TENSOR ALGEBRAS OF C*-CORRESPONDENCES
AND THEIR C*-ENVELOPES.

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Abstract. We show that the C*-envelope of the tensor algebra of an arbitrary C*-correspondence \( X \) coincides with the Cuntz-Pimsner algebra \( O_\mathcal{X} \), as defined by Katsura [7]. This improves earlier results of Muhly and Solel [13] and Fowler, Muhly and Raeburn [5], who came to the same conclusion under the additional hypothesis that \( X \) is strict and faithful.

1. Introduction

Fowler, Muhly and Raeburn have recently characterized [5, Theorem 5.3.] the C*-envelope of the tensor algebra \( T_X^+ \) of a faithful and strict C*-correspondence \( \mathcal{X} \), as the associated universal Cuntz-Pimsner algebra. Their proof is based on a gauge invariant uniqueness theorem and earlier elaborate results of Muhly and Solel [13]. Beyond faithful strict C*-correspondences, little is known: if \( \mathcal{X} \) is strict, but not necessary faithful, then the C*-envelope of \( T_X^+ \) is known to be a quotient of the associated Toeplitz-Cuntz-Pimsner algebra, without any further information (Theorem 6.4 in [13]). In [5, Remark 5.4], the authors ask whether the above mentioned conditions on \( \mathcal{X} \) are necessary for the validity of their Theorem 5.3 in [5].

In this note we answer the question of Fowler, Muhly and Raeburn [5] (and Muhly and Solel [13]) by showing that the C*-envelope of the tensor algebra of an arbitrary C*-correspondence \( \mathcal{X} \) coincides with the Cuntz-Pimsner algebra \( O_\mathcal{X} \), as defined by Katsura in [7]. Our proof does not require any of the results from [13] and is modelled upon the proof of our recent result [9] that identifies the C*-envelope of the tensor algebra of a directed graph. We also make use of the result of Muhly and Tomforde [16] that generalizes the process of adding tails to a graph to the context of C*-correspondences.

2000 Mathematics Subject Classification. 47L80, 47L55, 47L40, 46L05.

Key words and phrases. C*-envelope, Cuntz-Pimsner C*-algebra, tensor algebra, Fock space.

Second author was partially supported by an NSERC grant.
2. Preliminaries

Let $\mathcal{A}$ be a $C^*$-algebra and $\mathcal{X}$ be a (right) Hilbert $\mathcal{A}$-module, whose inner product is denoted as $\langle . , . \rangle$. Let $\mathcal{L}(\mathcal{X})$ be the adjointable operators on $\mathcal{X}$ and let $\mathcal{K}(\mathcal{X})$ be the norm closed subalgebra of $\mathcal{L}(\mathcal{X})$ generated by the operators $\theta_{\xi,\eta}$, $\xi, \eta \in \mathcal{X}$, where $\theta_{\xi,\eta}(\zeta) = \xi \langle \eta|\zeta \rangle$, $\zeta \in \mathcal{X}$.

A Hilbert $\mathcal{A}$-module $\mathcal{X}$ is said to be a $C^*$-correspondence over $\mathcal{A}$ provided that there exists a $*$-homomorphism $\phi_{\mathcal{X}} : \mathcal{A} \to \mathcal{L}(\mathcal{X})$. We refer to $\phi_{\mathcal{X}}$ as the left action of a $C^*$-correspondence $\mathcal{X}$. A $C^*$-correspondence $\mathcal{X}$ over $\mathcal{A}$ is said to be essential (resp. faithful) if and only if $\phi_{\mathcal{X}}(\mathcal{A})(\mathcal{X}) = \mathcal{X}$ (resp. $\phi_{\mathcal{X}}$ is faithful).

From a given $C^*$-correspondence $\mathcal{X}$ over $\mathcal{A}$, one can form new $C^*$-correspondences over $\mathcal{A}$, such as the $n$-fold ampliation or direct sum $\mathcal{X}^{(n)}$ ([10, page 5]) and the $n$-fold interior tensor product $\mathcal{X}^\otimes_n \equiv \mathcal{X} \otimes_{\phi_{\mathcal{X}}} \mathcal{X} \otimes_{\phi_{\mathcal{X}}} \cdots \otimes_{\phi_{\mathcal{X}}} \mathcal{X}$ ([10, page 39], $n \in \mathbb{N}$, $(\mathcal{X}^\otimes_0 \equiv \mathcal{A}$). These operations are defined within the category of $C^*$-correspondences over $\mathcal{A}$. (See [10] for more details.)

A representation $(\pi, t)$ of a $C^*$-correspondence $\mathcal{X}$ over $\mathcal{A}$ consists of a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and a linear map $t : \mathcal{X} \to \mathcal{B}(\mathcal{H})$ so that

1. $t(\xi)^* t(\eta) = \pi(\langle \xi|\eta \rangle)$, for $\xi, \eta \in \mathcal{X}$,
2. $\pi(a) t(\xi) = t(\phi_{\mathcal{X}}(a) \xi)$, for $a \in \mathcal{A}$, $\xi \in \mathcal{X}$.

For a representation $(\pi, t)$ of a $C^*$-correspondence $\mathcal{X}$ there exists a $*$-homomorphism $\psi_t : \mathcal{K}(\mathcal{X}) \to \mathcal{B}(\mathcal{H})$ so that $\psi_t(\theta_{\xi,\eta}) = t(\xi) t^*(\eta)$, for $\xi, \eta \in \mathcal{X}$. Following Katsura [7], we say that the representation $(\pi, t)$ is covariant iff $\psi_t(\phi_{\mathcal{X}}(a)) = \pi(a)$, for all $a \in \mathcal{J}_X$, where

$$\mathcal{J}_X \equiv \phi_{\mathcal{X}}^{-1}(\mathcal{K}(\mathcal{X})) \cap (\ker \phi_{\mathcal{X}})^\perp.$$

If $(\pi, t)$ is a representation of $\mathcal{X}$ then the $C^*$-algebra (resp. norm closed algebra) generated by the images of $\pi$ and $t$ is denoted as $C^*(\pi, t)$ (resp. alg$(\pi, t)$). There is a universal representation $(\bar{\pi}_X, \bar{t}_X)$ for $\mathcal{X}$ and the $C^*$-algebra $C^*(\bar{\pi}_X, \bar{t}_X)$ is the Toeplitz-Cuntz-Pimsner algebra $\mathcal{T}_X$. Similarly, the Cuntz-Pimsner algebra $\mathcal{O}_X$ is the $C^*$-algebra generated by the image of the universal covariant representation $(\pi_X, t_X)$ for $\mathcal{X}$.

A concrete presentation of both $\mathcal{T}_X$ and $\mathcal{O}_X$ can be given in terms of the generalized Fock space $\mathcal{F}_X$ which we now describe. The Fock space $\mathcal{F}_X$ over the correspondence $\mathcal{X}$ is defined to be the direct sum of the $\mathcal{X}^\otimes_n$ with the structure of a direct sum of $C^*$-correspondences over $\mathcal{A}$,

$$\mathcal{F}_X = \mathcal{A} \bigoplus \mathcal{X} \bigoplus \mathcal{X}^\otimes_2 \bigoplus \cdots.$$
Given $\xi \in \mathcal{X}$, the (left) creation operator $t_\infty(\xi) \in \mathcal{L}(\mathcal{F}_\mathcal{X})$ is defined by the formula

$$t_\infty(\xi)(a, \zeta_1, \zeta_2, \ldots) = (0, \xi a, \xi \otimes \zeta_1, \xi \otimes \zeta_2, \ldots),$$

where $\zeta_n \in \mathcal{X}^{\otimes n}$, $n \in \mathbb{N}$. Also, for $a \in \mathcal{A}$, we define $\pi_\infty(a) \in \mathcal{L}(\mathcal{F}_\mathcal{X})$ to be the diagonal operator with $\pi_\mathcal{X}(a) \otimes id_{n-1}$ at its $\mathcal{X}^{\otimes n}$-th entry. It is easy to verify that $(\pi_\infty, t_\infty)$ is a representation of $\mathcal{X}$ which is called the Fock representation of $\mathcal{X}$. Fowler and Raeburn [4] (resp. Katsura [7]) have shown that the $C^*$-algebra $C^*(\pi_\infty, t_\infty)$ (resp. $C^*(\pi_\infty, t_\infty)/\mathcal{K}(\mathcal{F}_\mathcal{X}\mathcal{F}_\mathcal{X})$) is isomorphic to $\mathcal{T}_\mathcal{X}$ (resp. $\mathcal{O}_\mathcal{X}$).

**Definition 2.1.** The tensor algebra of a $C^*$-correspondence $\mathcal{X}$ over $\mathcal{A}$ is the norm-closed algebra $\text{alg}(\pi_\mathcal{X}, \tilde{t}_\mathcal{X})$ and is denoted as $\mathcal{T}_\mathcal{X}^+$.

According to [4], the algebras $\mathcal{T}_\mathcal{X}^+ \equiv \text{alg}(\pi_\mathcal{X}, \tilde{t}_\mathcal{X})$ and $\text{alg}(\pi_\infty, t_\infty)$ are completely isometrically isomorphic and we will therefore identify them. The main result of this paper implies that $\mathcal{T}_\mathcal{X}^+$ is also completely isometrically isomorphic to $\text{alg}(\pi_\mathcal{X}, t_\mathcal{X})$.

3. **Main Result**

We begin with a useful description of the norm in $\mathcal{X}^{(n)}$.

**Lemma 3.1.** Let $\mathcal{X}$, $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and let $\phi : \mathcal{A} \to \mathcal{L}(\mathcal{Y})$ be an injective $*$-homomorphism. If $(\xi_i)_{i=1}^n \in \mathcal{X}^{(n)}$, then

$$(1) \quad \|(\xi_i)_{i=1}^n\| = \sup\{\|(\xi_i \otimes \phi \ u)_{i=1}^n\| \mid u \in \mathcal{Y}, \|u\| = 1\}.$$  

**Proof.** Let us denote by $M$ the supremum in (1). Then, using the fact that $\phi$ is injective and therefore isometric,

$$M^2 = \sup\{\|\sum_{i=1}^n \langle u|\phi(\langle \xi_i|\xi_i\rangle)u\rangle\| \mid u \in \mathcal{Y}, \|u\| = 1\}$$

$$= \sup\{\|\phi(\langle \xi_i|\xi_i\rangle^{1/2})u\|_2 \mid u \in \mathcal{Y}, \|u\| = 1\}$$

$$= \|\phi(\sum_{i=1}^n \langle \xi_i|\xi_i\rangle)\|$$

$$= \|\phi(\sum_{i=1}^n \|\xi_i\|^2)\|$$

and the conclusion follows. \blacksquare
In the proof of our next lemma we make use of the right creation operators. If \( Y \) be a \( \mathcal{C}^* \)-correspondence over \( A \) and \( \xi \in Y \otimes^k \), then define the right creation operator \( R_\xi \) by the formula
\[
R_\xi(a, \zeta_1, \zeta_2, \ldots) = (0, 0, \ldots, 0, \phi_X(a) \otimes id_{k-1}(\xi), \zeta_1 \otimes \xi, \zeta_2 \otimes \xi, \ldots),
\]
\( \zeta_n \in Y \otimes^n, \ n \in \mathbb{N} \). The operator, \( R_\xi \) may not be adjointable but it is nevertheless bounded by \( \|\xi\| \) and commutes with \( \mathcal{C}^*(\pi_\infty, t_\infty) \).

**Lemma 3.2.** If \( X \) be a faithful \( \mathcal{C}^* \)-correspondence over \( A \), then
\[
\|A\| = \inf \{\|A + K\| | K \in M_n(\mathcal{K}(\mathcal{F}_X))\}
\]
for all \( A \in M_n(T^+_X), \ n \in \mathbb{N} \).

**Proof.** Let \( K \in M_n(\mathcal{K}(\mathcal{F}_X)) \) be an \( n \times n \) matrix with entries in \( \mathcal{K}(\mathcal{F}_X) \) and let \( \epsilon > 0 \). We choose unit vector \( \xi \in \mathcal{F}_X^{(n)} \) so that \( \|A\xi\| \geq \|A\| - \epsilon \).

Since \( K \in M_n(\mathcal{K}(\mathcal{F}_X)) \), there exists \( k \in \mathbb{N} \) so that \( \|KR_u^{(n)}\| \leq \epsilon \), for all unit vectors \( u \in \mathcal{X} \otimes^k \). (Here \( R_u^{(n)} \) denotes the the \( n \)-th ampliation of the right creation operator \( R_u \).) Note that for any vector \( u \in \mathcal{X} \otimes^k \) we have
\[
\|R_u^{(n)}A\xi\| = \|A\xi \otimes u\|.
\]
Therefore, using Lemma 3.1, we choose unit vector \( u \in \mathcal{X} \otimes^k \) so that
\[
\|R_u^{(n)}A\xi\| \geq \|A\xi\| - \epsilon \\
\quad \geq \|A\| - 2\epsilon.
\]
We compute,
\[
\|A + K\| \geq \|(A + K)R_u^{(n)}\\xi\| \\
\quad \geq \|AR_u^{(n)}\xi\| - \epsilon \\
\quad = \|R_u^{(n)}A\xi\| - \epsilon \\
\quad \geq \|A\| - 3\epsilon.
\]
Since \( \epsilon \) and \( K \) are arbitrary, the proof is complete. \( \blacksquare \)

**Corollary 3.3.** Let \( X \) be a faithful \( \mathcal{C}^* \)-correspondence over \( A \), and let \( (\pi_X, t_X) \) be the universal covariant representation of \( X \). Then, there exists a complete isometry
\[
\tau_X : T^+_X \longrightarrow \text{alg}(\pi_X, t_X)
\]
so that \( \tau_X(\pi_\infty(a)) = \pi_X(a), \) for all \( a \in A \), and \( \tau_X(t_\infty(\xi)) = t_X(\xi), \) for all \( \xi \in \mathcal{X} \).

In particular, the algebra \( \text{alg}(\pi_X, t_X) \) is completely isometrically isomorphic to the tensor algebra \( T^+_X \).
**Proof.** Let $\tau_X$ be the restriction of the natural quotient map
\[ C^*(\pi_{\infty}, t_{\infty}) \longrightarrow C^*(\pi_{\infty}, t_{\infty})/\mathcal{K}(\mathcal{F}_{\mathcal{X}J}) \]
on the non-selfadjoint subalgebra $\text{alg}(\pi_{\infty}, t_{\infty})$. By Lemma 3.2, this map is a complete isometry. □

**Remark 3.4.** Note that the above lemma already implies the result of Fowler, Muhly and Raeburn [5, Theorem 5.3.] without their requirement of $\mathcal{X}$ being strict.

We now remove the requirement of $\mathcal{X}$ being faithful from the statement of the above Lemma. In the special case of a graph correspondence, this was done in [9] with the help of a well-known process called "adding tails to a graph". This process has been generalized to arbitrary correspondences by Muhly and Tomforde [16]. Indeed, let $\mathcal{X}$ be an arbitrary $C^*$-correspondence over $\mathcal{A}$ and let $\mathcal{Y} \equiv c_0(\ker \phi_\mathcal{X})$ consist of all null sequences in $\ker \phi_\mathcal{X}$. Muhly and Tomforde show that there exists a well defined left action of $\mathcal{B} \equiv \mathcal{A} \oplus \mathcal{Y}$ on $\mathcal{X} \oplus \mathcal{Y}$ so that $\mathcal{Y}$ becomes a **faithful** $C^*$-correspondence over $\mathcal{B}$. One can view $\mathcal{A}$ and the $C^*$-correspondence $\mathcal{X}$ as subsets of $\mathcal{B}$ and $\mathcal{Y}$ respectively, via the identifications
\[ \mathcal{A} \ni a \longrightarrow (a, 0) \in \mathcal{A} \oplus 0, \]
\[ \mathcal{X} \ni \xi \longrightarrow (\xi, 0) \in \mathcal{X} \oplus 0. \]
and by noting that the action of $\phi_\mathcal{Y}$ on $\mathcal{A} \oplus 0$ coincides with that of $\phi_\mathcal{X}$ on $\mathcal{A}$. (The restriction of a representation $(\pi, t)$ of $\mathcal{Y}$ on that subset of $\mathcal{Y}$ will be denoted as $(\pi|\mathcal{X}, t|\mathcal{Y})$ and is indeed a representation of $\mathcal{X}$.) In [16, Theorem 4.3.(b)] it is shown that if $(\pi, t)$ is a covariant representation of $\mathcal{Y}$, then $(\pi|\mathcal{X}, t|\mathcal{Y})$ is a covariant representation of $\mathcal{X}$.

**Lemma 3.5.** Let $\mathcal{X}$ be a $C^*$-correspondence over $\mathcal{A}$, and let $(\pi_\mathcal{X}, t_\mathcal{X})$ be the universal covariant representation of $\mathcal{X}$. Then, there exists a complete isometry
\[ \tau_\mathcal{X} : \mathcal{T}_\mathcal{X}^+ \longrightarrow \text{alg}(\pi_\mathcal{X}, t_\mathcal{X}) \]
such that $\tau_\mathcal{X}(\pi_{\infty}(a)) = \pi_\mathcal{X}(a)$, for all $a \in \mathcal{A}$, and $\tau_\mathcal{X}(t_{\infty}(\xi)) = t_\mathcal{X}(\xi)$, for all $\xi \in \mathcal{X}$.

**Proof.** Let $(\pi_{\infty}, t_{\infty})$ be the Fock representation of $\mathcal{Y}$ and note that [7, Corollary 4.5] shows that
\[ \pi_{\infty}(\mathcal{B}) \cap \psi_{t_{\infty}}(\mathcal{K}(\mathcal{Y})) = \{0\}. \]
Therefore, the restriction $(\pi_{\infty}|_{\mathcal{X}}, t_{\infty}|_{\mathcal{X}})$ satisfies the same property and so [7, Theorem 6.2] implies that the integrated representation $\pi_{\infty}|_{\mathcal{X}} \times
$t_{\infty}|_\mathcal{X}$ is a $C^*$-isomorphism from the universal Toeplitz algebra $\mathcal{T}_\mathcal{X}$ onto $C^*(\pi_{\infty}|_\mathcal{X}, t_{\infty}|_\mathcal{X})$. We therefore view $\mathcal{T}_\mathcal{X}^+$ as a subalgebra of $\mathcal{T}_\mathcal{Y}^+$.

Corollary 3.3 shows now that there exists a complete isometry

$$\tau_\mathcal{Y} : \mathcal{T}_\mathcal{Y}^+ \longrightarrow \operatorname{alg}(\pi_\mathcal{Y}, t_\mathcal{Y})$$

so that $\tau_\mathcal{Y}(\phi_\infty(b)) = \phi_\mathcal{Y}(b)$, for all $b \in \mathcal{B}$, and $\tau_\mathcal{Y}(\phi_\infty(\xi)) = \phi_\mathcal{Y}(\xi)$, for all $\xi \in \mathcal{Y}$. As we discussed earlier, [16, Theorem 4.3,(b)] shows that the restriction $(\pi_\mathcal{Y}|_\mathcal{X}, t_\mathcal{Y}|_\mathcal{X})$ is covariant for $\mathcal{X}$. Since it is also injective, the gauge invariant uniqueness theorem [7, Theorem 6.4] shows that the restriction $\tau_\mathcal{X} \equiv \tau_\mathcal{Y}|_{\mathcal{T}_\mathcal{X}}$ has range isomorphic to $\operatorname{alg}(\pi_\mathcal{X}, t_\mathcal{X})$ and satisfies the desired properties.

Let $\mathcal{B}$ be a $C^*$-algebra and let $\mathcal{B}^+$ be a (nonselfadjoint) subalgebra of $\mathcal{B}$ which generates $\mathcal{B}$ as a $C^*$-algebra and contains a two-sided contractive approximate unit for $\mathcal{B}$, i.e., $\mathcal{B}^+$ is an essential subalgebra for $\mathcal{B}$. A two-sided ideal $J$ of $\mathcal{B}^+$ is said to be a boundary ideal for $\mathcal{B}^+$ if and only if the quotient map $\pi : \mathcal{B} \to \mathcal{B}/J$ is a complete isometry when restricted to $\mathcal{B}^+$. It is a result of Hamana [6], following the seminal work of Arveson [1], that there exists a boundary ideal $J_{sh}(\mathcal{B}^+)$, the Shilov boundary ideal, that contains all other boundary ideals. In that case, the quotient $\mathcal{B}/J_{sh}(\mathcal{B}^+)$ is called the $C^*$-envelope of $\mathcal{B}^+$ and it is denoted as $C^*_{env}(\mathcal{B}^+)$. The $C^*$-envelope is unique in the following sense: Assume that $\phi' : \mathcal{B}^+ \to \mathcal{B}'$ is a completely isometric isomorphism of $\mathcal{B}^+$ onto an essential subalgebra of a $C^*$-algebra $\mathcal{B}'$ and suppose that the Shilov boundary for $\phi'(\mathcal{B}^+) \subseteq \mathcal{B}'$ is zero. Then $\mathcal{B}$ and $\mathcal{B}'$ are $\ast$-isomorphic, via an isomorphism $\phi$ so that $\phi(\pi(x)) = \phi'(x)$, for all $x \in \mathcal{B}$.

In the case where an operator algebra $\mathcal{B}^+$ has no contractive approximate identity, the $C^*_{env}(\mathcal{B}^+)$ is defined by utilizing the unitization [11] $(\mathcal{B}^+)_{1}$ of $\mathcal{B}^+$: the $C^*$-envelope of $\mathcal{B}^+$ is the $C^*$-subalgebra of $C^*_{env}(\mathcal{B}^+)_{1}$ generated by $\mathcal{B}^+$. (See [2, 3] for a comprehensive discussion regarding the implications of [11] on the theory of $C^*$-envelopes.)

**Lemma 3.6.** Let $\mathcal{B}$ be a non-unital $C^*$-algebra and let $J \subseteq \mathcal{B}_1$ be a closed two-sided ideal in its unitization. If $J \cap \mathcal{B} = \{0\}$ then $J = \{0\}$.

**Proof.** Assume that $J \neq \{0\}$. Since $\mathcal{B}_1 \subseteq \mathcal{B}$ has codimension 1, $J$ is of the form $J = \{\{B + \lambda I\}\}$, for some $B \in \mathcal{B}$ and non-zero $\lambda \in \mathbb{C}$. Then, easy manipulations show that there is no loss of generality assuming that $\lambda \in \mathbb{R}$ (because $J J^* \neq 0$), $A$ is selfadjoint (because $J \cap J^* \neq 0$) and

$$ (A + \lambda I)^2 = A + \lambda, $$

where

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where
after perhaps scaling (since $\mathcal{J}^2 \neq 0$). It is easy to see now that (2) implies that $A = -P$, for some projection $P \in \mathcal{B}$. But then, $(I - P)\mathcal{B} = 0$ and so $P$ is a unit for $\mathcal{B}$, a contradiction. ■

We have arrived to the main result of the paper.

**Theorem 3.7.** If $\mathcal{X}$ is a $C^*$-correspondence over $\mathcal{A}$, then the $C^*$-envelope of $\mathcal{T}_\mathcal{X}^+$ coincides with the universal Cuntz-Pimsner algebra $\mathcal{O}_X$.

**Proof.** According to Lemma 3.5, it suffices to show that the $C^*$-envelope of $\text{alg}(\pi_X, t_X)$ equals $\mathcal{O}_X$.

Assume first that $\text{alg}(\pi_X, t_X)$ is unital. In light of the above discussion, we need to verify that the Shilov boundary ideal $\mathcal{J}_S(\text{alg}(\pi_X, t_X))$ is zero. However, the maximality of $\mathcal{J}_S(\text{alg}(\pi_X, t_X))$ and the invariance of $\text{alg}(\pi_X, t_X)$ under the gauge action of $\mathbb{T}$ on $\mathcal{O}_X$ imply that $\mathcal{J}_S(\text{alg}(\pi_X, t_X))$ is a gauge-invariant ideal. By the gauge invariant uniqueness theorem [7, Theorem 6.4], any non-zero gauge-invariant ideal has non-zero intersection with $\pi_\mathcal{X}(\mathcal{A})$. Hence $\mathcal{J}_S(\text{alg}(\pi_X, t_X)) = \{0\}$, or otherwise the quotient map would not be faithful on $\text{alg}(\pi_X, t_X)$.

Assume now that $\text{alg}(\pi_X, t_X)$ is not unital. We distinguish two cases. If $\mathcal{O}_X$ has a unit $I \in \mathcal{O}_X$ then let

$$\text{alg}(\pi_X, t_X)_1 \equiv \text{alg}(\pi_X, t_X) + CI \subseteq \mathcal{O}_X.$$  

Clearly, $\text{alg}(\pi_X, t_X)_1$ is gauge invariant and so a repetition of the arguments in the second paragraph of the proof shows that

$$C^*_{\text{env}}(\text{alg}(\pi_X, t_X))_1 = \mathcal{O}_X.$$  

The $C^*$-subalgebra of $\mathcal{O}_X$ generated by $\text{alg}(\pi_X, t_X)$ equals $\mathcal{O}_X$, which by convention will be its $C^*$-envelope.

Finally, if $\mathcal{O}_X$ does not have a unit then unitize $\mathcal{O}_X$ by joining a unit $I$ and let

$$\text{alg}(\pi_X, t_X)_1 \equiv \text{alg}(\pi_X, t_X) + CI \subseteq \mathcal{O}_X + CI.$$  

Since the Shilov ideal $\mathcal{J}_S((\text{alg}(\pi_X, t_X))_1)$ is gauge invariant,

$$\mathcal{J}_S(\text{alg}(\pi_X, t_X)_1) \cap \mathcal{O}_X \subseteq \mathcal{O}_X$$  

is gauge invariant. Therefore,

$$\mathcal{J}_S(\text{alg}(\pi_X, t_X)_1) \cap \mathcal{O}_X = \{0\},$$

or else it meets $\pi_\mathcal{X}(\mathcal{A})$. By Lemma 3.6, $\mathcal{J}_S(\text{alg}(\pi_X, t_X)_1) = \{0\}$ and so $C^*_{\text{env}}((\text{alg}(\pi_X, t_X))_1) = \mathcal{O}_X + CI$. The $C^*$-subalgebra of $\mathcal{O}_X + CI$ generated by $\text{alg}(\pi_X, t_X)$ is $\mathcal{O}_X$, and the conclusion follows. ■
Remark 3.8. In [5, page 596], it is claimed that if a $\mathcal{X}$ is a C*-correspondence over $\mathcal{A}$, with universal Toeplitz representation $(\pi_X, \mathcal{T}_X)$, then $\pi_X$ maps an approximate unit of $\mathcal{A}$ to an approximate unit for both $\mathcal{T}_X$ and $\mathcal{T}_X^+$. It is not hard to see that this claim is valid if and only if $\phi_X$ is non-degenerate. Therefore, there is a gap in the proof of [5, Theorem 5.3] in the case where $\mathcal{X}$ is strict but not essential. Nevertheless, our Theorem 3.7 incorporates all possible cases and hence completes the proof of [5, Theorem 5.3].

We now obtain one of the main results of [9] as a corollary.

Corollary 3.9. [9, Theorem 2.5] If $G$ is a countable directed graph then the C*-envelope of $\mathcal{T}_+(G)$ coincides with the universal Cuntz-Krieger algebra associated with $G$.

Note that in [9], the proof of the above corollary is essentially self-contained and avoids the heavy machinery used in this paper. The reader would actually benefit from reading that proof and then making comparisons with the proof of Theorem 3.7 here.

Acknowledgement. We are grateful to David Blecher for directing us to the work of Meyer [11] and its impact on the theory of C*-envelopes [2].

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