APPLICATIONS OF THE WOLD DECOMPOSITION TO THE STUDY OF ROW CONTRACTIONS ASSOCIATED WITH DIRECTED GRAPHS

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Abstract. Based on a Wold decomposition for families of partial isometries and projections of Cuntz-Krieger-Toeplitz-type, we extend several fundamental theorems from the case of single vertex graphs to the general case of countable directed graphs with no sinks. We prove a Szego-type factorization theorem for CKT families, which leads to information on the structure of the unit ball in free semigroupoid algebras, and show that joint similarity implies joint unitary equivalence for such families. For each graph we prove a generalization of von Neumann's inequality which applies to row contractions of operators on Hilbert space which are related to the graph in a natural way. This yields a functional calculus determined by quiver algebras and free semigroupoid algebras. We establish a generalization of Coburn's theorem for the C*-algebra of a CKT family, and prove a universality theorem for C*-algebras generated by these families. In both cases, the C*-algebras generated by quiver algebras play the universal role.

1. Introduction

In [**20**], the second author and Jury derived a Wold decomposition for families of partial isometries which satisfy the 'Cuntz-Krieger-Toeplitz' directed graph relations, what we call the (†) relations. This theorem may also be obtained as a special case of the Wold decomposition recently established by Muhly and Solel [**29**] for the more general setting of induced representations of tensor algebras. In [**20**] it was used to investigate the internal structure of free semigroupoid algebras [**18, 20, 21, 25, 26**] and a dilation theory was built around it in [**19**]. In this paper, we apply this Wold decomposition to establish extensions of a number of fundamental theorems to the case of general countable

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directed graphs with no sinks, whereby the extended results can be regarded as occurring for single vertex graphs.

The next section (§2) contains background information. In §3 we include an expanded discussion of the Wold decomposition and extend a folklore result for isometries on Hilbert space: We prove that CKT families $\{S_e, P_x\}$ of partial isometries and projections are jointly similar precisely when they are jointly unitarily equivalent. We also establish a Szego-type factorization theorem [**36, 34**] for CKT families. In §4 we use the Szego theorem to obtain detailed information [**11**] on the unit ball of semigroupoid algebras which are partly free [**25, 26**].

Given a countable directed graph G with no sinks, in §5 we show there is a von Neumann inequality for the set of row contractions $T = (T_1, \ldots, T_n)$ of operators on Hilbert space such that the joint actions of the T_i are related to G in a natural way. In the case of single vertex graphs, we recover the original form of the von Neumann inequality [**41, 38**] (when $n = 1$) and Popescu's version [**33**] (when $2 \leq n \leq \infty$). If $n \geq 2$, the estimates obtained are sharper than the estimates of [**33**] in general. In fact, we show these estimates respect the partial ordering on directed graphs induced by graph deformations. An immediate consequence of the von Neumann inequality is the existence of functional calculus (§6) for row contractions based on quiver algebras [**21, 25, 26, 28, 29, 30**] and free semigroupoid algebras. There is a long history of both positive and negative results associated with von Neumann inequalities for the multivariable setting, and this work fits in with such efforts. See [**1, 2, 3, 6, 9, 12, 17, 30, 33**] for some of the different perspectives. In connection with the current work, see [**30**] where a more general form of the von Neumann inequality is obtained.

In the final section, we apply the Wold decomposition to obtain a pair of universality theorems for directed graph C*-algebras. In particular, a generalization of Coburn's theorem [**8, 32**] for isometries on Hilbert space is established: The C*-algebra generated by any CKT family which satisfies a natural non-unitary condition is isomorphic to the C*-algebra generated by the corresponding quiver algebra. Further, we prove that the C*-algebras generated by quiver algebras are the universal C*-algebras for the CKT relations.

2. Preliminaries

Let G be a countable directed graph with directed edges $\mathcal{E}(G)$ and vertices $\mathcal{V}(G)$. Given a finite path w in G, we write $w = ywx$ when the source vertex of w is $s(w) = x$ and the range vertex of w is $r(w) = y$. (For vertices $x \in \mathcal{V}(G)$ put $r(x) = x = s(x)$.) Say that G has a *source*

at $x \in \mathcal{V}(G)$ if there are no edges which finish at x; that is, $r(e) \neq x$ for $e \in \mathcal{E}(G)$. Similarly, G has a *sink* at $x \in \mathcal{V}(G)$ if $s(e) \neq x$ for $e \in \mathcal{E}(G)$. Define the *free semigroupoid* $\mathbb{F}^+(G)$ to be the set of all vertices $\mathcal{V}(G)$ and all finite paths in G, written in reduced form, with the natural operations of concatenation of paths allowed by the graph structure. A path $w = e_k \cdots e_1 \in \mathbb{F}^+(G)$ is said to be a *loop* if $s(w) = r(w)$. If in addition, $r(e_i) \neq r(e_j)$, for $i \neq j$, then w is said to be *vertex-simple*. A vertex-simple loop $w = e_k \cdots e_1$, $e_i \in \mathcal{E}(G)$, has an *entrance* if there exists an edge $f \in \mathcal{E}(G)$ such that $r(f) = r(e_i)$, for some i, but $f \neq e_i$. Finally, let $\mathcal{P}_G^+$ be the *path algebra* generated by G; this is the set of polynomials with complex coefficients and monomials belonging to $\mathbb{F}^+(G)$.

Given a family of non-zero partial isometries $\{S_e : e \in \mathcal{E}(G)\}$ and projections $\{P_x : x \in \mathcal{V}(G)\}$ which act on a common Hilbert space, consider the following relations:

$$(\dagger) \begin{cases} (i) & P_x P_y = 0 & \text{for all } x, y \in \mathcal{V}(G),\ x \neq y \\ (ii) & S_e^* S_f = 0 & \text{for all } e, f \in \mathcal{E}(G),\ e \neq f \\ (iii) & S_e^* S_e = P_{s(e)} & \text{for all } e \in \mathcal{E}(G) \\ (iv) & \sum_{r(e)=x} S_e S_e^* \leq P_x & \text{for all } x \in \mathcal{V}(G) \end{cases}$$

If equality is achieved in (iv) for all $x \in \mathcal{V}(G)$, then we refer to the relations as $(\ddagger)$. Observe that $(\ddagger)$ can only occur for G with no sources. We mention that the operator algebras generated by families $\{S_e, P_x\}$ which satisfy $(\dagger)$ have been the focus of intense recent interest. They include so-called Cuntz-Krieger directed graph C*-algebras, free semigroupoid algebras, quiver algebras, etc. (The references **[4, 10, 18, 20, 21, 22, 23, 24, 25, 26, 28, 29, 30, 37, 39]** give a starting point for the interested reader.) Let us mention at this point the existence of the universal C*-algebra $\mathrm{C}^*(G)$ **[22, 23, 24]** for the relations $(\ddagger)$ for a given directed graph G. From these operator algebra motivations, we refer to $(\ddagger)$ as the Cuntz-Krieger (CK) relations, and $(\dagger)$ as the Cuntz-Krieger-Toeplitz relations (CKT).

We next describe the prototypical 'pure model' **[19, 20]** in the Wold decomposition used here. (More details are given in the next section.) Let G be a countable directed graph (with no sinks) and define $\mathcal{H}_G$ to be the Hilbert space with orthonormal basis given by $\{\xi_w : w \in \mathbb{F}^+(G)\}$. This generalized 'Fock space' associated with a directed graph was introduced by Muhly **[28]**. For $e \in \mathcal{E}(G)$ define partial isometries L_e on $\mathcal{H}_G$ by

$$L_e \xi_w = \begin{cases} \xi_{ew} & \text{if } r(w) = s(e) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Observe that $L_e^* L_e = P_{s(e)}$ is the projection onto $\text{span}\{\xi_w : r(w) = s(e)\}$, and that the family $\{L_e, P_x\}$ satisfies (†). For $x \in \mathcal{V}(G)$, we define the 'tree component' subspace $\mathcal{H}_x = \text{span}\{\xi_w : w = wx\}$. Notice that these subspaces are reducing for the L_e. In fact, the subspaces $\{\mathcal{H}_x : x \in \mathcal{V}(G)\}$ form the *unique* family of minimal non-zero joint reducing subspaces for the L_e [**25**].

Given a directed graph G, the associated *quiver algebra* $\mathcal{A}_G$ is the norm-closed algebra generated by $\{L_e, P_x : e \in \mathcal{E}(G), x \in \mathcal{V}(G)\}$, where the P_x here are the vertex projections associated with the pure model (1). The associated *free semigroupoid algebra* $\mathfrak{L}_G$ is the weak operator topology closure of $\mathcal{A}_G$. The commutant of $\mathfrak{L}_G$ coincides with $\mathfrak{L}'_G = \mathfrak{R}_G$ [**25**], the WOT-closed algebra generated by partial isometries R_e, defined on $\mathcal{H}_G$ (as in (1)) by $R_e \xi_w = \xi_{we}$, together with their initial projections which we denote by Q_x, $x \in \mathcal{V}(G)$. The projections Q_x are precisely the projections onto the tree component subspaces $\mathcal{H}_x$.

We mention that the graph G was proved to be a complete invariant for $\mathcal{A}_G$ (and $\mathfrak{L}_G$) up to unitary equivalence by Kribs and Power [**25**]. More recently, Solel [**37**] proved G is an invariant for isometric isomorphism and, independently, the authors [**21**] proved that G is a banach algebra isomorphism invariant for $\mathcal{A}_G$. (In fact, when G has no sources or no sinks it was proved in [**21**] that G is an algebraic isomorphism invariant for $\mathcal{A}_G$.) This is a true departure from the C*-algebra case, where, for instance, it is not hard to see that G is not a $*$-isomorphism invariant for $\text{C}^*(G)$. (e.g. For many non-isomorphic graphs, $\text{C}^*(G)$ is $*$-isomorphic to the set of compact operators.)

3. Wold Decomposition

The Wold decomposition from [**20**] (which assumed G had no sinks) asserts that every set of partial isometries $\{S_e : e \in \mathcal{E}(G)\}$ which satisfies (†) on a Hilbert space $\mathcal{H}$ is jointly unitarily equivalent to a direct sum

$$S_e \simeq V_e \oplus \Big(\sum_{x \in V(G)} \oplus L_e^{(\alpha_x)} \Big|_{\mathcal{H}_x^{(\alpha_x)}} \Big) \quad \text{for} \quad e \in E(G), \tag{2}$$

where $V = (V_e)_{e \in \mathcal{E}(G)}$ satisfies (‡), and hence determines a representation of a Cuntz-Krieger directed graph C*-algebra. As indicated above, this result can be derived from a special case of the Wold decomposition from [**29**]. The restricted ampliations of the L_e in this decomposition are said to form the *pure part* of the dilation. (Notice that the pure model (1) is captured with $V_e \equiv 0$ and $\alpha_x \equiv 1$.) The α_x are called the *vertex multiplicities* in the dilation and are computed from $\{S_e, P_x\}$ via

the equations

$$\alpha_x = \operatorname{rank}\Big(P_x\big(I - \sum_e S_e S_e^*\big)\Big) \quad \text{for} \quad x \in \mathcal{V}(G). \tag{3}$$

More specifically, the joint unitary equivalence (2) arises from the spatial decomposition $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_p$ where $\mathcal{H}_c = \mathcal{H}_p^\perp$ and $\mathcal{H}_p = \sum_{w\in\mathbb{F}^+(G)} \oplus w(S)\mathcal{W}$ are joint reducing subspaces for the S_e, with $\mathcal{W}$ the wandering subspace for S given by

$$\mathcal{W} = \operatorname{Ran}\big(I - \sum_e S_e S_e^*\big) = \bigcap_e \ker S_e^*.$$

Observe for $x \in \mathcal{V}(G)$ that $P_x\mathcal{W} = \operatorname{Ran}\big(P_x - \sum_{r(e)=x} S_e S_e^*\big)$ is a subspace of $\mathcal{W}$. The subspaces $\mathcal{H}_c$ and $\mathcal{H}_p$ have the alternate descriptions

$$\mathcal{H}_p = \Big\{\xi \in \mathcal{H} \ : \ \lim_{d\to\infty} \sum_{w\in\mathbb{F}^+(G)_d} ||w(S)^*\xi||^2 = 0\Big\}$$

$$\mathcal{H}_c = \bigcap_{d\geq 1} \Big\{\xi \in \mathcal{H} : \sum_{w\in\mathbb{F}^+(G)_d} w(S)w(S)^*\xi = \xi\Big\}, \tag{4}$$

where $\mathbb{F}^+(G)_d$ is the set of paths inside $\mathbb{F}^+(G)$ of length d.

As a direct consequence of the Wold decomposition, we prove a generalization of the well-known fact that similar isometries are unitarily equivalent. This result also generalizes Popescu's theorem **[32]** for the case of n-tuples of isometries with orthogonal ranges, which occurs below when G is a single vertex graph with $n \geq 2$ edges.

Theorem 3.1. *Let G be a countable directed graph with no sinks. If $\{S_e, P_x\}$ and $\{S'_e, P'_x\}$ both satisfy (†) for G and are jointly similar, then they are jointly unitarily equivalent. That is, if there is an invertible operator A with*

$$\begin{aligned} AS_eA^{-1} &= S'_e \qquad \text{for} \quad e \in \mathcal{E}(G) \\ AP_xA^{-1} &= P'_x \qquad \text{for} \quad x \in \mathcal{V}(G), \end{aligned}$$

then there is a unitary operator U such that

$$\begin{aligned} US_eU^* &= S'_e \qquad \text{for} \quad e \in \mathcal{E}(G) \\ UP_xU^* &= P'_x \qquad \text{for} \quad x \in \mathcal{V}(G). \end{aligned}$$

Proof. Without loss of generality assume both families of operators act on $\mathcal{H}$. Let $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_p = \mathcal{H}'_c \oplus \mathcal{H}'_p$ be the spatial decompositions associated with the Wold decompositions (2) for the S_e and S'_e respectively.

For $e \in \mathcal{E}(G)$ let

$$S_e|_{\mathcal{H}_c} \equiv V_e, \quad S_e|_{\mathcal{H}_p} \equiv W_e, \quad S'_e|_{\mathcal{H}'_c} \equiv V'_e, \quad S'_e|_{\mathcal{H}'_p} \equiv W'_e.$$

By hypothesis, for $e \in \mathcal{E}(G)$ we have

$$\ker S_e^* = \ker(A^* S_e'^* (A^*)^{-1}) = A^*\big(\ker(S_e'^*)\big).$$

Let $\mathcal{W}$ and $\mathcal{W}'$ be the wandering subspaces for S and S' respectively. Then

$$A^*(\mathcal{W}') = A^*\Big(\bigcap_e \ker S_e'^*\Big) = \bigcap_e \ker S_e^* = \mathcal{W}.$$

In particular, $\dim \mathcal{W} = \dim \mathcal{W}'$. But more is true. Recall that the vertex multiplicities in the pure parts of the Wold decompositions for S and S' are given by $\alpha_x = \dim P_x\mathcal{W}$ and $\alpha'_x = \dim P_x\mathcal{W}'$. By hypothesis and the previous identity we have

$$P_x\mathcal{W} = P_x A^*\mathcal{W}' = A^* P'_x \mathcal{W}' \quad \text{for} \quad x \in \mathcal{V}(G),$$

and hence the invertibility of A yields $\alpha_x = \alpha'_x$ for $x \in \mathcal{V}(G)$.

Thus, it is easily seen from the Wold decomposition that there is a unitary $U_1 : \mathcal{H}_p \to \mathcal{H}'_p$ which intertwines the pure parts of S and S';

$$U_1 W_e U_1^* = W'_e \quad \text{for} \quad e \in \mathcal{E}(G). \tag{5}$$

Since G has no sinks, every vertex projection P_x is the initial projection for some S_e. Further, recall that $\mathcal{H}_p$ (respectively $\mathcal{H}'_p$) is a reducing subspace for the S_e (respectively S'_e). It follows that the restricted vertex projections $P_x|_{\mathcal{H}_p}$ and $P'_x|_{\mathcal{H}'_p}$ are also intertwined by U_1.

We now show that the restrictions to $\mathcal{H}_c, \mathcal{H}'_c$ may be spatially intertwined as well. As $AS_e = S'_e A$ and $S_e A^{-1} = A^{-1} S'_e$ for $e \in \mathcal{E}(G)$, equation (4) applied for the S_e and the S'_e implies that $A\mathcal{H}_c = \mathcal{H}'_c$. Let $A_0 = A|_{\mathcal{H}_c} : \mathcal{H}_c \to \mathcal{H}'_c$ and observe that A_0 is invertible. Then we have

$$A_0 V_e = V'_e A_0 \quad \text{for} \quad e \in \mathcal{E}(G).$$

In particular, equation (4) gives

$$A_0 = A_0(I_{\mathcal{H}_c}) = A_0\Big(\sum_e V_e V_e^*\Big) = \sum_e V'_e A_0 V_e^*.$$

Thus the (†) relations and the hypothesis implies that for $e \in \mathcal{E}(G)$,

$$V_e'^* A_0 = V_e'^* V'_e A_0 V_e^* = P'_{s(e)} A_0 V_e^* = A_0 P_{s(e)} V_e^* = A_0 V_e^*.$$

It follows from the adjoint of this identity that

$$V_e A_0^* A_0 = A_0^* V'_e A_0 = A_0^* A_0 V_e \quad \text{for} \quad e \in \mathcal{E}(G). \tag{6}$$

Consider the polar decomposition of $A_0 = U_2 B$, where $U_2 : \mathcal{H}_c \to \mathcal{H}'_c$ is unitary and $B = (A_0^* A_0)^{1/2}$ is an invertible operator on $\mathcal{H}_c$. By (6) we have

$$V_e B = B V_e \quad \text{for} \quad e \in \mathcal{E}(G),$$

and it follows that

$$\begin{aligned} V'_e U_2 = V'_e U_2 B B^{-1} &= V'_e A_0 B^{-1} \\ &= A_0 V_e B^{-1} = U_2 B V_e B^{-1} = U_2 V_e. \end{aligned}$$

Thus

$$U_2 V_e = V'_e U_2 \quad \text{for} \quad e \in \mathcal{E}(G). \tag{7}$$

Therefore, we may define a unitary $U = U_1 \oplus U_2$ on $\mathcal{H}$ so that (5) and (7) imply

$$U S_e = S'_e U^* \quad \text{for} \quad e \in \mathcal{E}(G).$$

As G has no sinks, it follows that the projections P_x and P'_x are also intertwined by the unitary U. ∎

We now present another application of the Wold decomposition that allows us to prove a Szego-type factorization theorem. This theorem generalizes earlier results of Rosenblum and Rovnyak [**36**, Theorem 3.4] and Popescu [**34**].

Theorem 3.2. *Let G be a countable directed graph with no sinks and let $\{P_x\}_{x\in\mathcal{V}(G)}$ and $\{S_e\}_{e\in\mathcal{E}(G)}$ be sets of projections and partial isometries which satisfy (†). Assume that $\{S_e\}_{e\in\mathcal{E}(G)}$ is pure. If $Y \in B(\mathcal{H})$ is a positive invertible operator then the following assertions are equivalent:*

(i) *$Y = A^*A$ for some $A \in B(\mathcal{H})$ which commutes with $\{P_x\}_{x\in\mathcal{V}(G)}$ and $\{S_e\}_{e\in\mathcal{E}(G)}$.*
(ii) *$S_e^* Y S_f = Y S_e^* S_f = S_e^* S_f Y$ for all $e, f \in \mathcal{E}(G)$.*

Proof. We need to prove that (ii) implies (i). Let $T_u \equiv Y^{1/2} S_u Y^{-1/2}$ for $u \in \mathbb{F}^+(G)$. We will show that the family $\{T_e\}_{e\in\mathcal{E}(G)}$ is jointly unitarily equivalent to $\{S_e\}_{e\in\mathcal{E}(G)}$. Towards this end notice that $T_e^* T_e = P_{s(e)}$ and $T_e T_e^* \leq P_{r(e)}$. Indeed,

$$\begin{aligned} T_e^* T_e &= Y^{-1/2} S_e^* Y S_e Y^{-1/2} \\ &= Y^{-1/2} Y Y^{-1/2} P_{s(e)} = P_{s(e)} = S_e^* S_e, \end{aligned}$$

and a similar calculation shows the claimed inequality. Therefore, the collections $\{P_x\}_{x\in\mathcal{V}(G)}$ and $\{T_e\}_{e\in\mathcal{E}(G)}$ satisfy the requirements of the Wold Decomposition; i.e., the relations (†), and therefore the earlier

discussion applies. First, we determine that the Cuntz-Krieger part of $\{T_e\}_{e\in\mathcal{E}(G)}$ is trivial. Indeed,

$$\begin{aligned}\mathcal{H}_c &= \bigwedge_{k\in\mathbb{N}} \bigvee_{|u|=k} T_u(\mathcal{H}_G) \\ &= \bigwedge_{k\in\mathbb{N}} \bigvee_{|u|=k} Y^{1/2} S_u(\mathcal{H}) \\ &= Y^{1/2}\Big(\bigwedge_{k\in\mathbb{N}} \bigvee_{|u|=k} S_u(\mathcal{H})\Big) = \{0\}.\end{aligned}$$

Thus $\{T_e\}_{e\in\mathcal{E}(G)}$ is pure. Let $\widetilde{\mathcal{W}}$ be its wandering subspace and let $\widetilde{\mathcal{W}}_x \equiv P_x\widetilde{\mathcal{W}} \subseteq \widetilde{\mathcal{W}}$. Then by the Wold decomposition there exists a unitary operator

$$V : \mathcal{H} \longrightarrow \sum_{x\in\mathcal{V}(G)} \oplus (Q_x\mathcal{H}_G)^{(\beta_x)}$$

such that for any $e \in \mathcal{E}(G)$,

$$VT_eV^* = \sum_{x\in\mathcal{V}(G)} \oplus L_e^{(\beta_x)}\Big|_{(Q_x\mathcal{H}_G)^{(\beta_x)}},$$

where, $\beta_x = \dim \widetilde{\mathcal{W}}_x$, $x \in \mathcal{V}(G)$. Similarly, there exists a unitary

$$U : \mathcal{H} \longrightarrow \sum_{x\in\mathcal{V}(G)} \oplus (Q_x\mathcal{H}_G)^{(\alpha_x)}$$

such that for any $e \in \mathcal{E}(G)$,

$$US_eU^* = \sum_{x\in\mathcal{V}(G)} \oplus L_e^{(\alpha_x)}\Big|_{(Q_x\mathcal{H}_G)^{(\alpha_x)}}.$$

In order to establish the joint unitary equivalence between the families $\{T_e\}_{e\in\mathcal{E}(G)}$ and $\{S_e\}_{e\in\mathcal{E}(G)}$, we will prove that $VT_eV^* = U^*S_eU$, $e \in \mathcal{E}(G)$. From what we have proved so far, it follows that we need only to to verify $\beta_x = \alpha_x$, for all $x \in \mathcal{V}(G)$. Fix an $x \in \mathcal{V}(G)$ and let $\xi \in \widetilde{\mathcal{W}}_x$. Then for all $e \in \mathcal{E}(G)$ and $\zeta \in \mathcal{H}_G$, we have

$$(Y^{1/2}\xi\,,\, S_e\zeta) = (\xi\,,\, Y^{1/2}S_e\zeta) = (\xi\,,\, T_eY^{1/2}\zeta) = 0.$$

Thus by the invertibility of Y we have $Y^{1/2}\widetilde{\mathcal{W}} = \mathcal{W}$. But $Y^{1/2}T_e^*T_e = S_e^*S_eY^{1/2}$ for $e \in \mathcal{E}(G)$ since the same is true for Y. Hence, as G has no sinks, every P_x is equal to some initial projection $S_e^*S_e = T_e^*T_e$ and it follows that $Y^{1/2}\widetilde{\mathcal{W}}_x = \mathcal{W}_x$ for $x \in \mathcal{V}(G)$ since $Y^{1/2}$ is invertible. Therefore, the identity $\beta_x = \alpha_x$ holds for all $x \in \mathcal{V}(G)$, as required.

We have shown that $VT_eV^* = US_eU^*$ for $e \in \mathcal{E}(G)$. In particular, this implies that $U^*VY^{1/2}S_e = S_eU^*VY^{1/2}$, for all $e \in \mathcal{E}(G)$, and the operator $U^*VY^{1/2}$ commutes with $\{S_e\}_{e\in\mathcal{E}(G)}$. Moreover, notice that for $e \in \mathcal{E}(G)$,

$$\begin{aligned} S_e^*S_eU^*VY^{1/2} &= S_e^*U^*VT_eV^*VY^{1/2} \\ &= U^*VT_e^*V^*VT_eY^{1/2} = U^*VY^{1/2}T_e^*T_e, \end{aligned}$$

and it follows that $U^*VY^{1/2}$ commutes with $\{P_x\}_{x\in\mathcal{V}(G)}$ as well. The desired operator for (i) is $A = U^*VY^{1/2}$. ∎

Using the fact that $\mathfrak{R}_G' = \mathfrak{L}_G$ we obtain the following.

Corollary 3.3. *Let G be a countable directed graph with no sinks. If $Y \in B(\mathcal{H})$ is a positive invertible operator then the following assertions are equivalent:*

(i) *$Y = A^*A$ for some $A \in \mathfrak{L}_G$.*
(ii) *$R_e^*YR_f = YR_e^*R_f = R_e^*R_fY$ for all $e, f \in \mathcal{E}(G)$.*

4. An Application To The Geometry Of The Unit Ball

In [**11**], Davidson, Pitts and the first author proved that every operator in the open unit ball of a w*-closed operator algebra generated by n isometries with orthogonal ranges, is the average of isometries. This result applies in particular to the noncommutative Toeplitz algebra $\mathfrak{L}_n$, thus extending a classical result of Marshall [**27**] from function theory to the noncommutative setting. Using the Szego type factorization theorem we proved earlier, we extend the result of [**11**] to a broad class of free semigroupoid algebras.

The techniques of [**11**] yield a more general result than the one quoted in the introduction. In order to state it, we need the following.

Definition 4.1. An operator algebra $\mathcal{A}$ satisfies the *factorization property* if for any $A \in \mathcal{A}$ with $\|A\| < 1$, the equation $X^*X = I - A^*A$ has a solution in $X \in \mathcal{A}$.

It is clear from Corollary 3.3 that any free semigroupoid algebra for a graph with no sinks satisfies the factorization property.

Theorem 4.2. [**11**] *Let $\mathcal{A}$ be a norm closed operator algebra satisfying the following two properties:*

(i) *$\mathcal{A}$ satisfies the factorization property.*
(ii) *$\mathcal{A}$ contains two isometries with orthogonal ranges.*

If $\|A\| < 1 - \frac{1}{k}$, $A \in \mathcal{A}$, $k \in \mathbb{N}$, then A is the average of $6k$ isometries.

Remark 4.3. The above result is not explicitly stated as a theorem in [**11**] but its validity is ascertained on [**11**, page 118].

On the other hand, in [**26**] Power and the second author identified a property for a directed graph G that is equivalent to the existence of two isometries in $\mathfrak{L}_G$ with orthogonal ranges.

Definition 4.4. A countable directed graph satisfies the *uniform aperiodic path property* if the saturation of each vertex $x \in \mathcal{V}(G)$ either contains two distinct loops or an infinite proper path.

Theorem 4.5. [**26**] *Let G be a countable directed graph. Then G satisfies the uniform aperiodic path property if and only if $\mathfrak{L}_G$ contains a pair of isometries with orthogonal ranges.*

In light of Theorems 4.2 and 3.3, the previous result provides the last step for establishing the main result of this section.

Theorem 4.6. *Let G be a countable directed graph which satisfies the uniform aperiodic path property and let $A \in \mathfrak{L}_G$. If $\|A\| < 1 - \frac{1}{k}$, $k \in \mathbb{N}$, then A is the average of $6k$ isometries.*

Remark 4.7. What about other free semigroupoid algebras? Clearly, H^∞ fails the aperiodic path property and yet the convex combinations of the isometries cover the open unit ball. On the other hand, not all free semigroupoid algebras satisfy that property. Indeed, if G is a graph with two vertices x, y and an edge $e = xey$, then $\mathfrak{L}_G$ is the direct sum of the 2×2 upper triangular matrices with $\mathbb{C}$. The convex hull of the isometries in this algebra consists of the diagonal matrices.

5. Von Neumann Inequality

Given a row contraction $T = (T_1, \ldots, T_n)$ of (non-zero) operators acting on a Hilbert space $\mathcal{H}$, consider a family of mutually orthogonal projections $\mathcal{P} = \{P_x\}_{x \in \mathcal{J}}$ on $\mathcal{H}$ which sum to the identity operator and *stabilize* T in the following sense:

$$T_i P_x,\ P_x T_i \in \{T_i, 0\} \quad \text{for} \quad 1 \leq i \leq n \quad \text{and} \quad x \in \mathcal{J}. \tag{8}$$

Observe that these relations determine a directed graph G with vertex set $\mathcal{V}(G) \equiv \mathcal{J}$ and n directed edges $e_1, \ldots e_n$, where $r(e_i)$ and $s(e_i)$ are the unique vertices with $P_{r(e_i)} T_i P_{s(e_i)} = T_i$. When there is no confusion, we shall write $T = (T_e)_{e \in \mathcal{E}(G)}$ if an ordering of the T_i has been induced by a fixed projection set $\mathcal{P}$.

Let $T = (T_e)_{e \in E(G)}$ and $\mathcal{P} = \{P_x\}_{x \in \mathcal{V}(G)}$ satisfy (8) on $\mathcal{H}$ for a graph G with no sinks. Then it was shown in [**19**] (and previously in a more

abstract form [**30**] for the general setting of tensor algebras over C*-correspondences) that there are partial isometries $\{S_e : e \in \mathcal{E}(G)\}$ on a space $\mathcal{K} \supseteq \mathcal{H}$ such that

(*a*) $S = (S_e)_{e\in E(G)}$ satisfy (†).
(*b*) $\mathcal{H}$ reduces each $S_e^* S_e$, $e \in \mathcal{E}(G)$, and $\{S_e^* S_e|_{\mathcal{H}}\} = \mathcal{P}$.
(*c*) $\mathcal{H}$ is invariant for each S_e^* with $S_e^*|_{\mathcal{H}} = T_e^*$, $e \in \mathcal{E}(G)$.
(*d*) $\mathcal{K} = \bigvee_{w\in\mathbb{F}^+(G)} w(S)\mathcal{H}$.

Such a *minimal partially isometric dilation* of T is unique up to a unitary equivalence which fixes $\mathcal{H}$. The vertex multiplicities α_x in such a dilation are determined by (3) and, in fact, it was shown in [**19**] that they may be computed from $T, \mathcal{P}$ by

$$\alpha_x = \operatorname{rank}\Big(P_x\big(I_{\mathcal{H}} - \sum_e T_e T_e^*\big)\Big) \quad \text{for} \quad x \in \mathcal{V}(G). \tag{9}$$

For a fixed directed graph G we shall define the supremum norm $||p||_\infty$ of a polynomial $p \in \mathcal{P}_G^+$ by $||p||_\infty \equiv ||p(L_G)||$, where $L_G = (L_e)_{e\in E(G)}$ is the canonical pure model (1) for G. This definition is in line with other settings, where the norm $||p||_\infty$ is determined by the pure model in this manner. As the simplest example, recall that the norm of a polynomial p on the unit complex disk satisfies $||p||_\infty = ||p(U_+)||$, where U_+ is the unilateral forward shift [**38**].

Theorem 5.1. *Let G be a countable directed graph with no sinks. Let $T = (T_e)_{e\in\mathcal{E}(G)}$ be a row contraction on a Hilbert space $\mathcal{H}$ which satisfies (8) for $\{P_x : x \in \mathcal{V}(G)\}$. Then*

$$||p(T)|| \le ||p||_\infty \quad \textit{for all} \quad p \in \mathcal{P}_G^+. \tag{10}$$

Proof. Let $S = (S_e)_{e\in\mathcal{E}(G)}$ be the minimal partially isometric dilation of T associated with the projections $\{P_x\}$. Let p belong to $\mathcal{P}_G^+$. Then $p(T) = P_{\mathcal{H}}\, p(V)|_{\mathcal{H}}$ by virtue of (*a*), (*b*) and (*c*) above, and hence the Wold decomposition implies that

$$||p(T)|| \le ||p(S)|| = \max\{||p(V)||, ||p(W)||\},$$

where $V = (V_e)_{e\in\mathcal{E}(G)}$ satisfies (‡) and $W = (W_e)_{e\in\mathcal{E}(G)}$ is pure and satisfies (†).

The decomposition (2) of the pure part W of the minimal dilation S clearly implies that

$$||p(W)|| \le ||p(L_G)|| \equiv ||p||_\infty.$$

Thus, we shall finish the proof by showing that $||p(S)|| \le ||p(L_G)||$ for *every* row contraction $S = (S_e)_{e\in\mathcal{E}(G)}$ which satisfies (‡) for G.

To see this, let $S = (S_e)_{e\in\mathcal{E}(G)}$ satisfy (‡) and let $0 < r < 1$. Then $rS = (rS_e)_{e\in\mathcal{E}(G)}$ is a pure row contraction, and hence the minimal

partially isometric dilation of rS with respect to $\{S_e^* S_e : e \in \mathcal{E}(G)\}$ is given by the form (2) with $V_e \equiv 0$. We claim that the vertex multiplicities for this pure dilation $S' = (S'_e)_{e\in\mathcal{E}(G)}$ satisfy $\alpha_x \geq 1$ for $x \in \mathcal{V}(G)$. Indeed, from (9) these multiplicities are determined by

$$\alpha_x = \text{rank}\left(P_x - \sum_{e=xe} (rS_e)(rS_e)^*\right) \quad \text{for} \quad x \in \mathcal{V}(G).$$

Thus if $\alpha_x = 0$, then the (‡) relations give

$$\sum_{e=xe} S_e S_e^* = P_x = r^2 \sum_{e=xe} S_e S_e^*,$$

which is clearly a contradiction since the P_x are assumed to be non-zero in the (†) relations.

As $\alpha_x \geq 1$ for $x \in \mathcal{V}(G)$, we have

$$||p(rS)|| \leq ||p(S')|| = ||p(L_G)||.$$

Since $0 < r < 1$ was arbitrary, it follows that $||p(S)|| \leq ||p(L_G)||$, as required. ■

Remark 5.2. In the case of single vertex graphs with n edges, the above theorem recovers the classical von Neumann inequality **[41, 38]** (when $n = 1$) and Popescu's version **[6, 33]** (when $2 \leq n \leq \infty$). We also mention that the conclusion of Theorem 5.1 can be proved through an application of Theorem 3.10 from **[30]**. However, we note that our direct proof via the Wold decomposition is novel and provides a new perspective on the problem.

5.1. **Partially Ordered Sets of Directed Graphs.** An alternative perspective on Theorem 5.1 is the following: Given a countable directed graph G with no sinks, the theorem shows there is a corresponding von Neumann inequality with estimate given by (for this discussion) $||p||_{G,\infty} \equiv ||p(L_G)||$ for $p \in \mathcal{P}_G^+$, recalling that $L_G = (L_e)_{e\in\mathcal{E}(G)}$. On the other hand, every directed graph determines a partially ordered set of directed graphs through its deformations. A directed graph G_2 is a *deformation* of G_1 if G_2 is obtained from G_1 by identifying certain vertices in G_1. The partial order is defined by: $G_1 \leq G_2$ if and only if G_2 is a deformation of G_1. (This perspective was also discussed in **[19]** in the context of dilation theory, where it was shown that every row contraction T generates such a partially ordered set of directed graphs through its family of minimal partially isometric dilations.) We observe below that the von Neumann inequality estimates respect the ordering in such a partially ordered set of directed graphs.

Let C_n be the directed graph with a single vertex and n directed loop edges. The Sz.-Nagy and Popescu estimates are given by $||p(L_{C_n})||$, where p is allowed to be any polynomial in n noncommuting variables when $n \geq 2$. Let G be a countable directed graph with n edges. Then $L_G = (L_e)_{e \in E(G)}$ is a pure row contraction (in the n-tuple sense **[35]**) and hence its (unique) minimal isometric dilation **[7, 15, 35]** is given by a multiple of $L_{C_n} = (L_1, \ldots, L_n)$, where the L_i are the creation isometries on the full n-variable Fock space. Thus, given $L_G = (L_e)_{e \in E(G)}$ and such a polynomial p, we shall write $p(L_G)$ for the evaluation of p at the L_e which arises from the ordering of the L_e induced by the minimal isometric dilation of L_G.

Corollary 5.3. *If $G_1 \leq G_2$ are directed graphs with no sinks and n directed edges, then for every polynomial p in n noncommuting variables*

$$||p||_{G_1,\infty} \leq ||p||_{G_2,\infty} \leq ||p||_{C_n,\infty}.$$

Proof. The relation $G_1 \leq G_2$ can be seen to induce a G_2-ordering on L_{G_1}, where the corresponding projections which stabilize L_{G_1} are given by the sums of the initial projections for L_e, $e \in \mathcal{E}(G_1)$, which are naturally induced by the deformation. Hence the first inequality is a consequence of Theorem 5.1. The last inequality follows from Popescu's von Neumann inequality since L_{G_1} and L_{G_2} are both row contractions and n-tuples. ∎

It is not hard to see that $||p||_{G,\infty}$ gives a sharper estimate than $||p(L_{C_n})||$ in general when $n \geq 2$. Consider the following simple illustration of this fact.

Example 5.4. Let T_1, T_2 be contractions on a Hilbert space $\mathcal{H}$ and define operators on $\mathcal{H}^{(2)}$ by

$$V_1 = \begin{pmatrix} 0 & 0 \\ T_1 & 0 \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} 0 & T_2 \\ 0 & 0 \end{pmatrix}.$$

Then $V = (V_1, V_2)$ is a row contraction which is stabilized by $\mathcal{P} = \{P_1, P_2\}$ where $P_i \equiv P_{\mathcal{H}}$ are the projections of $H^{(2)}$ onto its coordinate spaces. Let G be the graph with two vertices x, y and edges $e = yex$, $f = xfy$. Then $L_G = (L_e, L_f)$ in this case. Observe that C_2 is a deformation (the only one) of G. Let $L_{C_2} = (L_1, L_2)$, where L_1, L_2 are the creation isometries on the full 2-variable Fock space. Then $L_e + L_f$ is an isometry and $L_1 + L_2 = \sqrt{2}L$ where L is an isometry, and hence

$$||V_1 + V_2|| \leq ||L_e + L_f|| = 1 < \sqrt{2} = ||L_1 + L_2||.$$

Further, the $L_e + L_f$ estimate is best possible in this example since $||V_1 + V_2|| = 1$ if $||T_i|| = 1$ for $i = 1$ or 2.

6. Functional Calculus

The von Neumann inequality leads to a natural functional calculus determined by quiver algebras and, in the case of pure row contractions, by free semigroupoid algebras. Observe that when $T = (T_e)_{e\in\mathcal{E}(G)}$ is a row contraction with G-ordering induced by projections $\mathcal{P} = \{P_x\}$ which satisfy (8) for T, we may consider all polynomials $\{p(T) : p \in \mathcal{P}_G^+\}$ in the generators T_e, P_x. The following result is an immediate consequence of Theorem 5.1 and may also be obtained as a special case of Theorem 3.10 of [**30**].

Theorem 6.1. *Let G be a countable directed graph with no sinks. Let $T = (T_e)_{e\in\mathcal{E}(G)}$ be a row contraction on $\mathcal{H}$ which satisfies (8) for $\{P'_x : x \in \mathcal{V}(G)\}$. Then there is a contractive homomorphism*

$$\Psi : \mathcal{A}_G \longrightarrow \mathcal{B}(\mathcal{H})$$

defined by $\Psi(L_e) = T_e$ and $\Psi(P_x) = P'_x$ for $e \in \mathcal{E}(G)$ and $x \in \mathcal{V}(G)$.

When the row contraction satisfies an extra condition, the functional calculus can be extended to $\mathfrak{L}_G$.

Theorem 6.2. *Let G be a countable directed graph with no sinks. Let $T = (T_e)_{e\in\mathcal{E}(G)}$ be a row contraction on $\mathcal{H}$ which satisfies (8) for $\{P'_x : x \in \mathcal{V}(G)\}$, and suppose that*

$$\lim_{d\to\infty} \Big(\sum_{w\in\mathbb{F}^+(G)_d} ||w(T)^*\xi|| \Big) = 0 \quad \text{for} \quad \xi \in \mathcal{H}. \tag{11}$$

Then there is a contractive homomorphism

$$\Psi : \mathfrak{L}_G \longrightarrow \mathcal{B}(\mathcal{H})$$

defined by $\Psi(L_e) = T_e$ and $\Psi(P_x) = P'_x$ for $e \in \mathcal{E}(G)$ and $x \in \mathcal{V}(G)$.

Proof. Condition (11) means precisely that the minimal partially isometric dilation $V = (V_e)_{e\in\mathcal{E}(G)}$ of T with respect to $\{P'_x\}$ is pure [**19**]. By the Wold decomposition for V, we may define operators $f(V)$ for all $f \in \mathfrak{L}_G$, where $f(V) = \text{WOT}-\lim_k p_k(V)$ for some sequence $p_k \in \mathcal{P}_G^+$. From the properties of the minimal dilation we have $p_k(T) = P_{\mathcal{H}}\, p_k(V)|_{\mathcal{H}}$. Thus, $p_k(T)$ converges to an operator $f(T) \equiv \text{WOT}-\lim_k p_k(T) \in \mathcal{B}(\mathcal{H})$. Finally, by the von Neumann inequality and a standard approximation argument we have $||f(T)|| \leq ||f(V)|| \leq ||f(L)||$, and the result follows. ∎

Remark 6.3. It should be possible to prove an $\mathfrak{L}_G$ functional calculus for more general row contractions. For instance, as in [**38, 5, 32**] this could be accomplished by first identifying 'characteristic functions' for an appropriate notion of *completely non-unitary* row contractions here.

7. A Generalization of Coburn's Theorem

In [8] Coburn showed that the C*-algebra generated by a non-unitary isometry V is isomorphic to the Toeplitz C*-algebra, via a map that sends V to the forward shift. This result was generalized by Popescu to Cuntz-Toeplitz C*-algebras [32]. In this section we consider the generalization of Coburn's Theorem to C*-algebras generated by partial isometries associated with more general directed graphs.

The proof of Coburn's theorem depends on the universality of the algebra of continuous functions on the circle as the C*-algebra of a graph G consisting of one vertex and one loop-edge. The following result of Szymanski [39] generalizes this fact to arbitrary graph C*-algebras.

Theorem 7.1. [39] *Let G be a countable directed graph with no sources and $\rho : \mathrm{C}^*(G) \longrightarrow \mathcal{A}$ be a $*$-homomorphism into a C*-algebra $\mathcal{A}$. Then ρ is injective if and only if the following two conditions are satisfied:*

(i) *$\rho(P_x) \neq 0$ for each vertex $x \in \mathcal{V}(G)$.*
(ii) *For each vertex simple loop $w \in \mathbb{F}^+(G)$ without entrances, we have $\mathbb{T} \subseteq \sigma(\rho(S_w))$.*

Corollary 7.2. *Let G be a countable directed graph with no sources and no sinks. Let $\mathrm{C}^*(\mathcal{A}_G)$ be the C*-algebra generated by the quiver algebra $\mathcal{A}_G$, let $\mathcal{K}$ denote the set of compact operators on $\mathcal{H}_G$ and let $\mathrm{C}^*(\mathcal{A}_G)/\mathcal{K}$ denote the image of $\mathrm{C}^*(\mathcal{A}_G)$ under the Calkin map. Then there exists a $*$-isomorphism*

$$\rho : \mathrm{C}^*(G) \to \mathrm{C}^*(\mathcal{A}_G)/\mathcal{K}$$

such that $\rho(S_e) = L_e + \mathcal{K}$, for all $e \in \mathcal{E}(G)$.

Proof. Since G has no sources, the projections L_x, $x \in \mathcal{V}(G)$, are infinite dimensional and thus $L_x + \mathcal{K} \neq 0$. Furthermore,

$$\sum_{r(e)=x} L_e L_e^* = L_x - \xi_x \otimes \xi_x$$

and so the collections $\{L_x + \mathcal{K}\}_{x \in \mathcal{V}(G)}$ and $\{L_e + \mathcal{K}\}_{e \in \mathcal{E}(G)}$ satisfy the relations (‡). Hence there exists a $*$-homomorphism $\rho : \mathrm{C}^*(G) \longrightarrow \mathrm{C}^*(\mathcal{A}_G)/\mathcal{K}$ with $\rho(P_x) = L_x + \mathcal{K}$ and $\rho(S_e) = L_e + \mathcal{K}$ for all $x \in \mathcal{V}(G)$ and $e \in \mathcal{E}(G)$. We now use Szymanski's result to show that ρ is injective.

Towards this end, notice that we have already verified condition (i) in Theorem 7.1. To verify the other condition let $w \in \mathbb{F}^+(G)$ be any loop. Consider the direct sum decomposition $\mathcal{H}_G = L_x(\mathcal{H}_G) \oplus L_x(\mathcal{H}_G)^\perp$

and notice that with respect to this decomposition, L_w admits the 2×2 matrix form

$$\begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix},$$

with L unitarily equivalent to some ampliation $S^{(\alpha)}$ of the forward shift S. It is well known that if $\lambda \in \mathbb{T}$, then the range of $S-\lambda I$ is not closed. Therefore the same is true for $S^{(\alpha)}-\lambda I$ and so λ belongs to the essential spectrum of $S^{(\alpha)}$; i.e., $\lambda \in \sigma(\rho(S_w))$, as desired. ■

With a standard 'diagram chasing' argument, Corollary 7.2 can be used to show the essential norm coincides with the operator norm for any element of $\mathcal{A}_G$. However, we can prove this directly for a larger algebra, the free semigroupoid algebra $\mathfrak{L}_G$, hence we include a proof of this fact.

Proposition 7.3. *The following assertions are equivalent for a countable directed graph G:*

(*i*) *G has no sources.*
(*ii*) *$\|A\| = \|A\|_e$ for all $A \in \mathfrak{L}_G$.*

Proof. If x is a source vertex in G, then P_x is a rank one projection inside $\mathcal{A}_G$ (and $\mathfrak{L}_G$). Conversely, suppose G has no sources. Let $d \geq 1$ be a positive integer. For all $x \in \mathcal{V}(G)$ we may choose a path $w_x \in \mathbb{F}^+(G)$ of length d such that $r(w_x) = x$. As the w_x are distinct paths of the same length, notice that the partial isometries R_{w_x}, $x \in \mathcal{V}(G)$, have pairwise orthogonal ranges. Thus, we may define an isometry $R^{(d)} = \sum_{x\in\mathcal{V}(G)} R_{w_x}$ in the commutant $\mathfrak{L}_G' = \mathfrak{R}_G$ (where the sum converges WOT in the infinite vertex case);

$$(R^{(d)})^* R^{(d)} = \sum_{x,y} R_{w_x}^* R_{w_y} = \sum_x R_{w_x}^* R_{w_x} = \sum_x Q_x = I.$$

Hence, if A belongs to $\mathfrak{L}_G$ we have

$$||AR^{(d)}\xi|| = ||R^{(d)}A\xi|| = ||A\xi|| \quad \text{for} \quad \xi \in \mathcal{H}_G \quad \text{and} \quad d \geq 1.$$

On the other hand, $\{R^{(d)}\xi\}_{d\geq 1}$ converges weakly to zero for any choice of vector $\xi \in \mathcal{H}_G$, and thus $||A|| = ||A||_e$ as claimed. ■

Corollary 7.4. *If G be a directed graph with no sources, then the compact perturbations $\mathfrak{L}_G + \mathcal{K}$ are norm closed.*

The next result shows that $\mathrm{C}^*(\mathcal{A}_G)$ is the universal C^*-algebra for the relations (†) and it is known to the specialists (compare with **[31, 16]**). Nevertheless, we feel that the short proof presented here is new and we therefore include it.

Theorem 7.5. *Let G be a countable directed graph with no sources and no sinks. Let $\{P'_x\}_{x\in\mathcal{V}(G)}$ and $\{S'_e\}_{e\in\mathcal{E}(G)}$ be families of projections and partial isometries acting on a Hilbert space $\mathcal{H}$ which satisfy (†). Then there exists a $*$-epimorphism*

$$\tau : \mathrm{C}^*(\mathcal{A}_G) \longrightarrow \mathrm{C}^*(\{S'_e\}_{e\in\mathcal{E}(G)})$$

such that $\tau(L_e) = S'_e$, for all $e \in \mathcal{E}(G)$.

Proof. From the discussion in Section 3, it is clear that the Wold decomposition (which was proved for G with no sinks) implies the existence of C*-homomorphisms

$$\phi_1 : \mathrm{C}^*(G) \longrightarrow B(\mathcal{H}_1)$$

and

$$\phi_2 : \mathrm{C}^*(\mathcal{A}_G) \longrightarrow B(\mathcal{H}_2)$$

such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and

$$S'_u \simeq (\phi_1 \oplus \phi_2)(S_u \oplus L_u) \quad \text{for} \quad u \in \mathbb{F}^+(G).$$

Consider the diagram

$$\mathrm{C}^*(\mathcal{A}_G) \xrightarrow{\pi} \mathrm{C}^*(\mathcal{A}_G)/\mathcal{K} \xrightarrow{\rho} \mathrm{C}^*(G),$$

where ρ is the $*$-isomorphism identified by Corollary 7.2. If $\psi \equiv \rho \circ \pi$, then the desired map here is

$$\tau = (\phi_1 \oplus \phi_2) \circ (\psi \oplus id) : \mathrm{C}^*(\mathcal{A}_G) \longrightarrow \mathrm{C}^*(\{S'_e\}_{e\in\mathcal{E}(G)}).$$

Tracing the definitions of these maps shows that $\tau(L_u) = S'_u$ for all $u \in \mathbb{F}^+(G)$. ∎

The previous theorem leads to a natural C*-algebra extension of the von Neumann inequality Theorem 5.1 (see also **[32]** for the case of isometries with orthogonal ranges). Though note that the proof of this result relies on Szymanski's Theorem, whereas our proof of Theorem 5.1 is direct. Let $\mathcal{P}^0_G$ be the set of polynomials generated by all indeterminant monomials of the form $v\overline{w}$ with $v, w \in \mathbb{F}^+(G)$ and where $x = \overline{x}$ for $x \in \mathcal{V}(G)$. Given $T = (T_e)_{e\in E(G)}$ and $p \in \mathcal{P}^0_G$, we write $p(T, T^*)$ for the polynomial p evaluated at the generators T_e (for $e \in \mathcal{P}^0_G$) and T^*_e (for $\overline{e} \in \mathcal{P}^0_G$) and P_x (for $x = \overline{x} \in \mathcal{P}^0_G \cap \mathcal{V}(G)$).

Corollary 7.6. *Let G be a countable directed graph with no sources and no sinks. Let $T = (T_e)_{e\in E(G)}$ be a row contraction on $\mathcal{H}$ stabilized as in (8) by a given family of projections. Then*

$$||p(T, T^*)|| \leq ||p(L_G, L^*_G)|| \quad \textit{for} \quad p \in \mathcal{P}^0_G. \tag{12}$$

Proof. Let $S = (S_e)_{e\in E(G)}$ be the minimal partially isometric dilation of T with respect to the given projection family. From the basic properties of the minimal dilation, it follows that

$$p(T, T^*) = P_{\mathcal{H}}\, p(S, S^*)|_{\mathcal{H}} \quad \text{for} \quad p \in \mathcal{P}_G^0.$$

Thus, Theorem 7.5 gives $||p(T, T^*)|| \leq ||p(S, S^*)|| \leq ||p(L_G, L_G^*)||$ for all $p \in \mathcal{P}_G^0$. ∎

The next theorem is the promised generalization of Coburn's Theorem. Given a collection $\{S'_e\}_{e\in\mathcal{E}(G)}$ of isometries, we isolate a condition which is the appropriate extension of an isometry being non-unitary here (condition (13) below). We show that any collection of partial isometries satisfying this condition generates a C*-algebra isomorphic to $\mathrm{C}^*(\mathcal{A}_G)$.

Theorem 7.7. *Let G be a countable directed graph with no sources and no sinks and let $\mathrm{C}^*(\mathcal{A}_G)$ be the C*-algebra generated by the quiver algebra $\mathcal{A}_G$. Let $\{P'_x\}_{x\in\mathcal{V}(G)}$ and $\{S'_e\}_{e\in\mathcal{E}(G)}$ be families of projections and partial isometries respectively, acting on a Hilbert space $\mathcal{H}$ and satisfying (†). If,*

$$\sum_{r(e)=x} S'_e(S'_e)^* \neq P'_x \quad \textit{for} \quad x \in \mathcal{V}(G), \tag{13}$$

then there exists an injective $*$-homorphism $\phi : \mathrm{C}^*(\mathcal{A}_G) \to B(\mathcal{H})$ such that $\phi(L_e) = S'_e$ for all $e \in \mathcal{E}(G)$.

In particular, the C-algebra $\mathrm{C}^*(\{S'_e\}_{e\in\mathcal{E}(G)})$, generated by the collection $\{S'_e\}_{e\in\mathcal{E}(G)}$, is isomorphic to $\mathrm{C}^*(\mathcal{A}_G)$.*

Proof. We utilize the Wold decomposition for the collection $\{S'_e\}_{e\in\mathcal{E}(G)}$. Recall the existence of reducing subspaces $\mathcal{H}_c$ and $\mathcal{H}_p$ for $\{S'_e\}_{e\in\mathcal{E}(G)}$ such that $\mathcal{H} = \mathcal{H}_p \oplus \mathcal{H}_c$ and the restriction of $\{S'_e\}_{e\in\mathcal{E}(G)}$ on $\mathcal{H}_p$ (respectively $\mathcal{H}_c$) is pure (respectively fully coisometric). Moreover, if

$$\mathcal{W} \equiv \mathrm{Ran}\left(I - \sum_e S'_e(S'_e)^*\right)$$

then $\mathcal{W}$ is wandering for $\{S'_e\}_{e\in\mathcal{E}(G)}$ and the smallest invariant subspace for $\{S'_e\}_{e\in\mathcal{E}(G)}$ containing $\mathcal{W}$ equals $\mathcal{H}_p$. In addition, the restrictions $S'_e|_{\mathcal{H}_p}$, $e \in \mathcal{E}(G)$, are jointly unitarily equivalent to

$$S'_e|_{\mathcal{H}_p} \simeq \sum_{x\in\mathcal{V}(G)} \oplus L_e^{(\alpha_x)}|_{(Q_x\mathcal{H}_G)^{(\alpha_x)}},$$

where,

$$\alpha_x = \text{rank}\left(P_x^{'} - \sum_{r(e)=x} S_e'(S_e')^*\right).$$

Let $\mathcal{W}_x = P_x'\mathcal{W} \neq 0$ and let $\zeta_x \in \mathcal{W}_x$ be a unit vector. Notice that the subspaces

$$\mathcal{M}_x \equiv \text{span}\{S_w\zeta_x : w \in \mathbb{F}^+(G)\}$$

are reducing for $\{S_e'\}_{e\in\mathcal{E}(G)}$ and the restrictions $S_e'|_{M_x}$, $e \in \mathcal{E}(G)$, are jointly unitarily equivalent to $L_e|_{Q_x}$, $e \in \mathcal{E}(G)$, respectively. Therefore, the subspace $\mathcal{M} = \sum \oplus_{x\in\mathcal{V}(G)}\mathcal{M}_x$ is reducing for $\{S_e'\}_{e\in\mathcal{E}(G)}$ and the restrictions $S_e'|_{\mathcal{M}_x}$, $e \in \mathcal{E}(G)$, are jointly unitarily equivalent to L_e, $e \in \mathcal{E}(G)$. We have therefore established the existence of an epimorphism

$$\psi : \mathrm{C}^*(\{S_e'\}_{e\in\mathcal{E}(G)}) \longrightarrow \mathrm{C}^*(\mathcal{A}_G)$$

mapping generators to generators. It remains to show that ψ is injective.

Consider the diagram

$$\mathrm{C}^*(\mathcal{A}_G) \xrightarrow{\pi} \mathrm{C}^*(\mathcal{A}_G)/\mathcal{K} \xrightarrow{\rho} \mathrm{C}^*(G) \xrightarrow{\tau} \mathrm{C}^*(\{S_e'|_{\mathcal{H}_c}\}_{e\in\mathcal{E}(G)}),$$

where π is the natural quotient map, ρ is as in Lemma 7.2 and τ exists because $\mathrm{C}^*(G)$ is universal. Let

$$\phi_1 \equiv \tau\circ\rho\circ\pi : \mathrm{C}^*(\mathcal{A}_G) \longrightarrow \mathrm{C}^*(\{S_e'|_{\mathcal{H}_c}\}_{e\in\mathcal{E}(G)})$$

and note that $\phi_1(L_e) = S_e'|_{\mathcal{H}_c}$, for all $e \in \mathcal{E}(G)$. Consider now the map $\phi : \mathrm{C}^*(\mathcal{A}_G) \longrightarrow \mathrm{C}^*(\{S_e'\}_{e\in\mathcal{E}(G)})$, defined as

$$p(L_e, L_e^*) \longmapsto p(S_e', (S_e')^*)|_{\mathcal{H}_p} \oplus \phi_1(p(L_e, L_e^*)),$$

for any non-commutative polynomial p. The Wold decomposition shows that ϕ is a $*$-homomorphism and the earlier discussion shows that $\phi(L_e) = S_e'$, for all $e \in \mathcal{E}(G)$. Therefore, $\phi\circ\psi = id$ and so ψ is injective, which proves the theorem. ∎

Remark 7.8. It was pointed to us by the referee that Theorem 7.7 has been previously obtained by Fowler and Raeburn [**13**, Theorem 4.1]. Their approach is quite different from ours since they make an extensive use of Hilbert C*-modules as well as their own methods [**14**] for analysing semigroup crossed products. Our proof is shorter and elementary; we only require Szymanski's result, whose proof in [**39**] is essentially self-contained.

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