NEST REPRESENTATIONS
OF DIRECTED GRAPH ALGEBRAS

KENNETH R. DAVIDSON AND ELIAS KATSOULIS

ABSTRACT. This paper is a comprehensive study of the nest representations for the free semigroupoid algebra $\mathfrak{L}_G$ of countable directed graph G as well as its norm-closed counterpart, the tensor algebra $\mathcal{T}^+(G)$.

We prove that the finite dimensional nest representations separate the points in $\mathfrak{L}_G$, and a fortiori, in $\mathcal{T}^+(G)$. The irreducible finite dimensional representations separate the points in $\mathfrak{L}_G$ if and only if G is transitive in components (which is equivalent to being semisimple). Also the upper triangular nest representations separate points if and only if for every vertex $x \in \mathcal{V}(G)$ supporting a cycle, x also supports at least one loop edge.

We also study *faithful* nest representations. We prove that $\mathfrak{L}_G$ (or $\mathcal{T}^+(G)$) admits a faithful irreducible representation if and only if G is strongly transitive as a directed graph. More generally, we obtain a condition on G which is equivalent to the existence of a faithful nest representation. We also give a condition that determines the existence a faithful nest representation for a maximal type $\mathbb{N}$ nest.

1. INTRODUCTION

The representation theory for non-selfadjoint operator algebras, even though it is still in its infancy, it is already playing an important role. Given a non-selfadjoint algebra, its nest representations are thought of as the proper generalization of the irreducible ones and a general theory around them is developing [**26, 6, 21**]. This paper contributes to the further development of that theory by analyzing such representations for algebras associated to directed graphs.

The study of operator algebras associated to directed graphs is fairly recent in both C*-algebras and non-selfadjoint algebras. While we do not use the C*-algebraic theory here, we refer to a few papers for the interested reader [**24, 25, 2, 13**]. The non-selfadjoint theory is even

more recent. The finite graphs yield quiver algebras, which have been studied by algebraists for some time and were introduced to operator algebras by Muhly [**27**]. Much of the motivations for questions in this area have come from the case of free semigroup algebras, which corresponds to a graph consisting of a single vertex and n loop edges [**10, 11, 12, 8, 7, 9**]. Non-selfadjoint algebras of infinite graphs have been studied by a number of authors beginning with Kribs and Power [**22**]. See also [**20, 18, 19, 23**].

In [**20**], the quiver and free semigroupoid algebras were classified by making an essential use of their two dimensional nest representations (see also [**34**]). The analysis there raises the question: to what extend do the finite dimensional nest representations determine the algebra. Questions of this kind have been investigated for other classes of operator algebras but apparently little has been done here, even for the free semigroup algebras $\mathfrak{L}_n$, $n \geq 2$.

In this paper, we prove that for an arbitrary countable directed graph G, the finite dimensional nest representations of $\mathcal{T}^+(G)$ or $\mathfrak{L}_G$ separate points (Theorem 4.7). If G is transitive in components, then Theorem 4.4 shows that the *irreducible* finite dimensional representations alone are sufficient to separate the points. Indeed this property characterizes transitivity in components for G. At the other extreme, Theorem 4.10 shows that the upper triangular nest representations separate the points if and only if for every vertex $x \in \mathcal{V}(G)$ supporting a cycle, x also supports at least one loop edge. Both these results apply to the non commutative disc algebra $\mathfrak{A}_n$ and the non-commutative Toeplitz algebra $\mathfrak{L}_n$, for $n \geq 2$, thus showing that these algebras are strongly semisimple, i.e. the maximal ideals have zero intersection.

Recently, Read [**33**] proved that $B(\mathcal{H})$ is a free semigroup algebra generated by two isometries with orthogonal ranges. (See also [**5**].) In the language of representation theory, his result asserts that the non-commutative disc algebra $\mathfrak{A}_2$ admits a faithful *-extendible irreducible representation. If one drops the *-extendibility, things simplify considerably. We show that $\mathfrak{L}_G$ (or $\mathcal{T}^+(G)$) admits a faithful irreducible representation if and only if the graph G is strongly transitive. This result is new even for the free semigroup algebras $\mathfrak{L}_n$, $n \geq 2$.

The irreducible representations are a special case of nest representations. The above considerations suggest the problem of characterizing which graph algebras admit faithful nest representations. It turns out that not all graph algebras admit faithful nest representations. Indeed, in Theorem 5.9 we identify an intricate set of conditions on the graph which determine the existence of a faithful nest representation. In particular, the transitive quotient of such a graph G should inherit a total

ordering from the order of G. We also characterize (Theorem 5.11) when there exists a faithful nest representation on the upper triangular operators, with respect to a basis ordered as $\mathbb{N}$. If G has no sinks, this characterization simply requires that the graph is transitive and each one of its edges supports at least one loop edge. In particular, the free semigroup algebras $\mathfrak{L}_n$, $n \geq 2$, admit such representations.

Remark 1.1. In what follows, we state and prove the above mentioned results only in the case of a free semigroupoid algebra. All these results are valid in the tensor algebra case with identical statements and proofs. See Proposition 3.3. We therefore omit them for brevity of the presentation.

2. Notation and Background

A countable directed graph G is given by $(\mathcal{V}(G), \mathcal{E}(G), r, s)$ where $\mathcal{V}(G)$ is a countable set of objects called vertices, $\mathcal{E}(G)$ is a countable set of objects called edges, and r and s are maps from $\mathcal{E}(G)$ into $\mathcal{V}(G)$. The maps are called the range and source maps, and we think of G as a set of points $x \in \mathcal{V}(G)$ with directed edges $e \in \mathcal{E}(G)$ connecting the vertex $s(e)$ to $r(e)$. A vertex x is called a sink of there is no edge e with $s(e) = x$, and a source if there is no edge e with $r(e) = x$.

A path in G is a finite sequence of edges which connect, range to source, within the graph, that is, a finite sequence $p = e_k e_{k-1} \ldots e_2 e_1$ such that $r(e_i) = s(e_{i+1})$ for $1 \leq i < k$. We will write these paths from right to left because this will coincide with operator products in the representations of the graphs. The length of a path is the number of edges, and is written $|p|$. A vertex x will be considered to be a path of zero length with source and range maps $s(x) = r(x) = x$.

If $p = e_k e_{k-1} \ldots e_2 e_1$ is a path, we set $r(p) = r(e_k)$ and $s(p) = s(e_1)$. A cycle is a path p such that $r(p) = s(p)$. A loop edge is an edge e which is a cycle, that is $r(e) = s(e)$.

A graph G is *transitive* if there is a path from each vertex in the graph to every other. Say that a graph G is *strongly transitive* if it is transitive and is not a single cycle nor is trivial (no edges, one vertex). The transpose of G, G^t, has the same same vertices and edges as G, but the source and range maps are reversed: $r_{G^t} = s_G$ and $s_{G^t} = r_G$. The graph G is connected if the undirected graph determined by G is transitive. A connected component of G is a maximal connected subset of G. We say that G is transitive on components is each component of G is transitive. We will also talk of the *transitive component* of a vertex x to mean the largest transitive subgraph of G containing x.

The path space $\mathbb{F}^+(G)$ consists of all finite paths including the vertices. We may consider $\mathbb{F}^+(G)$ as a semigroupoid with a product pq defined if $r(q) = s(p)$. In particular, $p = ps(p) = r(p)p$.

A representation of G is a norm continuous homomorphism ρ from $\mathbb{F}^+(G)$ into $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Since the vertices are idempotents in $\mathbb{F}^+(G)$ which annihilate each other, a well-known result of Dixmier implies that there is no loss of generality assuming that $\rho(x)$ are pairwise orthogonal *projections* for $x \in \mathcal{V}(G)$. The representation is non-degenerate if $\mathcal{H}$ is spanned by the ranges of these projections. If p is an edge, then it is easy to verify that $\rho(p) = \rho(r(p))\rho(p)\rho(s(p))$. It turns out that the representations constructed in this paper are *completely contractive*: this is equivalent to the row operator

$$\begin{bmatrix} \rho(e_1) & \rho(e_2) & \cdots \end{bmatrix}_{e_i \in \mathcal{E}(G)}$$

being a contraction. Of course one could demand instead that $\begin{bmatrix} \rho(e_1) & \cdots & \rho(e_k) \end{bmatrix}$ be a contraction for every finite subset of distinct edges of $\mathcal{E}(G)$.

Of special interest is the left regular representation λ of G on $\mathcal{H}_G = l^2(\mathbb{F}^+(G))$. This space has an orthonormal basis $\{\xi_w : w \in \mathbb{F}^+(G)\}$. We define

$$\lambda(p)\xi_w = \begin{cases} \xi_{pw} & \text{if} \quad s(p) = r(w) \\ 0 & \text{otherwise} \end{cases}$$

We will write L_p for $\lambda(p)$ when p is a path of positive length and P_x or L_x for $\lambda(x)$ when $x \in \mathcal{V}(G)$ as is convenient. Each edge e is sent to a partial isometry with domain $P_{s(e)}\mathcal{H}_G$ and has range contained in $P_{r(e)}\mathcal{H}_G$. Moreover the ranges of L_e are pairwise orthogonal for $e \in \mathcal{E}(G)$. Therefore λ is a completely contractive representation.

For $x \in \mathcal{V}(G)$, define the subspace $\mathcal{H}_x = P_x\mathcal{H}_G$ to be the span of $\{\xi_w : w \in \mathbb{F}^+(G), \ w = wx\}$. It is clear that $\mathcal{H}_G = \sum_{x \in \mathcal{V}(G)}^{\oplus} \mathcal{H}_x$ and that each $\mathcal{H}_x$ reduces λ. Let λ_x denote the restriction of λ to $\mathcal{H}_x$. Also observe that

$$\text{SOT--}\sum_{r(e)=x} L_e L_e^* = P_x - \xi_x\xi_x^*$$

is a proper subprojection of P_x, orthogonal to the unit vector ξ_x. Indeed

$$\text{SOT--}\lim_{d\to\infty} \text{SOT--} \sum_{r(p)=x, \ |p|=d} L_p L_p^* = 0$$

because for fixed d, this sum is a projection onto basis vectors

$$\{\xi_w : w \in \mathbb{F}^+(G), \ s(w) = x, \ |w| \geq d\},$$

and these sets decrease to the empty set.

The norm closed algebra generated by $\lambda(\mathbb{F}^+(G))$ is the *tensor algebra* of G and is denoted as $\mathcal{T}^+(G)$. The tensor algebras associated with graphs were introduced under the name *quiver algebras* by Muhly and Solel in [27, 28]. The algebras $\mathcal{T}^+(G)$ generalize Popescu's non-commutative disc algebras [32] and they are the motivating example in the theory of tensor algebras of Hilbert bimodules [27, 28].

The free semigroupoid algebra $\mathfrak{L}_G$ is the WOT-closed algebra generated by $\lambda(\mathbb{F}^+(G))$ introduced by Kribs and Power [22]. This generalizes the free semigroup algebras $\mathfrak{L}_n$ which correspond to the graphs with one vertex and n loop edges [30, 10]. Each element A of $\mathfrak{L}_G$ has a Fourier series representation $\sum_{p \in \mathbb{F}^+(G)} a_p L_p$ defined as follows. For each path $p \in \mathbb{F}^+(G)$, let $a_p = \langle A\xi_{s(p)}, \xi_p \rangle$. Then the Cesaro means

$$\Sigma_k(A) = \sum_{|p| < k} \left(1 - \frac{|p|}{k} \right) a_p L_p$$

converge in the strong operator topology to A. If, in addition, $A \in \mathcal{T}^+(G)$, then the Cesaro sums converge in norm.

A nest algebra is the algebra of all operators leaving invariant a nest, i.e. a complete chain of subspaces including $\{0\}$ and $\mathcal{H}$. It is said to have multiplicity one if von Neumann algebra generated by the projections onto these subspaces is maximal abelian. An atom of a nest is the difference of an element of the nest and its immediate predecessor. A nest is atomic if the Hilbert space is spanned by all of the atoms. Such nests are multiplicity free precisely when the atoms are one-dimensional. The finite multiplicity free nest algebras are just the upper triangular matrices $\mathcal{T}_n$ acting on the n-dimensional Hilbert space $\mathbb{C}^n$. A type $\mathbb{N}$ nest will indicate that there is an orthonormal basis $\{h_i : i \geq 1\}$ with subspaces $\mathcal{N}_k = \operatorname{span}\{h_i : i \leq k\}$ for $k \geq 0$ and $\mathcal{H}$. We refer to [4] for more information about nest algebras.

A nest representation of an algebra $\mathfrak{A}$ is a representation ρ of $\mathfrak{A}$ on $\mathcal{H}$ such that $\rho(\mathfrak{A})$ is WOT-dense in a nest algebra. Our terminology differs somewhat from the original definition of Lamoureux [26] where he asks only that the invariant subspaces of $\rho(\mathfrak{A})$ be a nest, and not that the algebra be WOT-dense in the nest algebra. In the case where the WOT-closure of $\rho(\mathfrak{A})$ contains a masa, these two notions are equivalent.

3. Partially isometric dilations

In this section we describe a dilation theory for n-tuples of operators which generalizes the Frazho-Bunce-Popescu dilation Theorem [16, 3, 29] and is useful in producing representations for $\mathfrak{L}_G$. The results below can be found in two recent works. Muhly and Solel [28] have obtained

a dilation theory for covariant representations of C*-correspondences which implies Theorem 3.1 below. Jury and Kribs [**19**] obtained self-contained proofs for dilations of n-tuples of operators associated with directed graphs. We follow their approach only because it avoids the technical language of Hilbert bimodules. However we reformulate it to better reflect our interest in representations of graphs.

Let G be a countable directed graph. A representation σ of $\mathbb{F}^+(G)$ will be called *partially isometric* if $S_e := \sigma(e)$ is a partial isometry with domain $\sigma(s(e))\mathcal{H}$ for each edge $e \in \mathcal{E}(G)$. We will write $P_x := \sigma(x)$ for the vertex projections. It is easy to see that such a representation will satisfy the relations (†) below, where (1) is a consequence of being a representation, (3) is the partial isometry condition, and (2) and (4) follow because ρ is completely contractive.

$$(\dagger) \begin{cases} (1) & P_x P_y = 0 & \text{for all } x, y \in \mathcal{V}(G),\ x \neq y \\ (2) & S_e^* S_f = 0 & \text{for all } e, f \in \mathcal{E}(G),\ e \neq f \\ (3) & S_e^* S_e = P_{s(e)} & \text{for all } e \in \mathcal{E}(G) \\ (4) & \sum_{r(e)=x} S_e S_e^* \leq P_x & \text{for all } x \in \mathcal{V}(G) \end{cases}$$

Clearly, the left regular representation is partially isometric, but there are many others. If a partially isometric representation σ satisfies

$$\text{SOT--}\sum_{\substack{e \in \mathcal{E}(G) \\ r(e)=x}} \sigma(e)\sigma(e)^* = P_x \quad \text{for all} \quad x \in \mathcal{V}(G),$$

then we say that σ is *fully coisometric.* >From the operator algebra perspective, fully coisometric representations generate C*-algebras which generalize the Cuntz-Krieger algebras.

At the other extreme, we say a partially isometric representation σ is *pure* if

$$\text{SOT--}\lim_{d \to \infty} \text{SOT--}\sum_{\substack{p \in \mathbb{F}^+(G) \\ |p|=d}} \sigma(p)\sigma(p)^* = 0$$

We have already observed that λ and its subrepresentations λ_x are pure.

Jury and Kribs [**19**] prove the converse: that a pure partially isometric representation σ is equivalent to the restriction of $\lambda^{(\infty)}$ to a reducing subspace. More precisely, $\sigma \simeq \sum_{x \in V(G)}^{\oplus} \lambda_x^{(\alpha_x)}$ where the multiplicities are determined by $\alpha_x = \text{rank}\left(P_x(I - \sum_e S_e S_e^*)\right)$ for $x \in \mathcal{V}(G)$. They prove an analogue of the Wold decomposition which states that every partially isometric representation of G decomposes as a direct sum of a pure representation and a fully coisometric representation.

Let ρ be a completely contractive representation of $\mathbb{F}^+(G)$. A *partially isometric dilation* of ρ is a partially isometric representation σ of $\mathbb{F}^+(G)$ on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ such that $\mathcal{H}$ is invariant for $\sigma(\mathbb{F}^+(G))^*$ and $\sigma(p)^*|_{\mathcal{H}} = \rho(p)^*$ for every path $p \in \mathbb{F}^+(G)$. Say that σ is a *minimal* partially isometric dilation if in addition $\mathcal{H}$ is cyclic for $\sigma(\mathbb{F}^+(G))$, namely $\mathcal{K} = \bigvee_{p \in \mathbb{F}^+(G)} \sigma(p)\mathcal{H}$. Since $\sigma(x)$ is self-adjoint, it follows that $\sigma(x)$ commutes with $P_{\mathcal{H}}$ for all $x \in \mathcal{V}(G)$.

Given a row contraction $T = \begin{bmatrix} T_1 & \ldots & T_n \end{bmatrix}$ or $T = \begin{bmatrix} T_1 & T_2 & \ldots \end{bmatrix}$ of operators acting on a Hilbert space $\mathcal{H}$, consider a family of mutually orthogonal projections $\mathcal{P} = \{P_x\}_{x \in \mathcal{J}}$ on $\mathcal{H}$ which sum to the identity operator and *stabilize* T in the following sense:

$$T_i P_x, \ P_x T_i \in \{T_i, 0\} \quad \text{for all} \quad i \quad \text{and all} \quad x \in \mathcal{J}.$$

Observe that these relations lead to a directed graph G with vertex set $\mathcal{V}(G) = \mathcal{J}$ and directed edges e_i, where $r(e_i)$ and $s(e_i)$ are vertices with $P_{r(e_i)} T_i P_{s(e_i)} = T_i$. Indeed these vertices are unique when $T_i \neq 0$, but are arbitrary if $T_i = 0$. We shall fix a choice of G and rewrite the indices as edges of G, $T = (T_e)_{e \in \mathcal{E}(G)}$. It is clear that this determines a unique completely contractive representation of $\mathbb{F}^+(G)$ with $\rho(e) = T_e$ and $\rho(x) = P_x$.

Let $T = (T_e)_{e \in E(G)}$ be a row contraction stabilized by $\mathcal{P} = \{P_x\}_{x \in \mathcal{V}(G)}$ and indexed as above by a directed graph G. A *partially isometric G-dilation* of T is a partially isometric dilation σ of the induced representation ρ of G determined by T.

With this latter viewpoint, it was shown in [**19**, Theorem 3.2] that if G has no sinks, then T has a unique minimal partially isometric G-dilation σ. They also show that σ is fully coisometric if and only if T is a coisometry; that is,

$$I = TT^* = \sum_{e \in \mathcal{E}(G)} T_e T_e^*.$$

Moreover, they show that σ is pure if and only if

$$\text{SOT--}\lim_{d \to \infty} \text{SOT--}\sum_{\substack{p \in \mathbb{F}^+(G) \\ |p| = d}} \rho(p)\rho(p)^* = 0$$

In particular, this holds when $\|T\| < 1$.

Observe that when σ is pure, there is a WOT-continuous extension of σ to $\mathfrak{L}_G$. This can then be compressed to $\mathcal{H}$ to obtain a WOT-continuous representation of $\mathfrak{L}_G$ extending ρ. This provides an $\mathfrak{L}_G$ functional calculus for the row contraction T.

We require this dilation result for graphs that may have sinks. Using a technique of [**2**] of adding a tail to each sink, we are able to extend the Jury–Kribs result to arbitrary countable graphs with little effort.

Proposition 3.1. *Let G be a directed graph and let ρ be a completely contractive representation on a Hilbert space $\mathcal{H}$. Then ρ has a minimal partly isometric dilation σ which is unique up to unitary equivalence fixing $\mathcal{H}$.*

Proof. The trick of [**2**] is to add an infinite tail to each sink. Briefly, if v_0 is a sink, add vertices v_i for $i \geq 1$ and edges e_i with $s(e_i) = e_{i-1}$ and $r(e_i) = v_i$ for $i \geq 1$. In this manner, we can embed G into a larger graph H with no sinks. Extend ρ to a representation $\bar{\rho}$ of $\mathbb{F}^+(H)$ by setting $\bar{\rho}(e_i) = \bar{\rho}(v_i) = 0$ for $i \geq 1$ on every tail. It is evident that this does not affect the condition of complete contractivity, and $\bar{\rho}$ is pure or fully coisometric exactly when ρ is. Moreover $\mathbb{F}^+(G)$ is a subalgebra of $\mathbb{F}^+(H)$ and $\mathfrak{L}_G$ is a WOT-closed subalgebra of $\mathfrak{L}_H$.

By the Jury–Kribs dilation, there is a unique minimal partially isometric dilation τ of $\bar{\rho}$ on a Hilbert space $\bar{\mathcal{K}}$ containing $\mathcal{H}$. Let $\mathcal{K} = \bigvee \tau(\mathbb{F}^+(G))\mathcal{H}$ and let $\sigma(p) = \tau(p)|_{\mathcal{K}}$ for all paths $p \in \mathbb{F}^+(G)$. It is easy to verify that σ is a partly isometric dilation of ρ, and minimality is easy to verify as well.   ∎

The following corollary is immediate from earlier comments.

Corollary 3.2. *Let G be a directed graph and let ρ be a completely contractive representation on a Hilbert space $\mathcal{H}$. Then the minimal partly isometric dilation σ of ρ is fully coisometric if and only if the row operator T with entries $(\rho(e))_{e \in \mathcal{E}(G)}$ is a coisometry. And σ is pure if and only if*

$$\underset{d \to \infty}{\text{SOT–}\lim} \ \text{SOT–}\sum_{\substack{p \in \mathbb{F}^+(G) \\ |p|=d}} \rho(p)\rho(p)^* = 0.$$

In particular, this holds if $\|T\| < 1$. When ρ is pure, it extends uniquely to a WOT-continuous completely contractive representation of $\mathfrak{L}_G$.

We give one routine result about the tensor algebra case in light of Remark 1.1. This will show that the assumption of a bounded representation of $\mathcal{T}^+(G)$ with a particular property implies the existence of a WOT-continuous completely contractive representation of $\mathfrak{L}_G$ with the same property. In the converse direction, we will always be constructing WOT-continuous completely contractive representations of $\mathfrak{L}_G$. In particular, this applies to the property of irreducibility, being a nest representation and being faithful. The similarity used does not affect

any of these. If we are concerned only with contractive representations, then the restriction to the diagonal is already a $*$-representation and that part of the proof is unnecessary.

Proposition 3.3. *Suppose that ρ is a bounded, norm continuous representation of the tensor algebra $\mathcal{T}^+(G)$ on a Hilbert space $\mathcal{H}$. Then there is a representation similar to ρ and a WOT-continuous completely contractive representation ϕ of $\mathfrak{L}_G$ on the same space such that the WOT-closure of the ranges of the two representations coincide.*

Proof. Note that the representation of ρ restricted to the diagonal algebra span$\{P_x : x \in \mathcal{V}(G)\}$ is similar to a $*$-representation. So we adjust ρ by a similarity to achieve this. Then there is a bound

$$C := \sup\{\|\rho(e)\| : e \in \mathcal{E}(G)\}.$$

Define an injective homomorphism τ of $\mathfrak{L}_G$ into $\mathcal{T}^+(G)$ by setting $\tau(P_x) = P_x$ for $x \in \mathcal{V}(G)$ and $\tau(L_{e_i}) = 2^{-i-1}C^{-1}L_{e_i}$ for any enumeration $\{e_i : i \geq 1\}$ of $\mathcal{E}(G)$. Then Corollary 3.2 shows that $\rho\tau$ extends to a WOT-continuous completely contractive representation ϕ of $\mathfrak{L}_G$. Clearly this has the desired properties. ∎

4. Finite dimensional representations for free semigroupoid algebras

First we will describe a class of representations of $\mathfrak{L}_G$ onto full matrix algebras. Let u be a cycle in G of length k and $\lambda \in \mathbb{T}$. Write $u = e_{i_k} e_{i_{k-1}} \ldots e_{i_1}$. Let $\mathcal{H}_k$ be a k-dimensional Hilbert space with orthonormal basis $h_1, h_2, \ldots, h_k$. Define a representation $\phi_{u,\lambda}$ of $\mathbb{F}^+(G)$ on $\mathcal{H}_k$ as follows. Set

$$\phi_{u,\lambda}(x) = \sum_{s(e_{i_j})=x} h_j h_j^* \quad \text{and}$$

$$\phi_{u,\lambda}(e) = \frac{1}{2} \sum_{e_{i_j}=e} h_{j+1} h_j^*$$

where h_{k+1} denotes λh_1. Since the row operator with entries

$$\big[\phi_{u,\lambda}(e)\big]_{e \in \mathcal{E}(G)}$$

has norm $1/2$, it follows from Corollary 3.2 that this extends to a completely contractive WOT-continuous representation of $\mathfrak{L}_G$.

Definition 4.1. A cycle $u \in \mathbb{F}^+(G)$ is *primitive* if it is not a nontrivial power of another cycle.

Clearly every cycle factors uniquely as $u = v^p$ where v is a primitive cycle. A primitive cycle v has no rotational symmetry, meaning that v is not invariant under any cyclic permutation. Indeed we have:

Lemma 4.2. *If u is a cycle which can be non-trivially factored as $u = u_1 u_2 = u_2 u_1$, then it is not primitive.*

Proof. The hypothesis shows that the cyclic rotation of the letters in u by $|u_1|$ steps leaves u invariant. The set of cyclic permutations leaving the cycle u invariant is a subgroup of the cyclic group $C_{|u|}$, and hence is C_d for some divisor d of $|u|$. Since this subgroup is non-trivial, we obtain $u = v^d$ where v is the cycle formed by the first $|u|/d$ edges of u. In particular, u is not primitive.   ∎

If $u = v^p$, then the representation $\phi_{u,\lambda}$ will not be irreducible. To see this, define a unitary U by $U h_j = h_{j+k/p}$ where we identify h_{k+j} with λh_j. Observe that U commutes with the range of $\phi_{u,\lambda}$. We now establish the converse.

Lemma 4.3. *When u is a primitive cycle and $\lambda \in \mathbb{T}$, the representation $\phi_{u,\lambda}$ is irreducible.*

Proof. Let $u_j = e_{i_{j-1}} \ldots e_{i_2} e_{i_1} e_{i_k} e_{i_{k-1}} \ldots e_{i_j}$ be the cycle that follows u around beginning with the jth path. Primitivity ensures that the cycles u_j are distinct. Observe that $\phi_{u,\lambda}(u) = 2^{-k} \lambda h_1 h_1^*$. Indeed, if you apply $\phi_{u,\lambda}(u)$ to each basis vector h_j, one sees that h_1 returns to a multiple of itself. But starting with any other h_j, the result must be 0 unless at each step the edge e_{i_s} coincides with the edge $e_{i_{s+j}}$, which cannot occur for all k edges by primitivity. Similarly $\phi_{u,\lambda}(u_j) = 2^{-k} \lambda h_j h_j^*$.

Hence we see that $\phi_{u,\lambda}(e_{i_j} u_j) = 2^{-k-1} \lambda h_{j+1} h_j^*$. So the range of $\phi_{u,\lambda}$ contains a set of matrix units for $\mathcal{B}(\mathcal{H}_k)$, whence it is irreducible.   ∎

Recall that G is transitive if there is a path from each vertex in the graph to every other. It is not hard to see that each connected component of G is transitive if and only if every edge of G lies on a cycle.

Theorem 4.4. *If G is a countable directed graph, then the finite dimensional irreducible representations of $\mathfrak{L}_G$ separate points if and only if G is transitive in each component.*

Proof. If the finite dimensional irreducible representations of $\mathfrak{L}_G$ separate the points, then $\mathfrak{L}_G$ is semisimple and therefore G is transitive in each component [**22**, Theorem 5.1].

Conversely, assume that G is transitive in each component. For any path $w \in \mathbb{F}^+(G)$, the transitivity in each component of G implies that

there exists a path $v \in \mathbb{F}^+(G)$ so that vw is a cycle. Hence there is a primitive cycle $u = e_{i_k} e_{i_{k-1}} \ldots e_{i_1}$ so that $vw = u^p$. Consequently there exist $0 \leq k' < k$ and $n \geq 0$ so that $w = e_{i_{k'}} \ldots e_{i_1} u^n$, where we mean u^n if $k' = 0$. Let us write $p_r = e_{i_{k'}} \ldots e_{i_1} u^r$ for $r \geq 0$.

We will make use of the irreducible representations $\phi_{u,\lambda}$ for $\lambda \in \mathbb{T}$. Let us evaluate $g(p, \lambda) := \langle \phi_{u,\lambda}(p)h_1, h_{k'} \rangle$ for paths $p \in \mathbb{F}^+(G)$. Arguing as in Lemma 4.3, one sees that $g(p, \lambda) = 0$ unless $p = p_r$ for some $r \geq 0$; and $g(p_r) = \lambda^r 2^{-k'-rk}$. This power of λ will allow us to distinguish the p_r by integration.

For each $A \in \mathfrak{L}_G$, let $g(A, \lambda) = \langle \phi_{u,\lambda}(A)h_1, h_{k'} \rangle$. If A has the Fourier series $\sum_{p \in \mathbb{F}^+(G)} a_p L_p$, then,

$$(1) \qquad g(A, \lambda) = \sum_{r \geq 0} 2^{-k'-rk} a_{p_r} \lambda^r.$$

The above equality requires a short argument. The calculation of the previous paragraph shows that $g(A, \lambda)$ equals the limit of the Cesaro partial sums of the series in display. However, the coefficients $2^{-k'-rk} a_p$ are dominated by a geometric series. This is easily seen to imply that the Cesaro partial sums as well as the series in (1) converge absolutely and uniformly over the circle $\mathbb{T}$ to the same limit, i.e., $g(A, \lambda)$. We may now compute

$$a_w = \frac{2^{k'+rn}}{2\pi i} \int_{\mathbb{T}} \bar{\lambda}^{n+1} g(A, \lambda) \, d\lambda.$$

For any non-zero element $A \in \mathfrak{L}_G$, there exists a least one non-zero Fourier coefficient a_w for some path w. The argument above shows that $g(A, \lambda)$ is a non-zero function, and hence $\phi_{u,\lambda}(A) \neq 0$ for some $\lambda \in \mathbb{T}$. Therefore these representations separate points. ∎

Recall that the strong radical of a Banach algebra is the intersection of all maximal ideals. If this intersection is $\{0\}$, the algebra is strongly semisimple. The following corollary is new even for the free semigroup algebras $\mathfrak{L}_k$, $k \geq 2$.

Corollary 4.5. *The following are equivalent:*
  (1) $\mathfrak{L}_G$ *is semisimple*
  (2) $\mathfrak{L}_G$ *is strongly semisimple*
  (3) G *is transitive in each component.*
If $\mathcal{V}(G)$ is finite, then the Jacobson radical of $\mathfrak{L}_G$ is equal to the strong radical.

Proof. The kernels of finite dimensional irreducible representations are maximal ideals. Thus (3) implies (2) by Theorem 4.4. Clearly (2) implies (1). By Kribs and Power [**22**, Theorem 5.1], (1) implies (3).

If $\mathcal{V}(G)$ is finite, Kribs and Power also show that the Jacobson radical is generated by $\lambda(e)$ for those edges $e \in \mathcal{E}(G)$ which are not contained in any cycle. Indeed, a little thought shows that this ideal is the WOT-closed span of $\lambda(w)$ of all paths $w \in \mathbb{F}^+(G)$ which are not contained in any cycle (in the sense that there is no path v such that vw is a cycle). This means that $A \in \mathfrak{L}_G$ lies in the radical if and only if the Fourier coefficients a_w vanish whenever w is contained in a cycle. On the other hand, suppose that $A \in \mathfrak{L}_G$ and $a_w \neq 0$ for some $w \in \mathbb{F}^+(G)$ which is contained in a cycle vw. Then factor $vw = u^k$ as a power of a primitive cycle u. The proof of Theorem 4.4 shows that A is not in the maximal ideal $\ker \phi_{u,\lambda}$ for some values of $\lambda \in \mathbb{T}$; whence A is not in the strong radical. So the two radicals coincide. $\blacksquare$

The previous theorem excludes a variety of interesting graphs. In order to cover more cases, we replace irreducible representations with nest representations, where we seek representations that map $\mathfrak{L}_G$ onto a WOT-dense subalgebra of a nest algebra. When restricting attention to finite dimensional representations, the nest algebras are the set of block upper triangular operators (with respect to some decomposition of a finite dimensional space into orthogonal subspaces). Moreover, density implies surjectivity in this case; and the WOT-density of the span of $\mathbb{F}^+(G)$ means that its image spans the nest algebra. Finite dimensional irreducible representations are a special case because $\mathcal{M}_n$ is a nest algebra.

Lemma 4.6. *Let G be a countable directed graph. Any path $w \in \mathbb{F}^+(G)$ decomposes uniquely as a product*

$$w = w_l v_l w_{l-1} v_{l-1} \ldots w_1 v_1 w_0$$

where all the edges involved in a path w_i lie in a single transitive component of G (or the term is a vertex) and each v_i is an edge with source and range in distinct transitive components.

Proof. If there is an edge from one transitive component to another, there can be no path back. The directed graph obtained by mapping each transitive component to a single vertex has no (directed) cycles. It follows that $s(v_i)$ lie in distinct transitive components, as do $r(v_i)$. If there is no path w_i, set it to be the vertex $w_i = r(v_i) = s(v_{i+1})$. $\blacksquare$

Theorem 4.7. *If G is a countable directed graph, then the finite dimensional nest representations separate the points in $\mathfrak{L}_G$.*

Proof. Once again, we will construct a family of nest representations which allow us to recover the Fourier coefficients in $\mathfrak{L}_G$.

Let w be a path in $\mathbb{F}^+(G)$. Factor w using Lemma 4.6 above as $w = w_l v_l w_{l-1} v_{l-1} \ldots w_1 v_1 w_0$. Since each w_i is contained in a single transitive component, arbitrarily complete it to a cycle $w_i' w_i$, which we then factor as a power of a primitive cycle, say $w_i' w_i = u_i^{k_i}$ and $w_i = p_i u_i^{n_i}$ for some proper subpath p_i of u_i. If w_i is a vertex, set $u_i = w_i$. Let ϕ_{u_i, λ_i} be the irreducible representations defined at the beginning of this section, each acting on a finite dimensional Hilbert space $\mathcal{H}_i$. If u_i is a vertex, take $\mathcal{H}_i$ to be one-dimensional and set $\phi_{u_i, \lambda_i}(u_i) = 1$, and $\phi_{u_i, \lambda_i}(p) = 0$ for all other paths $p \in \mathbb{F}^+(G)$.

Set $\mathcal{H} = \sum_i^{\oplus} \mathcal{H}_i$. For each path $w \in \mathbb{F}^+(G)$ and $\lambda = (\lambda_i)_{i=0}^l \in \mathbb{T}^{l+1}$, define a representation $\rho_{w, \lambda}$ as follows. First we set

$$P_{\mathcal{H}_i} \rho_{w, \lambda}|_{\mathcal{H}_i} = \phi_{u_i, \lambda_i} \quad \text{for} \quad 0 \leq i \leq l$$

Observe that ϕ_{u_i, λ_i} is zero except on paths which lie entirely in the transitive component of w_i. In particular, this annihilates all edges except those with both source and range in this transitive component. Now $\phi_{u_i, \lambda_i}(w_i)$ has the form $2^{-|w_i|} \lambda_i^{n_i} k_i h_i^*$ for certain vectors $k_i, h_i \in \mathcal{H}_i$. Set $\rho_{w, \lambda}(v_i) = \frac{1}{2} h_i k_{i-1}^*$. All other edges and vertices are sent to 0. The image of a path is determined by the product of the edges. As before, we verify that the row operator with entries $\left[\rho_{w, \lambda}(e) \right]_{e \in \mathcal{E}(G)}$ has norm $1/2$, and thus by Corollary 3.2 this representation extends from $\mathbb{F}^+(G)$ to $\mathfrak{L}_G$ to be a completely contractive WOT-continuous representation.

To see that this is a nest representation, observe that the proof of Theorem 4.4 shows that the blocks $\mathcal{B}(\mathcal{H}_i)$ lie in the image. The only other non-zero edges are $\rho_{w, \lambda}(v_i) = \frac{1}{2} h_i k_{i-1}^*$. These provide the connection from $\mathcal{H}_i$ to $\mathcal{H}_{i+1}$. It is evident that each element is block lower triangular, and that these operators generate the block lower triangular nest algebra as an algebra.

Now consider the function $g(A, \lambda) = \langle \rho_{w, \lambda}(A) h_0, k_l \rangle$ for $A \in \mathfrak{L}_G$ and $\lambda \in \mathbb{T}^{l+1}$. The idea now is much as before. We observe that if p is a path, then $g(p, \lambda) = 0$ unless

$$p_m := p_l u_l^{m_l} v_l p_{l-1} u_{l-1}^{m_{l-1}} \ldots v_1 p_0 u_0^{m_0}$$

for $m = (m_0, \ldots, m_l) \in \mathbb{Z}^{l+1}$; and $g(p_m, \lambda) = 2^{-|p_m|} \prod_{i=0}^l \lambda_i^{m_i}$. If A has Fourier series $A \sim \sum_{p \in \mathbb{F}^+(G)} a_p L_p$, then

$$g(A, \lambda) = \sum_{m \in \mathbb{Z}^{l+1}} a_{p_m} 2^{-|p_m|} \prod_{i=0}^l \lambda_i^{m_i}.$$

This series converges absolutely and uniformly on a neighbourhood of $\mathbb{T}^{l+1}$ to a holomorphic function because the Fourier coefficients are

bounded by $\|A\|$. Thus we recover the coefficients by integration:

$$a_w = 2^{|w|} \frac{1}{(2\pi i)^{l+1}} \int_{\mathbb{T}^{l+1}} g(A, \lambda) \prod_{i=0}^{l} \bar{\lambda}_i^{n_i+1} \, d\lambda_i.$$

Therefore there exists some word w and λ so that $\rho_{w,\lambda}(A) \neq 0$,

We conclude that the collection of all nest representations separates points. ∎

Since the functions $g(A, \lambda)$ are continuous, it suffices to use a countable set of λ's for each (of countably many) words $w \in \mathbb{F}^+(G)$ in order to separate points.

Corollary 4.8. *Let G be a countable directed graph. Then is a countable family of nest representations that separates the points in $\mathfrak{L}_G$.*

A similar conclusion about the countability of a separating family of irreducible representations holds when G is transitive in each component and in the case to follow of upper triangular nest representations.

An ideal is meet irreducible if it is not the intersection of two strictly larger ideals. The kernel of a nest representation is always meet irreducible. In particular in the finite dimensional case, this holds because a nest algebra on a finite dimensional space has a unique minimal ideal. We can now state the appropriate generalization of Corollary 4.5 to arbitrary free semigroupoid algebras.

Corollary 4.9. *The meet irreducible ideals of a free semigroupoid algebra $\mathfrak{L}_G$ have zero intersection.*

Finally in this section, we consider representations onto the upper triangular matrices $\mathcal{T}_n$ with respect to a fixed basis $h_1, \ldots, h_n$ of $\mathcal{H}_n$. We call such representation an *upper triangular nest representation*. This constraint puts special requirements on the graph. Note that the result below is essential in the proof of Theorem 5.11.

Theorem 4.10. *If G is a countable directed graph, then the upper triangular nest representations of $\mathfrak{L}_G$ separate points if and only if every vertex $x \in \mathcal{V}(G)$ which supports a cycle also supports at least one loop edge.*

Proof. Suppose that there exists a cycle w with $r(w) = s(w) = x$ but x supports no loop edges. We claim that any upper triangular nest representation ψ of $\mathbb{F}^+(G)$ satisfies $\psi(w) = 0$.

Since $\psi(x)$ is a projection in $\mathcal{T}_n$, it is diagonal. We will show that it always has rank at most one. Notice that $x\mathbb{F}^+(G)x$ consists of all cycles $u \in \mathbb{F}^+(G)$ with $r(u) = s(u) = x$. Since x supports no loop edges, this

cycle must pass through another vertex y. Therefore we can factor u as $u = xu_1yu_2x$. Hence

$$\psi(u) = \psi(x)\psi(u_1)\psi(y)\psi(u_2)\psi(x).$$

Since $\psi(x)$ and $\psi(y)$ are orthogonal diagonal projections, this means that $\psi(u)$ is strictly upper triangular. Let Δ_n denote the expectation of $\mathcal{M}_n$ onto the diagonal $\mathcal{D}_n$, and recall that the restriction of Δ_n to $\mathcal{T}_n$ is a homomorphism. So

$$\psi(x)\mathcal{D}_n = \psi(x)\Delta_n(\operatorname{span}\psi(\mathbb{F}^+(G)))\psi(x)$$
$$= \operatorname{span}\Delta_n(\psi(x\mathbb{F}^+(G)x)) = \mathbb{C}\psi(x)$$

because $\Delta_n(\psi(u)) = 0$ all non-trivial cycles u beginning at x. Therefore $\psi(x)\mathcal{D}_n$ is one dimensional; whence so is $\psi(x)$. Consequently $\psi(x)\mathcal{T}_n\psi(x) = \mathbb{C}\psi(x)$ supports no strictly upper triangular operators. So $\psi(u) = 0$ for all cycles u beginning at x. Thus these representations cannot separate points.

Conversely, suppose that every vertex $x \in \mathcal{V}(G)$ which supports a cycle also supports a loop edge. We proceed as in the proof of Theorem 4.4 to construct a family of upper triangular nest representations of $\mathfrak{L}_G$ from which we can recover the Fourier coefficients of each element $A \in \mathfrak{L}_G$.

Let $\mathcal{L}$ denote the set of vertices $x \in \mathcal{V}(G)$ which supports a loop edge. For each vertex $x \in \mathcal{L}$, select one loop edge at x and label it f_x. Call the remaining vertices $\mathcal{R} = \mathcal{V}(G) \setminus \mathcal{L}$ the *rank one* vertices.

Let $w = e_{i_{k-1}}e_{i_{k-2}}\ldots e_{i_2}e_{i_1}$ be a path in $\mathbb{F}^+(G)$ which does not involve any of the designated loop edges. Denote the vertices involved by $x_j = s(e_{i_j})$ for $1 \leq j < k$ and $x_k = r(e_{i_{k-1}})$. Let $J = \{j : x_j \in \mathcal{L}\}$. For each $\lambda = (\lambda_j)_{j \in J} \in \mathbb{T}^J$ with *distinct* coefficients, we will define a representation $\psi_{w,\lambda}$ of $\mathbb{F}^+(G)$ on $\mathcal{H}_k$. Set

$$\psi_{w,\lambda}(x) = \sum_{x_j=x} h_j h_j^* \qquad \text{for } x \in \mathcal{V}(G)$$

$$\psi_{w,\lambda}(f_x) = \frac{1}{2}\sum_{x_j=x} \lambda_j h_j h_j^* \quad \text{for } x \in \mathcal{L}$$

$$\psi_{w,\lambda}(e) = \frac{1}{2}\sum_{e_{i_j}=e} h_{j+1} h_j^* \quad \text{for } e \in \mathcal{E}(G) \setminus \{f_x : x \in \mathcal{L}\}$$

The row operator E with entries $(\psi_{w,\lambda}(e))_{e \in \mathcal{E}(G)}$ satisfies

$$EE^* = \frac{1}{4}\Big((I - h_1 h_1^*) + \sum_{j \in J} h_j h_j^*\Big) \leq \frac{1}{2}I.$$

This is a strict contraction, and thus by Corollary 3.2, the representation $\psi_{w,\lambda}$ extends uniquely to a completely contractive WOT-continuous representation of $\mathfrak{L}_G$.

It is clear that the range of $\psi_{w,\lambda}$ is contained in the algebra $\mathcal{T}_n^*$ of lower triangular matrices. We will show that it is surjective. If $j \notin J$, then x_j is a rank one vertex. Since x does not lie on any cycle, it can occur only once on the path w. Therefore $\psi_{w,\lambda}(x_j) = h_j h_j^*$. Now consider a vertex x such that $x = x_j$ for some $j \in J$ and let $J_x = \{j \in J : x_j = x\}$. Then

$$\psi_{w,\lambda}(2f_x) = \sum_{j \in J_x} \lambda_j h_j h_j^*$$

is a diagonal operator with distinct eigenvalues λ_j. Therefore the algebra it generates contains each $h_j h_j^*$, Thus the image of $\psi_{w,\lambda}$ contains the diagonal. Moreover $\psi_{w,\lambda}(e_{i_j}) h_j h_j^* = h_{j+1} h_j^*$. Therefore the range of $\psi_{w,\lambda}$, which is an algebra, is all of $\mathcal{T}_n^*$. So by reversing the order of the basis, we obtain an upper triangular nest representation.

Now let $v \in \mathbb{F}^+(G)$ be given. We distinguish between edges of the form f_x and the rest. Deleting all occurrences of the f_x's leaves a path w as above. Then putting these loops back yields a factorization of the form

$$v = f_{x_k}^{m_k} e_{i_{k-1}} f_{x_{k-1}}^{m_{k-1}} e_{i_{k-2}} \ldots e_{i_2} f_{x_2}^{m_2} e_{i_1} f_{x_1}^{m_1} =: p_{w,m}$$

where for notational convenience we allow only $m_j = 0$ when $j \notin J$ and ignore the fact that f_x does not exist in this case.

We now proceed as in the proof of Theorem 4.4. For each path w above and for each $A \in \mathfrak{L}_G$, define a function almost everywhere on $\mathbb{T}^J$ (i.e. when the coefficients are distinct) by

$$f_w(A, \lambda) = \langle \psi_{w,\lambda}(A) h_1, h_k \rangle.$$

Observe that as in the earlier proof that for a path p, one will have $f_w(p, \lambda) = 0$ unless p has the form $p_{w,m}$ for some $m = (m_1, \ldots, m_k)$; and

$$f_w(p_{w,m}) = 2^{-|p_{w,m}|} \prod_{j \in J} \lambda_j^{m_j}$$

where $|p_{w,m}| = (k-1) + |m|$. (Here $|m| = \sum m_j$.) Therefore if A has Fourier expansion $A \sim \sum_{p \in \mathbb{F}^+(G)} a_p L_p$, then

$$f_w(A, \lambda) = \sum_{m \in \mathbb{N}^J} 2^{1-k-|m|} a_{p_{w,m}} \prod_{j \in J} \lambda_j^{m_j}.$$

As in the previous proof, observe that this series converges absolutely and uniformly on a neighbourhood of $\mathbb{T}^J$ to a holomorphic function

because the Fourier coefficients are bounded by $\|A\|$. Thus we can recover

$$a_v = a_{p_{w,m}} = 2^{k-1+|m|} \frac{1}{(2\pi i)^{|J|}} \int_{\mathbb{T}^J} f_w(A, \lambda) \prod_{j \in J} \bar{\lambda}_j^{m_j+1} \, d\lambda_j$$

In particular, if $a_v \neq 0$, then there is a set of positive measure on which $f_w(A, \lambda)$ is non-zero. So there are representations with $\psi_{w,\lambda}(A) \neq 0$. Thus they separate points. ∎

5. Faithful nest representations

In this section, we consider the question of when a free semigroupoid algebra $\mathfrak{L}_G$ has a faithful nest representation. The first case to consider is the nest algebra $\mathcal{B}(\mathcal{H})$—in other words, a faithful irreducible representation.

It is useful to first describe some faithful irreducible representations of the free semigroup algebras $\mathfrak{L}_n$. The argument of Read [**33**] provides a much stronger statement, namely that there is a $*$-representation of $\mathcal{O}_2$ which carries the non-commutative disk algebra onto an algebra which is weak-$*$ dense in $\mathcal{B}(\mathcal{H})$. See [**5**] for a simpler proof which works for all $n \geq 2$. However these examples are not so easy to work into our context; and indeed, our purpose may be achieved much more easily anyhow.

This first lemma is likely well-known, but the best we could find in the literature doesn't quite do it, although it comes close. Indeed in [**14**, Corollary 1 of Theorem 3.6], they construct a unitary group $(U_t)_{t \in \mathbb{R}}$ for a given non-closed operator range $A\mathcal{H}$ so that $\mathrm{Ran}(U_t A U_t^*)$ are pairwise disjoint.

Lemma 5.1. *There is an injective compact operator*

$$K = \begin{bmatrix} K_1 & K_2 & \ldots \end{bmatrix}$$

mapping $\mathcal{H}^{(\infty)}$ into $\mathcal{H}$ so that the range of each K_i is dense in $\mathcal{H}$.

Proof. We follow the proof of [**14**, Theorem 3.6]. They work with $\mathcal{H} = L^2(0, 2\pi)$ and begin with $A = \mathrm{diag}(e^{-n^2})$ with respect to the basis $e_n = e^{inx}$ for $n \in \mathbb{Z}$. They observe that the rapid decrease of the Fourier coefficients of functions in the range means that each agrees with an entire function restricted to $[0, 2\pi]$ (almost everywhere, of course). Then they define a unitary U as multiplication by the function $\chi_{(0,\pi)} - \chi_{(\pi,2\pi)}$, and observe that conjugating A by U yields an operator range with trivial intersection.

For $k \geq 1$, let U_k be the multiplication operator on $L^2(0, 2\pi)$ by

$$\chi_k := \chi_{(0,2\pi/k)} - \chi_{(2\pi/k,2\pi)}.$$

Let $K_k = 2^{-k} U_k A U_k^*$ for $k \geq 1$ and $K = \begin{bmatrix} K_1 & K_2 & \dots \end{bmatrix}$.

A function in the range of K is an absolutely convergent sum of functions which will then be the restrictions of entire functions on each interval $[2\pi/(k+1), 2\pi/k]$ for each $k \geq 1$. The change at the point $2\pi/k$ depends only on the image of K_k for $k \geq 2$. Suppose that a vector $x = \sum_{k \geq 1}^{\oplus} x_k$ satisfies $Kx = 0$. As noted above, the image $K_k x_k = \chi_k f_k$, where f_k is the restriction of an entire function. So if y_k is the vector x with the kth coordinate replaced by 0. Then $g_k = K y_k$ agrees with an entire function on $[2\pi/(k+1), 2\pi/(k-1)]$. So $\chi_k f_k$ must agree with g_k on this interval. This forces f_k and all of its derivatives to vanish at $2\pi/k$, whence $f_k = 0$ and thus $x_k = 0$. We conclude that $x_k = 0$ for $k \geq 2$; and since K_1 is injective, so also is $x_1 = 0$. Therefore K is injective. ∎

Lemma 5.2. *Suppose that K_1 and K_2 are injective compact operators with dense range. Then there are invertible operators S_i so that the algebra generated by $K_1 S_1$ and $K_2 S_2$ is the whole algebra $\mathfrak{K}$ of compact operators.*

Proof. Observe that the injectivity and the range of K_i is unchanged if it is replaced by $K_i S_i$ when S_i is invertible. There is a unitary U so that $K_1 U$ is positive, and thus is diagonalizable with respect to some basis $\{e_i : i \geq 1\}$. Then choosing $S_1 = UD$ for some invertible diagonal operator D, we may arrange that $K_1 S_1$ has distinct eigenvalues.

We wish to modify K_2 to $K_2 S_2$ so that the matrix entries $\langle K_2 S_2 e_1, e_i \rangle$ and $\langle K_2 S_2 e_i, e_1 \rangle$ are non-zero for all $i \geq 2$. Since the range of K_2 is dense, it is routine to verify that the range contains a vector $y = K_2 x$ so that $\alpha_i = \langle y, e_i \rangle \neq 0$ for all $i \geq 1$; and in addition, we may easily arrange that x is not a multiple of $K_2^* e_1$. Let $z = P_{\mathbb{C}x}^{\perp} K_2^* e_1$. Then pick a vector w so that $\langle w, e_1 \rangle = 0$ and $\langle w, e_i \rangle \neq 0$ for all $i \geq 2$. Finally let V be any isometry mapping span$\{e_1, w\}^{\perp}$ onto span$\{x, z\}^{\perp}$. We now define an invertible operator

$$S_2 = x e_1^* + \|z\|^{-2} z w^* + V.$$

It now follows that for all $i \geq 1$,

$$\langle K_2 S_2 e_1, e_i \rangle = \langle K_2 x, e_i \rangle = \langle y, e_i \rangle = \alpha_i \neq 0$$

and

$$\langle K_2 S_2 e_i, e_1 \rangle = \langle e_i, S_2^* K_2^* e_1 \rangle$$
$$= \langle e_i, S_2^* (\bar{\alpha}_1 \|x\|^{-2} x + z) \rangle$$
$$= \langle e_i, \bar{\alpha}_1 e_1 + w \rangle \neq 0.$$

We may replace K_2 by $K_2 S_2$ without changing its range. However we now have that $\mathrm{Alg}\{K_1, K_2\}$ is the whole algebra of compact operators. This is because K_1 generates the diagonal algebra with respect to the basis $\{e_i : i \geq 1\}$, and the non-zero matrix entries of K_2 ensure that the matrix units $e_i e_1^*$ and $e_1 e_i^*$ lie in the algebra for all $1 \geq 1$. This guarantees that the whole set of compact operators is generated as an algebra by K_1 and K_2. ∎

Recall that for the free semigroup algebras $\mathfrak{L}_n$, $2 \leq n \leq \infty$, are the free semigroupoid algebras for the graph P_n with one vertex and n loop edges. We write $\mathbb{F}_n^+$ for the free semigroup on n letters, $\mathfrak{L}_n$ for the algebra generated by the left regular representation, and $\mathcal{H}_n$ for the Hilbert space rather than $\mathbb{F}^+(P_n)$, $\mathfrak{L}(P_n)$ and $\mathcal{H}_{P_n}$.

Theorem 5.3. *The free semigroup algebras $\mathfrak{L}_n$ for $2 \leq n \leq \infty$, have faithful irreducible representations.*

Proof. We will use notation as if n is finite, but the argument works equally well if $n = \infty$. Let $\mathcal{H}$ be a separable Hilbert space. Use Lemma 5.1 to obtain an injective compact operator $K = \begin{bmatrix} K_1 & \ldots & K_n \end{bmatrix}$ mapping $\mathcal{H}^{(n)}$ into $\mathcal{H}$, so that the range of each K_i is dense. Use Lemma 5.2 to replace K_i by $K_i S_i$ for $i = 1, 2$ so that the algebra that they generate is all of the compact operators. Scale each K_i by a constant in $(0, 1]$ so that $\sum_{i \geq 1} \|K_i\| = r < 1$. This does not affect the properties that we have established.

In particular, $\|K\| \leq r < 1$. By [**31**] or by Corollary 3.2, there is a unique completely contractive WOT-continuous representation ρ of $\mathfrak{L}_n$ into $\mathcal{B}(\mathcal{H})$ such that $\rho(L_i) = K_i$ for all $i \leq n$. Since $\mathrm{Alg}\{K_1, K_2\} = \mathfrak{K}(\mathcal{H})$, it follows that this representation is irreducible.

Since K is compact and has dense range, $\mathrm{Ran}\, K$ is not closed. Pick a unit vector $x_\emptyset$ which is not in this range. Define vectors $x_w = \rho(L_w) x_\emptyset$ for all $w \in \mathbb{F}_n^+$. We will show that $V := \sum_{w \in \mathbb{F}_n^+} x_w \xi_w^*$ is a bounded

operator from $\mathcal{H}_n$ into $\mathcal{H}$. Indeed setting $\alpha = (\alpha_i)_{i=1}^n$ where $\alpha_i = \|K_i\|$,

$$\|V\| \leq \sum_{w \in \mathbb{F}_n^+} \|x_w\| \leq \sum_{w \in \mathbb{F}_n^+} \|\rho(L_w)\| \leq \sum_{w \in \mathbb{F}_n^+} w(\alpha)$$

$$\leq \sum_{k \geq 0} \sum_{|w|=k} w(\alpha) = \sum_{k \geq 0} \Big(\sum_{i=1}^n \alpha_i \Big)^k \leq \sum_{k \geq 0} r^k = \frac{1}{1-r}.$$

Next we claim that V is injective. Suppose, to the contrary, that $V\zeta = 0$ where $\zeta = \sum_{w \in \mathbb{F}_n^+} a_w \xi_w$ in non-zero. Let w_0 be a word of minimal length k_0 with $a_{w_0} \neq 0$. We may write this vector uniquely as a sum

$$\zeta = \sum_{|w|=k_0} \zeta_w = \zeta_{w_0} + \sum_{|w|=k_0,\ w \neq w_0} \zeta_w =: \zeta_{w_0} + \eta$$

where

$$\zeta_w = \sum_{v \in \mathbb{F}_n^+} a_{wv} \xi_{wv} \quad \text{and} \quad \eta = \sum_{|w|=k_0,\ w \neq w_0} \zeta_w.$$

Note that $V\zeta_w$ lies in the range of $w(K_1, K_2, \dots)$. Observe that the row operator J_{k_0} from $\mathcal{H}^{n^{k_0}}$ into $\mathcal{H}$ with entries $\big(w(K_1, K_2, \dots)\big)_{|w|=k}$ is injective. In particular, the range of $w_0(K_1, K_2, \dots)$ does not intersect the range of the row operator with entries $\big(w(K_1, K_2, \dots)\big)_{|w|=k,\ w \neq w_0}$. In particular, $0 = V\zeta = V\zeta_{w_0} + V\eta$ implies that $V\zeta_{w_0} = V\eta = 0$.

Now $\zeta_{w_0} = a_{w_0} \xi_{w_0} + \sum_{i \geq 1} \zeta_{w_0 i}$. Arguing as before, $V \sum_{i \geq 1} \zeta_{w_0 i}$ lies in the range of $w_0(K_1, K_2, \dots) \operatorname{Ran} K$ while

$$a_{w_0} x_{w_0} = V a_{w_0} \xi_{w_0} = w_0(K_1, K_2, \dots) a_{w_0} x_\emptyset$$

does not. The injectivity of $w_0(K_1, K_2, \dots)$ shows that $a_{w_0} = 0$, contrary to our assumption. Therefore V is injective.

Clearly V is an intertwining map for ρ, meaning that $\rho(A)V = VA$ for all $A \in \mathfrak{L}_n$, as this holds for $\{L_w : w \in \mathbb{F}_n^+\}$ and extends by WOT-continuity to the whole algebra. We can now show that ρ is faithful. If it is not, then the kernel is a non-zero WOT-closed ideal of $\mathfrak{L}_n$. By [**11**, Theorem 2.1], such an ideal must contain an isometry L. But then $0 = \rho(L)V = VL$ contradicts the injectivity of V. So ρ is faithful. ■

Remark 5.4. There is an interesting and unexpected consequence of this proof, which is that the irreducible representations constructed have a separating vector, namely $x_\emptyset$. Indeed, if $A \in \mathfrak{L}_n$ is non-zero, then

$$\rho(A)x_\emptyset = \rho(A)V\xi_\emptyset = VA\xi_\emptyset.$$

As $A\xi_\emptyset \neq 0$ and V is injective, the result follows. Of course $x_\emptyset$ is also cyclic, as is every non-zero vector. For future reference, we note that

the only condition used in the choice of $x_\emptyset$ was that it was not in the range of K.

We now extend the argument to the general case.

Definition 5.5. A graph C is a *cycle graph* if it has a finite set $\mathcal{V}(C) = \{x_1, \ldots, x_n\}$ of vertices and edges $\mathcal{E}(C) = \{e_1, \ldots, e_n\}$ with $s(e_i) = x_i$ and $r(e_i) = x_{i+1 \ (\mathrm{mod}\ n)}$. The *trivial graph* consists of one vertex and no edges. A non-trivial transitive graph which is not a cycle is called *strongly transitive*.

Note that a single vertex with one loop edge is transitive but not strongly transitive.

Theorem 5.6. *Let G be a non-trivial countable directed graph. Then $\mathfrak{L}_G$ has a faithful irreducible representation if and only if it is strongly transitive.*

Proof. Transitivity is an obvious necessary condition: for if x and y are vertices and ϕ is a faithful representation, the projections $\phi(x)$ and $\phi(y)$ are non-zero and $\phi(y)\phi(\mathfrak{L}_G)\phi(x)$ will be non-zero if and only if there are paths from x to y. Also G cannot be a cycle for then $x\mathbb{F}^+(G)x$ is abelian and $P_x\mathfrak{L}_G P_x$ is isomorphic to H^∞. Clearly any representation will restrict to an abelian algebra on $\rho(P_x)\mathcal{H}$. If this is dense, then $\mathrm{rank}\,\rho(P_x) = 1$. But then ρ is not injective unless there are no cycles at all, which would mean that G is the trivial graph.

We will establish the converse. Let G be a strongly transitive graph. Observe first that if each vertex in a transitive graph is the source of exactly one edge, then G is a cycle. So there is a least one vertex x_0 and two distinct edges e_1 and e_2 with $s(e_1) = s(e_2) = x_0$.

Let $\mathcal{E}(G)$ be enumerated as $\{e_1, e_2, \ldots\}$. Using Lemma 5.1, we select an injective compact operator

$$K = \begin{bmatrix} K_1 & K_2 & \ldots \end{bmatrix}$$

so that each K_i has dense range. This property is unaffected if each K_i is multiplied by non-zero scalars bounded by 1, so we may also assume that $\sum_{i \geq 1} \|K_i\| = r < 1$. Therefore $\|K\| \leq r$.

We will define a representation ρ on a Hilbert space

$$\mathcal{K} = \mathcal{H}^{|\mathcal{V}(G)|} = \sum_{x \in \mathcal{V}(G)} \oplus \mathcal{H}_x$$

so that $Q_x := \rho(x) = P_{\mathcal{H}_x}$ for each vertex $x \in \mathcal{V}(G)$. Let J_x be a fixed unitary mapping $\mathcal{H}$ onto $\mathcal{H}_x$. By Corollary 3.2, there is a WOT-continuous representation ρ of $\mathfrak{L}_G$ on such that $\rho(e_i) = J_{r(e_i)}K_i J^*_{s(e_i)}$.

Pick two minimal cycles with source x_0 of the form $c_1 = p_1 e_1$ and $c_2 = p_2 e_2$. Observe that the edges e_1 and e_2 do not occur in either of the paths p_1 or p_2: in order to use either in p_i, the path would first have to return to x_0 and then leave again on e_i, contradicting minimality. By Lemma 5.2, there are (contractive) injective operators S_1 and S_2 on $\mathcal{B}(\mathcal{H})$ so that $\rho(c_1)S_1$ and $\rho(c_2)S_2$ generate the compact operators as an algebra. Replace K_1 and K_2 with $K_1 S_1$ and $K_2 S_2$ and redefine ρ with the new values. We still have the WOT-continuous extension to $\mathfrak{L}_G$. But in addition, we now have that

$$\rho(P_{x_0}\mathfrak{L}_G P_{x_0}) = Q_{x_0}\rho(\mathfrak{L}_G)Q_{x_0}$$

contains generators for $\mathfrak{K}(\mathcal{H}_{x_0})$.

Let us show that ρ is irreducible. The WOT-closure of the range of ρ contains $\mathcal{B}(\mathcal{H}_{x_0})$. If x is any other vertex of G, select a path p from x_0 to x and a path q from x to x_0. Then $\rho(p)$ is a compact operator which is injective on $\mathcal{H}_{x_0}$ and maps this onto a dense subspace of $\mathcal{H}_x$. Likewise $\rho(q)$ is a compact operator which is injective on $\mathcal{H}_x$ and maps it onto a dense subspace of $\mathcal{H}_{x_0}$. Therefore $\rho(P_x\mathfrak{L}_G P_{x_0})$ contains $Q_x\rho(p)\mathfrak{K}(\mathcal{H}_{x_0})$ which is norm-dense in $Q_k\mathfrak{K}Q_{x_0}$, and thus WOT-dense in $Q_x\mathcal{B}(\mathcal{K})Q_{x_0}$. Similarly $\rho(P_{x_0}\mathfrak{L}_G P_x)$ is WOT-dense in $Q_{x_0}\mathcal{B}(\mathcal{K})Q_x$. Whence $\rho(P_x\mathfrak{L}_G P_x)$ is WOT-dense in $Q_x\mathcal{B}(\mathcal{K})Q_x$ as well. It follows that ρ is irreducible.

Next we verify that ρ is faithful. We wish first to show that the restriction to $Q_{x_0}\mathcal{K}$ is faithful on $P_{x_0}\mathfrak{L}_G P_{x_0}$. Enumerate all minimal cycles with source x_0 as $\mathcal{C} = \{c_1, c_2, \dots\}$ beginning with the two cycles c_1 and c_2 used above. Now $x_0\mathbb{F}^+(G)x_0$ is isomorphic to $\mathbb{F}^+(G_{x_0})$ where G_{x_0} is the graph with a single vertex x_0 and loop edges given by $\mathcal{C}$. In particular, $P_{x_0}\mathfrak{L}_G P_{x_0}$ is isomorphic to $\mathfrak{L}_{G_{x_0}}$, which is in turn isomorphic to $\mathfrak{L}_n$ where n is the cardinality of $\mathcal{C}$. Following the proof of Theorem 5.3, as we have the conditions of dense range and generation of the compact operators, it suffices to verify that $\sum_{i\geq 1}\|\rho(c_i)\| = r < 1$. There are a number of ways to see this. One is this: the calculation of $\|V\|$ in the proof of Theorem 5.3 bounds the sum of the upper bounds of $\|\rho(w)\|$ over all paths by $(1-r)^{-1}$. The corresponding calculation using the same norm estimates for $\rho(c_i)$ must yield a smaller sum, as the collection of paths built from $\mathcal{C}$ is a proper subset. But if $\sum_{i\geq 1}\|\rho(c_i)\| > r$, this estimate would be strictly greater. We can conclude that ρ is faithful on $P_{x_0}\mathfrak{L}_G P_{x_0}$.

Suppose that A is a non-zero element in $\ker\rho$. Then for some vertices x and y, $P_y A P_x \neq 0$. Choose a path p from x_0 to x and a path q from y to x_0. Then $L_q A L_p$ is a non-zero element of $P_{x_0}\mathfrak{L}_G P_{x_0}$ in $\ker\rho$, contrary to fact. Hence ρ is faithful. ∎

Remark 5.7. *The transpose graph.* We need to make an observation about representations of the transposed graph G^t which has the same vertices and edges as G, but the edges have the range and source maps reversed. The semigroupoids $\mathbb{F}^+(G)$ and $\mathbb{F}^+(G^t)$ are anti-isomorphic but generally this does not extend to the algebras $\mathfrak{L}_G$ and $\mathfrak{L}_{G^t}$. Kribs and Power [**22**] show that $\mathfrak{L}_{G^t}$ is unitarily equivalent to $\mathfrak{R}_G$, the algebra generated by the right regular representation. However the fact that we are in a strictly contractive situation makes things easier.

Observe that the diagonal algebra $\mathcal{D} = \text{WOT–span}\{P_x : x \in \mathcal{V}(G)\}$ is mapped isometrically onto the diagonal $\text{WOT–span}\{\rho(x) : x \in \mathcal{V}(G)\}$. Consider the WOT-closed ideal $\mathfrak{L}_G^0$ generated by $\{L_e :\in \mathcal{E}(G)\}$. Note that $\mathfrak{L}_G = \mathcal{D} + \mathfrak{L}_G^0$ as a topological direct sum. Let $A(\mathbb{F}^+(G))$ denote the (separable) algebra of absolutely convergent sums of $\{L_w : w \in \mathbb{F}^+(G)\}$ with norm $\|\sum a_w L_w\|_1 = \sum_{w \in \mathbb{F}^+(G)} |a_w|$. The restriction of ρ to $\mathfrak{L}_G^0$ maps factors the ideal through $A(\mathbb{F}^+(G))$. Indeed, set $r_e = \|K_e\|$. The map determined by $\sigma(L_e) = r_e L_e$ maps $\mathfrak{L}_G^0$ into $A(\mathbb{F}^+(G))$ because Fourier coefficients are bounded and

$$\sum_{k \geq 1} \sum_{|w|=k} \|\sigma(L_w)\| \leq \sum_{k \geq 1} \sum_{|w|=k} r_{e_{i_k}} \ldots r_{e_{i_1}}$$

$$\leq \sum_{k \geq 1} \Big(\sum_{e \in \mathcal{E}(G)} r_e \Big)^k \leq \frac{r}{1-r}$$

where we write $w = e_{i_k} \ldots e_{i_1}$. Then define a representation τ of $A(\mathbb{F}^+(G))$ by setting $\tau(L_e) = \|\rho(L_e)\|^{-1}\rho(L_e)$ and $\tau(x) = \rho(x)$. Clearly this determines a contractive representation of $A(\mathbb{F}^+(G))$. Moreover the factorization $\rho = \tau\sigma$ is evident.

The representation τ inherits certain properties from ρ. For example, since the WOT-closure of the range of τ equals the WOT-closure of the range of $\rho(\mathfrak{L}_G)$, it will be irreducible if and only if ρ is. Moreover in the construction used here, it will be faithful if ρ is faithful. This essentially comes down to a calculation as in the proof of Theorem 5.3 that shows that the intertwining operator V is injective.

Now observe that $A(\mathbb{F}^+(G^t))$ is anti-isomorphic to $A(\mathbb{F}^+(G))$ in the natural way by identifying a path with the reverse. The representation

$$\tau^t\Big(\sum_{w \in \mathbb{F}^+(G^t)} a_w L_w \Big) = \sum_{w \in \mathbb{F}^+(G^t)} a_w \tau(L_{w^t})^*$$

is continuous, and has range equal to $\tau(A(\mathbb{F}^+(G)))^*$. In particular, if τ is injective, then so is τ^t; and if τ is irreducible, so is τ^t.

Now consider the representation $\rho^t = \tau^t\sigma^t$ of $\mathfrak{L}_{G^t}$. This is the representation of $\mathfrak{L}_{G^t}$ which satisfies $\rho^t(L_w) = \rho(L_{w^t})^*$. This extends

to a WOT-continuous completely contractive representation of $\mathfrak{L}_{G^t}$ by Corollary 3.2. Now σ^t maps $\mathfrak{L}_{G^t}$ injectively into a dense subalgebra of $A(\mathbb{F}^+(G^t))$. If ρ is faithful, then so is τ; whence τ^t and so ρ^t are also faithful. Likewise ρ^t will be irreducible if ρ is irreducible. For the next application, we observe that if the range of ρ is a nest representation, then so is ρ^t for the complementary nest.

We can now establish a complete characterization of when a graph algebra has a faithful nest representation.

Definition 5.8. If G is a countable directed graph, the *transitive quotient* of G is the graph obtained by replacing each transitive component with a vertex τ, each edge mapping from one component to another becomes an edge between the corresponding vertices of the quotient graph, and the number of loop edges at a vertex τ will be the number of primitive cycles through any fixed vertex in the corresponding component.

Observe that there is no loop on a vertex τ in the transitive quotient only when the transitive component is trivial. There is a single loop edge precisely when the component is a cycle. When a component is strongly transitive, there will be countably many loop edges in the quotient.

Theorem 5.9. *A directed graph G has a faithful nest representation if and only if*

(1) *The transitive quotient of G is totally ordered.*
(2) *No transitive component of G is a cycle.*
(3) *The trivial components of G form an interval in the total order order equivalent to a subset of $\mathbb{Z}$, and the restriction of the graph to this set consists of a single path with one edge from each vertex to the next in the order.*

Proof. First observe that these conditions are necessary for the existence of a faithful nest representation ρ into $\mathcal{T}(\mathcal{N})$. It is evident that each transitive component must map onto an interval of the nest, and that this restriction must be faithful. So cycle components are eliminated as they yield infinite dimensional abelian algebras. Moreover there must be edges from one component to another if and only if the image interval dominates the other interval in the order on the nest. Hence the transitive quotient is totally ordered.

If x and y represent trivial components, then $\rho(x)$ and $\rho(y)$ are one dimensional. Suppose that $\rho(x) \prec \rho(y)$ in the order on the nest. Then $\rho(x)\mathcal{T}(\mathcal{N})\rho(y) = \rho(P_x\mathfrak{L}_G P_y)$ is one dimensional. Hence $P_x\mathfrak{L}_G P_y$ is

one dimensional. Thus there is a unique path from y to x in G. So this path cannot pass through any strongly transitive component. So the path passes only through trivial components. Moreover, there can only be one edge from a trivial component leading to another trivial component (for otherwise from the total order, we would deduce the existence of two paths from one vertex to another). In particular, the order structure on the trivial components must coincide with a subset of $\mathbb{Z}$.

For the converse, first suppose that every transitive component is strongly transitive and that the transitive quotient is totally ordered. We will make a modification in the construction of an irreducible representation as follows. Add some extra edges to G to form a (strongly) transitive graph H. Use Lemma 5.1 to define an injective compact operator $K = \begin{bmatrix} K_1 & K_2 & \dots \end{bmatrix}$ so that each K_i has dense range and $\sum_{i \geq 1} \|K_i\| = r < 1$. As in the proof of Theorem 5.6, we can now define a faithful irreducible representation ϕ of $\mathfrak{L}_H$. This is not affected if certain of the operators K_i are replaced by $K_i S_i$ if S_i is an invertible contraction. So using Lemma 5.2 as in the proof of Theorem 5.6, we can modify some of these operators so that the restriction to each transitive component of G is WOT-dense in the algebra of operators on its range. Now restrict the representation to $\mathfrak{L}_G$.

Clearly the restriction is faithful. We claim that $\phi(\mathfrak{L}_G)$ is WOT-dense in a nest algebra $\mathcal{T}(\mathcal{N})$ which has an infinite dimensional atom for each transitive component of G and is ordered as the transitive quotient. Indeed, for each transitive component τ, let

$$P_\tau = \text{SOT--}\sum_{x \in \mathcal{V}(\tau)} \phi(x).$$

Then by the construction above, we have arranged that $\phi(\tau)$ is WOT-dense in $\mathcal{B}(P_\tau \mathcal{H})$. Form the nest $\mathcal{N}$ obtained by ordering the subspaces $P_\tau \mathcal{H}$ by the total order on the transitive quotient of G. Observe that the range of ϕ is WOT-dense in the diagonal algebra

$$\mathcal{T}(\mathcal{N}) \cap \mathcal{T}(\mathcal{N})^* = \sum_\tau \oplus \mathcal{B}(P_\tau \mathcal{H}).$$

Finally, from the fact that the induced order on the transitive quotient is total, there will be a path from each component to every other lower in the total order. The image of this path yields a non-zero operator between the corresponding atoms of the nest. Together with the diagonal, it is straightforward to check that the algebra generated is WOT-dense in $\mathcal{T}(\mathcal{N})$.

To deal with an interval of trivial components, proceed as follows. Let $\mathcal{V}_t$ denote the vertices of G which are trivial components. Let $\mathcal{V}_+$ denote the vertices of G which are greater than $\mathcal{V}_t$ in the total order, and $\mathcal{V}_-$ be those which are less than $\mathcal{V}_t$. Let G_-, G_t and G_+ be the restriction of G to $\mathcal{V}_-$, $\mathcal{V}_t$ and $\mathcal{V}_+$ respectively. In addition, the restriction of G to $\mathcal{V}_t \cup V_-$ and $\mathcal{V}_t \cup V_+$ will be denoted as $G_{-,t}$ and $G_{t,+}$ respectively.

Form a Hilbert space $\mathcal{H} = \sum_{x \in \mathcal{V}(G)}^{\oplus} \mathcal{H}_x$ where $\mathcal{H}_x$ is infinite dimensional for $x \in \mathcal{V}_- \cup \mathcal{V}_+$ and $\mathcal{H}_x = \mathbb{C}h_x$ is one dimensional for $x \in \mathcal{V}_t$. The representation will have $Q_x := \phi(x) = P_{\mathcal{H}_x}$ for $x \in \mathcal{V}(G)$. As before, identify each $\mathcal{H}_x$ for $x \in \mathcal{V}_- \cup \mathcal{V}_+$ with $\mathcal{H}$ via a unitary J_x of $\mathcal{H}$ onto $\mathcal{H}_x$. Again start with an injective compact operator $K = [K_e]_{e \in \mathcal{E}(G)}$ such that each component K_e has dense range and

$$\|K\| \leq \sum_{e \in \mathcal{E}(G)} \|K_e\| \leq r < 1.$$

We define the representation ϕ in several stages. First, we define ϕ on $G_{-,t}$. Use the construction above for the case of all strongly transitive components to define a faithful nest representation ϕ^- of $\mathfrak{L}(G^-)$. This involved some modification of the K_es.

Enumerate the vertices x_j in $\mathcal{V}_t$ in their given order by a subset $J \subset \mathbb{Z}$; and let e_j be the unique edge in G_t with $s(e_j) = x_j$ and $r(e_j) = x_{j-1}$ for all $j \in J$ except the least element (if there is one). Define

$$\phi(e_j) = r_j h_{j-1} h_j^*$$

where $r_j = \|K_{e_j}\|$. Here the role of the operator K_{e_j} is merely to control the norm of $\phi(e_j)$.

Set

$$\mathcal{E}_- = \{e \in \mathcal{E}(G) : s(e) \in \mathcal{V}_t \text{ and } r(e) \in \mathcal{V}_-\}$$

and dually

$$\mathcal{E}_+ = \{e \in \mathcal{E}(G) : s(e) \in \mathcal{V}_+ \text{ and } r(e) \in \mathcal{V}_t\}.$$

Fix a unit vector $u_0 \in \mathcal{H}$. Define a vector $k_e = J_{r(e)} K_e u_0$ for each $e \in \mathcal{E}_-$; and let $\phi(e) = k_e h_x^*$. This defines ϕ on $G_{-,t}$.

For $G_{t,+}$, we will instead use the same construction for the transposed graph $G_{t,+}^t$, and take its transpose as in Remark 5.7. Finally, if e is an edge with $s(e) \in \mathcal{V}_+$ and $r(e) \in \mathcal{V}_-$, define $\phi(e) = J_{r(e)} K_e J_{s(e)}^*$. This defines $\phi(e)$ for all edges of G. Since

$$\sum_{e \in \mathcal{E}(G)} \|\phi(e)\| \leq \sum_{e \in \mathcal{E}(G)} \|K_e\| \leq r < 1,$$

this defines a WOT-continuous completely contractive representation of $\mathfrak{L}_G$ by Corollary 3.2.

The nest $\mathcal{N}$ is built from these atoms using the total order on the transitive quotient as before.

Observe that this is a nest representation. The restriction of ϕ to G_- is a faithful nest representation of $\mathfrak{L}_{G_-}$ by the earlier construction. The restriction of ϕ to G_+ is a faithful nest representation of $\mathfrak{L}_{G_+}$ because of Remark 5.7. The restriction of ϕ to G_t maps $\mathfrak{L}_{G_t}$ into a WOT-dense subalgebra of the upper triangular operators on the basis $\{h_{x_j} : j \in J\}$. This is also faithful because there are no extra edges involved. Moreover there are edges mapping G^+ into G^t and from G^t into G^- so that there is a path from every vertex in $\mathcal{V}_+$ to any vertex in $\mathcal{V}_t$ and from every vertex in $\mathcal{V}_t$ to every vertex in $\mathcal{V}_-$. Consequently, the fact that the edges are sent to non-zero operators is sufficient to see that the algebra generated is WOT-dense in the nest algebra.

It remains to verify that the map ϕ is faithful. Clearly, $\ker \phi$ is generated by elements of the form $A = P_y A P_x$ where $x, y \in \mathcal{E}(G)$. So it suffices to show that ϕ does not annihilate elements of this form. Since the restrictions to the three subgraphs are faithful, we need only consider three possibilities.

Consider a non-zero element $A \in \mathfrak{L}_G$ of the form $A = P_y A P_x$ for $x \in \mathcal{V}_t$ and $y \in \mathcal{V}_-$. Let $A \sim \sum_{w=ywx} a_w L_w$ be its Fourier expansion. Each w in the above sum factors uniquely as $w = w'ew''$, with $w' \in \mathbb{F}^+(G_-)$, $e \in \mathcal{E}_-$ and $w'' \in \mathbb{F}^+(G_t)$. Let k be the length of the shortest path w' appearing in these factorizations, say $\tilde{w} = w_0 e_0 w_0''$.

Set

$$W = \{we \in \mathbb{F}^+(G) : w \in \mathbb{F}^+(G_-), \ |w| = k, \ r(w) = y, \ e \in \mathcal{E}(G_-) \cup \mathcal{E}_-\}.$$

Then A factors as:

$$A = \sum_{we \in W} L_{we} A_{we} = \left[L_{we}\right]_{we \in W} \left[A_{we}\right]_{we \in W}^t$$

where $A_{we} = L_{we}^* A$. This is a bounded factorization in which $\left[L_{we}\right]_{we \in W}$ is a partial isometry and $\|\left[A_{we}\right]_{we \in W}^t\| = \|A\|$. The term $L_{w_0 e_0} A_{w_0 e_0}$ is non-zero. We may suppose that $A_{w_0 e_0} = P_{s(e_0)} A_{w_0 e_0} P_x$, and this is a non-zero element of $\mathfrak{L}_{G_t}$.

But then, $\phi(A_{w_0 e_0}) h_x$ is a non-zero multiple of $h_{s(e_0)}$, which is send by $\phi(L_{e_0})$ to a non-zero multiple of k_{e_0}, and then by $\phi(L_{w_0})$ to a non-zero vector in the range of $J_y K_{w_0 e_0}$. On the other hand,

$$\phi(A - L_{w_0 e_0} A_{w_0 e_0}) = \left[\phi(L_{we})\right]_{we \in W \setminus \{w_0 e_0\}} \left[\phi(A_{we})\right]_{we \in W \setminus \{w_0 e_0\}}^t$$

lies in the range of $J_y\big[K_{we}\big]_{we\in W\setminus\{w_0e_0\}}$. The intersection is $\{0\}$ since $J_y\big[K_{we}\big]_{we\in W}$ is injective. Therefore $\phi(A)\neq 0$. So ϕ is injective on the graph restricted to $\mathcal{V}_-\cup\mathcal{V}_t$.

A similar argument shows that ϕ is injective on the graph restricted to $\mathcal{V}_-\cup\mathcal{V}_t$. Indeed, taking adjoints puts us in the same case as above for the transpose algebra. This is why we used the transpose of a representation of $G^t_{t,+}$. To return to $G_{t,+}$, we use Remark 5.7.

Finally we consider non-zero elements $A = P_y A P_x$ with $x\in\mathcal{V}_+$ and $y\in\mathcal{V}_-$. The argument is much the same. Again there is a least integer k so that A has a non-zero Fourier coefficient $a_{w'ew''}$ where $w'\in\mathbb{F}^+(G_-)$ and $s(e)\in\mathcal{V}_+\cup\mathcal{V}_t$. Set

$$W=\{we\in\mathbb{F}^+(G): w\in\mathbb{F}^+(G_-),\ |w|=k,\ r(w)=y,\ s(e)\in\mathcal{V}_+\cup\mathcal{V}_t\}.$$

Consider once again factorizations of the form

$$A = \sum_{we\in W} L_{we}A_{we} = \big[L_{we}\big]_{we\in W}\big[A_{we}\big]^t_{we\in W}$$

where $A_{we} = L^*_{we}A$. This is a bounded factorization in which $\big[L_{we}\big]_{we\in W}$ is a partial isometry and $\|\big[A_{we}\big]^t_{we\in W}\| = \|A\|$. The term $L_{w_0e_0}A_{w_0e_0}$ is non-zero. We may suppose that $A_{w_0e_0} = P_{s(e_0)}A_{w_0e_0}P_x$, and this is a non-zero element of $\mathfrak{L}_G$.

But then, $\phi(A_{w_0e_0})$ has non-zero range in $\mathcal{H}_{s(e_0)}$, which is send by the injective operator $\phi(L_{w_0e_0})$ to a non-zero vector in the range of $J_y K_{w_0e_0}$. On the other hand,

$$\phi(A - L_{w_0e_0}A_{w_0e_0}) = \big[\phi(L_{we})\big]_{we\in W\setminus\{w_0e_0\}}\big[\phi(A_{we})\big]^t_{we\in W\setminus\{w_0e_0\}}$$

lies in the range of $J_y\big[K_{we}\big]_{we\in W\setminus\{w_0e_0\}}$. The intersection is $\{0\}$ since $J_y\big[K_{we}\big]_{we\in W}$ is injective. Therefore $\phi(A)\neq 0$.

This completes all of the cases and establishes that ϕ is faithful. $\blacksquare$

Example 5.10. The nest used in the this theorem need not have a simple order. For example, consider a graph G with vertices indexed by the rationals $\mathbb{Q}$. Put two loop edges at each vertex and an edge from x_s to x_r if $r < s$. Then the theorem establishes a faithful representation into the Cantor nest with infinite dimensional atoms. This nest is similar to a nest with a large continuous part. Thus there are WOT-continuous completely bounded faithful nest representations into nests which are not atomic.

We do not know which graphs have a faithful representation into a continuous nest algebra.

The final result determines when a free semigroupoid algebra $\mathfrak{L}_G$ admits a nest representation for a multiplicity-free nest which has basis of atoms ordered by $\mathbb{N}$. It is remarkable that in spite of the large radical in the nest algebra, if G has no sinks, then such a free-semigroupoid algebra has to be semisimple. This result can be modified to yield a characterization for nests of order type $-\mathbb{N}$ and $\mathbb{Z}$ as well, but details are omitted.

Theorem 5.11. *Let G be a countable directed graph and let $\mathcal{N}$ be the multiplicity-free nest which is ordered as $\mathbb{N} \cup \{+\infty\}$. The free semi-groupoid algebra $\mathfrak{L}_G$ admits a faithful nest representation into $\mathrm{Alg}\,\mathcal{N}$ if and only if one of the following conditions is satisfied:*

(1) *G is strongly transitive and every vertex $x \in \mathcal{V}(G)$ supports a loop edge.*

(2) *$\mathcal{V}(G) = \{x_i : i \geq 1\}$ and $\mathcal{E}(G) = \{e_i : i \geq 1\}$ where $s(e_i) = x_{i+1}$ and $r(e_i) = x_i$ for all $i \geq 1$.*

(3) *$\mathcal{V}(G) = \mathcal{V}_0 \cup \{x_i : 1 \leq i \leq n\}$. The restriction of G to $\mathcal{V}_0$ is a strongly transitive graph G_0 such that every vertex $x \in \mathcal{V}_0$ supports a loop edge. The restriction of G to $\{x_i : 1 \leq i \leq n\}$ consists of $\{e_i : 1 \leq i \leq n-1\}$ where where $s(e_i) = x_{i+1}$ and $r(e_i) = x_i$ for all $1 \leq i \leq n-1$. Plus there is an edge e_n with $r(e_n) = x_n$ and $s(e_n) \in \mathcal{V}_0$. There may in addition be other edges with source in $\mathcal{V}_0$ and range in $\{e_i : 1 \leq i \leq n-1\}$.*

Proof. The nest $\mathcal{N}$ is unique up to unitary equivalence, so let $\mathcal{H} = \ell^2$ with orthonormal basis $\{h_i : i \geq 1\}$, and set

$$N_k = \mathrm{span}\{h_i : 1 \leq i \leq k\} \quad \text{for} \quad k \geq 0 \quad \text{and} \quad N_\infty = \mathcal{H}.$$

The nest is

$$\mathcal{N} = \big\{ N_k : k \in \mathbb{N}_0 \cup \{\infty\} \big\}.$$

The atoms of the nest are the projections $P_i = h_i h_i^* = P_{N_i} - P_{N_{i-1}}$ for $i \geq 1$.

Assume that there exists a faithful nest representation ϕ mapping $\mathfrak{L}_G$ onto a WOT-dense subalgebra of $\mathrm{Alg}\,\mathcal{N}$. Then vertices are sent to diagonal projections $Q_x := \phi(x)$ for $x \in \mathcal{V}(G)$.

As in the proof of Theorem 4.10 about upper triangular nest representations, we find that if x supports no loop edge, then Q_x will have rank at most one (thus equal to one since ϕ is faithful). But then this forces $x\mathbb{F}^+(G)x$ to be one dimensional, which implies that there are no cycles through x. Thus each transitive component that consists of more than one point has a loop edge at each vertex.

In the case that the transitive component of x is not the trivial graph, there are cycles through x. Therefore $x\mathbb{F}^+(G)x$ is infinite and $P_x \mathfrak{L}_G P_x$

is infinite dimensional. Therefore $\phi(P_x)$ has infinite rank. This is a diagonal projection, so its support must be unbounded. Therefore it follows that $\phi(P_y) \operatorname{Alg}\mathcal{N}\phi(P_x)$ is non-zero for every edge y. Consequently there is a path in G from x to y. So the infinite rank points form a single transitive component.

There is a difference from the situation of Theorem 4.10 regarding a transitive component consisting of exactly one point. We claim that in this instance, there are at least two loop edges or none at all. In other words, it cannot be a cycle consisting of a single point and single loop edge. Indeed, if x is itself a transitive component with a single loop edge u, then $x\mathbb{F}^+(G)x$ consists of powers of u and $P_x\mathfrak{L}_G P_x$ is isomorphic to H^∞. This is an infinite dimensional abelian algebra. As in the previous paragraph, $\phi(x)$ has infinite rank. But then $\phi(x)\operatorname{Alg}\mathcal{N}\phi(x)$ is not abelian, so it cannot be contained in the range of $\phi(P_x\mathfrak{L}_G P_x)$. This establishes our claim.

So if G contains no rank one vertices, it consists of a single strongly transitive component in which each vertex supports a loop edge. This is case (1).

In the event that $\phi(x)$ has rank one, it must equal some atom P_n of the nest algebra. Hence $\operatorname{Alg}\mathcal{N}\phi(x)$ is finite dimensional. So $\mathfrak{L}_G P_x$ is finite dimensional. This implies that x has no paths into any cycles, and thus the path component beginning at x is finite with no loops. Thus no atom P_i for $i < n$ is dominated by $\phi(y)$ for any y in a strongly transitive component. So all correspond to rank one vertices. The vertices x_i which have rank one are linearly ordered in the transitive relation determined by $\operatorname{Alg}\mathcal{N}|_{\sum \phi(P_{x_i})\mathcal{H}}$ because exactly one of $\phi(P_{x_j})\operatorname{Alg}\mathcal{N}\phi(P_{x_i})$ or $\phi(P_{x_i})\operatorname{Alg}\mathcal{N}\phi(P_{x_j})$ is non-zero for $i \neq j$.

Say that an initial segment $P_1, \ldots, P_n$ of $\mathcal{N}$ are atoms of the nest (corresponding to unit vectors $h_1, \ldots, h_n$) which are the image of rank one vertices $x_1, \ldots, x_n$. For $2 \leq k \leq n$, There is an element $a_k \in \mathfrak{L}_G$ such that $\langle \phi(a_k)h_k, h_{k-1}\rangle = 1$. We replace a_k by $P_{x_{k-1}}aP_{x_k}$ so that $\phi(a_k) = h_{k-1}h_k^*$. It follows that there is an edge e_k in the graph from x_k to x_{k-1}, for any other path will factor through other vertices and so would not be supported on the $(k-1, k)$ entry.

Moreover it follows that there are no other edges between rank one vertices. To see this, suppose that there is an edge e from x_k to x_j where $j \leq k - 2$. Then arguing as above, we see that $\phi(e) = \alpha h_j h_k^*$ for some $\alpha \in \mathbb{C}$. Now $a = \alpha a_{j+1} \ldots a_k$ is a word of order at least two. Note that $\phi(a - e) = 0$ and $a \neq e$, contradicting the faithfulness of ϕ. It follows that the graph on $x_1, \ldots, x_n$ is a simple directed chain.

If G consists entirely of rank one vertices, then G is the graph corresponding to case (2).

If there is a non-empty finite collection $x_1, \ldots, x_k$ of rank one vertices, then there is a single strongly transitive component which is mapped into a dense subalgebra of $P_{N_k}^{\perp} \operatorname{Alg} \mathcal{N}$. A particular vertex y_1 must satisfy $\phi(y) > P_{n+1}$. Arguing as above, there must be an edge in G from y to x_n. But there may also be other edges from the transitive component to the rank one vertices. This is case (3).

Thus we have shown that one of the three possible types of graph above are necessary for a nest representation onto a dense subalgebra of $\operatorname{Alg} \mathcal{N}$.

For the converse, we deal separately with the three cases.

Case (2) is simplest. Just define $\phi(x_i) = h_i h_i^*$ and $\phi(e_i) = \frac{1}{2} h_i h_{i+1}^*$. This extends to a WOT-continuous representation of $\mathfrak{L}_G$ by Corollary 3.2. The range is evidently upper triangular and contains the matrix units. Hence it is WOT-dense in $\operatorname{Alg} \mathcal{N}$.

Now suppose that (1) holds. We use the notation and terminology from the proofs of Theorem 4.4 and Theorem 4.10.

As in the proof of Theorem 4.10, we select a loop edge f_x for each vertex $x \in \mathcal{V}(G)$. Let $\mathcal{L} = \{f_x : x \in \mathcal{V}(G)\}$. As we have observed in Theorem 4.10 and Corollary 4.8, there exists a countable family $\{\psi_{w_n, \lambda_n}\}_{n \geq 1}$ of upper triangular representations of $\mathfrak{L}_G$ which separates the points in $\mathfrak{L}_G$. These paths w_n do not involve the edges f_x. A moments reflection shows that the scalars $\lambda_{n,i} \in \mathbb{T}$ used in these representations may be chosen to be distinct.

Choose an infinite path in G of the form $w = e_{j_1} e_{j_2} e_{j_3} \ldots$ such that $x_0 = r(e_{j_1})$ and $x_k = r(e_{j_k}) = s(e_{j_{k-1}})$ for $k \geq 1$. We require that no edge of the form f_x is used, and that each path w_n should occur separately somewhere within the path w. Assign distinct scalars $\lambda_k \in \mathbb{T}$ for $k \geq 0$ such that on each segment corresponding to one of the words w_n, we use the scalars $\lambda_{n,i}$ selected above for the separating family of representations. Write $\lambda = (\lambda_k)_{k \geq 0} \in \mathbb{T}^{\mathbb{N}_0}$.

Define a representation $\psi_{w,\lambda}$ of $\mathbb{F}^+(G)$ into $\mathcal{T}(\mathcal{N})$ as follows:

$$\psi_{w,\lambda}(x) = \sum_{x_k = x} h_k h_k^* \qquad \text{for } x \in \mathcal{V}(G)$$

$$\psi_{w,\lambda}(f_x) = \frac{1}{2} \sum_{x_k = x} \lambda_k h_k h_k^* \quad \text{for } x \in \mathcal{V}(G)$$

$$\psi_{w,\lambda}(e) = \frac{1}{2} \sum_{e_{j_k} = e} h_{k-1} h_k^* \quad \text{for } e \in \mathcal{E}(G) \setminus \mathcal{L}$$

Arguing as in Theorem 4.10, the row operator

$$E = \big[\psi_{w,\lambda}(e)\big]_{e \in \mathcal{E}(G)}$$

satisfies $EE^* \leq \frac{1}{2}I$. So by Corollary 3.2, this representation extends to a WOT-continuous completely contractive representation of $\mathfrak{L}_G$.

Observe that the image of each vertex is diagonal and the image of each edge is upper triangular, so the range of this representation is contained in $\mathcal{T}(\mathcal{N})$. The algebra generated by $\{\psi_{w,\lambda}(f_x) : x \in \mathcal{V}(G)\}$ contains a family of diagonal operators with distinct non-zero entries in each coordinate. Consequently, the WOT-closed algebra that they generate is the full diagonal algebra $\mathcal{D}$. This together with the rank one partial isometries

$$h_{k-1}h_k^* = \psi_{w,\lambda}(e_{j_k})$$

generates $\mathcal{T}(\mathcal{N})$ as a WOT-closed algebra. Therefore this is a nest representation.

Finally observe that for each n, there is an interval of the nest corresponding to the representation ψ_{w_n,λ_n}. The compression to this interval is equivalent to it. As the direct sum of these representations is faithful, it follows that $\psi_{w,\lambda}$ is also faithful.

It remains to deal with case (3). There is a finite initial segment of the graph with vertices $x_1, \ldots, x_n$ and edges $e_1, \ldots, e_n$ where $r(e_i) = x_i$ and $s(e_i) = x_{i+1}$ for $1 \leq i < n$, $r(e_n) = x_n$ and $s(e_n) = y_0$ belongs to the strongly transitive component. Let G_0 be the graph G restricted to $\mathcal{V}_0$.

There is a faithful upper triangular nest representation of $\mathfrak{L}_{G_0}$ determined by the previous paragraphs. We will need to specify some special features of this representation. Let $\mathcal{L} = \{f_x : x \in \mathcal{V}(G_0)\}$ be a choice of a loop edge at each vertex of G_0. For each word v_s built from edges in $\mathcal{E}(G_0) \setminus \mathcal{L}$, choose a countable family of points $\lambda_{v_s,k} = (\lambda_{v_s,k,i})_{i=0}^{|v_s|}$ dense in $\mathbb{T}^{|v_s|+1}$. We may also ensure that all $\lambda_{v_s,k,i}$ are distinct. For later use, we split the collection of all these pairs $(v_s, \lambda_{v_s,k})$ into countably many families U_j with the same properties.

Choose an infinite word w with $r(w) = y_0$ which contains each word $v_s \in \mathbb{F}^+(G_0) \setminus \mathcal{L}$ infinitely often, and denote the kth occurrence by $v_{s,k}$. Also choose $\lambda = (\lambda_j)_{j \geq 1} \in \mathbb{T}^{\mathbb{N}}$ so that the coordinates λ_j are distinct, and coincide with $\lambda_{v_s,k,i}$ at the occurrence of the word $v_{s,k}$ in w. Denote the pairs $v_{s,k}$ as u_i in order of occurrence. We may assume that they are sufficiently widely spaced that there are more than $4i$ edges between u_i and u_{i-1}. We obtain an upper triangular nest representation $\psi_{w,\lambda}$ as above. Denote the basis on which this acts as $\{h_{n+j} : j \geq 1\}$.

The plan is to extend this representation to all of G. Part of the extension is natural. Set

$$\psi(p) = \psi_{w,\lambda}(p) \quad \text{for } p \in \mathbb{F}(G_0)$$
$$\psi(x_i) = h_i h_i^* \quad \text{for } 1 \le i \le n$$
$$\psi(e_i) = \frac{1}{2} h_i h_{i+1}^* \quad \text{for } 1 \le i \le n-1$$

We also need to define ψ on edges in G with source in $\mathcal{V}_0$ and range in the initial segment including e_n. Let $\{g_j : j \in J\}$, $J \subset \mathbb{N}_0$, be an enumeration of these edges including $g_0 = e_n$. Let $r(g_j) = x_{i_j}$ and $s(g_j) = y_j \in \mathcal{V}_0$, We will define vectors $k_j = \psi(y_j)k_j$ and set

$$\psi(g_j) = h_{i_j} k_j^* \quad \text{for} \quad j \in J.$$

We need to define the vectors k_j in such a way that the representation is faithful. At this point, we specify only that $\alpha := \langle k_0, h_{n+1} \rangle \ne 0$.

Before dealing with that issue, let us observe that whatever the choice of vectors k_j, this representation will be a nest representation. Indeed, the compression to $\operatorname{span}\{h_{n+k} : k \ge 1\}$ is a faithful map of $\mathfrak{L}_{G_0}$ into a WOT-dense subalgebra of the upper triangular nest algebra. The restriction to $\operatorname{span}\{h_1 : 1 \le i \le n\}$ is a faithful map of the interval onto $\mathcal{T}_n$. Moreover $\rho(e_n)h_{n+1}h_{n+1}^* = \frac{\alpha}{2} h_n h_{n+1}^*$ provides an element in the range which now generates the full nest algebra. The addition of more edges only adds to this already dense subalgebra.

For each $j \in J$, set

$$V_j = \{i \ge 1 : u_i \in U_j, \ r(u_i) = y_j\}.$$

The vertex $r(u_i)$ is identified with a basis vector h_{r_i} in the representation ψ, and $s(u_i)$ is identified with h_{s_i} where $s_i = r_i + |u_i| - 1$. Define

$$k_0 = \frac{1}{2} h_{n+1} + \sum_{i \in V_0} 2^{-i-1} h_{r_i}$$

and

$$k_j = \sum_{i \in V_j} 2^{-i-1} h_{r_i} \quad \text{for} \quad j \in J \setminus \{0\}.$$

Observe that $k_j = \psi(y_j)k_j$ for each $j \in J$.

Again we can verify that the row operator $E = \big[\psi(e)\big]_{e \in \mathcal{E}(G)}$ satisfies $EE^* \le \frac{1}{2}I$. Therefore by Corollary 3.2, ψ extends to a completely contractive WOT-continuous representation of $\mathfrak{L}_G$.

The last step is to verify that ψ is faithful. We have already shown that it is faithful on $\mathfrak{L}_{G_0}$ and on the initial segment. So it suffices to consider elements with domain in G_0 and range in the tail. This is the ideal $\mathcal{J}$ generated by the $\{g_j : j \in J\}$.

To this end, suppose that $A \in \mathcal{J}$ satisfies $\psi(A) = 0$. This is equivalent to $h_i h_i^* \rho(A) = \rho(P_{x_i} A) = 0$ for $1 \leq i \leq n$. So it suffices to consider $A = \sum_{j \in J} q_j g_j A_j$ where q_j is the unique path in G from $r(g_j)$ to x_i (if $i_j \geq i$, else it is 0), and $A_j = P_{y_j} A_j$ lies in $P_{y_j} \mathcal{L}_{G_0}$. The range of $\psi(A)$ is contained in $\mathbb{C} h_i$. So $\psi(A) = 0$ if and only if

$$0 = \psi(A)^* h_i = \sum_{j \in J} \psi(A_j)^* k_j.$$

We can write the Fourier series of each term as $A_j \sim \sum_{r(v)=y_j} a_{v,j} L_v$. Also note that in $\mathcal{L}_G$, we have $\|A\|^2 = \|\sum_{j \in J} A_j^* A_j\|$. As in the previous proofs, we will recover the Fourier coefficients of each A_j, and in that manner, verify that ψ is faithful. Following the proof of Theorem 4.10, it suffices to show that the functions

$$f_{v_s,\lambda}(A_j) = \langle \psi_{v_s,\lambda}(A_j) h_1, h_k \rangle$$

can be recovered from $\psi(A)$ for all $j \in J$.

Fix $j_0 \in J$. Since $A_{j_0} = P_{y_{j_0}} A_{j_0}$, $f_{v_s,\lambda}(A_{j_0}) = 0$ if $r(v_s) \neq y_{j_0}$. If $r(v_s) = y_{j_0}$ and $\lambda \in \mathbb{T}^{|v|+1}$ is given, then there is a sequence $i_n \in V_{j_0}$ so that $u_{i_n} = v_{s,k}$ and $\lim_{n \to \infty} \lambda_{v_s,k} = \lambda$. The result will follow if we establish that

$$\overline{f_{v_s,\lambda}(A_{j_0})} = \lim_{n \to \infty} 2^{i_n} \langle \psi(A)^* h_i, h_{s_{i_n}} \rangle.$$

We have observed that $\psi(A)^* h_i = \sum_{j \in J} \psi(A_j)^* k_j$. So

$$\begin{aligned}
2^{i_n} \langle \psi(A)^* h_i, h_{s_{i_n}} \rangle &= 2^{i_n} \sum_{j \in J} \langle \psi(A_j)^* k_j, h_{s_{i_n}} \rangle \\
&= 2^{i_n} \sum_{j \in J} \langle k_j, \psi(A_j) h_{s_{i_n}} \rangle \\
&= \langle h_{r_{i_n}}, \psi(A_j) h_{s_{i_n}} \rangle + \sum_{j \in J} \sum_{\substack{1 \leq i < i_n \\ i \in V_j}} 2^{i_n - i} \langle h_{r_i}, \psi(A_j) h_{s_{i_n}} \rangle
\end{aligned}$$

The first term has the desired limit. So it remains to show that the second sum tends to zero. The key is an estimate of $|\langle h_{r_i}, \psi(A_j) h_{s_{i_n}} \rangle|$.

Observe that we can factor the representation $\psi_{w,\lambda}$ of $\mathcal{L}_{G_0}$ as $\sigma \rho$ where ρ replaces the maps

$$\psi_{w,\lambda}(e) = \frac{1}{2} \sum_{e_{j_k}=e} h_{k-1} h_k^* \quad \text{for} \quad e \in \mathcal{E}(G_0) \setminus \mathcal{L}$$

with

$$\rho(e) = \frac{1}{\sqrt{2}} \sum_{e_{j_k}=e} h_{k-1} h_k^* \quad \text{for} \quad e \in \mathcal{E}(G_0) \setminus \mathcal{L}.$$

and leaves

$$\rho(x) = \psi_{w,\lambda}(x) \quad \text{and} \quad \rho(f_x) = \psi_{w,\lambda}(f_x) \quad \text{for} \quad x \in \mathcal{V}(G_0).$$

Then we define σ to be the endomorphism of the nest algebra into itself by multiplying the nth diagonal entries by $2^{-n/2}$. Then ρ is completely contractive by Corollary 3.2.

We can now estimate $|\langle h_{r_i}, \psi(A_j)h_{s_{i_n}}\rangle|$ by observing that this is a matrix entry of $\psi_{w,\lambda}(A_j)$ lying on the $s_{i_n}-r_i$ diagonal, and it is therefore bounded by $\|A_j\|2^{(r_i-s_{i_n})/2}$. Since $\|A_j\| \leq \|A\|$ and $r_{i-1} - r_i \geq 4i$,

$$\sum_{j \in J} \sum_{\substack{1 \leq i < i_n \\ i \in V_j}} 2^{i_n-i}|\langle h_{r_i}, \psi(A_j)h_{s_{i_n}}\rangle| \leq \|A\| \sum_{i=1}^{i_n-1} 2^{i_n-i}2^{-\sum_{i<k\leq i_n} 2i}$$

$$\leq 2^{-i_n}\|A\|.$$

The desired limit is now evident. ∎

Remark 5.12. It is clear that taking the transpose of the graphs yields a characterization of which nests have a faithful representation onto the lower triangular nest algebra. Moreover it is not too difficult to formulate the corresponding result for maximal nests of order type $\mathbb{Z}$. In particular, a strongly transitive graph with a loop edge at every vertex has such a representation.

As we can see, the representations constructed in this section are not ∗-extendible. This is not a limitation of our construction when the nest is non-trivial. Indeed if $\mathfrak{A}_k$ is the norm closed algebra of the graph P_k with one vertex and k loop edges, the non-commutative disk algebra of order k, we have the following result.

Proposition 5.13. *Let $\mathcal{E}_k$ be the C*-algebra generated by $\lambda(\mathfrak{A}_k)$, $k \geq 2$, and let ϕ be a *-representation of $\mathcal{E}_k$ in $\mathcal{B}(\mathcal{H})$ that maps $\mathfrak{A}_k$ faithfully into a nest algebra $\mathrm{Alg}\,\mathcal{N}$, with $\mathcal{N} \neq \{0, I\}$. Then $\phi(\mathfrak{A}_k)$ is not dense in $\mathrm{Alg}\,\mathcal{N}$.*

Proof. Note that ϕ maps the creation operators onto isometries with orthogonal ranges. Therefore, the weakly closed algebra $\mathfrak{S} = \overline{\phi(\mathfrak{A}_k)}$ generated by $\phi(\mathfrak{A}_k)$ is a free semigroup algebra. Hence, [**7**, Corollary 2.9] shows that the Jacobson radical of $\mathfrak{S}$ consists of nilpotent operators of order at most two.

The only non-trivial nests with this property have exactly three elements, $\mathcal{N} = \{0, N, \mathcal{H}\}$. From the Structure Theorem for free semigroup algebras [**7**], this would imply that $P(N)^\perp$ is the structure projection P. But $P(N)$ is another projection with the property $P(N)\mathcal{T}(\mathcal{N})P(N)$

is self-adjoint. For free semigroup algebras, P is the unique maximal projection with this property. So a free semigroup algebra cannot be a nest algebra other than $\mathcal{B}(\mathcal{H})$.   ∎

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Pure Math. Dept., U. Waterloo, Waterloo, ON N2L–3G1, CANADA
E-mail address: krdavids@uwaterloo.ca

Dept. Math., East Carolina University, Greenville, NC 27858, USA
E-mail address: KatsoulisE@mail.ecu.edu