DILATING COVARIANT REPRESENTATIONS OF THE NON-COMMUTATIVE DISC ALGEBRAS

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Abstract. Let φ be an isometric automorphism of the non-commutative disc algebra $\mathfrak{A}_n$ for $n \geq 2$. We show that every contractive covariant representation of $(\mathfrak{A}_n, \varphi)$ dilates to a unitary covariant representation of $(\mathcal{O}_n, \varphi)$. Hence the C*-envelope of the semicrossed product $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$ is $\mathcal{O}_n \times_\varphi \mathbb{Z}$.

1. Introduction

In this paper, we continue our study of the crossed product $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$ of the non-commutative disk algebra $\mathfrak{A}_n$ by an isometric automorphism φ. These semicrossed products were introduced in **[7]** as universal algebras for the contractive covariant representations of $(\mathfrak{A}_n, \varphi)$, where we showed there that the isomorphic class of $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$ is determined by the analytic conjugacy class of φ. Recall that the isometric automorphisms of $\mathfrak{A}_n$ come from the natural action of the group $\mathrm{Aut}(\mathbb{B}_n)$ of conformal automorphisms of the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ on the character space $\overline{\mathbb{B}}_n$ of $\mathfrak{A}_n$.

The universality of $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$ allows for a rich representation theory and this was a key component for classifying these algebras. On the other hand, it is the universality of $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$ that raises the problem of finding concrete faithful representations. This is the main theme of this paper. As we shall see, the C*-envelope of $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$ is $\mathcal{O}_n \times_\varphi \mathbb{Z}$, where $\mathcal{O}_n$ denotes the Cuntz algebra, with generators going to generators. Using the theory of gauge invariant uniqueness for C*-crossed product C*-algebras, we obtain a concrete faithful representation for $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$.

The proof of this fact relies on a dilation theorem. We show first that every completely contractive representation of $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$ dilates to a unitary system in which the image of the row of generators $[L_1 \ \dots \ L_n]$

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of $\mathfrak{A}_n$ is a row isometry and the intertwining operator implementing the automorphism is unitary. This is not sufficient for our purposes, because the C*-algebra generated by a row isometry is either $\mathcal{O}_n$ or the Cuntz-Toeplitz algebra $\mathcal{E}_n$. We need to further dilate the latter representations to unitary dilations of Cuntz type. We thereby show that these are the only maximal representations of the semicrossed product. So using the Dritschel–McCullough approach **[13]** to the C*-envelope, we are able to obtain the desired conclusion.

Using a result of Kishimoto **[19]**, we show that in the case where φ is aperiodic, the C*-algebra $\mathcal{O}_n \times_\varphi \mathbb{Z}$ is simple. In this case, the quotient $\mathrm{C}^*(\mathcal{E}_n, U_\varphi)/\mathfrak{K}$, where U_φ is the Voiculescu unitary implementing φ on $\mathcal{E}_n$, is $*$-isomorphic to $\mathcal{O}_n \times_\varphi \mathbb{Z}$.

There is an extensive body of work studying dynamical systems via an associated operator algebras going back to work of von Neumann. The use of nonself-adjoint operator algebras in this area begins with seminal work of Arveson **[1]** and Arveson–Josephson **[4]**. This was put into the abstract setting of semi-crossed products by Peters **[24]**. See **[6]** for an overview of some of the recent work in this area.

There is also a large literature on dilation theory for various nonself-adjoint operator algebras going back to seminal work of Sz.Nagy. Arveson **[2]** established dilation theory as an essential tool for studying nonself-adjoint operator algebras. Work on abstract semicrossed products began with work of Peters **[24]**. He concentrated on the action of an endomorphism on a C*-algebra, and here the theory works well. Specifically one can define a family of natural orbit representations and show that these produce a faithful (completely isometric) representation of the semicrossed product. This can be used to find explicit information about the C*-envelope. See Peters **[25]** for the one variable case and **[12]** for the C*-envelope a multivariable dynamical system.

Muhly and Solel developed an extensive theory of certain nonself-adjoint operator algebras called tensor algebras of a C*-correspondences **[20, 21, 22]**. They showed, under certain hypotheses, that the C*-envelope of the tensor algebra is the Cuntz–Pimsner C*-algebra built from the correspondence. This result was extended by Fowler, Muhly and Raeburn **[16]** to the case when the left action is faithful and strict. Finally the second author and Kribs **[18]** removed those restrictions.

The semicrossed product of $\mathfrak{A}_n$ has a lot in common with these tensor algebras. However, since the semicrossed product is defined as the universal operator algebra for a family of covariant representations, one needs to prove a dilation theorem in order to decide whether or not this algebra sits inside a Cuntz–Pimsner algebra completely isometrically. This is, in fact, where all of the difficulty lies.

It is perhaps worth mentioning that classical counterexamples in dilation theory point to the difficulties that might arise in general. The bidisk algebra $A(\mathbb{D}^2)$ sits inside $\mathrm{C}(\mathbb{T}^2)$, which is its C*-envelope by Ando's theorem. Consider the identity automorphism id. Ando's theorem also shows that the completely contractive representations of $A(\mathbb{D}^2)$ are determined by an arbitrary pair T_1, T_2 of commuting contractions. A covariant representation of $(A(\mathbb{D}^2), \mathrm{id})$ is given by such a pair and a third contraction T_3 which commutes with T_1 and T_2. If it were true that the C*-envelope of this system was $\mathrm{C}(\mathbb{T}^2) \times_{\mathrm{id}} \mathbb{Z} \simeq \mathrm{C}(\mathbb{T}^3)$, then it would be true that every commuting triple of contractions has a unitary dilation. This was disproven in a famous paper by Varopolous. See Paulsen's book [**23**, Chapter 5] for a treatment of these topics. Thus when such unitary dilations are possible, we must see this as an important but special phenomenon.

2. Preliminaries

Consider the left regular representation λ of the free semigroup $\mathbb{F}_n^+$ acting on Fock space, $\ell^2(\mathbb{F}_n^+)$. Let $L_i = \lambda(i)$. The non-commutative disc algebra $\mathfrak{A}_n$, for $n \geq 2$, is the nonself-adjoint unital operator algebra generated by $L_1, \dots, L_n$. It sits as a subalgebra of the Cuntz–Toeplitz C*-algebra $\mathcal{E}_n = \mathrm{C}^*(\{L_1, \dots, L_n\})$. However the quotient map onto the Cuntz algebra $\mathcal{O}_n$ is completely isometric on $\mathfrak{A}_n$. So $\mathfrak{A}_n$ may be considered as the subalgebra of $\mathcal{O}_n$ generated by the standard generators $\mathfrak{s}_1, \dots, \mathfrak{s}_n$. Moreover the operator algebra generated by any n-tuple of isometries $S_1, \dots, S_n$ with pairwise orthogonal ranges is completely isometrically isomorphic to $\mathfrak{A}_n$. These algebras were introduced by Popesu [**27**] as a natural multivariable generalization of disc algebra $A(\mathbb{D})$. The Frahzo–Bunce dilation Theorem [**14, 5, 26**] shows that any row contractive n-tuple $T = [T_1, \dots, T_n]$ dilates to an n-tuple of isometries $S = [S_1, \dots, S_n]$ with pairwise orthogonal range. Hence given any such n-tuple T, there is a unique completely contractive homomorphism of $\mathfrak{A}_n$ onto the algebra $\mathcal{A}(T_1, \dots, T_n)$ taking generators to generators. Popescu [**27**] used this to establish a natural analogue of the von Neumann inequality for row contractive n-tuples.

If φ is an automorphism for an operator algebra $\mathcal{A}$, then a contractive (resp. isometric or unitary) covariant representation for $(\mathcal{A}, \varphi)$ consists of a completely contractive (resp. completely isometric) representation $\pi : \mathcal{A} \to B(\mathcal{H})$ and a contraction (resp. isometry or unitary) $U \in B(\mathcal{H})$ so that $\pi(A)U = U\pi(\varphi(A))$. If $\mathcal{A}$ happens to be a C*-algebra, then completely contractive maps are $*$-homomorphisms.

Each element $A \in \mathfrak{A}_n$ determines a function $\hat{A}$ on the character space, and this is a bounded holomorphic function on $\mathbb{B}_n$ which extends to a continuous function on $\overline{\mathbb{B}}_n$. An automorphism φ of $\mathfrak{A}_n$ induce an automorphism $\hat{\varphi}$ of the character space $\mathbb{B}_n$ via $\hat{A}(\hat{\varphi}(z)) = \widehat{\varphi(A)}(z)$. The map $\hat{\varphi}$ is biholomorphic, and thus is a conformal automorphism [**8**]. These maps are given by fractional linear transformations (c.f. Rudin [**30**]).

Each of these conformal maps is induced by a unitarily implemented automorphism of $\mathcal{E}_n$ which fixes the subalgebra $\mathfrak{A}_n$. Indeed, Voiculescu [**32**] constructs a unitary representation of the Lie group $U(n,1)$ which contains the scalar unitaries, and $U(n,1)/\mathbb{T} \simeq \mathrm{Aut}(\mathbb{B}_n)$, such that $\mathrm{ad}\, U$ implements the corresponding automorphism. In [**9**], the first author and Pitts study the automorphism of the weak operator closed algebra $\mathfrak{L}_n = \overline{\mathfrak{A}_n}^{\mathrm{WOT}}$. The case of $\mathfrak{A}_n$ is similar but more elementary. See Popescu [**28**] for another proof.

Definition 2.1. Let $\mathrm{Aut}(\mathfrak{A}_n)$ denote the group of completely isometric automorphisms of $\mathfrak{A}_n$, and let $\varphi \in \mathrm{Aut}(\mathfrak{A}_n)$. A *covariant representation* (π, K) of $(\mathfrak{A}_n, \varphi)$ is a completely contractive representation π of $\mathfrak{A}_n$ on a Hilbert space $\mathcal{H}$ and a contraction $K \in \mathcal{B}(\mathcal{H})$ so that

$$\pi(A)K = K\pi(\varphi(A)) \quad \text{for all} \quad A \in \mathfrak{A}_n.$$

The *semicrossed product* $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$ is the universal operator algebra generated by a copy of $\mathfrak{A}_n$ and a contraction $\mathfrak{u}$ so that $A\mathfrak{u} = \mathfrak{u}\varphi(A)$ for all $A \in \mathfrak{A}_n$.

In other words, $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$ is the operator algebra generated by a (completely isometric) copy of $\mathfrak{A}_n$ and a contraction $\mathfrak{u}$ with the property that every covariant representation of $(\mathfrak{A}_n, \varphi)$ induces a completely contractive representation $\pi \times K$ of $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$ on $\mathcal{H}$, which on polynomials is defined as

$$(\pi \times K)\big(\sum \mathfrak{u}^n A_n\big) = \sum K^n \pi(A_n).$$

The norm may be defined by as the supremum over all covariant representations:

$$\big\| \sum \mathfrak{u}^n A_n \big\| = \sup_{(\pi,K)} \big\| (\pi \times K)\big(\sum \mathfrak{u}^n A_n\big) \big\|.$$

A completely contractive representation of $\mathfrak{A}_n$ sends the generators $L = [L_1 \ \dots \ L_n]$ to a row contraction $A = [A_1 \ \dots \ A_n]$. Conversely, the Frahzo–Bunce dilation theorem [**14, 5**] shows that any row contraction dilates to a row isometry. Thus by Popescu's von Neumann inequality [**27**], there is a completely contractive representation π of $\mathfrak{A}_n$ with

$\pi(L_i) = A_i$ for $1 \le i \le n$. If we take $\pi = \mathrm{id}_{\mathfrak{A}_n}$ and $U = 0$, we see that the imbedding of $\mathfrak{A}_n$ into $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$ is completely isometric.

In [**7**], we identified several such representations which are worth repeating.

Example 2.2. For any $\varphi \in \mathrm{Aut}(\mathfrak{A}_n)$, consider the unitary U_φ constructed by Voiculescu [**32**] on the Fock space $\ell^2(\mathbb{F}_n^+)$ which implements the action of φ on the Cuntz–Toeplitz C*-algebra $\mathcal{E}_n$ by $U_\varphi^* A U_\varphi = \varphi(A)$, and fixes $\mathfrak{A}_n$. For any $\lambda \in \mathbb{T}$, this provides a covariant pair $(\mathrm{id}_{\mathcal{E}_n}, \lambda U_\varphi)$ for $(\mathcal{E}_n, \varphi)$. Therefore it produces a representation $\mathrm{id}_{\mathcal{E}_n} \times \lambda U_\varphi$ of $\mathcal{E}_n \times_\varphi \mathbb{Z}$. Since $\mathfrak{A}_n$ is invariant for $\mathrm{ad}\, U_\varphi$, this provides a covariant representation $(\mathrm{id}, \lambda U_\varphi)$ of $(\mathfrak{A}_n, \varphi)$ by restriction. This yields a representation $\mathrm{id} \times \lambda \hat{U}_\varphi$ of $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$ which is completely isometric on $\mathfrak{A}_n$ and λU_φ is unitary.

Similarly, by taking a quotient by $\mathcal{K}(\ell^2(\mathbb{F}_n^+))$, the compact operators on $\ell^2(\mathbb{F}_n^+)$, we obtain a covariant representation $(\pi_{\mathcal{O}_n}, \lambda \hat{U}_\varphi)$ for $(\mathcal{O}_n, \varphi)$ and therefore representations for both $\mathcal{O}_n \times_\varphi \mathbb{Z}$ and $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$, inside the Calkin algebra, which we denote as $\pi_{\mathcal{O}_n} \times \lambda \hat{U}_\varphi$.

Example 2.3. Let π be any completely contractive representation of $\mathfrak{A}_n$ on a Hilbert space $\mathcal{H}$. Define $\tilde{\pi}$ on $\mathcal{H} \otimes \ell^2$ by

$$\tilde{\pi}(a) = \sum_{k \ge 0}{}^{\oplus} \pi\varphi^k(a) \quad \text{and} \quad U = I_{\mathcal{H}} \otimes S$$

where S is the unilateral shift. This is easily seen to yield a completely contractive representation $\tilde{\pi}$ of $\mathfrak{A}_n$ and a contraction U so that $\tilde{\pi} \times_\varphi U$ yields a representation of $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$.

These are called *orbit representations*. When $\mathcal{A}$ is a C*-algebra, Peters [**24**] showed that the direct sum of all orbit representations $\tilde{\pi} \times_\varphi U$, as π runs over the $*$-representations of $\mathcal{A}$, yields a completely isometric representation of $\mathcal{A} \times_\varphi \mathbb{Z}^+$. For general operator algebras, this is not the case.

Example 2.4. Let $\varphi \in \mathrm{Aut}\, \mathfrak{A}_n$. Consider the non-commutative disc algebra $\mathfrak{A}_{n+1}$ acting on the Fock space $\ell^2(\mathbb{F}_{n+1}^+)$ and define an ideal

$$\mathfrak{J} = \big\langle L_i L_{n+1} - L_{n+1}\varphi(L_i) : 1 \le i \le n \big\rangle.$$

The WOT-closure $\overline{\mathfrak{J}}$ of $\mathfrak{J}$ is an ideal of $\mathfrak{L}_{n+1}$, and these ideals were studied in [**9, 10**]. In particular, it is shown in [**9**] that $\overline{\mathfrak{J}}$ is determined by its range, which is a subspace invariant for both $\mathfrak{L}_{n+1}$ and its commutant $\mathfrak{R}_{n+1}$. Then in [**10**], it is shown that $\mathfrak{L}_{n+1}/\overline{\mathfrak{J}}$ is completely isometrically isomorphic to the compression to $\mathcal{M}_\varphi = \mathrm{Ran}(\mathfrak{J})^\perp$. Since

$$\overline{\mathrm{Ran}\, \mathfrak{J}} = \mathrm{span}\big\{ A(L_i L_{n+1} - L_{n+1}\varphi(L_i))\ell^2(\mathbb{F}_{n+1}^+) : i = 1, 2, \dots, n \big\}$$

is evidently orthogonal to $\xi_\varnothing$, we see that $\mathcal{M}_\varphi$ is non-empty. The compression of $\mathfrak{A}_n$ to $\mathcal{M}_\varphi$ is a completely contractive homomorphism ρ, and the compression B of L_{n+1} is also a contraction. Therefore (ρ, B) is a covariant representation of $(\mathfrak{A}_n, \varphi)$, and thus determines a completely contractive representation $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$.

Example 2.5. Any representation of $\mathfrak{A}_n$ produces a representation of $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$ by simply taking $U = 0$. In [**7**], we contructed various finite dimensional representations of $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$ which allowed us to classify them as algebras.

3. Unitary Covariant Representations

The purpose of this section is to show that contractive representation of our covariant system always dilate to a unitary covariant system. The proof requires a number of known dilation theorems.

We call a representation π of an algebra $\mathcal{A}$ on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ an *extension* of a representation σ of $\mathcal{A}$ on $\mathcal{H}$ if $\mathcal{H}$ is invariant for $\pi(\mathcal{A})$ and $\pi(A)|_\mathcal{H} = \sigma(A)$ for $A \in \mathcal{A}$; i.e., $\pi(A) \simeq \begin{bmatrix} \sigma(A) & * \\ 0 & * \end{bmatrix}$. Likewise, π is a *co-extension* of σ if $\mathcal{H}$ is co-invariant for $\pi(\mathcal{A})$ and $P_\mathcal{H}\pi(A)|_\mathcal{H} = \sigma(A)$ for $A \in \mathcal{A}$; i.e., $\pi(A) \simeq \begin{bmatrix} * & * \\ 0 & \sigma(A) \end{bmatrix}$. Finally, we say that π is a *dilation* of σ if $P_\mathcal{H}\pi(A)|_\mathcal{H} = \sigma(A)$ for $A \in \mathcal{A}$. By a result of Sarason [**31**], $\mathcal{H}$ is semi-invariant and so $\pi(A) \simeq \begin{bmatrix} * & * & * \\ 0 & \sigma(A) & * \\ 0 & 0 & * \end{bmatrix}$.

The main result of this section will be established by a sequence of lemmas.

Theorem 3.1. *Let φ be an isometric automorphism of the non-commutative disc algebra $\mathfrak{A}_n$, $n \geq 2$. Then, any contractive covariant representation of $(\mathfrak{A}_n, \varphi)$ dilates to a unitary covariant representation of $(\mathcal{E}_n, \varphi)$, where $\mathcal{E}_n$ denotes the Cuntz-Toeplitz* C^**-algebra.*

According to the commutant lifting Theorem of Frazho [**15**] and Popescu [**26**], if S, T are row contractions and K another contraction intertwining them, i.e., $S^{(i)}K = KT^{(i)}$ for all $1 \leq i \leq n$, then K co-extends to a contraction K' that intertwines the minimal isometric dilations V_S and V_T of S and T respectively. A dual result can be obtained from a recent dilation of Solel [**29**]. (See also [**11**].) Solel's result says that if S and T are as above, then we can co-extend the contractions S, T and K to isometries W_S, W_T and W_K, which still satisfy $W_S^{(i)}W_K = W_KW_T^{(i)}$. This leads to the following.

Lemma 3.2. *Assume that* $S = [S^{(1)}, \ldots, S^{(n)}]$*,* $T = [T^{(1)}, \ldots, T^{(n)}]$ *are row contractions and* K *is a contraction on* $\mathcal{H}$ *so that*

$$S^{(i)}K = KT^{(i)}, \quad 1 \leq i \leq n.$$

Let V *be an isometric dilation of* K*, acting on a Hilbert space* $\mathcal{H}'$*. Then there exist row contractions* $S' = [S'^{(1)}, \ldots, S'^{(n)}]$*,* $T' = [T'^{(1)}, \ldots, T'^{(n)}]$ *on* $\mathcal{H}'$*, which co-extend* S *and* T *respectively, and satisfy*

$$S'^{(i)}V = VT'^{(i)}, \quad 1 \leq i \leq n.$$

Proof. Let V_K be the minimal isometric dilation of K. Then we can decompose $V \simeq V_K \oplus V'$. So if we can dilate S and T to S' and T' intertwining V_K, then we can extend further to all of $\mathcal{H}'$ by adding zero summands to S' and T'.

By Solel's result, S, T and K co-extend to intertwining isometries W_S, W_T and W_K, acting on a Hilbert space $\mathcal{H}''$. Let $\mathcal{H}' = \bigvee_{j \geq 0} W_K^j \mathcal{H}$ be the smallest invariant subspace of W_K containing $\mathcal{H}$. Clearly, $\mathcal{H}'$ is reducing for W_K and the restriction of W_K on $\mathcal{H}'$ is (unitarily equivalent to) the minimal dilation V_K. The result now follows by setting

$$S'^{(i)} = P_{\mathcal{H}'} S^{(i)} |_{\mathcal{H}'} \quad \text{and} \quad T'^{(i)} = P_{\mathcal{H}'} T^{(i)} |_{\mathcal{H}'} \quad \text{for} \quad 1 \leq i \leq n. \qquad \blacksquare$$

Lemma 3.3. *Let* $\varphi \in \mathrm{Aut}(\mathfrak{A}_n)$ *and let* $A = [A^{(1)}, \ldots A^{(n)}]$ *and* K *be contractions satisfying the covariance relations* $A^{(i)}K = K\varphi(A)^{(i)}$ *for* $1 \leq i \leq n$*. Then there exist isometries* $T_A = [T_A^{(1)} \ \ldots \ T_A^{(n)}]$ *and* T_K*, dilating* A *and* K *respectively, so that*

$$T_A^{(i)} T_K = T_K \varphi(T_A)^{(i)} \quad \textit{for} \quad 1 \leq i \leq n.$$

Proof. Notice that if V_A is the minimal isometric dilation of A, then $\varphi(V_A)$ is the minimal isometric dilation of $\varphi(A) = [\varphi(A)^{(1)} \ \ldots \ \varphi(A)^{(n)}]$. Therefore, by applying commutant lifting to the covariance relations, we obtain a contraction K_1 on a Hilbert space $\mathcal{H}_1$, satisfying

$$V_A^{(i)} K_1 = K_1 \varphi(V_A)^{(i)}$$

Let S_{K_1} be the Schaeffer dilation of K_1 on $\mathcal{H}_1^{(\infty)}$ by

$$S_{K_1} \simeq \begin{bmatrix} K_1 & 0 & 0 & 0 & \ldots \\ D_{K_1} & 0 & 0 & 0 & \ldots \\ 0 & I & 0 & 0 & \ldots \\ 0 & 0 & I & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where $D_{K_1} = (I - K_1^* K_1)^{1/2}$. We apply Lemma 3.2, with $S = V_A$, $T = \varphi(V_A)$, $K = K_1$ and its isometric dilation S_{K_1} to obtain row contractions $\hat{A}_1$ and $\hat{B}_1$, which co-extend V_A and $\varphi(V_A)$, and satisfy

$$\hat{A}_1^{(i)} S_{K_1} = S_{K_1} \hat{B}_1^{(i)} \quad \text{for} \quad 1 \le i \le n.$$

Because $V_a^{(i)}$ are already isometries, these dilations have the form

$$\hat{A}_1^{(i)} = \begin{bmatrix} V_A^{(i)} & 0 \\ 0 & [X_{jk}^{(i)}]_{j,k\ge 1} \end{bmatrix} \quad \text{and} \quad \hat{B}_1^{(i)} = \begin{bmatrix} \varphi(V_A^{(i)}) & 0 \\ 0 & [Y_{jk}^{(i)}]_{j,k\ge 1} \end{bmatrix}$$

By comparing $(2,1)$-entries in the covariance relation, we obtain

$$X_{11}^{(i)} D_{K_1} = D_{K_1} \varphi(V_A^{(i)}) \quad \text{for} \quad 1 \le i \le n.$$

For simplicity, write $X_i = X_{11}^{(i)}$. Note that $X = [X_1 \ \dots, \ X_n]$ is a row contraction, and so $\varphi(X)$ is meaningful. For $1 \le i \le n$, we now define

$$A_1^{(i)} = \begin{bmatrix} V_A^{(i)} & 0 & 0 & 0 & \dots \\ 0 & X_i & 0 & 0 & \dots \\ 0 & 0 & \varphi(X)^{(i)} & 0 & \dots \\ 0 & 0 & 0 & (\varphi\circ\varphi)(X)^{(i)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We obtain a row contraction $A_1 = [A_1^{(1)} \ \dots \ A_1^{(n)}]$ on a Hilbert space $\mathcal{H}_2$ so that

$$A_1^{(i)} S_{K_1} = S_{K_1} \varphi(A_1)^{(i)} \quad \text{for} \quad 1 \le i \le n.$$

Continuing in this fashion, we obtain a sequence

$$(A,K),\ (V_A,K_1),\ (A_1,S_{K_1}),\ (V_{A_1},K_2),\ (A_2,S_{K_2})\dots$$

of pairs of operators acting on Hilbert spaces $\mathcal{H} \subseteq \mathcal{H}_1 \subseteq \mathcal{H}_2 \dots$, co-extending A and K and satisfying the covariance relations. Let $\mathcal{H} = \bigvee_j \mathcal{H}_j$, and consider these pairs of operators as acting on $\mathcal{H}$ by extending them to be zero on the complement. Let

$$T_A = \textsc{sot}\text{–}\lim A_j = \textsc{sot}\text{–}\lim V_{A_j}$$

and

$$T_K = \textsc{sot}\text{–}\lim S_{K_j} = \textsc{sot}\text{–}\lim K_j.$$

These limits evidently exist as in each case, one of the sequences consists of isometries which decompose as infinite direct sums. In particular, T_A is a row isometry and T_K is an isometry. Multiplication is SOT-continuous on the ball, hence the covariance relations hold in the limit. ∎

We now extend this to a unitary representation. The proof uses the "one step extension" technique.

Lemma 3.4. *Let $\varphi \in \mathrm{Aut}(\mathfrak{A}_n)$ and let $S = [S^{(1)}, \dots S^{(n)}]$ be a row isometry and let V be an isometry acting on a Hilbert space $\mathcal{H}$ and satisfying the covariance relations*

$$S^{(i)}V = V\varphi(S)^{(i)} \quad \textit{for} \quad 1 \le i \le n.$$

Then there exist a row isometry $\tilde{S} = [\tilde{S}^{(1)} \ \dots \ \tilde{S}^{(n)}]$ and an isometry $\tilde{V}$, acting on a Hilbert space $\tilde{\mathcal{H}} \supset \mathcal{H}$, extending S and V respectively and satisfying

(i) $\tilde{S}^{(i)}\tilde{V} = \tilde{V}\varphi(\tilde{S})^{(i)}$ *for* $1 \le i \le n$

(ii) $\tilde{V}(\tilde{\mathcal{H}}) = \mathcal{H}$.

Proof. Let $\mathcal{K} = (I - VV^*)\mathcal{H}$ and set $\mathcal{H}' = \mathcal{H} \oplus \mathcal{K}$. Define a unitary operator $U \in B(\mathcal{H} \oplus \mathcal{K}, \mathcal{H})$ by

$$U(x, y) = Vx + y \quad \text{for} \quad x \in \mathcal{H} \text{ and } y \in \mathcal{K}.$$

Set

$$\tilde{V} = U^*VU \quad \text{and} \quad \tilde{S}^{(i)} = U^*\varphi^{-1}(S)^{(i)}U \quad \text{for} \quad i = 1, \dots, n.$$

Notice that $U^*(x) = (Vx\,, (I - VV^*)x)$ and so

$$\tilde{V}(x, y) = (Vx + y, 0) \quad \text{for} \quad x \in \mathcal{H} \text{ and } y \in \mathcal{K}.$$

Therefore $\tilde{V}$ extends V and maps $\mathcal{H}'$ onto $\mathcal{H}$.

To show that $\tilde{S}$ extends S, note that the covariance relations imply that

$$\varphi^{-1}(S)^{(i)}V = VS^{(i)} \quad \text{for} \quad 1 \le i \le n.$$

Hence, for any $x \in \mathcal{H}$ we have for $x \in \mathcal{H}$,

$$\varphi^{-1}(S)^{(i)}U(x, 0) = \varphi^{-1}(S)^{(i)}Vx = VS^{(i)}x = U(S^{(i)}x, 0).$$

Hence $\tilde{S}^{(i)}|_{\mathcal{H}} = S^{(i)}$. Finally, this same calculation shows that

$$\tilde{S}^{(i)}\tilde{V} = U^*\varphi^{-1}(S)^{(i)}VU = U^*VS^{(i)}U = \tilde{V}\varphi(\tilde{S})^{(i)}. \qquad \blacksquare$$

We can now complete the proof of the main result.

Proof of Theorem 3.1. Let $A = [A^{(1)}, \dots A^{(n)}]$ and K be contractions on a Hilbert space $\mathcal{H}$ satisfying

$$A^{(i)}K = K\varphi(A)^{(i)}, \quad 1 \le i \le n.$$

Using Lemma 3.3, we dilate A and K to isometries S and V satisfying $S^{(i)}U = U\varphi(S)^{(i)}$ for $1 \le i \le n$. Making repeated use of Lemma 3.4, we now produce a sequence $\{(S_j, V_j)\}_{j=1}^{\infty}$ of extensions consisting of a row isometry S_j extending S_{j-1} and an isometry V_j extending V_{j-1}, acting on an increasing sequence of Hilbert spaces $\mathcal{H}_j$, which satisfy the covariance relations and have $V_j\mathcal{H}_j = \mathcal{H}_{j-1}$. If we set $S = \textsc{sot}\text{–}\lim S_j$ and $U = \textsc{sot}\text{–}\lim V_j$, then condition (ii) in Lemma 3.4 implies that

U is a unitary while (i) shows that S and U satisfy the covariance relations. ∎

4. Maximal Covariant Representations and the C*-envelope of $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$.

There is a question left open in Theorem 3.1, which is whether the row isometry in the unitary dilation generates the Cuntz algebra or the Cuntz–Toeplitz algebra. It is not hard to see that in the former case, there is no sensible way to dilate further. But in the Cuntz–Toeplitz case, there is a gap, since $\sum_{i=1}^n S_i S_i^* < I$, that may allow a proper dilation. In fact this occurs, and in this section we will deal with this issue.

The Dritschel–McCullough proof **[13]** of Hamana's Theorem **[17]** proving the existence of Arveson's C*-envelope **[2]** is based on the notion of a *maximal representation*. This is a completely contractive representation ρ of an operator algebra with the property that the only (completely contractive) dilations have the form $\rho \oplus \sigma$. They establish that every representation dilates to a maximal one, and that maximal representations extend to $*$-representations of the C*-envelope. In this manner, they were able to establish the existence of the C*-envelope without taking Hamana's route via the injective envelope. The upshot for dilation theory is to focus attention on maximal dilations.

In our case, Theorem 3.1 shows that the maximal dilations must send the generators of $\mathfrak{A}_n$ to a row isometry S and the operator implementing the automorphism must be unitary. In the case when this representation is of Cuntz type, meaning that $SS^* = I$, it is evident that this representation is maximal. So we are left to deal with the other case.

We first show that the Wold decomposition of S decomposes U as well. Recall that the Wold decomposition uniquely splits the Hilbert space into $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ so that $S_i|_{\mathcal{H}_0} \simeq L_i^{(\alpha)}$ is pure, and $T_i := S_i|_{\mathcal{H}_1}$ has Cuntz type.

Lemma 4.1. *Suppose $S = [S_1 \ \dots \ S_n]$ is a row isometry and U is a unitary on a Hilbert space $\mathcal{H}$ satisfying the covariance relations $S_i U = U\varphi(S_i)$ for $1 \le i \le n$. Then the Wold decomposition reduces U, thereby decomposing the representation of $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$ into a pure part and a Cuntz part.*

Proof. Let σ be the representation of $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$ with $\sigma(L_i) = S_i$ and $\sigma(\mathfrak{u}) = U$. Let $M = \operatorname{Ran}(I - SS^*) = \operatorname{Ran}(I - \sum_{i=1}^n S_i S_i^*)$. Then $\mathcal{H}_0 = \overline{\sigma(\mathfrak{A}_n) M}$. Now $[\varphi(S_1) \ \dots \ \varphi(S_n)]$ is also a row isometry, and we let $N = \mathrm{Ran}(I - \sum_{i=1}^n \varphi(S_i)\varphi(S_i)^*)$. Since $S_i \simeq L_i^{(\alpha)} \oplus T_i$, we see that $\varphi(S_i) = \varphi(L_i)^{(\alpha)} \oplus \varphi(T_i)$. Thus the Wold decomposition of $\varphi(S)$ decomposes $\mathcal{H}$ in the same way as S. Therefore $\overline{\sigma(\mathfrak{A}_n)N} = \mathcal{H}_0$.

Now we use the fact that U implements φ to see that $N = UM$ and so

$$U\mathcal{H}_0 = U\overline{\sigma(\mathfrak{A}_n)M} = \overline{\sigma(\mathfrak{A}_n)UM} = \overline{\sigma(\mathfrak{A}_n)N} = \mathcal{H}_0.$$

Therefore $\mathcal{H}_0$ reduces U as claimed. ∎

Next we show how φ is implemented on $\mathcal{H}_0$.

Lemma 4.2. *Let $\varphi \in \mathrm{Aut}(\mathfrak{A}_n)$ and let U_φ be the Voiculescu unitary on $\ell^2(\mathbb{F}_n^+)$ which implements φ. Then the only unitaries on $\mathcal{B}(\ell^2(\mathbb{F}_n^+)^{(\alpha)}$ which implement φ on $\mathfrak{A}_n^{(\alpha)}$ have the form $U_\varphi \otimes W$.*

Proof. Clearly $U_\varphi \otimes I_\alpha$ implements φ. If V is another unitary implementing φ on $\mathfrak{A}_n^{(\alpha)}$, then $(U_\varphi^* \otimes I_\alpha)V$ commutes with $\mathfrak{A}_n^{(\alpha)}$. By Fuglede's Theorem, it commutes with $\mathrm{C}^*(\mathfrak{A}_n^{(\alpha)})'' = \mathcal{B}(\ell^2(\mathbb{F}_n^+)) \otimes \mathbb{C}I_\alpha$. Therefore it lies in $\mathrm{C}^*(\mathfrak{A}_n^{(\alpha)})' = \mathbb{C}I_{\ell^2(\mathbb{F}_n^+)} \otimes \mathcal{B}(\mathcal{H})$ where $\dim \mathcal{H} = \alpha$, say $(U_\varphi^* \otimes I_\alpha)V = I \otimes W$. ∎

Now W is unitary, and so has a spectral resolution. So essentially every pure representation $(\mathrm{id}^{(\alpha)}, U)$ of the covariance relations is a direct integral of the representations $(\mathrm{id}, \lambda U_\varphi)$ as λ runs over the unit circle $\mathbb{T}$. Thus it suffices to show how to dilate (id, U_φ) to a Cuntz type unitary dilation.

To accomplish this, we need to consider the map $\hat\varphi$ in $\mathrm{Aut}(\mathbb{B}_n)$. We refer to [**30**, Chapter 2] for details. We distinguish two cases. In the first case, $\hat\varphi$ has a fixed point inside $\mathbb{B}_n$. Because $\mathrm{Aut}(\mathbb{B}_n)$ acts transitively on $\mathbb{B}_n$, $\hat\varphi$ is biholomorphically conjugate to a map which fixes 0. Such an equivalence yields a completely isometric isomorphism of the semi-crossed products. So we may assume that $\hat\varphi(0) = 0$ without loss of generality. But then $\hat\varphi$ is a unitary matrix $U_0 \in \mathcal{U}(n)$, φ is the gauge automorphism it induces, and

$$U_\varphi = \sum\nolimits_{i\ge 0}^{\oplus} U_0^{\otimes i}.$$

In the second case, $\hat\varphi$ fixes one or two points on the unit sphere. Again $\mathrm{Aut}(\mathbb{B}_n)$ acts transitively on the sphere, so we may suppose that $e_1 = (1, 0, \dots, 0)$ is a fixed point. We will deal with these two cases separately.

In both cases, we will dilate to atomic representations of the Cuntz algebra. These are $*$-representations in which the generators permute an orthonormal basis up to scalar multiples. These representations were defined and classified in [**8**]. In the first case, we use representations of inductive type. Beginning with an infinite tail, i.e., an infinite word $\mathbf{x} = i_1 i_2 \dots$ in the alphabet $\{1, \dots, n\}$, define a sequence of Hilbert spaces $\mathcal{H}_k$, for $k \geq 0$, as follows. Each $\mathcal{H}_k$ naturally identified with Fock space $\ell^2(\mathbb{F}_n^+)$, and this determines the action of $\mathbb{F}_n^+$ on $\mathcal{H}_k$ by the left regular representation, which extends to a $*$-representation λ_k of the Cuntz–Toeplitz algebra $\mathcal{E}_n$. Imbed $\mathcal{H}_{k-1}$ into $\mathcal{H}_k$ by the isometry $V_k \xi_w^{k-1} = \xi_{wi_k}^k$, where with basis $\{\xi_w^k : w \in \mathbb{F}_n^+\}$ is the standard basis for $\mathcal{H}_k$. Effectively, V_k is unitarily equivalent to R_{i_k}, the right multiplication operator by the symbol i_k. Since this lies in the commutant of the left regular representation, it is evident that V_k intertwines λ_{k-1} and λ_k. The inductive limit of these representations, denoted $\lambda_{\mathbf{x}}$, on the Hilbert space $\mathcal{H}_{\mathbf{x}} = \lim \mathcal{H}_k$, is a $*$-representation of $\mathcal{E}_n$ onto the Cuntz algebra because in the limit, the sum of the ranges of $\lambda_{\mathbf{x}}(\mathfrak{s}_i)$ for $1 \leq i \leq n$ is the whole space.

Theorem 4.3. *Let $\varphi \in \mathrm{Aut}(\mathfrak{A}_n)$ such that $\hat{\varphi}$ has a fixed point in $\mathbb{B}_n$. Then (id, U_φ) has a unitary dilation of Cuntz type.*

Proof. As noted before the proof, φ is biholomorphically conjugate to an automorphism which fixes the origin, and hence is a gauge automorphism. So we start by assuming that φ has this form; so φ is determined by the unitary $\hat{\varphi} = U_0$ on $\mathrm{span}\{\xi_i : 1 \leq i \leq n\}$.

Since unitary matrices are diagonalizable, the map $\hat{\varphi}$ is biholomorphically conjugate to a diagonal unitary. Thus it suffices to assume that U_0 is diagonal, say $U_0 \xi_i = \mu_i \xi_i$ for scalars $\mu_i \in \mathbb{T}$. Let us write $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{T}^n$. It is easy to verify that U_φ is the diagonal operator $U_\varphi \xi_w = w(\mu) \xi_w$.

Now let $\mathbf{x} = i_1 i_2 \dots$ be any infinite tail, and consider the construction indicated before this proof. Set $x_k = i_1 i_2 \dots i_k$ for $k \geq 1$. Define unitaries V_k on $\mathcal{H}_k$ by $V_k \xi_w^k = \overline{x_k(\mu)} w(\mu) \xi_w^k$. It is easy to see that since this is a scalar multiple of U_φ, conjugation by V_k implements φ on $\lambda_k(\mathcal{E}_n)$. Moreover the scalar $\overline{x_k(\mu)}$ is chosen so that $V_k|_{\mathcal{H}_{k-1}} = V_{k-1}$ for $k \geq 1$. Thus the inductive limit yields the representation $\sigma_{\mathbf{x}}$ and a unitary operator V on $\mathcal{H}_{\mathbf{x}}$ implementing φ. Thus $(\sigma_{\mathbf{x}}, V)$ is the desired dilation.

Note that the discussion prior to the theorem implies now that *any* representation of $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$ dilates to a Cuntz-type representation, provided that φ is a gauge automorphism.

In case of an arbitrary φ, we want to prove the existence of a Cuntz-type dilation for (id, U_φ). As in the discussion prior to the theorem, there exists a biholomorphic automorphism α and a gauge automorphism φ' so that $\varphi' \circ \alpha = \alpha \circ \varphi$. By the previous paragraph, (id, φ')

has a unitary Cuntz dilation $(\sigma_{\mathbf{x}}, V)$. We claim that $(\sigma_{\mathbf{x}}\alpha, V)$ provides a unitary Cuntz dilation of (id, φ). It suffices to verify the covariance relations:

$$\sigma_{\mathbf{x}}\alpha(A)W = W\sigma_{\mathbf{x}}(\varphi'(\alpha(A))) = W\sigma_{\mathbf{x}}\alpha(\varphi(A)). \qquad \blacksquare$$

For the second case, we use a special case of the ring representations [**8**]. Let $\mathcal{H}_j = \ell^2(\mathbb{F}_n^+)$ with basis $\{\xi_w^j : w \in \mathbb{F}_n^+\}$ for $2 \le j \le n$. Let $\mathcal{H} = \mathbb{C}\zeta \oplus \sum_{j=2}^n \oplus \mathcal{H}_j$. Let σ_1 denote the representation determined by

$$\sigma_1(L_1)\zeta = \zeta, \qquad \sigma_1(L_j)\zeta = \xi_\varnothing^j \quad \text{for } 2 \le j \le n,$$
$$\sigma_1(L_i)\xi_w^j = \xi_{iw}^j \quad \text{for } 1 \le i \le n,\ 2 \le j \le n,\ w \in \mathbb{F}_n^+.$$

This is evidently a Cuntz representation. Moreover, $\mathbb{C}\zeta$ is coinvariant and thus the compression to $\mathbb{C}\zeta$ is a multiplicative functional ψ such that

$$\psi(A) = \langle \sigma_{1,\mu}(L_i)\zeta, \zeta\rangle = \delta_{i1} = \hat{L}_i(e_1).$$

Hence $\langle \sigma_1(A)\zeta, \zeta\rangle = \hat{A}(e_1)$ for all $A \in \mathfrak{A}_n$.

Theorem 4.4. *Let $\varphi \in \mathrm{Aut}(\mathfrak{A}_n)$ such that $\hat{\varphi}$ has a fixed point on the boundary of $\overline{\mathbb{B}}_n$. Then (id, U_φ) has a unitary dilation of Cuntz type.*

Proof. As in the previous proof, we may suppose that $\hat{\varphi}$ has e_1 as a fixed point.

First we show that φ is unitarily implemented on $(\sigma_1, \mathcal{H})$. Let $\psi(A) = \langle \sigma_1(A)\zeta, \zeta\rangle = \hat{A}(e_1)$. Define a unitary $W = 1 \oplus U_\varphi^{(n-1)}$, and consider $S_i = W^*\sigma_1(L_i)W$. Then

$$S_i|_{(\mathbb{C}\zeta)^\perp} = \varphi(L_i)^{(n-1)} \quad \text{for} \quad 1 \le i \le n.$$

Also

$$\langle S_i\zeta, \zeta\rangle = \psi(L_i) = \delta_{i1} = \hat{L}_i(e_1) = \hat{L}_i\hat{\varphi}(e_1) = \widehat{\varphi(L_i)}(e_1).$$

In particular,

$$\langle S_1\zeta, \zeta\rangle = 1 = \langle \sigma_1(\varphi(L_1))\zeta, \zeta\rangle.$$

Since both S_1 and $\sigma_1(\varphi(L_1)) = \varphi(\sigma_1(L_1))$ are isometries, we conclude that $S_1\zeta = \zeta = \sigma_1(\varphi(L_1))\zeta$. Both agree with $\varphi(L_1)^{(n-1)}$ on $(\mathbb{C}\zeta)^\perp$, and therefore

$$S_1 = 1 \oplus \varphi(L_1)^{(n-1)} = \sigma_1(\varphi(L_1)).$$

On the other hand, $S_j\zeta$ is orthogonal to ζ for $2 \le j \le n$. Because these are isometries with pairwise orthogonal ranges, $S_j\zeta$ is also orthogonal to

$$\Big(\sum_{i=1}^n \varphi(L_i)\ell^2(\mathbb{F}_n^+)\Big)^{(n-1)} = \Big(U_\varphi^* \sum_{i=1}^n L_i U_\varphi \ell^2(\mathbb{F}_n^+)\Big)^{(n-1)} = \big((\mathbb{C}\nu)^\perp\big)^{(n-1)}$$

where $\nu = U_\varphi^* \xi_\varnothing$. Observe that exactly the same is true for the isometries $\sigma_1(\varphi(L_j))$ because $\sigma_1(\varphi(L_j))|_{(\mathbb{C}\zeta)^\perp} = \varphi(L_i)^{(n-1)}$ also. Therefore there is a unitary V on $(\mathbb{C}\nu)^{(n-1)}$ so that

$$VS_j\zeta = \sigma_1(\varphi(L_j))\zeta \quad \text{for} \quad 2 \le j \le n.$$

Considering V as an operator on $\mathbb{C}^{n-1}$, we define $V' = I_{\ell^2(\mathbb{F}_n^+)} \otimes V$ in the commutant of $\mathfrak{A}_n^{(n-1)}$ extending V to all of $(\mathbb{C}\zeta)^\perp$. Define $W' = (1 \oplus V')^* W$. Then

$$W'^* \sigma_1(L_i) W' = \sigma_1(\varphi(L_i)) \quad \text{for} \quad 1 \le i \le n.$$

Pick a unit eigenvector $y \in \mathbb{C}^{n-1}$ for the unitary matrix V, say $Vy = \beta y$. Then $\mathcal{H} = \ell^2(\mathbb{F}_n^+) \otimes \mathbb{C}y$ is an invariant subspace for $\sigma_1(\mathfrak{A}_n)$ which is also invariant for W', and $W'|_{\mathcal{H}} = \overline{\beta} U_\varphi$. Thus it is clear that (σ_1, W') is a unitary dilation of $(\lambda, \overline{\beta} U_\varphi)$. Thus $(\sigma_1, \beta W')$ is a unitary dilation of (λ, U_φ). ∎

Remark 4.5. Arveson [**2**] defines a boundary representation of an operator algebra $\mathcal{A}$ to be an irreducible $*$-representation π of $\mathrm{C}^*(\mathcal{A})$ so that $\pi|_{\mathcal{A}}$ has a unique completely positive extension to $\mathrm{C}^*(\mathcal{A})$. These are just the maximal representations of $\mathcal{A}$ which are irreducible [**3**]. So it is of interest to know when we can obtain irreducible dilations. In Theorem 4.3, the representation $\lambda_{\mathbf{x}}$ is already irreducible provided that $\mathbf{x}$ is not eventually periodic, and the representation σ_1 is also irreducible [**8**]. So we obtain boundary representations.

An immediate consequence of these dilation theorems, Theorem 3.1 together with Theorems 4.3 and 4.4, are the following crucial facts.

Corollary 4.6. *Let $\varphi \in \mathrm{Aut}(\mathfrak{A}_n)$. Then every row contractive covariant representation has a unitary dilation of Cuntz type. Conversely, every covariant pair (σ, U), where σ is a $*$-extendible representation of $\mathfrak{A}_n$ such that $\sigma(L_1), \dots, \sigma(L_n)$ generate a copy of $\mathcal{O}_n$ and U is a unitary satisfying the covariance relations $\sigma(A)U = U\sigma(\varphi(A))$ for all $A \in \mathfrak{A}_n$ determines a maximal representation of $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$.*

Corollary 4.7. $\mathrm{C}^*_{env}(\mathfrak{A}_n \times_\varphi \mathbb{Z}^+) = \mathcal{O}_n \times_\varphi \mathbb{Z}$.

5. Concrete representations for $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$.

One of the motivations for the present paper was to provide concrete faithful representations for $\mathfrak{A}_n \times_\varphi \mathbb{Z}^+$. Corollary 4.7 essentially reduces this to the (selfadjoint) problem of finding faithful representations for $\mathcal{O}_n \times_\varphi \mathbb{Z}$. We know one construction of a representation of $\mathcal{O}_n \times_\varphi \mathbb{Z}$. Just take the canonical map onto $\mathrm{C}^*(\mathcal{E}_n, U_\varphi)/\mathfrak{K}$. When $\mathcal{O}_n \times_\varphi \mathbb{Z}$ is simple, this is an isomorphism. We show that this is the case when φ is aperiodic.

Theorem 5.1. *The only unitaries in $\mathcal{O}_n$ which conjugate $\mathfrak{A}_n$ into itself are scalars.*

Proof. Suppose that U is a unitary in $\mathcal{O}_n$ such that $U\mathfrak{A}_nU^* = \mathfrak{A}_n$.

Consider the atomic representation σ_i on $\mathcal{H} = \mathbb{C}\zeta \oplus \ell^2(\mathbb{F}_n^+)^{(n-1)}$, where the $\ell^2(\mathbb{F}_n^+)^{(n-1)} = \bigoplus\{\mathcal{H}_k : 1 \le k \ne i \le n\}$ and $\mathcal{H}_k$ has standard basis ξ_w^k for $w \in \mathbb{F}_n^+$. We define

$$\sigma_i(\mathfrak{s}_j)\zeta = \begin{cases} \zeta & \text{if} \quad j = i \\ \xi_\varnothing^j & \text{if} \quad j \ne i \end{cases}$$

and

$$\sigma_i(\mathfrak{s}_j)\xi_w^k = \xi_{jw}^k \quad \text{for} \quad k \ne i,\ w \in \mathbb{F}_n^+.$$

The significance of this representation is that $\mathbb{C}\zeta$ is the unique minimal invariant subspace for $\sigma_i(\mathfrak{A}_n^*)$. Hence it must be fixed by $\sigma_i(U)$. It follows that $\sigma_i(U\mathfrak{s}_iU^*)\zeta = \zeta$. But it is immediately apparent that the only elements of $\mathfrak{A}_n$ which take ζ to itself are of the form $h_i(\sigma_i(\mathfrak{s}_i))$ where $h_i \in A(\mathbb{D})$ and $h_i(1) = 1$. Thus $U\mathfrak{s}_iU^* = h_i(\mathfrak{s}_i)$.

Likewise there are representations σ_{ij} with a unique minimal minimal invariant subspace for $\sigma_i(\mathfrak{A}_n^*)$ which is one dimensional $\mathbb{C}\zeta$ satisfying $\sigma_{ij}((\mathfrak{s}_i + \mathfrak{s}_j)/\sqrt{2})\zeta = \zeta$. The same argument shows that there is an $h_{ij} \in A(\mathbb{D})$ so that $U\big((\mathfrak{s}_i + \mathfrak{s}_j)/\sqrt{2}\big)U^* = h_{ij}\big((\mathfrak{s}_i + \mathfrak{s}_j)/\sqrt{2}\big)$. Therefore

$$h_i(\mathfrak{s}_i) + h_j(\mathfrak{s}_j) = \sqrt{2}h_{ij}\big((\mathfrak{s}_i + \mathfrak{s}_j)/\sqrt{2}\big).$$

It is easy to see from this that $h_i = h_j = h_{ij} = \lambda z$. Since $h_i(1) = 1$, we see that $\lambda = 1$. Therefore U lies in the centre of $\mathcal{O}_n$; whence U is scalar. ∎

Corollary 5.2. *The non-trivial Voiculescu automorphisms of $\mathcal{O}_n$ are outer.*

Kishimoto [**19**, Theorem 3.1] showed that if $\mathfrak{A}$ is a simple C*-algebra and $\alpha \in \mathrm{Aut}(\mathfrak{A})$ such that α is *aperiodic*, i.e., α^n is outer for all $n \ne 0$, then $\mathfrak{A} \times_\alpha \mathbb{Z}$ is simple. Thus we obtain:

Corollary 5.3. *If $\varphi \in \mathrm{Aut}(\mathfrak{A}_n)$ is aperiodic, then $\mathcal{O}_n \times_\varphi \mathbb{Z}$ is simple, and thus is isomorphic to $\mathrm{C}^*(\mathcal{E}_n, U_\varphi)/\mathfrak{K}$. Therefore the representation $\pi_{\mathcal{O}_n} \times \hat{U}_\varphi$ of Example 2.2 is a faithful representation of $\mathfrak{A}_n \times_\varphi \mathbb{Z}$.*

We will now observe that the other representation of Example 2.2, i.e., $\mathrm{id} \times U_\varphi$ is also faithful for $\mathfrak{A}_n \times_\varphi \mathbb{Z}$, provided that φ is aperiodic. This is of course a feature of the non-selfadjoint theory, since $\mathrm{id} \times U_\varphi$ comes from a representation of $\mathcal{E}_n \times_\varphi \mathbb{Z}$.

Corollary 5.4. *If $\varphi \in \mathrm{Aut}(\mathfrak{A}_n)$ is aperiodic, then the representation $\mathrm{id} \times U_\varphi$ of Example 2.2 is a faithful representation of $\mathfrak{A}_n \times_\varphi \mathbb{Z}$.*

Proof. Consider the diagram

$$\mathfrak{A}_n \times_\varphi \mathbb{Z}^+ \xrightarrow{\mathrm{id}\times U_\varphi} \mathcal{B}(\ell^2(\mathbb{F}_n^+)) \xrightarrow{q} \mathcal{B}(\ell^2(\mathbb{F}_n^+))/\mathcal{K}(\ell^2(\mathbb{F}_n^+)),$$

where q denotes the Calkin map. By Corollary 5.3, the composition $q \circ (\mathrm{id} \times U_\varphi) = \pi_{\mathcal{O}_n} \times \hat{U}_\varphi$ is isometric, and therefore $\mathrm{id} \times U_\varphi$ is isometric as well. ∎

When φ is periodic, it may be necessary to use a family of representations. A natural choice are $(\mathrm{id}, \lambda U_\varphi)$ for $\lambda \in \mathbb{T}$. Form $\mathcal{H} = \ell^2(\mathbb{F}_n^+) \otimes L^2(\mathbb{T})$. Consider $(\mathrm{id}^{(\infty)}, U_\varphi \otimes M_z)$ where $\mathrm{id}^{(\infty)}(A) = A \otimes I$ and M_z is multiplication by z on $L^2(\mathbb{T})$. Clearly this is a covariant representation. Let R_μ denote the operator of rotation by $\mu \in \mathbb{T}$ on $L^2(\mathbb{T})$. Then $\mathrm{ad}\, I \otimes R_\mu$ fixes $\mathrm{id}^{(\infty)}(\mathcal{E}_n)$ and conjugates M_z to μM_z. Consequently integration with respect to μ yields a faithful expectation of $\mathfrak{B} = \mathrm{C}^*(\mathrm{id}^{(\infty)}(\mathcal{E}_n), U_\varphi \otimes M_z)$ onto the copy $\mathrm{id}^{(\infty)}(\mathcal{E}_n)$ of $\mathcal{E}_n$. A standard gauge invariant uniqueness argument shows that $\mathfrak{B} \simeq \mathcal{E}_n \times_\varphi \mathbb{Z}$. Modding out by the ideal $\mathfrak{J}$ generated by $\mathrm{id}^{(\infty)}(\mathfrak{K})$ yields a covariant representation of $\mathcal{O}_n \times_\varphi \mathbb{Z}$ which has a faithful expectation onto $\mathcal{O}_n$. Thus this also yields a faithful representation of the crossed product. To summarize, we have established that:

Proposition 5.5. *The crossed product $\mathcal{O}_n \times_\varphi \mathbb{Z}$ is isomorphic to $\mathfrak{B}/\mathfrak{J}$, where $\mathfrak{B} = \mathrm{C}^*(\mathrm{id}^{(\infty)}(\mathcal{E}_n), U_\varphi \otimes M_z)$ and $\mathfrak{J}$ is the ideal generated by $\mathrm{id}^{(\infty)}(\mathfrak{K})$.*

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