THE HYPERRIGIDITY OF TENSOR ALGEBRAS OF C*-CORRESPONDENCES

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Abstract. Given a C*-correspondence $X$, we give necessary and sufficient conditions for the tensor algebra $T_X$ to be hyperrigid. In the case where $X$ is coming from a topological graph we obtain a complete characterization.

1. Introduction

A not necessarily unital operator algebra $A$ is said to be hyperrigid if given any non-degenerate $*$-homomorphism
\[
\tau: C^*_{env}(A) \rightarrow B(H)
\]
then $\tau$ is the only completely positive, completely contractive extension of the restricted map $\tau|_A$. Arveson coined the term hyperrigid in [1] but he was not the only one considering properties similar to this at the time, e.g. [4].

There are many examples of hyperrigid operator algebras such as those which are Dirichlet but the situation was not very clear in the case of tensor algebras of C*-correspondences. It was known that the tensor algebra of a row-finite graph is hyperrigid [4], [5] and Dor-On and Salomon [3] showed that row-finiteness completely characterizes hyperrigidity for such graph correspondences. These approaches, while successful, did not lend themselves to a more general characterization.

The authors, in a previous work [11], developed a sufficient condition for hyperrigidity in tensor algebras. In particular, if Katsura’s ideal acts non-degenerately on the left then the tensor algebra is hyperrigid. The motivation was to provide a large class of hyperrigid C*-correspondence examples as crossed products of operator algebras behave in a very nice manner when the operator algebra is hyperrigid. This theory was in turn leveraged to provide a positive confirmation to the Hao-Ng isomorphism problem in the case of graph correspondences and arbitrary groups. For further reading on the subject please see [9, 10, 11].

In this paper, we provide a necessary condition for the hyperrigidity of a tensor algebra, that a C*-correspondence cannot be $\sigma$-degenerate, and show

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that this completely characterizes the situation where the $C^*$-correspondence is coming from a topological graph, which generalizes both the graph correspondence case and the semicrossed product arising from a multivariable dynamical system.

1.1. Regarding hyperrigidity. The reader familiar with the literature recognizes that in our definition of hyperrigidity, we are essentially asking that the restriction on $\mathcal{A}$ of any non-degenerate representation of $C^*_{\text{env}}(\mathcal{A})$ possesses the unique extension property (abbr. UEP). According to [3, Proposition 2.4] a representation $\rho : \mathcal{A} \to B(\mathcal{H})$, degenerate or not, has the UEP if and only if $\rho$ is a maximal representation of $\mathcal{A}$, i.e., whenever $\pi$ is a representation of $\mathcal{A}$ dilating $\rho$, then $\pi = \rho \oplus \pi'$ for some representation $\pi'$.

Our definition of hyperrigidity is in accordance with Arveson’s nomenclature [1], our earlier work [7, 11] and the works of Dor-On and Salomon [3] and Salomon [17], who systematized quite nicely the non-unital theory.

An alternative definition of hyperrigidity for $\mathcal{A}$ may ask that any representation of $C^*_{\text{env}}(\mathcal{A})$, not just the non-degenerate ones, possesses the UEP when restricted on $\mathcal{A}$. It turns out that for operator algebras with a positive contractive approximate unit $1$, such a definition would be equivalent to ours [17, Proposition 3.6 and Theorem 3.9]. However when one moves beyond operator algebras with an approximate unit, there are examples to show that the two definitions differ. One such example is the non-unital operator algebra $\mathcal{A}_V$ generated by the unilateral forward shift $V$. It is easy to see that $\mathcal{A}_V$ is hyperrigid according to our definition and yet the zero map, as a representation on $\mathcal{H} = \mathbb{C}$, does not have the UEP. (See for instance [17, Example 3.4].)

2. Main results

A $C^*$-correspondence $(X, \mathcal{C}, \varphi_X)$ (often just $(X, \mathcal{C})$) consists of a $C^*$-algebra $\mathcal{C}$, a Hilbert $\mathcal{C}$-module $(X, \langle , \rangle)$ and a (non-degenerate) $\ast$-homomorphism $\varphi_X : \mathcal{C} \to \mathcal{L}(X)$ into the $C^*$-algebra of adjointable operators on $X$.

An isometric (Toeplitz) representation $(\rho, t, \mathcal{H})$ of a $C^*$-correspondence $(X, \mathcal{C})$ consists of a non-degenerate $\ast$-homomorphism $\rho : \mathcal{C} \to B(\mathcal{H})$ and a linear map $t : X \to B(\mathcal{H})$, such that

$$\rho(c)t(x) = t(\varphi_X(c)(x)),$$

and

$$t(x)^*t(x') = \rho(\langle x, x' \rangle),$$

for all $c \in \mathcal{C}$ and $x, x' \in X$. These relations imply that the $C^*$-algebra generated by this isometric representation equals the closed linear span of

$$t(x_1) \cdots t(x_n)t(y_1)^* \cdots t(y_m)^*, \quad x_i, y_j \in X.$$

Moreover, there exists a $\ast$-homomorphism $\psi_t : \mathcal{K}(X) \to B$, such that

$$\psi_t(\theta_{x,y}) = t(x)t(y)^*,$$

1which includes all operator algebras appearing in this paper.
where $K(X) \subset L(X)$ is the subalgebra generated by the operators $\theta_{x,y}(z) = x(y,z)$, $x, y, z \in X$, which are called by analogy the compact operators.

The Cuntz-Pimsner-Toeplitz $C^*$-algebra $T_X$ is defined as the $C^*$-algebra generated by the image of $(\rho_\infty, t_\infty)$, the universal isometric representation. This is universal in the sense that for any other isometric representation there is a $\ast$-homomorphism of $T_X$ onto the $C^*$-algebra generated by this representation in the most natural way.

The tensor algebra $T_X^+$ of a $C^*$-correspondence $(X, C)$ is the norm-closed subalgebra of $T_X$ generated by $\rho_\infty(C)$ and $t_\infty(X)$. See [15] for more on these constructions.

Consider Katsura’s ideal

$$J_X \equiv \ker \varphi_X^{-1} \cap \varphi_X^{-1}(K(X)).$$

An isometric representation $(\rho, t)$ of $(X, C)$ is said to be covariant (or Cuntz-Pimsner) if and only if

$$\psi_t(\varphi_X(c)) = \rho(c),$$

for all $c \in J_X$. The Cuntz-Pimsner algebra $O_X$ is the universal $C^*$-algebra for all isometric covariant representations of $(X, C)$, see [13] for further details. Furthermore, the first author and Kribs [8, Lemma 3.5] showed that $O_X$ contains a completely isometric copy of $T_X^+$ and $C^*_{env}(T_X^+) \simeq O_X$.

We turn now to the hyperrigidity of tensor algebras. In [11] a sufficient condition for hyperrigidity was developed, Katsura’s ideal acting non-degenerately on the left of $X$. To be clear, non-degeneracy here means that $\varphi_X(J_X)X = X$ which by Cohen’s factorization theorem implies that we actually have $\varphi_X(J_X)X = X$.

**Theorem 2.1** (Theorem 3.1, [11]). Let $(X, C)$ be a $C^*$-correspondence. If $\varphi_X(J_X)$ acts non-degenerately on $X$, then $T_X^+$ is a hyperrigid operator algebra.

The proof shows that if $\tau': O_X \to B(H)$ is a completely contractive and completely positive map that agrees with a $\ast$-homomorphism of $O_X$ on $T_X^+$ then the multiplicative domain of $\tau'$ must be everything. This is accomplished through the multiplicative domain arguments of [2, Proposition 1.5.7] and the use of Kasparov’s Stabilization Theorem. In earlier versions of [11], Theorem 2.1 was claimed for countably generated $C^*$-correspondences only but a slight modification of the earlier proof makes it work for arbitrary $C^*$-correspondences.

A $C^*$-correspondence $(X, C)$ is called regular if and only if $C$ acts faithfully on $X$ by compact operators, i.e., $J_X = C$. We thus obtain the following which also appeared in [11].

**Corollary 2.2.** The tensor algebra of a regular $C^*$-correspondence is necessarily hyperrigid.

We seek a converse to Theorem 2.1.
Definition 2.3. Let \((X, \mathcal{C})\) be a C*-correspondence and let \(\mathcal{J}_X\) be Katsuura's ideal. We say that \(\varphi_X(\mathcal{J}_X)\) acts \(\sigma\)-degenerately on \(X\) if there exists a representation \(\sigma: \mathcal{C} \to B(\mathcal{H})\) so that
\[
\varphi_X(\mathcal{J}_X) X \otimes_{\sigma} \mathcal{H} \neq X \otimes_{\sigma} \mathcal{H}.
\]

Remark 2.4. In particular, if there exists \(n \in \mathbb{N}\) so that
\[
(\varphi_X(\mathcal{J}_X) \otimes \text{id}) X^{\otimes n} \otimes_{\sigma} \mathcal{H} \neq X^{\otimes n} \otimes_{\sigma} \mathcal{H}.
\]
then by considering the Hilbert space \(\mathcal{K} := X^{\otimes n-1} \otimes_{\sigma} \mathcal{H}\), we see that
\[
\varphi_X(\mathcal{J}_X) X \otimes_{\sigma} \mathcal{K} \neq X \otimes_{\sigma} \mathcal{K}.
\]
and so \(\varphi_X(\mathcal{J}_X)\) acts \(\sigma\)-degenerately on \(X\).

The following gives a quick example of a \(\sigma\)-degenerate action. Note that this is possibly stronger than having a not non-degenerate action.

Proposition 2.5. Let \((X, \mathcal{C})\) be a C*-correspondence. If \((\varphi_X(\mathcal{J}_X) X)^{\perp} \neq \{0\}\), then \(\varphi_X(\mathcal{J}_X)\) acts \(\sigma\)-degenerately on \(X\).

Proof. Let \(0 \neq f \in (\varphi_X(\mathcal{J}_X) X)^{\perp}\). Let \(\sigma: \mathcal{C} \to B(\mathcal{H})\) be a \(*\)-representation and \(h \in \mathcal{H}\) so that \(\sigma((f, f))^{1/2})h \neq 0\). Then,
\[
\langle f \otimes_{\sigma} h, f \otimes_{\sigma} h \rangle = \langle h, \sigma((f, f))h \rangle = \|\sigma((f, f))^{1/2})h\| \neq 0.
\]
A similar calculation shows that
\[
0 \neq f \otimes_{\sigma} h \in (\varphi_X(\mathcal{J}_X) X \otimes_{\sigma} \mathcal{H})^{\perp}
\]
and we are done. \(\blacksquare\)

We need the following

Lemma 2.6. Let \((X, \mathcal{C})\) be a C*-correspondence and \((\rho, t)\) an isometric representation of \((X, \mathcal{C})\) on \(\mathcal{H}\).

(i) If \(\mathcal{M} \subseteq \mathcal{H}\) is an invariant subspace for \((\rho \times t)(T_X^{\perp})\), then the restriction \((\rho|_{\mathcal{M}}, t|_{\mathcal{M}})\) of \((\rho, t)\) on \(\mathcal{M}\) is an isometric representation.

(ii) If \(\rho(c)h = \psi_t(\varphi_X(c))h\), for all \(c \in \mathcal{J}_X\) and \(h \in \langle t(X) \mathcal{H} \rangle^{\perp}\), then \((\rho, t)\) is a Cuntz-Pimsner representation.

Proof. (i) If \(p\) is the orthogonal projection on \(\mathcal{M}\), then \(p\) commutes with \(\rho(\mathcal{C})\) and so \(\rho|_{\mathcal{M}}(\cdot) = pp(\cdot)p\) is a \(*\)-representation of \(\mathcal{C}\).

Furthermore, for \(x, y \in X\), we have
\[
t|_{\mathcal{M}}(x)^* t|_{\mathcal{M}}(y) = pt(x)^* pt(y)p = pt(x)^* t(y)p = pp([x, y])p = \rho|_{\mathcal{M}}([x, y])
\]
and the conclusion follows.

(ii) It is easy to see on rank-one operators and therefore by linearity and continuity on all compact operators \(K \in \mathcal{K}(X)\) that
\[
t(Kx) = \psi_t(K)t(x), \quad x \in X.
\]
Now if \( c \in \mathcal{J}_X \), then for any \( x \in X \) and \( h \in \mathcal{H} \) we have
\[
\rho(c)t(x)h = t(\varphi_X(c)x)h = \psi_t(\varphi_X(c))t(x)h.
\]
By assumption \( \rho(c)h = \psi_t(\varphi_X(c))h \), for any \( h \in [t(X)\mathcal{H}]^\perp \) and the conclusion follows. \( \blacksquare \)

**Theorem 2.7.** Let \((X, \mathcal{C})\) be a \( \mathbb{C}^*\)-correspondence. If Katsura’s ideal \( \mathcal{J}_X \) acts \( \sigma\)-degenerately on \( X \) then the tensor algebra \( T^+_X \) is not hyperrigid.

**Proof.** Let \( \sigma: \mathcal{C} \to B(\mathcal{H}) \) so that
\[
\varphi_X(\mathcal{J}_X)X \otimes_\sigma \mathcal{H} \neq X \otimes_\sigma \mathcal{H}
\]
and let \( \mathcal{M}_0 := (\varphi_X(\mathcal{J}_X)X \otimes_\sigma \mathcal{H})^\perp \).

We claim that
\[
(\varphi_X(\mathcal{J}_X) \otimes I)\mathcal{M}_0 = \{0\}.
\]
Indeed for any \( f \in \mathcal{M}_0 \) and \( j \in \mathcal{J}_X \) we have
\[
\langle (\varphi_X(j) \otimes I) f, (\varphi_X(j) \otimes I) f \rangle = \langle (\varphi_X(j^*) \otimes I) f, (\varphi_X(j^*) \otimes I) f \rangle = 0
\]
since \( f \in (\varphi_X(\mathcal{J}_X)X \otimes_\sigma \mathcal{H})^\perp \). This proves the claim.

We also claim that
\[
(\varphi_X(\mathcal{C}) \otimes I)\mathcal{M}_0 = \mathcal{M}_0.
\]
Indeed this follows from the fact that
\[
(\varphi_X(\mathcal{C}) \otimes I)(\varphi_X(\mathcal{J}_X)X \otimes_\sigma \mathcal{H}) = \varphi_X(\mathcal{J}_X)X \otimes_\sigma \mathcal{H},
\]
which is easily verified.

Using the subspace \( \mathcal{M}_0 \) we produce a Cuntz-Pimsner representation \((\rho, t)\) of \((X, \mathcal{C})\) as follows. Let \((\rho_\infty, t_\infty)\) be the universal representation of \((X, \mathcal{C})\) on the Fock space \( F(X) = \oplus_{n=0}^{\infty} X^\otimes n, X^\otimes 0 := \mathcal{C} \). Let
\[
\rho_0: \mathcal{C} \to B(F(X) \otimes_\sigma \mathcal{H}); c \mapsto \rho_\infty(c) \otimes I
\]
\[
t_0: X \to B(F(X) \otimes_\sigma \mathcal{H}); x \mapsto t_\infty(x) \otimes I.
\]
Define
\[
\mathcal{M} := 0 \oplus \mathcal{M}_0 \oplus (X \otimes \mathcal{M}_0) \oplus (X^\otimes 2 \otimes \mathcal{M}_0) \oplus \ldots
\]
\[
= (\rho_0 \times t_0)(T^+_X)(0 \oplus \mathcal{M}_0 \oplus 0 \oplus 0 \oplus \ldots) \subseteq F(X) \otimes_\sigma \mathcal{H},
\]
with the second equality following from (2). Clearly, \( \mathcal{M} \) is an invariant subspace for \((\rho_0 \times t_0)(T^+_X)\).

Let \( \rho := \rho_{0|\mathcal{M}} \) and \( t := t_{0|\mathcal{M}} \). By Lemma 2.6(i), \((\rho, t)\) is a representation of \((X, \mathcal{C})\). We claim that \((\rho, t)\) is actually Cuntz-Pimsner.

Indeed by Lemma 2.6(ii) it suffices to examine whether \( \psi_t(\varphi_X(j))h = \rho(j)h \), for any \( h \in \mathcal{M} \oplus t(X)\mathcal{M} \). Note that since
\[
t(X)\mathcal{M} = 0 \oplus 0 \oplus (X \otimes \mathcal{M}_0) \oplus (X^\otimes 2 \otimes \mathcal{M}_0) \oplus \ldots,
\]
we have that
\[
\mathcal{M} \oplus t(X)\mathcal{M} = 0 \oplus \mathcal{M}_0 \oplus 0 \oplus 0 \oplus \ldots.
\]
From this it follows that for any $h \in M \ominus t(X)M$ we have
\[ t_0(x)^*h \in (C \otimes \sigma H) \oplus 0 \oplus 0 \oplus \ldots, \quad x \in X \]
and so in particular for any $j \in J_X$ we obtain
\[ \psi_t(\varphi_X(j))h \in t_0|M(X)(t_0|M)(X)^*h = \{0\}. \]
On the other hand,
\[ \rho(j)h \in 0 \oplus (\varphi_X(J_X) \otimes I)M_0 \oplus 0 \oplus 0 \oplus \cdots = \{0\}, \]
because of (2). Hence $(\rho, t)$ is Cuntz-Pimsner.

At this point by restricting on $T^+_X$, we produce the representation $\rho \oplus t|T^+_X$ of $T^+_X$ coming from a $\sigma$-representation of its $C^*$-envelope $\mathcal{O}_X$, which admits a dilation, namely $\rho_0 \times t_0|T^+_X$. If we show now that $\rho_0 \times t_0|T^+_X$ is a non-trivial dilation of $\rho \times t|T^+_X$, i.e. $M_0$ is not reducing for $(\rho_0 \times t_0)(T^+_X)$, then $\rho \times t|T^+_X$ is not a maximal representation of $T^+_X$. Proposition 2.4 [3] shows $\rho \times t|T^+_X$ does not have the UEP and so $T^+_X$ is not hyperrigid, as desired.

Towards this end, note that
\[ M \perp = C \oplus (\varphi_X(J_X)X \otimes \sigma H) \oplus (X \otimes M_0) \perp \oplus \ldots \]
and so
\[ t_0(X)M \perp = 0 \oplus (XC \otimes \sigma H) \oplus 0 \oplus 0 \oplus \cdots \not\subseteq M \perp \]
Therefore $M \perp$ is not an invariant subspace for $(\rho_0 \times t_0)(T^+_X)$ and so $M$ is not a reducing subspace for $(\rho_0 \times t_0)(T^+_X)$. This completes the proof. 

3. Topological graphs

A broad class of $C^*$-correspondences arises naturally from the concept of a topological graph. For us, a topological graph $G = (G^0, G^1, r, s)$ consists of two $\sigma$-locally compact spaces $G^0, G^1$, a continuous proper map $r : G^1 \to G^0$ and a local homeomorphism $s : G^1 \to G^0$. The set $G^0$ is called the base (vertex) space and $G^1$ the edge space. When $G^0$ and $G^1$ are both equipped with the discrete topology, we have a discrete countable graph.

With a given topological graph $G = (G^0, G^1, r, s)$ we associate a $C^*$-correspondence $X_G$ over $C_0(G^0)$. The right and left actions of $C_0(G^0)$ on $C_c(G^1)$ are given by
\[ (fFg)(e) = f(r(e))F(e)g(s(e)) \]
for $F \in C_c(G^1)$, $f, g \in C_0(G^0)$ and $e \in G^1$. The inner product is defined for $F, H \in C_c(G^1)$ by
\[ \langle F \mid H \rangle(v) = \sum_{e \in s^{-1}(v)} \overline{F(e)}H(e) \]
for \( v \in \mathcal{G}^0 \). Finally, \( X_G \) denotes the completion of \( C_r(\mathcal{G}^1) \) with respect to the norm
\[
\| F \| = \sup_{v \in \mathcal{G}^0} \langle F | F \rangle(v)^{1/2}.
\]

When \( \mathcal{G}^0 \) and \( \mathcal{G}^1 \) are both equipped with the discrete topology, then the tensor algebra \( \mathcal{T}_G^+ = \mathcal{T}_{XG}^+ \) associated with \( G \) coincides with the quiver algebra of Muhly and Solel \[15\]. See \[16\] for further reading.

Given a topological graph \( G = (\mathcal{G}^0, \mathcal{G}^1, r, s) \), we can describe the ideal \( \mathcal{J}_{X_G} \) as follows. Let
\[
\mathcal{G}_{\text{sce}}^0 = \{ v \in \mathcal{G}^0 \mid v \text{ has a neighborhood } V \text{ such that } r^{-1}(V) = \emptyset \}
\]
\[
\mathcal{G}_{\text{fin}}^0 = \{ v \in \mathcal{G}^0 \mid v \text{ has a neighborhood } V \text{ such that } r^{-1}(V) \text{ is compact} \}
\]
Both sets are easily seen to be open and in \[12\], Proposition 1.24 Katsura shows that
\[
\ker \varphi_{XG} = C_0(\mathcal{G}_{\text{sce}}^0) \quad \text{and} \quad \varphi_{XG}^{-1}(\mathcal{K}(X_G)) = C_0(\mathcal{G}_{\text{fin}}^0).
\]
From the above it is easy to see that \( \mathcal{J}_{X_G} = C_0(\mathcal{G}_{\text{reg}}^0) \), where
\[
\mathcal{G}_{\text{reg}}^0 := \mathcal{G}_{\text{fin}}^0 \backslash \overline{\mathcal{G}_{\text{sce}}^0}.
\]

We need the following

**Lemma 3.1.** Let \( G = (\mathcal{G}^0, \mathcal{G}^1, r, s) \) be a topological graph. Then \( r^{-1}(\mathcal{G}_{\text{reg}}^0) = \mathcal{G}^1 \) if and only if \( r : \mathcal{G}^1 \to \mathcal{G}^0 \) is a proper map satisfying \( r(\mathcal{G}^1) \subseteq (r(\mathcal{G}^1))^c \).

**Proof.** Notice that
\[
\mathcal{G}_{\text{reg}}^0 = r^{-1}(\mathcal{G}_{\text{fin}}^0) \cap r^{-1}(\mathcal{G}_{\text{sce}}^0)^c
\]
and so \( r^{-1}(\mathcal{G}_{\text{reg}}^0) = \mathcal{G}^1 \) is equivalent to \( r^{-1}(\mathcal{G}_{\text{fin}}^0) = r^{-1}(\mathcal{G}_{\text{sce}}^0) = \mathcal{G}^1 \).

First we claim that \( r^{-1}(\mathcal{G}_{\text{fin}}^0) = \mathcal{G}^1 \) if and only if \( r \) is a proper map. Indeed, assume that \( r^{-1}(\mathcal{G}_{\text{fin}}^0) = \mathcal{G}^1 \) and let \( K \subseteq r(\mathcal{G}^1) \) compact in the relative topology. For every \( x \in K \), let \( V_x \) be a compact neighborhood of \( x \) such that \( r^{-1}(V_x) \) is compact and so \( r^{-1}(V_x \cap K) \) is also compact. By compactness, there exist \( x_1, x_2, \ldots, x_n \in K \) so that \( K = \cup_{i=1}^n (V_{x_i} \cap K) \) and so
\[
r^{-1}(K) = \cup_{i=1}^n r^{-1}(V_{x_i} \cap K)
\]
and so \( r^{-1}(K) \) is compact.

Conversely, if \( r \) is proper then any compact neighborhood \( V \) of any point in \( \mathcal{G}^0 \) is inverted by \( r^{-1} \) to a compact set and so \( r^{-1}(\mathcal{G}_{\text{fin}}^0) = \mathcal{G}^1 \).

We now claim that \( r^{-1}(\mathcal{G}_{\text{sce}}^0) = \emptyset \) if and only if \( r(\mathcal{G}^1) \subseteq (r(\mathcal{G}^1))^c \).

Indeed, \( e \in r^{-1}(\mathcal{G}_{\text{sce}}^0) \) is equivalent to \( r(e) \in (r(\mathcal{G}^1))^c \) and so \( r^{-1}(\mathcal{G}_{\text{sce}}^0) = \emptyset \) is equivalent to
\[
r(\mathcal{G}^1) \subseteq \left((r(\mathcal{G}^1))^c\right)^c = (r(\mathcal{G}^1))^c,
\]
as desired. \( \square \)
If $G = (G^0, G^1, r, s)$ is a topological graph and $S \subseteq G^1$, then $N(S)$ denotes the collection of continuous functions $F \in X_G$ with $F|_S = 0$, i.e., vanishing at $S$. The following appears as Lemma 4.3(ii) in [6].

**Lemma 3.2.** Let $G = (G^0, G^1, r, s)$ be a topological graph. If $S_1 \subseteq G^0$, $S_2 \subseteq G^1$ closed, then

$$N(r^{-1}(S_1) \cup S_2) = \overline{\operatorname{span}}\{(f \circ r)F \mid f|_{S_1} = 0, F|_{S_2} = 0\}$$

**Theorem 3.3.** Let $G = (G^0, G^1, r, s)$ be a topological graph and let $X_G$ the $C^*$-correspondence associated with $G$. Then the following are equivalent

(i) the tensor algebra $T_{X_G}^+$ is hyperrigid
(ii) $\varphi(J_{X_G})$ acts non-degenerately on $X_G$
(iii) $r : G^1 \to G^0$ is a proper map satisfying $r(G^1) \subseteq (\overline{r(G^1)})^0$

**Proof.** If $\varphi(J_{X_G})$ acts non-degenerately on $X_G$, then Theorem 2.1 shows that $T_{X_G}^+$ is hyperrigid. Thus (ii) implies (i).

For the converse, assume that $\varphi(J_{X_G})$ acts degenerately on $X_G$. If we verify that $\varphi(J_{X_G})$ acts $\sigma$-degenerately on $X_G$, then Theorem 2.7 shows that $T_{X_G}^+$ is not hyperrigid and so (i) implies (ii).

Towards this end note that $J_{X_G} = C_0(U)$ for some proper open set $U \subseteq G^0$. (Actually we know that $U = G^0_{\text{reg}}$ but this is not really needed for this part of the proof!) Hence

$$\varphi(J_{X_G})X_G = \overline{\operatorname{span}}\{(f \circ r)F \mid f|_U = 0\}$$

(4)

$$= N(r^{-1}(U)^c),$$

according to Lemma 3.2.

Since $\varphi(J_{X_G})$ acts degenerately on $X_G$, (4) shows that $r^{-1}(U)^c \neq \emptyset$. Let $e \in r^{-1}(U)^c$ and let $F \in C_e(G^1) \subseteq X_G$ with $F(e) = 1$ and $F(e') = 0$, for any other $e' \in G^1$ with $s(e') = s(e)$. Consider the one dimensional representation $\sigma : C_0(G_0) \to \mathbb{C}$ coming from evaluation at $s(e)$. We claim that

$$\varphi_{X_G}(J_{X_G})X_G \otimes_{\sigma} \mathbb{C} \neq X_G \otimes_{\sigma} \mathbb{C}.$$

Indeed for any $G \in \varphi(J_{X_G})X_G = N(r^{-1}(U)^c)$ we have

$$\langle F \otimes_{\sigma} 1, G \otimes_{\sigma} 1 \rangle = \langle 1, \sigma((F,G)1) \rangle = \langle F,G \rangle s(e)$$

$$= \sum_{s(e') = s(e)} F(e')G(e')$$

$$= F(e)G(e) = 0.$$

Furthermore,

$$\langle F \otimes_{\sigma} 1, F \otimes_{\sigma} 1 \rangle s(e) = |F(e)|^2 = 1$$

and so $0 \neq F \otimes_{\sigma} 1 \in (\varphi_{X_G}(J_{X_G})X_G \otimes_{\sigma} \mathbb{C})^\perp$. This establishes the claim and finishes the proof of (i) implies (ii).
Finally we need to show that (ii) is equivalent to (iii). Notice that (4) implies that \( \varphi(J_{X,G}) \) acts degenerately on \( X_G \) if and only if
\[
\nu^{-1}(\mathcal{U})^c = \nu^{-1}(G^0_{\text{reg}})^c = \emptyset.
\]
The conclusion now follows from Lemma 3.1.

The statement of the previous Theorem takes its most pleasing form when \( G^0 \) is a compact space. In that case \( T^+_X \) is hyperrigid if and only is \( G^1 \) is compact and \( r(G^1) \subseteq G^0 \) is clopen.

**Remark 3.4.** In a recent preprint [14], the author claims a different condition than ours as equivalent to the hyperrigidity of the tenor algebra of a topological graph, namely the density of \( G^0_{\text{fin}} \) in \( G^0 \). (See [14, Theorem 3.6].) Apparently this is false as it implies that any semicrossed product on a compact Hausdorff space is hyperrigid. Examples to refute this have already appeared in [11, Example 3.4].

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**References**


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