TENSOR ALGEBRAS OF C*-CORRESPONDENCES AND THEIR C*-ENVELOPES.

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Abstract. We show that the C*-envelope of the tensor algebra of an arbitrary C*-correspondence $\mathcal{X}$ coincides with the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{X}}$, as defined by Katsura [**7**]. This improves earlier results of Muhly and Solel [**13**] and Fowler, Muhly and Raeburn [**5**], who came to the same conclusion under the additional hypothesis that $\mathcal{X}$ is strict and faithful.

1. Introduction

Fowler, Muhly and Raeburn have recently characterized [**5**, Theorem 5.3.] the C*-envelope of the tensor algebra $\mathcal{T}_{\mathcal{X}}^{+}$ of a *faithful and strict* C*-correspondence $\mathcal{X}$, as the associated universal Cuntz-Pimsner algebra. Their proof is based on a gauge invariant uniqueness theorem and earlier elaborate results of Muhly and Solel [**13**]. Beyond faithful strict C*-correspondences, little is known: if $\mathcal{X}$ is strict, but not necessary faithful, then the C*-envelope of $\mathcal{T}_{\mathcal{X}}^{+}$ is known to be a quotient of the associated Toeplitz-Cuntz-Pimsner algebra, without any further information (Theorem 6.4 in [**13**]). In [**5**, Remark 5.4], the authors ask whether the above mentioned conditions on $\mathcal{X}$ are necessary for the validity of their Theorem 5.3 in [**5**].

In this note we answer the question of Fowler, Muhly and Raeburn [**5**] (and Muhly and Solel [**13**]) by showing that the C*-envelope of the tensor algebra of an *arbitrary* C*-correspondence $\mathcal{X}$ coincides with the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{X}}$, as defined by Katsura in [**7**]. Our proof does not require any of the results from [**13**] and is modelled upon the proof of our recent result [**9**] that identifies the C*-envelope of the tensor algebra of a directed graph. We also make use of the result of Muhly and Tomforde [**16**] that generalizes the process of adding tails to a graph to the context of C*-correspondences.

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2. Preliminaries

Let $\mathcal{A}$ be a C*-algebra and $\mathcal{X}$ be a (right) Hilbert $\mathcal{A}$-module, whose inner product is denoted as $\langle\, . \mid . \,\rangle$. Let $\mathcal{L}(\mathcal{X})$ be the adjointable operators on $\mathcal{X}$ and let $\mathcal{K}(\mathcal{X})$ be the norm closed subalgebra of $\mathcal{L}(\mathcal{X})$ generated by the operators $\theta_{\xi,\eta}$, $\xi, \eta \in \mathcal{X}$, where $\theta_{\xi,\eta}(\zeta) = \xi\langle \eta | \zeta\rangle$, $\zeta \in \mathcal{X}$.

A Hilbert $\mathcal{A}$-module $\mathcal{X}$ is said to be a C*-*correspondence* over $\mathcal{A}$ provided that there exists a $*$-homomorphism $\phi_{\mathcal{X}} : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{X})$. We refer to $\phi_{\mathcal{X}}$ as the left action of a C*-correspondence $\mathcal{X}$. A C*-correspondence $\mathcal{X}$ over $\mathcal{A}$ is said to be *essential* (resp. *faithful*) if and only if $\phi_{\mathcal{X}}(\mathcal{A})(\mathcal{X}) = \mathcal{X}$ (resp. $\phi_{\mathcal{X}}$ is faithful).

From a given C*-correspondence $\mathcal{X}$ over $\mathcal{A}$, one can form new C*-correspondences over $\mathcal{A}$, such as the *n-fold ampliation* or direct sum $\mathcal{X}^{(n)}$ ([**10**, page 5]) and the *n-fold interior tensor product* $\mathcal{X}^{\otimes n} \equiv \mathcal{X} \otimes_{\phi_{\mathcal{X}}} \mathcal{X} \otimes_{\phi_{\mathcal{X}}} \cdots \otimes_{\phi_{\mathcal{X}}} \mathcal{X}$ ([**10**, page 39], $n \in \mathbb{N}$, ($\mathcal{X}^{\otimes 0} \equiv \mathcal{A}$). These operation are defined within the category of C*-correspondences over $\mathcal{A}$. (See [**10**] for more details.)

A *representation* (π, t) of a C*-correspondence $\mathcal{X}$ over $\mathcal{A}$ consists of a $*$-homomorphism $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ and a linear map $t : \mathcal{X} \rightarrow \mathcal{B}(\mathcal{H})$ so that

(i) $t(\xi)^* t(\eta) = \pi(\langle \xi | \eta \rangle)$, for $\xi, \eta \in \mathcal{X}$,
(ii) $\pi(a)t(\xi) = t(\phi_{\mathcal{X}}(a)\xi)$, for $a \in \mathcal{A}$, $\xi \in \mathcal{X}$.

For a representation (π, t) of a C*-correspondence $\mathcal{X}$ there exists a $*$-homomorphism $\psi_t : \mathcal{K}(\mathcal{X}) \rightarrow \mathcal{B}(\mathcal{H})$ so that $\psi_t(\theta_{\xi,\eta}) = t(\xi)t(\eta)^*$, for $\xi, \eta \in \mathcal{X}$. Following Katsura [**7**], we say that the representation (π, t) is *covariant* iff $\psi_t(\phi_{\mathcal{X}}(a)) = \pi(a)$, for all $a \in \mathcal{J}_{\mathcal{X}}$, where

$$\mathcal{J}_{\mathcal{X}} \equiv \phi_{\mathcal{X}}^{-1}(\mathcal{K}(\mathcal{X})) \cap (\ker \phi_{\mathcal{X}})^{\perp}.$$

If (π, t) is a representation of $\mathcal{X}$ then the C*-algebra (resp. norm closed algebra) generated by the images of π and t is denoted as $\mathrm{C}^*(\pi, t)$ (resp. $\mathrm{alg}((\pi, t))$. There is a universal representation $(\overline{\pi}_{\mathcal{X}}, \overline{t}_{\mathcal{X}})$ for $\mathcal{X}$ and the C*-algebra $\mathrm{C}^*(\overline{\pi}_{\mathcal{X}}, \overline{t}_{\mathcal{X}})$ is the Toeplitz-Cuntz-Pimsner algebra $\mathcal{T}_{\mathcal{X}}$. Similarly, the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{X}}$ is the C*-algebra generated by the image of the universal covariant representation $(\pi_{\mathcal{X}}, t_{\mathcal{X}})$ for $\mathcal{X}$.

A concrete presentation of both $\mathcal{T}_{\mathcal{X}}$ and $\mathcal{O}_{\mathcal{X}}$ can be given in terms of the generalized Fock space $\mathcal{F}_{\mathcal{X}}$ which we now describe. The *Fock space* $\mathcal{F}_{\mathcal{X}}$ over the correspondence $\mathcal{X}$ is defined to be the direct sum of the $\mathcal{X}^{\otimes n}$ with the structure of a direct sum of C*-correspondences over $\mathcal{A}$,

$$\mathcal{F}_{\mathcal{X}} = \mathcal{A} \bigoplus \mathcal{X} \bigoplus \mathcal{X}^{\otimes 2} \bigoplus \cdots .$$

Given $\xi \in \mathcal{X}$, the (left) creation operator $t_\infty(\xi) \in \mathcal{L}(\mathcal{F}_\mathcal{X})$ is defined by the formula

$$t_\infty(\xi)(a, \zeta_1, \zeta_2, \dots) = (0, \xi a, \xi \otimes \zeta_1, \xi \otimes \zeta_2, \dots),$$

where $\zeta_n \in \mathcal{X}^{\otimes n}$, $n \in \mathbb{N}$. Also, for $a \in \mathcal{A}$, we define $\pi_\infty(a) \in \mathcal{L}(\mathcal{F}_\mathcal{X})$ to be the diagonal operator with $\pi_\mathcal{X}(a) \otimes id_{n-1}$ at its $\mathcal{X}^{\otimes n}$-th entry. It is easy to verify that (π_∞, t_∞) is a representation of $\mathcal{X}$ which is called the *Fock representation* of $\mathcal{X}$. Fowler and Raeburn [**4**] (resp. Katsura [**7**]) have shown that the C*-algebra $\mathrm{C}^*(\pi_\infty, t_\infty)$ (resp $\mathrm{C}^*(\pi_\infty, t_\infty)/\mathcal{K}(\mathcal{F}_{\mathcal{X}\mathcal{J}_\mathcal{X}})$) is isomorphic to $\mathcal{T}_\mathcal{X}$ (resp. $\mathcal{O}_\mathcal{X}$).

Definition 2.1. The *tensor algebra* of a C*-correspondence $\mathcal{X}$ over $\mathcal{A}$ is the norm-closed algebra $\mathrm{alg}(\overline{\pi}_\mathcal{X}, \overline{t}_\mathcal{X})$ and is denoted as $\mathcal{T}_\mathcal{X}^+$.

According to [**4**], the algebras $\mathcal{T}_\mathcal{X}^+ \equiv \mathrm{alg}(\overline{\pi}_\mathcal{X}, \overline{t}_\mathcal{X})$ and $\mathrm{alg}(\pi_\infty, t_\infty)$ are completely isometrically isomorphic and we will therefore identify them. The main result of this paper implies that $\mathcal{T}_\mathcal{X}^+$ is also completely isometrically isomorphic to $\mathrm{alg}(\pi_\mathcal{X}, t_\mathcal{X})$.

3. Main Result

We begin with a useful description of the norm in $\mathcal{X}^{(n)}$.

Lemma 3.1. *Let $\mathcal{X}$, $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and let $\phi : \mathcal{A} \to \mathcal{L}(\mathcal{Y})$ be an injective $*$-homomorphism. If $(\xi_i)_{i=1}^n \in \mathcal{X}^{(n)}$, then*

$$\|(\xi_i)_{i=1}^n\| = \sup\{\|(\xi_i \otimes_\phi u)_{i=1}^n\| \mid u \in \mathcal{Y}, \|u\| = 1\}. \tag{1}$$

Proof. Let us denote by M the supremum in (1). Then, using the fact that ϕ is injective and therefore isometric,

$$\begin{aligned}
M^2 &= \sup\{\|\sum_{i=1}^n \langle u|\phi(\langle \xi_i|\xi_i\rangle)u\rangle\| \mid u \in \mathcal{Y}, \|u\| = 1\} \\
&= \sup\{\|(\phi(\langle \xi_i|\xi_i\rangle^{1/2})u)_i\|^2 \mid u \in \mathcal{Y}, \|u\| = 1\} \\
&= \left\| \begin{pmatrix} 0 & 0 & \dots & \phi(\langle \xi_1|\xi_1\rangle^{1/2}) \\ 0 & 0 & \dots & \phi(\langle \xi_2|\xi_2\rangle^{1/2}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi(\langle \xi_n|\xi_n\rangle^{1/2}) \end{pmatrix} \right\|^2 \\
&= \|\phi(\sum_{i=1}^n \langle \xi_i|\xi_i\rangle)\| \\
&= \|(\xi)_i\|^2
\end{aligned}$$

and the conclusion follows. ■

In the proof of our next lemma we make use of the right creation operators. If $\mathcal{Y}$ be a C*-correspondence over $\mathcal{A}$ and $\xi \in \mathcal{Y}^{\otimes k}$, then define the right creation operator R_ξ by the formula

$$R_\xi(a, \zeta_1, \zeta_2, \dots) = (\underbrace{0, 0, \dots, 0}_{k}, \phi_{\mathcal{X}}(a) \otimes id_{k-1}(\xi), \zeta_1 \otimes \xi, \zeta_2 \otimes \xi, \dots),$$

$\zeta_n \in \mathcal{Y}^{\otimes n}$, $n \in \mathbb{N}$. The operator, R_ξ may not be adjointable but it is nevertheless bounded by $\|\xi\|$ and commutes with C$^*(\pi_\infty, t_\infty)$.

Lemma 3.2. *If $\mathcal{X}$ be a faithful* C**-correspondence over $\mathcal{A}$, then*

$$\|A\| = \inf\{\|A + K\| \mid K \in M_n(\mathcal{K}(\mathcal{F}_{\mathcal{X}}))\}$$

for all $A \in M_n(\mathcal{T}_{\mathcal{X}}^+)$, $n \in \mathbb{N}$.

Proof. Let $K \in M_n(\mathcal{K}(\mathcal{F}_{\mathcal{X}}))$ be an $n \times n$ matrix with entries in $\mathcal{K}(\mathcal{F}_{\mathcal{X}})$ and let $\epsilon > 0$. We choose unit vector $\xi \in \mathcal{F}_{\mathcal{X}}^{(n)}$ so that $\|A\xi\| \geq \|A\| - \epsilon$. Since $K \in M_n(\mathcal{K}(\mathcal{F}_{\mathcal{X}}))$, there exists $k \in \mathbb{N}$ so that $\|KR_u^{(n)}\| \leq \epsilon$, for all unit vectors $u \in \mathcal{X}^{\otimes k}$. (Here $R_u^{(n)}$ denotes the the n-th ampliation of the right creation operator R_u.) Note that for any vector $u \in \mathcal{X}^{\otimes k}$ we have

$$\|R_u^{(n)} A\xi\| = \|A\xi \otimes u\|.$$

Therefore, using Lemma 3.1, we choose unit vector $u \in \mathcal{X}^{\otimes k}$ so that

$$\begin{aligned} \|R_u^{(n)} A\xi\| &\geq \|A\xi\| - \epsilon \\ &\geq \|A\| - 2\epsilon. \end{aligned}$$

We compute,

$$\begin{aligned} \|A + K\| &\geq \|(A + K)R_u^{(n)}\xi\| \\ &\geq \|AR_u^{(n)}\xi\| - \epsilon \\ &= \|R_u^{(n)} A\xi\| - \epsilon \\ &\geq \|A\| - 3\epsilon. \end{aligned}$$

Since ϵ and K are arbitrary, the proof is complete. ∎

Corollary 3.3. *Let $\mathcal{X}$ be a faithful* C**-correspondence over $\mathcal{A}$, and let $(\pi_{\mathcal{X}}, t_{\mathcal{X}})$ be the universal covariant representation of $\mathcal{X}$. Then, there exists a complete isometry*

$$\tau_{\mathcal{X}} : \mathcal{T}_{\mathcal{X}}^+ \longrightarrow \mathrm{alg}(\pi_{\mathcal{X}}, t_{\mathcal{X}})$$

so that $\tau_{\mathcal{X}}(\pi_\infty(a)) = \pi_{\mathcal{X}}(a)$, for all $a \in \mathcal{A}$, and $\tau_{\mathcal{X}}(t_\infty(\xi)) = t_{\mathcal{X}}(\xi)$, for all $\xi \in \mathcal{X}$.

In particular, the algebra $\mathrm{alg}(\pi_{\mathcal{X}}, t_{\mathcal{X}})$ *is completely isometrically isomorphic to the tensor algebra $\mathcal{T}_{\mathcal{X}}^+$.*

Proof. Let $\tau_{\mathcal{X}}$ be the restriction of the natural quotient map

$$\mathrm{C}^*(\pi_\infty, t_\infty) \longrightarrow \mathrm{C}^*(\pi_\infty, t_\infty)/\mathcal{K}(\mathcal{F}_{\mathcal{X}\mathcal{J}_{\mathcal{X}}})$$

on the non-selfadjoint subalgebra $\mathrm{alg}(\pi_\infty, t_\infty)$. By Lemma 3.2, this map is a complete isometry. ∎

Remark 3.4. Note that the above lemma already implies the result of Fowler, Muhly and Raeburn [**5**, Theorem 5.3.] without their requirement of $\mathcal{X}$ being strict.

We now remove the requirement of $\mathcal{X}$ being faithful from the statement of the above Lemma. In the special case of a graph correspondence, this was done in [**9**] with the help of a well-known process called "adding tails to a graph". This process has been generalized to arbitrary correspondences by Muhly and Tomforde [**16**]. Indeed, let $\mathcal{X}$ be an arbitrary C*-correspondence over $\mathcal{A}$ and let $\mathfrak{T} \equiv c_0(\ker \phi_{\mathcal{X}})$ consist of all null sequences in $\ker \phi_{\mathcal{X}}$. Muhly and Tomforde show that there exists a well defined left action of $\mathcal{B} \equiv \mathcal{A} \oplus \mathfrak{T}$ on $\mathcal{Y} \equiv \mathcal{X} \oplus \mathfrak{T}$ so that $\mathcal{Y}$ becomes a *faithful* C*-correspondence over $\mathcal{B}$. One can view $\mathcal{A}$ and the C*-correspondence $\mathcal{X}$ as a subsets of $\mathcal{B}$ and $\mathcal{Y}$ respectively, via the identifications

$$\mathcal{A} \ni a \longrightarrow (a, 0) \in \mathcal{A} \oplus 0,$$
$$\mathcal{X} \ni \xi \longrightarrow (\xi, 0) \in \mathcal{X} \oplus 0.$$

and by noting that the action of $\phi_{\mathcal{Y}}$ on $\mathcal{A} \oplus 0$ coincides with that of $\phi_{\mathcal{X}}$ on $\mathcal{A}$. (The restriction of a representation (π, t) of $\mathcal{Y}$ on that subset of $\mathcal{Y}$ will be denoted as $(\pi_{|\mathcal{X}}, t_{|\mathcal{X}})$ and is indeed a representation of $\mathcal{X}$.) In [**16**, Theorem 4.3.(b)] it is shown that if (π, t) is a covariant representation of $\mathcal{Y}$, then $(\pi_{|\mathcal{X}}, t_{|\mathcal{X}})$ is a covariant representation of $\mathcal{X}$.

Lemma 3.5. *Let $\mathcal{X}$ be a* C*-*correspondence over $\mathcal{A}$, and let $(\pi_{\mathcal{X}}, t_{\mathcal{X}})$ be the universal covariant representation of $\mathcal{X}$. Then, there exists a complete isometry*

$$\tau_{\mathcal{X}} : \mathcal{T}^+_{\mathcal{X}} \longrightarrow \mathrm{alg}(\pi_{\mathcal{X}}, t_{\mathcal{X}})$$

so that $\tau_{\mathcal{X}}(\pi_\infty(a)) = \pi_{\mathcal{X}}(a)$, for all $a \in \mathcal{A}$, and $\tau_{\mathcal{X}}(t_\infty(\xi)) = t_{\mathcal{X}}(\xi)$, for all $\xi \in \mathcal{X}$.

Proof. Let (π_∞, t_∞) be the Fock representation of $\mathcal{Y}$ and note that [**7**, Corollary 4.5] shows that

$$\pi_\infty(\mathcal{B}) \bigcap \psi_{t_\infty}(\mathcal{K}(\mathcal{Y})) = \{0\}.$$

Therefore, the restriction $(\pi_{\infty|\mathcal{X}}, t_{\infty|\mathcal{X}})$ satisfies the same property and so [**7**, Theorem 6.2] implies that the integrated representation $\pi_{\infty|\mathcal{X}} \times$

$t_{\infty|\mathcal{X}}$ is a C*-isomorphism from the universal Toeplitz algebra $\mathcal{T}_{\mathcal{X}}$ onto $\mathrm{C}^*(\pi_{\infty|\mathcal{X}}, t_{\infty|\mathcal{X}})$. We therefore view $\mathcal{T}_{\mathcal{X}}^+$ as a subalgebra of $\mathcal{T}_{\mathcal{Y}}^+$.

Corollary 3.3 shows now that there exists a complete isometry

$$\tau_{\mathcal{Y}} : \mathcal{T}_{\mathcal{Y}}^+ \longrightarrow \mathrm{alg}(\pi_{\mathcal{Y}}, t_{\mathcal{Y}})$$

so that $\tau_{\mathcal{Y}}(\phi_\infty(b)) = \phi_{\mathcal{Y}}(b)$, for all $b \in \mathcal{B}$, and $\tau_{\mathcal{Y}}(\phi_\infty(\xi)) = \phi_{\mathcal{Y}}(\xi)$, for all $\xi \in \mathcal{Y}$. As we discussed earlier, [**16**, Theorem 4.3.(b)] shows that the restriction $(\pi_{\mathcal{Y}|\mathcal{X}}, t_{\mathcal{Y}|\mathcal{X}})$ is covariant for $\mathcal{X}$. Since it is also injective, the gauge invariant uniqueness theorem [**7**, Theorem 6.4] shows that the restriction $\tau_{\mathcal{X}} \equiv \tau_{\mathcal{Y}}|_{\mathcal{T}_{\mathcal{X}}}$ has range isomorphic to $\mathrm{alg}(\pi_{\mathcal{X}}, t_{\mathcal{X}})$ and satisfies the desired properties. ∎

Let $\mathcal{B}$ be a C*-algebra and let $\mathcal{B}^+$ be a (nonselfadjoint) subalgebra of $\mathcal{B}$ which generates $\mathcal{B}$ as a C*-algebra and contains a two-sided contractive approximate unit for $\mathcal{B}$, i.e., $\mathcal{B}^+$ is an essential subalgebra for $\mathcal{B}$. A two-sided ideal $\mathcal{J}$ of $\mathcal{B}^+$ is said to be a *boundary ideal* for $\mathcal{B}^+$ if and only if the quotient map $\pi : \mathcal{B} \to \mathcal{B}/\mathcal{J}$ is a complete isometry when restricted to $\mathcal{B}^+$. It is a result of Hamana [**6**], following the seminal work of Arveson [**1**], that there exists a boundary ideal $\mathcal{J}_S(\mathcal{B}^+)$, the *Shilov boundary ideal*, that contains all other boundary ideals. In that case, the quotient $\mathcal{B}/\mathcal{J}_S(\mathcal{B}^+)$ is called the C*-*envelope* of $\mathcal{B}^+$ and it is denoted as $\mathrm{C}^*_{\mathrm{env}}(\mathcal{B}^+)$. The C*-envelope is unique in the following sense: Assume that $\phi' : \mathcal{B}^+ \to \mathcal{B}'$ is a completely isometric isomorphism of $\mathcal{B}^+$ onto an essential subalgebra of a C*-algebra $\mathcal{B}'$ and suppose that the Shilov boundary for $\phi'(\mathcal{B}^+) \subseteq \mathcal{B}'$ is zero. Then $\mathcal{B}$ and $\mathcal{B}'$ are $*$-isomorphic, via an isomorphism ϕ so that $\phi(\pi(x)) = \phi'(x)$, for all $x \in \mathcal{B}$.

In the case where an operator algebra $\mathcal{B}^+$ has no contractive approximate identity, the $\mathrm{C}^*_{\mathrm{env}}(\mathcal{B}^+)$ is defined by utilizing the unitization [**11**] $(\mathcal{B}^+)_1$ of $\mathcal{B}^+$: the C*-envelope of $\mathcal{B}^+$ is the C*-subalgebra of $\mathrm{C}^*_{\mathrm{env}}(\mathcal{B}^+)_1$ generated by $\mathcal{B}^+$. (See [**2, 3**] for a comprehensive discussion regarding the implications of [**11**] on the theory of C*-envelopes.)

Lemma 3.6. *Let $\mathcal{B}$ be a non-unital* C*-*algebra and let $\mathcal{J} \subseteq \mathcal{B}_1$ be a closed two-sided ideal in its unitization. If $\mathcal{J} \cap \mathcal{B} = \{0\}$ then $\mathcal{J} = \{0\}$.*

Proof. Assume that $\mathcal{J} \neq \{0\}$. Since $\mathcal{B}_1 \subseteq \mathcal{B}$ has codimension 1, $\mathcal{J}$ is of the form $\mathcal{J} = [\{B + \lambda I\}]$, for some $B \in \mathcal{B}$ and non-zero $\lambda \in \mathbb{C}$. Then, easy manipulations show that there is no loss of generality assuming that $\lambda \in \mathbb{R}$ (because $\mathcal{J}\mathcal{J}^* \neq 0$), A is selfadjoint (because $\mathcal{J} \cap \mathcal{J}^* \neq 0$) and

$$(A + \lambda I)^2 = A + \lambda, \tag{2}$$

after perhaps scaling (since $\mathcal{J}^2 \neq 0$). It is easy to see now that (2) implies that $A = -P$, for some projection $P \in \mathcal{B}$. But then, $(I-P)\mathcal{B} = 0$ and so P is a unit for $\mathcal{B}$, a contradiction. ■

We have arrived to the main result of the paper.

Theorem 3.7. *If $\mathcal{X}$ is a* C*-*correspondence over $\mathcal{A}$, then the* C*-*envelope of $\mathcal{T}_{\mathcal{X}}^{+}$ coincides with the universal Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{X}}$.*

Proof. According to Lemma 3.5, it suffices to show that the C*-envelope of $\mathrm{alg}(\pi_{\mathcal{X}}, t_X)$ equals $\mathcal{O}_{\mathcal{X}}$.

Assume first that $\mathrm{alg}(\pi_{\mathcal{X}}, t_X)$ is unital. In light of the above discussion, we need to verify that the Shilov boundary ideal $\mathcal{J}_S(\mathrm{alg}(\pi_{\mathcal{X}}, t_X))$ is zero. However, the maximality of $\mathcal{J}_S(\mathrm{alg}(\pi_{\mathcal{X}}, t_X))$ and the invariance of $\mathrm{alg}(\pi_{\mathcal{X}}, t_X)$ under the gauge action of $\mathbb{T}$ on $\mathcal{O}_{\mathcal{X}}$ imply that $\mathcal{J}_S(\mathrm{alg}(\pi_{\mathcal{X}}, t_X)\,)$ is a gauge-invariant ideal. By the gauge invariant uniqueness theorem [**7**, Theorem 6.4], any non-zero gauge-invariant ideal has non-zero intersection with $\pi_{\mathcal{X}}(\mathcal{A})$. Hence $\mathcal{J}_S(\mathrm{alg}(\pi_{\mathcal{X}}, t_X)\,) = \{0\}$, or otherwise the quotient map would not be faithful on $\mathrm{alg}(\pi_{\mathcal{X}}, t_X)$.

Assume now that $\mathrm{alg}(\pi_{\mathcal{X}}, t_X)$ is not unital. We distinguish two cases.

If $\mathcal{O}_{\mathcal{X}}$ has a unit $I \in \mathcal{O}_{\mathcal{X}}$ then let

$$\mathrm{alg}(\pi_{\mathcal{X}}, t_X)_1 \equiv \mathrm{alg}(\pi_{\mathcal{X}}, t_X) + \mathbb{C}I \subseteq \mathcal{O}_{\mathcal{X}}.$$

Clearly, $\mathrm{alg}(\pi_{\mathcal{X}}, t_X)_1$ is gauge invariant and so a repetition of the arguments in the second paragraph of the proof shows that

$$\mathrm{C}^*_{\mathrm{env}}(\mathrm{alg}(\pi_{\mathcal{X}}, t_X))_1 = \mathcal{O}_{\mathcal{X}}.$$

The C*-subalgebra of $\mathcal{O}_{\mathcal{X}}$ generated by $\mathrm{alg}(\pi_{\mathcal{X}}, t_X)$ equals $\mathcal{O}_{\mathcal{X}}$, which by convention will be its C*-envelope.

Finally, if $\mathcal{O}_{\mathcal{X}}$ does not have a unit then unitize $\mathcal{O}_{\mathcal{X}}$ by joining a unit I and let

$$\mathrm{alg}(\pi_{\mathcal{X}}, t_X)_1 \equiv \mathrm{alg}(\pi_{\mathcal{X}}, t_X) + \mathbb{C}I \subseteq \mathcal{O}_{\mathcal{X}} + \mathbb{C}I.$$

Since the Shilov ideal $\mathcal{J}_S((\mathrm{alg}(\pi_{\mathcal{X}}, t_X))_1)$ is gauge invariant,

$$\mathcal{J}_S(\mathrm{alg}(\pi_{\mathcal{X}}, t_X)_1) \cap \mathcal{O}_{\mathcal{X}} \subseteq \mathcal{O}_{\mathcal{X}}$$

is gauge invariant. Therefore,

$$\mathcal{J}_S(\mathrm{alg}(\pi_{\mathcal{X}}, t_X)_1) \cap \mathcal{O}_{\mathcal{X}} = \{0\},$$

or else it meets $\pi_{\mathcal{X}}(\mathcal{A})$. By Lemma 3.6, $\mathcal{J}_S(\mathrm{alg}(\pi_{\mathcal{X}}, t_X)_1) = \{0\}$ and so $\mathrm{C}^*_{\mathrm{env}}((\mathrm{alg}(\pi_{\mathcal{X}}, t_X))_1) = \mathcal{O}_{\mathcal{X}} + \mathbb{C}I$. The C*-subalgebra of $\mathcal{O}_{\mathcal{X}} + \mathbb{C}I$ generated by $\mathrm{alg}(\pi_{\mathcal{X}}, t_X)$ is $\mathcal{O}_{\mathcal{X}}$, and the conclusion follows. ■

Remark 3.8. In [**5**, page 596], it is claimed that if a $\mathcal{X}$ is a C^*-correspondence over $\mathcal{A}$, with universal Toeplitz representation $(\overline{\pi}_{\mathcal{X}}, \overline{t}_{\mathcal{X}})$, then $\overline{\pi}_{\mathcal{X}}$ maps an approximate unit of $\mathcal{A}$ to an approximate unit for both $\mathcal{T}_{\mathcal{X}}$ and $\mathcal{T}^+_{\mathcal{X}}$. It is not hard to see that this claim is valid if and only if $\phi_{\mathcal{X}}$ is non-degenerate. Therefore, there is a gap in the proof of [**5**, Theorem 5.3] in the case where $\mathcal{X}$ is strict but not essential. Nevertheless, our Theorem 3.7 incorporates all possible cases and hence completes the proof of [**5**, Theorem 5.3].

We now obtain one of the main results of [**9**] as a corollary.

Corollary 3.9. [**9**, Theorem 2.5] *If G is a countable directed graph then the* C^**-envelope of $\mathcal{T}_+(G)$ coincides with the universal Cuntz-Krieger algebra associated with G.*

Note that in [**9**], the proof of the above corollary is essentially self-contained and avoids the heavy machinery used in this paper. The reader would actually benefit from reading that proof and then making comparisons with the proof of Theorem 3.7 here.

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