ISOMORPHISMS OF ALGEBRAS ASSOCIATED WITH DIRECTED GRAPHS

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Abstract. Given countable directed graphs G and G', we show that the associated tensor algebras $\mathcal{T}_+(G)$ and $\mathcal{T}_+(G')$ are isomorphic as Banach algebras if and only if the graphs G are G' are isomorphic. For tensor algebras associated with graphs having no sinks or no sources, the graph forms an invariant for algebraic isomorphisms. We also show that given countable directed graphs G, G', the free semigroupoid algebras $\mathfrak{L}_G$ and $\mathfrak{L}_{G'}$ are isomorphic as dual algebras if and only if the graphs G are G' are isomorphic. In particular, spatially isomorphic free semigroupoid algebras are unitarily isomorphic. For free semigroupoid algebras associated with locally finite directed graphs with no sinks, the graph forms an invariant for algebraic isomorphisms as well.

1. introduction and Preliminaries

Let G be a countable directed graph with vertex set $\mathcal{V}(G)$, edge set $\mathcal{E}(G)$ and range and source maps r and s respectively. The *Toeplitz algebra of* G, denoted as $\mathcal{T}(G)$, is the universal C*-algebra generated by a set of partial isometries $\{S_e\}_{e\in\mathcal{E}(G)}$ and projections $\{P_x\}_{x\in\mathcal{V}(G)}$ satisfying the relations

$$(\dagger)\begin{cases} (1) & P_xP_y = 0 & \text{for all } x, y \in \mathcal{V}(G),\ x \neq y \\ (2) & S_e^*S_f = 0 & \text{for all } e, f \in \mathcal{E}(G),\ e \neq f \\ (3) & S_e^*S_e = P_{s(e)} & \text{for all } e \in \mathcal{E}(G) \\ (4) & \sum_{r(e)=x} S_eS_e^* \leq P_x & \text{for all } x \in \mathcal{V}(G). \end{cases}$$

The existence of such a universal object is implicit in [**21**, Theorem 3.4] and [**18**, Theorem 2.12] and was made explicit in [**9**, Proposition 1.3 and Theorem 4.1].

Definition 1.1. Given a countable directed graph G, the *tensor algebra of* G, denoted as $\mathcal{T}_+(G)$, is the norm closed subalgebra of $\mathcal{T}(G)$ generated by the partial isometries $\{S_e\}_{e\in\mathcal{E}(G)}$ and projections $\{P_x\}_{x\in\mathcal{V}(G)}$.

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The tensor algebras associated with graphs were introduced under the name *quiver algebras* by Muhly and Solel in [**18**] and were further studied in [**19**]. They were defined rather differently, in the setting of C*-correspondences, but by [**8**, Corollary 2.2] our Definition 1.1 is equivalent to the original one.

The primary objective in this paper is to relate the structure of a tensor algebra to its graph and show that the graph forms an invariant for bicontinuous isomorphisms between tensor algebras of graphs (Theorem 2.11). For tensor algebras associated with graphs having no sinks or no sources, we show the graph forms an invariant for algebraic isomorphisms as well (Corollary 2.15). For this purpose we prove an automatic continuity theorem, Theorem 2.14, of independent interest.

It is easy to see that if two graphs G, G' are isomorphic, then the tensor algebras $\mathcal{T}_+(G)$ and $\mathcal{T}_+(G')$ are isomorphic by the restriction of a C*-isomorphism between the associated Toeplitz algebras. Therefore the focus will be on proving that if $\mathcal{T}_+(G)$ is isomorphic to $\mathcal{T}_+(G')$ as a Banach algebra, then G and G' are isomorphic. The proof of Theorem 2.11 rests on analyzing certain representations for a tensor algebra $\mathcal{T}_+(G)$. The one dimensional representations (characters) of $\mathcal{T}_+(G)$ are parametrized by $\mathcal{V}(G)$ and points in the unit ball of a complex n-space for various n (Proposition 2.1). The two dimensional nest representations of $\mathcal{T}_+(G)$ allow us to identify the edges in a way that reveals how they are "connected" and so reveals the graph as an artifact of the representation theory of $\mathcal{T}_+(G)$ (Theorem 2.6). Independently, similar ideas were used by Solel [**26**] to prove an analogue of Theorem 2.11 for *isometric* algebra isomorphisms. (The two papers overlap on the identification of the character spaces, Proposition 2.1 and Theorem 3.5, and the use of nest representations. Note however that the representations in [**26**] are assumed to be contractive while we are forced to make no assumptions on ours, apart from continuity.) Our weaker hypothesis leads to complications when "counting" edges between distinct vertices that need to be addressed (Theorem 2.10).

Let $\lambda_{G,0}$ be the multiplication representation of $c_0(\mathcal{V}(G))$ on $l^2(\mathcal{V}(G))$, determined by the counting measure on $\mathcal{V}(G)$, and let λ_G denote the representation of $\mathcal{T}(G)$ induced by $\lambda_{G,0}$, in the sense of [**19**] and [**8**]. It is easily seen that the Hilbert space $\mathcal{H}_G$ of λ_G is $l^2(\mathbb{F}^+(G))$, where $\mathbb{F}^+(G)$ denotes the *free semigroupoid* of the graph G (also called the *path space* of G). This consists of all vertices $v \in \mathcal{V}(G)$ and all paths $w = e_k e_{k-1} \dots e_1$, where the e_i are edges satisfying $s(e_i) = r(e_{i-1})$, $i = 2, 3, \dots, k$, $k \in \mathbb{N}$. (Paths of the form $w = e_k e_{k-1} \dots e_1$ are said to have length k, denoted as $|w| = k$, and vertices are called paths of length 0.) The maps r and s extend to $\mathbb{F}^+(G)$ in the obvious way, two paths w_1 and w_2 are composable precisely when $s(w_2) = r(w_1)$ and, in that case, the composition $w_2 w_1$ is just the concatenation of w_1 and w_2. Let $\{\xi_w\}_{w\in\mathbb{F}^+(G)}$ denote the usual orthonormal basis of $l^2(\mathbb{F}^+(G))$, where ξ_w is the characteristic function of $\{w\}$. Then, $\lambda_G(S_e)$, $e \in \mathcal{E}(G)$, equals the left creation operator $L_e \in \mathcal{B}(l^2(\mathbb{F}^+(G)))$ defined by

$$L_e \xi_w = \begin{cases} \xi_{ew} & \text{if } s(e) = r(w) \\ 0 & \text{otherwise.} \end{cases}$$

(We shall write P_v for $\lambda_G(P_v)$.) By [**8**, Corollary 2.2], the representation λ_G is a faithful representation of $\mathcal{T}_+(G)$, being the restriction of a faithful representation of the corresponding Toeplitz algebra. For the rest of the paper, we will identify $\mathcal{T}_+(G)$ with its image under λ_G, i.e., we consider $\mathcal{T}_+(G)$ to be the norm closed algebra generated by the operators L_e and P_v defined above.

Definition 1.2. The weak closure of $\lambda_G(\mathcal{T}_+(G)) \subseteq \mathcal{B}(l^2(\mathbb{F}^+(G)))$ is called the *free semigroupoid algebra* of G and is denoted as $\mathfrak{L}_G$.

The second author and Power began a systematic study for the free semigroupoid algebras in [**14, 15**]. These algebras include as special cases the space H^∞ realized as the analytic Toeplitz algebra [**7, 10**] and the *non commutative analytic Toeplitz algebras* $\mathfrak{L}_n$, introduced by Popescu [**22, 23, 24**] and studied by Davidson and Pitts [**3, 4, 5**], as well as Arias [**1**] and the authors [**2, 16**]. However, this is a broad enough class to include various nest algebras, inflation algebras and infinite matrix function algebras as special cases.

The second main result of the paper, Theorem 3.9, asserts that given countable directed graphs G and G', the free semigroupoid algebras $\mathfrak{L}_G$ and $\mathfrak{L}_{G'}$ are isomorphic as dual algebras if and only if G are G' are isomorphic as graphs. This improves one of the main results of [**14**], which showed that the graph forms an invariant for unitary isomorphism between free semigroupoid algebras, and provided the initial motivation for this paper. Indepedently, Solel [**26**] proved a similar result for *isometric* $w*$-bicontinuous isomorphisms. (Our result can be thought as an extension of Solel's result to Hardy algebras [**20**] associated with finite graphs.) In particular, we show that spatially isomorphic free semigroupoid algebras are unitarily isomorphic. Our methods show, in fact, that with a slight additional hypothesis, the graph forms an invariant for algebraic isomorphisms. While the arguments in Section 3 follow the general line used in Section 2, the proofs are more involved.

2. The classification of tensor algebras of graphs

Given a countable directed graph G, the character space of $\mathcal{T}_+(G)$ is denoted as $\mathfrak{M}_G$. It consists of all non-zero multiplicative linear functionals on $\mathcal{T}_+(G)$ and is equipped with the w^*-topology. Given $x \in \mathcal{V}(G)$, the set of all $\rho \in \mathfrak{M}_G$ so that $\rho(P_x) = 1$ is denoted as $\mathfrak{M}_{G,x}$. Notice that the finite sums from $\{P_x\}_{x\in\mathcal{V}(G)}$ form an approximate identity for $\mathcal{T}_+(G)$. Therefore, given $\rho \in \mathfrak{M}_G$ there exists at least one $x \in \mathcal{V}(G)$ so that $\rho(P_x) = 1$; the orthogonality of $\{P_x\}_{x\in\mathcal{V}(G)}$ guarantees that such an x is unique. Given any edge e with distinct source and range, the partial isometry L_e is nilpotent and so $\rho(L_e) = 0$, for any $\rho \in \mathfrak{M}_G$. Hence, given $\rho \in \mathfrak{M}_{G,x}$ and a path w, we have $\rho(L_w) = 0$ whenever w passes through a vertex $y \neq x$.

Proposition 2.1. *Let G be a countable directed graph. Let $x \in \mathcal{V}(G)$ and assume that there exist exactly n distinct loop edges whose source is x. Then, $\mathfrak{M}_{G,x}$ equipped with the w^*-topology is homeomorphic to the closed unit ball $\mathbb{B}_n$ inside $\mathbb{C}^n$.*

Proof. Let $e_1, e_2, \dots, e_n$ be the distinct loop edges with source $x \in \mathcal{V}(G)$. We define a map $\phi_x : \mathfrak{M}_{G,x} \longrightarrow \mathbb{C}^n$ by the formula

$$\phi_x(\rho) = (\rho(L_{e_1}), \rho(L_{e_2}), \dots, \rho(L_{e_n})), \quad \rho \in \mathfrak{M}_{G,x}.$$

Since any multiplicative form is completely contractive, and the L_{e_i} are partial isometries with mutually orthogonal ranges, it follows that the range of ϕ_x is contained in $\mathbb{B}_n$. We will show that ϕ_x is the desired homeomorphism.

Let $\lambda = (\lambda_1, \lambda_2, \dots \lambda_n)$ be an n-tuple satisfying $\|\lambda\|_2^2 = \sum_{i=1}^n |\lambda_i|^2 < 1$; that is, λ belongs to the interior of $\mathbb{B}_n$. We define

$$v_{\lambda,x} = (1 - \|\lambda\|_2^2) \sum_w \overline{w(\lambda)} \xi_w,$$

where w in the above sum ranges over all monomials in $e_1, e_2, \dots, e_n$. (This construction originates from **[1, 4]** and was reiterated in **[14]**). It is easily seen that $v_{\lambda,x}$ is an eigenvalue for $\mathcal{T}_+(G)^*$ satisfying $P_x v_{\lambda,x} = v_{\lambda,x}$ and $L_{e_i}^* v_{\lambda,x} = \overline{\lambda_i} v_{\lambda,x}$. Therefore, the linear form $\rho_{\lambda,x}$ defined as

$$\rho_{\lambda,x}(A) = \langle A v_{\lambda,x}, v_{\lambda,x} \rangle, \quad A \in \mathcal{T}_+(G),$$

belongs to $\mathfrak{M}_{G,x}$.

The above paragraph, combined with a compactness argument, shows that $\phi_x(\mathfrak{M}_{G,x}) = \mathbb{B}_n$. Now we show that ϕ_x is injective. If $\rho_1, \rho_2 \in \mathfrak{M}_{G,x}$ so that $\phi_x(\rho_1) = \phi_x(\rho_2)$, then $\rho_1(L_{e_i}) = \rho_2(L_{e_i})$, $i = 1, 2, \dots, n$. As we have mentioned earlier, $\rho_1(L_f) = \rho_2(L_f) = 0$ for all other $f \in \mathcal{E}(G)$ and so ρ_1 and ρ_2 agree on the generators of $\mathcal{T}_+(G)$. By continuity, $\rho_1 = \rho_2$.

We have established that ϕ_x is an continuous injective map between compact Hausdorff spaces. Hence ϕ_x^{-1} is also continuous. ∎

Remark 2.2. If $n = \infty$, then a easy modification of the above proof shows that $\mathfrak{M}_{G,x}$ is isomorphic to the closed unit ball of l_2, equipped with the weak topology.

Corollary 2.3. *If G is a countable directed graph, then $\mathfrak{M}_G$ is a locally compact Hausdorff space whose connected components coincide with $\mathfrak{M}_{G,x}$, $x \in V(G)$.*

Proof. The space $\mathfrak{M}_G \cup \{0\}$ is a compact Hausdorff space and so $\mathfrak{M}_G$ is locally compact. Fix an $x \in \mathcal{V}(G)$ and consider the map $\psi_x : \mathfrak{M}_G \longrightarrow \mathbb{C}$ defined as

$$\psi_x(\rho) = \rho(P_x), \quad \rho \in \mathfrak{M}_G.$$

Clearly, ψ_x is continuous and $\mathfrak{M}_{G,x} = \psi_x^{-1}(\{1\}) = \psi_x^{-1}((0,2))$. Hence, $\mathfrak{M}_{G,x}$ is clopen. Proposition 2.1 shows that $\mathfrak{M}_{G,x}$ is also connected and the conclusion follows. ∎

The character space $\mathfrak{M}_G$ allows us to identify the vertices of the graph G with the connected components of $\mathfrak{M}_G$. In order to decide whether there exists a directed edge between two given vertices we use nest representation theory.

Let $\pi : \mathcal{T}_+(G) \longrightarrow \operatorname{Alg}\mathcal{N}$ be a two dimensional nest representation of $\mathcal{T}_+(G)$, where G is a countable directed graph. Assume that the nest of projections $\mathcal{N} = \{0, N, I\}$ is acting on a Hilbert space $\mathcal{H}$ and so there exists a basis $\{h_1, h_2\}$ of $\mathcal{H}$ with $N = [h_2]$ and $N^{\perp} = [h_1]$. We may associate with π two multiplicative linear forms $\rho_\pi^{(1)}$ and $\rho_\pi^{(2)}$, defined as

$$\rho_\pi^{(i)}(A) = \langle \pi(A)h_i, h_i \rangle, \quad A \in \mathcal{T}_+(G). \tag{1}$$

Lemma 2.4. *Let G be a countable directed graph, let $\pi : \mathcal{T}_+(G) \longrightarrow \operatorname{Alg}\mathcal{N}$ be a two dimensional nest representation and let $\rho_\pi^{(i)} \in \mathfrak{M}_{G,x_i}$, $i = 1, 2$, be as in (1). If $x_1 \neq x_2$, then there exists an edge $e \in \mathcal{E}(G)$ so that $e = x_2 e x_1$.*

Proof. By way of contradiction assume that there are no edges in G with source x_1 and range x_2. There exist $a, b \in \mathbb{C}$ so that

$$\pi(P_{x_1}) = \begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \pi(P_{x_2}) = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}.$$

(However, $\pi(P_{x_2})\pi(P_{x_1}) = 0$ and this implies that $a = -b$.)

Let $S \in \mathcal{T}_+(G)$ so that $\pi(S) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Without loss of generality we may assume that $S = P_{x_2} S P_{x_1}$ since

$$\pi(P_{x_2} S P_{x_1}) = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Hence, S can be approximated in norm by polynomials in L_w, with $w = x_2 w x_1 \in \mathbb{F}^+(G)$. This implies the existence of one such $w = x_2 w x_1 \in \mathbb{F}^+(G)$ with $\pi(L_w) = \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right)$, for some non-zero $a \in \mathbb{C}$. However, by assumption in this case we have

$$L_w = A L_e B L_f C,$$

where $A, B, C \in \mathcal{T}_+(G)$ and $e, f \in \mathcal{E}(G)$ satisfy, $s(e) \neq r(e)$ and $s(f) \neq r(f)$. Both $\pi(A L_e)$ and $\pi(B L_f C)$ are then nilpotent of order two and so their product is zero. So, $\pi(L_w) = 0$, which is a contradiction. ∎

Definition 2.5. Given two vertices $x_1, x_2 \in \mathcal{V}(G)$ define $\mathrm{rep}_{x_1,x_2}(\mathcal{T}_+(G))$ to be the collection of all two dimensional nest representations $\pi : \mathcal{T}_+(G) \longrightarrow \mathrm{Alg}\,\mathcal{N}$ determined as in Lemma 2.4 by edges $e \in \mathcal{E}(G)$ with $e = x_2 e x_1$; i.e., so that $\rho_\pi^{(i)} \in \mathfrak{M}_{G,x_i}$, $i = 1, 2$.

Theorem 2.6. *Let G be a countable directed graph and let $x_1 \neq x_2$ be two distinct vertices in $\mathcal{V}(G)$. Then $\mathrm{rep}_{x_1,x_2}(\mathcal{T}_+(G)) \neq \emptyset$ if and only if there exists an edge $e \in \mathcal{E}(G)$ with $s(e) = x_1$ and $r(e) = x_2$.*

Proof. One direction follows from Lemma 2.4. For the other direction assume that there exists an edge $e = x_2 e x_1$. Consider the Hilbert space $\mathcal{H}_e \subseteq \mathcal{H}_G$ generated by the orthogonal vectors ξ_{x_1}, ξ_e and let E_e be the orthogonal projection on $\mathcal{H}_e$. Notice that for any edge $f \in \mathcal{E}(G)$, $L_f^* \xi_{x_1} = 0$. In addition, either $L_f^* \xi_e = \xi_{x_1}$ or $L_f^* \xi_e = 0$, for all $f \in \mathcal{E}(G)$. In any case, the space $\mathcal{H}_e$ is co-invariant by $\mathcal{T}_+(G)$ and so the mapping $\pi_e(A) = E_e A E_e$, $A \in \mathcal{T}_+(G)$, defines a representation for $\mathcal{T}_+(G)$ on $\mathcal{H}_e$.

Let N be the subspace generated by ξ_e and let $\mathcal{N} = \{0, N, I\}$. The previous considerations show that N is invariant by $E_e \mathcal{T}_+(G) E_e$ and so π_e maps into $\mathrm{Alg}\,\mathcal{N}$. In addition, $\pi_e(P_{x_1}) = N^\perp$, $\pi_e(P_{x_2}) = N$ and

$$\pi_e(L_e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Hence, π_e is surjective. Finally, it is clear that $\rho_{\pi_e}^{(i)} \in \mathfrak{M}_{G,x_i}$, $i = 1, 2$, and so $\mathrm{rep}_{x_1,x_2}(\mathcal{T}_+(G)) \neq \emptyset$. ∎

In order to calculate the number of edges between two vertices x, y of G, we need to gain a better understanding of $\mathrm{rep}_{x,y}(\mathcal{T}_+(G))$. We define

$$\mathcal{K}_{x,y} = \bigcap \big\{ \ker \pi \mid \pi \in \mathrm{rep}_{x,y}(\mathcal{T}_+(G)) \big\}.$$

Notice that each $\pi \in \mathrm{rep}_{x,y}(\mathcal{T}_+(G))$ induces a natural representation of $\mathcal{T}_+(G)/\mathcal{K}_{x,y}$, which we also denote as π, and is defined as

$$\pi(A + \mathcal{K}_{x,y}) = \pi(A), \quad A \in \mathcal{T}_+(G).$$

Recall that the *Jacobson radical* $\mathrm{rad}(\mathcal{A})$ of a complex algebra $\mathcal{A}$ is defined as the intersection of the kernels of the irreducible representations of $\mathcal{A}$. It is known that for any complex algebra $\mathcal{A}$, the Jacobson radical $\mathrm{rad}(\mathcal{A})$ coincides with the largest ideal of $\mathcal{A}$ consisting of quasinilpotent elements.

Proposition 2.7. *Let G be a countable directed graph and let $x, y \in \mathcal{V}(G)$ with $x \neq y$. Then $A + \mathcal{K}_{x,y} \in \mathrm{rad}(\mathcal{T}_+(G)/\mathcal{K}_{x,y})$ if and only if $\pi(A)^2 = 0$, for all $\pi \in \mathrm{rep}_{x,y}(\mathcal{T}_+(G))$.*

Proof. Assume first that $A + \mathcal{K}_{x,y} \in \mathrm{rad}(\mathcal{T}_+(G)/\mathcal{K}_{x,y})$. Then given any $\pi \in \mathrm{rep}_{x,y}(\mathcal{T}_+(G))$, $\pi(A) = \pi(A + \mathcal{K}_{x,y})$ is quasinilpotent. Therefore, $\pi(A)$ is a strictly upper triangular 2×2 matrix and so $\pi(A)^2 = 0$.

Conversely, it is easily seen that the collection of all cosets $A + \mathcal{K}_{x,y} \in \mathcal{T}_+(G)/\mathcal{K}_{x,y}$ satisfying $\pi(A + \mathcal{K}_{x,y})^2 = 0$, $\forall \pi \in \mathrm{rep}_{x,y}(\mathcal{T}_+(G))$, forms an ideal of $\mathcal{T}_+(G)/\mathcal{K}_{x,y}$. Therefore, any such coset is contained in $\mathrm{rad}(\mathcal{T}_+(G)/\mathcal{K}_{x,y})$ and the conclusion follows. ■

Lemma 2.8. *Let G be a countable directed graph and let $x, y \in \mathcal{V}(G)$ with $x \neq y$. Then*

$$A + \mathcal{K}_{x,y} = P_y A P_x + \mathcal{K}_{x,y}$$

for any $A + \mathcal{K}_{x,y} \in \mathrm{rad}(\mathcal{T}_+(G)/\mathcal{K}_{x,y})$.

Proof. Let $\pi \in \mathrm{rep}_{x,y}(\mathcal{T}_+(G))$. Proposition 2.7 implies the existence of a $c \in \mathbb{C}$ so that $\pi(A) = \left(\begin{smallmatrix} 0 & c \\ 0 & 0 \end{smallmatrix}\right)$. Arguing as in the proof of Theorem 2.4 we obtain

$$\pi(P_y A P_x) = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}.$$

and so

$$\pi(A - P_y A P_x) = 0.$$

Since $\pi \in \mathrm{rep}_{x,y}(\mathcal{T}_+(G))$ was arbitrary, the conclusion follows. ■

Lemma 2.9. *Let G be a countable directed graph and let $x, y \in \mathcal{V}(G)$ with $x \neq y$. Assume that there exist n distinct edges $e_1, e_2, \dots, e_n$ with source x and range y. Let $A = P_y A P_x \in \mathcal{K}_{x,y}$ and let $A \sim \sum a_w L_w$ be the Fourier expansion of A [**14**]. Then, $a_{e_i} = 0$, for all $i = 1, 2, \dots, n$.*

Proof. Let $\pi_{e_i} \in \operatorname{rep}_{x,y}(\mathcal{T}_+(G))$, $i = 1, 2, \dots, n$ be as in the proof of Theorem 2.6. As we have seen, $\pi_{e_i}(L_{e_i}) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, while $\pi_{e_i}(L_f) = 0$, for all other $f \in \mathcal{E}(G)$. Hence,

$$\pi_{e_i}(A) = a_{e_i} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0,$$

and so $a_{e_i} = 0$ for $i = 1, 2, \dots, n$. ∎

Let $\mathcal{I}$ be a closed ideal of a Banach algebra $\mathcal{B}$ and let $\{B_i\}_{i\in\mathbb{I}}$ be a subset of $\mathcal{I}$. The set $\{B_i\}_{i\in\mathbb{I}}$ is said to be a *generating set* for $\mathcal{I}$ if the closed ideal generated by $\{B_i\}_{i\in\mathbb{I}}$ equals $\mathcal{I}$. (In that case, the B_i's are said to be the *generators* of $\mathcal{I}$.) The ideal $\mathcal{I}$ is said to be n*-generated* when n is the smallest cardinality of a generating set. If $n = \infty$, we understand n-generated to mean that there is no finite generating set for $\mathcal{I}$.

Theorem 2.10. *Let G be a countable directed graph and let $x, y \in \mathcal{V}(G)$ with $x \neq y$. Assume that there exist n distinct edges with source x and range y. Then $\operatorname{rad}(\mathcal{T}_+(G)/\mathcal{K}_{x,y}) \subseteq \mathcal{T}_+(G)/\mathcal{K}_{x,y}$ is n-generated.*

Proof. Let $e_1, e_2, \dots, e_n$ be the edges with source x and range y. First we show that the collection,

$$L_{e_i} + \mathcal{K}_{x,y}, \quad i = 1, 2, \dots, n,$$

is a generating set for $\operatorname{rad}(\mathcal{T}_+(G)/\mathcal{K}_{x,y})$.

Indeed, let $A+\mathcal{K}_{x,y} = P_y A P_x + \mathcal{K}_{x,y} \in \operatorname{rad}(\mathcal{T}_+(G)/\mathcal{K}_{x,y})$ by Lemma 2.8, and let

$$p(L) = \sum_{w_k = y w_k x} a_{w_k} L_{w_k} \; + \; \mathcal{K}_{x,y}$$

be a polynomial which is ϵ-close to $A+\mathcal{K}_{x,y}$. We may assume that each L_{w_k} in the sum above factors as $L_{w_k} = L_{u_k} L_{e_i} L_{v_k}$, for some $u_k, v_k \in \mathbb{F}^+(G)$ and $i \in \{1, 2, \dots, n\}$. (An L_{w_k} not of this form is easily seen to belong to $\mathcal{K}_{x,y}$ since $\pi(P_z) = 0$ for $\pi \in \operatorname{rep}_{x,y}(\mathcal{T}_+(G))$ and $z \notin \{x, y\}$.) Hence each $L_{w_k} + \mathcal{K}_{x,y}$ belongs to the ideal generated by $L_{e_i} + \mathcal{K}_{x,y}$, $i = 1, 2, \dots, n$, and so does $p(L)+\mathcal{K}_{x,y}$. But the polynomials $p(L)+\mathcal{K}_{x,y}$ approximate $A + \mathcal{K}_{x,y}$ and so $L_{e_i} + \mathcal{K}_{x,y}$, $i = 1, 2, \dots, n$, is a generating set for $\operatorname{rad}(\mathcal{T}_+(G)/\mathcal{K}_{x,y})$.

By way of contradiction assume that there exists a generating set

$$A_j + \mathcal{K}_{x,y}, \quad j = 1, 2, \dots, m,$$

for $\operatorname{rad}(\mathcal{T}_+(G)/\mathcal{K}_{x,y})$ with $m < n$. By Lemma 2.8, we may assume that $A_j = P_y A_j P_x$. Therefore, there exist scalars $a_{j,i}$ so that

$$\Phi_1(A_j) = a_{j,1} L_{e_1} + a_{j,2} L_{e_2} + \cdots + a_{j,n} L_{e_n},$$

where Φ_1 is the contractive idempotent on $\mathcal{T}_+(G)$ defined by the formula

$$\Phi_1\Big(\sum_w a_w L_w\Big) = \sum_{|w|=1} a_w L_w.$$

Let $\mathcal{M}$ be the subspace of $\mathcal{T}_+(G)$ generated by L_{e_i}, $i = 1, 2, \ldots, n$, and let $\mathcal{M}_0 \subset \mathcal{M}$ be the linear span of

$$a_{j,1}L_{e_1} + a_{j,2}L_{e_2} + \cdots + a_{j,n}L_{e_n}, \quad j = 1, 2, \ldots, m.$$

Let $X \in \mathcal{M}\backslash\mathcal{M}_0$. Clearly, $X + \mathcal{K}_{x,y}$ belongs to $\operatorname{rad}(\mathcal{T}_+(G)/\mathcal{K}_{x,y})$. Since the collection $A_j+\mathcal{K}_{x,y}$, $j = 1, 2, \ldots, m$, is generating for $\operatorname{rad}(\mathcal{T}_+(G)/\mathcal{K}_{x,y})$, the operator X can be approximated by finite sums of the form

$$\sum_k B_k \hat{A}_k C_k + D, \tag{2}$$

where, $B_k, C_k \in \mathcal{T}_+(G)$, and $D = P_y D P_x \in \mathcal{K}_{x,y}$ and $\hat{A}_k \in \{A_1, A_2, \ldots A_m\}$. Lemma 2.9 shows now that $\Phi_1(D) = 0$. It is also easy to see that

$$\Phi_1(BA_jC) = \mu_j\big(a_{j,1}L_{e_1} + a_{j,2}L_{e_2} + \cdots + a_{j,n}L_{e_n}\big), \quad \mu_j \in \mathbb{C},$$

for all $j = 1, 2, \ldots, m$ and $B, C \in \mathcal{T}_+(G)$. Hence,

$$\Phi_1\left(\sum_k B_k \hat{A}_k C_k + D\right) \in \mathcal{M}_0. \tag{3}$$

A simple approximation argument with the contractive map Φ_1 shows now that $\Phi_1(X) = X$ belongs to $\mathcal{M}_0$, which is a contradiction. ∎

We are in position now to prove the classification theorem for the tensor algebras of graphs. Given a directed graph G, we associate a graph $\mathfrak{G}$, which is constructed using the algebraic structure of $\mathcal{T}_+(G)$ as follows. The vertices of $\mathfrak{G}$ are the connected components of the character space $\mathfrak{M}_G$ of $\mathcal{T}_+(G)$. By Corollary 2.3 there is a natural one-to-one correspondence $g : \mathcal{V}(\mathfrak{G}) \longrightarrow \mathcal{V}(G)$ so that $\rho(P_{g(x)}) = 1$, for all $\rho \in x$. To each vertex $x \in \mathcal{V}(\mathfrak{G})$, we attach as many loop edges as the homeomorphic class of x. By Proposition 2.1, the number of loop edges starting at x and $g(x)$ are equal. Given two distinct vertices $x_1, x_2 \in \mathcal{V}(\mathfrak{G})$, we create as many directed edges from x_1 to x_2 as the least number of generators for $\operatorname{rad}(\mathcal{T}_+(G)/\mathcal{K}_{g(x_1),g(x_2)}) \subseteq \mathcal{T}_+(G)/\mathcal{K}_{g(x_1),g(x_2)}$, provided that $\operatorname{rep}_{g(x_1),g(x_2)}(G) \neq \emptyset$. Theorem 2.10 shows that the number of directed edges from x_1 to x_2 and $g(x_1)$ to $g(x_2)$ coincide. Hence, G and $\mathfrak{G}$ are isomorphic as graphs.

Theorem 2.11. *Let G, G' be countable directed graphs. The algebras $\mathcal{T}_+(G)$, $\mathcal{T}_+(G')$ are isomorphic as Banach algebras if and only if the graphs G are G' are isomorphic.*

Proof. Assume that the algebras $\mathcal{T}_+(G)$, $\mathcal{T}_+(G')$ are isomorphic as Banach algebras. It suffices to show that the graphs $\mathfrak{G}$ and $\mathfrak{G}'$ are isomorphic. Let

$$\tau : \mathcal{T}_+(G) \longrightarrow \mathcal{T}_+(G')$$

be a bicontinuous isomorphism between $\mathcal{T}_+(G)$ and $\mathcal{T}_+(G')$. Then, τ induces a homeomorphism between the character spaces of $\mathcal{T}_+(G)$ and $\mathcal{T}_+(G')$, which we still denote as τ, and is defined by the formula $\tau(\rho) = \rho \circ \tau^{-1}$, $\rho \in \mathfrak{M}_G$. Since homeomorphisms preserve connected components, τ establishes a one-to-one correspondence between the vertices of $\mathfrak{G}$ and $\mathfrak{G}'$, which we, once again, denote as τ. It is clear that the number of loop edges attached between x and $\tau(x)$, $x \in \mathcal{V}(\mathfrak{G})$, coincide.

Let x_1, x_2 be two distinct connected components of $\mathfrak{M}_G$.

Claim: $\pi \in \mathrm{rep}_{x_1,x_2}(G)$ if and only if $\pi \circ \tau^{-1} \in \mathrm{rep}_{\tau(x_1),\tau(x_2)}(G')$.

Assume that $\pi \in \mathrm{rep}_{x_1,x_2}(G)$. Let $\mathcal{N}$ be a nest such that π maps $\mathcal{T}_+(G)$ onto $\mathrm{Alg}\,\mathcal{N}$ and let h_1, h_2 be as in (1) so that

$$\rho_\pi^{(i)}(A) = \langle \pi(A)h_i, h_i\rangle, \quad A \in \mathcal{T}_+(G).$$

Assume that, $\rho_\pi^{(i)} \in x_i$, $i = 1, 2$. Now $\pi \circ \tau^{-1}$ is a representation of $\mathcal{T}_+(G')$ and by definition,

$$\rho_{\pi\circ\tau^{-1}}^{(i)}(A') = \langle \pi \circ \tau^{-1}(A')h_i, h_i\rangle, \quad A' \in \mathcal{T}_+(G').$$

Therefore, $\rho_{\pi\circ\tau^{-1}}^{(i)}(\tau(A)) = \rho_\pi^{(i)}(A)$ for $A \in \mathcal{T}_+(G)$ and so

$$\rho_{\pi\circ\tau^{-1}}^{(i)} = \rho_\pi^{(i)} \circ \tau^{-1} = \tau(\rho_\pi^{(i)}) \in \tau(x_i)$$

This proves one direction of the claim. By reversing the above argument, we obtain the other direction.

The claim above implies now that $\tau(\mathcal{K}_{x_1,x_2}) = \mathcal{K}_{\tau(x_1),\tau(x_2)}$ and so τ induces an isomorphism between $\mathcal{T}_+(G)/\mathcal{K}_{x,y}$ and $\mathcal{T}_+(G')/\mathcal{K}_{\tau(x_1),\tau(x_2)}$. Furthermore,

$$\tau\left(\mathrm{rad}(\mathcal{T}_+(G)/\mathcal{K}_{x_1,x_2})\right) = \mathrm{rad}(\mathcal{T}_+(G')/\mathcal{K}_{\tau(x_1),\tau(x_2)})$$

and so $\mathrm{rad}(\mathcal{T}_+(G)/\mathcal{K}_{x_1,x_2})$ is n-generated whenever $\mathrm{rad}(\mathcal{T}_+(G')/\mathcal{K}_{\tau(x_1),\tau(x_2)})$ is. Hence τ preserves the number of directed edges between distinct vertices of the graphs $\mathfrak{G}$ and $\mathfrak{G}'$. This concludes the proof. ■

Given an (algebraic) isomorphism or an epimorphism between two Banach algebras, it is always desirable to know whether or not it is continuous. This is the problem of *automatic continuity* and it has attracted much attention since the early stages of Banach algebra theory. An important tool for the study of this problem is Rickart's notion of a separating space. Let $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be an epimorphism between Banach algebras and let $\mathcal{S}(\phi)$ be the *separating space* of ϕ. This is the two-sided closed ideal of $\mathcal{B}$ defined as

$$\mathcal{S}(\phi) = \big\{ b \in \mathcal{B} \mid \exists \{a_n\}_n \subseteq \mathcal{A} \text{ such that } a_n \to 0 \text{ and } \phi(a_n) \to b \big\}.$$

One can easily see that the graph of ϕ is closed if and only if $\mathcal{S}(\phi) = \{0\}$. Thus by the closed graph theorem, ϕ is continuous if and only if $\mathcal{S}(\phi) = \{0\}$.

The following is an adaption of [**25**, Lemma 2.1] and was used in [**6**] for the study of isomorphisms between limit algebras.

Lemma 2.12 (Sinclair)**.** *Let $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be an epimorphism between Banach algebras and let $\{B_n\}_{n\in\mathbb{N}}$ be any sequence in $\mathcal{B}$. Then there exists $n_0 \in \mathbb{N}$ so that for all $n \geq n_0$,*

$$\overline{B_1 B_2 \dots B_n \mathcal{S}(\phi)} = \overline{B_1 B_2 \dots B_{n+1} \mathcal{S}(\phi)}$$

and

$$\overline{\mathcal{S}(\phi) B_n B_{n-1} \dots B_1} = \overline{\mathcal{S}(\phi) B_{n+1} B_n \dots B_1}.$$

The above lemma will allow us to classify a large class of tensor algebras up to algebraic isomorphism.

Definition 2.13. A vertex x of a countable directed graph is said to be a *sink* (resp. *source*) if it emits (resp. receives) no edges.

Theorem 2.14. *Let G be a countable directed graph which has either no sinks or no sources. If $\mathcal{A}$ is any Banach algebra and $\phi : \mathcal{A} \longrightarrow \mathcal{T}_+(G)$ is an epimorphism of algebras then ϕ is automatically continuous.*

Proof. Consider the extension $\hat{\phi} : \mathcal{A} + \mathbb{C}I \longrightarrow \mathcal{T}_+(G) + \mathbb{C}I$ of ϕ. It suffices to show that $\hat{\phi}$ is continuous.

Assume that G has no sinks. By way of contradiction assume that $\hat{\phi}$ is not continuous and so $\mathcal{S}(\hat{\phi}) \neq \{0\}$. If there are infinitely many distinct $x_n \in \mathcal{V}(G)$, $n \in \mathbb{N}$, so that $P_{x_n} \mathcal{S}(\hat{\phi}) \neq 0$, then the sequence

$$B_n = I - \sum_{i=1}^{n} P_{x_i}$$

contradicts Lemma 2.12. Hence there exists a vertex $x \in \mathcal{V}(G)$ and a path $w = ywx$, $y \in \mathcal{V}(G)$, so that $P_x \mathcal{S}(\hat{\phi}) \neq 0$ and the vertex y supports at least one loop, say $u \in \mathbb{F}^+(G)$. (Indeed, otherwise we could start from x, move forward on an infinite path with no loops, by multiplying on the left with the corresponding creation operators, and therefore produce infinitely many distinct P_x with $P_x\mathcal{S}(\phi) \neq 0$.) Hence $P_y\mathcal{S}(\hat{\phi}) \neq 0$ and we have

$$\overline{(L_u)^n\mathcal{S}(\hat{\phi})} = (L_u)^n\mathcal{S}(\hat{\phi}) \supset (L_u)^{n+1}\mathcal{S}(\hat{\phi}) = \overline{(L_u)^{n+1}\mathcal{S}(\hat{\phi})}.$$

This shows that the sequence $B_n = (L_u)^n$, $n \geq 1$, contradicts Lemma 2.12 and so ϕ is necessarily continuous.

An argument similar to the one above shows that $\hat{\phi}$ is also continuous when G has no sources. ■

Corollary 2.15. *Let G, G' be countable directed graphs which which have no sinks or no sources. Then $\mathcal{T}_+(G)$ and $\mathcal{T}_+(G')$ are algebraically isomorphic if and only if the graphs G and G' are isomorphic.*

In order to show that the graph of any tensor algebra is an invariant for algebraic isomorphisms, one has to extend Theorem 2.14 to arbitrary countable graphs. At the present, we do not know how to do this. Nevertheless, there are cases where the automatic continouity of the algebraic isomorphism is not needed, as the following result shows.

Corollary 2.16. *Let G, G' be countable directed graphs with no loop edges. Then $\mathcal{T}_+(G)$ and $\mathcal{T}_+(G')$ are algebraically isomorphic if and only if the graphs G and G' are isomorphic.*

Proof. In this case we assume that $\operatorname{rep}_{x,y}(G)$ consists of not necessarily continuous representations and we proceed as earlier. The difference here is that $\operatorname{rad}(A_G/\mathcal{K}_{x,y})$ is an n-dimensional vector space, provided that there exist exactly n-directed edges from x to y. This property is preserved by arbitrary isomorphisms and the conclusion follows. ■

3. An Application to Free Semigroupoid Algebra Theory

The free semigroupoid algebras have been classified by Kribs and Power [**14**] up to unitary equivalence. Once again, the graph forms a complete invariant. In light of Theorem 2.11, it is natural to ask if the graph forms a complete invariant for bicontinuous or even algebraic isomorphisms. In Theorem 3.9 we show that two free semigroupoid algebras are isomorphic as dual algebras if and only if the associated graphs are isomorphic. In particular, our result classifies the free semigroupoid algebras up to similarity and shows that two such algebras are similar if and only if they are unitarily equivalent. For free semigroupoid algebras associated with locally finite directed graphs with no sinks, we show that the graph forms an invariant for algebraic isomorphisms as well.

The proof of Theorem 3.9 follows the same line of reasoning as that of Theorem 2.11. All the results of Section 2, as well as their proofs, adopt easily to the w^* context. The only exceptions are Proposition 2.1 and Theorem 2.10.

Let $\mathfrak{M}_G^{w^*}$ denote the set of all w^*-continuous multiplicative linear functionals on $\mathfrak{L}_G$. If $x \in \mathcal{V}(G)$ then $\mathfrak{M}_{G,x}^{w^*}$ denotes the collection of all functionals $\rho \in \mathfrak{M}_G^{w^*}$ so that $\rho(P_x) = 1$. Clearly, the (disjoint) union of all $\mathfrak{M}_{G,x}^{w^*}$, $x \in \mathcal{V}(G)$, equals $\mathfrak{M}_G^{w^*}$.

Definition 3.1. Let G be a countable directed graph and let $x \in \mathcal{V}(G)$. We say that a directed loop $u = xux$ is *primitive* if it does not factor as $u = wv$, where $w, v \in \mathbb{F}^+(G)$, $r(w) = s(w) = r(v) = s(v) = x$, and $1 \leq |w| < |u|$. Clearly any loop edge supported at x is a primitive loop but there may be many more. The collection of all primitive loops w with $r(x) = s(x) = x$ is denoted as $\mathcal{P}(\mathbb{F}^+(G), x)$.

Lemma 3.2. *Let G be a countable directed graph, $x \in \mathcal{V}(G)$ and let $u_1, u_2, \dots, u_n$, with $n = \infty$ possibly, be the primitive loops with source x. Then there exists a w^*-bicontinuous isomorphism from $P_x\mathfrak{L}_G P_x$ onto $\mathfrak{L}_n$, which maps L_{u_i} to L_i for $i = 1, 2, \dots, n$.*

Proof. It suffices to show that $P_x\mathfrak{L}_G P_x$ is unitarily equivalent to an ampliation of $\mathfrak{L}_n$, such that generators are mapped to generators. Clearly $P_x\mathfrak{L}_G P_x$ is generated by L_w with $w \in \mathcal{P}(\mathbb{F}^+(G), x)$. Let $\mathcal{T}_x$ be the collection of all vectors ξ_w, $w = xw$, such that $w \neq uv$, $u \in \mathcal{P}(\mathbb{F}^+(G), x)$, $v \in \mathbb{F}^+(G)$. The subspaces

$$\mathcal{V}_w = \operatorname{span}\left\{A\xi_w : A \in P_x\mathfrak{L}_G P_x\right\} \quad \text{for} \quad w \in \mathcal{T}_x \cup \{\xi_x\}$$

are mutually orthogonal, reducing for $P_x\mathfrak{L}_G P_x$, and satisfy

$$\sum_{w \in \mathcal{T}_x \cup \{\xi_x\}} \oplus\, \mathcal{V}_w = P_x(\mathcal{H}_G).$$

For each $w \in \mathcal{T}_x \cup \{\xi_x\}$ consider the unitary operator

$$U_w : \mathcal{V}_w \longrightarrow \mathcal{H}_n$$

defined by $U_w(\xi_w) = \xi_\emptyset$ and

$$U_w\left(\xi_{u_{i_1}u_{i_2}\dots u_{i_m}w}\right) = \xi_{i_1 i_2 \dots i_m},$$

for any $u_{i_1}, u_{i_2}, \dots, u_{i_m} \in \mathcal{P}(\mathbb{F}^+(G), x)$. Then it is evident that a desired unitary operator is given by $\oplus_{w \in \mathcal{T}_x \cup \{\xi_x\}} U_w$. ■

If w is a loop with source $x \in \mathcal{V}(G)$, then $\|w\|$ denotes the number of primitive loops whose product equals w.

Lemma 3.3. *Let G be a countable directed graph, $x \in \mathcal{V}(G)$ and let $e_1, e_2, \dots, e_n \in \mathcal{E}(G)$ with $r(e_i) = s(e_i) = x$, $i = 1, 2, \dots, n$, be the distinct loop edges starting at x. Suppose $\rho \in \mathfrak{M}_{G,x}^{w^*}$ satisfies $\sum_{i=1}^n |\rho(L_{e_i})|^2 = r < 1$. If $A = P_x A P_x \in \mathfrak{L}_G$ has Fourier expansion of the form*

$$A \sim \sum_{\|w\| \geq k} a_w L_w \quad \text{for some} \quad k \geq 1,$$

then $|\rho(A)| \leq r^k \|A\|$.

Proof. The proof follows from the identification of $P_x \mathfrak{L}_G P_x$ with $\mathfrak{L}_n$ (Lemma 3.2) and an application of [**3**, Lemma 3.1]. ∎

We also need the following lemma, whose proof is similar to that of Lemma 3.2. (Compare also with [**26**, Proposition 3.1]).

Lemma 3.4. *Let G be a countable directed graph, $x \in \mathcal{V}(G)$ and let $e_1, e_2, \dots, e_n \in \mathcal{E}(G)$ so that $r(e_i) = s(e_i) = x$, $i = 1, 2, \dots, n$. Let $\mathrm{alg}(P_x, L_{e_1}, L_{e_2}, \dots, L_{e_n})$ be the w^*-closed algebra generated by P_x and $L_{e_1}, L_{e_2}, \dots, L_{e_n}$. Then there exists a w^*-bicontinuous isomorphism from $\mathrm{alg}(P_x, L_{e_1}, L_{e_2}, \dots, L_{e_n})$ onto $\mathfrak{L}_n$, which maps L_{e_i} to L_i, $i = 1, 2, \dots, n$.*

Theorem 3.5. *Let G be a countable directed graph. Let $x \in \mathcal{V}(G)$ and assume there exists exactly n distinct loop edges whose source is x. Then $\mathfrak{M}_{G,x}^{w^*}$ equipped with the relative w^*-topology is homeomorphic to $\mathbb{B}_n^\circ$, the open unit ball of $\mathbb{C}^n$.*

Proof. Let $e_1, e_2, \dots, e_n$, $e_i = x e_i x$, $i = 1, 2, \dots, n$, be the distinct loop edges starting at $x \in \mathcal{V}(G)$ and define $\phi_x : \mathfrak{M}_{G,x}^{w^*} \longrightarrow \mathbb{C}^n$ by the formula

$$\phi_x(\rho) = (\rho(L_{e_1}), \rho(L_{e_2}), \dots, \rho(L_{e_n})), \quad \rho \in \mathfrak{M}_{G,x}^{w^*}.$$

Since any multiplicative form is completely contractive, the range of ϕ_x is contained in $\mathbb{B}_n$. We will show that ϕ_x is the desired homeomorphism.

Arguing as in the proof of Proposition 2.1, we obtaint that $\phi_x(\mathfrak{M}_{G,x}) \subseteq \mathbb{B}_n^\circ$. Now Lemma 3.4 shows that if there were w^*-continuous multiplicative forms $\rho \in \mathfrak{M}_{G,x}^{w^*}$ with $\rho \in \vartheta \mathbb{B}_n$, then such forms would also exist on $\mathfrak{L}_n$. But this contradicts [**3**, Theorem 2.3] and so $\phi_x(\mathfrak{M}_{G,x}) = \mathbb{B}_n^\circ$.

We now show that ϕ_x^{-1} is continuous. Let $A = P_x A P_x \in \mathfrak{L}_G$ with Fourier expansion $A \sim \sum_{w = xwx} a_w L_w$ and let $\rho, \mu \in \mathfrak{M}_{G,x}^{w^*}$ satisfying $\|\phi_x(\rho)\|_2, \|\phi_x(\mu)\|_2 \leq r < 1$. Then given $k \geq 1$, Lemma 3.3 shows that

$$\begin{aligned} |\rho(A) - \mu(A)| &\leq \Big|(\rho - \mu)\Big(\sum_{\|w\| < k} a_w L_w\Big)\Big| + 2r^k ||A - \sum_{||w|| < k} a_w L_w|| \\ &\leq \Big|(\rho - \mu)\Big(\sum_{\|w\| < k} a_w L_w\Big)\Big| + 2(k+1) r^k ||A|| \end{aligned}$$

Notice however that if $w \in \mathcal{P}(\mathbb{F}^+(G), x)$ contains an edge different from $e_1, e_2, \dots, e_n$, then $\rho(L_w) = 0$. Hence,

$$\begin{aligned} |\rho(A) - \mu(A)| \ \leq \ & \sum_{1 \leq i_1,, i_2, \dots, i_k \leq n} |a_{e_{i_1} e_{i_2} \dots e_{i_k}} (\rho - \mu)(L_{e_{i_1} e_{i_2} \dots e_{i_k}})| \\ & + 2(k+1) r^k \|A\|. \end{aligned}$$

Thus, since k was arbitrary and $0 \leq r < 1$, the result follows from a standard approximation argument. ■

Corollary 3.6. *If G is a countable directed graph, then $\mathfrak{M}_G^{w^*}$ is a locally compact Hausdorff space whose connected components coincide with $\mathfrak{M}_{G,x}^{w^*}$, $x \in V(G)$.*

We now proceed as in Section 2. Given two distinct vertices x_1, x_2 in a directed graph G, we define $\mathrm{rep}_{x_1,x_2}^{w^*}(\mathfrak{L}_G)$ to be the collection of all w^*-continuous, two dimensional nest representations π for which $\rho_\pi^{(i)} \in \mathfrak{M}_{G,x_i}^{x*}$, $i = 1, 2$. Arguing as in Corollary 2.6, one shows that $\mathrm{rep}_{x_1,x_2}^{w^*}(\mathfrak{L}_G) \neq \emptyset$ exactly when there exists $e \in \mathcal{E}(G)$ such that $e = x_2 e x_1$. Let

$$\mathcal{K}_{x_1,x_2}^{w^*} = \bigcap \{\ker \pi \mid \pi \in \mathrm{rep}_{x_1,x_2}^{w^*}(\mathfrak{L}_G)\}$$

and let $^\perp(\mathcal{K}_{x_1,x_2}^{w^*})$ denote the collection of all w^*-continuous functionals on $\mathfrak{L}_G$ that vanish on $\mathcal{K}_{x_1,x_2}^{w^*}$, equiped with the usual norm. It is a standard result in Functional Analysis that the dual Banach space of $^\perp(\mathcal{K}_{x_1,x_2}^{w^*})$ is isometrically isomorphic to $\mathfrak{L}_G / \mathcal{K}_{x_1,x_2}^{w^*}$. Therefore $\mathfrak{L}_G / \mathcal{K}_{x_1,x_2}^{w^*}$ can be equiped with a w^*-topology so that it becomes a dual Banach algebra.

Definition 3.7. Let $\mathcal{I}$ be a w^*-closed ideal of a dual Banach algebra $\mathcal{B}$ and let $\{B_i\}_{i \in \mathbb{I}}$ be a subset of $\mathcal{I}$. The set $\{B_i\}_{i \in \mathbb{I}}$ is said to be a *w^*-generating set* for $\mathcal{I}$ if the w^*-closed ideal generated by $\{B_i\}_{i \in \mathbb{I}}$ equals $\mathcal{I}$. (In that case, the B_i's are said to be the *w^*-generators* of $\mathcal{I}$.) The ideal $\mathcal{I}$ is said to be *n-generated* with respect to the w^*-topology iff n is the smallest cardinality of a w^*-generating set. If in addition, the sequential w^*-closure, i.e., w^*-limits of sequences, of the algebraic ideal generated by the n generators equals $\mathcal{I}$, then $\mathcal{I}$ is said to be *sequentially n-generated.*

As in Section 2, let $\mathrm{rad}(\mathfrak{L}_G / \mathcal{K}_{x,y}^{w^*})$ be the Jacobson radical of $\mathfrak{L}_G / \mathcal{K}_{x,y}^{w^*}$. An argument similar to that of Proposition 2.7 shows that $\mathrm{rad}(\mathfrak{L}_G / \mathcal{K}_{x,y}^{w^*})$ consists of all cosets $A + \mathcal{K}_{x,y}^{w^*} \in \mathfrak{L}_G / \mathcal{K}_{x,y}^{w^*}$ which satisfy $\pi(A)^2 = 0$, for all $\pi \in rep_{x,y}^{w^*}(\mathfrak{L}_G)$.

Theorem 3.8. *Let G be a countable directed graph and let $x, y \in \mathcal{V}(G)$ with $x \neq y$. Assume that there exist n distinct edges with source x and range y. Then the ideal* $\mathrm{rad}(\mathfrak{L}_G/\mathcal{K}^{w^*}_{x,y}) \subseteq \mathfrak{L}_G/\mathcal{K}^{w^*}_{x,y}$ *is sequentially n-generated with respect to the w^*-topology of $\mathfrak{L}_G/\mathcal{K}^{w^*}_{x,y}$.*

Proof. Let $e_1, e_2, \dots, e_n$ be the edges with source x and range y. An argument similar to that of Theorem 2.10 shows that the collection,

$$L_{e_i} + \mathcal{K}^{w^*}_{x,y}, \quad i = 1, 2, \dots, n,$$

is a generating set for $\mathrm{rad}(\mathfrak{L}_G/\mathcal{K}^{w^*}_{x,y})$. Furthermore, the w^*-convergence of the Cesaro-type sums shows that $\mathrm{rad}(\mathfrak{L}_G/\mathcal{K}^{w^*}_{x,y})$ is sequentially n-generated.

By way of contradiction assume that there exists a generating set

$$A_j + \mathcal{K}^{w^*}_{x,y}, \quad j = 1, 2, \dots, m,$$

for $\mathrm{rad}(\mathfrak{L}_G/\mathcal{K}^{w^*}_{x,y})$ with $m < n$. By Lemma 2.8, we may assume that $A_j = P_y A_j P_x$. Therefore, there exist scalars $a_{j,i}$ so that

$$\Phi_1(A_j) = a_{j,1} L_{e_1} + a_{j,2} L_{e_2} + \dots + a_{j,n} L_{e_n},$$

where Φ_1 is the contractive idempotent on $\mathfrak{L}_G$ defined by the formula

$$\Phi_1\Big(\sum_w a_w L_w\Big) = \sum_{|w|=1} a_w L_w.$$

Let $\mathcal{M}$ be the subspace of $\mathfrak{L}_G$ generated by L_{e_i}, $i = 1, 2, \dots, n$, and let $\mathcal{M}_0 \subset \mathcal{M}$ be the linear span of

$$a_{j,1} L_{e_1} + a_{j,2} L_{e_2} + \dots + a_{j,n} L_{e_n}, \quad j = 1, 2, \dots, m.$$

Let $X \in \mathcal{M} \backslash \mathcal{M}_0$. Clearly, $X + \mathcal{K}^{w^*}_{x,y}$ belongs to $\mathrm{rad}(\mathfrak{L}_G/\mathcal{K}^{w^*}_{x,y})$. Since the collection $A_j + \mathcal{K}^{w^*}_{x,y}$, $j = 1, 2, \dots, m$, is generating for $\mathrm{rad}(\mathfrak{L}_G/\mathcal{K}^{w^*}_{x,y})$, the coset $X + \mathcal{K}^{w^*}_{x,y}$ can be approximated in the w^*-topology by a sequence $\{X_n + \mathcal{K}^{w^*}_{x,y}\}_{n=1}^{\infty}$ consisting of finite sums of the form

$$X_n + \mathcal{K}^{w^*}_{x,y} = \sum_k B^{(n)}_k \hat{A}^{(n)}_k C^{(n)}_k + \mathcal{K}^{w^*}_{x,y},$$

where, $B^{(n)}_k, C^{(n)}_k \in \mathfrak{L}_G$, and $\hat{A}^{(n)}_k \in \{A_1, A_2, \dots A_m\}$. Since $\{X_n + \mathcal{K}^{w^*}_{x,y}\}_{n=1}^{\infty}$ is w^*-convergent, it is bounded. Hence we can choose $D_n \in \mathcal{K}^{w^*}_{x,y}$, $n \in \mathbb{N}$, so that the sequence $\{X_n + D_n\}_{n=1}^{\infty}$ is bounded as well. Passing to a subsequence if necessary, we may assume that the sequence $\{X_n + D_n\}_{n=1}^{\infty}$ is w^*-convergent to $X + D$, for some $D \in \mathcal{K}^{w^*}_{x,y}$. Lemma 2.9 shows now that $\Phi_1(P_y(D_n - D)P_x) = 0$, for all $n \in \mathbb{N}$. It is also easy to see that

$$\Phi_1(B A_j C) = \mu_j\big(a_{j,1} L_{e_1} + a_{j,2} L_{e_2} + \dots + a_{j,n} L_{e_n}\big), \quad \mu_j \in \mathbb{C},$$

for all $j = 1, 2, \ldots, m$ and $B, C \in \mathfrak{L}_G$. Hence,

$$\Phi_1\left(P_y\left(\sum_k B_k^{(n)}\hat{A}_k^{(n)}C_k^{(n)} + D_n - D\right)P_x\right) \in \mathcal{M}_0.$$

A simple approximation argument with the contractive map Φ_1 shows now that $\Phi_1(X) = X$ belongs to $\mathcal{M}_0$, which is a contradiction. ∎

A verbatim repetition of the proof of Theorem 2.11 proves now the following.

Theorem 3.9. *Let G, G' be countable directed graphs. Then there exists a w^*-bicontinuous isomorphism $\tau : \mathfrak{L}_G \longrightarrow \mathfrak{L}_{G'}$ if and only if the graphs G and G' are isomorphic.*

Definition 3.10. If $\mathcal{A}$ (resp. $\mathcal{B}$) is an algebra on a Hilbert space $\mathcal{H}_{\mathcal{A}}$ (resp. $\mathcal{H}_{\mathcal{B}}$) then $\mathcal{A}$ and $\mathcal{B}$ are said to be *spatially isomorphic* if there exists an invertible operator $S : \mathcal{H}_{\mathcal{A}} \longrightarrow \mathcal{H}_{\mathcal{B}}$ so that $S\mathcal{A}S^{-1} = \mathcal{B}$. If S is a unitary operator then $\mathcal{A}$ and $\mathcal{B}$ are said to be *unitarily isomorphic.*

Corollary 3.11. *Two free semigroupoid algebras $\mathfrak{L}_G$ and $\mathfrak{L}_{G'}$ are spatially isomorphic if and only if they are unitarily isomorphic.*

Proof. A similarity between two free semigroupoid algebras $\mathfrak{L}_G$ and $\mathfrak{L}_{G'}$ is a map satisfying the requirements of Theorem 3.9. Therefore, the graphs G and G' are isomorphic and so $\mathfrak{L}_G$ and $\mathfrak{L}_{G'}$ are unitarily isomorphic [**14**, Theorem 9.1]. The other direction is trivial. ∎

Remark 3.12. It should be emphasized that a spatial isomorphism between two free semigroupoid algebras $\mathfrak{L}_G$ and $\mathfrak{L}_{G'}$ need not transform λ_G into $\lambda_{G'}$. The relation between the spatial isomorphism and the unitary isomorphism guaranteed by our analysis can be quite mysterious. This is partly due to the fact that our understanding of the structure of the automorphism group of a free semigroup algebra is still limited.

In order to prove that the graph forms an invariant for algebraic isomorphisms of free semigroupoid algebras, it remains to show that algebraic isomorphisms between such algebras are w^*-continuous. In what follows we do this for a special class of graphs. The general case remains open.

Theorem 3.13. *Let G be a countable directed graph with no sinks and let $\mathcal{A}$ be any Banach algebra. If $\tau : \mathcal{A} \longrightarrow \mathfrak{L}_G$ is an epimorphism of algebras then τ is automatically continuous.*

Proof. The proof is identical to that of Theorem 2.14. ∎

One of the main results in [**13**] asserts that the algebra $\mathfrak{L}_G$ satisfies the property $\mathbb{A}_1$, provided that G has no sinks. Therefore for the free semigroupoid algebras considered in the rest of this section, the WOT and w^*-topology coincide.

Lemma 3.14. *Let G, G' be locally finite directed graphs with no sinks. If $\tau : \mathfrak{L}_G \longrightarrow \mathfrak{L}_{G'}$ is an algebraic isomorphism, then $\tau(\mathfrak{L}_G^{0,k})$, for $k \geq 1$, is w^*-closed.*

Proof. Let $\mathfrak{L}_{G,x}^{0,k}$ be the collection of all $A \in \mathfrak{L}_G$ which can be written as a sum

$$A = (I - P_x)A_1 + P_x A_2,$$

where $A_2 \sim \sum_{|w| \geq k} a_w L_w$. It is easy to verify that $\mathfrak{L}_{G,x}^{0,k}$ is WOT-closed and that $\mathfrak{L}_G^{0,k} = \bigcap_{x \in \mathcal{V}(G)} \mathfrak{L}_{G,x}^{0,k}$. Therefore it suffices to show that for each $x \in \mathcal{V}(G)$, the set $\tau(\mathfrak{L}_{G,x}^{0,k})$ is WOT-closed.

Let $\{B_\alpha\}_{\alpha \in \mathbb{A}}$ be a net in $\tau(\mathfrak{L}_{G,x}^{0,k})$ converging to B. The Krein-Smulian Theorem implies that there is no loss of generality assuming that $\{B_\alpha\}_{\alpha \in \mathbb{A}}$ is bounded in norm. Let $\{A_\alpha\}_{\alpha \in \mathbb{A}}$ be a bounded net in $\mathfrak{L}_G$ so that $B_\alpha = \tau(A_\alpha)$, for all $\alpha \in \mathbb{A}$ (recall that τ^{-1} is continuous). Corollary 4.8 in [**13**] implies that A_α can be written as a sum

$$A_\alpha = (I - P_x)A_1^{(\alpha)} + \sum_{|w|=k} L_{xw} A_w^{(\alpha)}, \tag{4}$$

where $A_1^{(\alpha)}, A_w^{(\alpha)} \in \mathfrak{L}_G$, for all $\alpha \in \mathbb{A}$ and $w \in \mathbb{F}^+(G)$. (Note that the local finiteness of G guarantees that the sum in (4) is finite.) Since $\{A_\alpha\}_{\alpha \in \mathbb{A}}$ is a bounded net, the nets $\{A_1^{(\alpha)}\}_{\alpha \in \mathbb{A}}$ and $\{A_w^{(\alpha)}\}_{\alpha \in \mathbb{A}}$ are also bounded. Since τ is continuous, the nets $\{\tau(A_1^{(\alpha)})\}_{\alpha \in \mathbb{A}}$ and $\{\tau(A_w^{(\alpha)})\}_{\alpha \in \mathbb{A}}$ are bounded as well. The compactness of the closed ball in the w^*-topology, combined with a diagonal argument, shows that the net $\{B_\alpha\}_{\alpha \in \mathbb{A}}$ has a subnet converging to an element of the form

$$\tau(I - P_x)\tau(A_1) + \sum_{|w|=k} \tau(L_{xw})\tau(A_w)$$

and so $B = \tau\left((I - P_x)A_1 + \sum_{|w|=k} L_{xw} A_w\right) \in \mathfrak{L}_{G,x}^{0,k}$, as desired. ■

Lemma 3.15. *Let G, G' be countable directed graphs with no sinks and let $\tau : \mathfrak{L}_G \longrightarrow \mathfrak{L}_{G'}$ be an algebraic isomorphism. Then $\sum_{x \in \mathcal{V}(G)} \tau(P_x) = I$ in the strong* topology.*

Proof. Let $\mathcal{A}$ be the unital C*-algebra generated by P_x, $x \in \mathcal{V}(G)$. Since $\mathcal{A}$ is abelian and τ can be thought as a representation of $\mathcal{A}$, there exists an invertible operator $S \in B(\mathcal{H}_{G'})$ so that the mapping $\hat{\tau} : \mathfrak{L}_G \longrightarrow \mathcal{B}(\mathcal{H}_{G'})$, defined as $\hat{\tau}(A) = S\tau(A)S^{-1}$, $A \in \mathfrak{L}_G$, is a $*$-representation on $\mathcal{A}$. It suffices to show that $\sum_{x\in\mathcal{V}(G)} \hat{\tau}(P_x) = I$, in the strong topology.

Note that since the projections P_x, $x \in \mathcal{V}(G)$, are mutually orthogonal, the same is true for $\hat{\tau}(P_x)$, $x \in \mathcal{V}(G)$. Consider the projection

$$Q = \sum_{x\in\mathcal{V}(G)} \hat{\tau}(P_x) \in \hat{\tau}(\mathfrak{L}_G).$$

Then $(I - Q)\hat{\tau}(P_x) = 0$, and so $\hat{\tau}^{-1}(I - Q)P_x = 0$, for all $x \in \mathcal{V}(G)$. Since, $\sum_{x\in\mathcal{V}(G)} P_x = I$, we obtain that $\hat{\tau}^{-1}(I - Q) = 0$. Therefore, $\sum_{x\in\mathcal{V}(G)} \tau(P_x) = I$, as desired. ■

The proof of the following theorem is modelled on the proof from [**3**] for the special case of $\mathfrak{L}_n$.

Theorem 3.16. *Let G, G' be locally finite directed graphs with no sinks. Then every algebraic isomorphism $\tau : \mathfrak{L}_G \longrightarrow \mathfrak{L}_{G'}$ is w^*-continuous.*

Proof. In [**13**] it was shown that the w^* and WOT topologies on $\mathfrak{L}_G$ coincide when G has no sinks. Thus, by an application of the Krein-Smulian Theorem, it follows that τ is w^*-continuous if and only if τ is WOT-continuous on every closed ball of $\mathfrak{L}_G$.

Let A_α be a bounded net of operators in $\mathfrak{L}_G$ which converge WOT to zero. Let $x_1, x_2, \dots$ be an enumeration of G and let $\xi_\phi = \sum_{n\in\mathbb{N}} 2^{-n}\xi_{x_n}$. By an elementary argument (see for instance [**3**, Lemma 4.4]), it suffices to show that

$$\lim_\alpha \langle \tau(A_\alpha)\xi_\phi, \zeta\rangle = 0,$$

for all ζ in a dense subset of $\mathcal{H}_{G'}$. We shall obtain a distinguished dense subset as follows. From Lemma 3.14 we know that

$$\mathcal{J}_k \equiv \tau(\mathfrak{L}_G^{0,k})$$

is WOT-closed. Observe that $\cap_{k\geq 1}\mathcal{J}_k = \{0\}$ since

$$\tau^{-1}(\cap_{k\geq 1}\mathcal{J}_k) \subseteq \tau^{-1}(\mathcal{J}_k) = \mathfrak{L}_G^{0,k} \quad \text{for} \quad k \geq 1.$$

It follows from [**13**, Theorem 5.2] that

$$\cap_{k\geq 1}\overline{\mathcal{J}_k\mathcal{H}_{G'}} = \{0\},$$

and hence $\cup_{k\geq 1}(\mathcal{J}_k\mathcal{H}_{G'})^\perp$ is dense inside $\mathcal{H}_{G'}$.

Now let $\zeta \in (\mathcal{J}_k\mathcal{H}_{G'})^\perp$ and let $\epsilon > 0$. Using Lemma 3.15 we can choose a subset $\mathcal{S} \subseteq \mathcal{V}(G)$ so that $\mathcal{V}(G)\backslash\mathcal{S}$ is finite and

$$\| \sum_{x\in\mathcal{S}} \tau(P_x)^*\zeta \| \leq \epsilon.$$

Therefore, it suffices to identify α_0 such that

$$|\langle\tau(P_xA_\alpha)\xi_\phi,\zeta\rangle| \leq \frac{\epsilon}{|\mathcal{S}|}, \tag{5}$$

for all $\alpha \geq \alpha_0$ and $x \in \mathcal{V}(G)\backslash\mathcal{S}$. Suppose $A_\alpha \sim \sum_w a_w^{(\alpha)} L_w$ and decompose $P_xA_\alpha = B_{\alpha,x} + C_{\alpha,x}$ where

$$B_{\alpha,x} = \sum_{|w|<k} a_{xw}^{(\alpha)} L_{xw}$$

and

$$C_{\alpha,x} = A_{\alpha,x} - B_{\alpha,x} = \sum_{|w|=k} L_w A_w^{(\alpha)} \in \mathfrak{L}_{G'}^{0,k}.$$

By construction, $\langle\tau(C_\alpha)\xi_\phi,\zeta\rangle = 0$ since $\tau(C_\alpha)\xi_\phi$ belongs to $\mathcal{J}_k\mathcal{H}_{G'}$. On the other hand, $a_w^{(\alpha)} = \langle A_\alpha\xi_{s(w)},\xi_w\rangle$, so that $\lim_\alpha a_w^{(\alpha)} = 0$ for all $w \in \mathbb{F}^+(G)$, $|w| < k$. Since the sum $B_{\alpha,x} = \sum_{|w|<k} a_{xw}^{(\alpha)} L_{xw}$ is finite (G is locally finite), the desired inequality (5) follows. ■

Corollary 3.17. *Let G, G' be locally finite directed graphs with no sinks. Then $\mathfrak{L}_G$ and $\mathfrak{L}_{G'}$ are isomorphic as algebras if and only if the graphs G and G' are isomorphic.*

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