SEMICROSSED PRODUCTS OF OPERATOR ALGEBRAS AND THEIR C^*-ENVELOPES

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Abstract. Let $\mathcal{A}$ be a unital operator algebra and let α be an automorphism of $\mathcal{A}$ that extends to a $*$-automorphism of its C^*-envelope $\mathrm{C}^*_{\text{env}}(\mathcal{A})$. In this paper we introduce the isometric semicrossed product $\mathcal{A} \times_\alpha^{\text{is}} \mathbb{Z}^+$ and we show that $\mathrm{C}^*_{\text{env}}(\mathcal{A} \times_\alpha^{\text{is}} \mathbb{Z}^+) \simeq \mathrm{C}^*_{\text{env}}(\mathcal{A}) \times_\alpha \mathbb{Z}$. In contrast, the C^*-envelope of the familiar contractive semicrossed product $\mathcal{A} \times_\alpha \mathbb{Z}^+$ may not equal $\mathrm{C}^*_{\text{env}}(\mathcal{A}) \times_\alpha \mathbb{Z}$. Our main tool for calculating C^*-envelopes for semicrossed products is the concept of a relative semicrossed product of an operator algebra, which we explore in the more general context of injective endomorphisms.

As an application, we extend the main result of **[9]** to tensor algebras of C^*-correspondences. We show that if $\mathcal{T}_\mathcal{X}^+$ is the tensor algebra of a C^*-correspondence $(\mathcal{X}, \mathfrak{A})$ and α a completely isometric automorphism of $\mathcal{T}_\mathcal{X}^+$ that fixes the diagonal elementwise, then the contractive semicrossed product satisfies $\mathrm{C}^*_{\text{env}}(\mathcal{T}_\mathcal{X}^+ \times_\alpha \mathbb{Z}^+) \simeq \mathcal{O}_\mathcal{X} \times_\alpha \mathbb{Z}$, where $\mathcal{O}_\mathcal{X}$ denotes the Cuntz-Pimsner algebra of $(\mathcal{X}, \mathfrak{A})$.

1. Introduction and preliminaries

In this paper, we offer three choices for defining the semicrossed product of an operator algebra $\mathcal{A}$ by a unital, completely contractive endomorphism α of $\mathcal{A}$ (Definitions 1.1 and 1.2.) In all cases, the resulting algebras contain a completely isometric copy of $\mathcal{A}$ and a "universal" operator that implements the covariance relations. In the case where $\mathcal{A}$ is a C^*-algebra and α preserves adjoints, all three choices produce the same operator algebra, Peters' semicrossed product of a C^*-algebra **[25]** by an endomorphism. (Semicrossed products of C^*-algebras have been under investigation by various authors **[1, 2, 4, 6, 7, 8, 14, 22, 26]**, starting with the work of Arveson **[2]** in the late sixties.) In the general (non-selfadjoint) case however, the semicrossed products we introduce

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here may lead to non-isomorphic operator algebras. The main objective of this paper is to clarify the relation between the three semicrossed products and calculate their C*-envelope, whenever possible.

The present paper is a continuation of the recent work of Davidson and the second named author in [**9**]. In the language of the present paper, the main objective of [**9**] was to show that in the special case where $\mathcal{A}$ is Popescu's non-commutative disc algebra [**27**] and α a completely isometric automorphism of $\mathcal{A}$, all three semicrossed products coincide. One of the main results of this paper, Theorem 2.6, shows that two of these semicrossed products, the isometric and the relative one, coincide for *any* operator algebra $\mathcal{A}$ and *any* completely isometric automorphism α of $\mathcal{A}$. To prove this, we had to abandon the rather intricate but ad-hoc arguments of the second half of [**9**] and instead adopt an abstract approach. Theorem 2.6 focuses now any further research on semicrossed products to the study of the other two, the isometric and the contractive semicrossed product. For these two, there are examples to show that they do not coincide in general (Remark 2.8). Nevertheless, with Theorem 2.6 in hand, we show that they do coincide in the case of a tensor algebra of a C*-correspondence and and a completely isometric isomorphism of the algebra that fixes its diagonal (Corollary 2.6). This not only generalizes the main result of [**9**] to a broader context but also paves the way for additional results of this kind to come in the future.

The various semicrossed products we define in this paper are actually closed images of the following universal semicrossed product under concrete representations.

Definition 1.1. Let α be a unital, completely contractive endomorphism of an operator algebra $\mathcal{A}$. A contractive (isometric) covariant representation (π, K) of $(\mathcal{A}, \alpha)$ is a completely contractive representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and a contraction (resp. isometry) $K \in \mathcal{B}(\mathcal{H})$ so that

$$\pi(A)K = K\pi(\alpha(A)) \quad \text{for all } A \in \mathcal{A}.$$

The contractive (resp. isometric) semicrossed product $\mathcal{A} \times_\alpha \mathbb{Z}^+$ (resp. $\mathcal{A} \times_\alpha^{\text{is}} \mathbb{Z}^+$) for the system $(\mathcal{A}, \alpha)$ is the universal operator algebra generated by a copy of $\mathcal{A}$ and a contraction (resp. isometry) $\mathfrak{V}$ so that $A\mathfrak{V} = \mathfrak{V}\alpha(A)$, for all $A \in \mathcal{A}$.

The contractive semicrossed product has, by definition, a rich representation theory which unfortunately makes it very intractable. This was first observed in [**9**] based on the famous example of Varopoulos [**28**] regarding three commuting contractions that do not satisfy the usual von Neumann inequality. Nevertheless, there are significant cases where the contractive semicrossed product has been completely identified. These include the case where $\mathcal{A}$ is a C*-algebra **[17, 25]** and the case where $\mathcal{A}$ is the non-commutative disc algebra $\mathfrak{A}_n$ and α is an isometric automorphism **[9]**.

The isometric semicrossed product is the (closed) image of $\mathcal{A} \times_\alpha \mathbb{Z}^+$ under the representation which restricts to the entries where the contractions K are actually isometries. We believe that this is a more tractable object and as we shall see, in the case where α is a completely isometric automorphism, i.e., it extends to an automorphism of the C*-envelope $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A})$ of $\mathcal{A}$, we can identify the C*-envelope of $\mathcal{A} \times_\alpha \mathbb{Z}^+$ as the crossed product C*-algebra $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A}) \times_\alpha \mathbb{Z}$. The main tool for establishing this result is the concept of a relative semicrossed product.

Recall that a C* algebra $\mathcal{C}$ is said to be a C*-*cover* for a subalgebra $\mathcal{A} \subseteq \mathcal{C}$ provided that $\mathcal{A}$ generates $\mathcal{C}$ as a C*-algebra, i.e., $\mathcal{C} = C^*(\mathcal{A})$. If $\mathcal{C}$ is a C*-cover for $\mathcal{A}$, then $\mathcal{J}_\mathcal{A}$ will denote the Šilov ideal of $\mathcal{A}$ in $\mathcal{C}$. Therefore, $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A}) = \mathcal{C}/\mathcal{J}_\mathcal{A}$ and the restriction of the natural projection $q : \mathcal{C} \to \mathcal{C}/\mathcal{J}_\mathcal{A}$ on $\mathcal{A}$ is a completely isometric representation of $\mathcal{A}$. (Any ideal $\mathcal{J} \subseteq \mathcal{C}$, with the property that the restriction of the natural projection $\mathcal{C} \to \mathcal{C}/\mathcal{J}$ on $\mathcal{A}$ is a complete isometry, is called a *boundary ideal* and $\mathcal{J}_\mathcal{A}$ is the largest such ideal.)

Definition 1.2. Let $\mathcal{A}$ be an operator algebra, $\mathcal{C}$ a C*-cover of $\mathcal{A}$ and let α be an $*$-endomorphism of $\mathcal{C}$ that leaves $\mathcal{A}$ invariant. The subalgebra of Peters' semicrossed product $\mathcal{C} \times_\alpha \mathbb{Z}^+$, which is generated by $\mathcal{A} \subseteq \mathcal{C} \subseteq \mathcal{C} \times_\alpha \mathbb{Z}^+$ and the universal isometry $\mathfrak{V} \in \mathcal{C} \times_\alpha \mathbb{Z}^+$, is denoted by $\mathcal{A} \times_{\mathcal{C},\alpha} \mathbb{Z}^+$ and is said to be a relative semicrossed product for the system $(\mathcal{A}, \alpha)$.

Therefore, the relative semicrossed product $\mathcal{A} \times_{\mathcal{C},\alpha} \mathbb{Z}^+$ comes from the representation of $\mathcal{A} \times_\alpha \mathbb{Z}^+$ that restricts to the entries where π and α are $*$-extendable to $\mathcal{C}$ and the contraction K satisfies the covariance relation with these extensions. It seems plausible that non-isomorphic C*-covers for $\mathcal{A}$ and varying extensions for the endomorphism α could produce non-isomorphic relative semicrossed products. It turns out that under a reasonable technical requirement, i.e., invariance of the Shilov ideal, all such relative semicrossed products are completely isometrically isomorphic (Proposition 2.3). In particular, this requirement is satisfied when α is a completely isometric automorphism of $\mathcal{A}$; in that case all relative semicrossed products for $(\mathcal{A}, \alpha)$ are completely isometrically isomorphic to each other.

2. The relative semicrossed product and its C*-envelope

We begin this section with some preliminary results. The first one is a standard result that shows how to lift an injective $*$-endomorphism of a C*-algebra to an automorphism of a possibly larger C*-algebra.

Proposition 2.1. *If α is an injective endomorphism of a C*-algebra $\mathfrak{A}$, then there is a unique triple $(\mathfrak{B}, \beta, j)$ (up to isomorphism) where $\mathfrak{B}$ is a C*-algebra, β is an automorphism of $\mathfrak{B}$ and j is a $*$-monomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ such that $\beta j = j\alpha$ and $\mathfrak{B} = \overline{\bigcup_{k\geq 0} \beta^{-k} j(\mathfrak{A})}$.*

To paraphrase, there is a unique minimal C-algebra $\mathfrak{B}$ containing $\mathfrak{A}$ with an automorphism β satisfying $\beta|_{\mathfrak{A}} = \alpha$.*

Proof. Consider the inductive limit $\mathfrak{B}$ of the system

$$\mathfrak{A}_1 \xrightarrow{\alpha_1} \mathfrak{A}_2 \xrightarrow{\alpha_2} \mathfrak{A}_3 \xrightarrow{\alpha_3} \mathfrak{A}_4 \xrightarrow{\alpha_4} \cdots,$$

where $\mathfrak{A}_i = \mathfrak{A}$ and $\alpha_i = \alpha$, for all $i \in \mathbb{N}$. Let j_i be the associated $*$-monomorphism from $\mathfrak{A} = \mathfrak{A}_i$ to $\mathfrak{B}$. This map is defined as $j_i(A) = (0, 0, \dots, 0, A, \alpha(A), \alpha^2(A), \dots)$, with the understanding that the infinite tuple in the definition signifies the appropriate equivalence class. Define $j = j_1$.

The system

$$\begin{array}{ccccccc}
\mathfrak{A}_1 & \xrightarrow{\alpha_1} & \mathfrak{A}_2 & \xrightarrow{\alpha_2} & \mathfrak{A}_3 & \xrightarrow{\alpha_3} & \cdots \\
{\scriptstyle\alpha}\downarrow & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\alpha}\downarrow & & \\
\mathfrak{A}_1 & \xrightarrow{\alpha_1} & \mathfrak{A}_2 & \xrightarrow{\alpha_2} & \mathfrak{A}_3 & \xrightarrow{\alpha_3} & \cdots
\end{array}$$

gives rise to an $*$-automorphism β of $\mathfrak{B}$ defined as

$$\beta(A_1, A_2, A_3, \dots) = (\alpha(A_1), \alpha(A_2), \alpha(A_3), \dots), \quad A_i \in \mathfrak{A}_i, i \in \mathbb{N}.$$

Clearly, $\beta j = j\alpha$. The inverse of β on $\bigcup_{k\geq 1} j_k(\mathfrak{A}_k)$ satisfies

$$\beta^{-1}(A_1, A_2, A_3, \dots) = (0, \alpha(A_1), \alpha(A_2), \alpha(A_3), \dots), \quad A_i \in \mathfrak{A}_i, i \in \mathbb{N}.$$

and so, if $A \in \mathfrak{A}$, then

$$\beta^{-k}(A, 0, 0, \dots) = (0, 0, \dots, 0, A, \alpha(A), \alpha^2(A), \dots).$$

Therefore, $\bigcup_{k\geq 0} \beta^{-k} j(\mathfrak{A})$ is dense in $\mathfrak{B}$. ∎

We now fix some notation and use the previous result to construct a useful embedding of $\mathfrak{A} \times_\alpha \mathbb{Z}^+$.

Let $\mathcal{A}$ be an operator algebra and let α be a completely contractive endomorphism of $\mathcal{A}$. If π is a completely contractive representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, we define

$$\widetilde{\pi} : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{Z}^+))$$

so that

$$\widetilde{\pi}(A) \equiv (\pi(A), \pi(\alpha(A)), \pi(\alpha^2(A)), \dots), \quad A \in \mathcal{A}. \tag{1}$$

Let $V_{\mathcal{H}} \equiv I \otimes V$, where V denotes the unilateral shift on $\ell^2(\mathbb{Z}^+)$. The pair $(\widetilde{\pi}, V_{\mathcal{H}})$ forms a contractive covariant representation of $(\mathcal{A}, \alpha)$ and the associated representation of $\mathcal{A} \times_\alpha \mathbb{Z}^+$ is denoted as $\widetilde{\pi} \times V_{\mathcal{H}}$. If α happens to be a completely isometric automorphism of $\mathcal{A}$, we also have the representation $\widehat{\pi} : \mathcal{A} \to \mathcal{B}(H \otimes \ell^2(\mathbb{Z}))$, such that

$$\widehat{\pi}(A) \equiv (\dots, \pi(\alpha^{-1}(A)), \pi(A), \pi(\alpha(A)), \pi(\alpha^2(A)), \dots), \quad A \in \mathcal{A},$$

the unitary $U_{\mathcal{H}} = I \otimes U$, where U is the bilateral shift on $\ell^2(\mathbb{Z})$ and the associated representation $\widehat{\pi} \times U_{\mathcal{H}}$ of $\mathfrak{A} \times_\alpha \mathbb{Z}^+$.

Proposition 2.2. *Let α is an injective endomorphism of a C*-algebra $\mathfrak{A}$ and let $(\mathfrak{B}, \beta, j)$ be the triple of Proposition 2.1. Then $\mathfrak{A} \times_\alpha \mathbb{Z}^+$ embeds completely isometrically in $\mathfrak{B} \times_\beta \mathbb{Z}$. Furthermore, $\mathfrak{B} \times_\beta \mathbb{Z}$ becomes a* C*-*cover for $\mathfrak{A} \times_\alpha \mathbb{Z}^+$.*

Proof. Let π be a faithful representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$. Since every representation of $\mathfrak{A}$ is a direct sum of cyclic representations, the GNS construction implies that there exists a representation π_β of $\mathfrak{B}$ on a Hilbert space $\mathcal{H}_\beta \supseteq \mathcal{H}$ so that $\mathcal{H}$ is reducing for $\pi_\beta(j(\mathfrak{A}))$ and $\pi_\beta(j(A)) \mid_{\mathcal{H}} = \pi(A)$ for all $A \in \mathfrak{A}$.

By gauge invariance, $\widetilde{\pi} \times V_{\mathcal{H}}$ is a completely isometric representation for $\mathfrak{A} \times_\alpha \mathbb{Z}^+$; therefore the same is true for the representation $\widehat{\pi_\beta}\, j \times U_{\mathcal{H}_\beta}$. Now notice that the representation $\widehat{\pi_\beta}$ is faithful on $\bigcup_{k \geq 0} \beta^{-k} j(\mathfrak{A})$ and so, by inductivity, on all of $\mathfrak{B}$. By gauge invariance, the representation $\widehat{\pi_\beta} \times U_{\mathcal{H}_\beta}$ is also faithful on $\mathfrak{B} \times_\beta \mathbb{Z}$. The proposition now follows by comparing the ranges of $\widehat{\pi_\beta}\, j \times U_{\mathcal{H}_\beta}$ and $\widehat{\pi_\beta} \times U_{\mathcal{H}_\beta}$. ∎

As we shall see in Theorem 2.5, $\mathfrak{B} \times_\beta \mathbb{Z}$ is actually the C*-envelope of $\mathfrak{A} \times_\alpha \mathbb{Z}^+$.

Let $\mathcal{A}$ be an operator algebra and let $\mathcal{C}$ be a C*-cover of $\mathcal{A}$. Let α be a $*$-endomorphism of $\mathcal{C}$ that leaves invariant both $\mathcal{A}$ and $\mathcal{J}_{\mathcal{A}}$ and let $\dot{\alpha} : \mathcal{C}/\mathcal{J}_{\mathcal{A}} \to \mathcal{C}/\mathcal{J}_{\mathcal{A}}$ be defined as $\dot{\alpha}(X + \mathcal{J}_{\mathcal{A}}) = \alpha(X) + \mathcal{J}_{\mathcal{A}}$, $X \in \mathcal{C}$. In this context, there are two relative semicrossed products to be considered, $\mathcal{A} \times_{\mathcal{C}, \alpha} \mathbb{Z}^+$ and $\mathcal{A}/\mathcal{J}_{\mathcal{A}} \times_{\mathcal{C}/\mathcal{J}_{\mathcal{A}}, \dot{\alpha}} \mathbb{Z}^+$. The following proposition, which clarifies the relation between these two semicrossed product, is an application of two significant recent results in the theory of maximal dilations for completely contractive maps. First, Dritschel and McCullough [**11**] have recently proven that every completely contractive map $\varphi : \mathcal{A} \to B(\mathcal{H})$ admits a maximal dilation $(\Phi, \mathcal{K})$, i.e., a dilation $\Phi : \mathcal{A} \to B(\mathcal{K})$ so that any further dilation of Φ has Φ as a direct summand. Furthermore, Muhly and Solel [**24**] have shown that any such maximal dilation Φ extends (uniquely) to a $*$-representation of any C*-cover of $\mathcal{A}$.

Proposition 2.3. *Let $\mathcal{A}$ be an operator algebra, $\mathcal{C}$ be a* C**-cover of $\mathcal{A}$ and let α be a $*$-endomorphism of $\mathcal{C}$ that leaves invariant both $\mathcal{A}$ and $\mathcal{J}_\mathcal{A}$. Then the relative semicrossed products $\mathcal{A}\times_{\mathcal{C},\alpha}\mathbb{Z}^+$ and $\mathcal{A}/\mathcal{J}_\mathcal{A}\times_{\mathcal{C}/\mathcal{J}_\mathcal{A},\dot{\alpha}}\mathbb{Z}^+$ are completely isometrically isomorphic.*

Proof. Let $F = \sum_{n=0}^k \mathfrak{V}^n A_n \in \mathcal{A}\times_{\mathcal{C},\alpha}\mathbb{Z}^+$ and $F' = \sum_{n=0}^k \mathfrak{V}^n(A_n + \mathcal{J}_\mathcal{A}) \in \mathcal{A}/\mathcal{J}_\mathcal{A}\times_{\mathcal{C}/\mathcal{J}_\mathcal{A},\dot{\alpha}}\mathbb{Z}^+$. We have to show that the homomorphism $F \mapsto F'$ is a completely isometric map.

Let π be a faithful representation of $\mathcal{C}$ on a Hilbert space $\mathcal{H}$ and let $(\widetilde{\pi}, V_\mathcal{H})$ be as in the beginning of the section (see (1)). Consider the completely isometric map

$$\varphi : \mathcal{A}/\mathcal{J}_\mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H}) : A + \mathcal{J}_\mathcal{A} \longmapsto \pi(A), \quad A \in \mathcal{A}.$$

According to our earlier discussion, there is a maximal dilation $(\Phi, \mathcal{K})$ of φ which extends uniquely to a representation of $\mathcal{C}/\mathcal{J}_\mathcal{A}$ such that

$$P_\mathcal{H}\Phi(A + \mathcal{J}_\mathcal{A})|_\mathcal{H} = \varphi(A + \mathcal{J}_\mathcal{A}) = \pi(A),$$

for all $A \in A$. Since $P_{\mathcal{H}\otimes\ell^2(\mathbb{Z}_+)} = P_\mathcal{H} \otimes I$, we have that

$$P_{\mathcal{H}\otimes\ell^2(\mathbb{Z}_+)}\widetilde{\Phi}(A + \mathcal{J}_\mathcal{A}))|_{\mathcal{H}\otimes\ell^2(\mathbb{Z}_+)} = \widetilde{\pi}(A + \mathcal{J}_\mathcal{A}),$$

for all $A \in A$. Also, $V_\mathcal{K}|_{\mathcal{H}\otimes\ell^2(\mathbb{Z}_+)} = V_\mathcal{H}$ and so

$$\begin{aligned}
\|F\| &= \|\sum_{n=0}^k V_\mathcal{H}^n \widetilde{\pi}(A_n)\| \\
&= \|P_{\mathcal{H}\otimes\ell^2(\mathbb{Z}_+)}\left(\sum_{n=0}^k V_\mathcal{K}^n \widetilde{\Phi}(A_n + \mathcal{J}_\mathcal{A})\right)|_{\mathcal{H}\otimes\ell^2(\mathbb{Z}_+)}\| \\
&\leq \|\sum_{n=0}^k V_\mathcal{K}^n \widetilde{\Phi}(A_n + \mathcal{J}_\mathcal{A})\| \leq \|F'\|.
\end{aligned}$$

The same is also true for all the matrix norms and so the map $F' \mapsto F$ is well defined and completely contractive. By reversing the roles of $\mathcal{A}$ and $\mathcal{A}/\mathcal{J}(\mathcal{A})$ in the previous arguments, we can also prove that $F \mapsto F'$ is completely contractive, and the conclusion follows. ∎

Now we wish to identify the C^*-envelope of $\mathcal{A}\times_{\mathcal{C},\alpha}\mathbb{Z}^+$. From the previous result we know that it coincides with the C^*-envelope of $\mathcal{A}/\mathcal{J}_\mathcal{A}\times_{\mathcal{C}/\mathcal{J}_\mathcal{A},\dot{\alpha}}\mathbb{Z}^+$. In the following we consider the case where $\alpha : \mathcal{C} \to \mathcal{C}$ is injective. This is easily seen to imply that $\dot{\alpha} : \mathcal{C}/\mathcal{J}_\mathcal{A} \to \mathcal{C}/\mathcal{J}_\mathcal{A}$ is an injective $*$-homomorphism. Indeed,

Lemma 2.4. *Let $\mathcal{A}$ be an operator algebra, $\mathcal{C}$ be a* C**-cover of $\mathcal{A}$ and let α be an injective $*$-endomorphism of $\mathcal{C}$ that leaves invariant both $\mathcal{A}$ and $\mathcal{J}_{\mathcal{A}}$. Then $\dot{\alpha} : \mathcal{C}/\mathcal{J}_{\mathcal{A}} \to \mathcal{C}/\mathcal{J}_{\mathcal{A}}$ is an injective $*$-homomorphism.*

Proof. In that case, α is a completely isometric map. Therefore,

$$\|q(A) + \ker \dot{\alpha}\| = \|\dot{\alpha}(A + \mathcal{J}_{\mathcal{A}})\| = \|\alpha(A) + \mathcal{J}_{\mathcal{A}}\| = \|\alpha(A)\| = \|A\| ,$$

since $\mathcal{J}_{\mathcal{A}}$ is a boundary ideal and $\alpha(\mathcal{A}) \subseteq \mathcal{A}$. The same argument holds for all the matrix norms. Thus $\ker \dot{\alpha}$ is a boundary ideal of $\mathcal{C}/\mathcal{J}_{\mathcal{A}}$. However, $\mathcal{C}/\mathcal{J}_{\mathcal{A}}$ is the C*-envelope of $\mathcal{A}$ and so it contains no non-trivial boundary ideals for $\mathcal{A}$. Thus $\ker \dot{\alpha} = (0)$. ∎

The following is the main technical result of the section.

Theorem 2.5. *Let $\mathcal{A}$ be an operator algebra, $\mathcal{C}$ be a* C**-cover of $\mathcal{A}$ and let $\mathcal{J}_{\mathcal{A}}$ be the Šilov ideal of $\mathcal{A}$ in $\mathcal{C}$. Let α be an injective $*$-endomorphism of $\mathcal{C}$ that leaves invariant both $\mathcal{A}$ and $\mathcal{J}_{\mathcal{A}}$. Then*

$$\mathrm{C}^*_{env}(\mathcal{A} \times_{\mathcal{C},\alpha} \mathbb{Z}^+) \simeq \mathfrak{B} \times_\beta \mathbb{Z} ,$$

where $(\mathfrak{B}, \beta, j)$ is the unique triple of Proposition 2.1 associated with the injective $*$-endomorphism $\dot{\alpha}$ of $\mathcal{C}/\mathcal{J}_{\mathcal{A}}$.

Proof. Proposition 2.3 shows that it suffices to identify the C^*-envelope of $\mathcal{A}/\mathcal{J}_{\mathcal{A}} \times_{\mathcal{C}/\mathcal{J}_{\mathcal{A}},\dot{\alpha}} \mathbb{Z}^+$.

If $(\mathfrak{B}, \beta, j)$ is the unique triple of Proposition 2.1 associated with the injective $*$-endomorphism $\dot{\alpha}$ of $\mathcal{C}/\mathcal{J}_{\mathcal{A}}$ then Proposition 2.2 shows that $\mathcal{C}/\mathcal{J}_{\mathcal{A}} \times_{\dot{\alpha}} \mathbb{Z}^+$, and therefore $\mathcal{A}/\mathcal{J}_{\mathcal{A}} \times_{\mathcal{C}/\mathcal{J}_{\mathcal{A}},\dot{\alpha}} \mathbb{Z}^+$, embeds completely isometrically in $\mathfrak{B} \times_\beta \mathbb{Z}$. Moreover, $\mathfrak{B} \times_\beta \mathbb{Z}$ is a C*-cover for $\mathcal{A}/\mathcal{J}_{\mathcal{A}} \times_{\mathcal{C}/\mathcal{J}_{\mathcal{A}},\dot{\alpha}} \mathbb{Z}^+$. Let $\mathcal{J}$ be the Šilov ideal of $\mathcal{A}/\mathcal{J}_{\mathcal{A}} \times_{\mathcal{C}/\mathcal{J}_{\mathcal{A}},\dot{\alpha}} \mathbb{Z}^+$ in $\mathfrak{B} \times_\beta \mathbb{Z}$. We are to show that $\mathcal{J} = \{0\}$.

Assume to the contrary that $\mathcal{J} \neq \{0\}$. Since $\mathcal{J}$ is invariant by automorphisms of the C*-cover, it remains invariant by the natural gauge action on $\mathfrak{B} \times_\beta \mathbb{Z}$. Therefore it has non-trivial intersection with the fixed point algebra of the natural gauge action, i.e., $\mathcal{J} \cap \mathfrak{B} \neq \{0\}$. However

$$\mathfrak{B} = \overline{\bigcup_{k \geq 0} \beta^{-k} j(\mathcal{C}/\mathcal{J}_{\mathcal{A}})}$$

and therefore by inductivity there exists $k \in \mathbb{N}$ so that

$$\mathcal{J} \cap \beta^{-k} j(\mathcal{C}/\mathcal{J}_{\mathcal{A}}) \neq \{0\}.$$

However, β acts by conjugating with a unitary in $\mathfrak{B} \times_\beta \mathbb{Z}$. Since $\mathcal{J}$ is an ideal of $\mathfrak{B} \times_\beta \mathbb{Z}$, the above implies that

$$\mathcal{J} \cap j(\mathcal{C}/\mathcal{J}_{\mathcal{A}}) \neq \{0\}.$$

But then $j^{-1}(\mathcal{J} \cap j(\mathcal{C}/\mathcal{J}_{\mathcal{A}}))$ is a non-zero boundary ideal for $\mathcal{A}$ in $\mathcal{C}/\mathcal{J}_{\mathcal{A}}$, a contradiction. ∎

In **[9]** Davidson and the second named author proved that

$$\mathrm{C}^*_{\mathrm{env}}(\mathcal{A}_n \times_\alpha \mathbb{Z}^+) = \mathcal{O}_n \times_\alpha \mathbb{Z},$$

where $\mathcal{A}_n$ is Popescu's non-commutative disc algebra **[27]**, α a is a (completely) isometric automorphism of $\mathcal{A}_n$ and $\mathcal{O}_n$ denotes the Cuntz algebra generated by n isometries. A dilation result in the first half of **[9]** reduces the problem of calculating the C*-envelope of $\mathcal{A}_n \times_\alpha \mathbb{Z}^+$ to essentially verifying that $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A}_n \times^{\mathrm{is}}_\alpha \mathbb{Z}^+) = \mathcal{O}_n \times_\alpha \mathbb{Z}$. It takes the second half of **[9]** and intricate use of the representation theory for $\mathcal{O}_n$ to verify that claim. The next result establishes the same claim for *arbitrary* operator algebras using only abstract arguments.

Theorem 2.6. *Let $\mathcal{A}$ be an operator algebra and α be an automorphism of $\mathcal{A}$ that extends to a $*$-automorphism of $\mathrm{C}^*_{env}(\mathcal{A})$. Then, any relative semicrossed product for $(\mathcal{A}, \alpha)$ is completely isometrically isomorphic to $\mathcal{A} \times^{\mathrm{is}}_\alpha \mathbb{Z}^+$. Hence,*

$$\mathrm{C}^*_{env}(\mathcal{A} \times^{\mathrm{is}}_\alpha \mathbb{Z}^+) \simeq \mathrm{C}^*_{env}(\mathcal{A}) \times_\alpha \mathbb{Z}.$$

Proof. In light of Theorem 2.5, it suffices to show that $\mathcal{A} \times^{\mathrm{is}}_\alpha \mathbb{Z}^+$ dilates to a relative semicrossed product. This is done as follows.

Let $\mathfrak{V} \in \mathcal{A} \times^{\mathrm{is}}_\alpha \mathbb{Z}^+$ be the universal isometry acting on a Hilbert space $\mathfrak{H}$ and let $\mathcal{H}$ be the direct limit Hilbert space of the inductive system

$$\mathfrak{H} \xrightarrow{\ \mathfrak{V}\ } \mathfrak{H} \xrightarrow{\ \mathfrak{V}\ } \mathfrak{H} \xrightarrow{\ \mathfrak{V}\ } \cdots .$$

For each $A \in \mathcal{A}$, the commutative diagram

$$\begin{array}{ccccccc}
\mathfrak{H} & \xrightarrow{\ \mathfrak{V}\ } & \mathfrak{H} & \xrightarrow{\ \mathfrak{V}\ } & \mathfrak{H} & \xrightarrow{\ \mathfrak{V}\ } & \cdots \\
{\scriptstyle A}\big\downarrow & & {\scriptstyle \alpha^{-1}(A)}\big\downarrow & & {\scriptstyle \alpha^{-2}(A)}\big\downarrow & & \\
\mathfrak{H} & \xrightarrow{\ \mathfrak{V}\ } & \mathfrak{H} & \xrightarrow{\ \mathfrak{V}\ } & \mathfrak{H} & \xrightarrow{\ \mathfrak{V}\ } & \cdots
\end{array}$$

defines an operator $\pi(A) \in \mathcal{B}(\mathcal{H})$. It is easily seen that π defines a completely isometric representation of $\mathcal{A}$ on $\mathcal{H}$. Consider now the unitary $U \in \mathcal{B}(\mathcal{H})$ defined as

$$U(h_1, h_2, h_3, \dots) = (h_2, h_3, \dots), \quad h_i \in \mathfrak{H}, i \in \mathbb{N}.$$

and notice that $\pi(\alpha(A)) = U^*\pi(A)U$, $A \in \mathcal{A}$. Therefore, the conjugation by U defines a $*$-automorphism of $\mathcal{C} \equiv \mathrm{C}^*(\pi(\mathcal{A}))$, which extends α and is denoted by the same symbol as well. Therefore,

$$\mathcal{A} \times^{\mathrm{is}}_\alpha \mathbb{Z}^+ \simeq \mathcal{A} \times_{\mathcal{C},\alpha} \mathbb{Z}^+$$

and the conclusion follows from Theorem 2.5. ∎

Remark 2.7. In light of Theorem 2.6, we wonder whether one can compute the C*-envelope of $\mathcal{A}\times_\alpha^{\mathrm{is}}\mathbb{Z}^+$ in the case where α is an endomorphism of $\mathcal{A}$ that extends to an injective $*$-endomorphism of $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A})$. To do this, one will have to prove an analogue of Theorem 2.5 in the case where α may not preserve the Šilov ideal $\mathcal{J}_\mathcal{A}$ of $\mathcal{A}$ in $\mathcal{C}$.

Remark 2.8. An observation from [**9**] shows that Theorem 2.6 fails for the contractive semicrossed product, thus showing that the isometric and the contractive semicrossed product are not completely isometrically isomorphic in general.

Indeed, the bidisk algebra $A(\mathbb{D}^2)$ sits inside $\mathrm{C}(\mathbb{T}^2)$, which is its C*-envelope by Ando's theorem. Consider the identity automorphism id. Ando's theorem also shows that the completely contractive representations of $A(\mathbb{D}^2)$ are determined by an arbitrary pair T_1, T_2 of commuting contractions. A covariant representation of $(A(\mathbb{D}^2), \mathrm{id})$ is given by such a pair and a third contraction T_3 which commutes with T_1 and T_2. If it were true that the C*-envelope of this system was $\mathrm{C}(\mathbb{T}^2) \times_{\mathrm{id}} \mathbb{Z} \simeq \mathrm{C}(\mathbb{T}^3)$, then it would be true that every commuting triple of contractions satisfies the usual von Neumann inequality. This has been disproved by Varopoulos [**28**].

3. An application to tensor algebras

In spite of Remark 2.8, there are special cases where the contractive and isometric semicrossed products coincide. The purpose of this section is to verify this in the case where $\mathcal{A}$ is the tensor algebra of a C*-correspondence and α a completely isometric isomorphism of $\mathcal{A}$ that fixes its diagonal elementwise (Corollary 2.6).

The tensor algebras for C*-correspondences were introduced by Muhly and Solel in [**23**]. This is a broad class of non-selfadjoint operator algebras which includes as special cases Peters' semicrossed products [**26**], Popescu's non-commutative disc algebras [**27**], the tensor algebras of graphs (introduced in [**23**] and further studied in [**19, 20**]) and the tensor algebras for multivariable dynamics [**10**], to mention but a few.

Let $\mathfrak{A}$ be a C*-algebra and $\mathcal{X}$ be a (right) Hilbert $\mathfrak{A}$-module, whose inner product is denoted as $\langle\, .\, |\, .\, \rangle$. Let $\mathcal{L}(\mathcal{X})$ be the adjointable operators on $\mathcal{X}$ and let $\mathcal{K}(\mathcal{X})$ be the norm closed subalgebra of $\mathcal{L}(\mathcal{X})$ generated by the operators $\theta_{\xi,\eta}$, $\xi, \eta \in \mathcal{X}$, where $\theta_{\xi,\eta}(\zeta) = \xi\langle\eta|\zeta\rangle$, $\zeta \in \mathcal{X}$.

A Hilbert $\mathfrak{A}$-module $\mathcal{X}$ is said to be a C*-*correspondence* over $\mathfrak{A}$ provided that there exists a $*$-homomorphism $\varphi_\mathcal{X} : \mathfrak{A} \to \mathcal{L}(\mathcal{X})$. We refer to $\varphi_\mathcal{X}$ as the left action of $\mathfrak{A}$ on $\mathcal{X}$. From a given C*-correspondence $\mathcal{X}$ over $\mathfrak{A}$, one can form new C*-correspondences over $\mathfrak{A}$, such as the *n-fold ampliation* or direct sum $\mathcal{X}^{(n)}$ ([**21**, page 5]) and the *n-fold interior tensor product* $\mathcal{X}^{\otimes n} \equiv \mathcal{X} \otimes_{\varphi_{\mathcal{X}}} \mathcal{X} \otimes_{\varphi_{\mathcal{X}}} \cdots \otimes_{\varphi_{\mathcal{X}}} \mathcal{X}$ ([**21**, page 39], $n \in \mathbb{N}$, ($\mathcal{X}^{\otimes 0} \equiv \mathfrak{A}$). These operation are defined within the category of C*-correspondences over $\mathfrak{A}$. (See [**21**] for more details.)

A *representation* (π, t) of a C*-correspondence $\mathcal{X}$ over $\mathfrak{A}$ on a C*-algebra $\mathcal{B}$ consists of a $*$-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}$ and a linear map $t : \mathcal{X} \to \mathcal{B}$ so that

(i) $t(\xi)^* t(\eta) = \pi(\langle \xi | \eta \rangle)$, for $\xi, \eta \in \mathcal{X}$,
(ii) $\pi(A) t(\xi) = t(\varphi_{\mathcal{X}}(A)\xi)$, for $A \in \mathfrak{A}$, $\xi \in \mathcal{X}$.

For a representation (π, t) of a C*-correspondence $\mathcal{X}$ there exists a $*$-homomorphism $\psi_t : \mathcal{K}(\mathcal{X}) \to \mathcal{B}$ so that $\psi_t(\theta_{\xi,\eta}) = t(\xi)t(\eta)^*$, for $\xi, \eta \in \mathcal{X}$. Following Katsura [**18**], we say that the representation (π, t) is *covariant* iff $\psi_t(\varphi_{\mathcal{X}}(A)) = \pi(A)$, for all $A \in \mathcal{J}_{\mathcal{X}}$, where

$$\mathcal{J}_{\mathcal{X}} \equiv \varphi_{\mathcal{X}}^{-1}(\mathcal{K}(\mathcal{X})) \cap (\ker \varphi_{\mathcal{X}})^{\perp}.$$

If (π, t) is a representation of $\mathcal{X}$ then the C*-algebra (resp. norm closed algebra) generated by the images of π and t is denoted as $\mathrm{C}^*(\pi, t)$ (resp. $\mathrm{Alg}((\pi, t))$. There is a universal representation $(\overline{\pi}_{\mathfrak{A}}, \overline{t}_{\mathcal{X}})$ for $\mathcal{X}$ and the C*-algebra $\mathrm{C}^*(\overline{\pi}_{\mathfrak{A}}, \overline{t}_{\mathcal{X}})$ is the Toeplitz-Cuntz-Pimsner algebra $\mathcal{T}_{\mathcal{X}}$. Similarly, the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{X}}$ is the C*-algebra generated by the image of the universal covariant representation $(\pi_{\mathfrak{A}}, t_{\mathcal{X}})$ for $\mathcal{X}$.

A concrete presentation of both $\mathcal{T}_{\mathcal{X}}$ and $\mathcal{O}_{\mathcal{X}}$ can be given in terms of the generalized Fock space $\mathcal{F}_{\mathcal{X}}$ which we now describe. The *Fock space* $\mathcal{F}_{\mathcal{X}}$ over the correspondence $\mathcal{X}$ is defined to be the direct sum of the $\mathcal{X}^{\otimes n}$ with the structure of a direct sum of C*-correspondences over $\mathfrak{A}$,

$$\mathcal{F}_{\mathcal{X}} = \mathfrak{A} \oplus \mathcal{X} \oplus \mathcal{X}^{\otimes 2} \oplus \dots .$$

Given $\xi \in \mathcal{X}$, the (left) creation operator $t_\infty(\xi) \in \mathcal{L}(\mathcal{F}_{\mathcal{X}})$ is defined by the formula

$$t_\infty(\xi)(A, \zeta_1, \zeta_2, \dots) = (0, \xi A, \xi \otimes \zeta_1, \xi \otimes \zeta_2, \dots),$$

where $\zeta_n \in \mathcal{X}^{\otimes n}$, $n \in \mathbb{N}$, and $A \in \mathfrak{A}$. Also, for $A \in \mathfrak{A}$, we define $\pi_\infty(A) \in \mathcal{L}(\mathcal{F}_{\mathcal{X}})$ to be the diagonal operator with $\varphi_{\mathcal{X}}(A) \otimes id_{n-1}$ at its $\mathcal{X}^{\otimes n}$-th entry. It is easy to verify that (π_∞, t_∞) is a representation of $\mathcal{X}$ which is called the *Fock representation* of $\mathcal{X}$. Fowler and Raeburn [**13**] (resp. Katsura [**18**]) have shown that the C*-algebra $\mathrm{C}^*(\pi_\infty, t_\infty)$ (resp $\mathrm{C}^*(\pi_\infty, t_\infty)/\mathcal{K}(\mathcal{F}_{\mathcal{X}}\mathcal{J}_{\mathcal{X}})$) is isomorphic to $\mathcal{T}_{\mathcal{X}}$ (resp. $\mathcal{O}_{\mathcal{X}}$).

Definition 3.1. The *tensor algebra* of a C*-correspondence $\mathcal{X}$ over $\mathfrak{A}$ is the norm-closed algebra $\mathrm{alg}(\overline{\pi}_{\mathfrak{A}}, \overline{t}_{\mathcal{X}})$ and is denoted as $\mathcal{T}_{\mathcal{X}}^{+}$.

According to [**13, 18**], the algebras $\mathcal{T}_{\mathcal{X}}^{+} \equiv \operatorname{alg}(\overline{\pi}_{\mathfrak{A}}, \overline{t}_{\mathcal{X}})$ and $\operatorname{alg}(\pi_{\infty}, t_{\infty})$ are completely isometrically isomorphic and we will therefore identify them.

In order to prove the main results of this section, we follow the strategy of the first half of [**9**]. However, the generality in which we are working with, presents new difficulties and requires innovation. One such innovation is the following.

Lemma 3.2. *Let $(\mathcal{X}, \mathfrak{A})$ be a* C*-correspondence, let α be a completely isometric automorphism of the associated tensor algebra $\mathcal{T}_{\mathcal{X}}^{+}$ and assume that $\alpha(A) = A$, for all $A \in \mathfrak{A}$. Let $\pi : \mathcal{T}_{\mathcal{X}}^{+} \to B(\mathcal{H})$ be a completely contractive representation of $\mathcal{T}_{\mathcal{X}}^{+}$ and let $X \in B(\mathcal{H})$ be a contraction satisfying,*

$$\pi(L)X = X\pi(\alpha(L)), \text{ for all } L \in \mathcal{T}_{\mathcal{X}}^{+}.$$

Then there exist isometric co-extensions π' and $(\pi \circ \alpha)'$, of π and $\pi \circ \alpha$ respectively, and an isometric co-extension X' of X, all acting on some Hilbert space $\mathcal{H}'$ and satisfying

$$\pi'(L)X' = X'(\pi \circ \alpha)'(L), \text{ for all } L \in \mathcal{T}_{\mathcal{X}}^{+}, \tag{2}$$

and

$$\pi'(A) = (\pi \circ \alpha)'(A), \text{ for all } A \in \mathfrak{A}. \tag{3}$$

Proof. First we construct isometric co-extensions $\widehat{\pi}_1$ and $\widehat{\pi}_2$, of π and $\pi \circ \alpha$ respectively, and an isometric co-extension $\widehat{X}$ of X, with the property that

$$\widehat{\pi}_1(A) = \widehat{\pi}_2(A) \tag{4}$$

and

$$\widehat{X}\widehat{\pi}_i(A) = \widehat{\pi}_i(A)\widehat{X}, \quad i = 1, 2, \tag{5}$$

for all $A \in \mathfrak{A}$.

To do this, notice that X commutes with $\pi(\mathfrak{A})$. co-extend X to its Schaeffer dilation

$$S_X \simeq \begin{bmatrix} K & 0 & 0 & 0 & \dots \\ D_K & 0 & 0 & 0 & \dots \\ 0 & I & 0 & 0 & \dots \\ 0 & 0 & I & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \in B(\mathcal{H}^{(\infty)}),$$

where $D_K = (I - K^*K)^{1/2}$. Let $\pi^{(\infty)}$ be the infinite ampliation of π and notice that S_X commutes with $\pi^{(\infty)}(\mathfrak{A})$. Subsequently, using [**23**, Theorem 3.3], we obtain some isometric co-extension $\widehat{\pi}$ of $\pi^{(\infty)}$, on some Hilbert space $\mathcal{K} = \mathcal{H}^{(\infty)} \oplus \mathcal{M}$, and let $\widehat{\pi}_1 = \widehat{\pi}$, $\widehat{\pi}_2 = \widehat{\pi} \circ \alpha$ and $\widehat{X} = S_X \oplus I_{\mathcal{M}}$. These $\widehat{\pi}_1, \widehat{\pi}_2$ and $\widehat{X}$ satisfy (4) and (5).

Since $\widehat{X}$ satisfies (5), the pairs $(\widehat{t}_i, \widehat{\pi}_i|_{\mathfrak{A}})$, $i = 1, 2$, where,

$$\widehat{t}_1(\xi) = \widehat{\pi}_1(t_\infty(\xi))\widehat{X},$$
$$\widehat{t}_2(\xi) = \widehat{X}\widehat{\pi}_2(t_\infty(\xi)), \quad \xi \in \mathcal{X},$$

define isometric representations of $(\mathcal{X}, \mathfrak{A})$ and so there exist $*$-representations $\rho_i : \mathcal{T}_{\mathcal{X}} \to B(\mathcal{K})$ which integrate $(\widehat{t}_i, \widehat{\pi}_i|_{\mathfrak{A}})$, $i = 1, 2$. Since,

$$P_{\mathcal{H}}\rho_1(L)|_{\mathcal{H}} = P_{\mathcal{H}}\rho_2(L)|_{\mathcal{H}}, \text{ for all } L \in \mathcal{T}_{\mathcal{X}}^+, \tag{6}$$

the representations ρ_i co-extend the same contractive representation of $\mathcal{T}_{\mathcal{X}}^+$ (appearing in (6)). By the uniqueness of the minimal isometric co-extension [**23**, Proposition 3.2], there exist projections Q_i commuting with $\rho_i(\mathcal{T}_{\mathcal{X}}^+)$, $i = 1, 2$, (hence commuting with $\widehat{\pi}_i(\mathfrak{A})$, $i = 1, 2$) and a unitary $W : Q_1(\mathcal{K}) \to Q_2(\mathcal{K})$, so that

$$W\rho_1(L)|_{Q_1(\mathcal{K})}W^* = \rho_2(L)|_{Q_2(\mathcal{K})}, \text{ for all } L \in \mathcal{T}_{\mathcal{X}}^+. \tag{7}$$

Furthermore, for each $i = 1, 2$, $\mathcal{H} \subseteq Q_i(\mathcal{K})$ and W fixes $\mathcal{H}$, because both ρ_1 and ρ_2 co-extend the same completely contractive representation of $\mathcal{T}_{\mathcal{X}}^+$, which acts on $\mathcal{H}$.

For $i = 1, 2$, let

$$\widetilde{\pi}_i(L) = \widehat{\pi}_i(L) \oplus \Big(\rho_1(L)|_{Q_1^\perp(\mathcal{K})}\Big)^{(\infty)} \oplus \Big(\rho_2(L)|_{Q_2^\perp(\mathcal{K})}\Big)^{(\infty)}, \ L \in \mathcal{T}_{\mathcal{X}}^+,$$

and let

$$\widetilde{X} = \widehat{X} \oplus I_{Q_1^\perp(\mathcal{K})}^{(\infty)} \oplus I_{Q_2^\perp(\mathcal{K})}^{(\infty)},$$

all of them acting on

$$\mathcal{H}' = \mathcal{K} \oplus Q_1^\perp(\mathcal{K})^{(\infty)} \oplus Q_2^\perp(\mathcal{K})^{(\infty)}.$$

Because of (7), there exists a unitary $U \in B(\mathcal{H}')$ which fixes $\mathcal{H}$, commutes with $\widetilde{\pi}_1(\mathfrak{A}) = \widetilde{\pi}_2(\mathfrak{A})$ and satisfies

$$U\widetilde{\pi}_1(L)\widetilde{X}U^* = \widetilde{X}\widetilde{\pi}_2(L), \text{ for all } L \in \mathcal{T}_{\mathcal{X}}^+.$$

Consider the isometric representations $(t_i, \widetilde{\pi}_i|_{\mathfrak{A}})$ of $(\mathcal{X}, \mathfrak{A})$, where,

$$t_1(\xi) = \widetilde{\pi}_1(t_\infty(\xi))U, \text{ and,}$$
$$t_2(\xi) = \widetilde{\pi}_2(t_\infty(\xi))U, \xi \in \mathcal{X},$$

and let π_i', $i = 1, 2$, be the $*$-representations of $\mathcal{T}_{\mathcal{X}}$ which integrate them. Let $X' = U^*\widetilde{X}$ and notice that for any $L \in \mathcal{T}_{\mathcal{X}}^+$ we have

$$\pi_1'(L)X' = \widehat{\pi}_1(L)UU^*\widetilde{X} = \widehat{\pi}_1(L)\widetilde{X}$$

while

$$X'\pi_2'(L) = U^*\widetilde{X}\widehat{\pi}_2(L)U = \widehat{\pi}_1(L)\widetilde{X},$$

and the conclusion follows. ∎

Remark 3.3. In the previous Lemma, one may take the isometric co-extension X' to be the minimal isometric co-extension X_m of X. In that case however, the co-extensions π' and $(\pi \circ \alpha)'$ can only be considered completely contractive and not necessarily isometric.

Indeed, using Lemma 3.2, we obtain isometric co-extensions $\pi', (\pi \circ \alpha)'$ and X' on some Hilbert space $\mathcal{H}'$ that satisfy (2) and (3). Let Q be the reducing projection for X' so that $QX'|_{Q(\mathcal{H}')} \simeq X_m$. By (3), the projection Q commutes with $\pi'(\mathfrak{A})$ and $(\pi \circ \alpha)'(\mathfrak{A})$ and so the completely contractive representations of $(\mathcal{X}, \mathfrak{A})$, determined by the representations π' and $(\pi \circ \alpha)'$ of $\mathfrak{A}$ and the mappings

$$\mathcal{X} \ni \xi \longmapsto Q\pi'(t_\infty(\xi))|_{Q(\mathcal{H}')}$$
$$\mathcal{X} \ni \xi \longmapsto Q(\pi \circ \alpha)'(t_\infty(\xi))|_{Q(\mathcal{H}')},$$

can be integrated to the desired contractive representations of $\mathcal{T}_{\mathcal{X}}^+$, satisfying the analogues of (2) and (3) with X_m instead of X'.

If we take $\alpha = \mathrm{id}$ in Lemma 3.2, then we obtain the commutant lifting Theorem of Muhly and Solel [**23**], without using the "one-step" extension in the proof. (In [**23**, page 418] the authors ask for such a proof.) Indeed

Corollary 3.4. *Let $(\mathcal{X}, \mathfrak{A})$ be a* C*-correspondence, let $\pi : \mathcal{T}_{\mathcal{X}}^+ \to B(\mathcal{H})$ be a completely contractive representation of $\mathcal{T}_{\mathcal{X}}^+$ and let $X \in B(\mathcal{H})$ be a contraction satisfying,*

$$\pi(L)X = X\pi(L), \text{ for all } L \in \mathcal{T}_{\mathcal{X}}^+.$$

If π_m is the minimal isometric co-extension of π, then there exists a contraction X' co-extending X and satisfying

$$\pi_m(L)X' = X'\pi_m(L), \text{ for all } L \in \mathcal{T}_{\mathcal{X}}^+.$$

Proof. Use Lemma 3.2 to obtain isometric co-extensions π', X'' on some Hilbert space $\mathcal{H}'$ that do the job. (Note however that π' may be "larger" than the minimal isometric co-extension.) There exists now a reducing subspace $\mathcal{K} \subseteq \mathcal{H}'$ for π' so that $\pi'|_{\mathcal{K}} \equiv \pi_m$. Letting X' be the compression of X'' on $\mathcal{K}$, the conclusion follows. ∎

A familiar 2×2 matrix trick also establishes the intertwining form of the commutant lifting theorem for minimal isometric co-extensions.

Theorem 3.5. *Let $(\mathcal{X}, \mathfrak{A})$ be a* C**-correspondence, let α be a completely isometric automorphism of the associated tensor algebra $\mathcal{T}_{\mathcal{X}}^+$ and assume that $\alpha(A) = A$, for all $A \in \mathfrak{A}$. Let $\pi : \mathcal{T}_{\mathcal{X}}^+ \to B(\mathcal{H})$ be a completely contractive representation of $\mathcal{T}_{\mathcal{X}}^+$ and let $X \in B(\mathcal{H})$ be a contraction satisfying,*

$$\pi(L)X = X\pi(\alpha(L)), \text{ for all } L \in \mathcal{T}_{\mathcal{X}}^+.$$

Then there exist an isometric co-extension π_1 of π and an isometric co-extension Z of X, so that

$$\pi_1(L)Z = Z\pi_1(\alpha(L)), \text{ for all } L \in \mathcal{T}_{\mathcal{X}}^+.$$

Proof. Notice that if π_m is the minimal isometric dilation of π, then $\pi_m \circ \alpha$ is the minimal isometric dilation of $\pi \circ \alpha$. Therefore, by applying commutant lifting to the covariance relations, we obtain a contraction X_1 on a Hilbert space $\mathcal{H}_1$, satisfying

$$\pi_m(L)X_1 = X_1(\pi_m \circ \alpha)(L), \text{ for all } L \in \mathcal{T}_{\mathcal{X}}^+.$$

Let $X_{1,m}$ be the minimal dilation of X_1, i.e.,

$$X_{1,m} \simeq \begin{bmatrix} X_1 & 0 & 0 & 0 & \dots \\ D_{X_1} & 0 & 0 & 0 & \dots \\ 0 & I & 0 & 0 & \dots \\ 0 & 0 & I & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{8}$$

where $D_{X_1} = (I - X_1^* X_1)^{1/2}$. We apply now Remark 3.3 to obtain completely contractive representations $\widehat{\pi}_m$ and $\widehat{\pi_m \circ \alpha}$, which co-extend π_m and $\pi_m \circ \alpha$, coincide on $\mathfrak{A}$, and satisfy

$$\widehat{\pi}_m(L)X_{1,m} = X_{1,m}\widehat{\pi_m \circ \alpha}(L) \quad \text{for} \quad L \in \mathcal{T}_{\mathcal{X}}^+. \tag{9}$$

Assume that these dilations have the form

$$\widehat{\pi}_m(L) = \begin{bmatrix} \pi_m(L) & 0 \\ Y^{(L)} & [Y_{jk}^{(L)}]_{j,k \geq 1} \end{bmatrix} \text{ and } \widehat{\pi_m \circ \alpha}(L) = \begin{bmatrix} \pi_m \circ \alpha(L) & 0 \\ Z^{(L)} & [Z_{jk}^{(L)}]_{j,k \geq 1} \end{bmatrix}$$

with regards to the decomposition of the Hilbert space that corresponds to the matricial form of $X_{1,m}$ in (8).

Claim: $Y^{(L)} = Z^{(L)} = 0$, *for all* $L \in \mathcal{T}_{\mathcal{X}}^+$.

Indeed, the claim is true in the case where $L = \pi_\infty(A)$, $A \in \mathfrak{A}$, since the restrictions of $\widehat{\pi}_m$ and $\widehat{\pi_m \circ \alpha}$ on $\mathfrak{A}$ are $*$-homomorphisms dilating the $*$-homomorphisms π_m and $\pi_m \circ \alpha$ respectively. Hence it suffices to prove the claim in the case where $L = t_\infty(\xi)$, $\xi \in \mathcal{X}$. We show that $Y^{(\xi)} = 0$; a similar argument will show that $Z^{(t_\infty(\xi))} = 0$. By the

Schwarz inequality for completely contractive maps on unital operator algebras we have

$$\widehat{\pi}_m\,(t_\infty(\xi))^*\,\widehat{\pi}_m(t_\infty(\xi)) \leq \widehat{\pi}_m\big(\pi_\infty(\langle\,\xi \mid \xi\,\rangle)\big).$$

By taking into account the matricial form of $\widehat{\pi}_m$ and comparing $(1,1)$-entries in the above inequality, we obtain

$$\pi_m\,(t_\infty(\xi))^*\,\pi_m(t_\infty(\xi)) + (Y^{(t_\infty(\xi))})^* Y^{(t_\infty(\xi))} \leq \pi_m\big(\pi_\infty(\langle\,\xi \mid \xi\,\rangle)\big). \tag{10}$$

However the map π_m is an isometric representation and so

$$\pi_m\,(t_\infty(\xi))^*\,\pi_m(t_\infty(\xi)) = \pi_m\big(\pi_\infty(\langle\,\xi \mid \xi\,\rangle)\big).$$

and so (10) obtains

$$(Y^{(t_\infty(\xi))})^* Y^{(t_\infty(\xi))} \leq 0,$$

which proves the claim.

By comparing $(2,i)$-entries, $i = 2, 3, \dots$, in the covariance relation (9) we also obtain

$$Y_{2,i}^{(L)} = 0, \quad \text{for all } i \geq 2.$$

In addition, by comparing $(i,1)$-entries $i = 3, 4, \dots$, in (9) we obtain

$$Y_{i,2}^{(L)} D_{X_1} = 0, \quad \text{for all } i \geq 3,$$

and so $Y_{i,2}^{(L)} = 0,$ for all $i \geq 3$. This combined with the Claim implies that the second row and column of $\widehat{\pi}_m(L)$, $L \in \mathcal{T}_{\mathcal{X}}^+$, are equal to zero, except perhaps from $Y_{2,2}^{(L)}$. Therefore, the map

$$\rho : \mathcal{T}_{\mathcal{G}}^+ \longrightarrow B(D_{X_1}(\mathcal{H}_1)),\ L \longmapsto Y_{2,2}^{(L)}$$

is a completely contractive representation of $\mathcal{T}_{\mathcal{X}}^+$. By comparing $(2,1)$-entries in the covariance relation (9), we now obtain

$$\rho(L) D_{X_1} = D_{X_1} \pi_m \circ \alpha(L), \quad \text{for} \quad L \in \mathcal{T}_{\mathcal{X}}^+. \tag{11}$$

For any $L \in \mathcal{T}_{\mathcal{X}}^+$, we now define

$$\pi_m'(L) = \begin{bmatrix} \pi_m(L) & 0 & 0 & 0 & \dots \\ 0 & \rho(L) & 0 & 0 & \dots \\ 0 & 0 & \rho(\alpha(L)) & 0 & \dots \\ 0 & 0 & 0 & \rho(\alpha^{(2)}(L)) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

By (11), this is a completely contractive representation π_m' on a Hilbert space $\mathcal{H}_2$ so that

$$\pi_m'(L) X_{1,m} = X_{1,m} \pi_m' \circ \alpha(L) \quad \text{for} \quad L \in \mathcal{T}_{\mathcal{X}}^+.$$

Continuing in this fashion, we obtain a sequence

$$(\pi, X),\ (\pi_m, X_1),\ (\pi'_m, X_{1,m}),\ ((\pi'_m)_m, X_2),\ (((\pi'_m)_m)', X_{2,m}) \dots$$

of pairs of operators and representations acting on Hilbert spaces $\mathcal{H} \subseteq \mathcal{H}_1 \subseteq \mathcal{H}_2 \dots$, co-extending π and X and satisfying the covariance relations. Let $\mathcal{H} = \bigvee_j \mathcal{H}_j$, and consider these pairs as acting on $\mathcal{H}$ by extending them to be zero on the complement. Let

$$Z = \textsc{sot}\text{–}\lim X_j = \textsc{sot}\text{–}\lim X_{m,j}$$

and define $\pi_1(L)$, $L \in \mathcal{T}_{\mathcal{X}}^+$, as a strong limit in a similar fashion. These limits evidently exist as in both cases the sequences consist of either isometries or isometric representations that decompose as infinite direct sums. Multiplication is SOT-continuous on the ball, hence the covariance relations hold in the limit. ∎

Combining the Proposition above with Theorem 2.6 we obtain the main result of the section,

Corollary 3.6. *Let $(\mathcal{X}, \mathfrak{A})$ be a C^*-correspondence, let α be a completely isometric automorphism of the associated tensor algebra $\mathcal{T}_{\mathcal{X}}^+$ and assume that $\alpha(A) = A$, for all $A \in \mathfrak{A}$. Then $\mathcal{T}_{\mathcal{X}}^+ \times_\alpha \mathbb{Z}^+$ and $\mathcal{T}_{\mathcal{X}}^+ \times_\alpha^{\mathrm{is}} \mathbb{Z}^+$ are completely isometrically isomorphic and*

$$\mathrm{C}^*_{env}(\mathcal{T}_{\mathcal{X}}^+ \times_\alpha \mathbb{Z}^+) \simeq \mathcal{O}_{\mathcal{X}} \times_\alpha \mathbb{Z}.$$

In particular, the above corollary recaptures the main result of **[9]** with a different proof.

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