ON THE ZEROS OF RIEMANN'S ZETA-FUNCTION ON THE CRITICAL LINE

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ABSTRACT. We combine the mollifier method with a zero detection method of Atkinson to prove in a new way that a positive proportion of the nontrivial zeros of the Riemann zeta-function $\zeta(s)$ are on the critical line. One of the main ingredients of the proof is an estimate for a mollified fourth moment of $\zeta(s)$. We deduce this estimate from the twisted fourth moment formula that has been recently developed by Hughes and Young.

1. Introduction and Results

Let $N(T)$ denote the number of zeros of the Riemann zeta-function $\zeta(s)$ with imaginary part between 0 and T, and let $N_0(T)$ denote the number of these zeros that are on the critical line $\mathrm{Re}(s) = 1/2$. In this paper we give a new proof of the following.

Theorem 1.1. *The quantity*

$$\kappa \ = \ \liminf_{T\to\infty} \frac{N_0(T)}{N(T)}$$

is positive.

We may interpret this theorem by saying that a positive proportion of the nontrivial zeros of $\zeta(s)$ are on the critical line. This theorem was first proved by Selberg [22] in 1942. The main new idea in Selberg's work is to replace $\zeta(s)$ by $\zeta(s)M(s)$ in a method of Hardy and Littlewood [15] for detecting zeros of $\zeta(s)$. Here, $M(s)$ is a certain function called a *mollifier*, and its purpose is to diminish losses incurred in estimations. The idea is that inequalities such as the Cauchy-Schwarz inequality become sharper when used with $\zeta(s)M(s)$ instead of the function $\zeta(s)$ without a mollifier. Selberg observed that introducing a mollifier in Hardy and Littlewood's method improves their lower bound $N_0(T) \gg T$ to $N_0(T) \gg T\log T$. This implies that $\kappa > 0$ since [24, §9.4]

$$N(T) \ \sim \ \frac{T}{2\pi}\log T$$

as $T \to \infty$. The lower bound for κ that is implicit in Selberg's proof is very small [24, §10.9].

Since Selberg's inception of the mollifier method, there has been a lot of progress towards improving the lower bound for κ. All recent advances in the subject are based on a second approach to the problem, developed by Levinson [20]. The key

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component in Levinson's technique is a method devised by Siegel [23] to detect zeros of $\zeta(s)$. Siegel's method alone leads only to $N_0(T) \gg T$, which is the same thing that Hardy and Littlewood proved. Levinson combined this method with the mollifier method, and the result is the second proof of Selberg's theorem with the stronger bound $\kappa \geq .3474$. This approach has been improved further by Conrey [6], who arrived at $\kappa \geq .4088$ by using deep techniques involving Kloosterman sums. Conrey's innovation is now the principal tool for pushing the lower bound of κ higher, and more recent developments involve using stronger mollifiers with Conrey's results. One improvement is due to Bui, Conrey, and Young [5], who showed that $\kappa \geq .4105$. Another is due to Feng [13], who has claimed to have proved $\kappa \geq .4128$. However, Bui, Conrey, and Young [5] have observed that there might be an error in his work, and Bui [4] has suggested a correction that leads to $\kappa \geq .4107$. A further step forward is due to Bui [4], who used Conrey's ideas together with a mollified third moment and a mollified fourth moment of $\zeta(s)$ to prove $\kappa \geq .4109$.

In this paper, we develop a third approach to the problem due to Atkinson [2]. The basic idea in Atkinson's technique is that one can estimate the number of zeros of a real-valued function $f(t)$ by counting the number of times it changes sign. This makes it similar to the method of Hardy and Littlewood, but Atkinson counts the sign changes in a different way. The approach needs two main ingredients. The first is an asymptotic formula for the mean-value

$$\int_T^{2T} e^{ia}\zeta(\tfrac{1}{2}+it')\zeta(\tfrac{1}{2}-it)\,dt, \tag{1.1}$$

where a is a fixed real number and t' is defined in a way that makes the integrand real-valued for all t. The second ingredient is an estimate for the fourth moment

$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^4\,dt. \tag{1.2}$$

Atkinson proves an asymptotic formula for (1.1) himself in [2], while he applies the theorem of Ingham [17] to estimate the fourth moment. He uses these two components together with the Cauchy-Schwarz inequality to show that the integrand in (1.1) is often negative. This allowed him to estimate the number of sign changes of the Hardy Z-function and deduce the lower bound $N_0(T) \gg T(\log T)^{-1}$. We give a more detailed description of Atkinson's method in Section 2.

The main idea in our proof of Theorem 1.1 is to combine the mollifier method with Atkinson's zero detection technique. We do this by inserting a mollifier into each of the mean-values (1.1) and (1.2). Using the resulting expressions in Atkinson's argument makes the Cauchy-Schwarz inequality sharper and thus reduces losses that occur from its use. The end result is an improvement of Atkinson's lower bound for $N_0(T)$ by two factors of $\log T$. To reach this conclusion, we need to evaluate both of the mollified mean-values. The first main result of the paper is an asymptotic formula for a smoothed version of (1.1) with a mollifier. We prove this using a method developed by Young [26] for estimating a smoothed second moment of $\zeta(s)$ times a mollifier. The result is Theorem 1.2 below.

The second main result of the paper is an asymptotic formula for a mollified fourth moment of $\zeta(s)$. This is considerably more difficult to evaluate than the second moment. The mollified fourth moment, or more generally the "twisted" fourth moment, has received considerable attention in the past few decades. One thing that makes the subject difficult is that any progress in it seems to depend on the theory of Kloosterman sums. Iwaniec [18] was the first to give a nontrivial upper bound for the twisted fourth moment

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4 \left|\sum_{n\le N} a_n n^{it}\right|^2 \, dt$$

with an arbitrary Dirichlet polynomial $\sum_{n\le N} a_n n^{it}$. His bound was improved by Deshouillers and Iwaniec [9, 10], and later by Watt [25]. Their bounds hold when N is small compared to T, and are of size $T^{1+\varepsilon}$. The first asymptotic formula for a twisted fourth moment was proved by Jose Gaggero Jara [14] in a Ph.D. thesis. The formula turns out to be quite complicated, and Gaggero did not use it to study the case when the Dirichlet polynomial is a mollifier. More recently, Hughes and Young [16] have found a different way of proving an asymptotic formula. Their method is based on the delta method of Duke, Friedlander, and Iwaniec [11], which uses Weil's bound for Kloosterman sums. Still another way of evaluating the twisted fourth moment has been developed by Motohashi [21]. His method uses the spectral theory of Kloosterman sums, and allows for a closer study of the error terms in the asymptotic formula.

Among the results we have described, the one we use to prove our formula for the mollified fourth moment is the theorem of Hughes and Young [16]. Their result reveals a permutational structure that is present in the main terms of the asymptotic formula for the twisted fourth moment. This allows us to use a lemma developed by Conrey, Farmer, Keating, Rubinstein, and Snaith [7] precisely for evaluating functions with this structure. This lemma is Lemma 2.5.1 in [7], and the special case we use is Lemma 6.1 below. To the author's knowledge, the mollified fourth moment that we consider in this paper has not been evaluated before. However, Conrey and Snaith [8] have formulated specific conjectures for it by using the Ratios Conjectures. Their hypotheses also apply to the fourth moment with a mollifier that is different from the one in this paper. Moreover, Bui [4] has proved a formula for this other fourth moment. To do this, he also used the theorem of Hughes and Young. Our result confirms a special case of one of the two conjectures of Conrey and Snaith, while Bui's verifies a special case of the other conjecture. Our asymptotic formula for the mollified fourth moment is Theorem 1.3 below.

To state our results, we need to make a few definitions. Let

$$Z(t) = e^{i\vartheta(t)}\zeta(\tfrac{1}{2}+it)$$

be the Hardy Z-function, where $\vartheta(t)$ is the Riemann-Siegel theta function defined by $e^{-2i\vartheta(t)} = \chi(\frac{1}{2}+it)$, where

$$\chi(s) = 2^s \pi^{s-1} \sin\left(\tfrac{\pi}{2}s\right)\Gamma(1-s).$$

Stirling's formula implies that $\vartheta(t)$ is strictly increasing for large t (see (3.8) below). Thus, if a is a fixed real number, then we can define t' for all large t by

$$\vartheta(t') - \vartheta(t) = a. \tag{1.3}$$

Let $M(t)$ be the mollifier defined by

$$M(t) \;=\; \sum_{n\le y} \frac{\mu(n)}{n^{1/2+it}} \left(\frac{\log(y/n)}{\log y}\right)^2, \tag{1.4}$$

where $y = T^\theta$ and $\mu(n)$ is the Möbius function. For a function $w(t)$, let

$$W \ = \ \int_{-\infty}^{\infty} w(t)\,dt. \tag{1.5}$$

We use ε to denote an arbitrarily small positive constant that is not necessarily the same for each instance.

Our first main result is the evaluation of a mollified version of Atkinson's mean-value (1.1). We prove it by using the method devised by Young [26].

Theorem 1.2. *Let $a \neq 0$ and $\theta > 0$ be fixed real numbers. Suppose $w(t)$ is a smooth, nonnegative function that is supported on $[T/2, 4T]$ and satisfies $w^{(j)}(t) \ll_j T_0^{-j}$ for all $j = 0, 1, 2, \ldots,$ where $T^{1/2+\varepsilon} \ll T_0 \ll T$. If $\ell \geq 0$ is a fixed integer, then*

$$\begin{aligned}\int_{-\infty}^{\infty} Z(t)Z(t')|M(t)|^2 w(t)\,dt \ &= \ W\cos a \ + \ \frac{4W}{3\theta}\left(\frac{\sin a}{a}\right) \\ &\quad + \ O\left(\frac{W}{\log T}\right) \ + \ O\left(y^{\ell+1}\frac{T^{\frac{3}{2}+\frac{\ell}{2}+\varepsilon}}{T_0^{\ell}}\right) \ + \ O\left(y^{1/2}T^{3/4}\log^3 T\right).\end{aligned} \tag{1.6}$$

Corollary 1.1. *Let $a \neq 0$ and $0 < \theta < \frac{1}{2}$ be real numbers. If $b \leq 1$ is a real number with $\theta < 2b - \frac{3}{2}$, then there exists a smooth function $w(t)$ that equals 1 on $[T, T+T^b]$ such that*

$$\int_{-\infty}^{\infty} Z(t)Z(t')|M(t)|^2 w(t)\,dt \ = \ \left(\cos a \ + \ \frac{4\sin a}{3a\theta} \ + \ o(1)\right)T^b \tag{1.7}$$

as $T \to \infty$.

We will not need the variants of the above results with $a = 0$. Those versions can be proved by taking the limiting case $\alpha = \beta = 0$ of a result of Young [26, Lemma 3].

Our second main result is the following asymptotic formula for a mollified fourth moment. We prove this by applying the recent theorem of Hughes and Young [16]. In fact, we generalize slightly and consider the fourth moment with a small "shift" ρ. We need to introduce this shift to be able to use our result to prove Theorem 1.1.

Theorem 1.3. *Let $0 < \theta < \frac{1}{22}$ and $a \in \mathbb{R}$ be fixed. Suppose $w(t)$ is a smooth, nonnegative function that is supported on $[T/2, 4T]$ and satisfies $w^{(j)}(t) \ll_j T_0^{-j}$ for all $j = 0, 1, 2, \ldots,$ where $T^{1/2+\varepsilon} \ll T_0 \ll T$. If $\rho = 2a/\log(T/2\pi)$, then*

$$\begin{aligned}\int_{-\infty}^{\infty} |\zeta(\tfrac{1}{2} + i\rho + it)|^4 |M(t)|^4 w(t)\,dt \ &= \ C(a,\theta)W \\ &\quad + \ O\left(\frac{W}{\log T}\right) \ + \ O(y^{11/2}T^{3/4+\varepsilon}(T/T_0)^{9/4}),\end{aligned} \tag{1.8}$$

where $C(a,\theta)$ is a computable constant that depends only on a and θ.

Corollary 1.2. *Let $a \in \mathbb{R}$, $0 < \theta < \frac{1}{22}$, and $\rho = 2a/\log(T/2\pi)$. If $b \leq 1$ is a real number with $\theta < \frac{13}{22}b - \frac{6}{11}$, then there exists a smooth function $w(t)$ that equals 1 on $[T, T+T^b]$ such that*

$$\int_{-\infty}^{\infty} |\zeta(\tfrac{1}{2} + i\rho + it)|^4 |M(t)|^4 w(t)\,dt \ = \ C(a,\theta)T^b \ + \ o(T^b)$$

as $T \to \infty$, where $C(a,\theta)$ is a computable constant that depends only on a and θ.

A few remarks about Theorem 1.3 are in order. First, note that we have left $C(a,\theta)$ unspecified. In principle, we can compute the exact value of $C(a,\theta)$. However, doing so by hand turns out to be a formidable task, and so we resort to numerical calculations via computer software. We calculate $C(a,\theta)$ by writing it in terms of the polynomial $\mathcal{R}(t;y,\rho)$ in (16.2) below. Hence we only need to find the coefficients of $\mathcal{R}(t;y,\rho)$, and we can use any computer software that can compute coefficients of polynomials. The resulting value we obtain for $C(a,\theta)$ is in Section 20. The equation (16.2) results from simplifying certain path integrals by writing their integrands as power series. Alternatively, we can evaluate these path integrals by using the method of Conrey and Snaith [8, §6] of using matrix determinants. This will lessen the amount of work needed to compute $C(a,\theta)$. However, it does not seem to do so to the extent that we are able to calculate by hand. Thus we adhere to our approach of using power series because it seems to be the most straightforward.

Aside from the main term $C(a,\theta)$, one may also take notice of the form (1.4) of the mollifier $M(t)$. In their proofs of Theorem 1.1, both Selberg [22] and Levinson [20] use the "weight" factor

$$\varpi \;=\; \frac{\log(y/n)}{\log y}$$

in their mollifiers. On the other hand, we use the weight ϖ^2, as one can see in the definition (1.4) of $M(t)$. Conrey [6] uses more general weights of the form $P(\varpi)$, where $P(x)$ is a polynomial with $P(0)=0$. This weight is also used in the above mentioned works of Bui [4], Bui, Conrey, and Young [5], and Feng [13]. We use the weight ϖ^2 so that we can deduce a useful formula, namely (1.8), for the mollified fourth moment in a way that is not too cumbersome. At one extreme, using the simpler weight ϖ leads to a different asymptotic formula that has a main term of size $W\log T$. Such a result cannot be applied to prove Theorem 1.1, because our proof needs a main term of size W. At the opposite extreme, working with a general weight $P(\varpi)$ makes the computations more complicated. Since our goal is merely to show that $\kappa > 0$, we do not need the most general version of Theorem 1.3, and we can focus on the simplest case with weight ϖ^2. Our method also works for the general case with weight $P(\varpi)$, where $P(x)$ is a polynomial satisfying $P(0) = P'(0) = 0$. It is left to the reader to make the slight changes necessary to prove the general case.

Theorem 1.3 does not overlap with the result of Bui [4]. Indeed, Bui uses a different form of a mollifier, namely

$$M_2(t) \;=\; \sum_{n\le y} \frac{\mu_2(n)}{n^{1/2+it}}\; P\left(\frac{\log(y/n)}{\log y}\right),$$

where $\mu_2(n)$ is the nth coefficient of the Dirichlet series for $\zeta^{-2}(s)$. To mollify the fourth moment of $\zeta(s)$ with this mollifier is to multiply $|\zeta(s)|^4$ by $|M_2(t)|^2$ in the integral. In our case, we multiply $|\zeta(s)|^4$ by $|M(t)|^4$.

Bettin, Bui, Li, and Radziwill [3] have recently improved the formula of Hughes and Young by increasing the range of h and k on which their formula is valid. We might be able to increase the size θ for which Theorem 1.3 is valid by applying the improved formula.

Having made our remarks about Theorem 1.3, we now return to the main problem of the paper. The lower bound for κ that we arrive at is very small; we only reach

$$\kappa \;>\; 0.0001049.$$

Moreover, we are only able to increase this by a small amount when we assume the (still unproven) assertion that it is possible to take any positive θ in the hypotheses of Theorems 1.2 and 1.3. Indeed, the largest lower bound for κ that we obtain through numerical computations is ≈ 0.0086729 when $\theta \approx 0.9967280$. This seems to indicate that Atkinson's method of detecting zeros has an inherent weakness that makes it difficult for us to apply it effectively. We deduced these estimates by using the weight ϖ^2. It is possible that these bounds can be improved by employing a more general weight $P(\varpi)$ and choosing an optimal polynomial $P(x)$. However, we do not expect any improvements to be significant.

One possible way of improving this approach is to modify it so that the lower bound for κ tends to 1 as $\theta \to \infty$. This phenomenon was observed to hold for Levinson's method by Farmer [12]. As we have found in the above paragraph, this does not hold for the current version of our approach. Sandro Bettin has suggested to the author that perhaps this might not be the case if one instead uses the mollifier

$$M_{1/2}(t) \;=\; \sum_{n\leq y} \frac{\lambda(n)}{n^{1/2+it}} \left(\frac{\log(y/n)}{\log y}\right)^2, \tag{1.9}$$

where $\lambda(n)$ is the nth coefficient of the Dirichlet series for $\zeta^{-1/2}(s)$. His suggestion is to use the factor $|M_{1/2}(t)M_{1/2}(t')|^2$ instead of $|M(t)|^2$ on the left-hand side of (1.6). Applying Cauchy-Schwarz as in (2.1) below would then result to inequalities involving $|Z(t)|^4|M_{1/2}(t)|^8$ and $|Z(t')|^4|M_{1/2}(t')|^8$ instead of $|Z(t)M(t)|^4$ and $|Z(t')M(t)|^4$. The idea is that $M(t)$ is an effective mollifier for $Z(t)$, but a poor one for $Z(t')$. Hence, using the mollifier (1.9) instead of (1.4) would eliminate losses caused by the incompatibility of $M(t)$ and $Z(t')$. However, evaluating the mollified mean-values of $\zeta(s)$ in this case is much more complicated. Thus this is left for future work. We do not make any claims that this suggestion will lead to any desirable results. In any case, it would be interesting to see how Atkinson's method can be further improved in this or in other ways.

2. Atkinson's method and our starting point

In this section, we describe the key idea in the method of Atkinson [2] for deducing a lower bound for the number of zeros of $\zeta(s)$ on the critical line. We present it in a form that enables us to use Theorems 1.2 and 1.3 to prove Theorem 1.1.

Since $|Z(t)| = |\zeta(\frac{1}{2}+it)|$, each real zero of $Z(t)$ corresponds to a unique zero of $\zeta(s)$ on the critical line. Thus we can prove Theorem 1.1 by showing that $Z(t)$ has many zeros as a function of the real variable t. To do this, we show that the sign of $Z(t)$ changes often. We define $\mathcal{S}$ to be the set of $t \in [T/2, 4T]$ for which $Z(t)Z(t') < 0$. Atkinson's approach is to prove a lower bound for the measure of $\mathcal{S}$. Using such an estimate, we can deduce a lower bound for the number of critical zeros of $\zeta(s)$ by an elementary argument (see Section 19).

To find a lower bound for the measure of $\mathcal{S}$, let $w(t)$ be a smooth nonnegative function supported on $[T/2, 4T]$. Apply the Cauchy-Schwarz inequality twice (or

Hölder's inequality once) to write

$$\begin{aligned}(2.1)\qquad &\left(\int_{\mathcal{S}} |Z(t)Z(t')||M(t)|^2 w(t)\,dt\right)^2 \\ &\le\ \mathcal{N}\ \left(\int_{-\infty}^{\infty} |Z(t)|^4|M(t)|^4 w(t)\,dt\right)^{1/2}\left(\int_{-\infty}^{\infty} |Z(t')|^4|M(t)|^4 w(t)\,dt\right)^{1/2},\end{aligned}$$

where

$$\mathcal{N}\ =\ \int_{-\infty}^{\infty} \mathbf{1}_{\mathcal{S}}(t)w(t)\,dt$$

and $\mathbf{1}_{\mathcal{S}}(t)$ is the characteristic function of $\mathcal{S}$. We can estimate the measure of $\mathcal{S}$ from the size of $\mathcal{N}$ (see Section 19), and so it suffices to bound $\mathcal{N}$ from below. We do this by dividing both sides of (2.1) by the two factors on its right-hand side that are different from $\mathcal{N}$. Hence, to prove a lower bound for $\mathcal{N}$, we need a lower bound for the mean-value on the left-hand side of (2.1) and an upper bound for each of the fourth power moments on the right-hand side.

We estimate the left-hand side of (2.1) as follows. Since $Z(t)Z(t') < 0$ for $t \in \mathcal{S}$, it is true that

$$|Z(t)Z(t')|\ =\ -Z(t)Z(t')$$

for such t. Moreover, if $t \in [T/2, 4T]$ is not in $\mathcal{S}$, then $Z(t)Z(t') \ge 0$. It follows from these observations that

$$\begin{aligned}(2.2)\qquad \int_{\mathcal{S}} |Z(t)Z(t')||M(t)|^2 w(t)\,dt\ &=\ -\int_{\mathcal{S}} Z(t)Z(t')|M(t)|^2 w(t)\,dt \\ &\ge\ -\int_{-\infty}^{\infty} Z(t)Z(t')|M(t)|^2 w(t)\,dt.\end{aligned}$$

Thus we can use Theorem 1.2 to bound the left-hand side of (2.1) from below. For the lower bound to be nontrivial, we will need to choose a value of a that makes the right-hand side of (1.6) negative.

We can use Theorem 1.3 to deduce upper bounds for the fourth power moments on the right-hand side of (2.1). The special case of Theorem 1.3 with $a = 0$ immediately gives a formula for the fourth moment that does not involve t'. For the other one, we approximate it by the left-hand side of (1.8). We fill-in the details and finish this argument in Section 19.

The structure of the rest of the paper is as follows. In Section 3, we state basic lemmas and develop the machinery we need in our proof of Theorem 1.2. In Section 4, we prove an asymptotic formula for a smoothed version of the mean-value (1.1) with a factor $(h/k)^{-it}$. We then use the formula to prove Theorem 1.2 in Section 5. The starting point of our proof of Theorem 1.3 is in Section 6. There, we apply Theorem 1.1 of Hughes and Young [16] and Lemma 2.5.1 of Conrey et al. [7] to write the mollified fourth moment as an iterated integral. Sections 7 through 16 are devoted to simplifying this iterated integral. We complete the proof of Theorem 1.3 in Section 16. In Section 17, we construct a specific smooth function $w(t)$ and prove Corollaries 1.1 and 1.2. In Section 18, we apply Theorem 1.3 to evaluate the integral with $Z(t')$ on the right-hand side of (2.1). We use our results from Sections 17 and 18 to finish our proof of Theorem 1.1 in Section 19. Finally, we describe the results of our numerical computations in Section 20.

3. Lemmata

Our first lemma is a generalization of the well-known Euler product formula. Special cases of this have been implicitly used in various places in the literature. We will need only the cases $k = 2$ and $k = 4$ in this paper.

Lemma 3.1. *Let $f(n_1, \ldots, n_k)$ be a complex-valued function such that*

$$f(h_1 j_1, \ldots, h_k j_k) \;=\; f(h_1, \ldots, h_k) f(j_1, \ldots, j_k)$$

whenever the products $h_1 \cdots h_k$ and $j_1 \cdots j_k$ are relatively prime. If

$$\sum_{1 \leq n_1, \ldots, n_k < \infty} |f(n_1, \ldots, n_k)| \;<\; \infty \tag{3.1}$$

or

$$\prod_p \left(\sum_{0 \leq \ell_1, \ldots, \ell_k < \infty} |f(p^{\ell_1}, \ldots, p^{\ell_k})| \right) \;<\; \infty, \tag{3.2}$$

then

$$\sum_{1 \leq n_1, \ldots, n_k < \infty} f(n_1, \ldots, n_k) \;=\; \prod_p \left(\sum_{0 \leq \ell_1, \ldots, \ell_k < \infty} f(p^{\ell_1}, \ldots, p^{\ell_k}) \right). \tag{3.3}$$

Proof. Assume first that (3.1) holds. Let

$$g(n) \;=\; \sum_{n_1 \cdots n_k = n} f(n_1, \ldots, n_k),$$

where the sum is over all k-tuples $(n_1, \ldots, n_k)$ of positive integers that have product equal to n. It follows from our hypothesis for $f(n_1, \ldots, n_k)$ that $g(n)$ is a multiplicative function, i.e. $g(mn) = g(m)g(n)$ for relatively prime m and n. This implies the Euler product formula (see, for example, Theorem 11.6 of [1])

$$\sum_{n=1}^{\infty} g(n) \;=\; \prod_p \left(\sum_{\ell=0}^{\infty} g(p^\ell) \right), \tag{3.4}$$

because the sum on the left-hand side is absolutely convergent by (3.1). By inserting the definition of $g(n)$ in (3.4) and rearranging the terms, we arrive at (3.3).

Now suppose that (3.2) holds. Let

$$g_0(n) \;=\; \sum_{n_1 \cdots n_k = n} |f(n_1, \ldots, n_k)|.$$

Our hypothesis for $f(n_1, \ldots, n_k)$ again implies that $g_0(n)$ is a multiplicative function. Therefore, for any integer $N \geq 2$, it is true that

$$\sum_{n=1}^{N} g_0(n) \;\leq\; \prod_{p \leq N} \left(\sum_{\ell=0}^{N} g_0(p^\ell) \right) \leq \prod_p \left(\sum_{0 \leq \ell_1, \ldots, \ell_k < \infty} |f(p^{\ell_1}, \ldots, p^{\ell_k})| \right).$$

Thus the sum $\sum_{n=1}^{\infty} g_0(n)$ converges absolutely by (3.2). Hence (3.1) holds, and so (3.3) is true by the above paragraph. □

The following are basic estimates for $\zeta(s)$ from chapter 3 of Titchmarsh [24].

Lemma 3.2. *Let $z = u + iv$. There is an absolute constant $c_0 > 0$ such that if $-\infty < v < \infty$ and*

$$-\frac{c_0}{\log(|v|+2)} \ \le\ u\ \le\ 1,$$

then $\zeta(1+z) \neq 0$ and the following bounds hold:

$$\zeta(1+z)\ \ll\ \max\left\{\frac{1}{|z|}\ ,\ \log(|v|+2)\right\},$$

$$\frac{1}{\zeta(1+z)}\ \ll\ \log(|v|+2),$$

and

$$\frac{\zeta'(1+z)}{\zeta(1+z)}\ \ll\ \log(|v|+2).$$

In the rest of this section, we develop the basic machinery used in our proof of Theorem 1.2. The implied constants in our estimates never depend on t, but may depend on other parameters, such as a, that we take to be fixed. The next lemma can be easily proved by induction.

Lemma 3.3. *If f and g are differentiable functions, then for $n \ge 2$,*

$$\frac{d^n}{dt^n}(f\circ g)(t)\ =\ (f'\circ g)(t)g^{(n)}(t)\ +\ (f^{(n)}\circ g)(t)(g'(t))^n\ +\ \Sigma,$$

where Σ is a sum of terms of the form

$$(f^{(m_1+m_2+\cdots+m_{n-1})}\circ g)(t)(g'(t))^{m_1}(g''(t))^{m_2}\cdots(g^{(n-1)}(t))^{m_{n-1}}$$

with $\sum_{j=1}^{n-1} jm_j = n$, $m_j \ge 0$ for all j, and $m_j \ge 1$ for some $j \ge 2$. The sum Σ is empty when $n = 2$.

Lemma 3.4. *Let a be a fixed real number. If t' is defined by* (1.3) *for large t, then the following formulas hold for large enough t:*

$$t' - t = \frac{2a}{\log\frac{t}{2\pi}} + O\left(\frac{1}{t\log^3 t}\right), \tag{3.5}$$

$$\frac{dt'}{dt}\ =\ 1\ +\ O\left(\frac{1}{t\log^2 t}\right), \tag{3.6}$$

and

$$\frac{d^n}{dt^n}t'\ \ll_n\ \frac{1}{t^n\log t}, \qquad n \ge 2. \tag{3.7}$$

Proof. Equations (3.5) and (3.6) are proved in Atkinson [2], but we include their proofs here for completeness. We will use the formulas

$$\vartheta'(t) = \frac{1}{2}\log\frac{t}{2\pi} + O\left(\frac{1}{t^2}\right), \tag{3.8}$$

and

$$\vartheta^{(n)}(t) = \frac{(-1)^n(n-2)!}{2t^{n-1}} + O\left(\frac{1}{t^n}\right), \qquad n \ge 2, \tag{3.9}$$

which both easily follow from Stirling's formula.

To prove (3.5), observe that the mean value theorem of differential calculus gives

$$t' - t \;=\; \frac{\vartheta(t') - \vartheta(t)}{\vartheta'(x_1)}$$

for some x_1 between t and t'. Hence $t' - t \ll (\log t)^{-1}$ by (1.3) and (3.8). This and the second-order Taylor expansion of ϑ at t imply that

$$\vartheta(t') \;-\; \vartheta(t) \;=\; (t'-t)\vartheta'(t) \;+\; O\left(\frac{\vartheta''(x_2)}{\log^2 t}\right)$$

for some x_2 between t and t'. We arrive at (3.5) upon dividing both sides by $\vartheta'(t)$ and using (1.3), (3.8), and (3.9) with $n = 2$.

To show (3.6), differentiate both sides of (1.3) to deduce that

$$\vartheta'(t')\frac{dt'}{dt} \;-\; \vartheta'(t) \;=\; 0.$$

From this and (3.8), it follows that

$$\frac{dt'}{dt} \;=\; \frac{\vartheta'(t)}{\vartheta'(t')} \;=\; \frac{\log\frac{t}{2\pi} + O\left(\frac{1}{t^2}\right)}{\log\frac{t'}{2\pi} + O\left(\frac{1}{t^2}\right)}.$$

We can approximate $\log t'$ by $\log t$ using (3.5), and the result is (3.6).

We next prove (3.7). Suppose, by way of induction, that $(d^j/dt^j)t' \ll (t^j \log t)^{-1}$ for $2 \le j \le n-1$. We differentiate both sides of (1.3) n times and use Lemma 3.3 with $f(t) = \vartheta(t)$ and $g(t) = t'$ to write

$$\vartheta'(t')\left(\frac{d^n}{dt^n}t'\right) \;+\; \vartheta^{(n)}(t')\left(\frac{dt'}{dt}\right)^n \;+\; \Sigma \;-\; \vartheta^{(n)}(t) \;=\; 0.$$

We apply (3.8), (3.9), (3.6), and our inductive assumption to estimate all the terms except for the first term on the left-hand side. After some rearrangement, the result is

$$\frac{d^n}{dt^n}t' \;=\; \frac{1}{\vartheta'(t')}\left(\frac{c_n}{t^{n-1}} - \frac{c_n}{(t')^{n-1}} + O\left(\frac{1}{t^n}\right)\right),$$

where $c_n = (-1)^n(n-2)!/2$. It follows from (3.8) and (3.5) that the right-hand side is $\ll (t^n \log t)^{-1}$. This proves (3.7). □

Let $H(t,s)$ be defined by

$$H(t,s) \;=\; \frac{\Gamma\left(\frac{\frac12 + it' + s}{2}\right)\Gamma\left(\frac{\frac12 - it + s}{2}\right)}{\Gamma\left(\frac{\frac12 + it'}{2}\right)\Gamma\left(\frac{\frac12 - it}{2}\right)}. \tag{3.10}$$

Lemma 3.5. *If* $\mathrm{Re}(s)$ *is fixed and* $> -\frac12$, *then for large enough* t,

$$H(t,s) \;\ll\; t^{1/2}e^{\frac{\pi}{2}t} \tag{3.11}$$

uniformly for $-\infty < \mathrm{Im}(s) < \infty$ *and*

$$H(t,s) \;=\; \left(\frac{t}{2}\right)^s\left(1 + O\left(\frac{1+|s|^2}{t}\right)\right) \tag{3.12}$$

uniformly for $|\mathrm{Im}(s)|^2 \ll t$.

Proof. We first prove (3.12). If t is large and $|\mathrm{Im}(s)|^2 \ll t$, then it follows from Stirling's formula that

(3.13)
$$\Gamma\left(\frac{\frac{1}{2}+it+s}{2}\right) = \sqrt{2\pi}\left(\frac{t}{2}\right)^{z-\frac{1}{2}} \exp\left(i\frac{\pi z}{2} - i\frac{\pi}{4} - i\frac{t}{2}\right)\left(1+O\left(\frac{1+|s|^2}{t}\right)\right),$$

where $z = \frac{1}{2}(\frac{1}{2}+it+s)$. We use this and its complex conjugate to estimate each gamma function in the definition (3.10) of $H(t,s)$. The result is

$$H(t,s) = \left(\frac{t'}{2}\right)^{s/2}\left(\frac{t}{2}\right)^{s/2}\left(1+O\left(\frac{1+|s|^2}{t}\right)\right).$$

The estimate (3.12) now follows from this and (3.5).

To prove (3.11), apply the trivial bound $\Gamma \ll 1$ to estimate the numerator of $H(t,s)$ and (3.13) to estimate the denominator. □

Lemma 3.6. *If* $\mathrm{Re}(s) > -\frac{1}{2}$ *and* $n \geq 1$ *are fixed, then for large enough* t,

(3.14)
$$\frac{d^n}{dt^n}\log H(t,s) \ll \log|s|$$

uniformly for $t \leq 2|\mathrm{Im}(s)|$ *and*

(3.15)
$$\frac{d^n}{dt^n}\log H(t,s) \ll \frac{1+|s|}{t^n}$$

uniformly for $t \geq 2|\mathrm{Im}(s)|$.

Proof. We prove (3.15) first. Observe that

(3.16)
$$\frac{d^n}{dt^n}\log H(t,s) = \frac{d^n}{dt^n}\Upsilon(t') + \frac{d^n}{dt^n}\Upsilon(-t),$$

where $\Upsilon(t)$ is defined by

$$\Upsilon(t) = \Upsilon(t,s) = \log\Gamma\left(\frac{\frac{1}{2}+it+s}{2}\right) - \log\Gamma\left(\frac{\frac{1}{2}+it}{2}\right).$$

If $t \geq 2|\mathrm{Im}(s)|$ and $m \geq 1$, then we see from Stirling's formula that

(3.17)
$$\begin{aligned}\frac{d^m}{dt^m}\Upsilon(t) &= \left(\frac{\Gamma'}{\Gamma}\right)^{(m-1)}\left(\frac{\frac{1}{2}+it+s}{2}\right)\left(\frac{i}{2}\right)^m - \left(\frac{\Gamma'}{\Gamma}\right)^{(m-1)}\left(\frac{\frac{1}{2}+it}{2}\right)\left(\frac{i}{2}\right)^m \\ &\ll \frac{1+|s|}{t^m}.\end{aligned}$$

To estimate the derivatives of $\Upsilon(t')$, we apply Lemma 3.3 with $f(t) = \Upsilon(t)$ and $g(t) = t'$. We use (3.17) to estimate the derivatives of $f(t)$ and (3.6) or (3.7) to bound the derivatives of $g(t)$. The result is

$$\frac{d^n}{dt^n}\Upsilon(t') \ll \frac{1+|s|}{t^n}$$

for $t \geq 2|\mathrm{Im}(s)|$ and $n \geq 1$. We now insert this and the bound (3.17), which also holds for $\Upsilon(-t)$, into (3.16) to deduce (3.15).

The proof of (3.14) is the same, except that we use the bound

$$\frac{d^m}{dt^m}\Upsilon(t) \ll \begin{cases}\log|s| & \text{if } m=1\\ 1 & \text{if } m\geq 2\end{cases}$$

instead of (3.17). This bound follows from Stirling's formula when $t \le 2|\mathrm{Im}(s)|$. □

Lemma 3.7. *If* $\mathrm{Re}(s) > -\frac{1}{2}$ *and* $n \ge 0$ *are fixed, then for large enough* t,

$$\frac{d^n}{dt^n}H(t,s) \ \ll\ t^{1/2}e^{\frac{\pi}{2}t}\log^n(|s|+2) \tag{3.18}$$

uniformly for $-\infty < \mathrm{Im}(s) < \infty$ *and*

$$\frac{d^n}{dt^n}H(t,s) \ \ll\ t^{Re(s)-n}(1+|s|)^n \tag{3.19}$$

uniformly for $|\mathrm{Im}(s)|^2 \ll t$.

Proof. By Lemma 3.3 with $f(t) = e^t$ and $g(t) = \log H(t,s)$, we see that $(d^n/dt^n)H(t,s)$ equals a sum of terms of the form

$$H(t,s)\left(\frac{d}{dt}\log H(t,s)\right)^{m_1}\cdots\left(\frac{d^n}{dt^n}\log H(t,s)\right)^{m_n} \tag{3.20}$$

with $m_1 + 2m_2 + 3m_3 + \cdots + nm_n = n$. To prove (3.18), observe that (3.14) and (3.15) together give

$$\frac{d^j}{dt^j}\log H(t,s) \ \ll_j\ \log(|s|+2)$$

for $-\infty < \mathrm{Im}(s) < \infty$ and $j \ge 1$. We use this and (3.11) to bound each term of the form (3.20) and thus deduce (3.18).

Now we show (3.19). If v and t are positive real numbers with $v^2 \le Bt$ and $t \ge 4B$, then $v \ge 2B$ implies $2v \le v^2/B \le t$, while $v \le 2B$ implies $2v \le 4B \le t$. Hence $|\mathrm{Im}(s)|^2 \ll t$ implies that $2|\mathrm{Im}(s)| \le t$ and so (3.15) holds for large enough t. Using (3.15) and (3.12) to bound each term of the form (3.20), we arrive at (3.19). □

Let $K(t,s)$ be defined by

$$K(t,s) \ =\ \pi^{it'-it}\frac{\Gamma\left(\frac{\frac{1}{2}-it'+s}{2}\right)\Gamma\left(\frac{\frac{1}{2}+it+s}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+it'}{2}\right)\Gamma\left(\frac{\frac{1}{2}-it}{2}\right)}. \tag{3.21}$$

Since $\Gamma(\overline{z}) = \overline{\Gamma(z)}$, we can write

$$K(t,s) \ =\ \pi^{it'-it}X(t)\overline{H(t,\overline{s})}, \tag{3.22}$$

where

$$X(t) \ =\ \frac{\Gamma\left(\frac{\frac{1}{2}-it'}{2}\right)\Gamma\left(\frac{\frac{1}{2}+it}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+it'}{2}\right)\Gamma\left(\frac{\frac{1}{2}-it}{2}\right)}. \tag{3.23}$$

The following lemma states that Lemma 3.7 holds verbatim when $H(t,s)$ is replaced by $K(t,s)$.

Lemma 3.8. *If* $\mathrm{Re}(s) > -\frac{1}{2}$ *and* $n \ge 0$ *are fixed, then for large enough* t,

$$\frac{d^n}{dt^n}K(t,s) \ \ll\ t^{1/2}e^{\frac{\pi}{2}t}\log^n(|s|+2)$$

uniformly for $-\infty < \mathrm{Im}(s) < \infty$ *and*

$$\frac{d^n}{dt^n}K(t,s) \ \ll\ t^{Re(s)-n}(1+|s|)^n$$

uniformly for $|\mathrm{Im}(s)|^2 \ll t$.

Proof. In view of (3.22) and Lemma 3.7, we only need to bound the derivatives of $\pi^{it'-it}$ and $X(t)$. First observe that from (3.6), (3.7), and Lemma 3.3 with $f(t) = e^t$ and $g(t) = i(t'-t)\log\pi$, it follows that

$$\frac{d^j}{dt^j}\pi^{it'-it} \ll_j \frac{1}{t^j} \tag{3.24}$$

for $j \geq 0$.

The next task is to estimate the derivatives of $X(t)$. We first consider $\log X(t)$. Write

$$\frac{d^j}{dt^j}\log X(t) = \frac{d^j}{dt^j}Y(t) - \frac{d^j}{dt^j}\overline{Y(t)}, \tag{3.25}$$

where

$$Y(t) = \log\Gamma\left(\frac{\frac{1}{2}+it}{2}\right) - \log\Gamma\left(\frac{\frac{1}{2}+it'}{2}\right).$$

It follows from Lemma 3.3 with $f(t) = \log\Gamma(t)$ and $g(t) = \frac{1}{2}(\frac{1}{2}+it')$ that

$$\begin{aligned}\frac{d^j}{dt^j}Y(t) = &\left(\frac{\Gamma'}{\Gamma}\right)^{(j-1)}\left(\frac{\frac{1}{2}+it}{2}\right)\left(\frac{i}{2}\right)^j - \left(\frac{\Gamma'}{\Gamma}\right)^{(j-1)}\left(\frac{\frac{1}{2}+it'}{2}\right)\left(\frac{i}{2}\right)^j\left(\frac{dt'}{dt}\right)^j \\ &- \frac{\Gamma'}{\Gamma}\left(\frac{\frac{1}{2}+it'}{2}\right)\left(\frac{i}{2}\right)\left(\frac{d^j}{dt^j}t'\right) - \Sigma\end{aligned}$$

for $j \geq 1$. The last two terms on the right-hand side do not appear when $j = 1$. Use (3.6), (3.7), and Stirling's formula to bound the last two terms on the right-hand side and deduce that

$$\frac{d^j}{dt^j}Y(t) = \left(\frac{\Gamma'}{\Gamma}\right)^{(j-1)}\left(\frac{\frac{1}{2}+it}{2}\right)\left(\frac{i}{2}\right)^j - \left(\frac{\Gamma'}{\Gamma}\right)^{(j-1)}\left(\frac{\frac{1}{2}+it'}{2}\right)\left(\frac{i}{2}\right)^j\left(\frac{dt'}{dt}\right)^j + O\left(\frac{1}{t^j}\right).$$

We estimate the first two terms on the right-hand side via Stirling's formula and (3.6). The result is

$$\frac{d^j}{dt^j}Y(t) = \begin{cases} \dfrac{i\log t}{2} - \dfrac{i\log t'}{2} + O\left(\dfrac{1}{t}\right) & \text{if } j = 1, \\[2ex] \dfrac{b_j}{(\frac{1}{2}+it)^{j-1}} - \dfrac{b_j}{(\frac{1}{2}+it')^{j-1}} + O\left(\dfrac{1}{t^j}\right) & \text{if } j \geq 2, \end{cases}$$

where $b_j = (-i)^j(j-2)!/2$. From this and (3.5), we arrive at $(d^j/dt^j)Y(t) \ll t^{-j}$ for $j \geq 1$. This bound and the equation (3.25) imply that

$$\frac{d^j}{dt^j}\log X(t) \ll_j \frac{1}{t^j} \tag{3.26}$$

for $j \geq 1$.

Now we can estimate the derivatives of $X(t)$. It follows from Lemma 3.3 with $f(t) = e^t$ and $g(t) = \log X(t)$ that $(d^j/dt^j)X(t)$ equals a sum of terms of the form

$$X(t)\left(\frac{d}{dt}\log X(t)\right)^{m_1}\cdots\left(\frac{d^j}{dt^j}\log X(t)\right)^{m_j} \tag{3.27}$$

with $m_1+2m_2+3m_3+\cdots+jm_j=j$. We apply the estimate (3.13) to each gamma function in the definition (3.23) of $X(t)$ to deduce that

$$X(t) \;=\; \left(\frac{t'}{2}\right)^{-it'}\left(\frac{t}{2}\right)^{it}e^{i(t'-t)}\left(1+O\left(\frac{1}{t}\right)\right). \tag{3.28}$$

Hence $X(t)\ll 1$. Use this and (3.26) to see that each term of the form (3.27) is $\ll t^{-j}$. Therefore

$$\frac{d^j}{dt^j}X(t) \;\ll_j\; \frac{1}{t^j} \tag{3.29}$$

for $j\geq 0$.

Lemma 3.8 now follows from (3.22), the product rule, (3.24), (3.29), and the bounds (3.18) and (3.19). □

Let $q(t,s)$ be defined by

$$q(t,s) \;=\; \frac{(t'-t)^2+4s^2}{(t'-t)^2}. \tag{3.30}$$

Using Lemmas 3.7 and 3.8, we can estimate the derivatives of $H(t,s)q(t,s)$ and $K(t,s)q(t,s)$, as follows.

Lemma 3.9. *If* $\mathrm{Re}(s)>-\frac{1}{2}$ *and* $n\geq 0$ *are fixed, then for large enough* t,

$$\frac{d^n}{dt^n}H(t,s)q(t,s) \;\ll\; t^{1/2}(\log t)^{n+2}e^{\frac{\pi}{2}t}(1+|s|)^2\log^n(|s|+2) \tag{3.31}$$

uniformly for $-\infty<\mathrm{Im}(s)<\infty$ *and*

$$\frac{d^n}{dt^n}H(t,s)q(t,s) \;\ll\; t^{Re(s)-n}(\log t)^{n+2}(1+|s|)^{n+2} \tag{3.32}$$

uniformly for $|\mathrm{Im}(s)|^2\ll t$. *The same bounds hold when the expression* $H(t,s)$ *is replaced by* $K(t,s)$.

Proof. We use Lemma 3.3 with $f(t)=(t^2+4s^2)/t^2$ and $g(t)=t'-t$ and the estimates (3.5), (3.6), and (3.7) to deduce that

$$\frac{d^m}{dt^m}q(t,s) \;\ll_m\; |s|^2\frac{(\log t)^{m+2}}{t^m}$$

for $m\geq 1$. For $m=0$, the bound is $q(t,s)\ll 1+|s|^2\log^2 t$. The lemma now follows from these, the product rule, and Lemmas 3.7 and 3.8. □

For a function $F(s)$, we use $\int_{(c)}F(s)\,ds$ to denote the path integral of $F(s)$ along the line from $c-i\infty$ to $c+i\infty$. Let $V_1(t,x)$ and $V_2(t,x)$ be defined by

$$V_1(t,x) \;=\; \frac{1}{2\pi i}\int_{(1)}H(t,s)q(t,s)\frac{\exp(s^2)}{x^s}\frac{ds}{s} \tag{3.33}$$

and

$$V_2(t,x) \;=\; \frac{1}{2\pi i}\int_{(1)}K(t,s)q(t,s)\frac{\exp(s^2)}{x^s}\frac{ds}{s}, \tag{3.34}$$

where $H(t,s)$ and $K(t,s)$ are defined by (3.10) and (3.21), respectively.

Lemma 3.10. *Let $n \geq 0$ and $B > 0$ be fixed. For $j = 1$ or 2, if t is large enough then*

$$\frac{d^n}{dt^n} V_j(t,x) \ll \frac{t^B}{x^B}\left(\frac{(\log t)^{n+2}}{t^n}\right)$$

uniformly for $x > 0$.

Proof. We first treat the case with $j = 1$. By Cauchy's theorem and the rapid decay of $\exp(s^2)$ as $|\mathrm{Im}(s)| \to \infty$, we can move the line of integration in the definition (3.33) of $V_1(t,x)$ to $\mathrm{Re}(s) = B$. Then we differentiate under the integral sign to deduce that

$$\frac{d^n}{dt^n} V_1(t,x) = \frac{1}{2\pi i}\int_{(B)} \frac{d^n}{dt^n} H(t,s) q(t,s) \frac{\exp(s^2)}{x^s}\frac{ds}{s}.$$

Write this integral as $I_1 + I_2$, where I_1 is the part with $|\mathrm{Im}(s)|^2 \leq 100t$ and I_2 is the rest. To bound I_1, use (3.32) to deduce that

$$I_1 \ll_{n,B} \frac{t^B}{x^B}\left(\frac{(\log t)^{n+2}}{t^n}\right).$$

On the other hand, we see from (3.31) that

$$I_2 \ll_{n,B} x^{-B} e^{-t}.$$

The conclusion of Lemma 3.10 for $j = 1$ now follows from these estimates for I_1 and I_2. We can reach the same conclusion for $j = 2$ in the exact same way because of the last sentence of Lemma 3.9. □

4. The twisted Atkinson mean value formula

In this section, we prove the following.

Lemma 4.1. *Let $w(t)$ be a smooth function with support in $[T/2, 4T]$ such that $w^{(j)}(t) \ll_j T_0^{-j}$ for $j \geq 0$, where $T^{1/2+\varepsilon} \ll T_0 \ll T$. If h and k are relatively prime positive integers, then*

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left(\frac{h}{k}\right)^{-it} \zeta(\tfrac12 + it')\zeta(\tfrac12 - it) w(t)\, dt \\
&= \frac{1}{\sqrt{hk}}\int_{-\infty}^{\infty} k^{-it'+it}\zeta(1 + it' - it) w(t)\, dt + \frac{e^{-i2a}}{\sqrt{hk}}\int_{-\infty}^{\infty} h^{it'-it}\zeta(1 - it' + it) w(t)\, dt \\
&\quad + O\left(\frac{T^{\frac32+\frac{\ell}{2}+\varepsilon}}{T_0^{\ell}}(hk)^{\ell/2}\right) + O\left((hk)^{-1/4}T^{3/4}\log^2 T\right)
\end{aligned}$$

for any fixed integer $\ell \geq 0$.

Our starting point in proving Lemma 4.1 is the following approximate functional equation. Its proof closely resembles the proof of Proposition 2.1 of Hughes and Young [16] (see also Theorem 5.3 of [19] and Lemma 4 of [26]).

Proposition 4.1. *If t is large enough, then*

$$\zeta(\tfrac12 + it')\zeta(\tfrac12 - it) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{V_1(t, \pi mn)}{m^{\frac12+it'} n^{\frac12 - it}} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{V_2(t,\pi mn)}{m^{\frac12+it}n^{\frac12-it'}} + O(e^{-t^2/4}).$$

Proof. Define $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, so that the functional equation for $\zeta(s)$ implies $\Lambda(s) = \Lambda(1-s)$. Let I be the integral

$$I \;=\; \frac{1}{2\pi i}\int_{(1)} \Lambda(\tfrac12 + it' + s)\Lambda(\tfrac12 - it + s)q(t,s)\exp(s^2)\frac{ds}{s}. \tag{4.1}$$

The poles of the integrand between the lines $\mathrm{Re}(s) = 1$ and $\mathrm{Re}(s) = -1$ are at $s = 0$, $s = \pm\frac12 - it'$, and $s = \pm\frac12 + it$. The residues of the poles different from $s = 0$ are of negligible size due to the rapid decay of $\exp(s^2)$ as $|\mathrm{Im}(s)| \to \infty$. More precisely, we can bound them by $O(e^{-t^2/2})$. Thus, the residue theorem gives

$$I \;=\; \Lambda(\tfrac12 + it')\Lambda(\tfrac12 - it) \;+\; J \;+\; O(e^{-t^2/2}), \tag{4.2}$$

where J is defined in the same way as I, except with $\mathrm{Re}(s) = -1$ as its line of integration. We can easily deduce from the functional equation $\Lambda(s) = \Lambda(1-s)$ and a change of variable that

$$J \;=\; -\frac{1}{2\pi i}\int_{(1)} \Lambda(\tfrac12 - it' + s)\Lambda(\tfrac12 + it + s)q(t,s)\exp(s^2)\frac{ds}{s}. \tag{4.3}$$

Now recall the definitions (3.10) of $H(t,s)$ and (3.21) of $K(t,s)$. Insert (4.1) and (4.3) into (4.2) and rearrange the result to deduce that

$$\begin{aligned}
&\zeta(\tfrac12 + it')\zeta(\tfrac12 - it)\\
&\quad= \frac{1}{2\pi i}\int_{(1)} \pi^{-s}H(t,s)\zeta(\tfrac12 + it' + s)\zeta(\tfrac12 - it + s)q(t,s)\exp(s^2)\frac{ds}{s}\\
&\quad+ \frac{1}{2\pi i}\int_{(1)} \pi^{-s}K(t,s)\zeta(\tfrac12 - it' + s)\zeta(\tfrac12 + it + s)q(t,s)\exp(s^2)\frac{ds}{s} \;+\; O(e^{-t^2/4}).
\end{aligned}$$

We expand each zeta-function into its absolutely convergent Dirichlet series, interchange the order of summation, and then use the definitions (3.33) and (3.34) to arrive at the conclusion of Proposition 4.1. □

We now begin our proof of Lemma 4.1. For brevity, we define

$$I(h,k) \;=\; \int_{-\infty}^{\infty}\left(\frac{h}{k}\right)^{-it}\zeta(\tfrac12 + it')\zeta(\tfrac12 - it)w(t)\,dt$$

We apply Proposition 4.1 to deduce that

(4.4)

$$\begin{aligned}
I(h,k) \;=\; &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\sqrt{mn}}\int_{-\infty}^{\infty}\left(\frac{hm}{kn}\right)^{-it} m^{-it'+it}V_1(t,\pi mn)w(t)\,dt\\
&+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\sqrt{mn}}\int_{-\infty}^{\infty}\left(\frac{hm}{kn}\right)^{-it} n^{it'-it}V_2(t,\pi mn)w(t)\,dt \quad+\quad O\left(e^{-T}\right),
\end{aligned}$$

where we have interchanged the order of summation of the right-hand side.

We will first estimate the contribution of the "off-diagonal" terms, which have $hm \neq kn$. Before we can do so, we need to bound the derivative

$$\frac{d^\ell}{dt^\ell}m^{-it'+it}V_1(t,\pi mn)w(t).$$

We restrict our attention to those t with $T/2 \le t \le 4T$ since we are assuming that the support of $w(t)$ is contained in $[T/2, 4T]$. Recall that we are also assuming that

$$w(t) \ll_j \frac{1}{T_0^j}, \tag{4.5}$$

where $T^{\frac{1}{2}+\varepsilon} \ll T_0 \ll T$. From (3.6), (3.7), and Lemma 3.3 with $f(t) = e^t$ and $g(t) = -i(t' - t)\log m$, it follows that

$$\frac{d^j}{dt^j} m^{-it'+it} \ll_j \frac{\log^j(m+2)}{T^j}$$

for $j \ge 0$. This implies

$$\frac{d^j}{dt^j} m^{-it'+it} \ll_j \frac{\log^j(m+2)}{T_0^j} \tag{4.6}$$

since $T_0 \ll T$. Moreover, from Lemma 3.10 and the fact that $T_0 \ll T$, it follows that

$$\frac{d^j}{dt^j} V_1(t, \pi mn) \ll_{j,B} \frac{T^B}{(mn)^B}\left(\frac{(\log T)^{j+2}}{T_0^j}\right)$$

for any integer $j \ge 0$ and real number $B > 0$. We apply this, (4.5), and (4.6) together with the product rule to deduce that

$$\frac{d^\ell}{dt^\ell} m^{-it'+it} V_1(t, \pi mn) w(t) \ll_{\ell,B} \frac{T^B}{T_0^\ell (mn)^B} (\log T)^{\ell+2} \log^\ell(m+2). \tag{4.7}$$

for $\ell \ge 0$ and $B > 0$. A similar argument leads to

$$\frac{d^\ell}{dt^\ell} n^{it'-it} V_2(t, \pi mn) w(t) \ll_{\ell,B} \frac{T^B}{T_0^\ell (mn)^B} (\log T)^{\ell+2} \log^\ell(m+2). \tag{4.8}$$

for any integer $\ell \ge 0$ and real number $B > 0$.

Using these, we can now estimate the contribution of the off-diagonal terms on the right-hand side of (4.4). By integrating by parts ℓ times and applying (4.7), we see that

$$\int_{-\infty}^{\infty} \left(\frac{hm}{kn}\right)^{-it} m^{-it'+it} V_1(t, \pi mn) w(t)\, dt \ll_{\ell,B} \frac{T}{|\log(hm/kn)|^\ell} \left\{ \frac{T^B}{T_0^\ell (mn)^B} (\log T)^{\ell+2} \log^\ell(m+2) \right\}$$

for any integer $\ell \ge 0$ and real number $B > 0$. We multiply both sides by $(mn)^{-1/2}$ and use the fact that $|\log(hm/kn)| \gg 1/\sqrt{hkmn}$ to arrive at

$$\begin{aligned} &\frac{1}{\sqrt{mn}} \int_{-\infty}^{\infty} \left(\frac{hm}{kn}\right)^{-it} m^{-it'+it} V_1(t, \pi mn) w(t)\, dt \\ &\qquad \ll_{\ell,B} (hk)^{\ell/2} \left\{ \frac{T^{B+1}}{T_0^\ell (mn)^{B-\frac{\ell}{2}+\frac{1}{2}}} (\log T)^{\ell+2} \log^\ell(m+2) \right\}. \end{aligned} \tag{4.9}$$

We choose $B = \frac{\ell}{2} + \frac{1}{2} + \varepsilon$ so that the sum of the right-hand side over all $m, n \ge 1$ converges. Summing (4.9) over all $m, n \ge 1$ with $hm \ne kn$, we now deduce that

$$\sum_{hm \ne kn} \frac{1}{\sqrt{mn}} \int_{-\infty}^{\infty} \left(\frac{hm}{kn}\right)^{-it} m^{-it'+it} V_1(t, \pi mn) w(t)\, dt \ll_{\ell,\varepsilon} (hk)^{\ell/2} \left(\frac{T^{\frac{3}{2}+\frac{\ell}{2}+\varepsilon}}{T_0^\ell} \right).$$

A similar argument using (4.8) leads to

$$\sum_{hm\neq kn}\frac{1}{\sqrt{mn}}\int_{-\infty}^{\infty}\left(\frac{hm}{kn}\right)^{-it}n^{it'-it}V_2(t,\pi mn)w(t)\,dt \ll_{\ell,\varepsilon} (hk)^{\ell/2}\left(\frac{T^{\frac{3}{2}+\frac{\ell}{2}+\varepsilon}}{T_0^{\ell}}\right).$$

From these estimates and (4.4), we conclude that

(4.10)

$$\begin{aligned} I(h,k) &= \sum_{hm=kn}\frac{1}{\sqrt{mn}}\int_{-\infty}^{\infty}m^{-it'+it}V_1(t,\pi mn)w(t)\,dt \\ &\quad + \sum_{hm=kn}\frac{1}{\sqrt{mn}}\int_{-\infty}^{\infty}n^{it'-it}V_2(t,\pi mn)w(t)\,dt + O\left(\frac{T^{\frac{3}{2}+\frac{\ell}{2}+\varepsilon}}{T_0^{\ell}}(hk)^{\ell/2}\right)\end{aligned}$$

for fixed $\ell\geq 0$ and $\varepsilon>0$.

Having bounded the contribution of the off-diagonal terms, our next task is to estimate the sums on the right-hand side of (4.10). Since h and k are relatively prime, the relation $hm=kn$ holds if and only if $m=k\nu$ and $n=h\nu$ for some integer ν. Thus we can write (4.10) as

(4.11)

$$\begin{aligned} I(h,k) &= \frac{1}{\sqrt{hk}}\sum_{\nu=1}^{\infty}\frac{1}{\nu}\int_{-\infty}^{\infty}(k\nu)^{-it'+it}V_1(t,\pi hk\nu^2)w(t)\,dt \\ &\quad + \frac{1}{\sqrt{hk}}\sum_{\nu=1}^{\infty}\frac{1}{\nu}\int_{-\infty}^{\infty}(h\nu)^{it'-it}V_2(t,\pi hk\nu^2)w(t)\,dt + O\left(\frac{T^{\frac{3}{2}+\frac{\ell}{2}+\varepsilon}}{T_0^{\ell}}(hk)^{\ell/2}\right).\end{aligned}$$

Let Σ_1 and Σ_2 denote the first and second terms, respectively, on the right-hand side of (4.11).

We first estimate Σ_1. Replacing $V_1(t,\pi hk\nu^2)$ by its definition (3.33) and interchanging the order of summation, we see that

$$\Sigma_1 = \frac{(hk)^{-1/2}}{2\pi i}\int_{-\infty}^{\infty}k^{-it'+it}\int_{(1)}H(t,s)q(t,s)\frac{\exp(s^2)}{(\pi hk)^s}\zeta(1+it'-it+2s)\frac{ds}{s}w(t)\,dt.$$

The interchange of order of summation is justified by absolute convergence. Observe that the pole of the zeta-function in the above integral is canceled by a zero of $q(t,s)$. Thus $s=0$ is the only pole of the integrand that lies in the region bounded by the lines $\mathrm{Re}(s)=-\frac{1}{4}$ and $\mathrm{Re}(s)=1$. Since $H(t,0)q(t,0)=1$ by (3.10) and (3.30), we therefore deduce from the residue theorem that

(4.12)

$$\begin{aligned}\Sigma_1 &= \frac{1}{\sqrt{hk}}\int_{-\infty}^{\infty}k^{-it'+it}\zeta(1+it'-it)w(t)\,dt \\ &\quad + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{k^{-it'+it}}{\sqrt{hk}}\int_{(-\frac{1}{4})}H(t,s)q(t,s)\frac{\exp(s^2)}{(\pi hk)^s}\zeta(1+it'-it+2s)\frac{ds}{s}w(t)\,dt.\end{aligned}$$

To complete our evaluation of Σ_1, we need to estimate the last term in (4.12). Write the s-integral in (4.12) as

$$\frac{1}{2\pi i}\int_{(-\frac{1}{4})}H(t,s)q(t,s)\frac{\exp(s^2)}{(\pi hk)^s}\zeta(1+it'-it+2s)\frac{ds}{s} = I_1 + I_2,$$

where I_1 is the part with $|\mathrm{Im}(s)|^2 \le 100t$ and I_2 is the rest. Note that it follows from (3.5) and a well-known bound for $\zeta(s)$ (see (2.12.2) of Titchmarsh [24]) that

$$\zeta(1 + it' - it + 2s) \ll 1 + |\mathrm{Im}(s)| \tag{4.13}$$

uniformly for $\mathrm{Re}(s) = -\frac{1}{4}$. To bound I_1, use (4.13) and (3.32) with $n = 0$ to deduce that

$$I_1 \ll \frac{\log^2 t}{t^{1/4}}(hk)^{1/4}.$$

On the other hand, we see from (4.13) and (3.31) with $n = 0$ that

$$I_2 \ll (hk)^{1/4}e^{-t}.$$

From these estimates for I_1 and I_2, it follows that

$$\frac{1}{2\pi i}\int_{(-\frac{1}{4})} H(t,s)q(t,s)\frac{\exp(s^2)}{(\pi hk)^s}\zeta(1+it'-it+2s)\frac{ds}{s} \ll \frac{\log^2 t}{t^{1/4}}(hk)^{1/4}.$$

Insert this into (4.12) and use our assumptions that $w(t) \ll 1$ and $w(t) = 0$ for $t \notin [T/2, 4T]$ to conclude that

$$\Sigma_1 = \frac{1}{\sqrt{hk}}\int_{-\infty}^{\infty} k^{-it'+it}\zeta(1+it'-it)w(t)\,dt + O\left((hk)^{-1/4}T^{3/4}\log^2 T\right). \tag{4.14}$$

We next estimate the second term Σ_2 on the right-hand side of (4.11). We repeat the arguments leading up to (4.12) to deduce from the definition (3.34) of $V_2(t, \pi hk\nu^2)$ and the residue theorem that

(4.15)

$$\begin{aligned}\Sigma_2 = &\frac{1}{\sqrt{hk}}\int_{-\infty}^{\infty} h^{it'-it}K(t,0)\zeta(1-it'+it)w(t)\,dt \\ &+ \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{h^{it'-it}}{\sqrt{hk}}\int_{(-\frac{1}{4})} K(t,s)q(t,s)\frac{\exp(s^2)}{(\pi hk)^s}\zeta(1-it'+it+2s)\frac{ds}{s}w(t)\,dt.\end{aligned}$$

By the last sentence in Lemma 3.9, we can carry out the same reasoning leading up to (4.14) to deduce from (4.15) that

(4.16)

$$\Sigma_2 = \frac{1}{\sqrt{hk}}\int_{-\infty}^{\infty} h^{it'-it}K(t,0)\zeta(1-it'+it)w(t)\,dt + O\left((hk)^{-1/4}T^{3/4}\log^2 T\right).$$

To evaluate $K(t,0)$, we apply (3.22) and the fact that $H(t,0) = 1$ to write $K(t,0) = \pi^{it'-it}X(t)$. Hence, by (3.28) and (3.5), it follows that

$$K(t,0) = e^{-i2a}\left(1 + O\left(\frac{1}{t}\right)\right).$$

We insert this into (4.16) and arrive at

$$\Sigma_2 = \frac{e^{-i2a}}{\sqrt{hk}}\int_{-\infty}^{\infty} h^{it'-it}\zeta(1-it'+it)w(t)\,dt + O\left((hk)^{-1/4}T^{3/4}\log^2 T\right), \tag{4.17}$$

where we have used (3.5) and the Laurent series expansion of $\zeta(s)$ near $s = 1$ to bound the contribution of $O(1/t)$.

Lemma 4.1 now follows from (4.11), (4.14), and (4.17).

5. Proof of Theorem 1.2

We now begin our proof of Theorem 1.2. For brevity, we define

$$\mathcal{I} \;=\; \int_{-\infty}^{\infty} Z(t')Z(t)|M(t)|^2 w(t)\,dt.$$

Since $Z(t)$ is real for real t, it follows that

$$Z(t) = e^{i\vartheta(t)}\zeta(\tfrac12 + it) = e^{-i\vartheta(t)}\zeta(\tfrac12 - it).$$

Use this and the definitions (1.3) and (1.4) to deduce that

$$\begin{aligned}\mathcal{I} &= e^{ia}\int_{-\infty}^{\infty}\zeta(\tfrac12+it')\zeta(\tfrac12-it)|M(t)|^2 w(t)\,dt\\ &= e^{ia}\sum_{m,n\le y}\frac{\mu(m)\mu(n)}{\sqrt{mn}}\left(\frac{\log(y/m)}{\log y}\right)^2\left(\frac{\log(y/n)}{\log y}\right)^2 \qquad (5.1)\\ &\quad\times\int_{-\infty}^{\infty}\left(\frac{m'}{n'}\right)^{-it}\zeta(\tfrac12+it')\zeta(\tfrac12-it)w(t)\,dt,\end{aligned}$$

where $m' = m/(m,n)$ and $n' = n/(m,n)$. We apply Lemma 4.1 to the inner integral. To estimate the contribution of the second error term in the conclusion of Lemma 4.1, we let $\varphi(d)$ be the Euler totient function and write

$$\begin{aligned}\sum_{m,n\le y}\frac{(m'n')^{-1/4}}{\sqrt{mn}} \;&=\; \sum_{m,n\le y}\frac{(m,n)^{1/2}}{(mn)^{3/4}} \;\le\; \sum_{m,n\le y}\frac{(m,n)}{(mn)^{3/4}}\\ &= \sum_{m,n\le y}\frac{1}{(mn)^{3/4}}\sum_{d|(m,n)}\varphi(d) \;=\; \sum_{d\le y}\frac{\varphi(d)}{d^{3/2}}\left(\sum_{j\le y/d}\frac{1}{j^{3/4}}\right)^2\\ &\ll y^{1/2}\log y.\end{aligned}$$

Therefore the result of applying Lemma 4.1 to the inner integral in (5.1) is

$$\begin{aligned}\mathcal{I} \;&=\; \frac{e^{ia}}{\log^4 y}\int_{-\infty}^{\infty}\zeta(1+it'-it)w(t)\mathbf{S}(t'-t)\,dt\\ &\quad+\;\frac{e^{-ia}}{\log^4 y}\int_{-\infty}^{\infty}\zeta(1-it'+it)w(t)\mathbf{S}(-t'+t)\,dt \qquad (5.2)\\ &\quad+\;O\left(y^{\ell+1}\frac{T^{\frac32+\frac{\ell}{2}+\varepsilon}}{T_0^{\ell}}\right)\;+\;O\left(y^{1/2}T^{3/4}\log^3 T\right),\end{aligned}$$

where

$$\mathbf{S}(\varrho) \;=\; \sum_{m,n\le y}\frac{\mu(m)\mu(n)}{\sqrt{mn}}\Big(\log(y/m)\Big)^2\Big(\log(y/n)\Big)^2\frac{(n')^{-i\varrho}}{\sqrt{m'n'}}.$$

Since $m' = m/(m,n)$ and $n' = n/(m,n)$, we can write this as

$$\mathbf{S}(\varrho) \;=\; \sum_{m,n\le y}\frac{\mu(m)\mu(n)}{mn^{1+i\varrho}}(m,n)^{1+i\varrho}\Big(\log(y/m)\Big)^2\Big(\log(y/n)\Big)^2. \qquad (5.3)$$

Our next task is to evaluate $\mathbf{S}(\varrho)$. To do this, we apply Perron's formula

$$\frac{2}{2\pi i}\int_{(\eta)}\frac{x^s}{s^3}\,ds=\begin{cases}\log^2 x & \text{if } x\ge 1\\ 0 & \text{if } 0<x\le 1,\end{cases} \tag{5.4}$$

which holds for any $\eta>0$, to the right-hand side of (5.3). The result is

$$\mathbf{S}(\varrho)=\frac{4}{(2\pi i)^2}\int_{(\eta)}\int_{(\eta)}\frac{y^{s_1+s_2}}{s_1^3s_2^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu(m)\mu(n)}{m^{1+s_1}n^{1+s_2+i\varrho}}(m,n)^{1+i\varrho}\,ds_2\,ds_1, \tag{5.5}$$

where the interchange of order of summation is valid by absolute convergence. To facilitate later discussions, we choose the lines of integration in (5.5) to have $\eta=1/L$, where $L=\log T$. Let

$$g(m,n)=g(m,n;s_1,s_2,\varrho)\;=\;\frac{\mu(m)\mu(n)}{m^{1+s_1}n^{1+s_2+i\varrho}}(m,n)^{1+i\varrho}$$

be the summand of the inner sum in (5.5). This function satisfies the multiplicative property

$$g(h_1j_1,h_2j_2)=g(h_1,h_2)g(j_1,j_2)$$

for $(h_1h_2,j_1j_2)=1$. Thus, it follows from Lemma 3.1 with $k=2$ that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}g(m,n)=\prod_p\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}g(p^m,p^n)\right), \tag{5.6}$$

so long as either side is absolutely convergent. We will see in what follows that the right-hand side is absolutely convergent when ϱ is real and both s_1 and s_2 have positive real parts. Indeed, observe that $g(p^m,p^n)=0$ when either $m\ge 2$ or $n\ge 2$. Thus if ϱ is real and s_1 and s_2 have positive real parts, then

$$\begin{aligned}\prod_p\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}g(p^m,p^n)\right)\;&=\;\prod_p\Big(g(1,1)+g(p,1)+g(1,p)+g(p,p)\Big)\\ &=\;\prod_p\left(1-\frac{1}{p^{1+s_1}}-\frac{1}{p^{1+s_2+i\varrho}}+\frac{1}{p^{1+s_1+s_2}}\right)\\ &=\;\frac{\zeta(1+s_1+s_2)}{\zeta(1+s_1)\zeta(1+s_2+i\varrho)}G(s_1,s_2,\varrho),\end{aligned} \tag{5.7}$$

where $G(s_1,s_2,\varrho)$ is an Euler product such that

$$G(s_1,s_2,z)\;=\;\prod_p\left(1+O\left(\frac{1}{p^{3/2}}\right)\right) \tag{5.8}$$

for $\mathrm{Re}(s_1)\ge -\frac{1}{8}$, $\mathrm{Re}(s_2)\ge -\frac{1}{8}$, and $|z|<\frac{1}{8}$. In view of (5.6) and (5.7), we now deduce from (5.5) that

$$\mathbf{S}(\varrho)=\frac{4}{(2\pi i)^2}\int_{(\eta)}\int_{(\eta)}\frac{y^{s_1+s_2}}{s_1^3s_2^3}\frac{\zeta(1+s_1+s_2)}{\zeta(1+s_1)\zeta(1+s_2+i\varrho)}G(s_1,s_2,\varrho)\,ds_2\,ds_1 \tag{5.9}$$

for real ϱ, where we have chosen $\eta=1/L$. For brevity, we define

$$\mathcal{G}(s_1,s_2,\varrho;y)\;=\;\frac{y^{s_1+s_2}}{s_1^3s_2^3}\frac{\zeta(1+s_1+s_2)}{\zeta(1+s_1)\zeta(1+s_2+i\varrho)}G(s_1,s_2,\varrho). \tag{5.10}$$

Hence we can write (5.9) as

$$\mathbf{S}(\varrho) = \frac{4}{(2\pi i)^2}\int_{(\eta)}\int_{(\eta)} \mathcal{G}(s_1,s_2,\varrho;y)\ \ ds_2\,ds_1 \tag{5.11}$$

where ϱ is real and $\eta = 1/L$.

To complete our evaluation of $\mathbf{S}(\varrho)$, the idea is to move the lines of integration in (5.11) to the left and use the residue theorem. In the rest of the paper, σ_j denotes the real part of s_j, and t_j the imaginary part. We assume that ϱ is a real number such that $\varrho \ll 1/L$. Also, we use A to denote some positive absolute constant that is not necessarily the same for each instance. The more trivial estimations will be left to the reader.

Let E_1 be the line segment from $\eta - iL^5$ to $\eta + iL^5$, and let E_2 be the line segment from $\eta - i2L^5$ to $\eta + i2L^5$. The exponent 5 of L^5 is arbitrary, and the discussions that follow remain valid if we replace 5 by any larger real number. Since $\eta = 1/L$ and $y = T^\theta$, it follows that $y^{s_j} \ll 1$ for $\sigma_j = \eta$. Using this, Lemma 3.2, and (5.8), we can show that the part of the integral in (5.11) that has large $|t_1|$ or $|t_2|$ is small. In other words,

$$\mathbf{S}(\varrho) = \frac{4}{(2\pi i)^2}\int_{E_1}\int_{E_2} \mathcal{G}(s_1,s_2,\varrho;y)\,ds_2\,ds_1 \ +\ O(L^{-A}). \tag{5.12}$$

Next, we let D_2 be the line segment from $-c(\log L)^{-1} - i2L^5$ to $-c(\log L)^{-1} + i2L^5$, where $c > 0$ is an absolute constant that is so small that we can apply Lemma 3.2 to bound the zeta-functions in (5.10) when $-c(\log L)^{-1} \le \sigma_2 \le 1$ and $s_1 \in E_1$. One can take $c = c_0/6$, say. We "move" E_2 to D_2. That is, we integrate $\mathcal{G}$ with respect to s_2 along the rectangle with right side E_2 and left side D_2 and deduce from the residue theorem that

$$\frac{1}{2\pi i}\int_{E_2}\mathcal{G}\,ds_2 \ =\ \operatorname*{Res}_{s_2=0}\mathcal{G} \ +\ \operatorname*{Res}_{s_2=-s_1}\mathcal{G} \ +\ \frac{1}{2\pi i}\int_{D_2}\mathcal{G}\,ds_2 \ +\ \frac{1}{2\pi i}\int_{\mathcal{H}}\mathcal{G}\,ds_2, \tag{5.13}$$

where $\mathcal{H}$ is the union of the horizontal sides of the rectangle. Here and in the sequel, $\operatorname{Res}_{s=\omega} f$ denotes the residue at $s = \omega$ of a function $f(s)$. We multiply both sides of (5.13) by $(2\pi i)^{-1}$ and integrate with respect to s_1 along E_1 to arrive at

(5.14)
$$\begin{aligned}\frac{1}{(2\pi i)^2}\int_{E_1}\int_{E_2}\mathcal{G}\,ds_2\,ds_1 \ &=\ \frac{1}{2\pi i}\int_{E_1}\left\{\operatorname*{Res}_{s_2=0}\mathcal{G} \ +\ \operatorname*{Res}_{s_2=-s_1}\mathcal{G}\right\} ds_1 \\ &\quad +\ \frac{1}{(2\pi i)^2}\int_{E_1}\int_{D_2}\mathcal{G}\,ds_2\,ds_1 \ +\ \frac{1}{(2\pi i)^2}\int_{E_1}\int_{\mathcal{H}}\mathcal{G}\,ds_2\,ds_1.\end{aligned}$$

Using Lemma 3.2, (5.8), and the fact that $|s_2| \ge |t_2| = 2L^5$ for $s_2 \in \mathcal{H}$, we can easily prove that

$$\frac{1}{(2\pi i)^2}\int_{E_1}\int_{\mathcal{H}}\mathcal{G}\,ds_2\,ds_1 \ \ll\ L^{-A}. \tag{5.15}$$

Similarly, applying Lemma 3.2, (5.8), and the bound $y^{s_2} \ll y^{-c/\log L}$ for $s_2 \in D_2$, we can show that

$$\frac{1}{(2\pi i)^2}\int_{E_1}\int_{D_2}\mathcal{G}\,ds_2\,ds_1 \ \ll\ L^A y^{-c/\log L} \ \ll\ L^{-A}. \tag{5.16}$$

It now follows from (5.12), (5.14), (5.15), and (5.16) that

$$\mathbf{S}(\varrho) = \frac{4}{2\pi i}\int_{E_1}\left\{\operatorname*{Res}_{s_2=0}\mathcal{G} \;+\; \operatorname*{Res}_{s_2=-s_1}\mathcal{G}\right\} ds_1 \;+\; O(L^{-A}). \tag{5.17}$$

Now we estimate the integral on the right-hand side of (5.17). We first bound the integral of the residue of $\mathcal{G}$ at $s_2 = -s_1$. From the definition (5.10) of $\mathcal{G}$ and the fact that $\mathrm{Res}_{s=1}\zeta = 1$, it follows that

$$\operatorname*{Res}_{s_2=-s_1}\mathcal{G} \;=\; -\frac{G(s_1,-s_1,\varrho)}{s_1^6\zeta(1+s_1)\zeta(1-s_1+i\varrho)}. \tag{5.18}$$

Let Γ_1 be the path defined by the equation

$$\Gamma_1(t_1) \;=\; \frac{c}{\log(|t_1|+3)} \;+\; it_1, \qquad -L^5 \le t_1 \le L^5.$$

We move E_1 to Γ_1, i.e. we integrate $\mathrm{Res}_{s_2=-s_1}\mathcal{G}$ along the closed curve obtained by connecting the endpoints of E_1 and Γ_1 with horizontal line segments. By Cauchy's theorem, the result is

$$\int_{E_1}\operatorname*{Res}_{s_2=-s_1}\mathcal{G}\;ds_1 \;=\; \int_{\Gamma_1}\operatorname*{Res}_{s_2=-s_1}\mathcal{G}\;ds_1 \;+\; O(L^{-A}), \tag{5.19}$$

where we have bounded the integrals along the horizontal line segments using (5.18), (5.8), Lemma 3.2, and the fact that $|s_1|^6 \ge |t_1|^6 = L^{30}$ for s_1 on those line segments. We can also bound the integral along Γ_1 via (5.18), (5.8), and Lemma 3.2 to deduce that

$$\int_{\Gamma_1}\operatorname*{Res}_{s_2=-s_1}\mathcal{G}\;ds_1 \;\ll\; 1.$$

From this and (5.19), it follows that

$$\int_{E_1}\operatorname*{Res}_{s_2=-s_1}\mathcal{G}\;ds_1 \;\ll\; 1. \tag{5.20}$$

We next estimate the integral of $\mathrm{Res}_{s_2=0}\mathcal{G}$ on the right-hand side of (5.17). By the residue theorem and the definition (5.10) of $\mathcal{G}$, we can write

$$\operatorname*{Res}_{s_2=0}\mathcal{G} \;=\frac{1}{2\pi i}\oint_{\mathcal{C}_2}\frac{y^{s_1+s_2}}{s_1^3s_2^3}\frac{\zeta(1+s_1+s_2)}{\zeta(1+s_1)\zeta(1+s_2+i\varrho)}G(s_1,s_2,\varrho)\,ds_2, \tag{5.21}$$

where $\mathcal{C}_2$ is the circle $|s_2| = 1/2L$, taken once in the positive direction. Let D_1 be the line segment from $-c(\log L)^{-1} - iL^5$ to $-c(\log L)^{-1} + iL^5$. We move E_1 to D_1. That is, we integrate $\mathrm{Res}_{s_2=0}\mathcal{G}$ along the rectangle determined by E_1 and D_1 and use the residue theorem. We bound the integral along the horizontal line segments using (5.21), (5.8), Lemma 3.2, and the fact that $|s_1| \ge |t_1| = L^5$, while we estimate the integral along D_1 via (5.21), (5.8), Lemma 3.2, and the bound $y^{s_1} \ll y^{-c/\log L} \ll L^{-A}$. The result is

$$\frac{1}{2\pi i}\int_{E_1}\operatorname*{Res}_{s_2=0}\mathcal{G}\;ds_1 \;=\; \operatorname*{Res}_{s_1=0}\operatorname*{Res}_{s_2=0}\mathcal{G} \;+\; O(L^{-A}).$$

From this, (5.20), and (5.17), it follows that

$$\mathbf{S}(\varrho) = 4\operatorname*{Res}_{s_1=0}\operatorname*{Res}_{s_2=0}\mathcal{G} \;+\; O(1). \tag{5.22}$$

It is left to evaluate the main term on the right-hand side. By (5.21) and the residue theorem, we can write

(5.23)

$$\operatorname*{Res}_{s_1=0}\operatorname*{Res}_{s_2=0}\mathcal{G} = \frac{1}{(2\pi i)^2}\oint_{\mathcal{C}_1}\oint_{\mathcal{C}_2}\frac{y^{s_1+s_2}}{s_1^3 s_2^3}\frac{\zeta(1+s_1+s_2)}{\zeta(1+s_1)\zeta(1+s_2+i\varrho)}G(s_1,s_2,\varrho)\,ds_2\,ds_1,$$

where $\mathcal{C}_1$ is the circle $|s_2|=1/L$, taken once in the positive direction. If $f(z)$ is analytic and bounded for $|z|<\frac{1}{8}$, then Cauchy's integral formula gives

$$f(s) = f(0) + \frac{s}{2\pi i}\oint_{|z|=\frac{3}{32}}\frac{f(z)}{z(z-s)}\,dz = f(0)+O(|s|) \tag{5.24}$$

uniformly for $|s|<1/16$. By (5.8) and uniform convergence over compact sets, we see that $G(s_1,s_2,z)$ is analytic in each of its variables for $|s_1|,|s_2|,|z|<\frac{1}{8}$. Thus we can apply (5.24) repeatedly to $G(s_1,s_2,\varrho)$ to deduce for $s_1,s_2,\varrho\ll 1/L$ that

$$G(s_1,s_2,\varrho)=G(0,s_2,\varrho)+O(L^{-1})=G(0,0,\varrho)+O(L^{-1})=G(0,0,0)+O(L^{-1}).$$

A straightforward computation using the definition (5.7) of $G(s_1,s_2,\varrho)$ shows us that $G(0,0,0)=1$. Thus

$$G(s_1,s_2,\varrho) \ = \ 1 \ + \ O(L^{-1})$$

for $s_1,s_2,\varrho\ll 1/L$. Also, for the same s_1, s_2, and ϱ, we have the Laurent series expansions

$$\zeta(1+s_1+s_2)=\frac{1}{s_1+s_2}+O(1),$$

$$\frac{1}{\zeta(1+s_1)}=s_1+O(|s_1|^2)=s_1+O(L^{-2}),$$

and

$$\frac{1}{\zeta(1+s_2+i\varrho)}=s_2+i\varrho+O(|s_2+i\varrho|^2)=s_2+i\varrho+O(L^{-2}).$$

We insert all these into (5.23) and arrive at

$$\operatorname*{Res}_{s_1=0}\operatorname*{Res}_{s_2=0}\mathcal{G} \ = \ \frac{1}{(2\pi i)^2}\oint_{\mathcal{C}_1}\oint_{\mathcal{C}_2}\frac{y^{s_1+s_2}}{s_1^2 s_2^3}\left(\frac{s_2+i\varrho}{s_1+s_2}\right)ds_2\,ds_1 \ + \ O(L^2) \tag{5.25}$$

for real ϱ such that $\varrho\ll 1/L$.

Next, we use the fact that

$$\frac{1}{s_1+s_2}=\frac{1}{s_1(1+s_2/s_1)}=\frac{1}{s_1}-\frac{s_2}{s_1^2}+\frac{s_2^2}{s_1^3}+\cdots$$

to deduce from (5.25) that

(5.26)

$$\operatorname*{Res}_{s_1=0}\operatorname*{Res}_{s_2=0}\mathcal{G}$$

$$= \ \frac{1}{(2\pi i)^2}\oint_{\mathcal{C}_1}\oint_{\mathcal{C}_2}y^{s_1+s_2}\left(\frac{1}{s_1^3s_2^2}-\frac{1}{s_1^4s_2}+\frac{i\varrho}{s_1^3s_2^3}-\frac{i\varrho}{s_1^4s_2^2}+\frac{i\varrho}{s_1^5s_2}+\cdots\right)ds_2\,ds_1 \ + \ O(L^2)$$

for $\varrho\ll 1/L$. For integers j and k, it follows from the residue theorem that

$$\frac{1}{(2\pi i)^2}\oint_{\mathcal{C}_1}\oint_{\mathcal{C}_2}\frac{y^{s_1+s_2}}{s_1^j s_2^k}\,ds_2\,ds_1 \ = \ \begin{cases}\dfrac{(\log y)^{j+k-2}}{(j-1)!(k-1)!} & \text{if } j,k\geq 1,\\[2ex] 0 & \text{else.}\end{cases}$$

From this and (5.26), we arrive at

$$\begin{aligned}\operatorname*{Res}_{s_1=0}\operatorname*{Res}_{s_2=0}\mathcal{G} &= \frac{(\log y)^3}{2}-\frac{(\log y)^3}{6}+\frac{i\varrho(\log y)^4}{4}-\frac{i\varrho(\log y)^4}{6}+\frac{i\varrho(\log y)^4}{24}+O(L^2)\\ &= \frac{\log^3 y}{3}+\frac{i\varrho\log^4 y}{8}\ +\ O(L^2)\end{aligned}$$

for real ϱ with $\varrho \ll 1/L$. We insert this into (5.22) to conclude that

$$\mathbf{S}(\varrho)=\frac{4\log^3 y}{3}+\frac{i\varrho\log^4 y}{2}+O(L^2) \tag{5.27}$$

for real ϱ with $\varrho \ll 1/L$. This completes our evaluation of $\mathbf{S}(\varrho)$.

We now finish our proof of Theorem 1.2. By (3.5), if $T/2 \le t \le 4T$ and $\varrho = t'-t$ or $\varrho = -t'+t$, then $\varrho \ll 1/L$ and so (5.27) holds for such ϱ. It follows from this and (5.2) that

$$\begin{aligned}\mathcal{I} \ =\ & \frac{e^{ia}}{\log^4 y}\int_{-\infty}^{\infty}\zeta(1+it'-it)w(t)\left(\frac{4\log^3 y}{3}+\frac{i(t'-t)\log^4 y}{2}+O(L^2)\right)dt\\ &+\ \frac{e^{-ia}}{\log^4 y}\int_{-\infty}^{\infty}\zeta(1-it'+it)w(t)\left(\frac{4\log^3 y}{3}-\frac{i(t'-t)\log^4 y}{2}+O(L^2)\right)dt\\ &+\ O\left(y^{\ell+1}\frac{T^{\frac{3}{2}+\frac{\ell}{2}+\varepsilon}}{T_0^{\ell}}\right)\ +\ O\left(y^{1/2}T^{3/4}\log^3 T\right).\end{aligned}$$

For brevity, write this as

$$\mathcal{I}\ =\ \mathcal{I}_1\ +\ \mathcal{I}_2\ +\ O\left(y^{\ell+1}\frac{T^{\frac{3}{2}+\frac{\ell}{2}+\varepsilon}}{T_0^{\ell}}\right)\ +\ O\left(y^{1/2}T^{3/4}\log^3 T\right). \tag{5.28}$$

By (3.5) and the Laurent series expansion of $\zeta(s)$ near $s=1$, it is true that

$$\zeta(1+it'-it)\ =\ \frac{\log\frac{t}{2\pi}}{i2a}\ +\ O(1)$$

and

$$i(t'-t)\zeta(1+it'-it)\ =\ 1\ +\ O(L^{-1}).$$

We insert these into the definition of $\mathcal{I}_1$ to deduce that

$$\mathcal{I}_1\ =\ \frac{2e^{ia}}{i3a\log y}\int_{-\infty}^{\infty}w(t)\log\frac{t}{2\pi}\,dt\ +\ \frac{e^{ia}}{2}W\ +\ O\left(\frac{W}{L}\right), \tag{5.29}$$

where W is defined by (1.5). Since the support of $w(t)$ is contained in $[T/2, 4T]$, we can write

$$\int_{-\infty}^{\infty}w(t)\log\frac{t}{2\pi}\,dt\ =\ W\log T\ +\ O(W).$$

From this and the definition $y = T^{\theta}$, we see that (5.29) simplifies to

$$\mathcal{I}_1\ =\ \frac{2e^{ia}}{i3a\theta}W\ +\ \frac{e^{ia}}{2}W\ +\ O\left(\frac{W}{L}\right). \tag{5.30}$$

A similar argument leads to

$$\mathcal{I}_2\ =\ -\frac{2e^{-ia}}{i3a\theta}W\ +\ \frac{e^{-ia}}{2}W\ +\ O\left(\frac{W}{L}\right).$$

From this, (5.30), and (5.28), we conclude that

$$\begin{aligned}\mathcal{I} \ &= \frac{4W}{3}\left(\frac{\sin a}{a\theta}\right) \ + \ W\cos a \ \ + \ \ O\left(\frac{W}{L}\right) \\ &\quad + \ O\left(y^{\ell+1}\frac{T^{\frac{3}{2}+\frac{\ell}{2}+\varepsilon}}{T_0^{\ell}}\right) \ + \ O\left(y^{1/2}T^{3/4}\log^3 T\right).\end{aligned}$$

This completes the proof of Theorem 1.2.

6. Application of the theorem of Hughes and Young

Our main tool in proving Theorem 1.3 is the formula for the twisted fourth moment of $\zeta(s)$ that is due to Hughes and Young [16, Theorem 1.1]. In this section, we apply their theorem to write the mollified fourth moment in terms of an integral that can be evaluated using residue theory. We assume that $w(t)$ is a function satisfying the properties in the hypothesis of Theorem 1.3.

We first express the formula of Hughes and Young in a form that is easier to handle. To do this, we make a few definitions. Let Ξ be the set of permutations σ of $\{1,2,3,4\}$ such that $\sigma(1)<\sigma(2)$ and $\sigma(3)<\sigma(4)$. For a function $K(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ of four variables, define

$$(\Psi K)(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \ = \ \sum_{\sigma\in\Xi} K(\alpha_{\sigma(1)},\alpha_{\sigma(2)},\alpha_{\sigma(3)},\alpha_{\sigma(4)}).$$

Let $\sigma_{\alpha,\beta}(\ell)=\sum_{mn=\ell} m^{-\alpha}n^{-\beta}$, and for a positive integer h, let h_p be the exponent of p in the prime factorization $h=\prod_p p^{h_p}$. As in Hughes and Young [16], we define

$$A_{\alpha,\beta,\gamma,\delta}(0)=\frac{\zeta(1+\alpha+\gamma)\zeta(1+\alpha+\delta)\zeta(1+\beta+\gamma)\zeta(1+\beta+\delta)}{\zeta(2+\alpha+\beta+\gamma+\delta)}, \tag{6.1}$$

$$\begin{aligned}B_{\alpha,\beta,\gamma,\delta,h,k}(0) &=\prod_{p\mid h}\left(\frac{\sum_{j=0}^{\infty}\sigma_{\alpha,\beta}(p^j)\sigma_{\gamma,\delta}(p^{j+h_p})p^{-j}}{\sum_{j=0}^{\infty}\sigma_{\alpha,\beta}(p^j)\sigma_{\gamma,\delta}(p^j)p^{-j}}\right) \\ &\qquad\times\prod_{p\mid k}\left(\frac{\sum_{j=0}^{\infty}\sigma_{\alpha,\beta}(p^{j+k_p})\sigma_{\gamma,\delta}(p^j)p^{-j}}{\sum_{j=0}^{\infty}\sigma_{\alpha,\beta}(p^j)\sigma_{\gamma,\delta}(p^j)p^{-j}}\right),\end{aligned} \tag{6.2}$$

and

$$Z_{\alpha,\beta,\gamma,\delta,h,k}(0)=A_{\alpha,\beta,\gamma,\delta}(0)B_{\alpha,\beta,\gamma,\delta,h,k}(0). \tag{6.3}$$

Since the value of $Z_{\alpha,\beta,\gamma,\delta,h,k}(0)$ does not change when α and β are interchanged or when γ and δ are interchanged, we can write the conclusion of Theorem 1.1 of Hughes and Young [16] in the form (note the minus signs before γ and δ)

$$\begin{aligned}&\int_{-\infty}^{\infty}\left(\frac{h}{k}\right)^{-it}\zeta(\tfrac12+\alpha+it)\zeta(\tfrac12+\beta+it)\zeta(\tfrac12-\gamma-it)\zeta(\tfrac12-\delta-it)w(t)\,dt \\ &\quad= \ \frac{1}{\sqrt{hk}}\int_{-\infty}^{\infty}w(t)\left(\frac{t}{2\pi}\right)^{-\frac{\alpha+\beta-\gamma-\delta}{2}}(\Psi K_0)(\alpha,\beta,\gamma,\delta)\,dt \\ &\qquad\qquad + \ O(T^{3/4+\varepsilon}(hk)^{7/8}(T/T_0)^{9/4}),\end{aligned} \tag{6.4}$$

where

$$K_0(\alpha,\beta,\gamma,\delta) \;=\; \left(\frac{t}{2\pi}\right)^{\frac{\alpha+\beta-\gamma-\delta}{2}} Z_{\alpha,\beta,-\gamma,-\delta,h,k}(0).$$

As stated in the hypothesis of Theorem 1.1 of Hughes and Young [16], the formula (6.4) holds when $\alpha,\beta,\gamma,\delta$ are complex numbers that are $\ll 1/L$ (recall that we use $L=\log T$) and h and k are relatively prime positive integers such that $hk \le T^{2/11-\varepsilon}$.

To be able to use (6.4) in our proof of Theorem 1.3, we replace each of the variables $\alpha,\beta,\gamma,\delta$ by itself plus $i\rho$, where

$$\rho \;=\; \frac{2a}{\log\frac{T}{2\pi}} \tag{6.5}$$

and a is a fixed real number. The result is

(6.6)

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\frac{h}{k}\right)^{-it} &\zeta(\tfrac12+\alpha+i\rho+it)\zeta(\tfrac12+\beta+i\rho+it)\\ &\times\zeta(\tfrac12-\gamma-i\rho-it)\zeta(\tfrac12-\delta-i\rho-it)w(t)\,dt\\ =\;& \frac{1}{\sqrt{hk}}\int_{-\infty}^{\infty} w(t)\left(\frac{t}{2\pi}\right)^{-\frac{\alpha+\beta-\gamma-\delta}{2}}(\Psi K_\rho)(\alpha,\beta,\gamma,\delta)\,dt\\ &+\; O(T^{3/4+\varepsilon}(hk)^{7/8}(T/T_0)^{9/4}),\end{aligned}$$

where $K_\rho(\alpha,\beta,\gamma,\delta)$ is defined by

(6.7)

$$K_\rho(\alpha,\beta,\gamma,\delta)=K_\rho(\alpha,\beta,\gamma,\delta;h,k)=\left(\frac{t}{2\pi}\right)^{\frac{\alpha+\beta-\gamma-\delta}{2}} Z_{\alpha+i\rho,\beta+i\rho,-\gamma-i\rho,-\delta-i\rho,h,k}(0).$$

We next express the factor $(\Psi K_\rho)(\alpha,\beta,\gamma,\delta)$ in (6.6) as an integral. To do this, we use the following lemma, which is a special case of Lemma 2.5.1 of Conrey et al. [7].

Lemma 6.1. *Let f be analytic in a neighborhood of $s=0$ except for a simple pole of residue 1 at $s=0$. Suppose that $F(\alpha,\beta,\gamma,\delta)$ is a function that is analytic in each variable in a neighborhood of $(0,0,0,0)$ such that $F(\alpha,\beta,\gamma,\delta)=F(\beta,\alpha,\gamma,\delta)$ and $F(\alpha,\beta,\gamma,\delta)=F(\alpha,\beta,\delta,\gamma)$. Let*

$$K(\alpha,\beta,\gamma,\delta)=F(\alpha,\beta,\gamma,\delta)f(\alpha-\gamma)f(\alpha-\delta)f(\beta-\gamma)f(\beta-\delta).$$

If $\alpha\ne\gamma,\delta$ and $\beta\ne\gamma,\delta$, then

$$\begin{aligned}&(\Psi K)(\alpha,\beta,\gamma,\delta)\\ &\quad=\;\frac{1}{4(2\pi i)^4}\oint\!\!\oint\!\!\oint\!\!\oint \frac{K(z_1,z_2,z_3,z_4)\prod_{1\le j<\ell\le 4}(z_j-z_\ell)^2}{\prod_{j=1}^4(z_j-\alpha)(z_j-\beta)(z_j-\gamma)(z_j-\delta)}\,dz_4\,dz_3\,dz_2\,dz_1,\end{aligned}$$

where the path of integration is the same for all four integrals and encircles each of the four points $\alpha,\beta,\gamma,\delta$ exactly once in the positive direction. In particular, ΨK is analytic at $(0,0,0,0)$.

Now let $\omega\ne 0$ and set $\alpha=\beta=\omega$ and $\gamma=\delta=2\omega$ in (6.6). We apply Lemma 6.1 to the right-hand side of the resulting equation, and then take the limit of both sides as $\omega\to 0$. The result we arrive at is stated in the following proposition.

Proposition 6.1. *Let $w(t)$ be a function satisfying the conditions in the hypothesis of Theorem 1.3. Suppose that h and k are relatively prime positive integers with $hk \leq T^{2/11-\varepsilon}$. If ρ is defined by* (6.5), *then*

(6.8)
$$\begin{aligned}\int_{-\infty}^{\infty}\left(\frac{h}{k}\right)^{-it}&\zeta(\tfrac12+i\rho+it)\zeta(\tfrac12+i\rho+it)\zeta(\tfrac12-i\rho-it)\zeta(\tfrac12-i\rho-it)w(t)\,dt\\ &= \frac{1}{\sqrt{hk}}\int_{-\infty}^{\infty}\frac{w(t)}{4(2\pi i)^4}\oint\oint\oint\oint K_\rho(z_1,z_2,z_3,z_4)\prod_{1\leq j<\ell\leq 4}(z_j-z_\ell)^2\prod_{j=1}^{4}\frac{dz_j}{z_j^4}\,dt\\ &\qquad + O(T^{3/4+\varepsilon}(hk)^{7/8}(T/T_0)^{9/4}),\end{aligned}$$

where the path of integration of each line integral encircles 0 exactly once in the positive direction.

In view of Proposition 6.1, we can now write the mollified fourth moment in terms of path integrals. For brevity, we define

$$\mathcal{J} = \int_{-\infty}^{\infty}|\zeta(\tfrac12+i\rho+it)|^4|M(t)|^4w(t)\,dt,$$

where $0<\theta<1/22$ in the definition (1.4) of $M(t)$. We start by inserting the definition of $M(t)$ into the above equation. Since $\theta<1/22$, we can apply Proposition 6.1 to the resulting expression and deduce that

$$\begin{aligned}\mathcal{J} &= \sum_{n_1,n_2,n_3,n_4\leq y}\left\{\prod_{j=1}^{4}\frac{\mu(n_j)}{n_j^{1/2}}\left(\frac{\log(y/n_j)}{\log y}\right)^2\right\}\frac{1}{\sqrt{(n_1n_2)'(n_3n_4)'}}\\ &\quad\times\int_{-\infty}^{\infty}\frac{w(t)}{4(2\pi i)^4}\oint\oint\oint\oint K_\rho(z_1,z_2,z_3,z_4)\prod_{1\leq j<\ell\leq 4}(z_j-z_\ell)^2\prod_{j=1}^{4}\frac{dz_j}{z_j^4}\,dt\\ &\qquad + O(y^{11/2}T^{3/4+\varepsilon}(T/T_0)^{9/4}),\end{aligned} \tag{6.9}$$

where $(n_1n_2)' = n_1n_2/(n_1n_2,n_3n_4)$ and $(n_3n_4)' = n_3n_4/(n_1n_2,n_3n_4)$. Note that we have trivially bounded the contribution of the error term in (6.8). Using the definition (6.7) of $K_\rho(z_1,z_2,z_3,z_4)$ with $h=(n_1n_2)'$ and $k=(n_3n_4)'$ and the definition (6.3) of Z, we rearrange the factors in (6.9) and write it as

(6.10)
$$\begin{aligned}\mathcal{J} &= \int_{-\infty}^{\infty}\frac{w(t)}{4(2\pi i)^4}\oint\oint\oint\oint\left(\frac{t}{2\pi}\right)^{\frac{z_1+z_2-z_3-z_4}{2}}A_{z_1,z_2,-z_3,-z_4}(0)\prod_{1\leq j<\ell\leq 4}(z_j-z_\ell)^2\\ &\quad\times\left(\frac{16}{(\log y)^8}\right)Q(z_1+i\rho,z_2+i\rho,-z_3-i\rho,-z_4-i\rho)\prod_{j=1}^{4}\frac{dz_j}{z_j^4}\,dt\\ &\qquad + O(y^{11/2}T^{3/4+\varepsilon}(T/T_0)^{9/4}),\end{aligned}$$

where $Q(\alpha,\beta,\gamma,\delta)=Q(\alpha,\beta,\gamma,\delta;y)$ is defined by

(6.11)
$$Q(\alpha,\beta,\gamma,\delta) = \sum_{n_1,n_2,n_3,n_4\le y}\left\{\prod_{j=1}^{4}\frac{\mu(n_j)}{n_j}\left(\frac{\log^2(y/n_j)}{2}\right)\right\}(n_1n_2,n_3n_4)\ B_{\alpha,\beta,\gamma,\delta,(n_1n_2)',(n_3n_4)'}(0).$$

To facilitate later discussions, we choose the path of integration of each line integral in (6.10) to be a circle with center 0 and radius $1/L$.

7. Evaluation of Q

The purpose of this and the next few sections is to evaluate the sum $Q(\alpha,\beta,\gamma,\delta)$ defined by (6.11). We may assume throughout that $\alpha,\beta,\gamma,\delta\ll 1/L$ since we will use our formula for $Q(\alpha,\beta,\gamma,\delta)$ only to estimate the right-hand side of (6.10).

We apply Perron's formula (5.4) to deduce from (6.11) that

(7.1)
$$Q(\alpha,\beta,\gamma,\delta) = \frac{1}{(2\pi i)^4}\int_{(\eta_1)}\int_{(\eta_2)}\int_{(\eta_3)}\int_{(\eta_4)}\frac{y^{s_1+s_2+s_3+s_4}}{s_1^3s_2^3s_3^3s_4^3}$$
$$\times\sum_{1\le n_1,n_2,n_3,n_4<\infty}\left\{\prod_{j=1}^{4}\frac{\mu(n_j)}{n_j^{1+s_j}}\right\}(n_1n_2,n_3n_4)\ B_{\alpha,\beta,\gamma,\delta,(n_1n_2)',(n_3n_4)'}(0)\ \prod_{j=1}^{4}ds_j,$$

where the interchange of order of summation is valid by absolute convergence. For ease in later discussions, we choose the lines of integration in (7.1) to have $\eta_j=c_j/L$ for $j=1,2,3,4$, where c_1, c_2, c_3, and c_4 are positive constants that satisfy the inequalities

$$|\alpha|,|\beta|,|\gamma|,|\delta|<\frac{c_1}{L}<\frac{c_2}{L}<\frac{c_3}{L}<\frac{c_4}{L}. \tag{7.2}$$

We next write the inner sum in (7.1) as an Euler product. Let

$$f(n_1,n_2,n_3,n_4) = \left\{\prod_{j=1}^{4}\frac{\mu(n_j)}{n_j^{1+s_j}}\right\}(n_1n_2,n_3n_4)\ B_{\alpha,\beta,\gamma,\delta,(n_1n_2)',(n_3n_4)'}(0). \tag{7.3}$$

It follows from the definition (6.2) and from the multiplicativity of the gcd function that f is multiplicative, i.e.

$$f(h_1j_1,h_2j_2,h_3j_3,h_4j_4)=f(h_1,h_2,h_3,h_4)\cdot f(j_1,j_2,j_3,j_4)$$

whenever $h_1h_2h_3h_4$ and $j_1j_2j_3j_4$ are relatively prime. Hence, it follows from Lemma 3.1 with $k=4$ that

(7.4)
$$\sum_{1\le n_1,n_2,n_3,n_4<\infty}f(n_1,n_2,n_3,n_4) = \prod_p\left(\sum_{0\le\ell_1,\ell_2,\ell_3,\ell_4<\infty}f(p^{\ell_1},p^{\ell_2},p^{\ell_3},p^{\ell_4})\right),$$

so long as either side is absolutely convergent. We will see in what follows that the right-hand side is absolutely convergent when α, β, γ, and δ are small and the real part σ_j of s_j is larger than each of $|\alpha|,|\beta|,|\gamma|,|\delta|$ for $j=1,2,3,4$. Observe that $f(n_1,n_2,n_3,n_4)$ is zero when at least one of n_1,n_2,n_3,n_4 is divisible by the square of a prime. Thus, to evaluate the right-hand side of (7.4), we only need to calculate $f(p^{\ell_1}, p^{\ell_2}, p^{\ell_3}, p^{\ell_4})$ for $0 \le \ell_1, \ell_2, \ell_3, \ell_4 \le 1$. To do this, we use the following lemma, which is a special case of Lemma 6.9 of Hughes and Young [16]. Recall the definition (6.2) of $B_{\alpha,\beta,\gamma,\delta,h,k}(0)$.

Lemma 7.1. *For small enough $\alpha, \beta, \gamma, \delta$, it is true that*

$$B_{\alpha,\beta,\gamma,\delta,h,k}(0) \;=\; B_{\alpha,\beta,\gamma,\delta,h}\; B_{\gamma,\delta,\alpha,\beta,k},$$

where $B_{\alpha,\beta,\gamma,\delta,h}$ is defined by

$$B_{\alpha,\beta,\gamma,\delta,h} \;=\; \prod_{p|h}\left(\frac{B^{(0)} - p^{-1}B^{(1)} + p^{-2}B^{(2)}}{(p^{-\gamma}-p^{-\delta})(1-p^{-2-\alpha-\beta-\gamma-\delta})}\right),$$

where

$$\begin{aligned} B^{(0)} &= p^{-\gamma(1+h_p)} \;-\; p^{-\delta(1+h_p)},\\ B^{(1)} &= (p^{-\alpha}+p^{-\beta})p^{-\gamma-\delta}(p^{-\gamma h_p}-p^{-\delta h_p}),\\ B^{(2)} &= p^{-\alpha-\beta-\gamma-\delta}(p^{-\delta-\gamma h_p}-p^{-\gamma-\delta h_p}), \end{aligned}$$

and $h = \prod_p p^{h_p}$ is the prime factorization of h.

Using Lemma 7.1 and the definition (7.3), we can evaluate each of the sixteen quantities $f(p^{\ell_1}, p^{\ell_2}, p^{\ell_3}, p^{\ell_4})$ with $0 \le \ell_1, \ell_2, \ell_3, \ell_4 \le 1$. For example, if $\alpha, \beta, \gamma, \delta$ are small enough, then a straightforward calculation shows that

$$\begin{aligned} f(p,1,1,1) \;&=\; \left\{\frac{-1}{p^{1+s_1}}\right\}\left\{\frac{p^{-2\gamma}-p^{-2\delta}-p^{-1}(p^{-\alpha}+p^{-\beta})p^{-\gamma-\delta}(p^{-\gamma}-p^{-\delta})}{(p^{-\gamma}-p^{-\delta})(1-p^{-2-\alpha-\beta-\gamma-\delta})}\right\}\\ &=\; -\frac{1}{p^{1+s_1+\gamma}} \;-\; \frac{1}{p^{1+s_1+\delta}} \;+\; O\left(\frac{1}{p^{3/2}}\right) \end{aligned}$$

for $\sigma_1 > -1/8$, say, with absolute implied constant. Similarly, we can evaluate each of the other fifteen quantities in a straightforward way. The details are left to the reader. From the results of these calculations, we deduce that if $\alpha, \beta, \gamma, \delta$ are small enough, then

$$\begin{aligned} &\sum_{0\le \ell_1,\ell_2,\ell_3,\ell_4<\infty} f(p^{\ell_1}, p^{\ell_2}, p^{\ell_3}, p^{\ell_4})\\ &=\; 1 - \frac{1}{p^{1+s_1+\gamma}} - \frac{1}{p^{1+s_1+\delta}} - \frac{1}{p^{1+s_2+\gamma}} - \frac{1}{p^{1+s_2+\delta}} - \frac{1}{p^{1+s_3+\alpha}} - \frac{1}{p^{1+s_3+\beta}}\\ &\quad - \frac{1}{p^{1+s_4+\alpha}} - \frac{1}{p^{1+s_4+\beta}} + \frac{1}{p^{1+s_1+s_3}} + \frac{1}{p^{1+s_1+s_4}} + \frac{1}{p^{1+s_2+s_3}} + \frac{1}{p^{1+s_2+s_4}} + O\left(\frac{1}{p^{3/2}}\right) \end{aligned}$$

for prime p and complex s_1, s_2, s_3, s_4 having real parts $\sigma_j > -1/8$. Therefore, if $\sigma_j > \max\{|\alpha|, |\beta|, |\gamma|, |\delta|\}$ for $j = 1, 2, 3, 4$, then

(7.5)

$$\begin{aligned} &\prod_p\left(\sum_{0\le \ell_1,\ell_2,\ell_3,\ell_4<\infty} f(p^{\ell_1}, p^{\ell_2}, p^{\ell_3}, p^{\ell_4})\right)\\ &= \zeta^{-1}(1+s_1+\gamma)\zeta^{-1}(1+s_1+\delta)\zeta^{-1}(1+s_2+\gamma)\zeta^{-1}(1+s_2+\delta)\\ &\quad\times \zeta^{-1}(1+s_3+\alpha)\zeta^{-1}(1+s_3+\beta)\zeta^{-1}(1+s_4+\alpha)\zeta^{-1}(1+s_4+\beta)\\ &\quad\times \zeta(1+s_1+s_3)\zeta(1+s_1+s_4)\zeta(1+s_2+s_3)\zeta(1+s_2+s_4)\; F(s_1,s_2,s_3,s_4), \end{aligned}$$

where $F(s_1, s_2, s_3, s_4) = F(s_1, s_2, s_3, s_4; \alpha, \beta, \gamma, \delta)$ is an Euler product satisfying

$$F(s_1, s_2, s_3, s_4) \; = \; \prod_p \left(1 \; + \; O\left(\frac{1}{p^{3/2}}\right)\right) \tag{7.6}$$

uniformly for all small enough $\alpha, \beta, \gamma, \delta$ and all complex s_1, s_2, s_3, s_4 that have real parts $\sigma_j > -1/8$.

In view of the assumption (7.2) and the definition (7.3), we now deduce from (7.1), (7.4), and (7.5) that

$$Q(\alpha, \beta, \gamma, \delta) \; = \; \frac{1}{(2\pi i)^4} \int_{(\eta_1)} \int_{(\eta_2)} \int_{(\eta_3)} \int_{(\eta_4)} \mathcal{F}(s_1, s_2, s_3, s_4) \; ds_4 \, ds_3 \, ds_2 \, ds_1, \tag{7.7}$$

where

(7.8)

$$\begin{aligned} \mathcal{F}(s_1, s_2, s_3, s_4) \; &= \; \mathcal{F}(s_1, s_2, s_3, s_4; \alpha, \beta, \gamma, \delta, y) \; = \; \frac{y^{s_1+s_2+s_3+s_4}}{s_1^3 s_2^3 s_3^3 s_4^3} \times \\ &\times \zeta^{-1}(1+s_1+\gamma)\zeta^{-1}(1+s_1+\delta)\zeta^{-1}(1+s_2+\gamma)\zeta^{-1}(1+s_2+\delta) \\ &\times \zeta^{-1}(1+s_3+\alpha)\zeta^{-1}(1+s_3+\beta)\zeta^{-1}(1+s_4+\alpha)\zeta^{-1}(1+s_4+\beta) \\ &\times \zeta(1+s_1+s_3)\zeta(1+s_1+s_4)\zeta(1+s_2+s_3)\zeta(1+s_2+s_4) \; F(s_1, s_2, s_3, s_4). \end{aligned}$$

Recall that we have chosen $\eta_j = c_j/L$ for $j = 1, 2, 3, 4$, where the constants c_j satisfy the assumption (7.2).

8. Initial shifting of contours

To continue with our evaluation of $Q(\alpha, \beta, \gamma, \delta)$, the idea is to move the lines of integration in (7.7) and write $Q(\alpha, \beta, \gamma, \delta)$ in terms of the residues of $\mathcal{F}(s_1, s_2, s_3, s_4)$ at its poles. This turns out to be a rather involved process, and will occupy us for the next few sections. Recall that we use the notation $s_j = \sigma_j + it_j$, that $L = \log T$, and that A denotes a positive absolute constant that is not necessarily the same for each instance. Recall also that we are assuming (7.2). The more trivial estimations will be left to the reader.

We first truncate the lines of integration in (7.7), as follows. For $m = 1, 2, 3, 4$, let E_m be the line segment from $\eta_m - imL^{17}$ to $\eta_m + imL^{17}$. The exponent 17 in L^{17} is arbitrary, and the discussions that follow remain valid if we replace 17 by any larger real number. Since $\eta_m = c_m/L$ and $y = T^\theta$, it follows that $y^{s_j} \ll 1$ when $\sigma_j = \eta_j$. Using this, Lemma 3.2, and (7.6), we can deduce from (7.7) and (7.8) that

$$Q \; = \; \frac{1}{(2\pi i)^4} \int_{E_1} \int_{E_2} \int_{E_3} \int_{E_4} \mathcal{F}(s_1, s_2, s_3, s_4) \quad ds_4 \, ds_3 \, ds_2 \, ds_1 \; + \; O(L^{-A}). \tag{8.1}$$

The next step is to move E_4 to the left and write Q in terms of an integral of residues of $\mathcal{F}$. Let $c > 0$ be an absolute constant that is so small that we can apply Lemma 3.2 to the zeta-functions in the definition (7.8) of $\mathcal{F}$ when $-c(\log L)^{-1} \le \sigma_4 \le 1$ and $s_j \in E_j$ for $j = 1, 2, 3$. We can take $c = c_0/99$, say, where c_0 is as in Lemma 3.2. Let D_4 be the line segment from $-c(\log L)^{-1} - i4L^{17}$ to $-c(\log L)^{-1} + i4L^{17}$. We move E_4 to D_4 and proceed in a way similar to that in the above discussion containing equations (5.13) through (5.16). That is, we integrate $\mathcal{F}$ with respect to s_4 along the rectangle determined by E_4 and D_4. We use the residue theorem to write that the integral equals a sum of residues of $\mathcal{F}$. Then we integrate both sides of the resulting equation with respect to s_3, s_2, and s_1 along E_3, E_2, and E_1, respectively. The contribution of the integrals along the horizontal sides of the rectangle is small because of Lemma 3.2, (7.6), the bound $y^{s_1+s_2+s_3+s_4} \ll 1$, and the small size of the factor s_4^{-3} of $\mathcal{F}$. Similarly, the contribution of the integral along D_4 is small because $y^{s_4} \ll y^{-c/\log L}$ for $s_4 \in D_4$. The result we arrive at from (8.1) and this process of moving E_4 to D_4 is

(8.2)
$$Q = \frac{1}{(2\pi i)^3} \int_{E_1} \int_{E_2} \int_{E_3} \left\{ \operatorname*{Res}_{s_4=0} \mathcal{F} + \operatorname*{Res}_{s_4=-s_1} \mathcal{F} + \operatorname*{Res}_{s_4=-s_2} \mathcal{F} \right\} ds_3 \, ds_2 \, ds_1 \; + \; O(L^{-A}).$$

In a similar way, we move E_3 to the line segment D_3 that runs from $-c(\log L)^{-1} - i3L^{17}$ to $-c(\log L)^{-1} + i3L^{17}$ to deduce from (8.2) that

$$\begin{aligned} Q = \frac{1}{(2\pi i)^2} \int_{E_1} \int_{E_2} \Bigg\{ & \operatorname*{Res}_{s_3=0} \operatorname*{Res}_{s_4=0} \mathcal{F} + \operatorname*{Res}_{s_3=0} \operatorname*{Res}_{s_4=-s_1} \mathcal{F} + \operatorname*{Res}_{s_3=0} \operatorname*{Res}_{s_4=-s_2} \mathcal{F} \\ & + \operatorname*{Res}_{s_3=-s_1} \operatorname*{Res}_{s_4=0} \mathcal{F} + \operatorname*{Res}_{s_3=-s_1} \operatorname*{Res}_{s_4=-s_1} \mathcal{F} + \operatorname*{Res}_{s_3=-s_1} \operatorname*{Res}_{s_4=-s_2} \mathcal{F} \\ & + \operatorname*{Res}_{s_3=-s_2} \operatorname*{Res}_{s_4=0} \mathcal{F} + \operatorname*{Res}_{s_3=-s_2} \operatorname*{Res}_{s_4=-s_1} \mathcal{F} + \operatorname*{Res}_{s_3=-s_2} \operatorname*{Res}_{s_4=-s_2} \mathcal{F} \Bigg\} ds_2 \, ds_1 \; + \; O(L^{-A}). \end{aligned}$$

Write this as

(8.3)
$$\begin{aligned} Q = \frac{1}{(2\pi i)^2} \int_{E_1} \int_{E_2} \Bigg\{ & R_{00} + R_{01} + R_{02} \\ & + R_{10} + R_{11} + R_{12} \\ & + R_{20} + R_{21} + R_{22} \Bigg\} ds_2 \, ds_1 \; + \; O(L^{-A}). \end{aligned}$$

We treat each of the nine integrals $\iint R_{\ell\nu}$ in (8.3) one by one in the following sections. As mentioned at the beginning of this section, the basic idea is to move the paths of integration in (8.3) and express Q as a sum of residues of $\mathcal{F}$. When we "move" one path of integration to another, we always do so by connecting the two paths with horizontal line segments and then applying the residue theorem to write one path integral in terms of the other. The contributions of the integrals along the horizontal line segments are always bounded by $\ll L^{-A}$. These bounds follow from using $y^{s_j} \ll 1$, Lemma 3.2, (7.6), and the fact that $|s_m| \geq |t_m| \asymp L^{17}$ for some $m = 1, 2, 3, 4$. The calculations involved in bounding the integrals of these horizontal segments are straightforward, and are thus left to the reader. From now on, we will move paths of integration without further mention of the horizontal line segments.

Another thing to keep in mind in the calculations below is that the factors $\zeta^{-1}(1 \pm s_j + \xi)$ with $\xi = \alpha, \beta, \gamma$, or δ are holomorphic functions of s_j in the domains we are concerned with. This is because we will only work within the known zero-free region of $\zeta(s)$ given by Lemma 3.2. Also, the euler product F defined in (7.5) is holomorphic by (7.6). Thus we can determine the poles of each $R_{\ell\nu}$ by looking only at the factors s_j^{-1} and $\zeta(1 + s_j \pm s_k)$ in our expression for $R_{\ell\nu}$.

9. Evaluation of $\iint R_{12}$ and $\iint R_{21}$

If $f(s)$ is a function with a simple pole at $s=\omega$, then

$$\operatorname*{Res}_{s=\omega} f \;=\; \lim_{s\to\omega}(s-\omega)f(s). \tag{9.1}$$

From this, the definition (7.8), and the equation $\lim_{s\to 0} s\zeta(1+s)=1$, it follows that

$$\begin{aligned} R_{12} = \frac{1}{s_1^6 s_2^6}&\zeta^{-1}(1+s_1+\gamma)\zeta^{-1}(1+s_1+\delta)\zeta^{-1}(1+s_2+\gamma)\zeta^{-1}(1+s_2+\delta)\\ &\times\zeta^{-1}(1-s_1+\alpha)\zeta^{-1}(1-s_1+\beta)\zeta^{-1}(1-s_2+\alpha)\zeta^{-1}(1-s_2+\beta)\\ &\times\zeta(1+s_1-s_2)\zeta(1+s_2-s_1)F(s_1,s_2,-s_1,-s_2)\end{aligned} \tag{9.2}$$

when $s_1\in E_1$ and $s_2\in E_2$. For a fixed $s_1\in E_1$, let Γ_0 be the path defined by

$$\Gamma_0(t_2) \;=\; \frac{c}{\log(|t_1|+|t_2|+3)} \;+\; it_2, \qquad -2L^{17}\le t_2\le 2L^{17}.$$

Note that Γ_0 lies to the right of E_2 for large enough L. Thus it also lies to the right of E_1 by our assumption (7.2). Hence, if $s_1\in E_1$, then R_{12} has no poles as a function of s_2 on the region between E_2 and Γ_0. Thus, the result of moving E_2 to Γ_0 is

$$\int_{E_1}\int_{E_2} R_{12}\;ds_2\,ds_1 = \int_{E_1}\int_{\Gamma_0} R_{12}\;ds_2\,ds_1 \;+\; O(L^{-A}). \tag{9.3}$$

To bound the integral on the right-hand side, we estimate the factors in (9.2) as follows. We take $s_1\in E_1$ and $s_2\in\Gamma_0$ in (9.2), and we recall our assumption (7.2). First, by Lemma 3.2, we see that

$$\zeta(1\pm s_1\mp s_2)\ll\max\left\{\frac{1}{\sigma_2-\sigma_1}\;,\;\log(|t_1|+|t_2|+2)\right\}\ll\log(|t_1|+|t_2|+3).$$

Second, if $|t_1|\le 1$ then we can use the power series of $\zeta^{-1}(s)$ near $s=1$ to write

$$\frac{1}{s_1\zeta(1\pm s_1+\alpha)} \;\ll\; 1.$$

Otherwise, we use Lemma 3.2. Thus, if $-\infty<t_1<\infty$ then

$$\frac{1}{s_1\zeta(1\pm s_1+\alpha)} \;\ll\; \min\left\{1,\frac{\log(|t_1|+3)}{|t_1|}\right\}.$$

The same estimate holds when s_1 is replaced by s_2 or α is replaced by β, γ, or δ. Therefore

$$\begin{aligned}&\frac{1}{s_1^4 s_2^4}\zeta^{-1}(1+s_1+\gamma)\zeta^{-1}(1+s_1+\delta)\zeta^{-1}(1+s_2+\gamma)\zeta^{-1}(1+s_2+\delta)\\ &\quad\times\zeta^{-1}(1-s_1+\alpha)\zeta^{-1}(1-s_1+\beta)\zeta^{-1}(1-s_2+\alpha)\zeta^{-1}(1-s_2+\beta)\\ &\ll\min\left\{1,\frac{\log(|t_1|+3)}{|t_1|}\right\}^4\min\left\{1,\frac{\log(|t_2|+3)}{|t_2|}\right\}^4.\end{aligned} \tag{9.4}$$

Third, it is true that

$$\frac{1}{s_1^2 s_2^2} \;\ll\; \frac{1}{\sigma_1^2\sigma_2^2} \;\ll\; L^2\log^2(|t_1|+|t_2|+3).$$

Fourth, it follows from (7.6) that $F(s_1,s_2,-s_1,-s_2)\ll 1$.

From all these bounds, we arrive at

$$\int_{E_1}\int_{\Gamma_0} R_{12}\ \ ds_2\, ds_1 \ \ll\ L^2.$$

It follows from this and (9.3) that

$$\int_{E_1}\int_{E_2} R_{12}\ \ ds_2\, ds_1 \ll L^2. \tag{9.5}$$

A similar argument gives

$$\int_{E_1}\int_{E_2} R_{21}\ \ ds_2\, ds_1 \ll L^2. \tag{9.6}$$

10. Evaluation of $\iint R_{22}$

From (9.1), the definition (7.8), and the equation $\lim_{s\to 0} s\zeta(1+s) = 1$, it follows that

(10.1)
$$\begin{aligned} R_{22} = \frac{y^{s_1-s_2}}{s_1^3 s_2^9}&\zeta^{-1}(1+s_1+\gamma)\zeta^{-1}(1+s_1+\delta)\zeta^{-1}(1+s_2+\gamma)\zeta^{-1}(1+s_2+\delta)\\ &\times \zeta^{-1}(1-s_2+\alpha)\zeta^{-1}(1-s_2+\beta)\zeta^{-1}(1-s_2+\alpha)\zeta^{-1}(1-s_2+\beta)\\ &\times \zeta(1+s_1-s_2)\zeta(1+s_1-s_2)F(s_1,s_2,-s_2,-s_2)\end{aligned}$$

when $s_1 \in E_1$ and $s_2 \in E_2$. Let Υ_2 be the line segment from $c(\log L)^{-1} - i2L^{17}$ to $c(\log L)^{-1} + i2L^{17}$. Note that Υ_2 lies to the right of E_2 for large enough L. Thus it also lies to the right of E_1 by (7.2). Hence, if $s_1 \in E_1$, then R_{22} has no poles as a function of s_2 inside the rectangle determined by E_2 and Υ_2. Therefore the result when we move E_2 to Υ_2 is

$$\int_{E_1}\int_{E_2} R_{22}\ \ ds_2\, ds_1 \ =\ \int_{E_1}\int_{\Upsilon_2} R_{22}\ \ ds_2\, ds_1 \ +\ O(L^{-A}). \tag{10.2}$$

Using (10.1), Lemma 3.2, (7.6), and the bound $y^{-s_2} \ll y^{-c/\log L}$, we can show that

$$\int_{E_1}\int_{\Upsilon_2} R_{22}\ \ ds_2\, ds_1 \ \ll\ L^A y^{-c/\log L} \ \ll\ L^{-A}. \tag{10.3}$$

It follows from this and (10.2) that

$$\int_{E_1}\int_{E_2} R_{22}\ \ ds_2\, ds_1 \ \ll\ L^{-A}. \tag{10.4}$$

11. Evaluation of $\iint R_{11}$

We use (9.1) to deduce from (7.8) that if $s_1 \in E_1$ and $s_2 \in E_2$, then

(11.1)
$$\begin{aligned} R_{11} = \frac{y^{s_2-s_1}}{s_1^9 s_2^3}&\zeta^{-1}(1+s_1+\gamma)\zeta^{-1}(1+s_1+\delta)\zeta^{-1}(1+s_2+\gamma)\zeta^{-1}(1+s_2+\delta)\\ &\times \zeta^{-1}(1-s_1+\alpha)\zeta^{-1}(1-s_1+\beta)\zeta^{-1}(1-s_1+\alpha)\zeta^{-1}(1-s_1+\beta)\\ &\times \zeta(1+s_2-s_1)\zeta(1+s_2-s_1)F(s_1,s_2,-s_1,-s_1).\end{aligned}$$

Let D_2 be the line segment from $-c(\log L)^{-1} - i2L^{17}$ to $-c(\log L)^{-1} + i2L^{17}$. We move E_2 to D_2 and arrive at

$$\begin{aligned} \frac{1}{2\pi i}\int_{E_1}\int_{E_2} R_{11}\ ds_2\, ds_1 \ &= \ \frac{1}{2\pi i}\int_{E_1}\int_{D_2} R_{11}\ ds_2\, ds_1 \\ &\quad + \int_{E_1}\operatorname*{Res}_{s_2=0} R_{11}\, ds_1 \ + \ \int_{E_1}\operatorname*{Res}_{s_2=s_1} R_{11}\, ds_1 \ + \ O(L^{-A}) \\ &= \ J_{11} \ + \ I_{110} \ + \ I_{111} \ + \ O(L^{-A}), \end{aligned} \tag{11.2}$$

say. We can bound J_{11} in the same way we showed (10.3). That is, we use (11.1) and $y^{s_2} \ll y^{-c/\log L}$ to write

$$J_{11} \ \ll \ L^{-A}. \tag{11.3}$$

To estimate the integral I_{110} in (11.2), let Υ_1 be the line segment from $c(\log L)^{-1} - iL^{17}$ to $c(\log L)^{-1} + iL^{17}$. Observe that Υ_1 lies to the right of E_1 when L is large enough. Thus, we see from (11.1) that $\mathrm{Res}_{s_2=0}R_{11}$ has no poles between E_1 and Υ_1. Hence we deduce from moving E_1 to Υ_1 that

$$I_{110} \ = \ \int_{\Upsilon_1}\operatorname*{Res}_{s_2=0} R_{11}\, ds_1 \ + \ O(L^{-A}).$$

The integral on the right-hand side is $\ll L^{-A}$ as in (10.3), because R_{11} has the factor y^{-s_1} in (11.1). Therefore

$$I_{110} \ \ll \ L^{-A}. \tag{11.4}$$

We next estimate the integral I_{111} in (11.2). If $f(s)$ is a function with a double pole at $s = \omega$, then

$$\operatorname*{Res}_{s=\omega} f \ = \ \lim_{s\to\omega}\frac{d}{ds}(s-\omega)^2 f(s).$$

It follows from this, (11.1), and straightforward differentiation that

$$\begin{aligned} \operatorname*{Res}_{s_2=s_1} R_{11} &= \frac{\log y}{s_1^{12}}\zeta^{-2}(1+s_1+\gamma)\zeta^{-2}(1+s_1+\delta) \\ &\quad \times \zeta^{-2}(1-s_1+\alpha)\zeta^{-2}(1-s_1+\beta)F(s_1,s_1,-s_1,-s_1) \ + \ X_0(s_1), \end{aligned} \tag{11.5}$$

where $X_0(s_1) = X_0(s_1;\alpha,\beta,\gamma,\delta)$ is an expression involving ζ^{-1}, ζ'/ζ, F, and $\partial F/\partial s_2$. The function $X_0(s_1)$ is independent of y. Now let Γ_1 be the path defined by

$$\Gamma_1(t_1) \ = \ \frac{c}{\log(|t_1|+3)} \ + \ it_1, \qquad -L^{17} \le t_1 \le L^{17}. \tag{11.6}$$

Note that Γ_1 lies to the right of E_1 when L is large. From (11.5), we can deduce that $\mathrm{Res}_{s_2=s_1}R_{11}$ has no poles between E_1 and Γ_1. Thus, the result of moving E_1 to Γ_1 is

$$I_{111} \ = \ \int_{\Gamma_1}\operatorname*{Res}_{s_2=s_1} R_{11}\, ds_1 \ + \ O(L^{-A}). \tag{11.7}$$

We use Lemma 3.2 and (7.6) to bound each term on the right-hand side of (11.5) with $s_1 \in \Gamma_1$ to see that

$$\int_{\Gamma_1}\operatorname*{Res}_{s_2=s_1} R_{11}\, ds_1 \ = \ O(\log y) \ + \ O(1) \ \ll \ L.$$

This and (11.7) imply

$$I_{111} \ll L.$$

From this, (11.4), (11.3), and (11.2), we arrive at

$$\int_{E_1}\int_{E_2} R_{11}\ ds_2\,ds_1 \ll L. \tag{11.8}$$

12. Evaluation of $\iint R_{02}$ and $\iint R_{20}$

Applying (9.1) and the residue theorem, we see for $s_1 \in E_1$ and $s_2 \in E_2$ that

$$\begin{aligned} R_{02} &= -\frac{1}{2\pi i}\oint_{\mathcal{C}_3} \frac{y^{s_1+s_3}}{s_1^3 s_2^6 s_3^3}\times \\ &\quad\times \zeta^{-1}(1+s_1+\gamma)\zeta^{-1}(1+s_1+\delta)\zeta^{-1}(1+s_2+\gamma)\zeta^{-1}(1+s_2+\delta) \\ &\quad\times \zeta^{-1}(1+s_3+\alpha)\zeta^{-1}(1+s_3+\beta)\zeta^{-1}(1-s_2+\alpha)\zeta^{-1}(1-s_2+\beta) \\ &\quad\times \zeta(1+s_1+s_3)\zeta(1+s_1-s_2)\zeta(1+s_2+s_3)F(s_1,s_2,s_3,-s_2)\,ds_3, \end{aligned} \tag{12.1}$$

where $\mathcal{C}_3$ is the circle $|s_3| = c_1/2L$, taken once in the positive direction. Let D_1 be the line segment from $-c(\log L)^{-1} - iL^{17}$ to $-c(\log L)^{-1} + iL^{17}$. We interchange the order of integration and then move E_1 to D_1 to deduce that

$$\begin{aligned} \frac{1}{2\pi i}\int_{E_1}\int_{E_2} R_{02}\ ds_2\,ds_1 &= \frac{1}{2\pi i}\int_{E_2}\int_{E_1} R_{02}\ ds_1\,ds_2 \\ &= \frac{1}{2\pi i}\int_{E_2}\int_{D_1} R_{02}\ ds_1\,ds_2 + \int_{E_2}\operatorname*{Res}_{s_1=0} R_{02}\ ds_2 + O(L^{-A}) \\ &= J_{02} + I_{02} + O(L^{-A}), \end{aligned} \tag{12.2}$$

say. Note that the pole of R_{02} at $s_1 = s_2$ does not leave a residue because it is not between E_1 and D_1 when $s_2 \in E_2$, by (7.2). Using (12.1), we can estimate the integral J_{02} in (12.2) in the same way we arrived at (10.3). In other words, we can write

$$J_{02} \ll L^A y^{-c/\log L} \ll L^{-A} \tag{12.3}$$

because of the presence of the factor y^{s_1} in (12.1).

To evaluate the expression I_{02} in (12.2), we use the residue theorem to write $\mathrm{Res}_{s_1=0}R_{02}$ as an integral along a circle. The result is

$$I_{02} = \frac{1}{2\pi i}\int_{E_2}\oint_{\mathcal{C}_1} R_{02}\ ds_1\,ds_2, \tag{12.4}$$

where $\mathcal{C}_1$ is the circle $|s_1| = c_1/L$, taken once in the positive direction. Define the path Γ_2 by the equation

$$\Gamma_2(t_2) = \frac{c}{\log(|t_2|+3)} + it_2, \qquad -2L^{17} \le t_2 \le 2L^{17}.$$

This path lies to the right of E_2 when L is large. If $s_1 \in \mathcal{C}_1$ and $s_3 \in \mathcal{C}_3$, then the integrand in (12.1) is holomorphic as a function of s_2 between E_2 and Γ_2, by (7.2). Therefore we deduce from (12.4) and moving E_2 to Γ_2 that

$$I_{02} = \frac{1}{2\pi i}\int_{\Gamma_2}\oint_{\mathcal{C}_1} R_{02}\ ds_1\,ds_2 + O(L^{-A}). \tag{12.5}$$

We need to estimate the integral on the right-hand side. To do this, we bound the factors in (12.1) as follows. Let $s_2 \in \Gamma_2$, $s_1 \in \mathcal{C}_1$, and $s_3 \in \mathcal{C}_3$, so that $|s_1| = c_1/L$

and $|s_3| = c_1/2L$. Also, recall the assumption (7.2). First, we see by the Laurent series of $\zeta(s)$ about $s = 1$ that

$$\zeta(1 + s_1 + s_3) \ll \frac{1}{|s_1 + s_3|} \ll L.$$

Second, we see from Lemma 3.2 that

$$\zeta(1 + s_1 - s_2) \ll \max\left\{\frac{1}{\sigma_2 - \sigma_1}, \log(|t_1| + |t_2| + 2)\right\} \ll \log(|t_2| + 3),$$

and similarly $\zeta(1 + s_2 + s_3) \ll \log(|t_2| + 3)$. Third, we can prove the inequality

$$\begin{aligned}
&\frac{1}{s_1^2 s_2^4 s_3^2}\zeta^{-1}(1+s_1+\gamma)\zeta^{-1}(1+s_1+\delta)\zeta^{-1}(1+s_2+\gamma)\zeta^{-1}(1+s_2+\delta)\\
&\quad\times \zeta^{-1}(1+s_3+\alpha)\zeta^{-1}(1+s_3+\beta)\zeta^{-1}(1-s_2+\alpha)\zeta^{-1}(1-s_2+\beta)\\
&\ll \min\left\{1, \frac{\log(|t_2|+3)}{|t_2|}\right\}^4
\end{aligned}$$

in the same way we showed (9.4). Fourth, it is true that

$$\frac{1}{s_1 s_2^2 s_3} \ll \frac{1}{|s_1|(\sigma_2)^2|s_3|} \asymp L^2 \log^2(|t_2| + 3).$$

Fifth, the estimate $y^{s_1+s_3} \ll 1$ follows from the fact that $s_1, s_3 \ll 1/L$. And finally, sixth, it follows from (7.6) that $F(s_1, s_2, s_3, -s_2) \ll 1$.

Using all these bounds, we deduce from (12.1) that

$$\frac{1}{2\pi i}\int_{\Gamma_2}\oint_{\mathcal{C}_1} R_{02}\ ds_1\, ds_2 \ll L. \tag{12.6}$$

From this, (12.5), (12.3), and (12.2), it follows that

$$\frac{1}{2\pi i}\int_{E_1}\int_{E_2} R_{02}\ ds_2\, ds_1 \ll L. \tag{12.7}$$

A similar argument gives

$$\frac{1}{2\pi i}\int_{E_1}\int_{E_2} R_{20}\ ds_2\, ds_1 \ll L. \tag{12.8}$$

13. Evaluation of $\iint R_{01}$ and $\iint R_{10}$

We use (9.1), (7.8), and the residue theorem to write

$$\begin{aligned}
R_{01} &= -\frac{1}{2\pi i}\oint_{\mathcal{C}_3} \frac{y^{s_2+s_3}}{s_1^6 s_2^3 s_3^3}\times\\
&\quad\times \zeta^{-1}(1+s_1+\gamma)\zeta^{-1}(1+s_1+\delta)\zeta^{-1}(1+s_2+\gamma)\zeta^{-1}(1+s_2+\delta)\\
&\quad\times \zeta^{-1}(1+s_3+\alpha)\zeta^{-1}(1+s_3+\beta)\zeta^{-1}(1-s_1+\alpha)\zeta^{-1}(1-s_1+\beta)\\
&\quad\times \zeta(1+s_1+s_3)\zeta(1+s_2+s_3)\zeta(1+s_2-s_1)F(s_1,s_2,s_3,-s_1)\, ds_3
\end{aligned} \tag{13.1}$$

for $s_1 \in E_1$ and $s_2 \in E_2$, where $\mathcal{C}_3$ is the circle $|s_3| = c_1/2L$, taken once in the positive direction. Let D_2 be the line segment from $-c(\log L)^{-1} - i2L^{17}$ to

$-c(\log L)^{-1} + i2L^{17}$. Move E_2 to D_2 to see that

$$\begin{aligned}\frac{1}{(2\pi i)^2}\int_{E_1}\int_{E_2} R_{01}\ ds_2\,ds_1 \ &= \ \frac{1}{(2\pi i)^2}\int_{E_1}\int_{D_2} R_{01}\ ds_2\,ds_1 \\ &+ \frac{1}{2\pi i}\int_{E_1}\operatorname*{Res}_{s_2=0} R_{01}\ ds_1 \ + \ \frac{1}{2\pi i}\int_{E_1}\operatorname*{Res}_{s_2=s_1} R_{01}\ ds_1 \ + \ O(L^{-A}) \\ &= \ J_{01} \ + \ I_{010} \ + \ I_{011} \ + \ O(L^{-A}),\end{aligned} \tag{13.2}$$

say. By the presence of the factor y^{s_2} in (13.1), it follows that

$$J_{01} \ \ll \ L^A y^{-c/\log L} \ \ll \ L^{-A} \tag{13.3}$$

in the same way that (10.3) holds.

To bound the integral I_{010} in (13.2), apply the residue theorem to write it as

$$I_{010} \ = \ \frac{1}{(2\pi i)^2}\int_{E_1}\oint_{\mathcal{C}_2} R_{01}\,ds_2\,ds_1, \tag{13.4}$$

where $\mathcal{C}_2$ is the circle $|s_2| = 3c_1/4L$, taken once in the positive direction. Let Γ_1 be the path defined by (11.6). Since there are no poles of $\operatorname{Res}_{s_2=0}R_{01}$ between E_1 and Γ_1, we deduce from (13.4) and moving E_1 to Γ_1 that

$$I_{010} \ = \ \frac{1}{(2\pi i)^2}\int_{\Gamma_1}\oint_{\mathcal{C}_2} R_{01}\,ds_2\,ds_1 \ + \ O(L^{-A}). \tag{13.5}$$

We can use (13.1) and arguments similar to those leading up to (12.6) to show that

$$\frac{1}{(2\pi i)^2}\int_{\Gamma_1}\oint_{\mathcal{C}_2} R_{01}\,ds_2\,ds_1 \ \ll \ L.$$

From this and (13.5), we arrive at

$$I_{010} \ \ll \ L. \tag{13.6}$$

The next step is to estimate the integral I_{011} in (13.2). We use (9.1) with (13.1) and pass the limit under the integral sign to write

$$\begin{aligned}\operatorname*{Res}_{s_2=s_1} R_{01} \ &= \ -\frac{1}{2\pi i}\oint_{\mathcal{C}_3}\frac{y^{s_1+s_3}}{s_1^9 s_3^3}\times \\ &\times \zeta^{-2}(1+s_1+\gamma)\zeta^{-2}(1+s_1+\delta)\zeta^{-1}(1+s_3+\alpha)\zeta^{-1}(1+s_3+\beta) \\ &\times \zeta^{-1}(1-s_1+\alpha)\zeta^{-1}(1-s_1+\beta)\ \zeta^2(1+s_1+s_3)F(s_1,s_1,s_3,-s_1)\,ds_3.\end{aligned} \tag{13.7}$$

Let D_1 be the line segment from $-c(\log L)^{-1} - iL^{17}$ to $-c(\log L)^{-1} + iL^{17}$. Move E_1 to D_1 to deduce that

$$I_{011} \ = \ \operatorname*{Res}_{s_1=0}\operatorname*{Res}_{s_2=s_1} R_{01} \ + \ \frac{1}{2\pi i}\int_{D_1}\operatorname*{Res}_{s_2=s_1} R_{01}\ ds_1 \ + \ O(L^{-A}).$$

The integral on the right-hand side is $\ll L^{-A}$ as in (10.3), since the factor y^{s_1} is present in (13.7). Therefore

$$I_{011} \ = \ \operatorname*{Res}_{s_1=0}\operatorname*{Res}_{s_2=s_1} R_{01} \ + \ O(L^{-A}).$$

It now follows from this, (13.6), (13.3), and (13.2) that

$$\frac{1}{(2\pi i)^2}\int_{E_1}\int_{E_2} R_{01}\ ds_2\,ds_1 \ = \ \operatorname*{Res}_{s_1=0}\operatorname*{Res}_{s_2=s_1} R_{01} \ + \ O(L). \tag{13.8}$$

By a similar argument, we can show that

$$\frac{1}{(2\pi i)^2}\int_{E_1}\int_{E_2} R_{10}\ ds_2\,ds_1 \ = \ \operatorname*{Res}_{s_1=0}\operatorname*{Res}_{s_2=s_1} R_{10} \ + \ O(L). \tag{13.9}$$

14. Evaluation of $\iint R_{00}$

By the definition (7.8) of $\mathcal{F}$ and the residue theorem, we can write

$$R_{00} \ = \ \frac{1}{(2\pi i)^2}\oint_{\mathcal{C}_3}\oint_{\mathcal{C}_4} \mathcal{F}\ ds_4\,ds_3 \tag{14.1}$$

for $s_1 \in E_1$ and $s_2 \in E_2$, where $\mathcal{C}_3$ is the circle $|s_3| = c_1/2L$ and $\mathcal{C}_4$ is the circle $|s_4| = c_1/2L$, both taken once along the positive direction. Let D_2 be the line segment from $-c(\log L)^{-1} - i2L^{17}$ to $-c(\log L)^{-1} + i2L^{17}$. Move E_2 to D_2 to see that

$$\begin{aligned} &\frac{1}{(2\pi i)^2}\int_{E_1}\int_{E_2} R_{00}\ ds_2\,ds_1 \\ &\quad = \ \frac{1}{(2\pi i)^2}\int_{E_1}\int_{D_2} R_{00}\ ds_2\,ds_1 \ + \ \frac{1}{2\pi i}\int_{E_1}\operatorname*{Res}_{s_2=0} R_{00}\ ds_1 \ + \ O(L^{-A}) \\ &\quad = \ J_{00} \ + \ I_{00} \ + \ O(L^{-A}), \end{aligned} \tag{14.2}$$

say. Because of the presence of the factor y^{s_2} of $\mathcal{F}$ in (14.1), we can show that

$$J_{00} \ \ll \ L^A y^{-c/\log L} \ \ll \ L^{-A}. \tag{14.3}$$

To evaluate the integral I_{00} in (14.2), let D_1 be the line segment from $-c(\log L)^{-1} - iL^{17}$ to $-c(\log L)^{-1} + iL^{17}$. We move E_1 to D_1 and write

$$I_{00} \ = \ \frac{1}{2\pi i}\int_{D_1}\operatorname*{Res}_{s_2=0} R_{00}\ ds_2\,ds_1 \ + \ \operatorname*{Res}_{s_1=0}\operatorname*{Res}_{s_2=0} R_{00} \ \ + \ O(L^{-A}).$$

The integral on the right-hand side is $\ll L^{-A}$ since $\mathcal{F}$ has the factor y^{s_1} in (14.1). Hence

$$I_{00} \ = \ \operatorname*{Res}_{s_1=0}\operatorname*{Res}_{s_2=0} R_{00} \ \ + \ O(L^{-A}).$$

From this, (14.3), and (14.2), it follows that

$$\frac{1}{(2\pi i)^2}\int_{E_1}\int_{E_2} R_{00}\ ds_2\,ds_1 \ = \ \operatorname*{Res}_{s_1=0}\operatorname*{Res}_{s_2=0} R_{00} \ \ + \ O(L^{-A}). \tag{14.4}$$

15. Putting together the estimates

We now collect together the nine estimates (9.5), (9.6), (10.4), (11.8), (12.7), (12.8), (13.8), (13.9), and (14.4), and then insert them into (8.3) to deduce that

$$Q(\alpha,\beta,\gamma,\delta) = \operatorname*{Res}_{s_1=0}\operatorname*{Res}_{s_2=0} R_{00} \ + \operatorname*{Res}_{s_1=0}\operatorname*{Res}_{s_2=s_1} R_{01} \ + \operatorname*{Res}_{s_1=0}\operatorname*{Res}_{s_2=s_1} R_{10} \ + O(L^2). \tag{15.1}$$

In this section, we simplify this further and prove an asymptotic formula for $Q(\alpha,\beta,\gamma,\delta)$ that has a polynomial main term. We continue to assume that $\alpha,\beta,\gamma,\delta \ll 1/L$.

We first write the main terms in (15.1) as path integrals. By (14.1) and the residue theorem, we can write

$$\operatorname*{Res}_{s_1=0}\operatorname*{Res}_{s_2=0} R_{00} \ = \ \frac{1}{(2\pi i)^4}\oint_{\mathcal{C}_1}\oint_{\mathcal{C}_2}\oint_{\mathcal{C}_3}\oint_{\mathcal{C}_4} \mathcal{F}\ ds_4\,ds_3\,ds_2\,ds_1, \tag{15.2}$$

where $\mathcal{C}_1$ is the circle $|s_1| = c_1/L$ and $\mathcal{C}_2$ is the circle $|s_2| = c_1/L$, both taken once in the positive direction. Similarly, from the residue theorem and (13.7), it follows that

(15.3)
$$\begin{aligned}\operatorname*{Res}_{s_1=0}\operatorname*{Res}_{s_2=s_1} R_{01} \ &= \ -\frac{1}{(2\pi i)^2}\oint_{\mathcal{C}_1}\oint_{\mathcal{C}_3}\frac{y^{s_1+s_3}}{s_1^9 s_3^3}\times \\ &\times \zeta^{-2}(1+s_1+\gamma)\zeta^{-2}(1+s_1+\delta)\zeta^{-1}(1+s_3+\alpha)\zeta^{-1}(1+s_3+\beta) \\ &\times \zeta^{-1}(1-s_1+\alpha)\zeta^{-1}(1-s_1+\beta)\ \zeta^2(1+s_1+s_3)F(s_1,s_1,s_3,-s_1)\,ds_3\,ds_1.\end{aligned}$$

Similarly, or by symmetry, the same equation holds with R_{10} in place of R_{01}.

We put these equations into simpler forms by using Laurent series expansions, as follows. By repeated applications of (5.24), we see that

$$F(s_1,s_2,s_3,s_4;\alpha,\beta,\gamma,\delta) \ = \ F(0,0,0,0;0,0,0,0) \ + \ O(L^{-1})$$

when each variable is $\ll 1/L$. We can show that $F(0,\dots,0) = \zeta(2)$ by a straightforward computation using the definition (7.5) of F. Thus

$$F(s_1,s_2,s_3,s_4;\alpha,\beta,\gamma,\delta) \ = \ \zeta(2) \ + \ O(L^{-1})$$

when each variable is $\ll 1/L$. We use this, the definition (7.8) of $\mathcal{F}$, and the Laurent series of $\zeta(s)$ and $\zeta^{-1}(s)$ near $s=1$ to simplify (15.2) to

(15.4)
$$\begin{aligned}\operatorname*{Res}_{s_1=0}\operatorname*{Res}_{s_2=0} R_{00} \ &= \ \frac{\zeta(2)}{(2\pi i)^4}\oint_{\mathcal{C}_1}\oint_{\mathcal{C}_2}\oint_{\mathcal{C}_3}\oint_{\mathcal{C}_4}\frac{y^{s_1+s_2+s_3+s_4}}{s_1^3 s_2^3 s_3^3 s_4^3}\times \\ &\times (s_1+\gamma)(s_1+\delta)(s_2+\gamma)(s_2+\delta)(s_3+\alpha)(s_3+\beta)(s_4+\alpha)(s_4+\beta) \\ &\times (s_1+s_3)^{-1}(s_1+s_4)^{-1}(s_2+s_3)^{-1}(s_2+s_4)^{-1}\,ds_4\,ds_3\,ds_2\,ds_1 \ + \ O(L^3).\end{aligned}$$

Similarly, (15.3) simplifies to

(15.5)
$$\begin{aligned}\operatorname*{Res}_{s_1=0}\operatorname*{Res}_{s_2=s_1} R_{01} \ &= \ -\frac{\zeta(2)}{(2\pi i)^2}\oint_{\mathcal{C}_1}\oint_{\mathcal{C}_3}\frac{y^{s_1+s_3}}{s_1^9 s_3^3}(s_1+\gamma)^2(s_1+\delta)^2(s_3+\alpha)(s_3+\beta) \\ &\qquad \times(-s_1+\alpha)(-s_1+\beta)(s_1+s_3)^{-2}\,ds_3\,ds_1 \ + \ O(L^3).\end{aligned}$$

The same formula holds with R_{10} in place of R_{01}.

We may express the factor $y^{s_1+s_2+s_3+s_4}$ in (15.4) as an infinite sum using the series expansion of $\exp(z)$. The same is true for $y^{s_1+s_3}$ in (15.5). Also, we may write each factor $(s_j+s_k)^{-1}$ in (15.4) and (15.5) as the infinite sum

$$\frac{1}{s_j(1+s_k/s_j)} \ = \ \frac{1}{s_j} \ - \ \frac{s_k}{s_j^2} \ + \ \frac{s_k^2}{s_j^3} \ + \cdots .$$

By Cauchy's theorem, all but finitely many terms from these sums contribute zero to the values of the integrals. With these observations, we now deduce from (15.1), (15.4), and (15.5) that

$$Q(\alpha,\beta,\gamma,\delta) \ = \ \zeta(2)P(\alpha,\beta,\gamma,\delta) \ + \ O(L^3), \qquad (15.6)$$

where $P(\alpha,\beta,\gamma,\delta)$ is defined by

$$\begin{aligned} P(\alpha,\beta,\gamma,\delta) &= \frac{1}{(2\pi i)^4}\oint_{\mathcal{C}_1}\oint_{\mathcal{C}_2}\oint_{\mathcal{C}_3}\oint_{\mathcal{C}_4}\left\{\sum_{n=0}^{12}\frac{(\log y)^n(s_1+s_2+s_3+s_4)^n}{n!\ s_1^5s_2^5s_3^3s_4^3}\right\} \\ &\quad\times(s_1+\gamma)(s_1+\delta)(s_2+\gamma)(s_2+\delta)(s_3+\alpha)(s_3+\beta)(s_4+\alpha)(s_4+\beta) \\ &\quad\times\left(1-\frac{s_3}{s_1}-\frac{s_3}{s_2}+\frac{s_3^2}{s_1^2}+\frac{s_3^2}{s_1s_2}+\frac{s_3^2}{s_2^2}\right)\left(1-\frac{s_4}{s_1}-\frac{s_4}{s_2}+\frac{s_4^2}{s_1^2}+\frac{s_4^2}{s_1s_2}+\frac{s_4^2}{s_2^2}\right) \\ &\quad\times ds_4\,ds_3\,ds_2\,ds_1 \\ &\quad-\frac{2}{(2\pi i)^2}\oint_{\mathcal{C}_1}\oint_{\mathcal{C}_3}\left\{\sum_{n=0}^{12}\frac{(\log y)^n(s_1+s_3)^n}{n!\ s_1^{11}s_3^3}\right\} \\ &\quad\times(s_1+\gamma)^2(s_1+\delta)^2(s_3+\alpha)(s_3+\beta)(-s_1+\alpha)(-s_1+\beta) \\ &\quad\times\left(1-2\frac{s_3}{s_1}+3\frac{s_3^2}{s_1^2}\right)ds_3\,ds_1. \end{aligned} \tag{15.7}$$

Note that it immediately follows from the residue theorem that $P(\alpha,\beta,\gamma,\delta)$ is a polynomial in $\alpha,\beta,\gamma,\delta$, and $\log y$.

16. Proof of Theorem 1.3

Having evaluated $Q(\alpha,\beta,\gamma,\delta)$, we can now complete the proof of Theorem 1.3. We first simplify (6.10). Observe that $|s_j|\asymp 1/L$ for each variable of integration s_j in (15.4) and (15.5). Therefore, it follows from (15.1), (15.4), and (15.5) that

$$Q(\alpha,\beta,\gamma,\delta)\ \ll\ L^4$$

for $\alpha,\beta,\gamma,\delta\ll 1/L$. Hence, if $|z_j|=1/L$ for $j=1,2,3,4$, then

$$Q(z_1+i\rho,z_2+i\rho,-z_3-i\rho,-z_4-i\rho)\ \ll\ L^4 \tag{16.1}$$

because $\rho\asymp 1/L$ by (6.5). Moreover, we see from the definition (6.1) and the Laurent series of $\zeta(s)$ near the points $s=1$ and $s=2$ that

$$A_{z_1,z_2,-z_3,-z_4}(0)\prod_{1\le j<\ell\le 4}(z_j-z_\ell)^2\ =\ \frac{1}{\zeta(2)}(z_1-z_2)(z_3-z_4)\prod_{1\le j<\ell\le 4}(z_j-z_\ell)+O\left(\frac{1}{L^9}\right)$$

when $z_j\ll 1/L$ for $j=1,2,3,4$. We insert this into (6.10). We bound the contribution of the error term $O(L^{-9})$ by applying (16.1) and the fact that $|z_j|=1/L$ and $t^{\pm z_j}\ll 1$ for each j. The result is

$$\begin{aligned} \mathcal{J}\ =\ &\int_{-\infty}^{\infty}\frac{w(t)}{(2\pi i)^4}\oint\oint\oint\oint\left(\frac{t}{2\pi}\right)^{\frac{z_1+z_2-z_3-z_4}{2}}(z_1-z_2)(z_3-z_4)\prod_{1\le j<\ell\le 4}(z_j-z_\ell) \\ &\times\left(\frac{4}{\zeta(2)(\log y)^8}\right)\ Q(z_1+i\rho,z_2+i\rho,-z_3-i\rho,-z_4-i\rho)\ \prod_{j=1}^{4}\frac{dz_j}{z_j^4}\ dt \\ &+\ O\left(\frac{W}{L}\right)\ +\ O(y^{11/2}T^{3/4+\varepsilon}(T/T_0)^{9/4}), \end{aligned}$$

where W is defined by (1.5).

We now insert into this formula the expression (15.6) with the appropriate values for $\alpha,\beta,\gamma,\delta$. We can easily bound the contribution of the error term $O(L^3)$ in (15.6) using the fact that $|z_j|=1/L$ and $t^{\pm z_j}\ll 1$ for each j. Furthermore, we can write the factor $(t/2\pi)^{(z_1+z_2-z_3-z_4)/2}$ as an infinite sum via the series expansion of $\exp(z)$. Only the first few terms from this infinite sum actually contribute to the value of the integral, because of Cauchy's theorem and the fact that P is a polynomial. Hence the result we arrive at is

$$\begin{aligned}\mathcal{J} \;=\; \int_{-\infty}^{\infty} \frac{w(t)}{(2\pi i)^4} \oint\oint\oint\oint \left\{ \sum_{m=0}^{4} \frac{\left(\log \frac{t}{2\pi}\right)^m}{m!\, 2^m} (z_1+z_2-z_3-z_4)^m \right\} \\ \times (z_1-z_2)(z_3-z_4) \prod_{1\le j<\ell\le 4} (z_j - z_\ell) \\ \times \left(\frac{4}{(\log y)^8} \right) P(z_1+i\rho, z_2+i\rho, -z_3-i\rho, -z_4-i\rho) \prod_{j=1}^{4} \frac{dz_j}{z_j^4}\, dt \\ +\; O\left(\frac{W}{L}\right) \;+\; O(y^{11/2}T^{3/4+\varepsilon}(T/T_0)^{9/4}).\end{aligned}$$

We next simplify this equation by evaluating the path integrals. Write the equation as

$$\mathcal{J} = \frac{4}{(\log y)^8} \int_{-\infty}^{\infty} w(t)\mathcal{R}(t;y,\rho)\,dt + O\left(\frac{W}{L}\right) + O(y^{11/2}T^{3/4+\varepsilon}(T/T_0)^{9/4}). \tag{16.2}$$

By the residue theorem and the definition (15.7) of P, it follows that $\mathcal{R}(t;y,\rho)$ equals the sum of the coefficients of certain monomials of the type

$$s_1^h s_2^k s_3^q s_4^r z_1^3 z_2^3 z_3^3 z_4^3, \qquad h+k+q+r=12, \tag{16.3}$$

in the polynomial $\mathcal{R}_0\mathcal{R}_1 - 2\mathcal{R}_0\mathcal{R}_2$, where

$$\mathcal{R}_0 = \left\{ \sum_{m=0}^{4} \frac{\left(\log \frac{t}{2\pi}\right)^m}{m!\, 2^m} (z_1+z_2-z_3-z_4)^m \right\} (z_1-z_2)(z_3-z_4) \prod_{1\le j<\ell\le 4} (z_j-z_\ell),$$

$$\begin{aligned}\mathcal{R}_1 \;=\; \left\{ \sum_{n=0}^{12} \frac{1}{n!} (\log y)^n (s_1+s_2+s_3+s_4)^n \right\} (s_1-z_3-i\rho)(s_1-z_4-i\rho) \\ \times (s_2-z_3-i\rho)(s_2-z_4-i\rho)(s_3+z_1+i\rho)(s_3+z_2+i\rho)(s_4+z_1+i\rho)(s_4+z_2+i\rho),\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}_2 \;=\; \left\{ \sum_{n=0}^{12} \frac{1}{n!} (\log y)^n (s_1+s_3)^n \right\} (s_1-z_3-i\rho)^2(s_1-z_4-i\rho)^2 \\ \times (s_3+z_1+i\rho)(s_3+z_2+i\rho)(-s_1+z_1+i\rho)(-s_1+z_2+i\rho).\end{aligned}$$

Observe that the exponents of the z_j in (16.3) add up to 12, and the same can be said of the s_k. Therefore $\mathcal{R}(t;y,\rho)$ is the sum of terms of the form

$$\Lambda_{m,n} \left(\log \frac{t}{2\pi} \right)^m (\log y)^n (i\rho)^{m+n-8},$$

where $0 \le m \le 4$, $0 \le n \le 12$, and each $\Lambda_{m,n}$ is an absolute constant that is computable. We write $\mathcal{R}(t;y,\rho)$ as this sum to express (16.2) as

$$\begin{aligned}\mathcal{J} \;=\; 4\sum_{m,n} \Lambda_{m,n} (\log y)^{n-8} (i\rho)^{m+n-8} \int_{-\infty}^{\infty} w(t) \left(\log \frac{t}{2\pi} \right)^m dt \\ +\; O\left(\frac{W}{L}\right) \;+\; O(y^{11/2}T^{3/4+\varepsilon}(T/T_0)^{9/4}),\end{aligned} \tag{16.4}$$

where the sum runs through all pairs m, n of integers satisfying $0 \le m \le 4$ and $0 \le n \le 12$.

We can further simplify (16.4) by using the properties of $w(t)$ and the definitions of y and ρ. Indeed, recall that $L = \log T$ and W is defined by (1.5). Since $w(t)$ is supported on $[T/2, 4T]$, it follows that

$$\int_{-\infty}^{\infty} w(t) \left(\log \frac{t}{2\pi}\right)^m dt \;=\; WL^m \left(1 \;+\; O\left(\frac{1}{L}\right)\right).$$

Also, the definition (6.5) of ρ implies that

$$\rho^{m+n-8} \;=\; \left(\frac{2a}{\log \frac{T}{2\pi}}\right)^{m+n-8} \;=\; \left(\frac{2a}{L}\right)^{m+n-8} \left(1 \;+\; O\left(\frac{1}{L}\right)\right).$$

Moreover, it follows from the definition $y = T^\theta$ that $\log y = \theta L$. We insert these into (16.4) and arrive at

$$\mathcal{J} \;=\; \sum_{m,n} \lambda_{m,n}\; \theta^{n-8} a^{m+n-8}\; W \;+\; O\left(\frac{W}{L}\right) \;+\; O(y^{11/2} T^{3/4+\varepsilon} (T/T_0)^{9/4}),$$

where $\lambda_{m,n} = 4\Lambda_{m,n}(2i)^{m+n-8}$ and the sum runs through all pairs m, n of integers with $0 \le m \le 4$ and $0 \le n \le 12$. This completes the proof of Theorem 1.3, with

$$C(a, \theta) \;=\; \sum_{m,n} \lambda_{m,n}\; \theta^{n-8} a^{m+n-8}. \tag{16.5}$$

17. Proofs of the corollaries

In this section, we choose a specific $w(t)$ to prove Corollaries 1.1 and 1.2. There may be many ways of doing this, and we do not claim that the method here is the best in any sense.

Let $h(x)$ be the smooth function

$$h(x) \;=\; \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

For each positive $\eta > 0$ and each interval $[u, \tau]$, define the function $\psi(x) = \psi(x; u, \tau, \eta)$ by

$$\psi(x) \;=\; \left(\frac{h(\tau - x + \eta)}{h(\tau - x + \eta) + h(x - \tau)}\right)\left(\frac{h(x - u + \eta)}{h(x - u + \eta) + h(u - x)}\right). \tag{17.1}$$

The properties $\psi(x)$ are summarized in the following proposition.

Proposition 17.1. *The function $\psi(x)$ is smooth and nonnegative on $(-\infty, \infty)$. It takes the value 1 on $[u, \tau]$ and is supported on $[u - \eta, \tau + \eta]$. Moreover,*

$$\psi^{(j)}(x) \;\ll_j\; \exp\left(\frac{6j}{\eta}\right) \tag{17.2}$$

for all $j = 0, 1, 2, \ldots$.

Proof. Since $h(x)$ is smooth and nonnegative, the first sentence of the proposition follows from the fact that the denominator in the definition (17.1) of $\psi(x)$ is never zero. The second sentence follows from the fact that $h(x) = 0$ for $x \le 0$ and $h(x) > 0$ for $x > 0$. Thus it is left to prove (17.2). Observe that

$$h^{(j)}(x) \ \ll_j \ 1 \tag{17.3}$$

for all $j \ge 0$. Since $h(x)$ is nonnegative, we can show that

$$h(\tau - x + \eta) + h(x - \tau) \ \ge \ \exp\left(-\frac{2}{\eta}\right) \tag{17.4}$$

for all real x by considering the two cases $x \ge \tau + \frac{\eta}{2}$ and $x \le \tau + \frac{\eta}{2}$ separately. Now apply Lemma 3.3 with $f(t) = 1/t$ and $g(t) = h(\tau - t + \eta) + h(t - \tau)$ to deduce from (17.3) and (17.4) that

$$\frac{d^j}{dt^j}\left(\frac{1}{h(\tau - t + \eta) + h(t - \tau)}\right) \ \ll_j \ \exp\left(\frac{2(j+1)}{\eta}\right)$$

for all $j \ge 0$. Similarly, we can prove that

$$\frac{d^j}{dt^j}\left(\frac{1}{h(t - u + \eta) + h(u - t)}\right) \ \ll_j \ \exp\left(\frac{2(j+1)}{\eta}\right)$$

for $j \ge 0$. With these bounds and (17.3), we can prove (17.2) for $j \ge 1$ by a straightforward computation using the product rule. The case $j = 0$ of (17.2) trivially follows from the fact that $\psi(x) \le 1$. □

We now start our proof of Corollary 1.1. Let $b \le 1$ be a real number with $\theta < 2b - \frac{3}{2}$. For large T, we take $\eta = L^{-1/2}$, $u = T/T^b$, and $\tau = (T + T^b)/T^b$ in the definition (17.1) of $\psi(x)$ and define

$$w(t) \ = \ \psi\left(\frac{t}{T^b}\right). \tag{17.5}$$

It immediately follows from Proposition 17.1 that $w(t)$ is smooth and nonnegative, and that

$$w(t) = 1 \quad \text{for} \ \ t \in [T, T + T^b] \tag{17.6}$$

and

$$w(t) = 0 \quad \text{for } t \notin \left[T - \frac{T^b}{\sqrt{\log T}}\ ,\ T + T^b + \frac{T^b}{\sqrt{\log T}}\right]. \tag{17.7}$$

Hence, since $b \le 1$, it follows that $w(t)$ is supported on $[T/2, 4T]$ for large T. Differentiating (17.5) j times and applying (17.2), we see that

$$w^{(j)}(t) \ \ll_j \ T_0^{-j} \tag{17.8}$$

for all $j \ge 0$, where

$$T_0 \ = \ T^b e^{-6\sqrt{\log T}}. \tag{17.9}$$

Note that $T^{1/2+\varepsilon} \ll T_0 \ll T$ since the inequalities $b \le 1$ and $0 < \theta < 2b - \frac{3}{2}$ imply that $\frac{3}{4} < b \le 1$. Hence $w(t)$ satisfies the conditions in the hypothesis of Theorem 1.2. Thus we can apply Theorem 1.2 with $w(t)$ defined by (17.5).

We need to simplify the right-hand side of (1.6) for our choice of $w(t)$. It follows from (17.6), (17.7), and the definition (1.5) of W that

$$W \ = \ (1 \ + \ o(1))T^b \tag{17.10}$$

as $T \to \infty$. This simplifies the main terms and shows that the first error term on the right-hand side of (1.6) is $o(T^b)$ for our specific $w(t)$. To simplify the second error term, observe that it follows from (17.9) that

$$T_0 \gg_\varepsilon T^{b-\varepsilon} \tag{17.11}$$

for arbitrarily small $\varepsilon > 0$. Thus the second error term in (1.6) is bounded by

$$\ll \frac{T^{\theta\ell+\theta}}{T^{(b-\varepsilon)\ell}} T^{\frac{3}{2}+\frac{\ell}{2}+\varepsilon} = T^{\ell\left(\theta-b+\varepsilon+\frac{1}{2}\right)+\theta+\frac{3}{2}+\varepsilon}.$$

Since $\theta < 2b - \frac{3}{2}$ and $b \le 1$, it follows that $\theta - b + \frac{1}{2} < 0$. Thus, if $A > 0$ is any constant, then we can choose ε small enough and ℓ large enough such that

$$\ell\left(\theta - b + \varepsilon + \frac{1}{2}\right) + \theta + \frac{3}{2} + \varepsilon < -A.$$

Hence the second error term in (1.6) is $O(T^{-A})$ for a suitably chosen ℓ. Next, to simplify the third error term, observe that our assumption $\theta < 2b - \frac{3}{2}$ implies that $\frac{\theta}{2} + \frac{3}{4} < b$ and so

$$y^{1/2} T^{3/4} \log^3 T = T^{\frac{\theta}{2}+\frac{3}{4}} \log^3 T = o(T^b)$$

as $T \to \infty$. This ends the proof of Corollary 1.1.

Now we prove Corollary (1.2). Let $b \le 1$ be a real number with $\theta < \frac{13}{22}b - \frac{6}{11}$. We continue to use the function $w(t)$ defined by (17.5). As we have proved above, $w(t)$ satisfies the conditions in the hypothesis of Theorem 1.3, and so we can apply Theorem 1.3. We can simplify the main term and the first error term in (1.8) using (17.10). To bound the second error term, observe that our assumption $\theta < \frac{13}{22}b - \frac{6}{11}$ implies

$$\frac{11}{2}\theta + 3 - \frac{9}{4}b < b.$$

It follows from this and (17.11) that the second error term in (1.8) is

$$\ll y^{11/2} T^{3/4+\varepsilon} (T/T_0)^{9/4} \ll T^{\frac{11}{2}\theta+3-\frac{9}{4}b+\varepsilon} = o(T^b) \tag{17.12}$$

for small enough ε. This proves Corollary 1.2.

18. The fourth moment of $Z(t')M(t)$

We are almost ready to complete the proof of Theorem 1.1. Before we can do so, we need to prove an asymptotic formula for each of the integrals on the right-hand side of (2.1). We do this for one of them in this section, and prove the following theorem.

Theorem 18.1. *Let $0 < \theta < \frac{1}{22}$ and $a \in \mathbb{R}$ be fixed. Suppose that $w(t)$ is defined by (17.5). If $\frac{12}{13} + \frac{22}{13}\theta < b < 1 - 2\theta$, then*

$$\int_{-\infty}^{\infty} |Z(t')|^4 |M(t)|^4 w(t)\, dt = C(a,\theta) T^b + o(T^b) \tag{18.1}$$

as $T \to \infty$, where $C(a,\theta)$ is the same as that in Theorem 1.3.

Proof. For brevity, let $\mathcal{J}_0$ denote the integral on the left-hand side of (18.1). Recall that $|Z(t)| = |\zeta(\frac12 + it)|$. We use the definition (1.4) of $M(t)$ to write

$$\mathcal{J}_0 = \sum_{n_1,n_2,n_3,n_4\le y} \left\{ \prod_{j=1}^{4} \frac{\mu(n_j)}{n_j^{1/2}} \left(\frac{\log(y/n_j)}{\log y}\right)^2 \right\} \int_{-\infty}^{\infty} \left(\frac{(n_3n_4)'}{(n_1n_2)'}\right)^{-it} |\zeta(\tfrac12 + it')|^4 w(t)\, dt, \tag{18.2}$$

where $(n_1n_2)' = n_1n_2/(n_1n_2, n_3n_4)$ and $(n_3n_4)' = n_3n_4/(n_1n_2, n_3n_4)$.

We first estimate the inner integral on the right-hand side. Recall that $w(t)$ satisfies the property (17.7). Thus, if $w(t) \neq 0$, then it follows from (3.5) that

$$-t \;=\; -t' \;+\; \frac{2a}{\log\frac{T}{2\pi}} \;+\; O\left(\frac{T^{b-1}}{\log^2 T}\right) \tag{18.3}$$

since $b < 1$. This formula implies that

$$\left(\frac{(n_3n_4)'}{(n_1n_2)'}\right)^{-it} \;=\; \left(\frac{(n_3n_4)'}{(n_1n_2)'}\right)^{-it'+i\rho} \left(1 \;+\; O\left(\frac{T^{b-1}}{\log T}\right)\right)$$

for $n_1, n_2, n_3, n_4 \le y$, where ρ is defined by (6.5). We insert this into (18.2) and apply the definition (1.4) of $M(t)$ to deduce that

$$\mathcal{J}_0 \;=\; \int_{-\infty}^{\infty} |\zeta(\tfrac12 + it')|^4 |M(t' - \rho)|^4 w(t)\, dt \;+\; \mathcal{E}, \tag{18.4}$$

where

$$\mathcal{E} \;\ll\; \frac{y^2 T^{b-1}}{\log T} \int_{-\infty}^{\infty} |\zeta(\tfrac12 + it')|^4 w(t)\, dt. \tag{18.5}$$

We need to bound the integral on the right-hand side of (18.5). To do this, we define $g(t)$ to be the inverse function of $t \mapsto t'$. Recall that t' is defined by (1.3). Hence $g(t)$ is defined by

$$\vartheta(g(t)) - \vartheta(t) \;=\; -a \tag{18.6}$$

for large enough t. Note that $g(t)' = g(t') = t$. Since a can be any fixed real number in the hypothesis of Lemma 3.4, $g(t)$ can take the role of t' in its statement. That is, we may replace a by $-a$ in the statement of Lemma 3.4 to deduce that

$$\frac{d}{dt} g(t) \;=\; 1 \;+\; O\left(\frac{1}{t\log^2 t}\right), \tag{18.7}$$

and

$$\frac{d^n}{dt^n} g(t) \;\ll_n\; \frac{1}{t^n \log t}, \qquad n \ge 2, \tag{18.8}$$

for large enough t.

Using the properties of $g(t)$, we can bound the integral on the right-hand side of (18.5) via Proposition 6.1. Indeed, make a change of variable and use the estimate (18.7) to deduce that

$$\begin{aligned} \int_{-\infty}^{\infty} |\zeta(\tfrac12 + it')|^4 w(t)\, dt \;&=\; \int_{-\infty}^{\infty} |\zeta(\tfrac12 + it)|^4 w(g(t)) \frac{dg}{dt}(t)\, dt \\ &\ll\; \int_{-\infty}^{\infty} |\zeta(\tfrac12 + it)|^4 w(g(t))\, dt. \end{aligned} \tag{18.9}$$

Note that $w(g(t))$ is smooth because both $w(t)$ and $g(t)$ are. Also, it follows from (17.7) and (3.5) that $w(g(t))$ is supported on $[T/2, 4T]$ for large T. To bound the derivatives of $w(g(t))$, we use Lemma 3.3 with $f(t) = w(t)$ and $g(t)$ defined by (18.6). We apply the estimates (17.8), (18.7), and (18.8) to the resulting equation for $(d^j/dt^j)w(g(t))$ to see that it is $\ll_j T_0^{-j}$ for all $j \geq 0$, where T_0 is defined by (17.9). Therefore the function $w(g(t))$ satisfies the conditions in the hypothesis of Theorem 1.3. Hence we can apply Proposition 6.1 with $h = k = 1$ and $\rho = 0$ to arrive at

$$\begin{aligned}&\int_{-\infty}^{\infty} |\zeta(\tfrac{1}{2}+it)|^4 w(g(t))\,dt \\ &= \int_{-\infty}^{\infty} \frac{w(g(t))}{4(2\pi i)^4} \oint\oint\oint\oint K(z_1,z_2,z_3,z_4) \prod_{1\leq j<\ell\leq 4}(z_j-z_\ell)^2 \prod_{j=1}^{4} \frac{dz_j}{z_j^4}\,dt \\ &\qquad + O(T^{3/4+\varepsilon}(T/T_0)^{9/4}),\end{aligned} \tag{18.10}$$

where

$$K(z_1,z_2,z_3,z_4) = \left(\frac{t}{2\pi}\right)^{\frac{z_1+z_2-z_3-z_4}{2}} A_{z_1,z_2,-z_3,-z_4}(0).$$

We choose each path of integration to be a circle with center 0 and radius $1/\log T$.

To estimate the right-hand side of (18.10), first observe that a change of variable and (3.6) give

$$\int_{-\infty}^{\infty} w(g(t))\,dt = \int_{-\infty}^{\infty} w(t)\frac{dt'}{dt}\,dt \ll W,$$

where W is defined by (1.5). Using this, the definition (6.1), and the Laurent series of $\zeta(s)$ near $s = 1$, we can bound the integral on the right-hand side of (18.10) by

$$\ll W \log^4 T.$$

This bound is $\ll T^b \log^4 T$ by (17.10). By (17.9) and our assumption $b > 12/13$, the error term in (18.10) is $o(T^b)$. From these bounds and (18.10), it follows that

$$\int_{-\infty}^{\infty} |\zeta(\tfrac{1}{2}+it)|^4 w(g(t))\,dt \ll T^b \log^4 T.$$

From this, (18.9), and (18.5), we arrive at

$$\mathcal{E} \ll y^2 T^{2b-1} \log^3 T = T^{2\theta+2b-1} \log^3 T.$$

Thus $\mathcal{E} = o(T^b)$ by our assumption $b < 1 - 2\theta$. This and (18.4) together give

$$\mathcal{J}_0 = \int_{-\infty}^{\infty} |\zeta(\tfrac{1}{2}+it')|^4 |M(t'-\rho)|^4 w(t)\,dt + o(T^b) \tag{18.11}$$

as $T \to \infty$.

The next step is to estimate the integral on the right-hand side of (18.11). By a change of variable, we see that

$$\begin{aligned}&\int_{-\infty}^{\infty} |\zeta(\tfrac{1}{2}+it')|^4 |M(t'-\rho)|^4 w(t)\,dt \\ &= \int_{-\infty}^{\infty} |\zeta(\tfrac{1}{2}+it+i\rho)|^4 |M(t)|^4 w(g(t+\rho))\,\frac{dg}{dt}(t+\rho)\,dt.\end{aligned} \tag{18.12}$$

It follows from (17.7) and (3.5) that $w(g(t+\rho))$ is supported on $[T/2,4T]$ for large enough T. Thus, by (18.7), if $w(g(t+\rho))$ is nonzero, then

$$\frac{dg}{dt}(t+\rho) \ = \ 1 \ + \ O\left(\frac{1}{T\log^2 T}\right).$$

Insert this into (18.12) to deduce that

$$\begin{aligned}&\int_{-\infty}^{\infty} |\zeta(\tfrac12+it')|^4|M(t'-\rho)|^4 w(t)\,dt \\ &\quad = \ \left(1 \ + \ O\left(\frac{1}{T\log^2 T}\right)\right)\int_{-\infty}^{\infty}|\zeta(\tfrac12+it+i\rho)|^4|M(t)|^4 w(g(t+\rho))\,dt.\end{aligned} \tag{18.13}$$

By arguments similar to those below (18.9), we can show that $w(g(t+\rho))$ satisfies the conditions in the statement of Theorem 1.3, with T_0 defined by (17.9). Therefore we can apply Theorem 1.3 to write

$$\begin{aligned}\int_{-\infty}^{\infty}|\zeta(\tfrac12+it+i\rho)|^4|M(t)|^4 w(g(t+\rho))\,dt \ &= \ C(a,\theta)W_g \\ &+ \ O\left(\frac{W_g}{\log T}\right) \ + \ O(y^{11/2}T^{3/4+\varepsilon}(T/T_0)^{9/4}),\end{aligned} \tag{18.14}$$

where

$$W_g \ = \ \int_{-\infty}^{\infty} w(g(t+\rho))\,dt.$$

By a change of variable and (3.6), we see that

$$W_g \ = \ \int_{-\infty}^{\infty} w(t)\frac{dt'}{dt}\,dt \ = \ (1+o(1))W,$$

where W is defined by (1.5). It follows from this and (17.10) that

$$W_g \ = \ (1 \ + \ o(1))T^b \tag{18.15}$$

as $T\to\infty$. Moreover, since $b > \frac{12}{13}+\frac{22}{13}\theta$, the bound (17.12) applies to the second error term in (18.14). By this, (18.15), and (18.14), we conclude that

$$\int_{-\infty}^{\infty}|\zeta(\tfrac12+it+i\rho)|^4|M(t)|^4 w(g(t+\rho))\,dt \ = \ C(a,\theta)T^b \ + \ o(T^b)$$

as $T\to\infty$. This together with (18.13) and (18.11) complete the proof of Theorem 18.1. □

19. Proof of Theorem 1.1

We now prove Theorem 1.1. To do this, we choose suitable values for the parameters θ, a, and b. Then we finish the arguments we started in Section 2.

Let $0<\theta<1/48$. This guarantees that we can choose a real number b with $\frac{12}{13}+\frac{22}{13}\theta < b < 1-2\theta$. Also, let $a>0$ such that the right-hand side of (1.6) is negative for large T. We may take $a=\pi$, say.

Our choice of the parameters θ, a, b allows us to apply Corollary 1.1, which we proved using the function $w(t)$ defined by (17.5). Hence, from (1.7) and (2.2), we deduce that

$$-\left(\cos a \ + \ \frac{4\sin a}{3a\theta} \ + \ o(1)\right)T^b \ \le \ \int_{\mathcal{S}}|Z(t)Z(t')||M(t)|^2 w(t)\,dt.$$

We may square both sides because both are positive by our choice of a. Thus

$$\left(\cos a \;+\; \frac{4\sin a}{3a\theta} \;+\; o(1)\right)^2 T^{2b} \;\le\; \left(\int_{\mathcal{S}} |Z(t)Z(t')||M(t)|^2 w(t)\,dt\right)^2. \tag{19.1}$$

The restrictions on θ, a, b also allow us to apply Corollary 1.2 and Theorem 18.1. We use the special case of Corollary 1.2 with $a = 0$ to write

$$\left(\int_{-\infty}^{\infty} |Z(t)|^4 |M(t)|^4 w(t)\,dt\right)^{1/2} \;=\; \left(C(0,\theta) \;+\; o(1)\right)^{1/2} T^{b/2}. \tag{19.2}$$

Also, the conclusion of Theorem 18.1 gives

$$\left(\int_{-\infty}^{\infty} |Z(t')|^4 |M(t)|^4 w(t)\,dt\right)^{1/2} \;=\; \left(C(a,\theta) \;+\; o(1)\right)^{1/2} T^{b/2}. \tag{19.3}$$

We now see from (2.1), (19.1), (19.2), and (19.3) that

$$(\Delta \;+\; o(1))T^b \;\le\; \mathcal{N}, \tag{19.4}$$

where $\mathcal{N} = \int_{-\infty}^{\infty} \mathbf{1}_{\mathcal{S}}(t) w(t)\,dt$ and

$$\Delta \;=\; \frac{\left(\cos a \;+\; \frac{4\sin a}{3a\theta}\right)^2}{C(0,\theta)^{1/2} C(a,\theta)^{1/2}}.$$

The properties (17.6) and (17.7) of $w(t)$ imply that

$$\begin{aligned} \mathcal{N} \;&=\; \int_{-\infty}^{\infty} \mathbf{1}_{\mathcal{S}}(t) w(t)\,dt \;=\; \int_{T}^{T+T^b} \mathbf{1}_{\mathcal{S}}(t)\,dt \;+\; \left(\frac{T^b}{\sqrt{\log T}}\right) \\ &=\; \mathcal{M} \;+\; o(T^b), \end{aligned} \tag{19.5}$$

where $\mathcal{M}$ is the measure of the set $\mathcal{S} \cap [T, T+T^b]$. Hence (19.4) gives a lower bound for the value of $\mathcal{M}$.

From this estimate for $\mathcal{M}$, we can deduce a lower bound for the number of zeros of $\zeta(s)$ on the critical line, as follows. Recall that $\mathcal{S}$ is the set of $t \in [T/2, 4T]$ for which $Z(t)Z(t') < 0$. Let $t \in \mathcal{S} \cap [T, T+T^b]$, so that $Z(t)$ and $Z(t')$ are of opposite sign. This implies that there is a γ such that $t < \gamma < t'$ and $Z(\gamma) = 0$. Since $t \in [T, T+T^b]$, the inequality $t < \gamma < t'$ implies $T < \gamma < (T+T^b)'$, which gives $T < \gamma < T + T^b + 1$ for large T by (3.5). Now (18.3) holds since $t \in [T, T+T^b]$. Thus it follows from the inequality $t < \gamma < t'$ that

$$t \;<\; \gamma \;<\; t \;+\; \frac{2a}{\log \frac{T}{2\pi}} \;+\; O\left(\frac{T^{b-1}}{\log^2 T}\right).$$

We can also write this as

$$\gamma \;-\; \frac{2a}{\log T}(1 + o(1)) \;<\; t \;<\; \gamma.$$

Hence we have proved that the set $\mathcal{S} \cap [T, T+T^b]$ is contained in the union of all intervals of length $(1 + o(1))2a/\log T$ that have right endpoints at zeros γ of $Z(t)$ satisfying $T < \gamma < T + T^b + 1$. The number of such intervals is at most $N_0(T + T^b + 1) - N_0(T)$ because $Z(\gamma) = 0$ if and only if $\zeta(\frac{1}{2} + i\gamma) = 0$. We thus conclude that

$$\mathcal{M} \;\le\; (1 + o(1))\frac{2a}{\log T}\left(N_0(T + T^b + 1) - N_0(T)\right).$$

It follows from this, (19.5), and (19.4) that

$$(\Delta \ + \ o(1))T^b \ \leq \ \frac{2a}{\log T}\bigg(N_0(T+T^b+1)-N_0(T)\bigg).$$

We rearrange this and arrive at

$$\Big(\frac{\pi}{a}\Delta \ + \ o(1)\Big)\frac{T^b}{2\pi}\log T \ \leq \ N_0(T+T^b+1)-N_0(T).$$

By the well-known bound $N(T+1)-N(T) \ll \log T$ (see Theorem 9.2 of Titchmarsh [24]) and the fact that $N_0(T) \leq N(T)$, it follows that

$$N_0(T+T^b+1) \ = \ N_0(T+T^b) \ + \ O(\log T).$$

Hence

$$\Big(\frac{\pi}{a}\Delta \ + \ o(1)\Big)\frac{T^b}{2\pi}\log T \ \leq \ N_0(T+T^b)-N_0(T). \tag{19.6}$$

The following elementary argument now shows that Theorem 1.1 follows from (19.6). For large T, let $U_1, U_2, U_3, \ldots$ be defined by $U_1 = T^b$ and the recursive relation

$$U_j \ = \ (T+U_1+U_2+\cdots+U_{j-1})^b, \qquad j=2,3,4,\ldots.$$

Let r be the unique integer for which

$$U_1+\cdots+U_r \ \leq \ T \ < \ U_1+\cdots+U_r+U_{r+1}.$$

Note that the definition of r implies that

$$U_1+\cdots+U_r \ > \ T-U_{r+1} \ = \ T-(T+U_1+U_2+\cdots+U_r)^b \ \geq \ T-(2T)^b.$$

Hence, since $T-(2T)^b = (1+o(1))T$, it follows that

$$(1+o(1))T \ < \ U_1+\cdots+U_r \ \leq \ T. \tag{19.7}$$

By (19.6), we can write

$$\Big(\frac{\pi}{a}\Delta \ + \ o(1)\Big)\frac{U_j}{2\pi}\log T \ \leq \ N_0(T+U_1+U_2+\cdots+U_j)-N_0(T+U_1+U_2+\cdots+U_{j-1})$$

for $j=1,2,3,\ldots,r$. Add the corresponding sides of these r inequalities to deduce that

$$\Big(\frac{\pi}{a}\Delta \ + \ o(1)\Big)\frac{U_1+\cdots+U_r}{2\pi}\log T \ \leq \ N_0(T+U_1+\cdots+U_r)-N_0(T).$$

It follows from this and (19.7) that

$$\Big(\frac{\pi}{a}\Delta \ + \ o(1)\Big)\frac{T}{2\pi}\log T \ \leq \ N_0(2T)-N_0(T).$$

Now replace T in turn by $\frac{T}{2}, \frac{T}{4}, \frac{T}{8}, \ldots$ and add the results to conclude that

$$\Big(\frac{\pi}{a}\Delta \ + \ o(1)\Big)\frac{T}{2\pi}\log T \ \leq \ N_0(T).$$

From this and the fact that $N(T) \sim \frac{T}{2\pi}\log T$ (see Theorem 9.4 of Titchmarsh [24]), we deduce Theorem 1.1 with

$$\kappa \ \geq \ \frac{\pi}{a}\Delta \ = \ \frac{\pi\left(\cos a \ + \ \frac{4\sin a}{3a\theta}\right)^2}{aC(0,\theta)^{1/2}C(a,\theta)^{1/2}} \ > \ 0. \tag{19.8}$$

This completes the proof of Theorem 1.1.

20. Numerical computations

We can use any standard computer package to compute the coefficients of the polynomial $\mathcal{R}(t;y,\rho)$ in (16.2). By such numerical computations we are able to determine the values of the coefficients $\lambda_{m,n}$ in (16.5) and see that

$$\begin{aligned} C(a,\theta) \; &= \frac{724}{31185}\theta^4 a^8 \; + \; \frac{14684}{42525}\theta^2 a^6 \; + \; \frac{2896}{10395}\theta^3 a^6 \; + \; \frac{64}{45}a^4 \\ &+ \; \frac{7136}{2835}\theta a^4 \; + \; \frac{362}{315}\theta^2 a^4 \; + \; \frac{724}{3465}\theta^3 a^4 \; + \; \frac{52}{45}\theta^{-2}a^2 \\ &+ \; \frac{272}{63}\theta^{-1}a^2 \; + \; \frac{88}{15}a^2 \; + \; \frac{416}{135}\theta a^2 \; + \; \frac{2}{9}\theta^{-4} \\ &+ \; \frac{64}{45}\theta^{-3} \; + \; \frac{10}{3}\theta^{-2} \; + \frac{68}{21}\theta^{-1} \; + \; 1. \end{aligned}$$

Note that

$$C(0,\theta) \; = \; \frac{2}{9}\theta^{-4} \; + \; \frac{64}{45}\theta^{-3} \; + \; \frac{10}{3}\theta^{-2} \; + \frac{68}{21}\theta^{-1} \; + \; 1.$$

This agrees with the conjecture (6.29) of Conrey and Snaith [8].

With the above explicit form of $C(a,\theta)$, we can numerically optimize the value of the lower bound (19.8) for κ. For fixed θ close to $1/48$, we find that the value $a \approx 4.3120526$ is optimal, and we arrive at the lower bound

$$\kappa > 0.0001049.$$

If we assume the (yet unproved) statement that Theorems 1.2 and 1.3 hold for *any* $\theta > 0$, then we can optimize numerically by varying both a and θ. We find that the values $a \approx 3.0407338$ and $\theta \approx 0.9967280$ are optimal, and they give

$$\kappa > 0.0086729.$$

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