THE REFLEXIVE CLOSURE OF THE ADJOINTABLE OPERATORS

E. G. KATSOULIS

ABSTRACT. Given a Hilbert module E over a C*-algebra $\mathcal{A}$, we show that the collection $\mathrm{End}_{\mathcal{A}}(E)$ of all bounded $\mathcal{A}$-module operators acting on E forms the reflexive closure for the algebra $\mathcal{L}(E)$ of the adjointable operators, i.e., $\mathrm{End}_{\mathcal{A}}(E) = \mathrm{alg}\,\mathrm{lat}\,\mathcal{L}(E)$. We also make an observation regarding the representation theory of the left centralizer algebra of a C^*-algebra and use it to give an intuitive proof of a related result of H. Lin.

1. INTRODUCTION

In this note, $\mathcal{A}$ denotes a C*-algebra and E a Hilbert C*-module over $\mathcal{A}$, i.e., a right $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued inner product $\langle\, , \,\rangle$ so that the norm $\|\xi\| \equiv \| \langle \xi, \xi \rangle^{1/2} \|$ makes E into a Banach space. The collection of all bounded $\mathcal{A}$-module operators acting on E is denoted as $\mathrm{End}_{\mathcal{A}}(E)$. A linear operator S acting on E is said to be adjointable iff given $x, y \in E$ there exists $y' \in E$ so that $\langle Sx, y \rangle = \langle x, y' \rangle$. Elementary examples of adjointable operators are the "rank one" operators $\theta_{\eta,\xi}$, defined by $\theta_{\eta,\xi}(x) \equiv \eta \langle \xi, x \rangle$, where $\eta, \xi, x \in E$. The collection of all adjointable operators acting on E will be denoted as $\mathcal{L}(E)$ while the norm closed subalgebra generated by the rank one operators will be denoted as $\mathcal{K}(E)$.

It is a well known fact that $\mathcal{L}(E) \subseteq \mathrm{End}_{\mathcal{A}}(E)$. However, the reverse inclusion is known to fail in general; this is perhaps the first obstacle one encounters when extending the theory of operators on a Hilbert space to that of operators on a Hilbert C^*-module. This problem has been addressed since the beginning of the theory [20, page 447] and has influenced its subsequent development. The first few chapters of the monograph of Manuilov and Troitsky [18] and the references therein provide the basics of the theory and give a good account of what is known regarding that issue. (See also [4, 16].) The purpose of this note is to demonstrate that the inequality between $\mathcal{L}(E)$ and $\mathrm{End}_{\mathcal{A}}(E)$ is intimately related to another area of continuing mathematical interest, the reflexivity of operator algebras.

If $\mathfrak{A}$ is a unital operator algebra acting on a Banach space $\mathfrak{X}$, then $\mathrm{lat}\,\mathfrak{A}$ will denote the collection of all closed subspaces $M \subseteq \mathfrak{X}$ which are left

2010 *Mathematics Subject Classification.* 46L08, 47L10.
Key words and phrases: Hilbert C^*-module, adjointable operator, reflexive operator algebra, reflexive closure, invariant subspace, left centralizer, left multiplier.

invariant by $\mathfrak{A}$, i.e., $A(m) \in M$, for all $A \in \mathfrak{A}$ and $m \in M$. Dually, for a collection $\mathfrak{L}$ of closed subspaces of $\mathfrak{X}$, we write $\operatorname{alg} \mathfrak{L}$ to denote the collection of all bounded operators on $\mathfrak{X}$ that leave invariant each element of $\mathfrak{L}$. The reflexive closure of an algebra $\mathfrak{A}$ of operators acting on $\mathfrak{X}$ is the algebra $\operatorname{alg} \operatorname{lat} \mathfrak{A}$; we say that $\mathfrak{A}$ is *reflexive* iff

$$\mathfrak{A} = \operatorname{alg} \operatorname{lat} \mathfrak{A}.$$

Similarly, the reflexive closure of a subspace lattice $\mathfrak{L}$ is the lattice $\operatorname{lat} \operatorname{alg} \mathfrak{L}$ and $\mathfrak{L}$ is said to be reflexive if $\mathfrak{L} = \operatorname{lat} \operatorname{alg} \mathfrak{L}$. A formal study of reflexivity for operator algebras and subspace lattices begun with the work of Halmos [10], after Ringrose's proof [22] that all nests on Hilbert space are reflexive. Since then, the concept of reflexivity for operator algebras and subspace lattices has been addressed by various authors on both Hilbert space [1, 2, 3, 6, 9, 13, 15, 19, 23, 24] and Banach space [5, 7, 8], including in particular investigations on a Hilbert C^*-module.

The main result of this short note, Theorem 2.6, provides a link between the two areas of inquiry discussed above. It shows that the presence of bounded but not adjointable module operators on a Hilbert C^*-module E is equivalent to the failure of reflexivity for $\mathcal{L}(E)$. (Here we think of $\mathcal{L}(E)$ as an operator algebra acting on E.) Actually, we do more: we completely describe $\operatorname{lat} \mathcal{L}(E)$ (Theorem 2.3) and we determine that $\operatorname{alg} \operatorname{lat} \mathcal{L}(E) = \operatorname{End}_{\mathcal{A}}(E)$. This shows in particular that $\operatorname{End}_{\mathcal{A}}(E)$ is always reflexive. A key step in the proof is a classical result of Barry Johnson [11, Theorem 1]. Actually, our Theorem 2.6 can also be thought as a generalization of Johnson's result, since its statement reduces to the statement of [11, Theorem 1], when applied to the case of the trivial (unital) Hilbert C^*-module.

Another interpretation for the inequality between $\mathcal{L}(E)$ and $\operatorname{End}_{\mathcal{A}}(E)$ comes from the work of H. Lin. Lin shows in [17, Theorem 1.5] that $\operatorname{End}_{\mathcal{A}}(E)$ is isometrically isomorphic as a Banach algebra to the left centralizer algebra of $\mathcal{K}(E)$. Furthermore, the isomorphism Lin constructs extends the familiar $*$-isomorphism between $\mathcal{L}(E)$ and the double centralizer algebra of $\mathcal{K}(E)$. This shows that the gap between $\mathcal{L}(E)$ and $\operatorname{End}_{\mathcal{A}}(E)$ is solely due to the presence of left centralizers for $\mathcal{K}(E)$ which fail to be double centralizers. In Proposition 3.3 we observe that the representation theory of the left centralizer algebra of a C^*-algebra is flexible enough to allow the use of representations on a Banach space. This leads to yet another short proof of Lin's Theorem, which we present in Theorem 3.4. The only prerequisite for our proof is the existence of a contractive approximate identity for an arbitrary C^*-algebra. (Compare also with [4, Proposition 8.1.16 (ii)].)

2. THE MAIN RESULT

We begin by identifying a useful class of subspaces of E.

Definition 2.1. *Let E a Hilbert C*-module over a C*-algebra $\mathcal{A}$. If $\mathcal{J} \subseteq \mathcal{A}$, then we define*

$$E(\mathcal{J}) := \overline{\text{span}}\{\xi a \mid \xi \in E, a \in \mathcal{J}\}.$$

The correspondence $\mathcal{J} \mapsto E(\mathcal{J})$ of Definition 2.1 is not bijective. Indeed, if $l(\mathcal{J})$ is the closed left ideal generated by $\mathcal{J} \subseteq \mathcal{A}$, then it is easy to see that $E(l(\mathcal{J})) = E(\mathcal{J})$. Therefore we restrict our attention to closed left ideals of $\mathcal{A}$. It turns out that an extra step is still required to ensure bijectivity. First we need the following.

Lemma 2.2. *Let E be a Hilbert C*-module over a C*-algebra $\mathcal{A}$ and let $\mathcal{J} \subseteq \mathcal{A}$ be a closed left ideal. Then*

$$E(\mathcal{J}) = \{\xi \in E \mid \langle \eta, \xi \rangle \in \mathcal{J} \text{ for all } \eta \in E\}.$$

Proof. The inclusion

$$E(\mathcal{J}) \subseteq \{\xi \in E \mid \langle \eta, \xi \rangle \in \mathcal{J} \text{ for all } \eta \in E\}$$

is obvious. The reverse inclusion follows from the well known fact [18, Lemma 1.3.9] that

$$\xi = \lim_{\epsilon \to 0} \xi \langle \xi, \xi \rangle [\langle \xi, \xi \rangle + \epsilon]^{-1}$$

for any $\xi \in E$. ∎

The following gives now a complete description for the lattice of invariant subspaces of the adjointable operators.

Theorem 2.3. *Let E a Hilbert C*-module over a C*-algebra $\mathcal{A}$. Then*

$$\operatorname{lat} \mathcal{L}(E) = \{E(\mathcal{J}) \mid \mathcal{J} \subseteq \overline{\langle E, E \rangle} \text{ closed left ideal } \}.$$

Furthermore, the association $\mathcal{J} \mapsto E(\mathcal{J})$ establishes a complete lattice isomorphism between the closed left ideals of $\overline{\langle E, E \rangle}$ and $\operatorname{lat} \mathcal{L}(E)$.

Proof. First observe that if $\mathcal{J} \subseteq \mathcal{A}$ is a closed left ideal, then the subspace $E(\mathcal{J})$ is invariant under $\mathcal{L}(E)$, because $\mathcal{L}(E)$ consists of $\mathcal{A}$-module operators.

Conversely assume that $M \in \operatorname{lat} \mathcal{L}(E)$ and let

$$J(M) \equiv \overline{\text{span}}\{\langle \eta, m \rangle \mid \eta \in E \text{ and } m \in M\}.$$

Clearly, $J(M) \subseteq \overline{\langle E, E \rangle}$ is a closed left ideal. We claim that $M = E(J(M))$. Indeed, if $m \in M$, then by the definition of $J(M)$ we have $\langle \eta, m \rangle \in J(M)$, for all $\eta \in E$, and so Lemma 2.2 implies that $m \in E(J(M))$. On the other hand, any ξa, with $\xi \in E$ and $a \in J(M)$ is the limit of finite sums of elements of the form $\xi \langle \eta, m \rangle$, where $\eta \in E$ and $m \in M$. However

$$\xi \langle \eta, m \rangle = \theta_{\xi, \eta}(m) \in M$$

and so $M = E(J(M))$. This shows that $\mathcal{J} \mapsto E(\mathcal{J})$ is surjective.

In order to prove that $\mathcal{J} \mapsto E(\mathcal{J})$ is also injective we need to verify that $\mathcal{J} = J(E(\mathcal{J}))$, for any closed ideal $\mathcal{J} \subseteq \overline{\langle E, E\rangle}$. Since $\mathcal{J} \subseteq \overline{\langle E, E\rangle}$ is a left ideal, $J(E(\mathcal{J})) \subseteq \mathcal{J}$. On the other hand, if $(e_i)_i$ is an approximate unit for $\mathcal{J}$, then any element of $\mathcal{J} \subseteq \overline{\langle E, E\rangle}$ can be approximated by elements of the form

$$\sum_k \langle \xi_k, \eta_k \rangle e_k = \sum_k \langle \xi_k, \eta_k e_k \rangle, \quad \xi_k, \eta_k \in E.$$

However, $\eta_k e_k \in E(\mathcal{J})$, by Definition 2.1, and so sums of the above form belong to $J(E(\mathcal{J}))$. Hence $\mathcal{J} \subseteq J(E(\mathcal{J}))$ and so $\mathcal{J} \mapsto E(\mathcal{J})$ is also injective with inverse $M \mapsto J(M)$.

The proof that $\mathcal{J} \mapsto E(\mathcal{J})$ respects the lattice operations follows from two successive applications of Lemma 2.2. Indeed, if $(\mathcal{J}_i)_i$ is a collection of closed ideals of $\overline{\langle E, E\rangle}$, then $\xi \in \cap_i E(\mathcal{J}_i)$ is equivalent by Lemma 2.2 to $\langle \eta, \xi \rangle \in \cap_i \mathcal{J}_i$ which, once again by Lemma 2.2, is equivalent to $\xi \in E(\cap_i \mathcal{J}_i)$. Therefore $\cap_i E(\mathcal{J}_i) = E(\cap_i \mathcal{J}_i)$. The proof of $\vee_i E(\mathcal{J}_i) = E(\vee_i \mathcal{J}_i)$ is immediate. ∎

The following result was proved by B. Johnson [11] for arbitrary semisimple Banach algebras. As stated below, it can be deduced from the GNS construction and a clever application of Kadison's Transitivity Theorem.

Theorem 2.4. *Let $\mathcal{A}$ be a C^*-algebra and let Φ be a linear operator acting on $\mathcal{A}$ that leaves invariant all closed left ideals of $\mathcal{A}$. Then $\Phi(ba) = \Phi(b)a$, $\forall\, a, b \in \mathcal{A}$. In particular, if $1 \in \mathcal{A}$ is a unit then Φ is the left multiplication operator by $\Phi(1)$.*

Note that the proof of Theorem 2.3 shows that any bounded $\mathcal{A}$-module map leaves invariant $\operatorname{lat} \mathcal{L}(E)$. This establishes one direction in the following, which is the main result of the paper.

Theorem 2.5. *Let E be a Hilbert module over a C^*-algebra $\mathcal{A}$. Then*

$$\operatorname{alg} \operatorname{lat} \mathcal{L}(E) = \operatorname{End}_{\mathcal{A}}(E).$$

In particular, $\operatorname{End}_{\mathcal{A}}(E)$ is a reflexive algebra of operators acting on E.

Proof. Let $S \in \operatorname{alg} \operatorname{lat} \mathcal{L}(E)$ and $\xi, \eta \in E$. Consider the linear operator

$$\Phi_{\eta,\xi} : \mathcal{A} \ni a \longmapsto \langle \eta, S(\xi a) \rangle \in \mathcal{A}$$

We claim that $\Phi_{\eta,\xi}$ leaves invariant any of the closed left ideals of $\mathcal{A}$. Indeed, if $\mathcal{J} \subseteq \mathcal{A}$ is such an ideal and $j \in \mathcal{J}$, then $\xi j \in E(\mathcal{J})$ and since $S \in \operatorname{alg} \operatorname{lat} \mathcal{L}$, $S(\xi j) \in E(\mathcal{J})$. By Theorem 2.3, we have

$$\Phi_{\eta,\xi}(j) = \langle \eta, S(\xi j) \rangle \in \mathcal{J}$$

and so $\Phi_{\eta,\xi}$ leaves $\mathcal{J}$ invariant, which proves the claim. Hence Theorem 2.4, implies now that $\Phi_{\eta,\xi}(ba) = \Phi_{\eta,\xi}(b)a$, $\forall\, a, b \in \mathcal{A}$.

Let (e_i) be an approximate unit for $\mathcal{A}$. By the above $\Phi_{\eta,\xi}(e_i a) = \Phi_{\eta,\xi}(e_i)a$, $\forall i$, and so

$$\begin{aligned}\langle \eta, S(\xi a)\rangle &= \lim_i \langle \eta, S(\xi e_i a)\rangle = \lim_i \Phi_{\eta,\xi}(e_i a)\\ &= \lim_i \Phi_{\eta,\xi}(e_i)a = \lim_i \langle \eta, S(\xi e_i)\rangle\, a\\ &= \langle \eta, S(\xi)\rangle\, a\end{aligned}$$

Hence

$$\langle \eta, S(\xi a)\rangle = \langle \eta, S(\xi)a\rangle\,, \quad \forall a \in \mathcal{A},$$

which establishes that S is an $\mathcal{A}$-module map. ∎

The above Theorem can also be thought as a generalization of Theorem 2.4 (Johnson's Theorem) since its statement reduces to the statement of Theorem 2.4 when applied to the case of the trivial unital Hilbert C^*-module. In order to incorporate the non-unital case as well, we need to reformulate our main result as follows.

Corollary 2.6. *Let E be a Hilbert module over a C^*-algebra $\mathcal{A}$. Then*

$$\operatorname{alg}\operatorname{lat}\mathcal{K}(E) = \operatorname{End}_{\mathcal{A}}(E).$$

Corollary 2.7. *If E is a selfdual Hilbert C^*-module, then $\mathcal{L}(E)$ is reflexive as an algebra of operators acting on E.*

In particular, the above Corollary shows that if $\mathcal{A}$ is a unital C^*-algebra, then $\mathcal{L}(\mathcal{A}^{(n)})$, $1 \leq n < \infty$, is a reflexive operator algebra. This is not necessarily true for $\mathcal{L}(\mathcal{A}^{(\infty)})$. Indeed in [18, Example 2.1.2] the authors give an example of a unital commutative C^*-algebra $\mathcal{A}$ for which $\mathcal{L}(\mathcal{A}^{(\infty)}) \neq \operatorname{End}_{\mathcal{A}}(\mathcal{A}^{(\infty)})$. By Theorem 2.6, $\mathcal{L}(\mathcal{A}^{(\infty)})$ is not reflexive.

3. Left Centralizers and a theorem of H. Lin

An alternative description for the inclusion $\mathcal{L}(E) \subseteq \operatorname{End}_{\mathcal{A}}(E)$ has been given by H. Lin in [17].

Definition 3.1. *If $\mathfrak{A}$ is a Banach algebra then a map $\Phi : \mathfrak{A} \to \mathfrak{A}$ is called a left centralizer if $\Phi(ab) = \Phi(a)b$, for all $a, b \in \mathfrak{A}$. If in addition there exists a map $\Psi : \mathfrak{A} \to \mathfrak{A}$ so that $\Psi(a)b = a\Phi(b)$, for all $a, b \in \mathfrak{A}$, then Φ is called a double centralizer.*

In the case where $\mathfrak{A}$ has an approximate unit, all centralizers are linear and bounded [12]. In that case, the collection of all left (resp. double) centralizers equipped with the supremum norm forms a Banach space which we denote as $\mathbf{LC}(\mathfrak{A})$ (resp. $\mathbf{DC}(\mathfrak{A})$).

In [17, Theorem 1.5] Lin shows that $\operatorname{End}_{\mathcal{A}}(E)$ is isometrically isomorphic as a Banach algebra to $\mathbf{LC}(\mathcal{K}(E))$. Furthermore, the isomorphism Lin constructs extends the familiar $*$-isomorphism of Kasparov [14] between $\mathcal{L}(E)$ and $\mathbf{DC}(\mathcal{K}(E))$. Lin's proof is similar in nature to that of Kasparov [14] for the double centralizers of $\mathcal{K}(E)$. However it is more elaborate and also requires some additional results of Paschke [20]. In what follows we give an elementary proof of Lin's Theorem. Our argument depends on the observation that the representation theory for the left centralizers of a C^*-algebra $\mathcal{A}$ is flexible enough to allow the use of representations on a Banach space.

Definition 3.2. *Let $\mathfrak{X}$ be a Banach space and let $\mathfrak{A}$ be a norm closed subalgebra of $B(\mathfrak{X})$, the bounded operators on $\mathfrak{X}$. The left multiplier algebra of $\mathfrak{A}$ is the collection*

$$\mathbf{LM}_{\mathfrak{X}}(\mathfrak{A}) \equiv \{b \in B(\mathfrak{X}) \mid ba \in \mathfrak{A}, \text{ for all } a \in \mathfrak{A}\}.$$

If $b \in \mathbf{LM}_{\mathfrak{X}}(\mathfrak{A})$, then $L_b \in B(\mathfrak{A})$ denotes the left multiplication operator by b.

The following has also a companion statement for double centralizers, which we plan to state and explore elsewhere.

Proposition 3.3. *Let $\mathcal{A}$ be a C^*-algebra and assume that $\mathcal{A}$ is acting isometrically and non-degenerately on a Banach space $\mathfrak{X}$. Then the mapping*

$$\mathbf{LM}_{\mathfrak{X}}(\mathcal{A}) \longrightarrow \mathbf{LC}(\mathcal{A})\colon b \longmapsto L_b \tag{1}$$

establishes an isometric Banach algebra isomorphism between $\mathbf{LM}_{\mathfrak{X}}(\mathcal{A})$ and $\mathbf{LC}(\mathcal{A})$.

Proof. The statement of this Proposition is a well-known fact, provided that $\mathfrak{X}$ is a Hilbert space. In that case, in order to establish the surjectivity of (1) one starts with a contractive approximate unit $(e_i)_i$ for $\mathcal{A}$. If $B \in \mathbf{LC}(\mathcal{A})$, then the net $(B(e_i))_i$ is bounded and therefore has at least one weak limit point $b \in B(\mathfrak{X})$. The conclusion then follows by showing that $b \in \mathbf{LM}_{\mathfrak{X}}(\mathcal{A})$. (See [21, Proposition 3.12.3] for a detailed argument.)

Bounded nets of operators on Banach space need not have weak limits. However, the non-degeneracy of the action guarantees that the net $(B(e_i))_i$ is Cauchy in the topology of pointwise convergence and so it converges pointwise[1] to a bounded operator $b \in B(\mathfrak{X})$, even when $\mathfrak{X}$ is assumed to be a Banach space. With this observation at hand, the rest of the proof now goes as in the Hilbert space case. ∎

We are in position now to give the promised proof for Lin's Theorem.

Theorem 3.4. *Let E be a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$. Then there exist an isometric isomorphism of Banach algebras*

$$\phi : \operatorname{End}_{\mathcal{A}}(E) \longrightarrow \mathbf{LC}(\mathcal{K}(E)),$$

whose restriction $\phi_{|\mathcal{L}(E)}$ establishes a $*$-isomorphism between $\mathcal{L}(E)$ and $\mathbf{DC}(\mathcal{K}(E))$.

[1]Note that this is not true in general for the right centralizers.

Proof. In light of Proposition 3.3, we need to verify that

$$\mathbf{LM}_E(\mathcal{K}(E)) = \mathrm{End}_{\mathcal{A}}(E).$$

Clearly $\mathrm{End}_{\mathcal{A}}(E) \subseteq \mathbf{LM}_E(\mathcal{K}(E))$. Conversely, let $S \in \mathbf{LM}_E(\mathcal{K}(E))$. If $a \in \mathcal{A}$ and $\eta, \xi, \zeta \in E$, then

$$\begin{aligned} S(\eta \langle \xi, \zeta \rangle a) &= S\theta_{\eta,\xi}(\zeta a) = S\theta_{\eta,\xi}(\zeta)a \\ &= S(\eta \langle \xi, \zeta \rangle)a. \end{aligned}$$

However vectors of the form $\eta \langle \xi, \zeta \rangle$, $\eta, \xi, \zeta \in E$, are dense in E by [18, Lemma 1.3.9] and so S is an $\mathcal{A}$-module map, as desired.

Specializing now the mapping of (1) to our setting, we obtain an isometric isomorphism

$$\phi\colon \mathrm{End}_{\mathcal{A}}(E) \longrightarrow \mathbf{LC}(\mathcal{K}(E))\colon S \longmapsto L_S. \tag{2}$$

Furthermore, the restriction $\phi_{|\mathcal{L}(E)}$ coincides with Kasparov's map and the conclusion follows. ∎

Acknowledgements. The present paper grew out of discussions between the author and Aristedes Katavolos during the International Conference on Operator Algebras, which was held at Nanjing University, China, June 20-23, 2013. The author would like to thank Aristedes for the stimulating conversations and is grateful to the organizers of the conference for the invitation to participate and their hospitality.

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Department of Mathematics, East Carolina University, Greenville, NC 27858, USA

E-mail address: katsoulise@ecu.edu