ON THE PAIR CORRELATION CONJECTURE AND THE ALTERNATIVE HYPOTHESIS

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ABSTRACT. We prove the equivalence of certain asymptotic formulas for (a) averages over intervals for the 2-point form factor $F(\alpha, T)$ for the zeros of the Riemann zeta-function, $\zeta(s)$, (b) the mean square of the logarithmic derivative of $\zeta(s)$, (c) a variance for the number of primes in short intervals, and (d) the number of pairs of zeros of $\zeta(s)$ with small gaps. The main result is a generalization of the fusion of a theorem of Goldston and a theorem of Goldston, Gonek, and Montgomery. We apply our result to deduce several consequences of the Alternative Hypothesis.

1. INTRODUCTION AND RESULTS

We assume the truth of the Riemann Hypothesis (RH) throughout this paper, and let $\frac{1}{2} + i\gamma$ denote a nontrivial zero of the Riemann zeta-function, $\zeta(s)$. In the early 1970's Montgomery [15] introduced a new method of studying the distribution of zeros of $\zeta(s)$. Assuming RH, he defined the function

$$F(\alpha) = F(\alpha, T) = \left(\frac{T}{2\pi} \log T \right)^{-1} \sum_{0 < \gamma, \gamma' \leq T} T^{i\alpha(\gamma - \gamma')} w(\gamma - \gamma'),$$

where the sum is over all pairs γ, γ' of ordinates of zeros counted according to multiplicity[1]. Here α is real, $T \geq 2$, and $w(u) = 4/(4 + u^2)$. He then observed that F is a real valued even function of α, and proved that

$$(1.1) \qquad F(\alpha, T) = (1 + o(1))T^{-2\alpha} \log T + \alpha + o(1)$$

uniformly for $0 \leq \alpha \leq 1 - \varepsilon$ for any fixed $\varepsilon > 0$. From this he deduced that if RH is true, then at least two-thirds of the zeros of $\zeta(s)$ are simple. He also showed that if $0 < \gamma_1 \leq \gamma_2 \leq \gamma_3 \cdots$ is a list of the ordinates of all the zeros above the real line, counted according to their multiplicities, then the differences

$$(1.2) \qquad \frac{\gamma_{n+1} \log \gamma_{n+1}}{2\pi} - \frac{\gamma_n \log \gamma_n}{2\pi}$$

are less than 0.68 infinitely often. Later, Mueller and Heath-Brown noted that F is nonnegative (see [4]), and Goldston and Montgomery [12, Lemma 8] showed that (1.1) holds uniformly for $0 \leq \alpha \leq 1$.

The usefulness of $F(\alpha)$ for deducing information on the distribution of the zeros is limited by the fact that the asymptotic behavior of $F(\alpha)$ is known only for $|\alpha| \leq 1$. To get around this difficulty, Montgomery [15] used a quantitative form of the Hardy-Littlewood twin prime hypothesis to conjecture that for any fixed $M \geq 1$,

$$(1.3) \qquad F(\alpha, T) = 1 + o(1)$$

[1]All sums over zeros in this paper will count multiplicity.

as $T \to \infty$, uniformly for $1 \leq \alpha \leq M$. This led him to make the following conjecture.

Conjecture (The Pair Correlation Conjecture). *For any fixed $\beta > 0$, we have*

$$N(T, \beta) \stackrel{def}{=} \left(\frac{T}{2\pi} \log T \right)^{-1} \sum_{\substack{0 < \gamma, \gamma' \leq T \\ 0 < \gamma - \gamma' \leq \frac{2\pi\beta}{\log T}}} 1 \quad \sim \quad \int_0^\beta 1 - \left(\frac{\sin \pi u}{\pi u} \right)^2 \, du$$

as $T \to \infty$.

The integrand above is called the *pair correlation function* of the zeros. It happens that the eigenvalues of large random Hermitian matrices (the Gaussian Unitary Ensemble or GUE), which are used in the study of particle physics, have exactly the same pair correlation function. Thus the pair correlation conjecture provided a surprising connection between number theory and random matrix theory, two fields that seemed unrelated at the time. Later, Odlyzko [16] found strong numerical evidence supporting the pair correlation conjecture. Bolanz [3], building on Montgomery's ideas, proved that (1.3) holds for $1 \leq M < \frac{3}{2}$ provided that a strong form of the Hardy-Littlewood twin prime conjecture is true. He later (unpublished) extended the range to $1 \leq M < 2$ under an additional assumption. Hejhal [14] proved results similar to Montgomery's for the triple correlation function of the zeros of $\zeta(s)$, and Rudnick and Sarnak [17] proved results for the n-correlation function, providing even more theoretical evidence that the zeros are distributed like the eigenvalues of matrices from the Gaussian Unitary Ensemble. Bogomolny and Keating [1, 2] used a prime-twin type conjecture to heuristically extend the range of the n-correlation result of Rudnick and Sarnak.

In spite of the overwhelming numerical and theoretical evidence, the pair correlation conjecture has yet to be proved. Thus, it is of interest to determine consequences of other conjectures about the spacings of the zeros. One well known alternative to the pair correlation conjecture is the *Alternative Hypothesis*. There are various formulations of it, but they all essentially say that almost all the differences (1.2) are close to half-integers. In a sense, the Alternative Hypothesis is antithetical to the pair correlation conjecture in that the latter says the zeros are randomly distributed, whereas the former says they are quite regular. Besides supporting the pair correlation conjecture, a disproof of the Alternative Hypothesis would be useful in showing that Landau-Siegel zeros do not exist. Indeed, Conrey and Iwaniec [6] have obtained a relation between the spacings (1.2) and the size of $L(1, \chi)$ for real primitive Dirichlet characters χ. A corollary of their Theorem 1.2 is that if the number of ordinates $\gamma_n \leq T$ for which the spacings (1.2) are less than 0.49, say, is $\gg T(\log T)^{4/5}$ (as $T \to \infty$), then

$$L(1, \chi) \gg (\log q)^{-90}.$$

The implied constant in their result is effectively computable.

Conrey [5] formulates the Alternative Hypothesis as the existence of a function $h(T)$ that goes to 0 as $T \to \infty$ such that if $\gamma_n \geq T_0$, then the difference (1.2) is within $h(T_0)$ of a half-integer. A more precise formulation is implicit in a lecture of Heath-Brown [13]. He observed that if $L(1, \chi) \ll q^{-1/4-\varepsilon}$, then there is a sequence of points t_n that are close to the zeros of $\zeta(s)L(s, \chi)$ such that if t_m and t_n are about the size of T, with T much larger than q, then there is an integer k with

$$(t_m - t_n) \log T = \pi k + O(|t_m - t_n|\{\log q + \log^{2/3} T\}).$$

We will base our formulation of the Alternative Hypothesis on this. We let

$$\widetilde{\gamma} = \frac{\gamma}{2\pi} \log \gamma$$

denote the normalized ordinate of a zero, so that the average spacing between consecutive $\widetilde{\gamma}_n$'s equals 1. Our version of the hypothesis is the following.

Hypothesis (The Alternative Hypothesis)**.** *For each n there is an integer k_n with*

$$(1.4) \qquad \widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n \; = \; \frac{1}{2}k_n \; + \; O(|\gamma_{n+1} - \gamma_n|\psi(\gamma_n)),$$

where $\psi(\gamma)$ is a function such that $\psi(\gamma) \to \infty$ and $\psi(\gamma) = o(\log \gamma)$ as $\gamma \to \infty$.

The aim of our study is to deduce consequences of the Alternative Hypothesis. To do this, we look at known consequences of the pair correlation conjecture and determine their corresponding forms under the Alternative Hypothesis.

Goldston [10] showed that the pair correlation conjecture is actually equivalent to a weaker variant of (1.3), namely

$$(1.5) \qquad \int_b^{b+\delta} F(\alpha, T) \sim \delta$$

as $T \to \infty$ for all fixed $b \geq 1$ and $\delta > 0$. He also showed that the pair correlation conjecture is equivalent to

$$J(\beta, T) \stackrel{\text{def}}{=} \int_1^{T^\beta} \left(\psi\left(x + \frac{x}{T} \right) - \psi(x) - \frac{x}{T} \right)^2 \frac{dx}{x^2} \sim \left(\beta - \frac{1}{2} \right) \frac{\log^2 T}{T}$$

as $T \to \infty$, for all fixed $\beta \geq 1$. Here, $\psi(x) = \sum_{n \leq x} \Lambda(n)$, where Λ is the von Mangoldt function defined by $\Lambda(p^m) = \log p$ for prime powers $p^m > 1$ and $\Lambda(n) = 0$ for all other n. Later, Goldston, Gonek, and Montgomery [11] showed that (1.5) is equivalent to

$$I(b, T) \stackrel{\text{def}}{=} \int_1^T \left| \frac{\zeta'}{\zeta}\left(\frac{1}{2} + \frac{b}{\log T} + it \right) \right|^2 dt \sim \left(\frac{1 - e^{-2b}}{4b^2} \right) T \log^2 T$$

as $T \to \infty$, for all fixed $b > 0$. Thus, collecting these results together, we have a four-way equivalence between the asymptotic formulas for $F(\alpha, T)$, $I(b, T)$, $J(\beta, T)$, and $N(T, \beta)$.

Theorem 1.1 (Goldston, Gonek, Montgomery)**.** *Assume RH. The following statements are equivalent.*

(A1) $\displaystyle\int_1^b F(\alpha, T)\, d\alpha \sim b - 1$ \hfill *as $T \to \infty$, for all fixed $b \geq 1$.*

(B1) $\displaystyle I(b, T) \sim \left(\frac{1 - e^{-2b}}{4b^2} \right) T \log^2 T$ \hfill *as $T \to \infty$, for all fixed $b > 0$.*

(C1) $\displaystyle J(\beta, T) \sim \left(\beta - \frac{1}{2} \right) \frac{\log^2 T}{T}$ \hfill *as $T \to \infty$, for all fixed $\beta \geq 1$.*

(D1) $\displaystyle N(T, \beta) \sim \int_0^\beta 1 - \left(\frac{\sin \pi u}{\pi u} \right)^2 du$ \hfill *as $T \to \infty$, for all fixed $\beta > 0$.*

Note that by (1.1) we can evaluate the integral of $F(\alpha, T)$ on any subinterval of $[0, 1]$ and write

$$\int_0^b F(\alpha, T)\, d\alpha \; \sim \; \frac{1}{2} + \frac{b^2}{2}$$

as $T \to \infty$, for all fixed $0 < b \leq 1$. This gives a version of the statement (A1) for $0 < b \leq 1$ that is true in any case (on RH). Also, a result of Gallagher and Mueller [8] states that

$$J(\beta, T) \; \sim \; \frac{\beta^2 \log^2 T}{2T} \quad \text{as } T \to \infty, \quad \text{for all fixed } 0 < \beta \leq 1.$$

This is a version of (C1) for $0 < \beta \leq 1$ that holds true in any case (unconditionally).

In this paper, we prove a generalization of Theorem 1.1 (Theorem 1.2). Afterwards, we apply our result to obtain an analogue of Theorem 1.1 for the Alternative Hypothesis (Theorem 1.3). To state our main result, we set

$$N^*(T) \stackrel{\text{def}}{=} \left(\frac{T}{2\pi} \log T \right)^{-1} \sum_{0 < \gamma \leq T} m_\gamma \ ,$$

where m_γ denotes the multiplicity of the zero $\frac{1}{2} + i\gamma$ and the sum again counts ordinates according to multiplicity. Also, for a general measure μ, we shall mean by $\int_a^b f(\alpha) d\mu(\alpha)$ the integral of f over the set $[a, b]$.

Theorem 1.2. *Assume RH. Let μ be a positive Borel measure on $[0, \infty)$ for which the function $\alpha \mapsto \min\{1, \alpha^{-2}\}$ is integrable over $[0, \infty)$. The following statements are equivalent*

(A) $\mu[0, b) + o(1) \leq \displaystyle\int_0^b F(\alpha, T) \, d\alpha \leq \mu[0, b] + o(1)$

as $T \to \infty$, for all fixed $b > 0$.

(B) $I(b, T) \sim \left(\displaystyle\int_0^\infty e^{-2b\alpha} \, d\mu(\alpha) - \frac{1}{2} \right) T \log^2 T$

as $T \to \infty$, for all fixed $b > 0$.

(C) $\mu[0, \beta) - \dfrac{1}{2} + o(1) \leq J(\beta, T) \dfrac{T}{\log^2 T} \leq \mu[0, \beta] - \dfrac{1}{2} + o(1)$

as $T \to \infty$, for all fixed $\beta > 0$.

(D) $\dfrac{\beta}{2} N^*(T) + \displaystyle\int_0^\beta N(T, u) \, du \sim \int_0^\infty \left(\frac{\sin \pi\beta\alpha}{\pi\alpha} \right)^2 d\mu(\alpha)$

as $T \to \infty$, for all fixed $\beta > 0$.

The measure that makes the statement (A) consistent with (1.1) and (1.5) is the measure with $\mu(0) = 1/2$, $d\mu(\alpha) = \alpha d\alpha$ for $0 < \alpha \leq 1$, and $d\mu(\alpha) = d\alpha$ for $\alpha > 1$. Thus we see that Theorem 1.1 corresponds to taking this measure in Theorem 1.2. Keeping in mind the remarks below Theorem 1.1, we see from a straightforward calculation that (A), (B), and (C) with this measure are equivalent to (A1), (B1), and (C1), respectively. However, to prove that (D) with this choice of μ is equivalent to (D1) is not as straightforward. We shall not carry this out here because the argument is similar to the one we shall use in Section 4 to prove Theorem 1.3.

Let $g(\alpha) = |\alpha|$ for $|\alpha| \leq 1$ and be extended to $\mathbb{R}$ by periodicity. In Section 5 we shall prove that the Alternative Hypothesis and the assumption that $N^*(T) \sim 1$ imply that

$$\frac{1}{2} + \sum_{n < b/2} 1 + \int_0^b g + o(1) \leq \int_0^b F(\alpha, T) \, d\alpha \leq \frac{1}{2} + \sum_{n \leq b/2} 1 + \int_0^b g + o(1)$$

for any fixed $b > 0$ as $T \to \infty$. Thus, the analogue of Theorem 1.1 for the Alternative Hypothesis is

Theorem 1.3. *Assume RH. The following statements are equivalent.*

(A2) $\quad \dfrac{1}{2} + \displaystyle\sum_{n < b/2} 1 + \int_0^b g + o(1) \leq \int_0^b F(\alpha, T) \, d\alpha \leq \frac{1}{2} + \sum_{n \leq b/2} 1 + \int_0^b g + o(1)$

<div align="right">*as $T \to \infty$, for all fixed $b > 0$.*</div>

(B2) $I(b,T) \sim \left(\dfrac{e^{2b} - 1}{4b^2(e^{2b} + 1)} + \dfrac{1}{e^{4b} - 1} \right) T \log^2 T$ *as $T \to \infty$, for all fixed $b > 0$.*

(C2) $\displaystyle\sum_{n < \beta/2} 1 + \int_0^\beta g \ + \ o(1) \ \leq J(\beta, T) \dfrac{T}{\log^2 T} \leq \sum_{n \leq \beta/2} 1 + \int_0^\beta g \ + \ o(1)$

<div align="right">*as $T \to \infty$, for all fixed $\beta > 0$.*</div>

(D2) $N(T, \beta) \sim \begin{cases} m \ - \ \dfrac{1}{2\pi^2} \displaystyle\sum_{n=0}^{m-1} \dfrac{4}{(2n+1)^2} & \text{if} \quad m < \beta < m + \frac{1}{2}, \\[3ex] m \ + \ \dfrac{1}{2} \ - \ \dfrac{1}{2\pi^2} \displaystyle\sum_{n=0}^{m} \dfrac{4}{(2n+1)^2} & \text{if} \ \ m + \frac{1}{2} < \beta < m + 1, \end{cases}$

as $T \to \infty$, for all fixed $\beta > 0$ not a half-integer. Here $m = [\beta]$, the greatest integer $\leq \beta$.

By Proposition 5.1 in Section 5, if the Alternative Hypothesis is true and $N^*(T) \sim 1$ holds, then the statement (A2) holds. An immediate consequence of this and Theorem 1.3 is

Corollary 1.1. *Assume RH, the Alternative Hypothesis, and that $N^*(T) \sim 1$ as $T \to \infty$. Then each of the statements (A2), (B2), (C2), and (D2) is true.*

If we assume the Alternative Hypothesis and that all zeros are simple, then we can use Corollary 1.1 to estimate the proportion of zeros γ_n for which $\widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n$ is near a fixed half-integer $k/2$. Let $B_{k/2}(T)$ be the set of zeros $\gamma_n \leq T$ such that $k/2$ is closest among all half-integers to $\widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n$. We also define

$$p_{k/2} = p_{k/2}(T) \stackrel{\text{def}}{=} \left(\dfrac{T}{2\pi} \log T \right)^{-1} |B_{k/2}(T)|.$$

Theorem 1.4. *Assume RH, the Alternative Hypothesis, and that all zeros of $\zeta(s)$ are simple. Then as $T \to \infty$ we have*

$$p_0 = o(1),$$

(1.6) $$p_{1/2} = \dfrac{1}{2} - \dfrac{2}{\pi^2} + o(1),$$

(1.7) $$\dfrac{4}{\pi^2} + o(1) \ \leq \ p_1 \ \leq \ \dfrac{1}{2} + o(1),$$

(1.8) $$p_{k/2} \leq \begin{cases} \dfrac{1}{2} + o(1) & \text{for} \ \ k \geq 4 \ \ \text{even}, \\[3ex] \dfrac{1}{2} - \dfrac{2}{\pi^2 k^2} + o(1) & \text{for} \ \ k \geq 3 \ \ \text{odd}. \end{cases}$$

Our estimates for p_0, $p_{1/2}$, and p_1 agree with those in Section 2 of Farmer, Gonek, and Lee [7]. However, one should note that the formulation of the Alternative Hypothesis in [7] is stronger than ours.

2. LEMMAS

In each of the following lemmas we assume there is a number T_0 such that the function $f(\alpha, T)$ is defined for $\alpha \geq 0$ and $T \geq T_0$, and $f(\cdot, T)$ is Lebesgue measurable for each such T.

Lemma 2.1. *Let $f(\alpha) = f(\alpha, T)$ be nonnegative and let μ be a positive Borel measure on $[0, \infty)$ such that $\int_0^\infty e^{-bx} d\mu(x) < \infty$ for all $b \geq 1$. If*

$$(2.1) \qquad \lim_{T \to \infty} \int_0^\infty f(\alpha, T) e^{-b\alpha} \, d\alpha = \int_0^\infty e^{-b\alpha} \, d\mu(\alpha)$$

for all fixed positive integers b, then

$$(2.2) \qquad \mu[0, d) \; + \; o(1) \; \leq \; \int_0^d f(\alpha, T) \, d\alpha \; \leq \; \mu[0, d] \; + \; o(1)$$

as $T \to \infty$, for all fixed $d > 0$.

Proof. The proof uses Karamata's method (see for example §7.53 of [18]). Let $0 < \delta < e^{-d}$, and set $\eta = -\log(e^{-d} - \delta)$. Define $k(u)$ by

$$k(u) = \begin{cases} 1/u & \text{if} \quad e^{-d} \leq u \leq 1, \\[2mm] (u - e^{-\eta}) \dfrac{e^d}{\delta} & \text{if} \quad e^{-\eta} \leq u \leq e^{-d}, \\[2mm] 0 & \text{otherwise.} \end{cases}$$

By the Weierstrass approximation theorem, given any $\varepsilon > 0$, there is a polynomial $P(u) = \sum_{n=0}^N a_n u^n$ such that

$$(2.3) \qquad k(u) < P(u) < k(u) + \varepsilon$$

for $u \in [0, 1]$. Now $uk(u) = 1$ for $e^{-d} \leq u \leq 1$. Thus, by (2.3), $1 < uP(u)$ for $e^{-d} \leq u \leq 1$. Also, since $k(u) \geq 0$, we have $0 < uP(u)$ for $0 < u \leq e^{-d}$. Hence $1 < e^{-\alpha} P(e^{-\alpha})$ for $0 \leq \alpha \leq d$ and $0 < e^{-\alpha} P(e^{-\alpha})$ for $\alpha \geq d$. Since f is nonnegative,

$$(2.4) \qquad \int_0^d f(\alpha) d\alpha \leq \int_0^d f(\alpha) e^{-\alpha} P(e^{-\alpha}) d\alpha \leq \int_0^\infty f(\alpha) e^{-\alpha} P(e^{-\alpha}) \, d\alpha.$$

By (2.1), we have

$$(2.5) \qquad \begin{aligned} \int_0^\infty f(\alpha) e^{-\alpha} P(e^{-\alpha}) \, d\alpha &= \sum_{n=0}^N a_n \int_0^\infty f(\alpha) e^{-\alpha(n+1)} \, d\alpha \\ &\sim \sum_{n=0}^N a_n \left(\int_0^\infty e^{-(n+1)\alpha} \, d\mu(\alpha) \right) = \int_0^\infty e^{-\alpha} P(e^{-\alpha}) \, d\mu(\alpha). \end{aligned}$$

It follows from (2.4) and (2.5) that

$$(2.6) \qquad \int_0^d f(\alpha) d\alpha \leq \int_0^\infty e^{-\alpha} P(e^{-\alpha}) \, d\mu(\alpha) + o(1)$$

as $T \to \infty$. By (2.3) we have $e^{-\alpha} P(e^{-\alpha}) < e^{-\alpha} k(e^{-\alpha}) + \varepsilon e^{-\alpha}$ for $\alpha \geq 0$. From this and (2.6) we see that

$$(2.7) \qquad \int_0^d f(\alpha) d\alpha \leq \int_0^\infty e^{-\alpha} k(e^{-\alpha}) \, d\mu(\alpha) + \varepsilon \int_0^\infty e^{-\alpha} \, d\mu(\alpha) + o(1).$$

If $e^{-\eta} \le u \le e^{-d}$, then by the definitions of $k(u)$ and η, we have

$$uk(u) = u(u - e^{-\eta})\frac{e^d}{\delta} \le e^{-d}(e^{-d} - e^{-\eta})\frac{e^d}{\delta} = e^{-d}(\delta)\frac{e^d}{\delta} = 1.$$

Thus $uk(u) \le 1$ for $e^{-\eta} \le u \le 1$. Hence $e^{-\alpha}k(e^{-\alpha}) \le 1$ for $0 \le \alpha \le \eta$. Also, since $k(u) = 0$ for $u \le e^{-\eta}$, we have $e^{-\alpha}k(e^{-\alpha}) = 0$ for $\alpha \ge \eta$. Thus

$$\int_0^\infty e^{-\alpha}k(e^{-\alpha})\,d\mu(\alpha) = \int_0^\eta e^{-\alpha}k(e^{-\alpha})\,d\mu(\alpha) \ + \ \int_\eta^\infty e^{-\alpha}k(e^{-\alpha})\,d\mu(\alpha)$$

$$\le \int_0^\eta d\mu(\alpha) \ + \ 0$$

$$= \mu[0, \eta).$$

We find from this and (2.7) that

(2.8) $$\int_0^d f(\alpha)d\alpha \le \mu[0, \eta) + \varepsilon \int_0^\infty e^{-\alpha}\,d\mu(\alpha) + o(1).$$

Now $\mu[0, \eta) \to \mu[0, d]$ as $\eta \to d^+$. Since $\eta = -\log(e^{-d} - \delta)$, we have that $\eta \to d^+$ as $\delta \to 0^+$. Thus, making δ and ε small enough in (2.8), we obtain the second inequality in (2.2).

To prove the first inequality in (2.2), define $\tau = -\log(e^{-d} + \delta)$ and define $\ell(u)$ by

$$\ell(u) = \begin{cases} 1/u & \text{if} \quad e^{-\tau} \le u \le 1, \\[2mm] (u - e^{-d})\dfrac{e^\tau}{\delta} & \text{if} \quad e^{-d} \le u \le e^{-\tau}, \\[2mm] 0 & \text{otherwise.} \end{cases}$$

By the Weierstrass approximation theorem, given any $\varepsilon > 0$, there is a polynomial $Q(u) = \sum_{n=0}^M b_n u^n$ such that

(2.9) $$\ell(u) - \varepsilon < Q(u) < \ell(u)$$

for $u \in [0, 1]$. Now $u\ell(u) = 1$ for $e^{-\tau} \le u \le 1$. Also, if $e^{-d} \le u \le e^{-\tau}$, then by the definition of τ,

$$u\ell(u) = u(u - e^{-d})\frac{e^\tau}{\delta} \le e^{-\tau}(e^{-\tau} - e^{-d})\frac{e^\tau}{\delta} = e^{-\tau}(\delta)\frac{e^\tau}{\delta} = 1.$$

Hence $u\ell(u) \le 1$ for $e^{-d} \le u \le 1$. Therefore, by (2.9), $1 > uQ(u)$ for $e^{-d} \le u \le 1$. Also, $\ell(u) = 0$ for $0 \le u \le e^{-d}$, so by (2.9) we have $0 > uQ(u)$ for $0 < u \le e^{-d}$. Hence, by a change of variable, $1 > e^{-\alpha}Q(e^{-\alpha})$ for $0 \le \alpha \le d$, and $0 > e^{-\alpha}Q(e^{-\alpha})$ for $\alpha \ge d$. Since f is nonnegative,

(2.10) $$\int_0^d f(\alpha)d\alpha \ge \int_0^d f(\alpha)e^{-\alpha}Q(e^{-\alpha})d\alpha \ge \int_0^\infty f(\alpha)e^{-\alpha}Q(e^{-\alpha})\,d\alpha.$$

By (2.1), we have

(2.11)
$$\int_0^\infty f(\alpha)e^{-\alpha}Q(e^{-\alpha})\,d\alpha = \sum_{n=0}^M b_n \int_0^\infty f(\alpha)e^{-\alpha(n+1)}\,d\alpha$$

$$\sim \sum_{n=0}^M b_n \left(\int_0^\infty e^{-(n+1)\alpha}\,d\mu(\alpha)\right) = \int_0^\infty e^{-\alpha}Q(e^{-\alpha})\,d\mu(\alpha).$$

It follows from (2.10) and (2.11) that

(2.12) $$\int_0^d f(\alpha)d\alpha \ge \int_0^\infty e^{-\alpha}Q(e^{-\alpha})\,d\mu(\alpha) + o(1)$$

as $T \to \infty$. By (2.9) we have $e^{-\alpha}Q(e^{-\alpha}) > e^{-\alpha}\ell(e^{-\alpha}) - \varepsilon e^{-\alpha}$ for $\alpha \geq 0$. It follows from this and (2.12) that

$$(2.13) \qquad \int_0^d f(\alpha)d\alpha \geq \int_0^\infty e^{-\alpha}\ell(e^{-\alpha})\,d\mu(\alpha) - \varepsilon \int_0^\infty e^{-\alpha}\,d\mu(\alpha) + o(1).$$

By the definition of $\ell(u)$, we have $u\ell(u) = 1$ for $e^{-\tau} \leq u \leq 1$. Also, $\ell(u) \geq 0$ for $0 \leq u \leq e^{-\tau}$. Thus $e^{-\alpha}\ell(e^{-\alpha}) = 1$ for $0 \leq \alpha \leq \tau$, and $e^{-\alpha}\ell(e^{-\alpha}) \geq 0$ for $\alpha \geq \tau$. We now see that

$$\int_0^\infty e^{-\alpha}\ell(e^{-\alpha})\,d\mu(\alpha) = \int_0^\tau e^{-\alpha}\ell(e^{-\alpha})\,d\mu(\alpha) \; + \; \int_\tau^\infty e^{-\alpha}\ell(e^{-\alpha})\,d\mu(\alpha)$$
$$\geq \int_0^\tau d\mu(\alpha) \; + \; 0$$
$$= \mu[0,\tau).$$

It follows from this and (2.13) that

$$(2.14) \qquad \int_0^d f(\alpha)d\alpha \geq \mu[0,\tau) - \varepsilon \int_0^\infty e^{-\alpha}\,d\mu(\alpha) + o(1).$$

Now $\mu[0,\tau) \to \mu[0,d)$ as $\tau \to d^-$. Furthermore, $\tau = -\log(e^{-d}+\delta)$, so $\tau \to d^-$ as $\delta \to 0^+$. Thus, making δ and ε small enough in (2.14), we obtain the first inequality in (2.2). $\qquad\square$

Lemma 2.2. *Suppose that*

$$(2.15) \qquad \int_0^x |f(\alpha,T)|\,d\alpha \ll x$$

uniformly for all large x and T, that μ is a positive Borel measure on $[0,\infty)$ such that $\mu[0,d] < \infty$ for $d > 0$, and that r is a real continuous function such that $\int_0^\infty |r(\alpha)|\,d\mu(\alpha) < \infty$. Assume that there exists a function $r_1(x)$ and an $x_0 > 0$ such that

> (i) $|r(x)| \leq r_1(x)$ for $x \geq x_0$;
> (ii) $r_1'(x)$ exists for $x \geq x_0$, and r_1' is Riemann integrable over closed subintervals of $[x_0,\infty)$;
> (iii) $xr_1(x) \to 0$ as $x \to \infty$; and
> (iv) $\int_{x_0}^\infty x|r_1'(x)|\,dx \; < \; \infty$.

If for each fixed $d > 0$, we have

$$(2.16) \qquad \mu[0,d) \; + \; o(1) \; \leq \int_0^d f(\alpha,T)\,d\alpha \; \leq \; \mu[0,d] \; + \; o(1)$$

as $T \to \infty$, then

$$(2.17) \qquad \lim_{T\to\infty} \int_0^\infty f(\alpha,T)r(\alpha)\,d\alpha = \int_0^\infty r(\alpha)\,d\mu(\alpha).$$

Proof. Let x_0 be as in our hypotheses and let $B' > B > x_0$. By condition (ii)
$$(2.18)$$
$$\int_B^{B'} |f(\alpha,T)|r_1(\alpha)\,d\alpha = \left.\left(r_1(\alpha)\int_0^\alpha |f(u,T)|\,du\right)\right|_{\alpha=B}^{\alpha=B'} - \int_B^{B'}\left(\int_0^\alpha |f(u,T)|\,du\right)r_1'(\alpha)\,d\alpha.$$

The assumptions (2.15), (iii), and (iv) allow us to let $B' \to \infty$ in (2.18) and obtain

$$(2.19) \qquad \int_B^\infty |f(\alpha,T)|r_1(\alpha)\,d\alpha = -r_1(B)\int_0^B |f(u,T)|\,du - \int_B^\infty\left(\int_0^\alpha |f(u,T)|\,du\right)r_1'(\alpha)\,d\alpha.$$

By the same assumptions, if $\varepsilon > 0$, then the right-hand side of (2.19) is less than ε for B and T large enough. Thus by (i),

$$(2.20) \qquad \left| \int_B^\infty f(\alpha, T) r(\alpha)\, d\alpha \right| \leq \int_B^\infty |f(\alpha, T)| r_1(\alpha)\, d\alpha < \varepsilon$$

for B and T large enough. Furthermore, since $\int_0^\infty |r(\alpha)|\, d\mu(\alpha) < \infty$, we see that

$$(2.21) \qquad \left| \int_B^\infty r(\alpha)\, d\mu(\alpha) \right| < \varepsilon$$

for B large enough. Since $\mu[a, b] < \infty$ for $b > a \geq 0$, not all points of $[0, \infty)$ can have positive measure. Thus, we may choose a B such that (2.20) and (2.21) hold and such that $\mu\{B\} = 0$. By (2.15) we can also choose an $\eta > 0$ so small that

$$(2.22) \qquad \eta\mu[0, B] < \varepsilon \qquad \text{and} \qquad \eta \int_0^B |f(\alpha, T)|\, d\alpha < \varepsilon$$

for all large T. Let $x_1, x_2, \ldots$ be the distinct points in $[0, B]$ with positive μ-measure. Since $\mu[0, B]$ is finite, there can only be countably many such points. For the same reason, a positive integer ν can be chosen such that

$$(2.23) \qquad 2\left(\max_{0 \leq u \leq B} |r(u)| \right) \sum_{k=\nu+1}^\infty \mu\{x_k\} \;<\; \varepsilon.$$

Since r is continuous, we can partition $[0, B]$ into small subintervals $[c, d]$ such that $|r(x) - r(y)| < \eta$ for all x, y in $[c, d]$. We may also choose the endpoints of each subinterval $[c, d]$ so that none, except possibly $c = 0$, is equal to $x_1, x_2, \ldots, x_\nu$. Now, for each subinterval $[c, d]$ we have

$$\left| \int_c^d f(\alpha) r(\alpha)\, d\alpha - \int_c^d r(\alpha)\, d\mu(\alpha) \right|$$

$$= \left| \int_c^d f(\alpha)(r(\alpha) - r(d))\, d\alpha + \int_c^d f(\alpha) r(d)\, d\alpha - \int_c^d r(d)\, d\mu(\alpha) - \int_c^d (r(\alpha) - r(d))\, d\mu(\alpha) \right|$$

$$\leq \int_c^d |f(\alpha)||r(\alpha) - r(d)|\, d\alpha + |r(d)| \left| \int_c^d f(\alpha)\, d\alpha - \int_c^d d\mu(\alpha) \right| + \int_c^d |r(\alpha) - r(d)|\, d\mu(\alpha).$$

Thus, since $|r(x) - r(y)| < \eta$ for x, y in $[c, d]$, we have
(2.24)
$$\left| \int_c^d f(\alpha) r(\alpha)\, d\alpha - \int_c^d r(\alpha)\, d\mu(\alpha) \right| \leq \eta \int_c^d |f(\alpha)|\, d\alpha + |r(d)| \left| \int_c^d f(\alpha)\, d\alpha - \int_c^d d\mu(\alpha) \right| + \eta \int_c^d d\mu(\alpha).$$

Now replace d by c in (2.16) and subtract the resulting formula from (2.16) to obtain

$$\mu(c, d) + o(1) \leq \int_c^d f(\alpha, T)\, d\alpha \leq \mu[c, d] + o(1).$$

Thus

$$-\mu\{c\} + o(1) \leq \int_c^d f(\alpha, T)\, d\alpha - \mu[c, d] \leq \mu\{d\} + o(1),$$

and we find that

$$\left| \int_c^d f(\alpha, T)\, d\alpha - \mu[c, d] \right| \leq \mu\{c\} + \mu\{d\} + o(1).$$

Hence

$$\left| \int_c^d f(\alpha)\, d\alpha - \int_c^d d\mu(\alpha) \right| = \left| \int_c^d f(\alpha)\, d\alpha - \mu[c, d] \right| \leq \mu\{c\} + \mu\{d\} + o(1).$$

It follows from this and (2.24) that
(2.25)
$$\left| \int_c^d f(\alpha) r(\alpha) \, d\alpha - \int_c^d r(\alpha) \, d\mu(\alpha) \right| \leq \eta \int_c^d |f(\alpha)| \, d\alpha + |r(d)|(\mu\{c\} + \mu\{d\} + o(1)) + \eta \int_c^d d\mu(\alpha).$$

We will use this estimate for subintervals $[c,d]$ with $c > 0$, but a different estimate when $c = 0$. When $c = 0$ we proceed as follows. By (2.16) we have

$$o(1) \leq \int_0^d f(\alpha, T) \, d\alpha - \mu[0, d) \leq \mu\{d\} + o(1).$$

Thus

$$\left| \int_0^d f(\alpha, T) \, d\alpha - \mu[0, d) \right| \leq \mu\{d\} + o(1).$$

That is,

$$\left| \int_0^d f(\alpha) \, d\alpha - \int_0^d d\mu(\alpha) \right| = \left| \int_0^d f(\alpha) \, d\alpha - \mu[0, d) \right| \leq \mu\{d\} + o(1).$$

It follows from this and (2.24) that

(2.26) $$\left| \int_0^d f(\alpha) r(\alpha) \, d\alpha - \int_0^d r(\alpha) \, d\mu(\alpha) \right| \leq \eta \int_0^d |f(\alpha)| \, d\alpha + |r(d)|(\mu\{d\} + o(1)) + \eta \int_0^d d\mu(\alpha).$$

Next we sum our estimates over all the subintervals $[c,d]$. Let $\sum_{[c,d]}$ denote a sum over all the subintervals $[c,d]$ of our partition of $[0, B]$, and let $\sum'_{[c,d]}$ denote a sum that excludes the term $\mu\{c\}$ if $c = 0$. Then by (2.25) and (2.26) we see that

$$\left| \int_0^B f(\alpha) r(\alpha) \, d\alpha - \int_0^B r(\alpha) \, d\mu(\alpha) \right| = \left| \sum_{[c,d]} \left(\int_c^d f(\alpha) r(\alpha) \, d\alpha - \int_c^d r(\alpha) \, d\mu(\alpha) \right) \right|$$

$$\leq \sum_{[c,d]} \left| \int_c^d f(\alpha) r(\alpha) \, d\alpha - \int_c^d r(\alpha) \, d\mu(\alpha) \right|$$

$$\leq \eta \sum_{[c,d]} \int_c^d |f(\alpha)| \, d\alpha + \sum_{[c,d]}' |r(d)|(\mu\{c\} + \mu\{d\} + o(1)) + \eta \sum_{[c,d]} \int_c^d d\mu(\alpha).$$

Hence
(2.27)
$$\left| \int_0^B f(\alpha) r(\alpha) \, d\alpha - \int_0^B r(\alpha) \, d\mu(\alpha) \right| \leq \eta \int_0^B |f(\alpha)| \, d\alpha + \sum_{[c,d]}' |r(d)|(\mu\{c\} + \mu\{d\} + o(1)) + \eta \int_0^B d\mu(\alpha).$$

To estimate the sum on the right-hand side, we first note that

$$\sum_{[c,d]}' |r(d)|(\mu\{c\} + \mu\{d\} + o(1)) \leq \left(\max_{0 \leq u \leq B} |r(u)| \right) \sum_{[c,d]}' (\mu\{c\} + \mu\{d\} + o(1))$$

(2.28)
$$= \left(\max_{0 \leq u \leq B} |r(u)| \right) \left(\sum_{[c,d]}' \mu\{c\} + \sum_{[c,d]} \mu\{d\} \right) + o(1).$$

Since none of the nonzero endpoints of our partition are among the points $x_1, \ldots, x_\nu$, we see that

$$\sum_{[c,d]}{}' \mu\{c\} \leq \sum_{k=\nu+1}^{\infty} \mu\{x_k\} \quad \text{and} \quad \sum_{[c,d]} \mu\{d\} \leq \sum_{k=\nu+1}^{\infty} \mu\{x_k\}.$$

Using these estimates in (2.28), we obtain

$$\sum_{[c,d]}{}' |r(d)|(\mu\{c\} + \mu\{d\} + o(1)) \leq 2 \left(\max_{0 \leq u \leq B} |r(u)| \right) \sum_{k=\nu+1}^{\infty} \mu\{x_k\} + o(1).$$

From this and (2.23) we now find that

$$\sum_{[c,d]}{}' |r(d)|(\mu\{c\} + \mu\{d\} + o(1)) < \varepsilon + o(1).$$

Inserting this in (2.27), we obtain

$$(2.29) \qquad \left| \int_0^B f(\alpha)r(\alpha)\, d\alpha - \int_0^B r(\alpha)\, d\mu(\alpha) \right| \leq \eta \int_0^B |f(\alpha)|\, d\alpha + \eta \int_0^B d\mu(\alpha) + \varepsilon + o(1).$$

Since $\int_0^B d\mu(\alpha) = \mu[0, B)$, it follows from (2.22) and (2.29) that

$$(2.30) \qquad \left| \int_0^B f(\alpha)r(\alpha)\, d\alpha - \int_0^B r(\alpha)\, d\mu(\alpha) \right| \leq 3\varepsilon + o(1).$$

Finally, by (2.20), (2.21), and (2.30), we see that

$$\left| \int_0^\infty f(\alpha)r(\alpha)\, d\alpha - \int_0^\infty r(\alpha)\, d\mu(\alpha) \right|$$

$$\leq \left| \int_B^\infty f(\alpha)r(\alpha)\, d\alpha \right| + \left| \int_0^B f(\alpha)r(\alpha)\, d\alpha - \int_0^B r(\alpha)\, d\mu(\alpha) \right| + \left| \int_B^\infty r(\alpha)\, d\mu(\alpha) \right|$$

$$< \varepsilon + 3\varepsilon + \varepsilon + o(1) = 5\varepsilon + o(1).$$

$$\square$$

Lemma 2.3. *Assume that $f(\alpha, T) \geq 0$ and that*

$$(2.31) \qquad \int_0^x f(\alpha, T)\, d\alpha \ll x$$

uniformly for $x \geq 1$ and large enough T. Let μ be a positive Borel measure on $[0, \infty)$ such that

$$\int_0^\infty \min\{1, \alpha^{-2}\}\, d\mu(\alpha) < \infty.$$

If

$$(2.32) \qquad \lim_{T \to \infty} \int_0^\infty \left(\frac{\sin \beta\alpha}{\alpha} \right)^2 f(\alpha, T)\, d\alpha = \int_0^\infty \left(\frac{\sin \beta\alpha}{\alpha} \right)^2 d\mu(\alpha)$$

for each fixed $\beta > 0$, then

$$(2.33) \qquad \mu[0, d) + o(1) \leq \int_0^d f(\alpha, T)\, d\alpha \leq \mu[0, d] + o(1)$$

as $T \to \infty$ for all fixed $d > 0$.

Proof. Our approach is similar to that used to prove Lemma 4 of [12]. Let $\eta > 0$, and define

$$K(x) = K_\eta(x) = \frac{\sin 2\pi x + \sin 2\pi(1+\eta)x}{2\pi x(1 - 4\eta^2 x^2)}.$$

Then

(2.34) $\qquad \hat{K}_\eta(t) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} K_\eta(x) e^{-2\pi i x t}\, dx = \begin{cases} 1 & \text{if} \quad |t| \leq 1, \\ \cos^2\left(\pi(|t|-1)/2\eta\right) & \text{if} \quad 1 \leq |t| \leq 1+\eta, \\ 0 & \text{if} \quad |t| \geq 1+\eta. \end{cases}$

Note that $K(x)$ is an even function and

(2.35) $$K^{(j)}(x) \ll_\eta \min\{1, x^{-3}\}$$

for $x \geq 0$ and $j = 0, 1, 2$. Integrating by parts twice, we have

(2.36) $$\hat{K}(t) = \int_0^\infty K''(x) \left(\frac{\sin \pi t x}{\pi t}\right)^2 dx$$

for all real t. To prove the lemma, we first show that

(2.37) $$\int_0^\infty f(\alpha, T) \hat{K}\left(\frac{\alpha}{d}\right) d\alpha \;=\; \int_0^\infty \hat{K}\left(\frac{\alpha}{d}\right) d\mu(\alpha) \;+\; o(1)$$

as $T \to \infty$, for all fixed $d > 0$. We substitute (2.36) for $\hat{K}$ and interchange the order of integration to obtain

(2.38) $\qquad \int_0^\infty f(\alpha, T) \hat{K}\left(\frac{\alpha}{d}\right) d\alpha \;-\; \int_0^\infty \hat{K}\left(\frac{\alpha}{d}\right) d\mu(\alpha) \;=\; \left(\frac{d}{\pi}\right)^2 \int_0^\infty K''(x) R\left(\frac{\pi x}{d}, T\right) dx,$

where

(2.39) $$R(\kappa, T) = \int_0^\infty f(\alpha, T) \left(\frac{\sin \kappa \alpha}{\alpha}\right)^2 d\alpha - \int_0^\infty \left(\frac{\sin \kappa \alpha}{\alpha}\right)^2 d\mu(\alpha).$$

The validity of the interchange in the order of integration will be justified below. From (2.38) we see that (2.37) will follow from

(2.40) $$\lim_{T \to \infty} \int_0^\infty K''(x) R\left(\frac{\pi x}{d}, T\right) dx = 0.$$

We write

(2.41) $$\int_0^\infty K''(x) R\left(\frac{\pi x}{d}, T\right) dx \;=\; \int_0^D + \int_D^\infty$$

and estimate the two integrals on the right-hand side separately.

In order to estimate the integral over $[0, D]$, we first show that

(2.42) $$R(\kappa, T) \;\ll\; 1 + \kappa^2$$

uniformly for $\kappa \geq 0$ and large enough T. By integration by parts and (2.31),

(2.43) $\qquad \int_1^\infty \frac{f(\alpha, T)}{\alpha^2} d\alpha \;=\; \frac{1}{\alpha^2} \int_0^\alpha f(\xi, T)\, d\xi \bigg|_{\alpha=1}^{\infty} + 2 \int_1^\infty \frac{1}{\alpha^3} \int_0^\alpha f(\xi, T)\, d\xi\, d\alpha \;\ll\; 1$

for T sufficiently large. It follows that

$$\int_1^\infty f(\alpha, T) \left(\frac{\sin \kappa \alpha}{\alpha}\right)^2 d\alpha \;\leq\; \int_1^\infty \frac{f(\alpha, T)}{\alpha^2} d\alpha \;\ll\; 1$$

for such T. Also, by (2.31) we have

$$\int_0^1 f(\alpha, T) \left(\frac{\sin \kappa \alpha}{\alpha}\right)^2 d\alpha \;=\; \kappa^2 \int_0^1 f(\alpha, T) \left(\frac{\sin \kappa \alpha}{\kappa \alpha}\right)^2 d\alpha \;\leq\; \kappa^2 \int_0^1 f(\alpha, T)\, d\alpha \;\ll\; \kappa^2$$

for all sufficiently large T. Hence

(2.44)
$$\int_0^\infty f(\alpha, T) \left(\frac{\sin \kappa \alpha}{\alpha} \right)^2 d\alpha = \int_0^1 f(\alpha, T) \left(\frac{\sin \kappa \alpha}{\alpha} \right)^2 + \int_1^\infty f(\alpha, T) \left(\frac{\sin \kappa \alpha}{\alpha} \right)^2 \ll \kappa^2 + 1$$

for T large enough. A similar computation shows that

(2.45)
$$\int_0^\infty \left(\frac{\sin \kappa \alpha}{\alpha} \right)^2 d\mu(\alpha) \ll \kappa^2 + 1.$$

The bound (2.42) now follows from (2.39), (2.44), and (2.45). Now by (2.32), we have

$$\lim_{T \to \infty} R(\kappa, T) = 0$$

for fixed $\kappa \geq 0$. By this, (2.42), and Lebesgue's dominated convergence theorem, we now see that

(2.46)
$$\lim_{T \to \infty} \int_0^D K''(x) R\left(\frac{\pi x}{d}, T \right) dx = 0$$

for fixed $D \geq 0$.

Next we show that the integral over $[D, \infty)$ in (2.41) is small. That is, we show that if $d > 0$ is fixed, then

(2.47)
$$\int_D^\infty K''(x) R\left(\frac{\pi x}{d}, T \right) dx \ll \frac{1}{D}$$

uniformly for $D \geq 1$ and all T sufficiently large. For $D \geq 1$ write

$$\int_D^\infty \int_0^\infty f(\alpha, T) K''(x) \left(\frac{\sin \pi \frac{\alpha}{d} x}{\alpha} \right)^2 d\alpha \, dx$$

$$= \int_D^\infty \int_1^\infty \dots d\alpha \, dx + \int_D^\infty \int_0^1 \dots d\alpha \, dx = J_1 + J_2,$$

say. By (2.35),

$$J_1 \ll \int_D^\infty \int_1^\infty \frac{f(\alpha, T)}{x^3 \alpha^2} d\alpha \, dx = \frac{1}{2D^2} \int_1^\infty \frac{f(\alpha, T)}{\alpha^2} d\alpha.$$

It follows from this and (2.43) that

(2.48)
$$J_1 \ll \frac{1}{D^2}$$

uniformly for large T. To estimate J_2, let $h(\alpha, x) = \left(\sin \pi \frac{\alpha}{d} x / \pi \frac{\alpha}{d} \right)^2$, and observe that

(2.49)
$$h(\alpha, x) \ll x^2 \quad \text{and} \quad h_x(\alpha, x) \stackrel{\text{def}}{=} \frac{\partial h}{\partial x}(\alpha, x) \ll x$$

uniformly for $\alpha \geq 0$. Using this and (2.31), we see that

$$\int_0^1 f(\alpha, T) h(\alpha, x) \, d\alpha \ll x^2 \int_0^1 f(\alpha, T) \, d\alpha \ll x^2 \quad \text{and}$$

$$\int_0^1 f(\alpha, T) h_x(\alpha, x) \, d\alpha \ll x \int_0^1 f(\alpha, T) \, d\alpha \ll x$$

uniformly for $x \geq 0$ and all sufficiently large T. It follows from these estimates, (2.35), and integration by parts that

$$
\begin{aligned}
\left(\frac{d}{\pi}\right)^2 J_2 &= \int_D^\infty \int_0^1 f(\alpha, T) K''(x) h(\alpha, x)\, d\alpha\, dx \\
&= K'(x) \int_0^1 f(\alpha, T) h(\alpha, x)\, d\alpha \Big|_{x=D}^\infty - \int_D^\infty K'(x) \int_0^1 f(\alpha, T) h_x(\alpha, x)\, d\alpha\, dx \\
&\ll \frac{1}{D}.
\end{aligned}
$$

Combining this and (2.48), we now find that

$$
\int_D^\infty \int_0^\infty f(\alpha, T) K''(x) \left(\frac{\sin \pi \frac{\alpha}{d} x}{\alpha} \right)^2 d\alpha\, dx \ll \frac{1}{D}
$$

for all sufficiently large T. A similar computation gives

$$
\int_D^\infty \int_0^\infty K''(x) \left(\frac{\sin \pi \frac{\alpha}{d} x}{\alpha} \right)^2 d\mu(\alpha)\, dx \ll \frac{1}{D}.
$$

Using these and the definition of $R(\kappa, T)$ from (2.39), we obtain (2.47).

By (2.46), (2.47), and (2.41), we obtain (2.40) upon taking D large enough. The formula (2.37) now follows from (2.38) and (2.40).

Having established (2.37), we now complete the proof of the lemma. If $\chi_{[a,b]}$ is the characteristic function of the interval $[a, b]$, then by (2.34) we have

$$
(2.50) \qquad\qquad \chi_{[0,1]}(t) \ \leq \ \hat{K}_\eta(t) \ \leq \ \chi_{[0,1+\eta]}(t)
$$

for $t \geq 0$. Thus

$$
(2.51) \qquad
\begin{aligned}
&\int_0^\infty f(\alpha, T) \hat{K}_\eta \left(\frac{(1+\eta)\alpha}{d} \right) d\alpha \\
&\qquad\qquad \leq \int_0^d f(\alpha, T)\, d\alpha \ \leq \ \int_0^\infty f(\alpha, T) \hat{K}_\eta \left(\frac{\alpha}{d} \right) d\alpha.
\end{aligned}
$$

Using (2.37) to replace the first and third integrals in (2.51), we obtain

$$
(2.52) \quad \int_0^\infty \hat{K}_\eta \left(\frac{(1+\eta)\alpha}{d} \right) d\mu(\alpha) \ + \ o(1) \ \leq \ \int_0^d f(\alpha, T)\, d\alpha \ \leq \ \int_0^\infty \hat{K}_\eta \left(\frac{\alpha}{d} \right) d\mu(\alpha) \ + \ o(1).
$$

By (2.50), we have

$$
\int_0^\infty \hat{K}_\eta \left(\frac{(1+\eta)\alpha}{d} \right) d\mu(\alpha) \geq \int_0^\infty \chi_{[0,1]} \left(\frac{(1+\eta)\alpha}{d} \right) d\mu(\alpha) = \mu[0, d/(1+\eta)]
$$

and

$$
\int_0^\infty \hat{K}_\eta \left(\frac{\alpha}{d} \right) d\mu(\alpha) \leq \int_0^\infty \chi_{[0,1+\eta]} \left(\frac{\alpha}{d} \right) d\mu(\alpha) = \mu[0, d(1+\eta)].
$$

It follows from this and (2.52) that

$$
\mu[0, d/(1+\eta)] \ + \ o(1) \ \leq \ \int_0^d f(\alpha, T)\, d\alpha \ \leq \ \mu[0, d(1+\eta)] \ + \ o(1)
$$

as $T \to \infty$, for all fixed $d > 0$ and $\eta > 0$. Since $\mu[0, \delta] \to \mu[0, d)$ as $\delta \to d^-$ and $\mu[0, \delta] \to \mu[0, d]$ as $\delta \to d^+$, making η small gives (2.33).

It remains to show that the interchange in the order of integration in (2.38) is valid. Write

$$\int_0^\infty \hat{K}\left(\frac{\alpha}{d}\right) d\mu(\alpha) = \int_0^\infty \int_0^\infty K''(x) \left(\frac{\sin \pi \frac{\alpha}{d} x}{\pi \frac{\alpha}{d}}\right)^2 dx \, d\mu(\alpha)$$

$$= \int_1^\infty \int_0^\infty \dots dx \, d\mu(\alpha) + \int_0^1 \int_0^\infty \dots dx \, d\mu(\alpha)$$

$$= I_1 + I_2, \quad \text{say.}$$

By (2.35) and the fact that $|\sin x| \leq 1$ for real x, the double integral I_1 converges absolutely. Thus the order of integration can be interchanged in I_1 by Fubini's theorem. For I_2 we use (2.35), (2.49), integration by parts (twice), and Fubini's theorem (twice) to see that

$$I_2 = \int_0^1 \int_0^\infty K''(x) h(\alpha, x) \, dx \, d\mu(\alpha) = -\int_0^1 \int_0^\infty K'(x) h_x(\alpha, x) \, dx \, d\mu(\alpha)$$

$$= -\int_0^\infty \int_0^1 K'(x) h_x(\alpha, x) \, d\mu(\alpha) \, dx = \int_0^\infty K''(x) \int_0^x \int_0^1 h_x(\alpha, \xi) \, d\mu(\alpha) \, d\xi \, dx$$

$$= \int_0^\infty K''(x) \int_0^1 \int_0^x h_x(\alpha, \xi) \, d\xi \, d\mu(\alpha) \, dx = \int_0^\infty K''(x) \int_0^1 h(\alpha, x) \, d\mu(\alpha) \, dx.$$

A similar computation using (2.43) justifies the interchange in the order of integration on the right-hand side of

$$\int_0^\infty f(\alpha, T) \hat{K}\left(\frac{\alpha}{d}\right) d\alpha = \int_0^\infty \int_0^\infty f(\alpha, T) K''(x) \left(\frac{\sin \pi \frac{\alpha}{d} x}{\pi \frac{\alpha}{d}}\right)^2 dx \, d\alpha.$$

This proves (2.38).

$\square$

3. Proof of Theorem 1.2

By a result of Goldston [10, (2.6)], on RH there exists a T_0 such that

$$(3.1) \qquad \int_c^d F(\alpha, T) \, d\alpha \ll \max\{1, d - c\}$$

uniformly for real $c \leq d$ and $T \geq T_0$. Thus (2.15) holds for $f(\alpha, T) = F(\alpha, T)$ (we can dispense with the absolute value sign because $F(\alpha, T)$ is nonnegative). We will use this fact repeatedly below without mention.

To prove Theorem 1.2 we show that (A) $\Rightarrow$ (B), (B) $\Rightarrow$ (A), (A) $\Rightarrow$ (C), (C) $\Rightarrow$ (A), (A) $\Rightarrow$ (D), and (D) $\Rightarrow$ (A).

First we show that (A) $\Rightarrow$ (B). Suppose that (A) holds. Take $r(x) = r_1(x) = e^{-2bx}$ in Lemma 2.2 to get

$$(3.2) \qquad \lim_{T \to \infty} \int_0^\infty F(\alpha, T) e^{-2b\alpha} \, d\alpha = \int_0^\infty e^{-2b\alpha} \, d\mu(\alpha).$$

By Theorem 1 of Goldston, Gonek, and Montgomery [11], stated in a slightly different form,

$$(3.3) \qquad I(b, T) = \left(\int_0^\infty F(\alpha, T) e^{-2b\alpha} \, d\alpha - \frac{1}{2}\right) (1 + o(1)) T \log^2 T$$

as $T \to \infty$, uniformly for $T^{-1} \log^3 T \leq b \ll 1$. Inserting (3.2) in (3.3), we obtain

$$I(b, T) = \left(\int_0^\infty e^{-2b\alpha} \, d\mu(\alpha) + o(1) - \frac{1}{2}\right) (1 + o(1)) T \log^2 T$$

as $T \to \infty$ for fixed $b > 0$, which is (B).

To prove (B) $\Rightarrow$ (A), suppose that (B) holds. Then combining (B) and (3.3), we see that (3.2) holds for all $b > 0$. Now apply Lemma 2.1 to obtain (A).

To show (A) $\Rightarrow$ (C), we make the change of variable $v = \frac{\log x}{\log T}$ in the definition of $J(\beta, T)$ and see that

$$
\begin{aligned}
J(\beta, T)\frac{T}{\log^2 T} &= \int_0^\beta (\psi(T^v + T^{v-1}) - \psi(T^v) - T^{v-1})^2 \frac{dv}{T^{v-1}\log T} \\
&= \int_0^\beta W_1(v, T)\, dv, \quad \text{say.}
\end{aligned}
\tag{3.4}
$$

A special case of a result of Goldston [10, Theorem 2] is[2]

$$
\int_1^\infty \left(\psi\left(x + \frac{x}{T}\right) - \psi(x) - \frac{x}{T}\right)^2 e^{-b\frac{\log x}{\log T}}\frac{dx}{x^2}
$$
$$
\sim \left(\int_0^\infty F(\alpha, T)e^{-b\alpha}\, d\alpha - \frac{1}{2}\right)\frac{\log^2 T}{T}
$$

as $T \to \infty$, for fixed $b > 0$. From this and our previous change of variable we find that

$$
\int_0^\infty W_1(v, T)e^{-bv}\, dv \sim \int_0^\infty F(\alpha, T)e^{-b\alpha}\, d\alpha - \frac{1}{2}.
\tag{3.5}
$$

Let $\omega(v, T)$ be a nonnegative function such that

$$
\int_0^\beta \omega(v, T)e^{-bv}\, dv \sim \frac{1}{2} \quad \text{as} \quad T \to \infty
\tag{3.6}
$$

for fixed $b \geq 0$, $\beta > 0$ (for instance we can take $\omega(v, T) = T^{-2v}\log T$). Let $W(v, T) = \omega(v, T) + W_1(v, T)$. Then combining (3.5) and (3.6), we obtain

$$
\int_0^\infty W(v, T)e^{-bv}\, dv \sim \int_0^\infty F(\alpha, T)e^{-b\alpha}\, d\alpha.
\tag{3.7}
$$

Now suppose that (A) holds. Take $f(\alpha, T) = F(\alpha, T)$ and $r(\alpha) = r_1(\alpha) = e^{-b\alpha}$ in Lemma 2.2 to see that

$$
\lim_{T \to \infty} \int_0^\infty F(\alpha, T)e^{-b\alpha}\, d\alpha = \int_0^\infty e^{-b\alpha}\, d\mu(\alpha).
\tag{3.8}
$$

It follows from this and (3.7) that

$$
\lim_{T \to \infty} \int_0^\infty W(v, T)e^{-bv}\, dv = \int_0^\infty e^{-b\alpha}\, d\mu(\alpha)
\tag{3.9}
$$

for all $b > 0$. Thus, by Lemma 2.1 with $f(\alpha, T) = W(\alpha, T)$, we have

$$
\mu[0, \beta) \; + \; o(1) \; \leq \int_0^\beta W(v, T)\, dv \leq \; \mu[0, \beta] \; + \; o(1)
\tag{3.10}
$$

as $T \to \infty$ for any fixed $\beta > 0$. Since $W(v, T) = W_1(v, T) + \omega(v, T)$, it follows from (3.10) and (3.6) that

$$
\mu[0, \beta) - \frac{1}{2} \; + \; o(1) \; \leq \int_0^\beta W_1(v, T)\, dv \leq \; \mu[0, \beta] - \frac{1}{2} \; + \; o(1)
$$

as $T \to \infty$ for any fixed $\beta > 0$. From this and (3.4) we obtain (C). This proves (A) $\Rightarrow$ (C).

---

[2]This can also be proved using (3.3) and an explicit formula due to Selberg, as in [11].

Next we prove that (C) $\Rightarrow$ (A). Let $W(v,T)$, $W_1(v,T)$, and $\omega(v,T)$ be as in the previous paragraph. By Theorem 1 of Goldston and Montgomery [12], if RH is true, then

$$\int_1^X \left(\psi\left(x + \frac{x}{T} \right) - \psi(x) - \frac{x}{T} \right)^2 \frac{dx}{x^2} \ll \frac{(\log X)(\log 2T)}{T}$$

for $T \geq 1$ and $X \geq 2$. Thus, letting $\beta \geq 1$ and $X = T^\beta$, we see that

$$J(\beta, T) \ll \frac{\beta(\log T)(\log 2T)}{T}$$

for $T \geq 2$. It follows from this and (3.4) that

$$\int_0^\beta W_1(v,T)\, dv \ll \beta$$

uniformly for $\beta \geq 1$ and $T \geq 2$. Since $W(v,T) = W_1(v,T) + \omega(v,T)$, by (3.6) we have

(3.11) $$\int_0^\beta W(v,T)\, dv \ll \beta \quad \text{uniformly for} \ \ \beta \geq 1 \ \text{ and } \ T \geq 2.$$

Thus (2.15) holds with $f(\alpha,T) = W(\alpha,T)$ (note that $W(\alpha,T)$ is nonnegative). Now assume that (C) is true. Since $W(v,T) = W_1(v,T) + \omega(v,T)$, we see from (3.4), (3.6), and (C) that (3.10) holds as $T \to \infty$ for any fixed $\beta > 0$. By (3.10) and (3.11) the hypotheses of Lemma 2.2 are satisfied with $f(\alpha,T) = W(\alpha,T)$ and $r(\alpha) = r_1(\alpha) = e^{-b\alpha}$. Hence we have (3.9). It now follows from (3.9) and (3.7) that (3.8) holds for all $b > 0$. Applying Lemma 2.1, we finally obtain (A).

To prove (A) $\Rightarrow$ (D), we use the formula (6.14) of [9] (see also (2.3) and (2.5) of [10]), namely

$$\int_{-\infty}^\infty F(\alpha,T) \left(\frac{\sin(\pi\beta\alpha)}{\pi\beta\alpha} \right)^2 d\alpha = \frac{1}{\beta} N^*(T) + \frac{2}{\beta^2} \int_0^\beta N(T,u)\, du + O\left(\frac{\beta(1+\beta)}{\log^2 T} \right).$$

Since $F(\alpha,T) = F(-\alpha,T)$, we can also write this as

(3.12) $$\frac{\beta}{2} N^*(T) + \int_0^\beta N(T,u)\, du = \int_0^\infty F(\alpha,T) \left(\frac{\sin(\pi\beta\alpha)}{\pi\alpha} \right)^2 d\alpha + O\left(\frac{\beta^3(1+\beta)}{\log^2 T} \right).$$

Suppose that (A) holds. By (3.1) and (A) we may use Lemma 2.2 with $f(\alpha) = F(\alpha)$, $r(\alpha) = (\sin(\pi\beta\alpha)/\pi\alpha)^2$ and $r_1(\alpha) = (\pi\alpha)^{-2}$ to obtain

(3.13) $$\lim_{T\to\infty} \int_0^\infty F(\alpha,T) \left(\frac{\sin(\pi\beta\alpha)}{\pi\alpha} \right)^2 d\alpha = \int_0^\infty \left(\frac{\sin(\pi\beta\alpha)}{\pi\alpha} \right)^2 d\mu(\alpha).$$

From this and (3.12) we obtain (D). This proves (A) $\Rightarrow$ (D).

To prove that (D) $\Rightarrow$ (A), suppose that (D) holds. Then by (3.12) we have (3.13). We may therefore apply Lemma 2.3 to obtain (A). This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.3

Recall that $g(\alpha) = |\alpha|$ for $|\alpha| \leq 1$, and is defined for all $\mathbb{R}$ by periodicity. Let ν be the measure on $[0,\infty)$ defined by $\nu(0) = \frac{1}{2}$, $\nu(2n) = 1$ for all integers $n \geq 1$, and $d\nu(\alpha) = g(\alpha)\, d\alpha$ for $2n < \alpha < 2n+2$, $n \geq 0$. When $\mu = \nu$, we can easily evaluate the expressions in (A), (B), and (C) involving μ and see that the statements (A), (B), and (C) are the same as (A2), (B2), and (C2), respectively. Thus (A2), (B2), and (C2) are equivalent statements by Theorem 1.2. However, showing that they are equivalent to (D2) is not as straightforward. To do this, we will define a new statement (D$'$) and show that, when $\mu = \nu$, the statement (D) is equivalent to (D$'$), which in turn is equivalent to (D2).

Lemma 4.1. *Assume RH. Let μ be a positive Borel measure on $[0, \infty)$ for which the function $\alpha \mapsto \min\{1, \alpha^{-2}\}$ is integrable over $[0, \infty)$, and define*

$$\varphi(\beta) = \varphi(\beta, \mu) = \int_0^\infty \left(\frac{\sin \pi\beta\alpha}{\pi\alpha} \right)^2 d\mu(\alpha).$$

If $\varphi(\beta)$ is absolutely continuous on each closed (finite) subinterval of $[0, \infty)$, then (D) is equivalent to

(D′) $\quad \dfrac{1}{2} N^*(T) + N(T, \beta) = \varphi'(\beta) + o(1)$

as $T \to \infty$, for all fixed $\beta > 0$ for which the derivative exists.

Proof. Suppose that $\varphi(\beta)$ is absolutely continuous on any closed subinterval of $[0, \infty)$. Then $\varphi'(\beta)$ exists almost everywhere. Also, $\varphi(\beta) = \int_0^\beta \varphi'(x)\, dx$ for all $\beta \geq 0$. To show that (D) $\Rightarrow$ (D′), assume that (D) holds and suppose that $\beta > 0$ is a real number such that $\varphi'(\beta)$ exists. Since $N(T, u)$ is an increasing function of u for fixed T, we have

$$\frac{1}{h} \int_{\beta-h}^\beta N(T, u)\, du \;\leq\; N(T, \beta) \;\leq\; \frac{1}{h} \int_\beta^{\beta+h} N(T, u)\, du$$

for all small enough $h > 0$. It follows from this and (D) that

$$\frac{\varphi(\beta) - \varphi(\beta - h)}{h} + o(1) \;\leq\; \frac{1}{2} N^*(T) + N(T, \beta) \;\leq\; \frac{\varphi(\beta + h) - \varphi(\beta)}{h} + o(1).$$

Making h small enough, we obtain (D′).

Now suppose that (D′) holds. To show that (D′) $\Rightarrow$ (D), we use the estimate

(4.1) $\quad\quad\quad\quad N(T, \beta) \ll 1 + \beta \quad$ uniformly for $\quad \beta > 0 \quad$ and large $\quad T$,

which follows from Lemma 9 of [12]. By (4.1) and (D′), if $\beta > 0$ is such that $\varphi'(\beta)$ exists, then

$$N^*(T) \ll 1 + \beta + |\varphi'(\beta)|.$$

Hence, since the left-hand side does not depend on β, we have

$$N^*(T) \ll 1.$$

This and (4.1) gives

$$\frac{N^*(T)}{2} + N(T, u) \ll 1 + u \quad \text{uniformly for} \quad u > 0 \quad \text{and large} \quad T.$$

Thus, by Lebesgue's dominated convergence theorem and (D′) we have

$$\lim_{T\to\infty} \int_0^\beta \left(\frac{N^*(T)}{2} + N(T, u) \right) du = \int_0^\beta \varphi'(u)\, du = \varphi(\beta),$$

which is (D). This proves that (D′) $\Rightarrow$ (D), thereby completing the proof of the lemma. $\qquad\square$

To show that Lemma 4.1 is applicable when $\mu = \nu$, we need to show that $\varphi(\beta, \nu)$ is absolutely continuous. We do this by explicitly calculating $\varphi(\beta, \nu)$ and computing its derivative. By the definition of ν (see the beginning of this section), we have

(4.2) $\quad \varphi(\beta, \nu) = \displaystyle\int_0^\infty \left(\frac{\sin \pi\beta\alpha}{\pi\alpha} \right)^2 d\nu(\alpha) = \frac{\beta^2}{2} + \sum_{n=1}^\infty \left(\frac{\sin 2\pi\beta n}{2\pi n} \right)^2 + \int_0^\infty g(\alpha) \left(\frac{\sin \pi\beta\alpha}{\pi\alpha} \right)^2 d\alpha.$

To evaluate the sum on the right-hand side of (4.2) we integrate the Fourier series expansion of $x - [x] - \frac{1}{2}$ term-by-term to see that

$$(4.3) \qquad \frac{(y - [y]) - (y - [y])^2}{2} = \sum_{n=1}^{\infty} \left(\frac{\sin \pi n y}{\pi n} \right)^2$$

for all real y. (This can also be obtained by directly computing the Fourier series expansion of the left-hand side of (4.3).) It follows from (4.3) that the sum on the right-hand side of (4.2) equals

$$(4.4) \qquad \sum_{n=1}^{\infty} \left(\frac{\sin 2\pi\beta n}{2\pi n} \right)^2 = \frac{(2\beta - [2\beta]) - (2\beta - [2\beta])^2}{8}.$$

To evaluate the integral on the right-hand side of (4.2), we expand g as a Fourier series:

$$(4.5) \qquad g(\alpha) = \sum_{n=-\infty}^{\infty} \frac{1}{2} \left(\frac{1 - e^{-\pi i n}}{\pi i n} \right)^2 e^{\pi i n \alpha}.$$

Use this and the Fourier transform pair

$$(4.6) \qquad r(\alpha) = \left(\frac{\sin(\pi\beta\alpha)}{\pi\beta\alpha} \right)^2 \quad \text{and} \quad \hat{r}(u) = \frac{1}{\beta} \max\left\{ 0, 1 - \frac{|u|}{\beta} \right\}$$

to write

$$(4.7) \qquad \begin{aligned} \int_{-\infty}^{\infty} g(\alpha) \left(\frac{\sin(\pi\beta\alpha)}{\pi\beta\alpha} \right)^2 d\alpha &= \sum_{n=-\infty}^{\infty} \frac{1}{2} \left(\frac{1 - e^{-\pi i n}}{\pi i n} \right)^2 \int_{-\infty}^{\infty} \left(\frac{\sin(\pi\beta\alpha)}{\pi\beta\alpha} \right)^2 e^{\pi i n \alpha} d\alpha \\ &= \frac{1}{2\beta} - \frac{1}{\pi^2\beta} \sum_{1 \leq n \leq 2\beta} \left(\frac{1 - (-1)^n}{n} \right)^2 \left(1 - \frac{n}{2\beta} \right). \end{aligned}$$

The interchange in order of summation is justified by absolute convergence. Inserting (4.4) and (4.7) in (4.2), we obtain

$$(4.8) \qquad \begin{aligned} \varphi(\beta, \nu) = {} & \frac{\beta}{4} + \frac{\beta^2}{2} + \frac{(2\beta - [2\beta]) - (2\beta - [2\beta])^2}{8} \\ & - \frac{1}{2\pi^2} \sum_{0 < n < 2\beta} \left(\frac{1 - (-1)^n}{n} \right)^2 \left(\beta - \frac{n}{2} \right). \end{aligned}$$

We see from this that $\varphi(\beta, \nu)$ is differentiable at each $\beta > 0$ that is not a half-integer. Thus $\varphi(\beta, \nu)$ is absolutely continuous on every closed subinterval of $[0, \infty)$.

It now follows that we may apply Lemma 4.1 with $\mu = \nu$. Using (4.8), we obtain

Lemma 4.2. *Assume RH. Denote $[\beta]$ by m. When $\mu = \nu$, the statement (D) is equivalent to*

$$(4.9) \qquad \frac{1}{2} N^*(T) + N(T, \beta) \sim \begin{cases} m + \dfrac{1}{2} - \dfrac{1}{2\pi^2} \sum_{n=0}^{m-1} \dfrac{4}{(2n+1)^2} & \text{if} \quad m < \beta < m + \frac{1}{2}, \\[2em] m + 1 - \dfrac{1}{2\pi^2} \sum_{n=0}^{m} \dfrac{4}{(2n+1)^2} & \text{if} \quad m + \frac{1}{2} < \beta < m + 1 \end{cases}$$

as $T \to \infty$, for all fixed $\beta > 0$ not a half-integer.

To complete the proof of Theorem 1.3, we need to show that (4.9) is equivalent to (D2). Observe that if both $N^*(T) \sim 1$ and (4.9) are true, then (D2) holds. Similarly, if both $N^*(T) \sim 1$ and (D2) hold, then (4.9) is true. Thus, to prove that (4.9) is equivalent to (D2), it suffices to show that each of them implies $N^*(T) \sim 1$.

First we prove that (4.9) implies $N^*(T) \sim 1$. Let $N(T)$ denote the number of zeros of $\zeta(s)$ with $0 < \gamma \le T$, counting multiplicity. If $\lambda > 0$, then by the definition of $N^*(T)$, the definition of $F(\alpha, T)$, and the Fourier transform pair (4.6), we have

(4.10)

$$
\left(\frac{T}{2\pi} \log T \right)^{-1} N(T) \le N^*(T) \le \left(\frac{T}{2\pi} \log T \right)^{-1} \sum_{0 < \gamma, \gamma' \le T} \left(\frac{\sin \frac{\lambda}{2}(\gamma - \gamma') \log T}{\frac{\lambda}{2}(\gamma - \gamma') \log T} \right)^2 w(\gamma - \gamma')
$$
$$
= \int_0^\infty F(\alpha, T) \cdot \frac{2}{\lambda} \max \left\{ 0, 1 - \frac{\alpha}{\lambda} \right\} d\alpha.
$$

Now suppose that (4.9) is true. Then by Lemma 4.2, the statement (D) holds with $\mu = \nu$. Hence, by Theorem 1.2, the statement (A) is true with $\mu = \nu$. It follows from this and Lemma 2.2 with $f(\alpha, T) = F(\alpha, T)$, $\mu = \nu$, $r(\alpha) = \frac{2}{\lambda} \max \left\{ 0, 1 - \frac{\alpha}{\lambda} \right\}$, and $r_1(\alpha) = 0$, that

(4.11) $$ \lim_{T \to \infty} \int_0^\infty F(\alpha, T) \cdot \frac{2}{\lambda} \max \left\{ 0, 1 - \frac{\alpha}{\lambda} \right\} d\alpha = \int_0^\infty \frac{2}{\lambda} \max \left\{ 0, 1 - \frac{\alpha}{\lambda} \right\} d\nu(\alpha). $$

By a straightforward calculation using the definition of ν, we may write the right-hand side of (4.11) as

(4.12)

$$
\int_0^\infty \frac{2}{\lambda} \max \left\{ 0, 1 - \frac{\alpha}{\lambda} \right\} d\nu(\alpha) =
$$
$$
= \begin{cases} \dfrac{\vartheta}{\lambda^2} + \dfrac{\vartheta^3}{3\lambda^2} + \displaystyle\sum_{n=0}^{[\lambda/2]-1} \left(\dfrac{4}{\lambda} - \dfrac{8n+4}{\lambda^2} \right) & \text{if} \quad 0 \le \vartheta \le 1, \\[2em] -\dfrac{\vartheta}{\lambda^2} + \dfrac{2\vartheta^2}{\lambda^2} - \dfrac{\vartheta^3}{3\lambda^2} + \dfrac{2}{3\lambda^2} + \displaystyle\sum_{n=0}^{[\lambda/2]-1} \left(\dfrac{4}{\lambda} - \dfrac{8n+4}{\lambda^2} \right) & \text{if} \quad 1 \le \vartheta < 2, \end{cases}
$$

where $\vartheta = 2 \left(\frac{\lambda}{2} - \left[\frac{\lambda}{2} \right] \right)$. Using the fact that the sum of the first m positive integers is $m(m+1)/2$ and the fact that ϑ is bounded, we easily see that the limit of (4.12) as $\lambda \to \infty$ is equal to 1. It follows from this and (4.11) that making λ large enough in (4.10) gives

$$
\left(\frac{T}{2\pi} \log T \right)^{-1} N(T) \le N^*(T) \le 1 + o(1)
$$

as $T \to \infty$. From this and the fact that

(4.13) $$ N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T) $$

(see [19], Theorem 9.4), we obtain $N^*(T) \sim 1$ as $T \to \infty$. This proves that (4.9) implies $N^*(T) \sim 1$.

Next we show that (D2) implies $N^*(T) \sim 1$. First split the integral

$$
\int_0^\infty F(\alpha, T) \left(\frac{\sin \pi \beta \alpha}{\pi \alpha} \right)^2 d\alpha
$$

into two integrals, one over $[0, 1]$, and the other over $[1, \infty]$. For the integral over $[1, \infty]$ we use (3.1) and integration by parts, as in (2.43), to see that

$$\int_1^\infty F(\alpha, T) \left(\frac{\sin \pi \beta \alpha}{\pi \alpha} \right)^2 d\alpha \ll \int_1^\infty F(\alpha, T) \frac{d\alpha}{\alpha^2} \ll 1.$$

For the integral over $[0, 1]$ we use (1.1) to see that

$$\int_0^1 F(\alpha, T) \left(\frac{\sin \pi \beta \alpha}{\pi \alpha} \right)^2 d\alpha \sim \frac{\beta^2}{2} + \int_0^1 \alpha \left(\frac{\sin \pi \beta \alpha}{\pi \alpha} \right)^2 d\alpha.$$

If $\beta \geq 1$, then we may write the latter integral as

$$\int_0^1 \alpha \left(\frac{\sin \pi \beta \alpha}{\pi \alpha} \right)^2 d\alpha = \int_0^{1/\beta} + \int_{1/\beta}^1 = \int_0^1 u \left(\frac{\sin \pi u}{\pi u} \right)^2 du + \int_{1/\beta}^1$$

$$\leq O(1) + \frac{1}{\pi^2} \int_{1/\beta}^1 \frac{d\alpha}{\alpha} = \frac{\log \beta}{\pi^2} + O(1).$$

Hence for a fixed $\beta \geq 2$, say, we have

$$(4.14) \qquad \int_0^\infty F(\alpha, T) \left(\frac{\sin \pi \beta \alpha}{\pi \alpha} \right)^2 d\alpha = \frac{1}{2} \beta^2 + O(\log \beta) + o(1).$$

as $T \to \infty$. The implied constant in the $O(\log \beta)$ term is absolute. We insert (4.14) into (3.12) and multiply through by $2/\beta$ to obtain

$$(4.15) \qquad N^*(T) + \frac{2}{\beta} \int_0^\beta N(T, u) \, du = \beta + O\left(\frac{\log \beta}{\beta} \right) + o(1)$$

as $T \to \infty$ for all fixed $\beta \geq 2$. Now suppose that (D2) holds. By (4.8), the right-hand side of (D2) is equal to the derivative of $\varphi(\beta, \nu) - \beta/2$. From this fact, (4.1), (D2), and Lebesgue's dominated convergence theorem we have

$$\int_0^\beta N(T, u) \, du \sim -\frac{\beta}{4} + \frac{\beta^2}{2} + \frac{(2\beta - [2\beta]) - (2\beta - [2\beta])^2}{8}$$

$$- \frac{1}{2\pi^2} \sum_{0 < n < 2\beta} \left(\frac{1 - (-1)^n}{n} \right)^2 \left(\beta - \frac{n}{2} \right).$$

Inserting this into (4.15), we obtain

$$(4.16) \qquad \begin{aligned} N^*(T) &\sim \frac{1}{2} - \frac{(2\beta - [2\beta]) - (2\beta - [2\beta])^2}{4\beta} \\ &\quad + \frac{1}{\pi^2} \sum_{0 < n < 2\beta} \left(\frac{1 - (-1)^n}{n} \right)^2 \left(1 - \frac{n}{2\beta} \right) + O\left(\frac{\log \beta}{\beta} \right) \end{aligned}$$

as $T \to \infty$ for all fixed $\beta \geq 2$. By the monotone convergence theorem and the special case $\alpha = 0$ of (4.5), we have

$$\lim_{\beta \to \infty} \frac{1}{\pi^2} \sum_{0 < n < 2\beta} \left(\frac{1 - (-1)^n}{n} \right)^2 \left(1 - \frac{n}{2\beta} \right)$$

$$= \frac{1}{\pi^2} \sum_{n=1}^\infty \left(\frac{1 - (-1)^n}{n} \right)^2 = \frac{1}{2} - g(0) = \frac{1}{2}.$$

We now see that $N^*(T) \sim 1$ on taking β large enough in (4.16). This proves that (D2) implies $N^*(T) \sim 1$. The proof of Theorem 1.3 is now complete.

Remark 1: As we mentioned in Section 1, Theorem 1.2 may be used to prove Theorem 1.1 by taking μ to be the measure defined by $\mu(0) = 1/2$, $d\mu(\alpha) = \alpha d\alpha$ for $0 < \alpha \leq 1$, and $d\mu(\alpha) = d\alpha$ for $\alpha > 1$. The process in doing this is similar to the one in this section.

Remark 2: A stronger conclusion holds in Lemma 2.2 in the special case when $r(\alpha) = (\sin \beta\alpha/\alpha)^2$ and $r_1(\alpha) = \alpha^{-2}$. Namely, the conclusion (2.17) of Lemma 2.2 holds uniformly for all β in any fixed closed interval. It follows from this and our proof of Theorem 1.2 that if (A) holds, then (D) holds uniformly for all β in any fixed, closed interval. Therefore, by our proof of Lemma 4.1, if (A) holds, then (D′) holds uniformly for all β in any fixed, closed subinterval of an open interval on which $\varphi'(\beta)$ exists and is continuous. Hence, by our proof of Theorem 1.3, if (D2) holds, then it holds uniformly for all β in any fixed closed subinterval of $[0, \infty)$ not containing half-integers. Similarly, if (D1) holds, then it holds uniformly on any fixed closed interval. This uniformity in (D1) was first observed by Gallagher and Mueller [8].

5. The Alternative Hypothesis

In this section we prove Proposition 5.1, which states that the Alternative Hypothesis and the assumption that $N^*(T) \sim 1$ imply (A2). We also prove Theorem 1.4. First, we need the following lemma.

Lemma 5.1. *Suppose the Alternative Hypothesis is true. There is a function $\Psi(T)$ such that $\Psi(T) \to \infty$ as $T \to \infty$, $\Psi(T) = o(\log T)$, and such that the zeros of $\zeta(s)$ have the following property: if γ and γ' are ordinates of zeros for which*

$$(5.1) \qquad \frac{T}{\log^2 T} < \gamma, \gamma' \leq T \qquad and \qquad \left| \frac{\gamma - \gamma'}{2\pi} \log T \right| \leq M,$$

then there is an integer k such that

$$(\gamma - \gamma') \log T \; = \; \pi k + O\left(|\gamma - \gamma'| \left(\Psi(T) + \frac{M \log T}{T} + \log\log T \right) \right).$$

Proof. Recall the definition of ψ from the statement of the Alternative Hypothesis. Let $\Psi(T) = \max\{\psi(\gamma) \; : \; T(\log T)^{-2} < \gamma \leq T\}$. The facts that $\Psi(T) \to \infty$ as $T \to \infty$ and $\Psi(T) = o(\log T)$ follow from the same properties of ψ. Let γ and γ' be as in the hypothesis of the Lemma. Suppose, without loss of generality, that $\gamma' = \gamma_n$ and $\gamma = \gamma_m$ with $m \geq n$. From (1.4) and the definition of $\Psi(T)$, it follows that

$$
\begin{aligned}
\widetilde{\gamma} - \widetilde{\gamma}' &= \sum_{\ell=n}^{m-1} (\widetilde{\gamma}_{\ell+1} - \widetilde{\gamma}_\ell) \\
(5.2) \qquad &= \sum_{\ell=n}^{m-1} \left(\frac{k_\ell}{2} + O((\gamma_{\ell+1} - \gamma_\ell)\psi(\gamma_\ell)) \right) \\
&= \frac{k}{2} \; + \; O(|\gamma - \gamma'|\Psi(T)),
\end{aligned}
$$

where $k = \sum_{\ell=n}^{m-1} k_\ell$. Now by (5.1), we have

$$\gamma \log \frac{\gamma}{\gamma'} \leq \gamma \left(\frac{\gamma - \gamma'}{\gamma'} \right) = \left(1 + \frac{\gamma - \gamma'}{\gamma'} \right) (\gamma - \gamma') \leq \left(1 + \frac{2\pi M \log T}{T} \right) (\gamma - \gamma').$$

Hence

$$\widetilde{\gamma} - \widetilde{\gamma}' = \frac{1}{2\pi}(\gamma - \gamma') \log \gamma' + \frac{\gamma}{2\pi} \log \frac{\gamma}{\gamma'}$$

$$= \frac{1}{2\pi}(\gamma - \gamma')(\log T + O(\log \log T)) + O\left(\left(1 + \frac{2\pi M \log T}{T}\right)(\gamma - \gamma')\right).$$

This and (5.2) complete the proof of the Lemma. $\qquad\square$

We will prove Proposition 5.1 by first showing that if ν is the measure defined at the beginning of Section 4, then

$$(5.3) \qquad \int_0^d \int_0^t F(\alpha, T) \, d\alpha \, dt \;=\; \int_0^d \nu[0, t] \, dt \;+\; o(1)$$

as $T \to \infty$ for any fixed $d > 0$. To prove this, we use the assumption that $N^*(T) \sim 1$ to handle the "diagonal terms" that have $\gamma = \gamma'$ in the definition of $F(\alpha, T)$. For the "off-diagonal terms" with $\gamma \neq \gamma'$, we use the Alternative Hypothesis, as follows.

Lemma 5.2. *Assume RH. Define*

$$G(\alpha, T) = \left(\frac{T}{2\pi} \log T\right)^{-1} \sum_{0 < \gamma' < \gamma \leq T} T^{i\alpha(\gamma - \gamma')} w(\gamma - \gamma').$$

If the Alternative Hypothesis is true, then

$$\int_{c+2}^{d+2} \int_0^t G(\alpha, T) \, d\alpha \, dt \;=\; \int_c^d \int_0^t G(\alpha, T) \, d\alpha \, dt \;+\; o(1)$$

as $T \to \infty$, for all fixed $c \leq d$.

Proof. By a change of variable, the conclusion of the Lemma is equivalent to

$$\int_c^d \int_t^{t+2} G(\alpha, T) \, d\alpha \, dt \;=\; o(1)$$

as $T \to \infty$, for all fixed $c \leq d$. To prove this formula, we integrate the definition of $G(\alpha, T)$ term-by-term to see that

$$\int_c^d \int_t^{t+2} G(\alpha, T) \, d\alpha \, dt$$

$$= \left(\frac{T}{2\pi} \log T\right)^{-1} \sum_{0 < \gamma' < \gamma \leq T} \frac{T^{i(d+2)(\gamma - \gamma')} - T^{i(c+2)(\gamma - \gamma')} - T^{id(\gamma - \gamma')} + T^{ic(\gamma - \gamma')}}{(i(\gamma - \gamma') \log T)^2} w(\gamma - \gamma').$$

Let $M > 1$ and write the above equation as

$$(5.4) \qquad \int_c^d \int_t^{t+2} G(\alpha, T) \, d\alpha \, dt \;=\; Z_0 + Z_1 + Z_2,$$

where Z_0 is the sum of the terms with $\gamma' \leq T(\log T)^{-2}$, Z_1 is the sum of those that satisfy the conditions (5.1), and Z_2 is the sum of the remaining terms.

To bound Z_0, observe that

$$\frac{T^{i(d+2)(\gamma - \gamma')} - T^{i(c+2)(\gamma - \gamma')} - T^{id(\gamma - \gamma')} + T^{ic(\gamma - \gamma')}}{(i(\gamma - \gamma') \log T)^2} \;=\; \int_c^d \int_t^{t+2} T^{i\alpha(\gamma - \gamma')} \, d\alpha \, dt,$$

which has absolute value at most $2(d-c)$ since $|T^{i\alpha(\gamma-\gamma')}| = 1$. Thus

$$(5.5) \qquad Z_0 \;\ll\; \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{\gamma' \le \frac{T}{\log^2 T}} \sum_{0 < \gamma \le T} w(\gamma - \gamma').$$

Using

$$(5.6) \qquad N(T+1) - N(T) \ll \log T$$

(see [19], Theorem 9.2), we see that

$$\sum_{0 < \gamma \le T} w(\gamma - t) \;\ll\; \log T$$

uniformly for all real t. From this, (4.13), and (5.5), it follows that

$$(5.7) \qquad Z_0 \;\ll\; \frac{1}{\log T}.$$

To estimate Z_1, let γ and γ' satisfy the conditions (5.1) and let α be a real number. By Lemma 5.1, we can write

$$
\begin{aligned}
T^{i(\alpha+2)(\gamma-\gamma')} - T^{i\alpha(\gamma-\gamma')} &= T^{i\alpha(\gamma-\gamma')}\Big(\exp(i2(\gamma-\gamma')\log T) - 1\Big) \\
&= T^{i\alpha(\gamma-\gamma')}\Big(\exp(i2\pi k + i\mathcal{E}) - 1\Big) \\
&= T^{i\alpha(\gamma-\gamma')}\Big(\exp(i\mathcal{E}) - 1\Big),
\end{aligned}
$$

where $\mathcal{E} = \mathcal{E}(\gamma, \gamma', T)$ is a real number such that

$$\mathcal{E} \;=\; O\left(|\gamma - \gamma'|\left(\Psi(T) + \frac{M\log T}{T} + \log\log T\right)\right).$$

From the identity

$$\exp(i\mathcal{E}) - 1 \;=\; i\int_0^{\mathcal{E}} e^{i\theta}\, d\theta$$

and the fact that $|e^{i\theta}| = 1$, we see that

$$\exp(i\mathcal{E}) - 1 \;\ll\; |\gamma - \gamma'|\left(\Psi(T) + \frac{M\log T}{T} + \log\log T\right).$$

Therefore

$$T^{i(\alpha+2)(\gamma-\gamma')} - T^{i\alpha(\gamma-\gamma')} \;\ll\; |\gamma - \gamma'|\left(\Psi(T) + \frac{M\log T}{T} + \log\log T\right).$$

We now express each term in Z_1 as an integral, and then insert the above estimate, as follows. Write

$$
\begin{aligned}
Z_1 &= \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{\substack{\frac{T}{\log^2 T} < \gamma' < \gamma \le T \\ 0 < \left|\frac{\gamma-\gamma'}{2\pi}\log T\right| \le M}} \int_c^d \frac{T^{i(t+2)(\gamma-\gamma')} - T^{it(\gamma-\gamma')}}{i(\gamma-\gamma')\log T}\, dt\; w(\gamma-\gamma') \\
&\ll \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{\substack{\frac{T}{\log^2 T} < \gamma,\gamma' \le T \\ 0 < \left|\frac{\gamma-\gamma'}{2\pi}\log T\right| \le M}} \left(\frac{\Psi(T)}{\log T} + \frac{M}{T} + \frac{\log\log T}{\log T}\right)\, w(\gamma-\gamma').
\end{aligned}
$$

Thus, since $w(\gamma - \gamma') \leq 1$, we see that

$$Z_1 \ll N(T, M) \left(\frac{\Psi(T)}{\log T} + \frac{M}{T} + \frac{\log \log T}{\log T} \right).$$

From this, (4.1), and the fact that $M > 1$, it follows that

(5.8)
$$Z_1 \ll \frac{M \Psi(T)}{\log T} + \frac{M^2}{T} + \frac{M \log \log T}{\log T}.$$

To bound Z_2, we use the trivial estimate

$$\left| T^{i(d+2)(\gamma - \gamma')} - T^{i(c+2)(\gamma - \gamma')} - T^{id(\gamma - \gamma')} + T^{ic(\gamma - \gamma')} \right| \leq 4$$

to write

$$
\begin{aligned}
Z_2 &\ll \left(\frac{T}{2\pi} \log T \right)^{-1} \sum_{\substack{\frac{T}{\log^2 T} < \gamma, \gamma' \leq T \\ M < \left| \frac{\gamma - \gamma'}{2\pi} \log T \right|}} \frac{1}{((\gamma - \gamma') \log T)^2} \\
&= \left(\frac{T}{2\pi} \log T \right)^{-1} \sum_{\ell=0}^{\infty} \sum_{\substack{\frac{T}{\log^2 T} < \gamma, \gamma' \leq T \\ 2^\ell M < \left| \frac{\gamma - \gamma'}{2\pi} \log T \right| \leq 2^{\ell+1} M}} \frac{1}{((\gamma - \gamma') \log T)^2} \\
&\ll \left(\frac{T}{2\pi} \log T \right)^{-1} \sum_{\ell=0}^{\infty} \sum_{\substack{0 < \gamma, \gamma' \leq T \\ 0 < \left| \frac{\gamma - \gamma'}{2\pi} \log T \right| \leq 2^{\ell+1} M}} \frac{1}{(2^\ell M)^2} \\
&= \sum_{\ell=0}^{\infty} \frac{2 N(T, 2^{\ell+1} M)}{(2^\ell M)^2}.
\end{aligned}
$$

Therefore, by (4.1), we obtain

(5.9)
$$Z_2 \ll \sum_{\ell=0}^{\infty} \frac{2^{\ell+1} M}{(2^\ell M)^2} \ll \frac{1}{M}.$$

It now follows from (5.4), (5.7), (5.8), and (5.9) that

$$\int_c^d \int_t^{t+2} G(\alpha, T) \, d\alpha \, dt \ll \frac{M \Psi(T)}{\log T} + \frac{M^2}{T} + \frac{M \log \log T}{\log T} + \frac{1}{M}.$$

We end the proof of the Lemma upon making M large and taking $T \to \infty$. $\qquad \square$

Now we can prove the claim we made at the beginning of this section.

Proposition 5.1. *Assume RH and suppose that $N^*(T) \sim 1$. If the Alternative Hypothesis is true, then (A2) holds.*

Proof. We first prove (5.3). If $0 \leq d \leq 1$, then we integrate (1.1) twice to see that

(5.10)
$$\int_0^d \int_0^t F(\alpha, T) \, d\alpha \, dt = \frac{d}{2} + \frac{d^3}{6} + o(1).$$

From the definition of $F(\alpha, T)$ in Section 1 and the definition of $G(\alpha, T)$ in Lemma 5.2, we have

(5.11)
$$F(\alpha, T) = N^*(T) + G(\alpha, T) + \overline{G(\alpha, T)},$$

where $\bar{z}$ denotes the complex conjugate of z. Inserting (5.11) in (5.10) and using our assumption that $N^*(T) \sim 1$, we obtain

$$(5.12) \qquad \int_0^d \int_0^t (G + \overline{G})(\alpha, T) \, d\alpha \, dt = \frac{d}{2} + \frac{d^3}{6} - \frac{d^2}{2} + o(1)$$

as $T \to \infty$ for $0 \le d \le 1$. Now let $d > 1$ and set $m = [(d-1)/2]$. Write

$$(5.13) \qquad \int_0^d \left(\int_0^t G + \overline{G} \, d\alpha \right) dt = \int_0^1 + \int_1^3 + \int_3^5 + \cdots + \int_{2m+1}^d .$$

We use Lemma 5.2 repeatedly on the right-hand side of (5.13) to obtain

$$\int_0^d \left(\int_0^t G + \overline{G} \, d\alpha \right) dt = \int_0^1 + \int_{-1}^1 + \int_{-1}^1 + \cdots + \int_{-1}^{d-2m-2} + o(1).$$

Since $G + \overline{G}$ is an even function of α, we have $\int_{-1}^1 = 0$ and

$$\int_0^1 + \int_{-1}^{d-2m-2} = \int_0^1 + \int_{-1}^0 + \int_0^{d-2m-2} = \int_0^{|d-2m-2|} .$$

Thus

$$\int_0^d \left(\int_0^t G + \overline{G} \, d\alpha \right) dt = \int_0^{|d-2m-2|} \left(\int_0^t G + \overline{G} \, d\alpha \right) dt + o(1).$$

Since $0 \le |d - 2m - 2| \le 1$, it follows from this and (5.12) that

$$\int_0^d \left(\int_0^t G + \overline{G} \, d\alpha \right) dt = \frac{|d - 2m - 2|}{2} + \frac{|d - 2m - 2|^3}{6} - \frac{|d - 2m - 2|^2}{2} + o(1).$$

By this, (5.11), and our assumption that $N^*(T) \sim 1$, we have

$$(5.14) \qquad \begin{aligned} &\int_0^d \int_0^t F(\alpha, T) \, d\alpha \, dt \\ &\qquad = \frac{d^2}{2} + \frac{|d - 2m - 2|}{2} + \frac{|d - 2m - 2|^3}{6} - \frac{|d - 2m - 2|^2}{2} + o(1) \end{aligned}$$

as $T \to \infty$, for all fixed $d > 1$. Notice that (5.14) reduces to (5.10) when $0 \le d \le 1$. Thus (5.14) is true for all fixed $d \ge 0$. By a straightforward computation using the definition of ν, we see that the right-hand side of (5.14) is equal to $\int_0^d \nu[0, t] \, dt + o(1)$. This proves (5.3).

Now we complete the proof of the proposition. Since $F(\alpha, T)$ is nonnegative, we may write

$$\frac{1}{h} \int_{d-h}^d \left(\int_0^t F \, d\alpha \right) dt \le \int_0^d F(\alpha, T) \, d\alpha \le \frac{1}{h} \int_d^{d+h} \left(\int_0^t F \, d\alpha \right) dt$$

for $d > 0$ and small enough $h > 0$. Thus by (5.3) we have

$$\frac{1}{h} \int_{d-h}^d \nu[0, t] \, dt + o(1) \le \int_0^d F(\alpha, T) \, d\alpha \le \frac{1}{h} \int_d^{d+h} \nu[0, t] \, dt + o(1)$$

as $T \to \infty$ for fixed $d > 0$ and fixed small $h > 0$. By making h small enough, we obtain (A) with $\mu = \nu$, that is, we obtain (A2). $\square$

Before we prove Theorem 1.4, we need to make a few definitions. Recall from Section 1 that we used $0 < \gamma_1 \le \gamma_2 \le \cdots$ to denote the sequence of ordinates of all the zeros above the real line, counting multiplicity. We also defined $B_{k/2}(T)$ as the set of $\gamma_n \le T$ such that $k/2$ is closest among all half-integers to $\widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n$, and we wrote

$$p_{k/2} = p_{k/2}(T) \stackrel{\text{def}}{=} \left(\frac{T}{2\pi} \log T \right)^{-1} |B_{k/2}(T)|.$$

It is possible that γ_n belongs to two of the sets $B_{k/2}(T)$; this happens when $\widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n$ equals the midpoint between two consecutive half-integers. Thus, for convenience, we will deal instead with the sets

$$B'_{k/2}(T) \stackrel{\text{def}}{=} \{\gamma_n \leq T : \tfrac{k}{2} - \tfrac{1}{4} \leq \widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n < \tfrac{k}{2} + \tfrac{1}{4}\}, \qquad k = 0, 1, 2, \ldots,$$

which are pairwise disjoint. If γ_n is in $B'_{k/2}(T)$, then, without loss of generality, we may take $k_n = k$ in (1.4). With this convention, we can write

(5.15) $$B'_{k/2}(T) = \{\gamma_n \leq T : k_n = k\}.$$

Note that $B'_{k/2}(T)$ is a subset of $B_{k/2}(T)$, and an ordinate γ_n that is in $B_{k/2}(T)$ is not in $B'_{k/2}(T)$ if and only if $\widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n = \tfrac{k}{2} + \tfrac{1}{4}$. Thus if γ_n is in $B_{k/2}(T)$ and not in $B'_{k/2}(T)$, then $k_n = k + 1$ and so by (1.4) we have

(5.16) $$\frac{1}{4} = |\widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n - \frac{k_n}{2}| \ll (\gamma_{n+1} - \gamma_n)\psi(\gamma_n).$$

We will use this and the following Lemma to show that $B'_{k/2}(T)$ has essentially the same size as $B_{k/2}(T)$.

Lemma 5.3. *Assume RH and suppose the Alternative Hypothesis is true. If $\delta > 0$ is fixed, then*

$$\#\{0 < \gamma_n \leq T : (\gamma_{n+1} - \gamma_n)\psi(\gamma_n) \geq \delta\} = o(T \log T).$$

Proof. Recall the properties of the function $\psi(\gamma)$ from the statement of the Alternative Hypothesis. Define

(5.17) $$\Psi_0(T) = \max_{\gamma_n \leq T} \psi(\gamma_n).$$

It is immediate from the properties of $\psi(\gamma)$ that $\Psi_0(T) \to \infty$ and $\Psi_0(T) = o(\log T)$ as $T \to \infty$. It follows from (5.17) that

(5.18) $$\#\{0 < \gamma_n \leq T : (\gamma_{n+1} - \gamma_n)\psi(\gamma_n) \geq \delta\} \leq \#\{0 < \gamma_n \leq T : \gamma_{n+1} - \gamma_n \geq \lambda/\log T\},$$

where $\lambda = \delta \log T/\Psi_0(T)$. By a result of Fujii (see §9.25 of [19]), we have

(5.19) $$\#\{0 < \gamma_n \leq T : \gamma_{n+1} - \gamma_n \geq \lambda/\log T\} \ll N(T)\exp(-A\lambda^{1/2}(\log \lambda)^{-1/4}).$$

The Lemma now follows from (5.18), (5.19), (4.13), and the fact that $\lambda \to \infty$ as $T \to \infty$. $\qquad\square$

By (5.16) and Lemma 5.3, we see that there are at most $o(T \log T)$ ordinates γ_n that are in $B_{k/2}(T)$ and not in $B'_{k/2}(T)$. Hence

(5.20) $$p_{k/2} = p'_{k/2} + o(1)$$

as $T \to \infty$.

We now prove Theorem 1.4. By (1.4), if $k_n = 0$ then

$$(\gamma_{n+1} - \gamma_n)\frac{\log \gamma_n}{2\pi} \leq \widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n \ll (\gamma_{n+1} - \gamma_n)\psi(\gamma_n).$$

This implies that $\log \gamma_n \ll \psi(\gamma_n)$ since $\gamma_{n+1} - \gamma_n > 0$ by our hypothesis that all the zeros are simple. Since $\psi(\gamma) = o(\log \gamma)$, it follows that $k_n = 0$ for at most finitely many n. Thus

(5.21) $$|B'_0(T)| = O(1),$$

and so $p_0 = o(1)$ as $T \to \infty$ by (5.20).

We next compute upper bounds for $p_{k/2}$ when $k \geq 1$. By (4.13) there exists a positive constant Q for which

(5.22) $$N(T + Q) - N(T) > 0$$

for all $T > 0$. Let $\delta > 0$ be small enough so that $k/2$ is the only half-integer in the interval $[k/2-\delta, k/2+\delta]$. Define $C_{k/2}(T)$ to be the subset of $B'_{k/2}(T)$ for which $T(\log T)^{-2} < \gamma_n \leq T-Q$ and $|\widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n - k_n/2| < \delta/2$. Note that by (4.13), (5.6), and Lemma 5.3 we have

$$(5.23) \qquad |B'_{k/2}(T)| \;=\; |C_{k/2}(T)| \;+\; o(T \log T).$$

Furthermore, define

$$(5.24) \quad E_{k/2}(T) \;=\; \left\{ (\gamma, \gamma') \;:\; 0 < \gamma, \gamma' \leq T, \; \frac{2\pi}{\log T}\left(\frac{k}{2} - \delta\right) < \gamma - \gamma' \leq \frac{2\pi}{\log T}\left(\frac{k}{2} + \delta\right) \right\}.$$

We will show for large T that if γ_n is in $C_{k/2}(T)$ then (γ_{n+1}, γ_n) is in $E_{k/2}(T)$. The mean value theorem of differential calculus gives that there is a real number x between γ_n and γ_{n+1} such that

$$\gamma_{n+1} \log \gamma_{n+1} - \gamma_n \log \gamma_n \;=\; (\gamma_{n+1} - \gamma_n)(\log x + 1).$$

Therefore

$$(5.25) \qquad (\gamma_{n+1} - \gamma_n)\frac{\log T}{2\pi} \;=\; (\widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n)\left(\frac{\log T}{\log x + 1}\right).$$

If γ_n is in $C_{k/2}(T)$, then $|\widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n - k/2| < \delta/2$ by the definition of $C_{k/2}(T)$ and (5.15); thus

$$\left(\frac{k}{2} - \frac{\delta}{2}\right)(1 - o(1)) \;<\; (\widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n)\left(\frac{\log T}{\log x + 1}\right) \;<\; \left(\frac{k}{2} + \frac{\delta}{2}\right)(1 + o(1))$$

as $T \to \infty$. From this and (5.25), it follows that

$$\frac{k}{2} - \delta < (\gamma_{n+1} - \gamma_n)\frac{\log T}{2\pi} < \frac{k}{2} + \delta$$

for large T. Moreover, from the definition of $C_{k/2}(T)$, we have $\gamma_n \leq T - Q$ and therefore $\gamma_{n+1} \leq T$ by (5.22). Hence, if γ_n is in $C_{k/2}(T)$ and T is large enough, then (γ_{n+1}, γ_n) is in $E_{k/2}(T)$. Thus $|C_{k/2}(T)| \leq |E_{k/2}(T)|$ for large enough T. It follows now from (5.24) and (D2), which holds by Corollary 1.1, that

$$|C_{k/2}(T)| \leq |E_{k/2}(T)| \sim \left(\frac{T}{2\pi}\log T\right) \cdot \begin{cases} \dfrac{1}{2} & \text{if } k \text{ is even,} \\[2ex] \dfrac{1}{2} - \dfrac{2}{\pi^2 k^2} & \text{if } k \text{ is odd.} \end{cases}$$

From this, (5.23), and (5.20), we now deduce (1.8), the upper bound for p_1 in (1.7), and that the right-hand side of (1.6) is an upper bound for $p_{1/2}$.

Next we prove the lower bound for $p_{1/2}$. To do this, we first consider the set $E_{1/2}(T)$ defined by (5.24) with $k = 1$. Write the set as a disjoint union

$$(5.26) \qquad E_{1/2}(T) \;=\; D_{1/2}(T) \cup V_{1/2}(T),$$

where $D_{1/2}(T)$ contains all the pairs (γ, γ') in $E_{1/2}(T)$ that have $\gamma' \leq T(\log T)^{-2}$ and $V_{1/2}(T)$ contains the rest. We claim that if (γ, γ') is in $V_{1/2}(T)$ then γ' is in $B'_{1/2}(T)$, provided T is large enough. To prove this, let (γ_m, γ_n) be in $V_{1/2}(T)$ and note that $m > n$ since $\gamma_m > \gamma_n$ by the definition of $E_{1/2}(T)$. By Lemma 5.1 and the fact that $\gamma_m - \gamma_n \ll 1/\log T$, we have

$$(\gamma_m - \gamma_n)\frac{\log T}{2\pi} \;=\; \frac{j}{2} \;+\; o(1)$$

for some integer j. Since (γ_m, γ_n) is in $E_{1/2}(T)$, we must have $j = 1$ for T large enough. From the proof of Lemma 5.1, we see that $j = \sum_{\ell=n}^{m-1} k_\ell$. By (5.21), (5.15), and the fact that $\gamma_n > T(\log T)^{-2}$, we see that if T is large enough then $k_\ell > 0$ for all $\ell \geq n$. Hence, since $j = 1$, the only possible values for m and the k_ℓ are $m = n + 1$ and $k_n = 1$. Therefore γ_n is in $B'_{1/2}(T)$ by (5.15), and we have proved our claim.

We have shown that the map $(\gamma, \gamma') \mapsto \gamma'$ from $V_{1/2}(T)$ is into $B'_{1/2}(T)$. This map is injective because of our assumption that all zeros are simple. Hence

$$(5.27) \qquad\qquad |V_{1/2}(T)| \leq |B'_{1/2}(T)|.$$

To estimate the size of the set $D_{1/2}(T)$ in (5.26), write

$$|D_{1/2}(T)| \leq \sum_{0 < \gamma' < \frac{T}{\log^2 T}} \sum_{\left(\frac{1}{2}-\delta\right)\frac{2\pi}{\log T} < (\gamma-\gamma') \leq \left(\frac{1}{2}+\delta\right)\frac{2\pi}{\log T}} 1$$

$$\leq \sum_{0 < \gamma' < \frac{T}{\log^2 T}} (N(\gamma'+1) - N(\gamma'))$$

$$\ll T$$

for large enough T, by (4.13) and (5.6). From this, (5.27), and (5.26), it follows that

$$|E_{1/2}(T)| \leq O(T) + |B'_{1/2}(T)|.$$

Hence, by (5.24) and (D2), we have

$$\left(\frac{T}{2\pi}\log T\right)\left(\frac{1}{2} - \frac{2}{\pi^2}\right) \sim |E_{1/2}(T)| \leq O(T) + |B'_{1/2}(T)|.$$

From this and (5.20), it follows that

$$\frac{1}{2} - \frac{2}{\pi^2} + o(1) \leq p_{1/2}.$$

Combining this with the upper bound for $p_{1/2}$, we obtain (1.6).

To prove the lower bound for p_1, we need the formulas

$$(5.28) \qquad\qquad \sum_{k=0}^{\infty} p'_{k/2} = 1 + o(1)$$

$$(5.29) \qquad\qquad \sum_{k=0}^{\infty} \left(\frac{k}{2}\right) p'_{k/2} = 1 + o(1).$$

The first formula follows from (4.13) and the disjoint union

$$\bigcup_{k=0}^{\infty} B'_{k/2}(T) = \{\gamma_n \ : \ \gamma_n \leq T\}.$$

To prove the second, observe that by (5.15) we have

$$(5.30) \qquad\qquad \sum_{k=0}^{\infty} \left(\frac{k}{2}\right) |B'_{k/2}(T)| = \sum_{\gamma_n \leq T} \frac{k_n}{2}.$$

Let $\Psi_0(T)$ be as defined in (5.17). By (1.4), we have

$$\sum_{\gamma_n \leq T} \frac{k_n}{2} = \sum_{\gamma_n \leq T} \left\{ \widetilde{\gamma}_{n+1} - \widetilde{\gamma}_n - O\left((\gamma_{n+1} - \gamma_n)\psi(\gamma_n)\right) \right\} = \widetilde{\gamma}_{m+1} - \widetilde{\gamma}_1 + O\left((\gamma_{m+1} - \gamma_1)\Psi_0(T)\right),$$

where m is the largest integer for which $\gamma_m \leq T$. It follows from (5.22) that $T < \gamma_{m+1} \leq T + Q$ and so

$$\widetilde{\gamma}_{m+1} \sim \frac{T}{2\pi}\log T \qquad \text{and} \qquad \gamma_{m+1} \sim T$$

as $T \to \infty$. Hence

$$\sum_{\gamma_n \leq T} \frac{k_n}{2} = (1 + o(1))\frac{T}{2\pi}\log T + O(T\Psi_0(T)).$$

Since $\Psi_0(T) = o(\log T)$, it follows that

$$\sum_{\gamma_n \leq T} \frac{k_n}{2} \;=\; (1 + o(1))\frac{T}{2\pi} \log T$$

as $T \to \infty$. From this and (5.30), we deduce (5.29).

Now we can prove the lower bound for p_1. The following elegant trick is due to Yoonbok Lee. Multiplying both sides of (5.28) by $3/2$ and subtracting the respective sides of (5.29) from the result, we obtain

$$\left(\frac{3}{2}\right) p_0' + p_{1/2}' + \left(\frac{1}{2}\right) p_1' - \left(\frac{1}{2}\right) p_2' - p_{5/2}' - \cdots \;=\; \frac{1}{2} + o(1).$$

Since $p_{k/2}' \geq 0$ for all k, it follows that

$$\left(\frac{3}{2}\right) p_0' + p_{1/2}' + \left(\frac{1}{2}\right) p_1' \;\geq\; \frac{1}{2} + o(1).$$

Therefore, by (5.20), we have

$$\left(\frac{3}{2}\right) p_0 + p_{1/2} + \left(\frac{1}{2}\right) p_1 \;\geq\; \frac{1}{2} + o(1).$$

The lower bound for p_1 in (1.7) now follows by inserting $p_0 = o(1)$ and our estimate (1.6) for $p_{1/2}$ into the inequality and rearranging. This completes the proof of Theorem 1.4.

Remark 3: The upper bounds in (1.6), (1.7), and (1.8) can be proved using the assumption that $N^*(T) \sim 1$ instead of the stronger assumption that all the zeros are simple. To do this, we only need to define the sets $E_{k/2}(T)$ so that they count multiplicity. However, we did use the hypothesis that all the zeros are simple to prove the lower bound for $p_{1/2}$. This is because we required $E_{k/2}(T)$ to count multiplicity to be able to use (D2), and at the same time we needed $E_{k/2}(T)$ to not count multiplicity in proving (5.27).

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