{D}_2$ (see Lemma 4.2) to write

$$(7.2.1) \qquad \begin{aligned} S_N^+ = \frac{8}{(\sqrt{2} - 1)^4} \sum_{\substack{d \le D \\ d \text{ odd}}} \mu^2(d) \lambda_d \sum_{\substack{m_1, m_2 \le M \\ m_1, m_2 \text{ odd}}} \frac{b_{m_1} b_{m_2}}{\sqrt{m_1 m_2}} \sum_{\substack{n \equiv 1 \,(\mathrm{mod}\, 8) \\ d | n}} N_Y(n) \Phi\left(\frac{n}{X}\right) \\ \times \sum_{\substack{\nu = 1 \\ \nu \text{ odd}}}^{\infty} \frac{d_2(\nu)}{\sqrt{\nu}} \omega_2\left(\frac{\nu \pi}{n}\right) \left(\frac{n}{m_1 m_2 \nu}\right). \end{aligned}$$

We next apply Poisson summation to evaluate the n-sum. Denote the n-sum in (7.2.1) by Z, i.e. define Z by

$$(7.2.2) \qquad Z = Z(d, \nu, m_1 m_2; X, Y) = \sum_{\substack{n \equiv 1 \,(\mathrm{mod}\, 8) \\ d | n}} N_Y(n) \Phi\left(\frac{n}{X}\right) \omega_2\left(\frac{\nu \pi}{n}\right) \left(\frac{n}{m_1 m_2 \nu}\right).$$

We insert the definition (3.7) of $N_Y(n)$ and interchange the order of summation to write Z as

$$(7.2.3) \qquad Z = \sum_{\substack{\alpha \le Y \\ \alpha \text{ odd}}} \mu(\alpha) \sum_{\substack{n \equiv 1 \,(\mathrm{mod}\, 8) \\ [\alpha^2, d] | n}} F_\nu\left(\frac{n}{X}\right) \left(\frac{n}{m_1 m_2 \nu}\right),$$

where $F_\nu(t)$ is defined by

$$(7.2.4) \qquad\qquad F_\nu(t) = \Phi(t)\omega_2\left(\frac{\nu\pi}{tX}\right).$$

If α and d are square-free, then $[\alpha^2, d] = \alpha^2 d_1$, where

$$(7.2.5) \qquad\qquad d_1 = \frac{d}{(d,\alpha)}.$$

We may thus relabel n as $\alpha^2 d_1 m$ in (7.2.3), and then split the resulting sum on m according to the congruence class of $m \pmod{m_1 m_2 \nu}$. We deduce from (7.2.3) that

$$Z = \sum_{\substack{\alpha \leq Y \\ (\alpha, 2m_1 m_2 \nu)=1}} \mu(\alpha)\left(\frac{d_1}{m_1 m_2 \nu}\right) \sum_{b \pmod{m_1 m_2 \nu}} \left(\frac{b}{m_1 m_2 \nu}\right) \sum_{\substack{m \equiv \overline{\alpha^2 d_1} \pmod 8 \\ m \equiv b \pmod{m_1 m_2 \nu}}} F_\nu\left(\frac{\alpha^2 d_1 m}{X}\right).$$

By the Chinese Remainder Theorem, we may write the congruence conditions on m as a single condition $m \equiv \gamma \pmod{8 m_1 m_2 \nu}$ for some integer γ depending on α, d, b. Thus, we may relabel m as $8jm_1 m_2 \nu + \gamma$, where j ranges over all integers, and arrive at

$$(7.2.6)$$
$$Z = \sum_{\substack{\alpha \leq Y \\ (\alpha, 2m_1 m_2 \nu)=1}} \mu(\alpha)\left(\frac{d_1}{m_1 m_2 \nu}\right) \sum_{b \pmod{m_1 m_2 \nu}} \left(\frac{b}{m_1 m_2 \nu}\right) \sum_{j \in \mathbb{Z}} F_\nu\left(\frac{\alpha^2 d_1 (8jm_1 m_2 \nu + \gamma)}{X}\right).$$

We apply Poisson summation to the j-sum to write

$$\sum_{j \in \mathbb{Z}} F_\nu\left(\frac{\alpha^2 d_1 (8jm_1 m_2 \nu + \gamma)}{X}\right) = \frac{X}{8\alpha^2 d_1 m_1 m_2 \nu} \sum_{k \in \mathbb{Z}} e\left(\frac{k\gamma}{8m_1 m_2 \nu}\right) \hat{F}_\nu\left(\frac{kX}{8\alpha^2 d_1 m_1 m_2 \nu}\right).$$

We insert this into (7.2.6), apply the reciprocity relation

$$e\left(\frac{k\gamma}{8m_1 m_2 \nu}\right) = e\left(\frac{k\overline{8}b}{m_1 m_2 \nu}\right) e\left(\frac{k\overline{\alpha^2 d_1 m_1 m_2 \nu}}{8}\right),$$

and then evaluate the b-sum using the definition (4.3) of the Gauss sum. Therefore

$$Z = \frac{X}{8m_1 m_2 \nu} \sum_{\substack{\alpha \leq Y \\ (\alpha, 2m_1 m_2 \nu)=1}} \frac{\mu(\alpha)}{\alpha^2 d_1}\left(\frac{2d_1}{m_1 m_2 \nu}\right)$$
$$\times \sum_{k \in \mathbb{Z}} e\left(\frac{k\overline{\alpha^2 d_1 m_1 m_2 \nu}}{8}\right) \hat{F}_\nu\left(\frac{kX}{8\alpha^2 d_1 m_1 m_2 \nu}\right) \tau_k(m_1 m_2 \nu).$$

Recalling (7.2.1) and (7.2.2), we arrive at

$$(7.2.7)$$
$$S_N^+ = \frac{X}{(\sqrt{2}-1)^4} \sum_{\substack{d \leq D \\ d \text{ odd}}} \mu^2(d)\lambda_d \sum_{\substack{m_1, m_2 \leq M \\ (m_1 m_2, 2d)=1}} \frac{b_{m_1} b_{m_2}}{(m_1 m_2)^{3/2}} \sum_{\substack{\nu=1 \\ (\nu, 2d)=1}}^{\infty} \frac{d_2(\nu)}{\nu^{3/2}} \sum_{\substack{\alpha \leq Y \\ (\alpha, 2m_1 m_2 \nu)=1}} \frac{\mu(\alpha)}{\alpha^2 d_1}$$
$$\times \left(\frac{2d_1}{m_1 m_2 \nu}\right) \sum_{k \in \mathbb{Z}} e\left(\frac{k\overline{\alpha^2 d_1 m_1 m_2 \nu}}{8}\right) \hat{F}_\nu\left(\frac{kX}{8\alpha^2 d_1 m_1 m_2 \nu}\right) \tau_k(m_1 m_2 \nu).$$

Note that we may impose the condition $(m_1 m_2 \nu, d) = 1$ because otherwise $(\frac{2d_1}{m_1 m_2 \nu}) = 0$. We write (7.2.7) as

$$(7.2.8) \qquad\qquad S_N^+ = \mathcal{T}_0 + \mathcal{B},$$

where $\mathcal{T}_0$ is the contribution from $k = 0$ in (7.2.7), while $\mathcal{B}$ is the contribution from $k \neq 0$ in (7.2.7). We evaluate $\mathcal{T}_0$ in the next subsection, and $\mathcal{B}$ in later subsections.

7.3. The contribution from $k = 0$.

By (4.3), $\tau_0(n) = \varphi(n)$ if n is a perfect square, and $\tau_0(n) = 0$ otherwise. Hence the term $\mathcal{T}_0$ in (7.2.7) is

$$(7.3.1) \quad \mathcal{T}_0 = \frac{X}{(\sqrt{2}-1)^4} \sum_{\substack{d \leq D \\ d \text{ odd}}} \mu^2(d) \lambda_d \sum_{\substack{m_1, m_2 \leq M \\ (m_1 m_2, 2d)=1}} \frac{b_{m_1} b_{m_2}}{(m_1 m_2)^{3/2}} \sum_{\substack{\nu=1 \\ (\nu, 2d)=1 \\ m_1 m_2 \nu = \square}}^{\infty} \frac{d_2(\nu)}{\nu^{3/2}} \sum_{\substack{\alpha \leq Y \\ (\alpha, 2m_1 m_2 \nu)=1}} \frac{\mu(\alpha)}{\alpha^2 d_1}$$

$$\times \hat{F}_\nu(0) \varphi(m_1 m_2 \nu).$$

We first extend the sum over α to infinity. Since $\varphi(n) \leq n$, the error introduced in doing so is

$$(7.3.2) \qquad \ll X \sum_{d \leq D} |\lambda_d| \sum_{\substack{m_1, m_2 \leq M \\ m_1 m_2 \nu = \square}} \frac{|b_{m_1} b_{m_2}|}{\sqrt{m_1 m_2}} \sum_{\substack{\nu=1}}^{\infty} \frac{d_2(\nu)}{\sqrt{\nu}} \sum_{\alpha > Y} \frac{1}{\alpha^2 d_1} |\hat{F}_\nu(0)|.$$

By Lemma 4.1, $\hat{F}_\nu(0) \ll 1$ uniformly for all $\nu > 0$, and $\hat{F}_\nu(0) \ll \exp(-\frac{\pi \nu}{8X})$ for $\nu > X^{1+\varepsilon}$. Moreover, (5.8) implies that $|\lambda_d| \ll d^\varepsilon$, while $|b_m| \ll 1$ by (3.5). It follows from these bounds that (7.3.2) is

$$(7.3.3) \qquad \ll X^{1+\varepsilon} \sum_{d \leq D} \sum_{m_1, m_2 \leq M} \frac{1}{\sqrt{m_1 m_2}} \sum_{\substack{\nu \leq X^{1+\varepsilon} \\ m_1 m_2 \nu = \square}} \frac{1}{\sqrt{\nu}} \sum_{\alpha > Y} \frac{1}{\alpha^2 d_1} + \exp\left(-X^\varepsilon\right).$$

Since $m_1 m_2 \nu$ is a perfect square, the sum over m_1, m_2, ν in (7.3.3) is $\ll X^\varepsilon$. Also, the definition (7.2.5) of d_1 implies that

$$\sum_{\alpha > Y} \frac{1}{\alpha^2 d_1} = \frac{1}{d} \sum_{j \mid d} \varphi(j) \sum_{\substack{\alpha > Y \\ j \mid \alpha}} \frac{1}{\alpha^2} \ll \frac{1}{d^{1-\varepsilon} Y}.$$

Therefore (7.3.3) is $O(X^{1+\varepsilon}/Y)$. This bounds the error in extending the sum over α in (7.3.1) to infinity, and we arrive at

$$\mathcal{T}_0 = \frac{X}{(\sqrt{2}-1)^4} \sum_{\substack{d \leq D \\ d \text{ odd}}} \mu^2(d) \lambda_d \sum_{\substack{m_1, m_2 \leq M \\ (m_1 m_2, 2d)=1}} \frac{b_{m_1} b_{m_2}}{(m_1 m_2)^{3/2}} \sum_{\substack{\nu=1 \\ (\nu, 2d)=1 \\ m_1 m_2 \nu = \square}}^{\infty} \frac{d_2(\nu)}{\nu^{3/2}} \sum_{\substack{\alpha=1 \\ (\alpha, 2m_1 m_2 \nu)=1}}^{\infty} \frac{\mu(\alpha)}{\alpha^2 d_1}$$

$$\times \hat{F}_\nu(0) \varphi(m_1 m_2 \nu) + O\left(\frac{X^{1+\varepsilon}}{Y}\right).$$

Writing the sum on α as an Euler product, we deduce that

$$\mathcal{T}_0 = \frac{4X}{3(\sqrt{2}-1)^4\zeta(2)} \sum_{\substack{d \leq D \\ d \text{ odd}}} \frac{\mu^2(d)\lambda_d}{d} \prod_{p|d}\left(\frac{p}{p+1}\right) \sum_{\substack{m_1,m_2 \leq M \\ (m_1 m_2, 2d)=1}} \frac{b_{m_1}b_{m_2}}{\sqrt{m_1 m_2}}$$

$$(7.3.4)$$
$$\times \sum_{\substack{\nu=1 \\ (\nu,2d)=1 \\ m_1 m_2 \nu = \square}}^{\infty} \frac{d_2(\nu)}{\sqrt{\nu}}\hat{F}_\nu(0) \prod_{p|m_1 m_2 \nu}\left(\frac{p}{p+1}\right) + O\left(\frac{X^{1+\varepsilon}}{Y}\right).$$

We next evaluate the sum over d. Lemma 5.3 implies

$$(7.3.5) \quad \sum_{\substack{d \leq D \\ (d,2m_1 m_2 \nu)=1}} \frac{\mu^2(d)\lambda_d}{d} \prod_{p|d}\left(\frac{p}{p+1}\right) = \frac{1+E_0(X)}{\log R} \prod_{\substack{p|2m_1 m_2 \nu \\ p \leq z_0}}\left(1+\frac{1}{p}\right) \prod_{p \leq z_0}\left(\frac{p^2}{p^2-1}\right)$$
$$+ O\left((\log R)^{-2018}\right).$$

Recall that $E_0(X) \to 0$, and depends only on X, G, and ϑ. Heretofore we just write $o(1)$ instead of $E_0(X)$.

We may omit the condition $p \leq z_0$ by trivial estimation and (5.1). It follows from (7.3.5) and (7.3.4) that

$$\mathcal{T}_0 = \frac{2X}{(\sqrt{2}-1)^4}\frac{1+o(1)}{\log R} \sum_{\substack{m_1,m_2 \leq M \\ (m_1 m_2, 2)=1}} \frac{b_{m_1}b_{m_2}}{\sqrt{m_1 m_2}} \sum_{\substack{\nu=1 \\ (\nu,2)=1 \\ m_1 m_2 \nu = \square}}^{\infty} \frac{d_2(\nu)}{\sqrt{\nu}}\hat{F}_\nu(0)$$

$$(7.3.6)$$
$$+ O\left(\frac{X}{(\log R)^{2018}} + \frac{X^{1+\varepsilon}}{Y}\right).$$

The next task is to carry out the summation over m_1, m_2, and ν. Let Υ_0 be defined by

$$(7.3.7) \quad \Upsilon_0 = \sum_{\substack{m_1,m_2 \leq M \\ (m_1 m_2, 2)=1}} \frac{b_{m_1}b_{m_2}}{\sqrt{m_1 m_2}} \sum_{\substack{\nu=1 \\ (\nu,2)=1 \\ m_1 m_2 \nu = \square}}^{\infty} \frac{d_2(\nu)}{\sqrt{\nu}}\hat{F}_\nu(0).$$

We insert into (7.3.7) the definition (3.5) of b_m and the definitions (7.2.4) and (4.1) of F_ν and ω_2, and then apply the Fourier inversion formula (6.1.2). After interchanging the order of summation, we arrive at

$$\Upsilon_0 = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(\frac{s}{2}+\frac{1}{4})^2}{\Gamma(\frac{1}{4})^2}\left(1-\frac{1}{2^{\frac{1}{2}-s}}\right)^2\left(\frac{X}{\pi}\right)^s \check{\Phi}(s)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(z_1)h(z_2)$$

$$(7.3.8)$$
$$\times \sum\sum\sum_{\substack{(m_1 m_2 \nu, 2)=1 \\ m_1 m_2 \nu = \square}} \frac{\mu(m_1)\mu(m_2)d_2(\nu)}{(m_1 m_2 \nu)^{\frac{1}{2}}m_1^{\frac{1+iz_1}{\log M}}m_2^{\frac{1+iz_2}{\log M}}\nu^s}\, dz_1 dz_2 \frac{ds}{s},$$

where we take $c = \frac{1}{\log X}$ to facilitate later estimations. We may write the sum on m_1, m_2, ν as an Euler product

$$\sum\sum\sum_{\substack{(m_1 m_2 \nu, 2)=1 \\ m_1 m_2 \nu = \square}} \frac{\mu(m_1)\mu(m_2)d_2(\nu)}{(m_1 m_2 \nu)^{\frac{1}{2}}m_1^{\frac{1+iz_1}{\log M}}m_2^{\frac{1+iz_2}{\log M}}\nu^s} = \prod_{p>2}\sum_{m_1=0}^{1}\sum_{m_2=0}^{1}\sum_{\substack{\nu=0 \\ m_1+m_2+\nu \text{ even}}}^{\infty} \frac{(-1)^{m_1+m_2}(\nu+1)}{p^{\frac{m_1+m_2+\nu}{2}+m_1\left(\frac{1+iz_1}{\log M}\right)+m_2\left(\frac{1+iz_2}{\log M}\right)+\nu s}}.$$

This can also be written as

$$(7.3.9) \quad \zeta^3(1+2s)\zeta\left(1+\tfrac{2+iz_1+iz_2}{\log M}\right)\zeta^{-2}\left(1+\tfrac{1+iz_1}{\log M}+s\right)\zeta^{-2}\left(1+\tfrac{1+iz_2}{\log M}+s\right)Q\left(\tfrac{1+iz_1}{\log M},\tfrac{1+iz_2}{\log M},s\right),$$

where $Q(w_1, w_2, s)$ is an Euler product that is uniformly bounded and holomorphic when each of $\mathrm{Re}(w_1)$, $\mathrm{Re}(w_2)$, and $\mathrm{Re}(s)$ is $\geq -\varepsilon$. From this definition of Q and a calculation, we see that

$$(7.3.10) \qquad\qquad\qquad\qquad Q(0,0,0) = 1,$$

a fact we use shortly. We insert the expression (7.3.9) for the m_1, m_2, ν-sum into (7.3.8) and arrive at

$$\Upsilon_0 = \frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(\tfrac{s}{2}+\tfrac{1}{4})^2}{\Gamma(\tfrac{1}{4})^2}\left(1-\frac{1}{2^{\frac{1}{2}-s}}\right)^2\left(\frac{X}{\pi}\right)^s\check{\Phi}(s)\zeta^3(1+2s)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(z_1)h(z_2)$$

$$\times\zeta\left(1+\tfrac{2+iz_1+iz_2}{\log M}\right)\zeta^{-2}\left(1+\tfrac{1+iz_1}{\log M}+s\right)\zeta^{-2}\left(1+\tfrac{1+iz_2}{\log M}+s\right)Q\left(\tfrac{1+iz_1}{\log M},\tfrac{1+iz_2}{\log M},s\right)\,dz_1 dz_2\frac{ds}{s}.$$

By (6.1.4) and the rapid decay of the gamma function, we may truncate the integrals to the region $|z_1|, |z_2| \leq \sqrt{\log M}$ and $|\mathrm{Im}(s)| \leq (\log X)^2$, introducing a negligible error. We then deform the path of integration of the s-integral to the path made up of the line segment L_1 from $\frac{1}{\log X}-i(\log X)^2$ to $-\frac{c'}{\log\log X}-i(\log X)^2$, followed by the line segment L_2 from $-\frac{c'}{\log\log X}-i(\log X)^2$ to $-\frac{c'}{\log\log X}+i(\log X)^2$, and then by the line segment L_3 from $-\frac{c'}{\log\log X}+i(\log X)^2$ to $\frac{1}{\log X}+i(\log X)^2$, where c' is a constant chosen so that

$$(7.3.11) \qquad\qquad \zeta(1+z) \ll \log|\mathrm{Im}(z)| \quad\text{and}\quad \frac{1}{\zeta(1+z)} \ll \log|\mathrm{Im}(z)|$$

for $\mathrm{Re}(z) \geq -c'/\log|\mathrm{Im}(z)|$ and $|\mathrm{Im}(z)| \geq 1$ (see, for example, Theorem 3.5 and (3.11.8) of Titchmarsh [42]). This leaves a residue from the pole at $s = 0$. The contributions of the integrals over L_1 and L_3 are negligible because of the rapid decay of the Γ function, while the contribution of the integral over L_2 is negligible because $X^s \ll \exp\left(-c'\frac{\log X}{\log\log X}\right)$ for s on L_2. Hence the main contribution arises from the residue of the pole at $s = 0$. Writing this residue as an integral along a circle centered at 0, we arrive at

$$\Upsilon_0 = \frac{1}{2\pi i}\oint_{|s|=\frac{1}{\log X}}\frac{\Gamma(\tfrac{s}{2}+\tfrac{1}{4})^2}{\Gamma(\tfrac{1}{4})^2}\left(1-\frac{1}{2^{\frac{1}{2}-s}}\right)^2\left(\frac{X}{\pi}\right)^s\check{\Phi}(s)\zeta^3(1+2s)$$

$$\times\int_{|z_i|\leq\sqrt{\log M}}\int h(z_1)h(z_2)\zeta\left(1+\tfrac{2+iz_1+iz_2}{\log M}\right)\zeta^{-2}\left(1+\tfrac{1+iz_1}{\log M}+s\right)\zeta^{-2}\left(1+\tfrac{1+iz_2}{\log M}+s\right)$$

$$\times Q\left(\tfrac{1+iz_1}{\log M},\tfrac{1+iz_2}{\log M},s\right)\,dz_1 dz_2\frac{ds}{s}+O\left(\frac{1}{(\log X)^{2018}}\right).$$

We may expand the zeta-functions and the function Q into Laurent series. The main contribution arises from the first terms of the Laurent expansions, and so we deduce using (7.3.10)

that

$$\Upsilon_0 = \frac{1}{16\pi i} \oint_{|s|=\frac{1}{\log X}} \frac{\Gamma(\frac{s}{2}+\frac{1}{4})^2}{\Gamma(\frac{1}{4})^2} \left(1 - \frac{1}{2^{\frac{1}{2}-s}}\right)^2 \left(\frac{X}{\pi}\right)^s \check{\Phi}(s) \iint_{|z_i|\le\sqrt{\log M}} h(z_1)h(z_2)$$

$$\times \left(\frac{\log M}{2+iz_1+iz_2}\right) \left(\frac{1+iz_1}{\log M}+s\right)^2 \left(\frac{1+iz_2}{\log M}+s\right)^2 dz_1 dz_2 \frac{ds}{s^4} + O\left(\frac{1}{(\log X)^{1-\varepsilon}}\right).$$

By (6.1.4), we may extend the integrals over z_1, z_2 to $\mathbb{R}^2$, introducing a negligible error. We then apply the formula

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(z_1)h(z_2) \frac{(1+iz_1)^j(1+iz_2)^k}{2+iz_1+iz_2} dz_1 dz_2$$

(7.3.12)
$$= \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(z_1)h(z_2)(1+iz_1)^j(1+iz_2)^k e^{-t(1+iz_1)-t(1+iz_2)} dz_1 dz_2 dt$$

$$= (-1)^{j+k} \int_0^{\infty} H^{(j)}(t)H^{(k)}(t)\, dt$$

to obtain

$$\Upsilon_0 = \frac{1}{16\pi i} \oint_{|s|=\frac{1}{\log X}} \frac{\Gamma(\frac{s}{2}+\frac{1}{4})^2}{\Gamma(\frac{1}{4})^2} \left(1 - \frac{1}{2^{\frac{1}{2}-s}}\right)^2 \left(\frac{X}{\pi}\right)^s \check{\Phi}(s) \Bigg\{ \frac{1}{(\log M)^3} \int_0^1 H''(t)^2\, dt$$

$$- \frac{4s}{(\log M)^2} \int_0^1 H'(t)H''(t)\, dt + \frac{2s^2}{\log M} \int_0^1 H(t)H''(t)\, dt + \frac{4s^2}{\log M} \int_0^1 H'(t)^2\, dt$$

$$- 4s^3 \int_0^1 H(t)H'(t)\, dt + s^4 \log M \int_0^1 H(t)^2\, dt \Bigg\} \frac{ds}{s^4} + O\left(\frac{1}{(\log X)^{1-\varepsilon}}\right).$$

We evaluate the s-integral as a residue using (6.1.6). The result is

$$\Upsilon_0 = \frac{\check{\Phi}(0)}{8} \left(1 - \frac{1}{\sqrt{2}}\right)^2 \Bigg\{ \frac{1}{6} \left(\frac{\log X}{\log M}\right)^3 \int_0^1 H''(t)^2\, dt - 2\left(\frac{\log X}{\log M}\right)^2 \int_0^1 H'(t)H''(t)\, dt$$

$$+ 2\frac{\log X}{\log M} \int_0^1 H(t)H''(t)\, dt + 4\frac{\log X}{\log M} \int_0^1 H'(t)^2\, dt - 4\int_0^1 H(t)H'(t)\, dt \Bigg\} + O\left(\frac{1}{(\log X)^{1-\varepsilon}}\right).$$

From this, (7.3.6), and the definition (7.3.7) of Υ_0, we arrive at

(7.3.13)
$$\mathcal{T}_0 = \frac{X}{8\left(1-\frac{1}{\sqrt{2}}\right)^2} \frac{1+o(1)}{\log R} \Bigg\{ \frac{1}{24} \left(\frac{\log X}{\log M}\right)^3 \int_0^1 H''(t)^2\, dt$$

$$- \frac{1}{2} \left(\frac{\log X}{\log M}\right)^2 \int_0^1 H'(t)H''(t)\, dt + \frac{\log X}{2\log M} \int_0^1 H(t)H''(t)\, dt + \frac{\log X}{\log M} \int_0^1 H'(t)^2\, dt$$

$$- \int_0^1 H(t)H'(t)\, dt \Bigg\} + O\left(\frac{X}{(\log X)^{1-\varepsilon}} + \frac{X^{1+\varepsilon}}{Y}\right).$$

7.4. The contribution from $k \neq 0$: splitting into cases.

Having estimated the term $\mathcal{T}_0$ in (7.2.8), we now begin our analysis of $\mathcal{B}$. The analysis of $\mathcal{B}$ is much more complicated than the analysis for $\mathcal{T}_0$.

The behavior of the additive character $e(k\overline{\alpha^2 d_1 m_1 m_2 \nu}/8)$ in (7.2.7) depends upon the residue class of k modulo 8. We therefore distinguish the following cases for k: k is odd, $k \equiv 2 \pmod 4$, $k \equiv 4 \pmod 8$, or $k \equiv 0 \pmod 8$. We split our analysis of the sum $\mathcal{B}$ in (7.2.8) according to these four cases. For the terms with odd k, we use the identity

$$e\left(\frac{h}{8}\right) = \frac{\sqrt{2}}{2}\left(\frac{2}{h}\right) + \frac{\sqrt{2}}{2}\left(\frac{-2}{h}\right)i, \quad h \text{ odd},$$

and treat separately the contributions of each term on the right-hand side. Moreover, for the terms with odd k or $k \equiv 2 \pmod 4$, we use the second expression in (4.3) for $\tau_k(n)$ and treat separately the contributions of the terms $\left(\frac{1+i}{2}\right)G_k(n)$ and $\left(\frac{-1}{n}\right)\left(\frac{1-i}{2}\right)G_k(n)$. We can treat these two contributions together as one combined sum for the terms with $k \equiv 0, 4 \pmod 8$, because, for those k, the additive character $e(k\overline{\alpha^2 d_1 m_1 m_2 \nu}/8)$ is constant and the conditions $k \equiv 0, 4 \pmod 8$ are invariant with respect to the substitution $k \mapsto -k$. Hence, in view of these considerations, (7.2.7), and (7.2.8), we write

$$(7.4.1) \quad \mathcal{B} = \frac{X}{(\sqrt{2}-1)^4} \sum_{\substack{d \leq D \\ d \text{ odd}}} \mu^2(d)\lambda_d \sum_{\substack{m_1, m_2 \leq M \\ (m_1 m_2, 2d)=1}} \frac{b_{m_1} b_{m_2}}{(m_1 m_2)^{3/2}} \sum_{\substack{\nu=1 \\ (\nu, 2d)=1}}^{\infty} \frac{d_2(\nu)}{\nu^{3/2}} \sum_{\substack{\alpha \leq Y \\ (\alpha, 2m_1 m_2 \nu)=1}} \frac{\mu(\alpha)}{\alpha^2 d_1}$$
$$\times (\mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3 + \mathcal{Q}_4 + \mathcal{U}_1 + \mathcal{U}_2 + \mathcal{V} + \mathcal{W}),$$

where

$$(7.4.2) \quad \mathcal{Q}_1 = \left(\frac{1+i}{2}\right)\frac{\sqrt{2}}{2}\left(\frac{2d_1}{m_1 m_2 \nu}\right)\sum_{\substack{k \in \mathbb{Z} \\ k \text{ odd}}} \left(\frac{2}{kd_1 m_1 m_2 \nu}\right)\hat{F}_\nu\left(\frac{kX}{8\alpha^2 d_1 m_1 m_2 \nu}\right)G_k(m_1 m_2 \nu),$$

$$(7.4.3) \quad \mathcal{Q}_2 = \left(\frac{1-i}{2}\right)\frac{\sqrt{2}}{2}\left(\frac{-2d_1}{m_1 m_2 \nu}\right)\sum_{\substack{k \in \mathbb{Z} \\ k \text{ odd}}} \left(\frac{2}{kd_1 m_1 m_2 \nu}\right)\hat{F}_\nu\left(\frac{kX}{8\alpha^2 d_1 m_1 m_2 \nu}\right)G_k(m_1 m_2 \nu),$$

$$(7.4.4)$$
$$\mathcal{Q}_3 = \left(\frac{1+i}{2}\right)i\frac{\sqrt{2}}{2}\left(\frac{2d_1}{m_1 m_2 \nu}\right)\sum_{\substack{k \in \mathbb{Z} \\ k \text{ odd}}} \left(\frac{-2}{kd_1 m_1 m_2 \nu}\right)\hat{F}_\nu\left(\frac{kX}{8\alpha^2 d_1 m_1 m_2 \nu}\right)G_k(m_1 m_2 \nu),$$

$$(7.4.5)$$
$$\mathcal{Q}_4 = \left(\frac{1-i}{2}\right)i\frac{\sqrt{2}}{2}\left(\frac{-2d_1}{m_1 m_2 \nu}\right)\sum_{\substack{k \in \mathbb{Z} \\ k \text{ odd}}} \left(\frac{-2}{kd_1 m_1 m_2 \nu}\right)\hat{F}_\nu\left(\frac{kX}{8\alpha^2 d_1 m_1 m_2 \nu}\right)G_k(m_1 m_2 \nu),$$

$$(7.4.6)$$
$$\mathcal{U}_1 = \left(\frac{1+i}{2}\right)\left(\frac{2d_1}{m_1 m_2 \nu}\right)\sum_{\substack{k \in \mathbb{Z} \\ k \equiv 2 \,(\text{mod } 4)}} e\left(\frac{k\overline{\alpha^2 d_1 m_1 m_2 \nu}}{8}\right)\hat{F}_\nu\left(\frac{kX}{8\alpha^2 d_1 m_1 m_2 \nu}\right)G_k(m_1 m_2 \nu),$$

(7.4.7)
$$\mathcal{U}_2 = \left(\frac{1-i}{2}\right)\left(\frac{-2d_1}{m_1 m_2 \nu}\right) \sum_{\substack{k \in \mathbb{Z} \\ k \equiv 2 \,(\mathrm{mod}\,4)}} e\left(\frac{k\overline{\alpha^2 d_1 m_1 m_2 \nu}}{8}\right) \hat{F}_\nu\left(\frac{kX}{8\alpha^2 d_1 m_1 m_2 \nu}\right) G_k(m_1 m_2 \nu),$$

(7.4.8)
$$\mathcal{V} = \left(\frac{2d_1}{m_1 m_2 \nu}\right) \sum_{\substack{k \in \mathbb{Z} \\ k \equiv 4 \,(\mathrm{mod}\,8)}} e\left(\frac{k\overline{\alpha^2 d_1 m_1 m_2 \nu}}{8}\right) \hat{F}_\nu\left(\frac{kX}{8\alpha^2 d_1 m_1 m_2 \nu}\right) \tau_k(m_1 m_2 \nu),$$

and

(7.4.9)
$$\mathcal{W} = \left(\frac{2d_1}{m_1 m_2 \nu}\right) \sum_{\substack{k \in \mathbb{Z} \\ k \equiv 0 \,(\mathrm{mod}\,8) \\ k \neq 0}} e\left(\frac{k\overline{\alpha^2 d_1 m_1 m_2 \nu}}{8}\right) \hat{F}_\nu\left(\frac{kX}{8\alpha^2 d_1 m_1 m_2 \nu}\right) \tau_k(m_1 m_2 \nu).$$

7.5. Evaluation of the sum with $\mathcal{Q}_1$.
In this subsection, we evaluate the sum

(7.5.1)
$$\mathcal{Q}_1^* := \sum_{\substack{\nu=1 \\ (\nu,2d)=1}}^{\infty} \frac{d_2(\nu)}{\nu^{3/2}} \sum_{\substack{\alpha \leq Y \\ (\alpha,2m_1 m_2 \nu)=1}} \frac{\mu(\alpha)}{\alpha^2 d_1} \mathcal{Q}_1,$$

with $\mathcal{Q}_1$ defined by (7.4.2). We may cancel the two Jacobi symbols $\left(\frac{2}{m_1 m_2 \nu}\right)$ in (7.4.2), insert the resulting expression into (7.5.1), and then apply the Mellin inversion formula to the ν-sum to deduce that

(7.5.2)
$$\mathcal{Q}_1^* = \left(\frac{1+i}{2}\right) \frac{\sqrt{2}}{2} \sum_{\substack{\alpha \leq Y \\ (\alpha,2m_1 m_2 \nu)=1}} \frac{\mu(\alpha)}{\alpha^2 d_1}\left(\frac{d_1}{m_1 m_2}\right) \sum_{\substack{k \in \mathbb{Z} \\ k \text{ odd}}} \left(\frac{2}{kd_1}\right)$$
$$\times \frac{1}{2\pi i}\int_{(c)}\int_0^\infty \hat{F}_t\left(\frac{kX}{8\alpha^2 d_1 m_1 m_2 t}\right) t^{w-1}\,dt \sum_{\substack{\nu=1 \\ (\nu,2d)=1}}^{\infty} \frac{d_2(\nu)}{\nu^{3/2+w}}\left(\frac{d_1}{\nu}\right) G_k(m_1 m_2 \nu)\,dw$$

for any $c > 1$. The interchange in the order of summation is justified by absolute convergence. The next step is to write the ν-sum as an Euler product, as follows.

Lemma 7.1. *Let d_1 be as defined by (7.2.5). For each nonzero integer k, define k_1 and k_2 uniquely by the equation*

(7.5.3)
$$4kd_1 = k_1 k_2^2,$$

where k_1 is a fundamental discriminant and k_2 is a positive integer. If ℓ is a positive integer and $\mathrm{Re}(s) > 1$, then

$$\sum_{\substack{\nu=1 \\ (\nu,2\alpha d)=1}}^{\infty} \frac{d_2(\nu)}{\nu^s}\left(\frac{d_1}{\nu}\right)\frac{G_k(\ell\nu)}{\sqrt{\nu}} \;=\; L(s,\chi_{k_1})^2 \prod_p \mathcal{G}_{0,p}(s;k,\ell,\alpha,d) \;=:\; L(s,\chi_{k_1})^2 \mathcal{G}_0(s;k,\ell,\alpha,d),$$

where $\chi_{k_1}(\cdot) = \left(\frac{k_1}{\cdot}\right)$ and $\mathcal{G}_{0,p}(s;k,\ell,\alpha,d)$ is defined by

$$\mathcal{G}_{0,p}(s;k,\ell,\alpha,d) \;=\; \left(1 - \frac{1}{p^s}\left(\frac{k_1}{p}\right)\right)^2 \qquad \text{if } p|2\alpha d, \quad \text{and}$$

$$\mathcal{G}_{0,p}(s;k,\ell,\alpha,d) = \left(1 - \frac{1}{p^s}\left(\frac{k_1}{p}\right)\right)^2 \sum_{r=0}^{\infty} \frac{r+1}{p^{rs}}\left(\frac{d_1}{p^r}\right)\frac{G_k(p^{r+\operatorname{ord}_p(\ell)})}{p^{r/2}} \qquad \text{if } p \nmid 2\alpha d.$$

The function $\mathcal{G}_0(s;k,\ell,\alpha,d)$ is holomorphic for $\operatorname{Re}(s) > \frac{1}{2}$. Moreover, if k_3 and k_4 are defined by the equation

(7.5.4)
$$k = k_3 k_4^2,$$

with k_3 square-free and k_4 a positive integer, then

$$\mathcal{G}_0(s;k,\ell,\alpha,d) \ll_\varepsilon (\alpha d|k|\ell)^\varepsilon \ell^{1/2}(\ell,k_4^2)^{1/2}$$

uniformly for $\operatorname{Re}(s) \geq \frac{1}{2} + \varepsilon$.

Proof. It follows from the definition of $\mathcal{G}_{0,p}(s;k,\ell,\alpha,d)$ and Lemma 4.3 that

$$\mathcal{G}_{0,p}(s;k,\ell,\alpha,d) = \left(1 - \frac{1}{p^s}\left(\frac{k_1}{p}\right)\right)^2\left(1 + \frac{2}{p^s}\left(\frac{d_1 k}{p}\right)\right) = 1 - \frac{3}{p^{2s}} + \frac{2}{p^{3s}}\left(\frac{k_1}{p}\right)$$

for each $p \nmid 2\alpha dk\ell$, since $\left(\frac{d_1 k}{p}\right) = \left(\frac{k_1}{p}\right)$ for odd primes p, by (7.5.3). The rest of the proof is similar to that of [39, Lemma 5.3]. $\qquad\square$

We also need some analytic properties of the function $h(\xi,w)$ defined for $\operatorname{Re}(w) > 0$ by

$$h(\xi,w) = \int_0^\infty \hat{F}_t\left(\frac{\xi}{t}\right) t^{w-1}\,dt.$$

These are embodied in the following lemma. As a bit of notation, for a real number x we define

$$\operatorname{sgn}(x) = \begin{cases} 1, & x \geq 0, \\ -1, & x < 0. \end{cases}$$

Lemma 7.2. *Let F_t be defined by (7.2.4). If $\xi \neq 0$ then*

$$h(\xi,w) = |\xi|^w \check{\Phi}(w)\int_0^\infty \omega_2\left(\frac{|\xi|\pi}{Xz}\right)(\cos(2\pi z) - i\operatorname{sgn}(\xi)\sin(2\pi z))\frac{dz}{z^{w+1}}.$$

The integral above may be expressed as

(7.5.5)
$$\frac{1}{2\pi i}\int_{(c)} \frac{\Gamma\left(\frac{s}{2}+\frac{1}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2}\left(1 - \frac{1}{2^{\frac{1}{2}-s}}\right)^2 \frac{X^s}{(\pi|\xi|)^s}(2\pi)^{-s+w}\Gamma(s-w)$$
$$\times \left\{\cos\left(\frac{\pi}{2}(s-w)\right) - i\operatorname{sgn}(\xi)\sin\left(\frac{\pi}{2}(s-w)\right)\right\}\frac{ds}{s}$$

for any c with $\operatorname{Re}(w)+1 > c > \max\{0, \operatorname{Re}(w)\}$. If $\xi \neq 0$, then $h(\xi,w)$ is an entire function of w. In the region $1 \geq \operatorname{Re}(w) > -1$, it satisfies the bound

$$h(\xi,w) \ll (1+|w|)^{-\operatorname{Re}(w)-\frac{1}{2}}\exp\left(-\frac{1}{10}\frac{\sqrt{|\xi|}}{\sqrt{X(|w|+1)}}\right)|\xi|^w|\check{\Phi}(w)|.$$

Proof. The proof is similar to that of [39, Lemma 5.2]. $\qquad\square$

Now, by these lemmas and the rapid decay of $\check{\Phi}(w)$ as $|\mathrm{Im}(w)| \to \infty$ in a fixed vertical strip, we may move the line of integration of the w-integral in (7.5.2) to $\mathrm{Re}(w) = -\frac{1}{2} + \varepsilon$. This leaves a residue from a pole at $w = 0$ only when χ_{k_1} is a principal character, which holds if and only if $k_1 = 1$. By (7.5.3), $k_1 = 1$ if and only if kd_1 is a perfect square. Hence

$$(7.5.6) \qquad\qquad \mathcal{Q}_1^* = \mathcal{P}_1 + \mathcal{R}_1,$$

where $\mathcal{P}_1$ is defined by

$$(7.5.7) \quad \mathcal{P}_1 = \operatorname*{Res}_{w=0} \left(\frac{1+i}{2}\right) \frac{\sqrt{2}}{2} \sum_{\substack{\alpha \leq Y \\ (\alpha, 2m_1 m_2)=1}} \frac{\mu(\alpha)}{\alpha^2 d_1} \left(\frac{d_1}{m_1 m_2}\right) \sum_{\substack{k \in \mathbb{Z} \\ k \text{ odd} \\ kd_1 = \square}} h\left(\frac{kX}{8\alpha^2 d_1 m_1 m_2}, w\right)$$
$$\times \zeta(1+w)^2 \mathcal{G}_0(1+w; k, m_1 m_2, \alpha, d)$$

and $\mathcal{R}_1$ is defined by

$$(7.5.8) \quad \mathcal{R}_1 = \left(\frac{1+i}{2}\right) \frac{\sqrt{2}}{2} \sum_{\substack{\alpha \leq Y \\ (\alpha, 2m_1 m_2)=1}} \frac{\mu(\alpha)}{\alpha^2 d_1} \left(\frac{d_1}{m_1 m_2}\right) \sum_{\substack{k \in \mathbb{Z} \\ k \text{ odd}}} \left(\frac{2}{kd_1}\right)$$
$$\times \frac{1}{2\pi i} \int_{(-\frac{1}{2}+\varepsilon)} h\left(\frac{kX}{8\alpha^2 d_1 m_1 m_2}, w\right) L(1+w, \chi_{k_1})^2 \mathcal{G}_0(1+w; k, m_1 m_2, \alpha, d)\, dw.$$

We bound $\mathcal{R}_1$ in Subsection 7.6. To estimate $\mathcal{P}_1$, observe that d_1 is square-free by its definition (7.2.5) and the fact that d is square-free. This implies that kd_1 is a perfect square if and only if k equals d_1 times a perfect square. Hence, in (7.5.7), we may relabel k as $d_1 j^2$, where j runs through all the odd positive integers. With this and Lemma 7.2, we deduce from (7.5.7) that

$$(7.5.9)$$
$$\mathcal{P}_1 = \operatorname*{Res}_{w=0} \left(\frac{1+i}{2}\right) \frac{\sqrt{2}}{2} \sum_{\substack{\alpha \leq Y \\ (\alpha, 2m_1 m_2)=1}} \frac{\mu(\alpha)}{\alpha^2 d_1} \zeta(1+w)^2 \check{\Phi}(w) X^w \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma\left(\frac{s}{2}+\frac{1}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2} \left(1 - \frac{1}{2^{\frac{1}{2}-s}}\right)^2$$
$$\times \pi^{-s} \Gamma_2(s-w)(8\alpha^2 m_1 m_2)^{s-w} \sum_{\substack{j=1 \\ j \text{ odd}}}^{\infty} j^{-2s+2w} \left(\frac{d_1}{m_1 m_2}\right) \mathcal{G}_0(1+w; d_1 j^2, m_1 m_2, \alpha, d) \frac{ds}{s},$$

where $\Gamma_2(u)$ is defined by

$$(7.5.10) \qquad \Gamma_2(u) = (2\pi)^{-u} \Gamma(u)(\cos\left(\tfrac{\pi}{2}u\right) - i\sin\left(\tfrac{\pi}{2}u\right)),$$

and where we take $c > \frac{1}{2}$ to guarantee the absolute convergence of the j-sum.

We next write the j-sum in (7.5.9) as an Euler product. By (ii) of Lemma 4.3, if j is a positive integer then

$$\left(\frac{d_1}{p^\beta}\right) G_{d_1 j^2}(p^\beta) = G_{j^2}(p^\beta)$$

for all $p \nmid 2\alpha d$ and $\beta \geq 1$. From this and the definition of $\mathcal{G}_0$ in Lemma 7.1, we see that

$$\left(\frac{d_1}{m_1 m_2}\right) \mathcal{G}_0(1+w; d_1 j^2, m_1 m_2, \alpha, d) = \mathcal{G}(1+w; j^2, m_1 m_2, \alpha d),$$

where $\mathcal{G}$ is defined by [39, (5.8)]. Hence we may write the inner j-sum in (7.5.9) as an Euler product

(7.5.11)
$$\sum_{\substack{j=1 \\ j \text{ odd}}}^{\infty} j^{-2s+2w}\mathcal{G}(1+w; j^2, m_1m_2, \alpha d) = \left(1 - \frac{1}{2^{1+w}}\right)^2 \prod_{p>2} \sum_{b=0}^{\infty} p^{2b(w-s)}\mathcal{G}_p(1+w; p^{2b}, m_1m_2, \alpha d)$$

$$= \left(1 - \frac{1}{4^{s-w}}\right) \prod_p \sum_{b=0}^{\infty} p^{2b(w-s)}\mathcal{G}_p(1+w; p^{2b}, m_1m_2, \alpha d).$$

This latter expression is [39, p. 471]

$$\left(1 - \frac{1}{4^{s-w}}\right)(m_1m_2)^{1-s+w}\ell_1^{s-w-\frac{1}{2}}\zeta(2s-2w)\zeta(2s+1)\mathcal{H}_1(s-w, 1+w; m_1m_2, \alpha d),$$

where ℓ_1 is the square-free integer defined by the equation

(7.5.12)
$$m_1m_2 = \ell_1\ell_2^2, \quad \mu^2(\ell_1) = 1, \ \ell_2 \in \mathbb{Z},$$

and $\mathcal{H}_1$ is defined by an Euler product

$$\mathcal{H}_1(s-w, 1+w; m_1m_2, \alpha d) = \prod_p \mathcal{H}_{1,p}.$$

The local factors $\mathcal{H}_{1,p}$ are
(7.5.13)
$$\mathcal{H}_{1,p} = \begin{cases} \left(1 - \frac{1}{p^{1+w}}\right)^2\left(1 - \frac{1}{p^{1+2s}}\right) & \text{if } p|2\alpha d \\[2ex] \frac{\left(1-\frac{1}{p^{1+w}}\right)^2}{\left(1-\frac{1}{p^{1+2s}}\right)}\left(1 + \frac{2}{p^{1+w}} - \frac{2}{p^{1+2s-w}} + \frac{1}{p^{1+2s}} - \frac{3}{p^{2+2s}} + \frac{1}{p^{3+4s}}\right) & \text{if } p \nmid 2\alpha dm_1m_2 \\[2ex] \frac{\left(1-\frac{1}{p^{1+w}}\right)^2}{\left(1-\frac{1}{p^{1+2s}}\right)}\left(1 - \frac{1}{p^{2s-2w}} + \frac{2}{p^{2s-w}} - \frac{2}{p^{1+2s-w}} + \frac{1}{p^{1+2s}} - \frac{1}{p^{1+4s-2w}}\right) & \text{if } p|\ell_1 \\[2ex] \frac{\left(1-\frac{1}{p^{1+w}}\right)^2}{\left(1-\frac{1}{p^{1+2s}}\right)}\left(1 - \frac{1}{p} + \frac{2}{p^{1+w}} - \frac{2}{p^{1+2s-w}} + \frac{1}{p^{1+2s}} - \frac{1}{p^{2+2s}}\right) & \text{if } p|m_1m_2, \ p \nmid \ell_1. \end{cases}$$

Inserting this expression for the j-sum in (7.5.9) into (7.5.9), we find that

(7.5.14)
$$\mathcal{P}_1 = \left(\frac{1+i}{2}\right)\frac{\sqrt{2}}{2} \sum_{\substack{\alpha \leq Y \\ (\alpha, 2m_1m_2)=1}} \frac{\mu(\alpha)}{\alpha^2 d_1}\mathcal{I},$$

where
(7.5.15)
$$\mathcal{I} = \operatorname*{Res}_{w=0} \zeta(1+w)^2\check{\Phi}(w)X^w \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma\left(\frac{s}{2}+\frac{1}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2}\left(1 - \frac{1}{2^{\frac{1}{2}-s}}\right)^2 \pi^{-s}\Gamma_2(s-w)(8\alpha^2)^{s-w}$$

$$\times \left(1 - \frac{1}{4^{s-w}}\right)m_1m_2\ell_1^{s-w-\frac{1}{2}}\zeta(2s-2w)\zeta(2s+1)\mathcal{H}_1(s-w, 1+w; m_1m_2, \alpha d)\frac{ds}{s}.$$

The next step is to extend the α-sum to infinity and show that the error introduced in doing so is small. To do this, we need to move the line of integration in (7.5.15) closer to

0 to guarantee the absolute convergence of the α-sum. We first evaluate the residue to see that (7.5.15) is the same as

$$
\begin{aligned}
\mathcal{I} = \frac{\check{\Phi}(0)}{2\pi i} \int_{(c)} & \frac{\Gamma\left(\frac{s}{2} + \frac{1}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2} \left(1 - \frac{1}{2^{\frac{1}{2}-s}}\right)^2 \pi^{-s} \Gamma_2(s)(8\alpha^2)^s \left(1 - \frac{1}{4^s}\right) m_1 m_2 \\
& \times \ell_1^{s-\frac{1}{2}} \zeta(2s)\zeta(2s+1)\mathcal{H}_1(s,1;m_1m_2,\alpha d) \left\{ 2\gamma + \frac{(\check{\Phi})'(0)}{\check{\Phi}(0)} + \log\left(\frac{X}{8\alpha^2\ell_1}\right) \right. \\
& \left. - \frac{\Gamma_2'}{\Gamma_2}(s) + \frac{\log 4}{(1-4^s)} - 2\frac{\zeta'}{\zeta}(2s) + \frac{\frac{\partial}{\partial w}\mathcal{H}_1(s-w,1+w;m_1m_2,\alpha d)}{\mathcal{H}_1(s-w,1+w;m_1m_2,\alpha d)}\bigg|_{w=0} \right\} \frac{ds}{s}.
\end{aligned}
$$

(7.5.16)

Here γ denotes the Euler-Mascheroni constant. The definition (7.5.13) of $\mathcal{H}_1(s-w,1+w;m_1m_2,\alpha d)$ implies that it is holomorphic for $\mathrm{Re}(s) > 0$ and $|w| < \max\{\frac{1}{2}, 2|s|\}$, and that it and its first partial derivatives at $w = 0$ are bounded by $\ll (\alpha X)^\varepsilon$ for $\mathrm{Re}(s) \geq \frac{1}{\log X}$. Thus, by the rapid decay of the gamma function, we may move the line of integration in (7.5.16) to $\mathrm{Re}(s) = \frac{1}{\log X}$. There is no residue because the poles of $\zeta(2s)$ and $\frac{\zeta'}{\zeta}(2s)$ at $s = \frac{1}{2}$ are canceled by the zero of the factor $(1 - 2^{s-\frac{1}{2}})^2$. Using well-known bounds for $\zeta(2s)$ and $\zeta'(2s)$ implied by the Phragmén-Lindelöf principle, we see that the new integral is now bounded by

$$
\ll m_1 m_2 \ell_1^{-\frac{1}{2}+\varepsilon} \alpha^\varepsilon X^\varepsilon \int_{\left(\frac{1}{\log X}\right)} \left|\Gamma\left(\frac{s}{2}+\frac{1}{4}\right)\right|^2 \max\{|\Gamma_2(s)|, |\Gamma_2'(s)|\}(1+|s|)^{\frac{1}{2}+\varepsilon} |ds|,
$$

which is $\ll m_1 m_2 \ell_1^{-\frac{1}{2}+\varepsilon} \alpha^\varepsilon X^\varepsilon$ by the rapid decay of the gamma function. Dividing this bound by $\alpha^2 d_1$ and summing the result over all $\alpha > Y$, we deduce that

$$
\sum_{\substack{\alpha > Y \\ (\alpha, 2m_1m_2)=1}} \frac{\mu^2(\alpha)}{\alpha^2 d_1} |\mathcal{I}| \ll \frac{m_1 m_2 \ell_1^{-\frac{1}{2}+\varepsilon} X^\varepsilon}{d^{1-\varepsilon} Y^{1-\varepsilon}}
$$

(7.5.17)

because, by (7.2.5), if $\varphi(j)$ is the Euler totient function, then

$$
\sum_{\alpha > Y} \frac{1}{\alpha^{2-\varepsilon} d_1} = \frac{1}{d} \sum_{j|d} \varphi(j) \sum_{\substack{\alpha > Y \\ j|\alpha}} \frac{1}{\alpha^{2-\varepsilon}} \ll \frac{1}{d^{1-\varepsilon} Y^{1-\varepsilon}}.
$$

From (7.5.14), (7.5.17), and (7.5.15) now with $c = \frac{1}{\log X}$, we arrive at

$$
\begin{aligned}
\mathcal{P}_1 = \operatorname*{Res}_{w=0} & \left(\frac{1+i}{2}\right) \frac{\sqrt{2}}{2} X^w \frac{1}{2\pi i} \int_{\left(\frac{1}{\log X}\right)} \Gamma_2(s-w) 8^{s-w} \left(1 - \frac{1}{4^{s-w}}\right) \\
& \times \mathcal{K}(s,w;m_1m_2,d) \frac{ds}{s} + O\left(\frac{m_1 m_2 \ell_1^{-\frac{1}{2}+\varepsilon} X^\varepsilon}{d^{1-\varepsilon} Y^{1-\varepsilon}}\right),
\end{aligned}
$$

(7.5.18)

with $\mathcal{K}(s, w; m_1 m_2, d)$ defined by

$$
\mathcal{K}(s, w; m_1 m_2, d) = \zeta(1+w)^2 \check{\Phi}(w) \frac{\Gamma\left(\frac{s}{2}+\frac{1}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2} \left(1 - \frac{1}{2^{\frac{1}{2}-s}}\right)^2 \pi^{-s} m_1 m_2 \ell_1^{s-w-\frac{1}{2}}
$$

(7.5.19)

$$
\times \zeta(2s - 2w)\zeta(2s+1) \sum_{\substack{\alpha=1 \\ (\alpha, 2m_1 m_2)=1}}^{\infty} \frac{\mu(\alpha)}{\alpha^{2-2s+2w} d_1} \mathcal{H}_1(s - w, 1 + w; m_1 m_2, \alpha d),
$$

where, as before, ℓ_1 is defined by (7.5.12), d_1 is defined by (7.2.5), and $\mathcal{H}_1$ is defined as the product of (7.5.13) over all primes.

It is convenient for later calculations to write $\mathcal{P}_1$ in terms of a residue, as in (7.5.18), rather than in terms of logarithmic derivatives as in (7.5.16).

7.6. Bounding the contribution of $\mathcal{R}_1$.

Having handled $\mathcal{P}_1$ in (7.5.6), we next turn to $\mathcal{R}_1$, defined by (7.5.8). It will be convenient to denote

$$
\mathcal{R}(\ell, d) = \frac{1}{\ell}\left(\frac{1+i}{2}\right)\frac{\sqrt{2}}{2} \sum_{\substack{\alpha \leq Y \\ (\alpha, 2\ell)=1}} \frac{\mu(\alpha)}{\alpha^2 d_1}\left(\frac{d_1}{\ell}\right) \sum_{\substack{k \in \mathbb{Z} \\ k \text{ odd}}}\left(\frac{2}{k d_1}\right)
$$

(7.6.1)

$$
\times \frac{1}{2\pi i}\int_{(-\frac{1}{2}+\varepsilon)} h\left(\frac{kX}{8\alpha^2 d_1 \ell}, w\right) L(1+w, \chi_{k_1})^2 \mathcal{G}_0(1+w; k, \ell, \alpha, d)\, dw,
$$

so that $\mathcal{R}_1 = m_1 m_2 \mathcal{R}(m_1 m_2, d)$. We will bound $|\mathcal{R}(\ell, d)|$ on average as ℓ and d each range over a dyadic interval.

Let $\beta_{\ell, d} = \overline{\mathcal{R}(\ell, d)}/|\mathcal{R}(\ell, d)|$ if $\mathcal{R}(\ell, d) \neq 0$, and $\beta_{\ell, d} = 1$ otherwise. Then $|\beta_{\ell, d}| = 1$ and $|\mathcal{R}(\ell, d)| = \beta_{\ell, d}\mathcal{R}(\ell, d)$. We sum this over all ℓ, d with $J \leq \ell < 2J$ and $V \leq d < 2V$, where $J, V \geq 1$. We then insert the definition (7.6.1) and bring the d, ℓ-sum inside the integral to deduce that

(7.6.2)
$$
\sum_{\substack{d=V \\ (d,2)=1}}^{2V-1} \sum_{\substack{\ell=J \\ (\ell, 2d)=1}}^{2J-1} |\mathcal{R}(\ell, d)| = \sum_{\substack{d=V \\ (d,2)=1}}^{2V-1} \sum_{\substack{\ell=J \\ (\ell, 2d)=1}}^{2J-1} \beta_{\ell, d}\mathcal{R}(\ell, d) \ll \sum_{\substack{\alpha \leq Y \\ (\alpha, 2)=1}} \frac{1}{\alpha^2} \sum_{\substack{k \in \mathbb{Z} \\ k \text{ odd}}} \int_{(-\frac{1}{2}+\varepsilon)} U(\alpha, k, w)\, |dw|,
$$

where for brevity we denote

$$
U(\alpha, k, w) = \sum_{\substack{d=V \\ (d,2)=1}}^{2V-1} \frac{1}{d_1}|L(1+w, \chi_{k_1})|^2 \left| \sum_{\substack{\ell=J \\ (\ell, 2\alpha d)=1}}^{2J-1} \frac{\beta_{\ell, d}}{\ell}\left(\frac{d_1}{\ell}\right)\mathcal{G}_0(1+w; k, \ell, \alpha, d) h\left(\frac{kX}{8\alpha^2 d_1 \ell}, w\right) \right|.
$$

We split the k-sum into dyadic blocks $K \le |k| < 2K$, with $K \ge 1$, and apply Cauchy's inequality to write

(7.6.3)

$$
\sum_{\substack{K \le |k| < 2K \\ k \text{ odd}}} U(\alpha, k, w) \ll \left(\sum_{\substack{d=V \\ (d,2)=1}}^{2V-1} \frac{1}{d_1} \sum_{\substack{K \le |k| < 2K \\ k \text{ odd}}} k_2 |L(1+w, \chi_{k_1})|^4 \right)^{\frac{1}{2}}
$$

$$
\times \left(\sum_{\substack{d=V \\ (d,2)=1}}^{2V-1} \frac{1}{d_1} \sum_{\substack{K \le |k| < 2K \\ k \text{ odd}}} \frac{1}{k_2} \left| \sum_{\substack{\ell=J \\ (\ell,2\alpha d)=1}}^{2J-1} \frac{\beta_{\ell,d}}{\ell} \left(\frac{d_1}{\ell} \right) \mathcal{G}_0(1+w; k, \ell, \alpha, d) h\left(\frac{kX}{8\alpha^2 d_1 \ell}, w \right) \right|^2 \right)^{\frac{1}{2}},
$$

where k_2 is defined by (7.5.3). To bound the first factor on the right-hand side of (7.6.3), we split the k-sum according to the values of k_1 and k_2 and interchange the order of summation. Then we use the fact that $d_1 \ge d/\alpha$ by (7.2.5) to deduce that

$$
\sum_{\substack{d=V \\ (d,2)=1}}^{2V-1} \frac{1}{d_1} \sum_{\substack{K \le |k| < 2K \\ k \text{ odd}}} k_2 |L(1+w, \chi_{k_1})|^4 \le \frac{\alpha}{V} \sum_{0 < |k_1| \ll KV} |L(1+w, \chi_{k_1})|^4 \sum_{k_2 \ll \sqrt{\frac{KV}{k_1}}} k_2 \sum_{\substack{d=V \\ (d,2)=1 \\ d_1 | k_1 k_2^2}}^{2V-1} 1.
$$

We estimate the inner sum using the divisor bound, and find that the above is

$$
\ll \alpha K^{1+\varepsilon} V^\varepsilon \sum_{0 < |k_1| \ll KV} \frac{1}{k_1} |L(1+w, \chi_{k_1})|^4 \ll \alpha K^{1+\varepsilon} V^\varepsilon (1+|w|)^{1+\varepsilon}
$$

by Lemma 4.5. It follows from this and (7.6.3) that

(7.6.4)

$$
\sum_{\substack{K \le |k| < 2K \\ k \text{ odd}}} U(\alpha, k, w) \ll \left(\alpha K^{1+\varepsilon} V^\varepsilon (1+|w|)^{1+\varepsilon} \right)^{\frac{1}{2}}
$$

$$
\times \left(\sum_{\substack{d=V \\ (d,2)=1}}^{2V-1} \frac{1}{d_1} \sum_{\substack{K \le |k| < 2K \\ k \text{ odd}}} \frac{1}{k_2} \left| \sum_{\substack{\ell=J \\ (\ell,2\alpha d)=1}}^{2J-1} \frac{\beta_{\ell,d}}{\ell} \left(\frac{d_1}{\ell} \right) \mathcal{G}_0(1+w; k, \ell, \alpha, d) h\left(\frac{kX}{8\alpha^2 d_1 \ell}, w \right) \right|^2 \right)^{\frac{1}{2}}.
$$

The next task is to bound the second factor on the right-hand side. To this end we prove the following two lemmas.

Lemma 7.3. *Let $\alpha \le Y$, d, K, and J be positive integers, and suppose w is a complex number with real part $-\frac{1}{2} + \varepsilon$. Then for any choice of complex numbers γ_ℓ with $|\gamma_\ell| \le 1$,*

$$
\sum_{\substack{K \le |k| < 2K \\ k \text{ odd}}} \frac{1}{k_2} \left| \sum_{\substack{\ell=J \\ (\ell,2\alpha d)=1}}^{2J-1} \frac{\gamma_\ell}{\ell} \mathcal{G}_0(1+w; k, \ell, \alpha, d) h\left(\frac{kX}{8\alpha^2 d_1 \ell}, w \right) \right|^2
$$

is bounded by

$$
\ll_\varepsilon |\check{\Phi}(w)|^2 \frac{d_1 \alpha^{2+\varepsilon} J^{2+\varepsilon} K^\varepsilon d^\varepsilon}{X^{1-\varepsilon}} \exp\left(-\frac{1}{20} \frac{\sqrt{K}}{\alpha \sqrt{d_1 J(1+|w|)}} \right).
$$

and also by

$$\ll_\varepsilon ((1+|w|)\alpha dJKX)^\varepsilon |\check{\Phi}(w)|^2 \frac{\alpha^2 d_1 (JK+J^2)}{KX}.$$

Lemma 7.4. *Let $\delta_\ell \ll \ell^\varepsilon$ be any sequence of complex numbers and let $\operatorname{Re}(w) = -\frac{1}{2} + \varepsilon$.*
Then

$$\sum_{K \le |k| < 2K} \frac{1}{k_2} \left| \sum_{\substack{\ell=J \\ (\ell,2\alpha d)=1}}^{2J-1} \frac{\delta_\ell}{\sqrt{\ell}} \mathcal{G}_0(1+w;k,\ell,\alpha,d) \right|^2 \ll_\varepsilon (\alpha dJK)^\varepsilon J(J+K).$$

Proof of Lemma 7.3 assuming Lemma 7.4. To prove the first bound, we use the triangle inequality and apply the bounds for $\mathcal{G}_0$ from Lemma 7.1 and $h(\xi,w)$ from Lemma 7.2 to deduce that the sum in question is

$$\ll |\check{\Phi}(w)|^2 \frac{d_1 \alpha^{2+\varepsilon} J^\varepsilon K^\varepsilon d^\varepsilon}{X^{1-\varepsilon}} \exp\left(-\frac{1}{20} \frac{\sqrt{K}}{\alpha\sqrt{d_1 J}(1+|w|)}\right) \sum_{\substack{K \le |k| < 2K \\ k \text{ odd}}} \frac{1}{|k|k_2} \left(\sum_{\substack{\ell=J \\ (\ell,2\alpha d)=1}}^{2J-1} (\ell,k_4^2)^{\frac{1}{2}}\right)^2.$$

We then estimate the k-sum by splitting it according to the values of k_1 and k_2 and using $(\ell,k_4^2) \le k_4^2 \le k_2^2$, which follows from (7.5.3) and (7.5.4). This leads to the first bound of the lemma.

To prove the second bound, we apply Lemma 7.2 and write the integral (7.5.5) as

$$\frac{1}{2\pi i} \int_{(c)} g(s,w;\operatorname{sgn}(\xi)) \left(\frac{X}{\pi|\xi|}\right)^s ds$$

with $c = \varepsilon$. We then bring the ℓ-sum inside the integral and use the triangle inequality to deduce that

$$\left| \sum_{\substack{\ell=J \\ (\ell,2\alpha d)=1}}^{2J-1} \frac{\gamma_\ell}{\ell} \mathcal{G}_0(1+w;k,\ell,\alpha,d) h\left(\frac{kX}{8\alpha^2 d_1 \ell}, w\right) \right|$$

$$\ll |\check{\Phi}(w)| \left(\frac{\alpha^{1+\varepsilon} d_1^{\frac{1}{2}+\varepsilon}}{|k|^{\frac{1}{2}-\varepsilon} X^{\frac{1}{2}-\varepsilon}}\right) \int_{(\varepsilon)} \left| g(s,w;\operatorname{sgn}(k)) \sum_{\substack{\ell=J \\ (\ell,2\alpha d)=1}}^{2J-1} \frac{\gamma_\ell}{\ell^{1+w-s}} \mathcal{G}_0(1+w;k,\ell,\alpha,d) \right| |ds|.$$

Thus, since $g(s,w;\operatorname{sgn}(k)) \ll_\varepsilon (1+|w|)^\varepsilon \exp(-(\frac{\pi}{2}-\varepsilon)|\operatorname{Im}(s)|)$ by Stirling's formula, it follows from Cauchy's inequality that

$$\left| \sum_{\substack{\ell=J \\ (\ell,2\alpha d)=1}}^{2J-1} \frac{\gamma_\ell}{\ell} \mathcal{G}_0(1+w;k,\ell,\alpha,d) h\left(\frac{kX}{8\alpha^2 d_1 \ell}, w\right) \right|^2 \ll (1+|w|)^\varepsilon |\check{\Phi}(w)|^2 \left(\frac{\alpha^{2+\varepsilon} d_1^{1+\varepsilon}}{|k|^{1-\varepsilon} X^{1-\varepsilon}}\right)$$

$$\times \int_{(\varepsilon)} \exp(-(\tfrac{\pi}{2}-\varepsilon)|\operatorname{Im}(s)|) \left| \sum_{\substack{\ell=J \\ (\ell,2\alpha d)=1}}^{2J-1} \frac{\gamma_\ell}{\ell^{1+w-s}} \mathcal{G}_0(1+w;k,\ell,\alpha,d) \right|^2 |ds|.$$

The second bound of the lemma follows from this and Lemma 7.4. $\qquad\square$

Proof of Lemma 7.4. For any integer $k = \pm \prod_{i,\ a_i \geq 1} p_i^{a_i}$, let $a(k)$ and $b(k)$ be defined by

$$(7.6.5) \qquad a(k) = \prod_i p_i^{a_i+1} \quad \text{and} \quad b(k) = \prod_{a_i=1} p_i \prod_{a_i \geq 2} p_i^{a_i-1}.$$

From the definition of $\mathcal{G}_0$ in Lemma 7.1, we see that $\mathcal{G}_0(1 + w; k, \ell, \alpha, d) = 0$ unless ℓ can be written as gm with $g|a(k)$ and m square-free and relatively prime to k. With this expression for ℓ, it follows from Lemma 4.3 that if $(\ell, 2\alpha d) = 1$ then

$$(7.6.6) \qquad \mathcal{G}_0(1 + w; k, \ell, \alpha, d) = \sqrt{m} \left(\frac{k}{m} \right) \prod_{p|m} \left(1 + \frac{2}{p^{1+w}} \left(\frac{k_1}{p} \right) \right)^{-1} \mathcal{G}_0(1 + w; k, g, \alpha, d).$$

From this and Cauchy's inequality, we arrive at
$$(7.6.7)$$

$$\sum_{K \leq |k| < 2K} \frac{1}{k_2} \left| \sum_{\substack{\ell = J \\ (\ell, 2\alpha d) = 1}}^{2J-1} \frac{\delta_\ell}{\sqrt{\ell}} \mathcal{G}_0(1 + w; k, \ell, \alpha, d) \right|^2 \ll_\varepsilon K^\varepsilon \sum_{K \leq |k| < 2K} \frac{1}{k_2} \sum_{\substack{g|a(k) \\ g < 2J}} (\Psi_1(k, g) + \Psi_2(k, g)),$$

where

$$\Psi_1(k, g) = \left| \sum_{\substack{\frac{J}{g} \leq m < \frac{2J}{g} \\ (m, 2\alpha d) = 1 \\ 3 \nmid m}} \frac{\mu^2(m) \delta_{gm}}{\sqrt{g}} \mathcal{G}_0(1 + w; k, g, \alpha, d) \left(\frac{k}{m} \right) \prod_{p|m} \left(1 + \frac{2}{p^{1+w}} \left(\frac{k_1}{p} \right) \right)^{-1} \right|^2$$

and $\Psi_2(k, g)$ is the same, but with the condition $3|m$ instead of $3 \nmid m$. We first bound the contribution of Ψ_1. We factor out $g^{-1/2} \mathcal{G}_0(1 + w; k, g, \alpha, d)$ and apply the bound from Lemma 7.1 to deduce that

$$(7.6.8) \qquad \Psi_1(k, g) \ll_\varepsilon (\alpha d K)^\varepsilon g^{1+\varepsilon} \left| \sum_{\substack{\frac{J}{g} \leq m < \frac{2J}{g} \\ (m, 6\alpha d) = 1}} \mu^2(m) \delta_{gm} \left(\frac{k}{m} \right) \prod_{p|m} \left(1 + \frac{2}{p^{1+w}} \left(\frac{k_1}{p} \right) \right)^{-1} \right|^2.$$

If $\left(\frac{k}{m} \right) \neq 0$, then

$$\prod_{p|m} \left(1 + \frac{2}{p^{1+w}} \left(\frac{k_1}{p} \right) \right)^{-1} = \prod_{p|m} \left(1 - \frac{4}{p^{2+2w}} \right)^{-1} \prod_{p|m} \left(1 - \frac{2}{p^{1+w}} \left(\frac{k_1}{p} \right) \right)$$

$$= \prod_{p|m} \left(1 - \frac{4}{p^{2+2w}} \right)^{-1} \sum_{j|m} \frac{\mu(j) d_2(j)}{j^{1+w}} \left(\frac{k_1}{j} \right).$$

We insert this into (7.6.8), interchange the order of summation, and apply Cauchy's inequality to see that

$$\Psi_1(k, g) \ll_\varepsilon (\alpha d K)^\varepsilon g^{1+\varepsilon} \sum_{j < \frac{2J}{g}} \left| \sum_{\substack{\frac{J}{g} \leq m < \frac{2J}{g} \\ (m, 6\alpha d) = 1 \\ j|m}} \mu^2(m) \delta_{gm} \left(\frac{k}{m} \right) \prod_{p|m} \left(1 - \frac{4}{p^{2+2w}} \right)^{-1} \right|^2.$$

We next relabel m as jm, factor out $\mu^2(j) \left(\frac{k}{j}\right) \prod_{p|j} \left(1 - \frac{4}{p^{2+2w}}\right)^{-1}$ from the m-sum, and observe that $\prod_{p|j} \left(1 - \frac{4}{p^{2+2w}}\right)^{-1} \ll_\varepsilon j^\varepsilon$ because $\operatorname{Re}(w) \geq -\frac{1}{2} + \varepsilon$ and $p > 3$ for all $p|m$. The result is

$$(7.6.9) \quad \Psi_1(k, g) \ll_\varepsilon (\alpha d J K)^\varepsilon g^{1+\varepsilon} \sum_{j < \frac{2J}{g}} \left| \sum_{\substack{\frac{J}{gj} \leq m < \frac{2J}{gj} \\ (m, 6\alpha d j) = 1}} \mu^2(m) \delta_{gjm} \left(\frac{k}{m}\right) \prod_{p|m} \left(1 - \frac{4}{p^{2+2w}}\right)^{-1} \right|^2.$$

Now, by (7.6.5), $g|a(k)$ implies $b(g)|k$. Thus we may interchange the order of summation to write

$$\sum_{\substack{K \leq |k| < 2K \\ g|a(k) \\ g < 2J}} \frac{1}{k_2} \sum \Psi_1(k, g) \leq \sum_{g < 2J} \sum_{\substack{K \leq |k| < 2K \\ b(g)|k}} \frac{1}{k_2} \Psi_1(k, g) = \sum_{g < 2J} \sum_{\frac{K}{b(g)} \leq |f| < \frac{2K}{b(g)}} \frac{1}{k_2} \Psi_1(fb(g), g),$$

where we have relabeled k in the last sum as $fb(g)$, so that, by (7.5.3), $k_2 > 0$ satisfies $4fb(g)d_1 = k_1 k_2^2$, with k_1 a fundamental discriminant. From this and (7.6.9), we arrive at

$$
\begin{aligned}
\sum_{\substack{K \leq |k| < 2K}} \frac{1}{k_2} \sum_{\substack{g|a(k) \\ g < 2J}} \Psi_1(k, g) &\ll_\varepsilon (\alpha d J K)^\varepsilon \sum_{g < 2J} g \sum_{\frac{K}{b(g)} \leq |f| < \frac{2K}{b(g)}} \frac{1}{k_2} \\
(7.6.10) \qquad\qquad & \times \sum_{j < \frac{2J}{g}} \left| \sum_{\substack{\frac{J}{gj} \leq m < \frac{2J}{gj} \\ (m, 6\alpha d j) = 1}} \mu^2(m) \delta_{gjm} \left(\frac{fb(g)}{m}\right) \prod_{p|m} \left(1 - \frac{4}{p^{2+2w}}\right)^{-1} \right|^2.
\end{aligned}
$$

If $4f = f_1 f_2^2$, with f_1 a fundamental discriminant and f_2 a positive integer, then the equation $4fb(g)d_1 = k_1 k_2^2$ implies that $f_2|2k_2$, and thus $k_2^{-1} \ll f_2^{-1}$. Hence it follows from (7.6.10) and Lemma 4.4 that

$$\sum_{\substack{K \leq |k| < 2K}} \frac{1}{k_2} \sum_{\substack{g|a(k) \\ g < 2J}} \Psi_1(k, g) \ll_\varepsilon (\alpha d J K)^\varepsilon J(J + K).$$

This proves the desired bound for the sum of $\Psi_1(k, g)$ in (7.6.7). To bound the sum of $\Psi_2(k, g)$, we argue in the same way, but instead of (7.6.6) we use

$$\mathcal{G}_0(1 + w; k, \ell, \alpha, d) = \sqrt{m} \left(\frac{k}{m}\right) \prod_{\substack{p|m \\ p > 3}} \left(1 + \frac{2}{p^{1+w}} \left(\frac{k_1}{p}\right)\right)^{-1} \mathcal{G}_0^*(1 + w; k, g, \alpha, d),$$

where

$$\mathcal{G}_0^*(1 + w; k, g, \alpha, d) = \left(1 - \frac{1}{3^{1+w}} \left(\frac{k_1}{3}\right)\right)^2 \prod_{p \neq 3} \mathcal{G}_{0,p}(1 + w; k, g, \alpha, d),$$

with $\mathcal{G}_{0,p}$ as defined in Lemma 7.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now estimate the contribution of $\mathcal{R}_1$. From the first bound of Lemma 7.3, we see that the sum of the right-hand side of (7.6.4) over all $K = 2^j > \alpha^2 V J(1 + |w|)(\log X)^4$ is

negligible. On the other hand, if $K \leq \alpha^2 V J (1 + |w|)(\log X)^4$ then it follows from (7.6.4) and the second bound in Lemma 7.3 that

$$
\sum_{\substack{K \leq |k| < 2K \\ k \text{ odd}}} U(\alpha, k, w) \ll_\varepsilon (1 + |w|)^{\frac{1}{2} + \varepsilon} |\check{\Phi}(w)| (\alpha J K V X)^\varepsilon \left(\frac{\alpha^3 V (JK + J^2)}{X} \right)^{\frac{1}{2}}
$$

$$
\ll_\varepsilon (1 + |w|)^{1 + \varepsilon} |\check{\Phi}(w)| (\alpha J K V X)^\varepsilon \frac{\alpha^{\frac{5}{2}} V J}{X^{\frac{1}{2}}}.
$$

We sum this over all $K = 2^j$, j a positive integer, with $K \leq \alpha^2 V J (1 + |w|)(\log X)^4$, and then multiply the resulting sum by α^{-2}. We then integrate over all w with $\mathrm{Re}(w) = -\frac{1}{2} + \varepsilon$ and sum over all integers $\alpha \leq Y$ to deduce from (7.6.2) that

$$
(7.6.11) \qquad \sum_{\substack{d=V \\ (d,2)=1}}^{2V-1} \sum_{\substack{\ell=J \\ (\ell,2d)=1}}^{2J-1} |\mathcal{R}(\ell, d)| \ll \frac{V^{1+\varepsilon} J^{1+\varepsilon} Y^{\frac{3}{2}+\varepsilon}}{X^{\frac{1}{2}-\varepsilon}}.
$$

Recall from (7.5.8) and (7.6.1) that $\mathcal{R}_1 = m_1 m_2 \mathcal{R}(m_1 m_2, d)$. Since $\lambda_d \ll d^\varepsilon$ by (5.8) and $b_m \ll 1$ by (3.5), it thus follows from (7.6.11) that

$$
(7.6.12) \qquad \sum_{\substack{d \leq D \\ d \text{ odd}}} \mu^2(d) \lambda_d \sum_{\substack{m_1, m_2 \leq M \\ (m_1 m_2, 2d)=1}} \frac{b_{m_1} b_{m_2}}{(m_1 m_2)^{3/2}} |\mathcal{R}_1| \ll \frac{D^{1+\varepsilon} M^{1+\varepsilon} Y^{\frac{3}{2}+\varepsilon}}{X^{\frac{1}{2}-\varepsilon}}.
$$

7.7. Conditions for the parameters.

From (7.5.1), (7.5.6), (7.5.18), and (7.6.12), we see that the total contribution of the sum with $\mathcal{Q}_1$ to $\mathcal{B}$ in (7.4.1) is

(7.7.1)

$$
\frac{X}{(\sqrt{2}-1)^4} \sum_{\substack{d \leq D \\ d \text{ odd}}} \mu^2(d) \lambda_d \sum_{\substack{m_1, m_2 \leq M \\ (m_1 m_2, 2d)=1}} \frac{b_{m_1} b_{m_2}}{(m_1 m_2)^{3/2}} \sum_{\substack{\nu=1 \\ (\nu, 2d)=1}}^{\infty} \frac{d_2(\nu)}{\nu^{3/2}} \sum_{\substack{\alpha \leq Y \\ (\alpha, 2 m_1 m_2 \nu)=1}} \frac{\mu(\alpha)}{\alpha^2 d_1} \mathcal{Q}_1
$$

$$
= \left(\frac{1+i}{2} \right) \frac{\sqrt{2} X}{2(\sqrt{2}-1)^4} \sum_{\substack{d \leq D \\ d \text{ odd}}} \mu^2(d) \lambda_d \sum_{\substack{m_1, m_2 \leq M \\ (m_1 m_2, 2d)=1}} \frac{b_{m_1} \overline{b_{m_2}}}{(m_1 m_2)^{3/2}} \operatorname*{Res}_{w=0} X^w \frac{1}{2\pi i} \int_{\left(\frac{1}{\log X} \right)} \Gamma_2(s-w)
$$

$$
\times 8^{s-w} \left(1 - \frac{1}{4^{s-w}} \right) \mathcal{K}(s, w; m_1 m_2, d) \frac{ds}{s} + O\left(\frac{X^{1+\varepsilon} D^\varepsilon M^\varepsilon}{Y^{1-\varepsilon}} + X^{\frac{1}{2}+\varepsilon} D^{1+\varepsilon} M^{1+\varepsilon} Y^{\frac{3}{2}+\varepsilon} \right).
$$

Recall the definition (3.4) of M. Also, recall the definitions (5.9) and (5.2) of D and R of D, respectively. So that the error terms in (7.7.1) are $O(X^{1-\varepsilon})$, we assume the parameters θ and ϑ satisfy

$$
\theta + 2\vartheta < \frac{1}{2},
$$

and we take the parameter Y in (3.6) to be

$$
Y = X^\delta
$$

with $\delta = \delta(\theta, \vartheta)$ sufficiently small.

7.8. **Evaluating the sums of the other terms with $k \neq 0$.** The procedure for evaluating the sum with $\mathcal{Q}_2$ in (7.4.1) is largely similar to the above process for $\mathcal{Q}_1$, with only a few differences. The main difference arises from the negative sign in the character $\left(\frac{-2d_1}{m_1 m_2 \nu}\right)$ in (7.4.3). This causes the residues in the versions of (7.5.6) and (7.5.7) for $\mathcal{Q}_2$ to have each $-kd_1$ equal to a perfect square instead of $kd_1 = \square$. This means $\mathrm{sgn}(k) = -1$. Hence, because of the factor $\mathrm{sgn}(\xi)$ in (7.5.5), the version of (7.5.9) for $\mathcal{Q}_2$ has the function

$$(2\pi)^{-u} \Gamma(u)\left(\cos\left(\tfrac{\pi}{2}u\right) + i \sin\left(\tfrac{\pi}{2}u\right)\right)$$

in place of the function $\Gamma_2(u)$ defined by (7.5.10). These lead to a version of (7.7.1) for $\mathcal{Q}_2$ that we may combine with (7.7.1) using the identity

(7.8.1) $$\left(\frac{1+i}{2}\right)(\cos u - i \sin u) + \left(\frac{1-i}{2}\right)(\cos u + i \sin u) = \cos u + \sin u.$$

The result is

(7.8.2)

$$\frac{X}{(\sqrt{2}-1)^4} \sum_{\substack{d \leq D \\ d \text{ odd}}} \mu^2(d)\lambda_d \sum_{\substack{m_1, m_2 \leq M \\ (m_1 m_2, 2d)=1}} \frac{b_{m_1} b_{m_2}}{(m_1 m_2)^{3/2}} \sum_{\substack{\nu=1 \\ (\nu, 2d)=1}}^{\infty} \frac{d_2(\nu)}{\nu^{3/2}} \sum_{\substack{\alpha \leq Y \\ (\alpha, 2m_1 m_2 \nu)=1}} \frac{\mu(\alpha)}{\alpha^2 d_1} (\mathcal{Q}_1 + \mathcal{Q}_2)$$

$$= \frac{\sqrt{2}X}{2(\sqrt{2}-1)^4} \sum_{\substack{d \leq D \\ d \text{ odd}}} \mu^2(d)\lambda_d \sum_{\substack{m_1, m_2 \leq M \\ (m_1 m_2, 2d)=1}} \frac{b_{m_1} b_{m_2}}{(m_1 m_2)^{3/2}} \operatorname*{Res}_{w=0} X^w \frac{1}{2\pi i} \int_{\left(\frac{1}{\log X}\right)} \Gamma_1(s-w)$$

$$\times 8^{s-w} \left(1 - \frac{1}{4^{s-w}}\right) \mathcal{K}(s, w; m_1 m_2, d) \frac{ds}{s} + O(X^{1-\varepsilon}),$$

where

(7.8.3) $$\Gamma_1(u) = (2\pi)^{-u} \Gamma(u)\left(\cos\left(\tfrac{\pi}{2}u\right) + \sin\left(\tfrac{\pi}{2}u\right)\right)$$

and the bound $O(X^{1-\varepsilon})$ for the error term is guaranteed by the conditions in Subsection 7.7.

The evaluation of the sums in (7.4.1) with $\mathcal{Q}_3$ and $\mathcal{Q}_4$ defined by (7.4.4) and (7.4.5) is similar. The version of (7.5.7) for $\mathcal{Q}_3$ has an extra -1 factor because the Kronecker symbol $\left(\frac{-2}{kd_1}\right)$ equals -1 when $-kd_1$ is an odd perfect square. The resulting expression for the sums in (7.4.1) with $\mathcal{Q}_3$ and $\mathcal{Q}_4$ is exactly the same as the right-hand side of (7.8.2). Therefore

$$\frac{X}{(\sqrt{2}-1)^4} \sum_{\substack{d \leq D \\ d \text{ odd}}} \mu^2(d)\lambda_d \sum_{\substack{m_1, m_2 \leq M \\ (m_1 m_2, 2d)=1}} \frac{b_{m_1} b_{m_2}}{(m_1 m_2)^{3/2}} \sum_{\substack{\nu=1 \\ (\nu, 2d)=1}}^{\infty} \frac{d_2(\nu)}{\nu^{3/2}} \sum_{\substack{\alpha \leq Y \\ (\alpha, 2m_1 m_2 \nu)=1}} \frac{\mu(\alpha)}{\alpha^2 d_1} \sum_{j=1}^{4} \mathcal{Q}_j$$

(7.8.4) $$= \frac{\sqrt{2}X}{(\sqrt{2}-1)^4} \sum_{\substack{d \leq D \\ d \text{ odd}}} \mu^2(d)\lambda_d \sum_{\substack{m_1, m_2 \leq M \\ (m_1 m_2, 2d)=1}} \frac{b_{m_1} b_{m_2}}{(m_1 m_2)^{3/2}} \operatorname*{Res}_{w=0} X^w \frac{1}{2\pi i} \int_{\left(\frac{1}{\log X}\right)} \Gamma_1(s-w)$$

$$\times 8^{s-w} \left(1 - \frac{1}{4^{s-w}}\right) \mathcal{K}(s, w; m_1 m_2, d) \frac{ds}{s} + O(X^{1-\varepsilon}).$$

To estimate the sum with $\mathcal{U}_1$ in (7.4.1), we first relabel k in (7.4.6) as $2k$, now with k odd, to write

$$(7.8.5) \quad \mathcal{U}_1 = \left(\frac{1+i}{2}\right)\left(\frac{2d_1}{m_1 m_2 \nu}\right) \sum_{\substack{k\in\mathbb{Z}\\ k \text{ odd}}} e\left(\frac{k\overline{\alpha^2 d_1 m_1 m_2 \nu}}{4}\right) \hat{F}_\nu\left(\frac{kX}{4\alpha^2 d_1 m_1 m_2 \nu}\right) G_{2k}(m_1 m_2 \nu).$$

From the definition (4.2) of $G_k(n)$, we see that $G_{2k}(n) = \left(\frac{2}{n}\right) G_k(n)$ for all odd integers n. Also, the orthogonality of Dirichlet characters modulo 4 implies that $e(\frac{h}{4}) = i(\frac{-1}{h})$ for odd h. It follows from these and (7.8.5) that

$$\mathcal{U}_1 = i\left(\frac{1+i}{2}\right)\left(\frac{-d_1}{m_1 m_2 \nu}\right) \sum_{\substack{k\in\mathbb{Z}\\ k \text{ odd}}} \left(\frac{-1}{kd_1}\right) \hat{F}_\nu\left(\frac{kX}{4\alpha^2 d_1 m_1 m_2 \nu}\right) G_k(m_1 m_2 \nu).$$

We then proceed as we did for $\mathcal{Q}_1$. We treat the sum with $\mathcal{U}_2$, defined by (7.4.7), in a similar way. We combine the resulting expressions using the identity (7.8.1), and we arrive at

$$(7.8.6)$$

$$\frac{X}{(\sqrt{2}-1)^4} \sum_{\substack{d\leq D\\ d \text{ odd}}} \mu^2(d)\lambda_d \sum_{\substack{m_1,m_2\leq M\\ (m_1 m_2, 2d)=1}} \frac{b_{m_1} b_{m_2}}{(m_1 m_2)^{3/2}} \sum_{\substack{\nu=1\\ (\nu,2d)=1}}^{\infty} \frac{d_2(\nu)}{\nu^{3/2}} \sum_{\substack{\alpha\leq Y\\ (\alpha,2m_1 m_2 \nu)=1}} \frac{\mu(\alpha)}{\alpha^2 d_1} (\mathcal{U}_1 + \mathcal{U}_2)$$

$$= \frac{X}{(\sqrt{2}-1)^4} \sum_{\substack{d\leq D\\ d \text{ odd}}} \mu^2(d)\lambda_d \sum_{\substack{m_1,m_2\leq M\\ (m_1 m_2, 2d)=1}} \frac{b_{m_1} b_{m_2}}{(m_1 m_2)^{3/2}} \operatorname*{Res}_{w=0} X^w \frac{1}{2\pi i} \int_{\left(\frac{1}{\log X}\right)} \Gamma_1(s-w)$$

$$\times 4^{s-w}\left(1 - \frac{1}{4^{s-w}}\right) \mathcal{K}(s,w;m_1 m_2, d) \frac{ds}{s} + O(X^{1-\varepsilon}).$$

Next, to evaluate the sum with $\mathcal{V}$ in (7.4.1), we relabel k in (7.4.8) as $4k$, now with k odd, to see that

$$\mathcal{V} = -\left(\frac{2d_1}{m_1 m_2 \nu}\right) \sum_{\substack{k\in\mathbb{Z}\\ k \text{ odd}}} \hat{F}_\nu\left(\frac{kX}{2\alpha^2 d_1 m_1 m_2 \nu}\right) \tau_k(m_1 m_2 \nu)$$

since $e(h/2) = -1$ for odd h and $\tau_{4k}(n) = \tau_k(n)$ for odd n by (4.3). Into this we insert the second expression for $\tau_k(n)$ in (4.3). Since $\left(\frac{-1}{n}\right) G_k(n) = G_{-k}(n)$ by (4.2), we may split our sum expression for $\mathcal{V}$ into two, one with $G_k(n)$ and the other with $G_{-k}(n)$. We relabel k as $-k$ in the latter and combine the result with the former to arrive at

$$(7.8.7) \quad \mathcal{V} = -\left(\frac{2d_1}{m_1 m_2 \nu}\right) \sum_{\substack{k\in\mathbb{Z}\\ k \text{ odd}}} \tilde{F}_\nu\left(\frac{kX}{2\alpha^2 d_1 m_1 m_2 \nu}\right) G_k(m_1 m_2 \nu),$$

where $\tilde{F}(\xi)$ is defined by

$$\tilde{F}(\xi) = \frac{1+i}{2}\hat{F}(\xi) + \frac{1-i}{2}\hat{F}(-\xi) = \int_{-\infty}^{\infty} (\cos(2\pi\xi x) + \sin(2\pi\xi x)) F(x)\, dx.$$

We then proceed as we did for $\mathcal{Q}_1$, using [39, Lemma 5.2] instead of Lemma 7.2. We arrive at versions of (7.5.6), (7.5.7), and (7.5.8) which show that the residue at $w = 0$ equals zero

because $2kd_1 \neq \square$ when kd_1 is odd. This leads to
(7.8.8)
$$\frac{X}{(\sqrt{2}-1)^4} \sum_{\substack{d \leq D \\ d \text{ odd}}} \mu^2(d)\lambda_d \sum_{\substack{m_1,m_2 \leq M \\ (m_1m_2,2d)=1}} \frac{b_{m_1}b_{m_2}}{(m_1m_2)^{3/2}} \sum_{\substack{\nu=1 \\ (\nu,2d)=1}}^{\infty} \frac{d_2(\nu)}{\nu^{3/2}} \sum_{\substack{\alpha \leq Y \\ (\alpha,2m_1m_2\nu)=1}} \frac{\mu(\alpha)}{\alpha^2 d_1} \mathcal{V} = O(X^{1-\varepsilon})$$

under the conditions in Subsection 7.7.

Lastly, to estimate the sum with $\mathcal{W}$ in (7.4.1), we relabel k in (7.4.9) as $8k$ to write

$$\mathcal{W} = \left(\frac{d_1}{m_1m_2\nu}\right) \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \hat{F}_\nu\left(\frac{kX}{\alpha^2 d_1 m_1 m_2 \nu}\right) \tau_k(m_1m_2\nu)$$

using the fact that $e(h) = 1$ for any integer h and $\tau_{8k}(n) = \left(\frac{2}{n}\right)\tau_k(n)$ for odd n by (4.3). Into this we insert the second expression for $\tau_k(n)$ in (4.3), apply $\left(\frac{-1}{n}\right) G_k(n) = G_{-k}(n)$, and recombine the k and $-k$ terms as we did for $\mathcal{V}$ in (7.8.7) to deduce that

$$\mathcal{W} = \left(\frac{d_1}{m_1m_2\nu}\right) \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \tilde{F}_\nu\left(\frac{kX}{\alpha^2 d_1 m_1 m_2 \nu}\right) G_k(m_1m_2\nu).$$

We then proceed as we did for $\mathcal{Q}_1$, using [39, Lemma 5.2] instead of Lemma 7.2. Since we are now summing over all nonzero integers k and not just the odd ones, instead of (7.5.11) we use

$$\sum_{j=1}^{\infty} j^{-2s+2w} \mathcal{G}(1+w; j^2, m_1m_2, \alpha d) = \prod_p \sum_{b=0}^{\infty} p^{2b(w-s)} \mathcal{G}_p(1+w; p^{2b}, m_1m_2, \alpha d)$$

$$= (m_1m_2)^{1-s+w} \ell_1^{s-w-\frac{1}{2}} \zeta(2s-2w)\zeta(2s+1)\mathcal{H}_1(s-w, 1+w; m_1m_2, \alpha d).$$

We arrive at

$$\frac{X}{(\sqrt{2}-1)^4} \sum_{\substack{d \leq D \\ d \text{ odd}}} \mu^2(d)\lambda_d \sum_{\substack{m_1,m_2 \leq M \\ (m_1m_2,2d)=1}} \frac{b_{m_1}b_{m_2}}{(m_1m_2)^{3/2}} \sum_{\substack{\nu=1 \\ (\nu,2d)=1}}^{\infty} \frac{d_2(\nu)}{\nu^{3/2}} \sum_{\substack{\alpha \leq Y \\ (\alpha,2m_1m_2\nu)=1}} \frac{\mu(\alpha)}{\alpha^2 d_1} \mathcal{W}$$

(7.8.9)
$$= \frac{X}{(\sqrt{2}-1)^4} \sum_{\substack{d \leq D \\ d \text{ odd}}} \mu^2(d)\lambda_d \sum_{\substack{m_1,m_2 \leq M \\ (m_1m_2,2d)=1}} \frac{b_{m_1}b_{m_2}}{(m_1m_2)^{3/2}} \operatorname*{Res}_{w=0} X^w \frac{1}{2\pi i} \int_{\left(\frac{1}{\log X}\right)} \Gamma_1(s-w)$$

$$\times \mathcal{K}(s,w; m_1m_2, d) \frac{ds}{s} + O(X^{1-\varepsilon}).$$

7.9. **Putting together the estimates.** From (7.4.1), (7.8.4), (7.8.6), (7.8.8), and (7.8.9), we deduce that

$$\mathcal{B} = \frac{X}{(\sqrt{2}-1)^4} \sum_{\substack{d \leq D \\ d \text{ odd}}} \mu^2(d)\lambda_d \sum_{\substack{m_1,m_2 \leq M \\ (m_1m_2,2d)=1}} \frac{b_{m_1}b_{m_2}}{(m_1m_2)^{3/2}} \operatorname*{Res}_{w=0} X^w \frac{1}{2\pi i} \int_{\left(\frac{1}{\log X}\right)} \Gamma_1(s-w)$$

$$\times \left(8^{s-w}\sqrt{2} + 4^{s-w} - 2^{s-w}\sqrt{2}\right) \mathcal{K}(s,w; m_1m_2, d) \frac{ds}{s} + O(X^{1-\varepsilon}).$$

We next evaluate the residue at $w = 0$. Note that, for fixed s, the integrand has a pole of order at most 2 at $w = 0$. We use (6.1.6) with $n = 2$ to write

(7.9.1)

$$
\mathcal{B} = \frac{X}{(\sqrt{2}-1)^4} \sum_{\substack{d \leq D \\ d \text{ odd}}} \mu^2(d)\lambda_d \sum_{\substack{m_1,m_2 \leq M \\ (m_1m_2,2d)=1}} \frac{b_{m_1}b_{m_2}}{(m_1m_2)^{3/2}} \frac{1}{2\pi i} \int_{\left(\frac{1}{\log X}\right)} \Gamma_1(s)\left(8^s\sqrt{2} + 4^s - 2^s\sqrt{2}\right)
$$

$$
\times \mathcal{K}(s,0;m_1m_2,d)\left\{ \log X - \frac{\Gamma_1'(s)}{\Gamma_1(s)} - (\log 2)\frac{3 \cdot 8^s\sqrt{2} + 2 \cdot 4^s - 2^s\sqrt{2}}{8^s\sqrt{2} + 4^s - 2^s\sqrt{2}} \right.
$$

$$
\left. + \frac{\frac{\partial}{\partial w}\mathcal{K}(s,w;m_1m_2,d)}{\mathcal{K}(s,w;m_1m_2,d)}\bigg|_{w=0} \right\} \frac{ds}{s} + O(X^{1-\varepsilon}).
$$

From the definitions (7.5.19) and (7.5.13) of $\mathcal{K}$ and $\mathcal{H}$, we see that, after some simplification,

$$
\left(8^s\sqrt{2} + 4^s - 2^s\sqrt{2}\right)\mathcal{K}(s,0;m_1m_2,d)
$$

$$
= \frac{\check{\Phi}(0)}{4}\frac{\Gamma\left(\frac{s}{2}+\frac{1}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2}\left(\frac{4}{\pi}\right)^s \zeta(2s)\zeta(2s+1)\left(1 - \frac{1}{2^{\frac{1}{2}+s}}\right)\left(1 - \frac{1}{2^{\frac{1}{2}-s}}\right)\left(\frac{5}{2} - 4^s - 4^{-s}\right)
$$

(7.9.2)
$$
\times \frac{\varphi(dm_1m_2)^2}{d^3m_1m_2\sqrt{\ell_1}} \sum_{ab=\ell_1}\left(\frac{a}{b}\right)^s \prod_{\substack{p|m_1m_2 \\ p\nmid \ell_1}}\left(1 + \frac{1}{p}\right)\prod_{p|d}\left(1 - \frac{1}{p^{1+2s}}\right)\left(1 - \frac{1}{p^{1-2s}}\right)
$$

$$
\times \prod_{p\nmid 2m_1m_2d}\left\{\left(1 - \frac{1}{p}\right)^2\left(1 + \frac{2}{p} + \frac{1}{p^3} - \frac{1}{p^{2-2s}} - \frac{1}{p^{2+2s}}\right)\right\},
$$

where ℓ_1 is defined by (7.5.12), and

$$
-(\log 2)\frac{3 \cdot 8^s\sqrt{2} + 2 \cdot 4^s - 2^s\sqrt{2}}{8^s\sqrt{2} + 4^s - 2^s\sqrt{2}} + \frac{\frac{\partial}{\partial w}\mathcal{K}(s,w;m_1m_2,d)}{\mathcal{K}(s,w;m_1m_2,d)}\bigg|_{w=0}
$$

$$
= 2\gamma + \frac{(\check{\Phi})'(0)}{\check{\Phi}(0)} - \log(2\ell_1) - 2\frac{\zeta'}{\zeta}(2s) + 2\frac{\zeta'}{\zeta}(2s+1) + \frac{\log 2}{\left(\sqrt{2}+2^s\right)\left(\sqrt{2}+2^{-s}\right)}
$$

(7.9.3)
$$
+ \sum_{p|d}\left(\frac{2\log p}{p-1} + \frac{2\log p}{p^{1+2s}-1} + \frac{2\log p}{p^{1-2s}-1}\right) + \sum_{p|m_1m_2}\frac{2\log p}{p-1} - \sum_{\substack{p|m_1m_2 \\ p\nmid\ell_1}}\frac{2\log p}{p+1}
$$

$$
+ \sum_{p\nmid 2m_1m_2d}\left(\frac{2\log p}{p-1} - \left(\frac{2\log p}{p}\right)\frac{1 + \frac{2}{p^2} - \frac{1}{p}(p^{2s}+p^{-2s})}{1 + \frac{2}{p} + \frac{1}{p^3} - \frac{1}{p^2}(p^{2s}+p^{-2s})}\right).
$$

Now the definition (7.8.3) of $\Gamma_1(u)$, the Legendre duplication formula, the functional equation of $\zeta(s)$, and the identity $\Gamma(z)\Gamma(1-z) = \pi\csc(\pi z)$ imply that the functions

$$
\frac{\Gamma^2\left(\frac{s}{2}+\frac{1}{4}\right)}{\Gamma^2\left(\frac{1}{4}\right)}\left(\frac{4}{\pi}\right)^s \Gamma_1(s)\zeta(2s)\zeta(2s+1)
$$

and

$$
-\frac{\Gamma_1'(s)}{\Gamma_1(s)} - 2\frac{\zeta'}{\zeta}(2s) + 2\frac{\zeta'}{\zeta}(2s+1)
$$

are even functions of s. Hence (7.9.2) and (7.9.3) are even functions of s. It follows that the integrand in (7.9.1) is an odd function of s. We move the line of integration in (7.9.1) to $\mathrm{Re}(s) = -\frac{1}{\log X}$, leaving a residue at $s = 0$. In the new integral, we make a change of variables $s \mapsto -s$ to see that, since its integrand is odd, it equals the negative of the original integral in (7.9.1). Therefore twice the original integral equals the residue at $s = 0$. We write this residue as an integral along the circle $|s| = \frac{1}{\log X}$, taken in the positive direction, and arrive at

(7.9.4)

$$
\mathcal{B} = \frac{X}{(\sqrt{2}-1)^4} \sum_{\substack{d \leq D \\ d \text{ odd}}} \mu^2(d)\lambda_d \sum_{\substack{m_1,m_2 \leq M \\ (m_1 m_2, 2d)=1}} \frac{b_{m_1} b_{m_2}}{(m_1 m_2)^{3/2}} \frac{1}{4\pi i} \oint_{|s|=\frac{1}{\log X}} \Gamma_1(s)\left(8^s\sqrt{2} + 4^s - 2^s\sqrt{2}\right)
$$

$$
\times \mathcal{K}(s,0;m_1 m_2, d)\left\{ \log X - \frac{\Gamma_1'(s)}{\Gamma_1(s)} - (\log 2)\frac{3 \cdot 8^s \sqrt{2} + 2 \cdot 4^s - 2^s\sqrt{2}}{8^s\sqrt{2} + 4^s - 2^s\sqrt{2}} \right.
$$

$$
\left. + \frac{\frac{\partial}{\partial w}\mathcal{K}(s,w;m_1 m_2, d)}{\mathcal{K}(s,w;m_1 m_2, d)}\bigg|_{w=0} \right\} \frac{ds}{s} + O(X^{1-\varepsilon}).
$$

The next step is to carry out the summation over d. From (7.9.2) and (7.9.3), we see that we need to evaluate the sums Σ_1 and Σ_2 defined by

(7.9.5)

$$
\Sigma_1 = \sum_{\substack{d \leq D \\ (d, 2m_1 m_2)=1}} \mu^2(d)\lambda_d \frac{\varphi(d)^2}{d^3} \prod_{p|d}\left(1 - \frac{1}{p^{1+2s}}\right)\left(1 - \frac{1}{p^{1-2s}}\right)
$$

$$
\times \prod_{p \nmid 2m_1 m_2 d}\left\{\left(1 - \frac{1}{p}\right)^2\left(1 + \frac{2}{p} + \frac{1}{p^3} - \frac{1}{p^{2-2s}} - \frac{1}{p^{2+2s}}\right)\right\}
$$

and

(7.9.6)

$$
\Sigma_2 = \sum_{\substack{d \leq D \\ (d, 2m_1 m_2)=1}} \mu^2(d)\lambda_d \frac{\varphi(d)^2}{d^3} \prod_{p|d}\left(1 - \frac{1}{p^{1+2s}}\right)\left(1 - \frac{1}{p^{1-2s}}\right)
$$

$$
\times \prod_{p \nmid 2m_1 m_2 d}\left\{\left(1 - \frac{1}{p}\right)^2\left(1 + \frac{2}{p} + \frac{1}{p^3} - \frac{1}{p^{2-2s}} - \frac{1}{p^{2+2s}}\right)\right\} \sum_{p|d} J(p,s),
$$

where

(7.9.7)
$$
J(p,s) = \frac{2\log p}{p^{1+2s} - 1} + \frac{2\log p}{p^{1-2s} - 1} + \left(\frac{2\log p}{p}\right)\frac{1 + \frac{2}{p^2} - \frac{1}{p}(p^{2s} + p^{-2s})}{1 + \frac{2}{p} + \frac{1}{p^3} - \frac{1}{p^2}(p^{2s} + p^{-2s})}
$$

and $|s| = \frac{1}{\log X}$. We only estimate Σ_1 since Σ_2 may be treated in the same way, except using Lemma 5.4 instead of Lemma 5.3. We rearrange the factors in (7.9.5) to write Σ_1 as

$$
\text{(7.9.8)} \quad \begin{aligned}
\Sigma_1 = &\prod_{p \nmid 2m_1 m_2} \left\{ \left(1 - \frac{1}{p}\right)^2 \left(1 + \frac{2}{p} + \frac{1}{p^3} - \frac{1}{p^{2-2s}} - \frac{1}{p^{2+2s}}\right) \right\} \sum_{\substack{d \leq D \\ (d, 2m_1 m_2) = 1}} \frac{\mu^2(d)\lambda_d}{d} \\
&\times \prod_{p \mid d} \left(1 - \frac{1}{p^{1+2s}}\right)\left(1 - \frac{1}{p^{1-2s}}\right)\left(1 + \frac{2}{p} + \frac{1}{p^3} - \frac{1}{p^{2-2s}} - \frac{1}{p^{2+2s}}\right)^{-1}.
\end{aligned}
$$

Now recall the definition (5.1) of z_0 and the definition (5.8) of λ_d. Factoring out the product over primes $p > z_0$, we see that

$$
\begin{aligned}
&\prod_{p \nmid 2m_1 m_2} \left\{ \left(1 - \frac{1}{p}\right)^2 \left(1 + \frac{2}{p} + \frac{1}{p^3} - \frac{1}{p^{2-2s}} - \frac{1}{p^{2+2s}}\right) \right\} \\
&= \left(1 + O\left(\frac{1}{z_0}\right)\right) \prod_{\substack{p \nmid 2m_1 m_2 \\ p \leq z_0}} \left\{ \left(1 - \frac{1}{p}\right)^2 \left(1 + \frac{2}{p} + \frac{1}{p^3} - \frac{1}{p^{2-2s}} - \frac{1}{p^{2+2s}}\right) \right\}.
\end{aligned}
$$

From this, (7.9.8), Lemma 5.3, and some simplification, we deduce that

(7.9.9)
$$
\Sigma_1 = \left(1 + O\left(\frac{1}{z_0}\right)\right) \frac{1 + o(1)}{\log R} \prod_{\substack{p \nmid 2m_1 m_2 \\ p \leq z_0}} \left(1 - \frac{1}{p^2}\right) \prod_{\substack{p \mid 2m_1 m_2 \\ p \leq z_0}} \left(1 - \frac{1}{p}\right)^{-1} + O\left(\frac{1}{(\log R)^{2018}}\right).
$$

The condition $p \leq z_0$ may be omitted because $\prod_{p > z_0}(1 + O(\frac{1}{p^2})) = 1 + O(\frac{1}{z_0})$ and

$$
\prod_{\substack{p \mid 2m_1 m_2 \\ p > z_0}} \left(1 - \frac{1}{p}\right)^{-1} = \left(1 + O\left(\frac{1}{z_0}\right)\right)^{O(\log X)} = 1 + O\left(\frac{\log X}{z_0}\right).
$$

The contributions of the error terms $O\left(\frac{1}{z_0}\right)$ and $O\left(\frac{\log X}{z_0}\right)$ are negligible. From these and (7.9.9), we arrive at

$$
\text{(7.9.10)} \quad \Sigma_1 = \frac{2m_1 m_2}{\varphi(m_1 m_2)} \prod_{p \nmid 2m_1 m_2} \left(1 - \frac{1}{p^2}\right) \frac{1 + o(1)}{\log R} + O\left(\frac{1}{(\log R)^{2018}}\right).
$$

In a similar way, but using Lemma 5.4 instead of Lemma 5.3, we deduce from (7.9.6) that

$$
\text{(7.9.11)} \quad \begin{aligned}
\Sigma_2 = &-\frac{2m_1 m_2}{\varphi(m_1 m_2)} \prod_{p \nmid 2m_1 m_2} \left(1 - \frac{1}{p^2}\right) \frac{1 + o(1)}{\log R} \\
&\times \sum_{p \nmid 2m_1 m_2} \frac{J(p, s)}{p + 1} \left(1 - \frac{1}{p^{1+2s}}\right)\left(1 - \frac{1}{p^{1-2s}}\right) + O\left(\frac{1}{(\log R)^{2018}}\right).
\end{aligned}
$$

In view of the expressions (7.9.2) and (7.9.3) and the definitions (7.9.5) and (7.9.6), it now follows from (7.9.4), (7.9.10) and (7.9.11) that

(7.9.12)

$$\mathcal{B} = \frac{X\check{\Phi}(0)}{3\zeta(2)(\sqrt{2}-1)^4} \frac{1+o(1)}{\log R} \sum_{\substack{m_1,m_2\leq M \\ (m_1m_2,2)=1}} \frac{b_{m_1}b_{m_2}}{\sqrt{m_1m_2\ell_1}} \prod_{p|\ell_1}\left(\frac{p}{p+1}\right)$$

$$\times \frac{1}{2\pi i}\oint_{|s|=\frac{1}{\log X}} \sum_{ab=\ell_1}\left(\frac{a}{b}\right)^s \Gamma_1(s) \frac{\Gamma\left(\frac{s}{2}+\frac{1}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2}\left(\frac{4}{\pi}\right)^s \zeta(2s)\zeta(2s+1)\left(1-\frac{1}{2^{\frac{1}{2}+s}}\right)\left(1-\frac{1}{2^{\frac{1}{2}-s}}\right)$$

$$\times \left(\frac{5}{2}-4^s-4^{-s}\right)\left\{\log\left(\frac{X}{2\ell_1}\right)+2\gamma+\frac{(\check{\Phi})'(0)}{\check{\Phi}(0)}-\frac{\Gamma_1'(s)}{\Gamma_1(s)}-2\frac{\zeta'}{\zeta}(2s)+2\frac{\zeta'}{\zeta}(2s+1)\right.$$

$$\left.+\frac{\log 2}{(\sqrt{2}+2^s)(\sqrt{2}+2^{-s})}+\sum_{p\neq 2}\eta_1(p,s)+\sum_{\substack{p|m_1m_2 \\ p\nmid\ell_1}}\eta_2(p,s)+\sum_{p|\ell_1}\eta_3(p,s)\right\}\frac{ds}{s}+O\left(\frac{X}{(\log R)^{2018}}\right),$$

where

$$\eta_1(p,s)=\frac{2\log p}{p-1}-\left(\frac{2\log p}{p}\right)\frac{1+\frac{2}{p^2}-\frac{1}{p}\left(p^{2s}+p^{-2s}\right)}{1+\frac{2}{p}+\frac{1}{p^3}-\frac{1}{p^2}\left(p^{2s}+p^{-2s}\right)}-\frac{J(p,s)}{p+1}\left(1-\frac{1}{p^{1+2s}}\right)\left(1-\frac{1}{p^{1-2s}}\right),$$

(7.9.13) $$\eta_2(p,s)=\frac{2\log p}{p-1}-\frac{2\log p}{p+1}-\eta_1(p,s),$$

and

$$\eta_3(p,s)=\frac{2\log p}{p-1}-\eta_1(p,s),$$

with $J(p,s)$ defined by (7.9.7).

Next, we carry out the summation over m_1, m_2. We see from (7.9.12) that we need to evaluate the sums Υ_1, Υ_2, Υ_3, and Υ_4 defined by

(7.9.14) $$\Upsilon_1 = \sum_{\substack{m_1,m_2\leq M \\ (m_1m_2,2)=1}} \frac{b_{m_1}b_{m_2}}{\sqrt{m_1m_2\ell_1}} \prod_{p|\ell_1}\left(\frac{p}{p+1}\right)\sum_{ab=\ell_1}\left(\frac{a}{b}\right)^s,$$

(7.9.15) $$\Upsilon_2 = -\sum_{\substack{m_1,m_2\leq M \\ (m_1m_2,2)=1}} \frac{b_{m_1}b_{m_2}}{\sqrt{m_1m_2\ell_1}} \prod_{p|\ell_1}\left(\frac{p}{p+1}\right)\sum_{ab=\ell_1}\left(\frac{a}{b}\right)^s\log\ell_1,$$

(7.9.16) $$\Upsilon_3 = \sum_{\substack{m_1,m_2\leq M \\ (m_1m_2,2)=1}} \frac{b_{m_1}b_{m_2}}{\sqrt{m_1m_2\ell_1}} \prod_{p|\ell_1}\left(\frac{p}{p+1}\right)\sum_{ab=\ell_1}\left(\frac{a}{b}\right)^s\sum_{p|\ell_1}\eta_2(p,s),$$

and

(7.9.17) $$\Upsilon_4 = \sum_{\substack{m_1,m_2\leq M \\ (m_1m_2,2)=1}} \frac{b_{m_1}b_{m_2}}{\sqrt{m_1m_2\ell_1}} \prod_{p|\ell_1}\left(\frac{p}{p+1}\right)\sum_{ab=\ell_1}\left(\frac{a}{b}\right)^s\sum_{\substack{p|m_1m_2 \\ p\nmid\ell_1}}\eta_3(p,s),$$

with $|s|=\frac{1}{\log X}$.

To estimate Υ_1, observe that if m_1 and m_2 are square-free then (7.5.12) implies

$$(7.9.18) \qquad \ell_1 = \frac{m_1 m_2}{(m_1, m_2)^2}$$

and

$$(7.9.19) \qquad \sum_{ab=\ell_1} \left(\frac{a}{b}\right)^s = \prod_{p|\ell_1}(p^s + p^{-s}).$$

From these, the definition (3.5) of b_m, and the Fourier inversion formula (6.1.2), we deduce from (7.9.14) that

$$\Upsilon_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(z_1)h(z_2) \sum_{(m_1 m_2, 2)=1}\sum \frac{\mu(m_1)\mu(m_2)(m_1, m_2)}{m_1^{1+\frac{1+iz_1}{\log M}} m_2^{1+\frac{1+iz_2}{\log M}}} \prod_{\substack{p|m_1 m_2 \\ p\nmid(m_1,m_2)}} (p^s + p^{-s})\left(\frac{p}{p+1}\right) dz_1 dz_2.$$

Thus, writing the sum as an Euler product, we see that

$$\Upsilon_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(z_1)h(z_2) \prod_{p>2}\left(1 - \frac{1}{p^{1+\frac{1+iz_1}{\log M}}}(p^s + p^{-s})\left(\frac{p}{p+1}\right)\right.$$
$$\left. - \frac{1}{p^{1+\frac{1+iz_2}{\log M}}}(p^s + p^{-s})\left(\frac{p}{p+1}\right) + \frac{1}{p^{1+\frac{2+iz_1+iz_2}{\log M}}}\right) dz_1 dz_2.$$

We write this as
(7.9.20)
$$\Upsilon_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{h(z_1)h(z_2)\zeta\left(1+\frac{2+iz_1+iz_2}{\log M}\right) W(s, z_1, z_2, \frac{1}{\log M})\, dz_1 dz_2}{\zeta\left(1+\frac{1+iz_1}{\log M}+s\right)\zeta\left(1+\frac{1+iz_1}{\log M}-s\right)\zeta\left(1+\frac{1+iz_2}{\log M}+s\right)\zeta\left(1+\frac{1+iz_2}{\log M}-s\right)},$$

where $W(s, z_1, z_2, \frac{1}{\log M})$ is an Euler product that is bounded and holomorphic for $|s| \le \varepsilon$ and complex z_1, z_2 with $|\mathrm{Im}(z_1)|, |\mathrm{Im}(z_2)| \le \varepsilon \log M$. Note that this definition of W implies

$$(7.9.21) \qquad W(0,0,0,0) = 8\prod_{p>2}\left(1 - \frac{4}{p+1} + \frac{1}{p}\right)\left(1 - \frac{1}{p}\right)^{-3} = 6\zeta(2),$$

a fact we use shortly. By (6.1.4), we may truncate the integrals in (7.9.20) to the range $|z_1|, |z_2| \le \sqrt{\log M}$, introducing a negligible error. On this range of z_1 and z_2, the function W and the zeta-functions in (7.9.20) may be written as Laurent series. The contributions of the terms other than the first terms of these Laurent expansions are a factor of $(\log X)^{1-\varepsilon}$ smaller than the contribution of the first terms. The first term of the Laurent expansion of W is given by (7.9.21). We thus arrive at

$$\Upsilon_1 = 6\zeta(2) \iint_{|z_i| \le \sqrt{\log M}} h(z_1)h(z_2)\left(\frac{\log M}{2+iz_1+iz_2}\right)\left(\frac{1+iz_1}{\log M}-s\right)\left(\frac{1+iz_1}{\log M}+s\right)$$
$$\times \left(\frac{1+iz_2}{\log M}+s\right)\left(\frac{1+iz_2}{\log M}-s\right) dz_1 dz_2 + O\left(\frac{1}{(\log X)^{4-\varepsilon}}\right).$$

By (6.1.4), we may extend the range of integration to $\mathbb{R}^2$, introducing a negligible error. We then apply (7.3.12) to deduce that

$$\Upsilon_1 = 6\zeta(2)\left(\frac{1}{\log^3 M}\int_0^1 H''(t)^2\,dt - \frac{2s^2}{\log M}\int_0^1 H(t)H''(t)\,dt\right.$$

(7.9.22)

$$\left. + s^4\log M\int_0^1 H(t)^2\,dt\right) + O\left(\frac{1}{(\log X)^{4-\varepsilon}}\right).$$

Having evaluated Υ_1, we next estimate Υ_2. Using the residue theorem, we write

$$-\log \ell_1 = \frac{1}{2\pi i}\oint_{|y|=\frac{1}{2\log X}} \ell_1^{-y}\,\frac{dy}{y^2}.$$

From this, (7.9.15), (7.9.18), (7.9.19), the definition (3.5) of b_m, and the Fourier inversion formula (6.1.2), it follows that

$$\Upsilon_2 = \frac{1}{2\pi i}\oint_{|y|=\frac{1}{2\log X}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(z_1)h(z_2)\sum_{(m_1m_2,2)=1}\sum \frac{\mu(m_1)\mu(m_2)(m_1,m_2)^{1+2y}}{m_1^{1+\frac{1+iz_1}{\log M}+y}m_2^{1+\frac{1+iz_2}{\log M}+y}}$$

$$\times \prod_{\substack{p|m_1m_2\\p\nmid(m_1,m_2)}}(p^s+p^{-s})\left(\frac{p}{p+1}\right)dz_1dz_2\frac{dy}{y^2}.$$

We express the sum as an Euler product to see that

$$\Upsilon_2 = \frac{1}{2\pi i}\oint_{|y|=\frac{1}{2\log X}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(z_1)h(z_2)\prod_{p>2}\left(1 - \frac{1}{p^{1+\frac{1+iz_1}{\log M}+y}}(p^s+p^{-s})\left(\frac{p}{p+1}\right)\right.$$

$$\left. - \frac{1}{p^{1+\frac{1+iz_2}{\log M}+y}}(p^s+p^{-s})\left(\frac{p}{p+1}\right) + \frac{1}{p^{1+\frac{2+iz_1+iz_2}{\log M}}}\right)dz_1dz_2\,\frac{dy}{y^2}.$$

Write this as

$$\Upsilon_2 = \frac{1}{2\pi i}\oint_{|y|=\frac{1}{2\log X}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(z_1)h(z_2)\zeta\left(1+\frac{2+iz_1+iz_2}{\log M}\right)V(s,z_1,z_2,\tfrac{1}{\log M},y)$$

$$\times \zeta^{-1}\left(1+\frac{1+iz_1}{\log M}+y+s\right)\zeta^{-1}\left(1+\frac{1+iz_1}{\log M}+y-s\right)$$

$$\times \zeta^{-1}\left(1+\frac{1+iz_2}{\log M}+y+s\right)\zeta^{-1}\left(1+\frac{1+iz_2}{\log M}+y-s\right)dz_1dz_2\frac{dy}{y^2},$$

where $V(s,z_1,z_2,\frac{1}{\log M},y)$ is an Euler product that is bounded and holomorphic for $|s|,|y|\leq\varepsilon$ and complex z_1,z_2 with $|\mathrm{Im}(z_1)|,|\mathrm{Im}(z_2)|\leq\varepsilon\log M$. This definition of V implies that $V(0,0,0,0,0)=6\zeta(2)$. As in our treatment of Υ_1, we use (6.1.4) to truncate the integrals. Then we write the function V and the zeta-functions as Laurent series. The main contribution arises from the first terms of the Laurent expansions, and we arrive at

$$\Upsilon_2 = \frac{6\zeta(2)}{2\pi i}\oint_{|y|=\frac{1}{2\log X}}\iint_{|z_i|\leq\sqrt{\log M}} h(z_1)h(z_2)\left(\frac{\log M}{2+iz_1+iz_2}\right)\left(\frac{1+iz_1}{\log M}+y-s\right)$$

$$\times \left(\frac{1+iz_1}{\log M}+y+s\right)\left(\frac{1+iz_2}{\log M}+y+s\right)\left(\frac{1+iz_2}{\log M}+y-s\right)dz_1dz_2\frac{dy}{y^2} + O\left(\frac{1}{(\log X)^{3-\varepsilon}}\right).$$

We carry out the integration over y by applying the formula (6.1.6) with $n = 2$ and deduce that

$$
\Upsilon_2 = 6\zeta(2) \iint\limits_{|z_i| \leq \sqrt{\log M}} h(z_1)h(z_2) \left(\frac{\log M}{2 + iz_1 + iz_2} \right)
$$

$$
\times \left\{ \left(\frac{1 + iz_1}{\log M} + s \right) \left(\frac{(1 + iz_2)^2}{(\log M)^2} - s^2 \right) + \left(\frac{1 + iz_1}{\log M} - s \right) \left(\frac{(1 + iz_2)^2}{(\log M)^2} - s^2 \right) \right.
$$

$$
\left. + \left(\frac{1 + iz_2}{\log M} + s \right) \left(\frac{(1 + iz_1)^2}{(\log M)^2} - s^2 \right) + \left(\frac{1 + iz_2}{\log M} - s \right) \left(\frac{(1 + iz_1)^2}{(\log M)^2} - s^2 \right) \right\} dz_1 dz_2
$$

$$
+ O\left(\frac{1}{(\log X)^{3-\varepsilon}} \right).
$$

We extend the integral and apply (7.3.12). After simplifying, we arrive at
(7.9.23)
$$
\Upsilon_2 = 6\zeta(2) \left(-\frac{4}{\log^2 M} \int_0^1 H'(t)H''(t)\,dt + 4s^2 \int_0^1 H(t)H'(t)\,dt \right) + O\left(\frac{1}{(\log X)^{3-\varepsilon}} \right).
$$

We next estimate Υ_3 defined by (7.9.16). We interchange the order of summation over m_1, m_2 and over p. From (7.9.18), we see for a prime q and square-free m_1 and m_2 that $q | \ell_1$ if and only if q divides exactly one of m_1 or m_2. If q divides m_2 and not m_1, then we may relabel m_1 as m_2 and vice versa. Hence

$$
\Upsilon_3 = 2 \sum_{2 < q \leq M} \eta_2(q, s) \sum_{\substack{m_1, m_2 \leq M \\ (m_1 m_2, 2) = 1 \\ q | m_1, \ q \nmid m_2}} \frac{b_{m_1} b_{m_2}}{\sqrt{m_1 m_2 \ell_1}} \prod_{p | \ell_1} \left(\frac{p}{p+1} \right) \sum_{ab = \ell_1} \left(\frac{a}{b} \right)^s.
$$

From this, the definition (3.5) of b_m, (7.9.18), and (7.9.19), it follows that

$$
\Upsilon_3 = 2 \sum_{2 < q \leq M} \eta_2(q, s) \sum_{\substack{m_1, m_2 \leq M \\ (m_1 m_2, 2) = 1 \\ q | m_1, \ q \nmid m_2}} \frac{\mu(m_1)\mu(m_2)}{[m_1, m_2]} \prod_{\substack{p | m_1 m_2 \\ p \nmid (m_1, m_2)}} \left(\frac{p}{p+1} \right) (p^s + p^{-s})
$$

$$
\times H\left(\frac{\log m_1}{\log M} \right) H\left(\frac{\log m_2}{\log M} \right).
$$

We relabel m_1 as $q m_1$ to write this as

$$
\Upsilon_3 = -2 \sum_{2 < q \leq M} \frac{\eta_2(q, s)}{q+1} (q^s + q^{-s}) \sum_{m_1 \leq \frac{M}{q}} \sum_{\substack{m_2 \leq M \\ (m_1 m_2, 2q) = 1}} \frac{\mu(m_1)\mu(m_2)}{[m_1, m_2]}
$$

$$
\times \prod_{\substack{p | m_1 m_2 \\ p \nmid (m_1, m_2)}} \left(\frac{p}{p+1} \right) (p^s + p^{-s}) H\left(\frac{\log q m_1}{\log M} \right) H\left(\frac{\log m_2}{\log M} \right).
$$

We insert the Fourier inversion formula (6.1.2), interchange the order of summation, and then write the m_1, m_2-sum as an Euler product to deduce that

$$\Upsilon_3 = -2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{2 < q \le M} \frac{\eta_2(q,s)(q^s + q^{-s})}{(q+1)q^{\frac{1+iz_1}{\log M}}} h(z_1)h(z_2) \prod_{p \nmid 2q} \left(1 - \frac{1}{p^{1+\frac{1+iz_1}{\log M}}}(p^s + p^{-s}) \left(\frac{p}{p+1} \right) \right.$$

$$\left. - \frac{1}{p^{1+\frac{1+iz_2}{\log M}}}(p^s + p^{-s}) \left(\frac{p}{p+1} \right) + \frac{1}{p^{1+\frac{2+iz_1+iz_2}{\log M}}} \right) dz_1 dz_2.$$

We may express the Euler product in terms of zeta-functions to write

(7.9.24)

$$\Upsilon_3 = -2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{2 < q \le M} \frac{\eta_2(q,s)(q^s + q^{-s})}{(q+1)q^{\frac{1+iz_1}{\log M}}} h(z_1)h(z_2) \zeta \left(1 + \frac{2+iz_1+iz_2}{\log M} \right) \zeta^{-1} \left(1 + \frac{1+iz_1}{\log M} + s \right)$$

$$\times \zeta^{-1} \left(1 + \frac{1+iz_1}{\log M} - s \right) \zeta^{-1} \left(1 + \frac{1+iz_2}{\log M} + s \right) \zeta^{-1} \left(1 + \frac{1+iz_2}{\log M} - s \right) U_q(s, z_1, z_2, \tfrac{1}{\log M}) dz_1 dz_2,$$

where $U_q(s, z_1, z_2, \frac{1}{\log M})$ is an Euler product that is uniformly bounded for $2 < q \le M$ prime, $|s| \le \varepsilon$, and real z_1, z_2. Using (6.1.4), we may truncate the integrals to the range $|z_1|, |z_2| \le \sqrt{\log M}$ and introduce only a negligible error. In this range, and for $|s| = \frac{1}{\log X}$, the quotient of zeta-functions in (7.9.24) is $(\log M)^{-3+\varepsilon}$. Moreover, (7.9.13) implies $\eta_2(q,s) \ll byq^{-1+\varepsilon}$ for $2 < q \le M$ and $|s| = \frac{1}{\log X}$. It thus follows that

(7.9.25)
$$\Upsilon_3 \ll \frac{1}{(\log X)^{3-\varepsilon}}.$$

A similar argument applies to Υ_4 defined by (7.9.17), except we use the fact that, for a prime q, $q | m_1 m_2$ and $q \nmid \ell_1$ both hold if and only if q divides both m_1 and m_2, by (7.9.18). This leads to

(7.9.26)
$$\Upsilon_4 \ll \frac{1}{(\log X)^{3-\varepsilon}}.$$

It now follows from (7.9.12), the definitions (7.9.14) through (7.9.17) of $\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4$, and the estimates (7.9.22), (7.9.23), (7.9.25), and (7.9.26) that

$$\mathcal{B} = \frac{2X\check{\Phi}(0)}{(\sqrt{2}-1)^4} \frac{1+o(1)}{\log R} \frac{1}{2\pi i} \oint_{|s|=\frac{1}{\log X}} \Gamma_1(s) \frac{\Gamma\left(\frac{s}{2}+\frac{1}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2} \left(\frac{4}{\pi}\right)^s \zeta(2s)\zeta(2s+1)$$

$$\times \left(1 - \frac{1}{2^{\frac{1}{2}+s}}\right) \left(1 - \frac{1}{2^{\frac{1}{2}-s}}\right) \left(\frac{5}{2} - 4^s - 4^{-s}\right) \left\{ \log\left(\frac{X}{2}\right) + 2\gamma + \frac{(\check{\Phi})'(0)}{\check{\Phi}(0)} - \frac{\Gamma_1'(s)}{\Gamma_1(s)} - 2\frac{\zeta'}{\zeta}(2s) \right.$$

$$+ 2\frac{\zeta'}{\zeta}(2s+1) + \frac{\log 2}{(\sqrt{2}+2^s)(\sqrt{2}+2^{-s})} + \sum_{p \ne 2} \eta_1(p,s) \Bigg\} \left\{ \frac{1}{\log^3 M} \int_0^1 H''(t)^2 \, dt \right.$$

$$- \frac{2s^2}{\log M} \int_0^1 H(t)H''(t) \, dt + s^4 \log M \int_0^1 H(t)^2 \, dt - \frac{4}{\log^2 M} \int_0^1 H'(t)H''(t) \, dt$$

$$+ 4s^2 \int_0^1 H(t)H'(t) \, dt \Bigg\} \frac{ds}{s} + O\left(\frac{X}{(\log X)^{2-\varepsilon}}\right).$$

Evaluating the s-integral as a residue, we deduce that

$$\mathcal{B} = \frac{X\check{\Phi}(0)}{4\left(1 - \frac{1}{\sqrt{2}}\right)^2}\frac{1+o(1)}{\log R}\left\{\frac{\log X}{2\log M}\int_0^1 H(t)H''(t)\,dt - \int_0^1 H(t)H'(t)\,dt\right\}$$
$$+ O\left(X(\log X)^{-2+\varepsilon}\right).$$

From this, (7.3.13), (7.2), (7.1.1), and (7.2.8), it now follows that

$$S^+ = \frac{X}{8\left(1 - \frac{1}{\sqrt{2}}\right)^2}\frac{1+o(1)}{\log R}\left\{\frac{1}{24}\left(\frac{\log X}{\log M}\right)^3\int_0^1 H''(t)^2\,dt\right.$$

$$-\frac{1}{2}\left(\frac{\log X}{\log M}\right)^2\int_0^1 H'(t)H''(t)\,dt + \frac{\log X}{\log M}\int_0^1 H(t)H''(t)\,dt + \frac{\log X}{\log M}\int_0^1 H'(t)^2\,dt$$

$$\left.- 2\int_0^1 H(t)H'(t)\,dt\right\} + O\left(\frac{X}{(\log X)^{2-\varepsilon}} + \frac{X^{1+\varepsilon}}{Y} + X^{\frac{1}{2}+\varepsilon}M\right).$$

The error terms are acceptable by the choices in Subsection 7.7, and this yields Proposition 7.1.

8. Choosing the mollifier: finishing the proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1 by making an optimal choice for the smooth function $H(x)$ (see (3.3),(3.5)).

By (3.2), Proposition 6.1, and Proposition 7.1, one derives the inequality

$$(8.1) \qquad \sum_{\substack{p \equiv 1 \,(\mathrm{mod}\ 8) \\ L(\frac{1}{2},\chi_p)\neq 0}} (\log p)\Phi\left(\frac{p}{X}\right) \geq \frac{X}{(1+\delta_0)8}\cdot\vartheta\frac{\left(H(0) - \frac{1}{2\theta}H'(0)\right)^2}{\mathfrak{I}},$$

where $\delta_0 > 0$ is sufficiently small and fixed. We also have the upper bound

$$\sum_{\substack{p \equiv 1 \,(\mathrm{mod}\ 8) \\ L(\frac{1}{2},\chi_p)\neq 0}} (\log p)\Phi\left(\frac{p}{X}\right) \leq (\log X)\sum_{\substack{X/2 < p \leq X \\ p \equiv 1 \,(\mathrm{mod}\ 8) \\ L(\frac{1}{2},\chi_p)\neq 0}} 1.$$

The right side of (8.1) is an increasing function of ϑ, and so ϑ should be as large as possible. The hypotheses of Proposition 7.1 allow $\vartheta = \frac{1}{2}(\frac{1}{2} - \theta) - \varepsilon$, and therefore

$$(8.2) \qquad \sum_{\substack{X/2 < p \leq X \\ p \equiv 1 \,(\mathrm{mod}\ 8) \\ L(\frac{1}{2},\chi_p)\neq 0}} 1 \geq \frac{X}{(1+2\delta_0)8\log X}\cdot\varrho,$$

where

$$\varrho := \frac{1}{2}\left(\frac{1}{2} - \theta\right)\frac{\left(H(0) - \frac{1}{2\theta}H'(0)\right)^2}{\mathfrak{I}}.$$

We seek a choice of $H(x)$ which maximizes ϱ.

As $H(x)$ is a smooth function supported in $[-1, 1]$, we have $H(1) = H'(1) = 0$. For notational simplicity we set $H(0) = A, -H'(0) = B$. Since

$$\int_0^1 H(x)H'(x)dx = -\frac{1}{2}A^2,$$

$$\int_0^1 H(x)H''(x)dx = AB - \int_0^1 H'(x)^2 dx,$$

$$\int_0^1 H'(x)H''(x)dx = -\frac{1}{2}B^2,$$

we have

$$\mathfrak{I} = \left(A + \frac{1}{2\theta}B\right)^2 + \frac{1}{24\theta^3}\int_0^1 H''(x)^2 dx.$$

We choose $H(x)$ such that on $[0, 1]$ it is a smooth approximation to the optimal function $H_*(x)$ which minimizes the integral

(8.3)
$$\int_0^1 H_*''(x)^2 dx$$

among all $H_1 \in \mathcal{C}^3([0, 1])$ satisfying the boundary conditions $H_1(0) = A, -H_1'(0) = B, H_1(1) = H_1'(1) = 0$. We may choose $H(x)$ such that

$$(1 + \delta_0)\int_0^1 H_*''(x)^2 dx \geq \int_0^1 H''(x)^2 dx.$$

By the Euler-Lagrange equation, we find that an $H_*(x)$ which minimizes (8.3) must satisfy

$$H_*^{(4)}(x) = 0.$$

Thus, $H_*(x)$ is a polynomial of degree at most three. Recalling the boundary conditions, we find

$$H_*(x) = (2A - B)x^3 + (2B - 3A)x^2 - Bx + A.$$

By direct computation we obtain

$$\int_0^1 H_*''(x)^2 dx = 3A^2 + (2B - 3A)^2,$$

and therefore

$$\varrho \geq \frac{1 - O(\delta_0)}{2}\left(\frac{1}{2} - \theta\right)\left(1 + \frac{3A^2 + (2B - 3A)^2}{24\theta^3(A + \frac{1}{2\theta}B)^2}\right)^{-1}.$$

It is now a straightforward, but tedious, calculus exercise to find that

$$A = \frac{B(4\theta + 3)}{6(\theta + 1)}$$

is an optimal choice. Thus

(8.4)
$$\varrho \geq \frac{1 - O(\delta_0)}{2}\left(\frac{1}{2} - \theta\right)\frac{2\theta(3 + 6\theta + 4\theta^2)}{(1 + 2\theta)^3}.$$

With this choice of A we have

$$H_*(x) = \frac{2B\theta}{6(\theta+1)}(1-x)^2\left(2+\frac{3}{2\theta}+x\right).$$

Since ϱ is invariant under multiplication of H by scalars, we arrive at the convenient expression

$$(8.5) \qquad\qquad H_*(x) = (1-x)^2\left(2+\frac{3}{2\theta}+x\right).$$

If we set $x = \frac{\log m}{\log M}$ in (8.5), we obtain that the mollifier coefficients b_m satisfy

$$b_m \approx \mu(m)\frac{\log^2(M/m)}{\log^2 M}\frac{\log(X^{3/2}M^2m)}{\log M}.$$

One might wish to compare this with the description of $\lambda(\ell)$ in [39, p. 449].

Define

$$\rho(\theta) := \frac{1}{2}\left(\frac{1}{2}-\theta\right)\frac{2\theta(3+6\theta+4\theta^2)}{(1+2\theta)^3} = \frac{1}{2}\left(\frac{1}{2}-\theta\right)\left(1-\frac{1}{(1+2\theta)^3}\right).$$

By (8.2) and (8.4), we obtain

$$(8.6) \qquad\qquad \sum_{\substack{X/2<p\leq X \\ p\equiv 1\,(\mathrm{mod}\,8) \\ L(\frac{1}{2},\chi_p)\neq 0}} 1 \;\geq\; \frac{X}{(1+O(\delta_0))8\log X}\cdot\rho(\theta).$$

The maximum of $\rho(\theta)$ on $(0,\frac{1}{2})$ occurs at the unique positive root θ_0 of the polynomial $16\theta^4+32\theta^3+24\theta^2+12\theta-3$. By numerical calculation we find

$$\theta_0 = 0.17409\ldots$$

and

$$(8.7) \qquad\qquad \rho(\theta_0) = 0.09645\ldots.$$

We then choose $\theta=\theta_0$. Since

$$\sum_{\substack{X/2<p\leq X \\ p\equiv 1\,(\mathrm{mod}\,8)}} 1 = (1+o(1))\frac{X}{8(\log X)},$$

we deduce Theorem 1.1 from (8.6) and (8.7) upon summing over dyadic intervals.

9. The second moment of $L(\frac{1}{2},\chi_p)$

In this section we prove Theorems 1.2 and 1.3. We first consider separately the upper and lower bounds for Theorem 1.2.

9.1. **The upper bound in Theorem 1.2.** We define

$$(9.1.1) \qquad M_2 := \sum_{p \equiv 1 \,(\text{mod } 8)} (\log p)\Phi\left(\frac{p}{X}\right) L\left(\tfrac{1}{2}, \chi_p\right)^2.$$

In this subsection we prove

$$(9.1.2) \qquad M_2 \le (4\mathfrak{c} + o(1))\frac{X}{8}(\log X)^3.$$

The upper bound of Theorem 1.2 then follows from (9.1.2) upon summation over dyadic intervals.

The proof of (9.1.2) follows the lines of the proof of Proposition 7.1, taking $M(p) = 1$. We employ positivity to replace $\log p$ by $\log X$ and then introduce an upper bound sieve. After applying the approximate functional equation we split $\mu^2(n) = N_Y(n) + R_Y(n)$, and employ the bound (7.1.1).

We follow the argument of Section 7 down to (7.2.8), obtaining

$$S_N^+ = \mathcal{T}_0 + \mathcal{B}.$$

Since we have no mollifier here, we find

$$\mathcal{T}_0 = \frac{2X}{(\sqrt{2}-1)^4}\frac{1+o(1)}{\log R}\sum_{\substack{\nu=1 \\ (\nu,2)=1 \\ \nu=\square}}^{\infty}\frac{d_2(\nu)}{\sqrt{\nu}}\hat{F}_\nu(0) + O\left(\frac{X}{(\log R)^{2018}}\right) + O\left(\frac{X^{1+\varepsilon}}{Y}\right).$$

We insert into this the definitions (7.2.4) and (4.1) of F_ν and ω_2, interchange the order of summation, and then write the sum on ν as an Euler product. The result is

$$\mathcal{T}_0 = \frac{2X}{(\sqrt{2}-1)^4}\frac{1+o(1)}{\log R}\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(\frac{s}{2}+\frac{1}{4})^2}{\Gamma(\frac{1}{4})^2}\left(1-\frac{1}{2^{\frac{1}{2}-s}}\right)^2\left(\frac{X}{\pi}\right)^s\check{\Phi}(s)\left(1-\frac{1}{2^{1+2s}}\right)^3$$
$$\times \zeta(1+2s)^3\left(1-\frac{1}{2^{2+4s}}\right)^{-1}\zeta(2+4s)^{-1}\frac{ds}{s} + O\left(\frac{X}{(\log R)^{2018}}+\frac{X^{1+\varepsilon}}{Y}\right).$$

As before, we truncate the integral to the range $|\text{Im}(s)| \le (\log X)^2$, and then deform the path of integration to the path made up of the line segments L_1, L_2, L_3 defined above (7.3.11) to see that the main contribution arises from the residue of the integrand at $s = 0$. We evaluate the residue using (6.1.6) and arrive at

$$\mathcal{T}_0 = \left(144\zeta(2)\left(1-\frac{1}{\sqrt{2}}\right)^2\right)^{-1}\frac{X\check{\Phi}(0)}{4}\frac{1+o(1)}{\log R}(\log X)^3 + O\left(X\log X + \frac{X^{1+\varepsilon}}{Y}\right).$$

Recalling the definition of $\mathfrak{c}$, we have

$$(9.1.3) \qquad \mathcal{T}_0 \le (\mathfrak{c}+\varepsilon)\frac{X}{8}\frac{(\log X)^3}{\log R} + O\left(X\log X + \frac{X^{1+\varepsilon}}{Y}\right).$$

Moreover, we see from (7.9.12) that if $M = 1$ and $b_1 = 1$, then

$$(9.1.4) \qquad \mathcal{B} \ll X\frac{\log X}{\log R} \ll X$$

since we may deform the path of integration in (7.9.12) to a circle $|s| = \varepsilon$. The condition $\theta + 2\vartheta < \frac{1}{2}$ in Subsection 7.7 with $\theta = 0$ allows us to take $\vartheta = \frac{1}{4} - \varepsilon$ in (9.1.3). We then set

$Y = X^\delta$, for some small, fixed $\delta > 0$. We see that the upper bound (9.1.2) then follows from (9.1.3) and (9.1.4) after sending ε to zero sufficiently slowly.

9.2. The lower bound in Theorem 1.2.

Recall the definition (9.1.1) of M_2. Our goal is to prove the following result.

Proposition 9.1. *For large X we have*

$$M_2 \geq \frac{1}{2}(\mathfrak{c} - o(1))\frac{X}{4}(\log X)^3,$$

where $\mathfrak{c}$ is the positive constant defined in Theorem 1.2, and $o(1)$ is some quantity that goes to zero as $X \to \infty$.

The lower bound for Theorem 1.2 easily follows from Proposition 9.1 by summing over dyadic intervals.

The main idea in the proof of Proposition 9.1 is a standard one. For any Dirichlet polynomial $A(p)$, the Cauchy-Schwarz inequality implies

$$(9.2.1) \qquad M_2 \geq \frac{\left(\sum_{p\equiv1\,(\mathrm{mod}\,8)}(\log p)\Phi\left(\frac{p}{X}\right) L\left(\frac{1}{2},\chi_p\right) A(p)\right)^2}{\sum_{p\equiv1\,(\mathrm{mod}\,8)}(\log p)\Phi\left(\frac{p}{X}\right) A(p)^2}.$$

Clearly, we should choose $A(p)$ to be an approximation to $L(\frac{1}{2},\chi_p)$. Our choice is inspired by the approximate functional equation in Lemma 4.2. For a positive real number α, we define

$$(9.2.2) \qquad A_\alpha(p) := \frac{2}{\left(1 - \frac{1}{\sqrt{2}}\right)^2} \sum_{n\text{ odd}} \frac{\chi_p(n)}{\sqrt{n}}\omega_1\left(n\sqrt{\frac{\pi}{p^\alpha}}\right).$$

With $\varepsilon_0 > 0$ small and fixed, we then choose $A(p)$ in (9.2.1) to be

$$(9.2.3) \qquad A(p) := A_{1-\varepsilon_0}(p).$$

Observe that taking $\alpha = 1$ in (9.2.2) yields

$$(9.2.4) \qquad A_1(p) = L\left(\frac{1}{2},\chi_p\right).$$

Proposition 9.2. *Let $\varepsilon_0 > 0$ be small. Let $\alpha_1 \leq \alpha_2$ be real numbers with $\alpha_1, \alpha_2 \in \{1-\varepsilon_0, 1\}$, and $(\alpha_1, \alpha_2) \neq (1, 1)$. Then*

$$M_{\alpha_1,\alpha_2} := \sum_{p\equiv1\,(\mathrm{mod}\,8)} (\log p)\Phi\left(\frac{p}{X}\right) A_{\alpha_1}(p)A_{\alpha_2}(p) = \frac{1}{2}(\mathfrak{c} + O(\varepsilon_0))\frac{X}{4}(\log X)^3.$$

Proof of Proposition 9.1 assuming Proposition 9.2. By (9.2.1), (9.2.3), and (9.2.4), we have

$$M_2 \geq \frac{M_{1-\varepsilon_0,1}^2}{M_{1-\varepsilon_0,1-\varepsilon_0}}.$$

We apply Proposition 9.2 to obtain

$$M_2 \geq \frac{1}{2}(\mathfrak{c} + O(\varepsilon_0))\frac{X}{4}(\log X)^3.$$

The proposition follows upon letting $\varepsilon_0 = \varepsilon_0(X)$ go to zero sufficiently slowly as $X \to \infty$. $\square$

We devote the rest of this subsection to the proof of Proposition 9.2.

Proof of Proposition 9.2. By definition,

$$M_{\alpha_1,\alpha_2} = \frac{4}{(1-\frac{1}{\sqrt{2}})^4} \sum_{p \equiv 1 \,(\mathrm{mod}\, 8)} (\log p)\Phi\left(\frac{p}{X}\right) \sum_{m,n \text{ odd}} \sum \frac{\chi_p(mn)}{\sqrt{mn}}\omega_1\left(m\sqrt{\frac{\pi}{p^{\alpha_1}}}\right)\omega_1\left(n\sqrt{\frac{\pi}{p^{\alpha_2}}}\right).$$

Let $M_{\neq}$ denote the contribution to M_{α_1,α_2} from $mn \neq \square$. An application of Lemma 6.1 shows that $M_{\neq} \ll X$, say. We note that for bounding $M_{\neq}$ it is crucial that $\alpha_1 = 1 - \varepsilon_0$.

We therefore have

$$M_{\alpha_1,\alpha_2} = \frac{4}{(1-\frac{1}{\sqrt{2}})^4} \sum_{p \equiv 1 \,(\mathrm{mod}\, 8)} (\log p)\Phi\left(\frac{p}{X}\right) \sum_{\substack{(mn,2p)=1 \\ mn=\square}} \sum \frac{1}{\sqrt{mn}}\omega_1\left(m\sqrt{\frac{\pi}{p^{\alpha_1}}}\right)\omega_1\left(n\sqrt{\frac{\pi}{p^{\alpha_2}}}\right)$$
$$+ O(X).$$

We use Lemma 4.1 to remove the condition $(mn,p) = 1$ at the cost of a negligible error. We then open ω_1 using its definition as an integral, and interchange the order of summation and integration. After some simplification we arrive at

$$M_{\alpha_1,\alpha_2} = \frac{4}{(1-\frac{1}{\sqrt{2}})^4} \frac{1}{(2\pi i)^2} \int_{(c_1)} \int_{(c_2)} K(s_1, s_2)\left(\frac{X^{\alpha_1}}{\pi}\right)^{\frac{s_1}{2}}\left(\frac{X^{\alpha_2}}{\pi}\right)^{\frac{s_2}{2}} \zeta(1+2s_1)\zeta(1+2s_2)$$
$$\times \zeta(1+s_1+s_2)\left(\sum_{p \equiv 1 \,(\mathrm{mod}\, 8)} (\log p)\Phi\left(\frac{p}{X}\right)\left(\frac{p}{X}\right)^{\frac{\alpha_1 s_1 + \alpha_2 s_2}{2}}\right)\frac{ds_1 ds_2}{s_1 s_2}$$
$$+ O(X),$$

where $c_\ell = \mathrm{Re}(s_\ell)$ is a positive real number, and

$$K(s_1, s_2) = \zeta^{-1}(2 + 2s_1 + 2s_2)\left(1 + \frac{1}{2^{1+s_1+s_2}}\right)^{-1}$$
$$\times \prod_{\ell=1}^{2} \frac{\Gamma\left(\frac{s_\ell}{2}+\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\left(1 - \frac{1}{2^{\frac{1}{2}-s_\ell}}\right)\left(1 - \frac{1}{2^{1+2s_\ell}}\right).$$

For the moment we choose $c_1 = c_2 = \frac{1}{\log X}$. By the rapid decay of $K(s_1, s_2)$ in vertical strips, we may truncate to $|\mathrm{Im}(s_\ell)| \leq (\log X)^2$ at the cost of a negligible error. With this condition in place, we use the prime number theorem in arithmetic progressions to obtain that the sum on p is

$$\frac{X}{4} \int_0^\infty \Phi(x)x^{\frac{\alpha_1 s_1 + \alpha_2 s_2}{2}} dx + O\left(X \exp(-c\sqrt{\log X})\right).$$

The error term clearly makes an acceptable contribution to M_{α_1,α_2}. We then remove the condition on $\mathrm{Im}(s_\ell)$ by the same means we installed it and obtain

$$M_{\alpha_1,\alpha_2} = \frac{4}{(1-\frac{1}{\sqrt{2}})^4} \frac{X}{4} \int_0^\infty \Phi(x)\frac{1}{(2\pi i)^2} \int_{(c_1)} \int_{(c_2)} K(s_1, s_2)\left(\frac{(xX)^{\alpha_1}}{\pi}\right)^{\frac{s_1}{2}}\left(\frac{(xX)^{\alpha_2}}{\pi}\right)^{\frac{s_2}{2}}$$
$$\times \zeta(1+2s_1)\zeta(1+2s_2)\zeta(1+s_2+s_2)\frac{ds_1 ds_2}{s_1 s_2}dx$$
$$+ O(X).$$

We wish to separate the variables s_1 and s_2. Since $c_\ell > 0$ we expand $\zeta(1 + s_1 + s_2)$ as an absolutely convergent Dirichlet series. Interchanging the order of summation and integration, we obtain

$$M_{\alpha_1,\alpha_2} = \frac{4}{(1 - \frac{1}{\sqrt{2}})^4} \frac{X}{4} \int_0^\infty \Phi(x) \sum_{n=1}^\infty \frac{1}{n} \frac{1}{(2\pi i)^2} \int_{(c_1)} \int_{(c_2)} K(s_1, s_2) \left(\frac{(xX)^{\alpha_1}}{\pi n^2}\right)^{\frac{s_1}{2}} \left(\frac{(xX)^{\alpha_2}}{\pi n^2}\right)^{\frac{s_2}{2}}$$
$$\times \zeta(1 + 2s_1)\zeta(1 + 2s_2) \frac{ds_1 ds_2}{s_1 s_2} dx + O(X).$$

To truncate the summation over n, first we move the contours of integration to the right to $c_1 = c_2 = 1$. By trivial estimation we deduce that the contribution from $n \gg X^{\frac{\alpha_1 + \alpha_2}{4}}$ is $O(X)$. For n in the range $X^{\frac{\alpha_1}{2}} \ll n \ll X^{\frac{\alpha_1 + \alpha_2}{4}}$, we move $\mathrm{Re}(s_2)$ to $c_2 = \frac{1}{\log X}$ and estimate trivially, getting an error term of $O(X(\log X)^2)$. With $n \ll X^{\frac{\alpha_1}{2}}$ we then move c_1 to $\frac{1}{\log X}$, obtaining

$$M_{\alpha_1,\alpha_2} = \frac{4}{(1 - \frac{1}{\sqrt{2}})^4} \frac{X}{4} \int_0^\infty \Phi(x) \sum_{n \le \sqrt{(xX)^{\alpha_1}/\pi}} \frac{1}{n}$$
$$\times \frac{1}{(2\pi i)^2} \int_{(\frac{1}{\log X})} \int_{(\frac{1}{\log X})} K(s_1, s_2) \left(\frac{(xX)^{\alpha_1}}{\pi n^2}\right)^{\frac{s_1}{2}} \left(\frac{(xX)^{\alpha_2}}{\pi n^2}\right)^{\frac{s_2}{2}}$$
$$\times \zeta(1 + 2s_1)\zeta(1 + 2s_2) \frac{ds_1 ds_2}{s_1 s_2} dx + O(X(\log X)^2).$$

The variables s_1 and s_2 are almost separated, except they are entangled inside of $K(s_1, s_2)$. We move the lines of integration to $\mathrm{Re}(s_1) = \mathrm{Re}(s_2) = -\delta$, for some small, fixed $\delta > 0$. In doing so we pick up contributions from the poles at $s_1, s_2 = 0$. The contribution from the integrals on $\mathrm{Re}(s_\ell) = -\delta$ is trivially bounded by $O(X \log X)$. We write the contributions from the poles at $s_\ell = 0$ as contour integrals around small circles, thereby obtaining

$$M_{\alpha_1,\alpha_2} = \frac{4}{(1 - \frac{1}{\sqrt{2}})^4} \frac{X}{4} \int_0^\infty \Phi(x) \sum_{n \le \sqrt{(xX)^{\alpha_1}/\pi}} \frac{1}{n}$$
$$\times \frac{1}{(2\pi i)^2} \oint \oint_{|s_\ell|=(\log X)^{-1}} K(s_1, s_2) \left(\frac{(xX)^{\alpha_1}}{\pi n^2}\right)^{\frac{s_1}{2}} \left(\frac{(xX)^{\alpha_2}}{\pi n^2}\right)^{\frac{s_2}{2}}$$
$$\times \zeta(1 + 2s_1)\zeta(1 + 2s_2) \frac{ds_1 ds_2}{s_1 s_2} dx + O(X(\log X)^2).$$

Since $|s_\ell| = (\log X)^{-1}$ we have

$$K(s_1, s_2) = K(0,0) + O\left(\frac{1}{\log X}\right) = \frac{1}{6\zeta(2)}\left(1 - \frac{1}{\sqrt{2}}\right)^2 + O\left(\frac{1}{\log X}\right),$$

and therefore

$$M_{\alpha_1,\alpha_2} = \frac{2}{3\zeta(2)(1 - \frac{1}{\sqrt{2}})^2} \frac{X}{4} \int_0^\infty \Phi(x) \sum_{n \le \sqrt{(xX)^{\alpha_1}/\pi}} \frac{1}{n} \frac{1}{(2\pi i)^2} \oint \oint_{|s_\ell|=(\log X)^{-1}} \left(\frac{(xX)^{\alpha_1}}{\pi n^2}\right)^{\frac{s_1}{2}} \left(\frac{(xX)^{\alpha_2}}{\pi n^2}\right)^{\frac{s_2}{2}}$$
$$\times \zeta(1 + 2s_1)\zeta(1 + 2s_2) \frac{ds_1 ds_2}{s_1 s_2} dx + O(X(\log X)^2).$$

Expanding in Laurent and power series yields

$$\frac{1}{2\pi i} \oint_{|s_\ell|=(\log X)^{-1}} \left(\frac{(xX)^{\alpha_\ell}}{\pi n^2}\right)^{\frac{s_\ell}{2}} \zeta(1+2s_\ell)\frac{ds_\ell}{s_\ell} = \frac{1}{2}\log\left(\frac{1}{n}\sqrt{\frac{(xX)^{\alpha_\ell}}{\pi}}\right) + O(1),$$

and hence

$$M_{\alpha_1,\alpha_2} = \frac{1}{6\zeta(2)(1-\frac{1}{\sqrt{2}})^2}\frac{X}{4}\int_0^\infty \Phi(x) \sum_{n\leq\sqrt{(xX)^{\alpha_1}/\pi}} \frac{1}{n}\log\left(\frac{1}{n}\sqrt{\frac{(xX)^{\alpha_1}}{\pi}}\right)\log\left(\frac{1}{n}\sqrt{\frac{(xX)^{\alpha_2}}{\pi}}\right)$$

$$+ O(X(\log X)^2).$$

Partial summation yields

$$\sum_{n\leq\sqrt{(xX)^{\alpha_1}/\pi}} \frac{1}{n}\log\left(\frac{1}{n}\sqrt{\frac{(xX)^{\alpha_1}}{\pi}}\right)\log\left(\frac{1}{n}\sqrt{\frac{(xX)^{\alpha_2}}{\pi}}\right) = \frac{1+O(\varepsilon_0)}{24}(\log X)^3,$$

and therefore

$$M_{\alpha_1,\alpha_2} = \frac{1}{2}(\mathfrak{c} + O(\varepsilon_0))\frac{X}{4}(\log X)^3.$$

$\square$

9.3. Proof of Theorem 1.3.

We turn now to the proof of Theorem 1.3. Throughout this subsection we set $\eta := 100\log\log X/\log X$. Recalling the definition (9.2.2) of $A_\alpha(p)$, we then have

$$L\left(\tfrac{1}{2},\chi_p\right) = A_{1-\eta}(p) + B(p),$$

say. Thus

$$(9.3.1) \qquad M_2 = \sum_{p\equiv 1\,(\mathrm{mod}\,8)} (\log p)\Phi\left(\frac{p}{X}\right)\left\{A_{1-\eta}(p)^2 + O(|A_{1-\eta}(p)B(p)| + |B(p)|^2)\right\}.$$

We prove, on GRH, that

$$(9.3.2) \qquad \sum_{p\equiv 1\,(\mathrm{mod}\,8)} (\log p)\Phi\left(\frac{p}{X}\right)A_{1-\eta}(p)^2 = \mathfrak{c}\frac{X}{8}(\log X)^3 + O(X(\log X)^{2+\varepsilon})$$

and

$$(9.3.3) \qquad \sum_{p\equiv 1\,(\mathrm{mod}\,8)} (\log p)\Phi\left(\frac{p}{X}\right)|B(p)|^2 \ll X(\log X)^{5/2}.$$

Theorem 1.3 then follows from (9.3.1), (9.3.2), and (9.3.3) after applying Cauchy-Schwarz and summing over dyadic ranges.

We may easily prove (9.3.2), since the treatment is substantially similar to the proof of Proposition 9.2. Applying the approximate functional equation, the main term of (9.3.2) is

$$\frac{4}{(1-\frac{1}{\sqrt{2}})^4} \sum_{p\equiv 1\,(\mathrm{mod}\,8)} (\log p)\Phi\left(\frac{p}{X}\right)\sum_{m,n\text{ odd}}\sum \frac{\chi_p(mn)}{\sqrt{mn}}\omega_1\left(m\sqrt{\frac{\pi}{p^{1-\eta}}}\right)\omega_1\left(n\sqrt{\frac{\pi}{p^{1-\eta}}}\right).$$

We argue as in Proposition 9.2 and obtain that the contribution from $mn = \square$ is

$$\mathfrak{c}\frac{X}{8}(\log X)^3 + O(X(\log X)^{2+\varepsilon}).$$

The following standard result implies that the contribution to (9.3.2) from $mn \neq \square$ is $O(X/\log X)$, say.

Lemma 9.1. *Let χ be a non-principal Dirichlet character modulo q. Let χ^* be the primitive character inducing χ, and assume that GRH holds for $L(s, \chi^*)$. If $q \leq X^M$ for some fixed positive constant M, then*

$$\sum_{p \leq X} \chi(p)(\log p) \ll_M X^{1/2}(\log X)^2.$$

The proof of (9.3.3) is more subtle. Here the method of proof is that of Soundararajan and Young [41]. As the arguments are very similar, our exposition will be sparse, and we refer the reader to [41] for more details. We perform some initial manipulations, and then we state the main proposition which will yield (9.3.3).

By definition, we have

$$(9.3.4) \qquad B(p) = \frac{1}{2\pi i} \int_{(c)} g(s) L\left(\frac{1}{2} + s, \chi_p\right) \frac{p^{s/2} - p^{(1-\eta)s/2}}{s} ds,$$

where $c > 0$ and

$$g(s) = \frac{2}{(1 - \frac{1}{\sqrt{2}})^2} \frac{\Gamma\left(\frac{s}{2} + \frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \left(1 - \frac{1}{2^{\frac{1}{2}-s}}\right)\left(1 - \frac{1}{2^{\frac{1}{2}+s}}\right) \pi^{-s/2}.$$

The function $(p^{s/2} - p^{(1-\eta)s/2})/s$ is entire, so we may move the line of integration in (9.3.4) to $\mathrm{Re}(s) = 0$. On the line $\mathrm{Re}(s) = 0$ we have the bound $|(p^{s/2} - p^{(1-\eta)s/2})/s| \ll \log\log X$, and hence the left side of (9.3.3) is

$$(9.3.5) \quad \ll (\log X)^{1+\varepsilon} \int_{\mathbb{R}} \int_{\mathbb{R}} |g(it_1)g(it_2)| \sum_{\substack{p \leq X \\ p \equiv 1 \,(\mathrm{mod}\, 8)}} \left| L\left(\tfrac{1}{2} + it_1, \chi_p\right) L\left(\tfrac{1}{2} + it_2, \chi_p\right)\right| dt_1 \, dt_2.$$

To state the proposition we need, we first establish some notation, following [41, Section 6]. Given $x \geq 10$, say, and a complex number z, we define

$$\mathcal{L}(z, x) = \begin{cases} \log\log x, & |z| \leq (\log x)^{-1}, \\ -\log|z|, & (\log x)^{-1} \leq |z| \leq 1, \\ 0, & |z| \geq 1. \end{cases}$$

For complex numbers z_1 and z_2 we define

$$\mathcal{M}(z_1, z_2, x) = \frac{1}{2}(\mathcal{L}(z_1, x) + \mathcal{L}(z_2, x)),$$

and

$$\mathcal{V}(z_1, z_2, x) = \frac{1}{2}[\mathcal{L}(2z_1, x) + \mathcal{L}(2z_2, x) + \mathcal{L}(2\mathrm{Re}(z_1), x) + \mathcal{L}(2\mathrm{Re}(z_2), x))$$
$$+ 2\mathcal{L}(z_1 + z_2, x) + 2\mathcal{L}(z_1 + \overline{z_2}, x)].$$

It is helpful to know that for the values of z_1 and z_2 we consider, we have $\log\log X \leq \mathcal{V}(z_1, z_2, X) \leq 4\log\log X$.

The following result, an analogue of [41, Theorem 6.1], is the key input we need.

Proposition 9.3. *Let X be large, and let z_1 and z_2 be complex numbers with $0 \leq \mathrm{Re}(z_i) \leq \frac{1}{\log X}$ and $|z_i| \leq X$. Assume the Riemann Hypothesis for the Riemann zeta function $\zeta(s)$ and for all Dirichlet L-functions $L(s, \chi_p)$ with $p \equiv 1 \pmod 8$. Then for any $r > 0$ in $\mathbb{R}$ and any $\varepsilon > 0$ we have*

$$\sum_{\substack{p \leq X \\ p \equiv 1 \,(\mathrm{mod}\,8)}} \left| L\left(\tfrac{1}{2} + z_1, \chi_p\right) L\left(\tfrac{1}{2} + z_2, \chi_p\right) \right|^r$$

$$\ll_{r,\varepsilon} \frac{X}{(\log X)^{1-\varepsilon}} \exp\left(r\mathcal{M}(z_1, z_2, X) + \frac{r^2}{2}\mathcal{V}(z_1, z_2, X) \right).$$

Proof of (9.3.3) assuming Proposition 9.3. Recall (9.3.5). If t_1 or t_2 satisfies $|t_i| > X$ we use Cauchy-Schwarz, Lemma 4.5, and the rapid decay of g to get a negligible error.

We may therefore assume that $|t_i| \leq X$. We then consider, for a parameter $0 < \alpha < 1$ at our disposal, two cases: (1) both t_1 and t_2 satisfy $|t_i| \leq (\log X)^{-\alpha}$, or (2) one of t_1, t_2 satisfies $|t_i| \geq (\log X)^{-\alpha}$. In case (1) we use the trivial bounds

$$\mathcal{M}(it_1, it_2, X) \leq \log \log X,$$
$$\mathcal{V}(it_1, it_2, X) \leq 4 \log \log X,$$

while in case (2) we use the bounds

$$\mathcal{M}(it_1, it_2, X) \leq \frac{1+\alpha}{2} \log \log X,$$
$$\mathcal{V}(it_1, it_2, X) \leq \frac{7+\alpha}{2} \log \log X + O(1).$$

Since $|g(it)| \ll (1+t^2)^{-1}$ we obtain by Proposition 9.3 that the quantity in (9.3.5) is

$$\ll X(\log X)^{\varepsilon} \left((\log X)^{3-2\alpha} + (\log X)^{9/4+3\alpha/4} \right) = X(\log X)^{27/11+\varepsilon} \leq X(\log X)^{5/2}$$

upon choosing $\alpha = 3/11$. $\qquad\qquad\square$

To prove Proposition 9.3 we establish estimates for how often $\left| L\left(\tfrac{1}{2} + z_1, \chi_p\right) L\left(\tfrac{1}{2} + z_2, \chi_p\right) \right|$ can be large. The following is very similar to [41, Proposition 6.2].

Proposition 9.4. *Assume the hypotheses of Proposition 9.3. Let $\mathcal{N}(V; z_1, z_2, X)$ denote the number of primes $p \leq X$, $p \equiv 1 \pmod 8$, such that $\log \left| L\left(\tfrac{1}{2} + z_1, \chi_p\right) L\left(\tfrac{1}{2} + z_2, \chi_p\right) \right| \geq V + \mathcal{M}(z_1, z_2, X)$. In the range $3 \leq V \leq 4r\mathcal{V}(z_1, z_2, X)$ we have*

$$\mathcal{N}(V; z_1, z_2, X) \ll \frac{X}{(\log X)^{1-o_r(1)}} \exp\left(-\frac{V^2}{2\mathcal{V}(z_1, z_2, X)} \right),$$

and for larger V we have

$$\mathcal{N}(V; z_1, z_2, X) \ll \frac{X}{(\log X)^{1-o_r(1)}} \exp(-4rV).$$

Proof of Proposition 9.3. We have

$$\sum_{\substack{p \leq X \\ p \equiv 1 \,(\mathrm{mod}\,8)}} \left| L\left(\tfrac{1}{2} + z_1, \chi_p\right) L\left(\tfrac{1}{2} + z_2, \chi_p\right) \right|^r$$

$$= r \int_{-\infty}^{\infty} \exp(rV + r\mathcal{M}(z_1, z_2, X)) \mathcal{N}(V; z_1, z_2, X) dV.$$

Then use Proposition 9.4.   □

We use the following lemma to determine how frequently a Dirichlet polynomial can be large. We write $\log_2 X$ for $\log \log X$.

Lemma 9.2. *Let X and y be real numbers and k a natural number with $y^k \leq X^{\frac{1}{2} - \frac{1}{\log_2 X}}$. For any complex numbers $a(q)$ we have*

$$\sum_{\substack{p \leq X \\ p \equiv 1 \,(\mathrm{mod}\ 8)}} \left| \sum_{2 < q \leq y} \frac{a(q)\chi_p(q)}{q^{\frac{1}{2}}} \right|^{2k} \ll \frac{X \log_2 X}{\log X} \frac{(2k!)}{2^k k!} \left(\sum_{q \leq y} \frac{|a(q)|^2}{q} \right)^k,$$

where the implied constant is absolute.

Proof. This result is similar to [41, Lemma 6.3], so we give only a sketch. Since we are assuming GRH we could use Lemma 9.1, but we get an unconditional result that is almost as good by appealing to sieve theory.

Since $p \equiv 1 \pmod 8$, we have $\chi_p(q) = \chi_{q^*}(p)$, where for an odd integer n we define $n^* = (-1)^{\frac{n-1}{2}} n$. Observe that χ_{q^*} is a primitive character with conductor $\leq 4q$. We then introduce an upper bound sieve supported on $d \leq D = X^{\frac{1}{\log_2 X}}$. With the upper bound sieve in place we drop the congruence condition modulo 8 and the condition that p is a prime. Opening the square and using the Pólya-Vinogradov inequality, the sum in question is then

$$\ll \sum_{n \leq X} \left(\sum_{d|n} \lambda_d \right) \left| \sum_{2 < q \leq p} \frac{a(q)\chi_{q^*}(n)}{q^{\frac{1}{2}}} \right|^{2k}$$

$$\ll \sum_{\substack{q_i \leq y \\ q_1 \cdots q_{2k} = \square}} \frac{|a(q_1) \cdots a(q_{2k})|}{\sqrt{q_1 \cdots q_{2k}}} \sum_{n \leq X} \left(\sum_{d|n} \lambda_d \right) + D \log(y^{2k}) \sum_{q_1,\ldots,q_{2k} \leq y} |a(q_1) \cdots a(q_{2k})|.$$

For the first term we obtain

$$\sum_{\substack{q_i \leq y \\ q_1 \cdots q_{2k} = \square}} \frac{|a(q_1) \cdots a(q_{2k})|}{\sqrt{q_1 \cdots q_{2k}}} \sum_{n \leq X} \left(\sum_{d|n} \lambda_d \right) \ll \frac{X \log_2 X}{\log X} \frac{(2k!)}{2^k k!} \left(\sum_{q \leq y} \frac{|a(q)|^2}{q} \right)^k,$$

and for the second term we use Cauchy-Schwarz to obtain

$$D \log(y^{2k}) \sum_{q_1,\ldots,q_{2k} \leq y} |a(q_1) \cdots a(q_{2k})| \ll X \frac{k \log X}{D} \left(\sum_{q \leq y} \frac{|a(q)|^2}{q} \right)^k.$$

□

Proof of Proposition 9.4. Assume GRH for $L(s, \chi_p)$. A modification of the proof of the Proposition in [40] then yields

$$\log \left| L\left(\tfrac{1}{2} + z_1, \chi_p\right) L\left(\tfrac{1}{2} + z_2, \chi_p\right) \right| \leq \mathrm{Re} \left(\sum_{q^\ell \leq x} \frac{\chi_p(q^\ell)}{\ell q^{\ell(\frac{1}{2} + \frac{1}{\log x})}} (p^{-\ell z_1} + p^{-\ell z_2}) \frac{\log(x/q^\ell)}{\log x} \right)$$

$$+ 2\frac{\log X}{\log x} + O\left(\frac{1}{\log x} \right).$$

The terms with $\ell \geq 3$ contribute $O(1)$. For $\ell = 2$ we use the Riemann hypothesis for $\zeta(s)$ (see [41, (6.4)]) and obtain

$$\frac{1}{2} \sum_{q \leq \sqrt{x}} \frac{1}{q^{1+\frac{2}{\log x}}} (q^{-2z_1} + q^{-2z_2}) \frac{\log(x/q^2)}{\log x} = \mathcal{M}(z_1, z_2, x) + O(\log \log \log X).$$

Since $\mathcal{M}(z_1, z_2, x) \leq \mathcal{M}(z_1, z_2, X) + 2\frac{\log X}{\log x}$, we obtain

$$(9.3.6) \qquad \log \left| L\left(\tfrac{1}{2} + z_1, \chi_p\right) L\left(\tfrac{1}{2} + z_2, \chi_p\right) \right| \leq \operatorname{Re} \sum_{2 < q \leq x} \frac{\chi_p(q)}{q^{\frac{1}{2}+\frac{1}{\log x}}} (q^{-z_1} + q^{-z_2})$$

$$+ \mathcal{M}(z_1, z_2, X) + 4\frac{\log X}{\log x} + O(\log \log \log X).$$

We put $\mathcal{V} = \mathcal{V}(z_1, z_2, X)$, and define

$$T = \begin{cases} \frac{1}{2} \log \log \log X, & V \leq \mathcal{V}, \\ \frac{\mathcal{V}}{2V} \log \log \log X, & \mathcal{V} < V \leq \frac{1}{16}\mathcal{V} \log \log \log X, \\ 8, & V > \frac{1}{16}\mathcal{V} \log \log \log X. \end{cases}$$

We take $x = X^{T/V}$, and $z = x^{1/\log \log X}$.

Taking $x = \log X$ in (9.3.6) and estimating trivially, we may assume $V \leq \frac{5 \log X}{\log \log X}$. In (9.3.6) we then have

$$\log \left| L\left(\tfrac{1}{2} + z_1, \chi_p\right) L\left(\tfrac{1}{2} + z_2, \chi_p\right) \right| \leq S_1 + S_2 + \mathcal{M}(z_1, z_2, X) + 5\frac{V}{T},$$

where S_1 is the sum on q truncated to $q \leq z$, and S_2 is the remainder of the sum. Since $\log \left| L\left(\tfrac{1}{2} + z_1, \chi_p\right) L\left(\tfrac{1}{2} + z_2, \chi_p\right) \right| \geq V + \mathcal{M}(z_1, z_2, X)$ we have

$$S_2 \geq \frac{V}{T} \qquad \text{or} \qquad S_1 \geq V\left(1 - \frac{6}{T}\right) =: V_1.$$

We take $k = \lfloor (\tfrac{1}{2} - \frac{1}{\log_4 X})\frac{V}{T} \rfloor - 1$ in Lemma 9.2 and apply the usual Chebyshev-type maneuver to deduce that the number of $p \leq X$ with $S_2 \geq V/T$ is

$$\ll \frac{X \log_2 X}{\log X} \exp\left(-\frac{V}{4T} \log V\right).$$

It remains to bound the number of p for which S_1 is large. By Lemma 9.2, for any $k \leq (\tfrac{1}{2} - \frac{1}{\log_2 X})\frac{V \log \log X}{T}$ the number of $p \leq X$ with $S_1 \geq V_1$ is

$$\ll \frac{X \log_2 X}{\log X} \left(\frac{2k\mathcal{V}(z_1, z_2, X) + O(\log \log \log X)}{eV_1^2}\right)^k.$$

For $V \leq (\log \log X)^2$ we take $k = \lfloor V_1^2/2\mathcal{V} \rfloor$, and for $V > (\log \log X)^2$ we take $k = \lfloor 10V \rfloor$. It follows that the number of p for which $S_1 \geq V_1$ is

$$\ll \frac{X \log_2 X}{\log X} \exp\left(-\frac{V_1^2}{2\mathcal{V}}\left(1 + O\left(\frac{\log \log \log X}{\log \log X}\right)\right)\right) + \frac{X \log_2 X}{\log X} \exp(-V \log V).$$

$\square$

10. Proof of Theorem 1.4

The proof of Theorem 1.4 breaks naturally into two parts: the lower bound, and the upper bound. The argument for the lower bound is very similar to that in [37], and we therefore give only a sketch. The argument for the upper bound is similar to that in Section 7. In either case, we crucially use the assumption that the central values are non-negative.

10.1. **The lower bound.** Let $d_{1/2}(n)$ be the multiplicative function with $(d_{1/2} \star d_{1/2})(n) = 1$. For a prime $p \equiv 1 \pmod 4$ and large X define

$$R(p) := \sum_{n \leq X^{1/500}} \frac{d_{1/2}(n)\chi_p(n)}{\sqrt{n}}.$$

By Hölder's inequality and the assumption $L(\frac{1}{2}, \chi_p) \geq 0$ we have

$$\sum_{p \equiv 1 \,(\mathrm{mod}\, 8)} (\log p)\Phi\left(\frac{p}{X}\right) L\left(\frac{1}{2}, \chi_p\right)^3 \geq \frac{T_1^3}{T_2^2},$$

where

$$T_1 := \sum_{p \equiv 1 \,(\mathrm{mod}\, 8)} (\log p)\Phi\left(\frac{p}{X}\right) L\left(\frac{1}{2}, \chi_p\right) R(p)^4,$$

$$T_2 := \sum_{p \equiv 1 \,(\mathrm{mod}\, 8)} (\log p)\Phi\left(\frac{p}{X}\right) R(p)^6.$$

In T_2 we open up $R(p)^6$, and obtain a sum over $n_1, \ldots, n_6$, and p. The terms with $n_1 \cdots n_6 = \square$ yield a main term of size $\ll X(\log X)^6$, and the terms with $n_1 \cdots n_6 \neq \square$ are shown to be an error term by using Lemma 6.1.

For T_1, we write $L(\frac{1}{2}, \chi_p)$ using Lemma 4.2. After opening $R(p)^4$, we have a sum over $n_1, \ldots, n_4, m$, and p, where m is the variable of summation in the approximate functional equation. The main term $mn_1 \cdots n_4 = \square$ is of size $\gg X(\log X)^6$, and the error term $mn_1 \cdots n_4 \neq \square$ is small by Lemma 6.1. This gives the lower bound.

10.2. **The upper bound.** Assuming that $L(\frac{1}{2}, \chi_n) \geq 0$ for all square-free $n \equiv 1 \pmod 8$, we can use an upper bound sieve and positivity to write

$$M_3 := \sum_{p \equiv 1 \,(\mathrm{mod}\, 8)} (\log p)\Phi\left(\frac{p}{X}\right) L\left(\tfrac{1}{2}, \chi_p\right)^3$$

$$\leq (\log X) \sum_{n \equiv 1 \,(\mathrm{mod}\, 8)} \mu^2(n) \left(\sum_{\substack{d|n \\ d \leq D}} \lambda_d \right) \Phi\left(\frac{n}{X}\right) L\left(\tfrac{1}{2}, \chi_n\right)^3$$

The coefficients λ_d of the sieve are given, as before, by (5.8). We take R to be a sufficiently small power of X.

We use the approximate functional equation

$$L(\tfrac{1}{2}, \chi_n)^3 = \frac{16}{(\sqrt{2}-1)^6} \sum_{\substack{\nu=1 \\ \nu \text{ odd}}}^{\infty} \frac{d_3(\nu)\left(\frac{\nu}{n}\right)}{\sqrt{\nu}} \omega_3\left(\nu\left(\frac{\pi}{n}\right)^{3/2}\right),$$

where $\omega_3(\xi)$ is defined by taking $j = 3$ in (4.1). Our function $\omega_3(\xi)$ is not the same as $\omega_3(\xi)$ in [39]. After using the approximate functional equation to represent $L(\frac{1}{2}, \chi_n)^3$, we write $\mu^2(n) = N_Y(n) + R_Y(n)$. The contribution from $R_Y(n)$ is bounded using arguments similar to those in Subsection 7.1. For $N_Y(n)$ we use Poisson summation as before. Up to negligible error, we therefore have the upper bound

$$M_3 \leq (\log X) \frac{16}{(\sqrt{2}-1)^6} \sum_{\substack{d \leq D \\ d|\overline{P}(z) \\ d \text{ odd}}} \lambda_d \sum_{\substack{\nu=1 \\ \nu \text{ odd}}}^{\infty} \frac{d_3(\nu)}{\sqrt{\nu}} \sum_{\substack{\alpha \leq Y \\ \alpha \text{ odd}}} \mu(\alpha)$$

$$\times \left(\frac{2[\alpha^2, d]}{\nu} \right) \frac{X}{[\alpha^2, d]8\nu} \sum_{k \in \mathbb{Z}} e\left(\frac{k\overline{[\alpha^2, d]\nu}}{8} \right) \hat{F}_\nu \left(\frac{kX}{[\alpha^2, d]8\nu} \right) \tau_k(\nu),$$

where

$$F_\nu(t) = \Phi(t)\omega_3 \left(\nu \left(\frac{\pi}{tX} \right)^{3/2} \right).$$

We treat separately the contributions from $k = 0$ and $k \neq 0$. The calculations are somewhat easier in that ultimately we seek only upper bounds, not asymptotic formulas.

The contribution from $k = 0$ is treated as in Subsection 7.3, and is

$$\ll X \frac{\log X}{\log R} (\log X)^6 \ll X(\log X)^6.$$

For $k \neq 0$ the presence of the additive character necessitates a splitting of k into residue classes modulo 8. When necessary, we write the additive character as a linear combination of multiplicative characters. We use the identity

$$\tau_k(n) = \left(\frac{1+i}{2} + \left(\frac{-1}{n} \right) \left(\frac{1-i}{2} \right) \right) G_k(n)$$

and treat the two terms separately. We then follow the method of Section 7 to obtain that the contribution from $k \neq 0$ is

$$\ll X \frac{\log X}{\log R} (\log X)^6 \ll X(\log X)^6.$$

One difference that arises is in proving analogues of Lemma 7.2. Here we have $\check{\Phi}(w + \frac{s}{2})$ inside of an integral, instead of just $\check{\Phi}(w)$ outside of an integral. It is helpful to use the bound

$$\check{\Phi}(y) \ll_j \left(\frac{\log X}{|y|} \right)^j.$$

Another difference is that we have a factor of $X^{s/2}$ in the integrals, whereas this factor disappeared for the $k \neq 0$ terms in Section 7. We therefore do not need to concern ourselves with any symmetry properties of the integrand (cf. the symmetry argument yielding (7.9.4)).

Acknowledgements

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