THE C*-ENVELOPE OF THE TENSOR ALGEBRA OF A DIRECTED GRAPH

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Abstract. Given an arbitrary countable directed graph G we prove the C*-envelope of the tensor algebra $\mathcal{T}_+(G)$ coincides with the universal Cuntz-Krieger algebra associated with G. Our approach is concrete in nature and does not rely on Hilbert module machinery. We show how our results extend to the case of higher rank graphs, where an analogous result is obtained for the tensor algebra of a row-finite k-graph with no sources.

1. Introduction

The fundamental nonselfadjoint operator algebra associated with a countable directed graph is its tensor algebra $\mathcal{T}_+(G)$ [**6, 19, 11, 14**]. Fowler, Muhly and Raeburn have recently characterized [**8**, Theorem 5.3.] the C*-envelope of the tensor algebra of a *faithful* strict Hilbert bimodule, as the associated universal Cuntz-Pimsner algebra. When applied to tensor algebras of graphs, their result shows that the C*-envelope of $\mathcal{T}_+(G)$ where G is a graph with *no sources*, coincides with the universal Cuntz-Krieger algebra associated with G. This generalizes the well-known fact that, for a single vertex graph with n loop edges, the C*-envelope of the corresponding disc algebra is the algebra of continuous functions on the unit circle ($n = 1$) and the Cuntz algebra ($n \geq 2$) [**22**]. On the other hand, tensor algebras of graphs that do have sources do not come from injective Hilbert bimodules and therefore [**8**, Theorem 5.3.] does not apply to such algebras. (In that case, Theorem 6.4 in [**18**] identifies the C*-envelope of the tensor algebra of such a graph as a quotient of the associated Toeplitz-Cuntz-Pimsner algebra, without any further information.) The class of tensor algebras of graphs with sources includes many motivating examples and therefore [**8**, Theorem 5.3.] raises the question whether or not the universal

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Cuntz-Krieger algebra of an *arbitrary* graph G is itself the C*-envelope of $\mathcal{T}_+(G)$.

The main objective of this article is to settle this question in the affirmative, thus providing a capstone to the investigations from [**17**, Theorem 6.8.] and [**8, 18, 19**] on the C*-envelope of the tensor algebra of a graph. As an added bonus, our approach does not require any familiarity with the heavy machinery of Hilbert bimodules. Actually, the only 'selfadjoint' prerequisite for reading this paper is the gauge-invariant uniqueness theorem for graph C*-algebras, a rather elementary result in the theory [**2, 3**]. Nevertheless, we hope the specialized techniques introduced here will contribute to the substantial theory of tensor algebras of Hilbert bimodules [**18, 19**].

Another objective is to begin here a systematic study of the C*-envelope of the tensor algebra of a higher rank graph. These were first introduced by Kumjian and Pask [**15**] as an abstraction of the combinatorial structure underlying the higher rank graph C*-algebras of Robertson and Steger [**26, 27**]. Their study seems in places similar to that of 1-graphs but there are significant complications here. One of the more notable is the limited understanding of what it should mean for a higher rank graph to have a source. In this note we concentrate on the original notion of Kumjian and Pask of a row-finite graph with no sources. We prove that if Λ is such a graph, then the C*-envelope of the tensor algebra of $\mathcal{T}_+(\Lambda)$ is the universal Cuntz-Krieger algebra $\mathrm{C}^*(\Lambda)$.

2. The C*-envelope of $\mathcal{T}_+(G)$

Let G be a countable directed graph with vertex set G^0, edge set G^1 and range and source maps r and s respectively. The *Toeplitz algebra of* G, denoted as $\mathcal{T}(G)$, is the universal C*-algebra generated by a set of partial isometries $\{S_e\}_{e\in G^1}$ and projections $\{P_x\}_{x\in G^0}$ satisfying the relations

$$(\dagger)\begin{cases} (1)\quad P_xP_y = 0 & \forall\, x,y\in G^0,\ x\neq y\\ (2)\quad S_e^*S_f = 0 & \forall\, e,f\in G^1,\ e\neq f\\ (3)\quad S_e^*S_e = P_{s(e)} & \forall\, e\in G^1\\ (4)\quad \sum_{r(e)=x} S_eS_e^* \leq P_x & \forall\, x\in G^0. \end{cases}$$

The existence of such a universal object is implicit in [**21**, Theorem 3.4] and [**18**, Theorem 2.12] and was made explicit in [**6**, Proposition 1.3 and Theorem 4.1].

Definition 2.1. Given a countable directed graph G, the *tensor algebra of* G, denoted as $\mathcal{T}_+(G)$, is the norm closed subalgebra of $\mathcal{T}(G)$ generated by the partial isometries $\{S_e\}_{e\in G^1}$ and projections $\{P_x\}_{x\in G^0}$.

The tensor algebras associated with graphs were introduced under the name *quiver algebras* by Muhly and Solel in [**17, 18**] as follows. Let $\lambda_{G,0}$ be the multiplication representation of $c_0(G^0)$ on $l^2(G^0)$, determined by the counting measure on G^0, and let λ_G denote the representation of $\mathcal{T}(G)$ induced by $\lambda_{G,0}$, in the sense of [**19**] and [**6**]. It is easily seen that the Hilbert space of λ_G is $\mathcal{H}_G = l^2(\mathbb{F}^+(G))$, where $\mathbb{F}^+(G)$ denotes the *free semigroupoid* [**14**] of the graph G (also called the *path space* of G). This consists of all vertices $v \in G^0$ and all paths $w = e_ke_{k-1}\dots e_1$, where the e_i are edges satisfying $s(e_i) = r(e_{i-1})$, $i = 2, 3, \dots, k$, $k \in \mathbb{N}$. (Paths of the form $w = e_ke_{k-1}\dots e_1$ are said to have length k, denoted as $|w| = k$, and vertices are called paths of length 0.) The maps r and s extend to $\mathbb{F}^+(G)$ in the obvious way, two paths w_1 and w_2 are composable precisely when $s(w_2) = r(w_1)$ and, in that case, the composition w_2w_1 is just the concatenation of w_1 and w_2. Let $\{\xi_w\}_{w\in\mathbb{F}^+(G)}$ denote the standard orthonormal basis of $\mathcal{H}_G$, where ξ_w is the characteristic function of $\{w\}$. Then, $\lambda_G(S_e)$, $e \in G^1$, is equal to the left creation operator $L_e \in \mathcal{B}(\mathcal{H}_G)$ defined by

$$L_e\xi_w = \begin{cases} \xi_{ew} & \text{if } s(e) = r(w) \\ 0 & \text{if } s(e) \neq r(w). \end{cases}$$

(We shall write P_x for $\lambda_G(P_x)$.) Further define R_e, $e \in G^1$, by the corresponding right actions on $\mathcal{H}_G$. By [**6**, Corollary 2.2], the representation λ_G is a faithful representation of the corresponding Toeplitz algebra and therefore a faithful representation of $\mathcal{T}_+(G)$. The algebra $\lambda_G(\mathcal{T}_+(G))$ is the *quiver algebra* of Muhly and Solel [**19**]. The WOT-closure of the quiver algebra is the *free semigroupoid algebra* of the second author and Power [**14**].

We now introduce the Cuntz-Krieger algebra of a directed graph G and one of its faithful representations that is convenient for the identification of the C*-envelope of $\mathcal{T}_+(G)$.

Recall that a family of partial isometries $\{S_e\}_{e\in G^1}$ and projections $\{P_x\}_{x\in G^0}$ is said to obey the Cuntz-Krieger relations associated with G if and only if they satisfy

$$(\ddagger)\begin{cases} (1) & P_xP_y = 0 & \forall\, x, y \in G^0,\ x \neq y \\ (2) & S_e^*S_f = 0 & \forall\, e, f \in G^1,\ e \neq f \\ (3) & S_e^*S_e = P_{s(e)} & \forall\, e \in G^1 \\ (4) & S_eS_e^* \leq P_{r(e)} & \forall\, e \in G^1 \\ (5) & \sum_{r(e)=x} S_eS_e^* = P_x & \forall\, x \in G^0 \text{ with } |r^{-1}(x)| \neq 0, \infty \end{cases}$$

The relations (‡) have been refined in a series of papers by the Australian school and reached the above form in [**2, 25**]. All refinements involved condition (5) and as it stands now, condition (5) gives the equality requirement for projections P_x such that x is not a source and receives finitely many edges. (Indeed, otherwise condition (5) would not be a C*-condition.)

It can been shown that there exists a universal C*-algebra, denoted as $\mathrm{C}^*(G)$, associated with the relations (‡). Indeed, one constructs a single family of partial isometries and projections obeying (‡). Then, $\mathrm{C}^*(G)$ is the C*-algebra generated by a 'maximal' direct sum of such families. See [**3**] for more details.

Let G_s be the graph resulting by 'adding tails' to G in the following sense. Let $x_1, x_2, \ldots \in G^0$ be the sources of G. For each source x_n, we add to G^0 and G^1 sequences $\{x_{n,i}\}_{i=1}^{\infty}$ and $\{e_{n,i}\}_{i=1}^{\infty}$ respectively, satisfying

$$s(e_{n,i}) = x_{n,i} \quad \text{and} \quad r(e_{n,i}) = x_{n,i-1}, \qquad i = 1, 2, \ldots,$$

with the convention $x_{n,0} \equiv x_n$. With these additions, the resulting graph is denoted as G_s and it is clear that G_s has no sources.

In the proof below, the faithfulness of our representation will follow from a gauge-invariant uniqueness theorem. For that reason, we need to construct a gauge action β of $\mathbb{T}$ on $\lambda_{G_s}(\mathcal{T}(G_s))$. For each $z \in \mathbb{T}$ we define a unitary operator $U_z \in B(\mathcal{H}_{G_s})$ via the formula

$$U_z \xi_w = \overline{z}^{|w|} \xi_w \quad \text{for} \quad w \in \mathbb{F}^+(G_s).$$

Note that the family $\{U_z\}_{z \in \mathbb{T}}$ induces by conjugation a gauge action β on the C*-algebra $\lambda_{G_s}(\mathcal{T}(G_s))$. Indeed, if we let

$$\beta_z(A) \equiv U_z^* A U_z \quad \text{for} \quad A \in B(\mathcal{H}_{G_s}),$$

then one easily verifies that $\beta_z(\lambda_{G_s}(P_x)) = \lambda_{G_s}(P_x)$, $x \in G_s^0$, and $\beta_z(\lambda_{G_s}(S_e)) = z\lambda_{G_s}(S_e)$, $e \in G_s^1$.

Theorem 2.2. *Let G be a countable directed graph and let G_s be the graph obtained from G by adding tails. Let $\mathrm{C}^*_{\mathcal{H}_{G_s}}(G) \subseteq \lambda_{G_s}(\mathcal{T}(G_s))$ be the C*-algebra generated by $\{\lambda_{G_s}(S_e)\}_{e \in G^1}$ and $\{\lambda_{G_s}(P_x)\}_{x \in G^0}$. Let π be the Calkin map on $\mathcal{B}(\mathcal{H}_{G_s})$. Then the algebra $\pi(\mathrm{C}^*_{\mathcal{H}_{G_s}}(G))$ is isomorphic to the universal Cuntz-Krieger C*-algebra $\mathrm{C}^*(G)$ associated with G.*

Proof. It is easy to see that the algebra $\pi(\mathrm{C}^*_{\mathcal{H}_{G_s}}(G))$ is generated by families of partial isometries $\{\pi(\lambda_{G_s}(S_e))\}_{e \in G^1}$ and projections $\{\pi(\lambda_{G_s}(P_x))\}_{x \in G^0}$ satisfying (‡), with respect to G. (By taking a quotient with the compacts, we ensure that we get an equality in (5), whenever applicable). Since each $\lambda_{G_s}(P_x)$ has infinite dimensional range (there are no sources in G_s), we obtain that $\pi(\lambda_{G_s}(P_x)) \neq 0$ for all $x \in G^0$.

Now notice that the gauge action β is spatially implemented and therefore passes to the Calkin algebra. It also preserves $\pi(\mathrm{C}^*_{\mathcal{H}_{G_s}}(G))$ and therefore it induces a gauge action on that algebra. Therefore, all the requirements of [**2**, Theorem 2.1.] are satisfied and so $\pi(\mathrm{C}^*_{\mathcal{H}_{G_s}}(G))$ is isomorphic to $\mathrm{C}^*(G)$. ■

Theorem 2.2 shows that the Calkin map

$$\lambda_{G_s}(\mathcal{T}(G_s)) \ni L_w \mapsto \pi(L_w) \in \pi(\lambda_{G_s}(\mathcal{T}(G_s)), \tag{1}$$

when restricted to the operators with symbols in G, has range equal to the universal Cuntz-Krieger algebra for G. The next lemma shows that actually (1) is a complete isometry when restricted to $\lambda_{G_s}(\mathcal{T}_+(G_s))$, thus providing a shorter proof for the main result of this section, provided that G has no sources. We note that the 'unampliated' version of this lemma has appeared as Proposition 7.3 in [**12**].

Lemma 2.3. *Let $\mathfrak{G}$ be a countable directed graph with no sources. Then,*

$$\|A\| = \|A\|_e,$$

for all $A \in M_n(\lambda_{\mathfrak{G}}(\mathcal{T}_+(\mathfrak{G}))$, $n \in \mathbb{N}$.

Proof. It suffices to construct a sequence of isometries in the commutant of $M_n(\lambda_{\mathfrak{G}}(\mathcal{T}_+(\mathfrak{G}))$ which converges weakly to 0.

Let $d \geq 1$ be a positive integer. For all $x \in \mathfrak{G}^0$ we may choose a path $w_x \in \mathbb{F}^+(\mathfrak{G})$ of length d such that $r(w_x) = x$. As the w_x are distinct paths of the same length, notice that the right creation operators R_{w_x}, $x \in \mathfrak{G}^0$, have pairwise orthogonal ranges. Thus, we may define an isometry $R_d = \sum_{x \in \mathfrak{G}^0} R_{w_x}$ in the commutant of $\lambda_{\mathfrak{G}}(\mathcal{T}_+(\mathfrak{G}))$ (where the sum converges WOT in the infinite vertex case). Indeed,

$$(R_d)^* R_d = \sum_{x,y} R^*_{w_x} R_{w_y} = \sum_{x} R^*_{w_x} R_{w_x} = I.$$

Let $R_d^{(n)} \equiv R_d \oplus R_d \oplus \cdots \oplus R_d \in M_n(\lambda_{\mathfrak{G}}(\mathcal{T}_+(\mathfrak{G})))'$ denote the n-th inflation of R_d. Then, for an $A \in M_n(\lambda_{\mathfrak{G}}(\mathcal{T}_+(\mathfrak{G})))$, a compact operator K and a vector ξ, we have

$$\begin{aligned}\|A\xi\| &= \|R_d^{(n)} A\xi\| = \|A R_d^{(n)} \xi\| \\ &\leq \|(A+K) R_d^{(n)} \xi\| + \|K R_d^{(n)} \xi\| \\ &\leq \|A+K\| + \|K R_d^{(n)} \xi\|.\end{aligned}$$

By taking limits we obtain $\|A\xi\| \leq \|A+K\|$, for all ξ, and so $||A|| = ||A||_e$, as required. ■

If G has sources, in order to make use of (1), we identify the quiver algebra $\lambda_G(\mathcal{T}_+(G))$ with a subalgebra of $\lambda_{G_s}(\mathcal{T}_+(G_s))$.

Lemma 2.4. *Let G be a countable directed graph. Then the map*

$$\lambda_G(\mathcal{T}_+(G)) \ni L_w \longmapsto L_w \in \lambda_{G_s}(\mathcal{T}_+(G_s)), \quad w \in \mathbb{F}^+(G)$$

extends to an injective C^**-homomorphism* $\phi : \lambda_G(\mathcal{T}(G)) \to \lambda_{G_s}(\mathcal{T}(G_s))$.

Proof. We begin by decomposing $\mathcal{H}_{G_s}$ into a direct sum of reducing subspaces for $\phi(L_w)$, $w \in \mathbb{F}^+(G)$. First make the natural identification of $\mathcal{H}_G$ inside $\mathcal{H}_{G_s}$. Then for all $n \geq 1$ and all $i \geq 1$ (note that we do not include $i = 0$ here) define subspaces of $\mathcal{H}_{G_s}$ by

$$\mathcal{H}_{n,i} \equiv \mathrm{span}\{\xi_w : w \in \mathbb{F}^+(G_s),\, s(w) = x_{n,i}\}.$$

Then each $\mathcal{H}_{n,i}$ is reducing for the range of $\phi(L_w)$, $w \in \mathbb{F}^+(G)$ and

$$\mathcal{H}_{G_s} \;=\; \mathcal{H}_G \bigoplus \Big(\bigoplus_{n,i\geq 1} \mathcal{H}_{n,i} \Big). \tag{2}$$

For each n, i further define $u_{n,i} \equiv e_{n,1}e_{n,2}\cdots e_{n,i} \in \mathbb{F}^+(G_s)$ and let $\mathcal{H}^0_{n,i}$ be the subspace of $\mathcal{H}_{n,i}$ given by

$$\mathcal{H}^0_{n,i} \equiv \mathrm{span}\left\{\xi_{wu_{n,i}} : w \in \mathbb{F}^+(G),\, s(w) = x_n = r(e_{n,1})\right\}.$$

Now let $U_{n,i} \in \mathcal{B}(\mathcal{H}^0_{n,i}, P_{x_n}\mathcal{H}_G)$ be the unitary defined by

$$U_{n,i}\,\xi_{wu_{n,i}} = \xi_w \quad \text{for} \quad w \in \mathbb{F}^+(G),\, s(w) = x_n.$$

Then for all n and $i \geq 1$ we have

$$\phi(p(L_w))|_{\mathcal{H}^0_{n,i}} = U^\dagger_{n,i} p(L_w) U_{n,i}$$

and

$$\phi(p(L_w))|_{\mathcal{H}_{n,i}\ominus\mathcal{H}^0_{n,i}} \equiv 0$$

for any noncommutative polynomial $p(L_w)$ with $w \in \mathbb{F}^+(G)$. In particular, this yields

$$||\phi(p(L_w))|_{\mathcal{H}_{n,i}}|| \leq ||p(L_w)||.$$

This, together with the reducing decomposition (2) of $\mathcal{H}_{G_s}$ and the fact that $\phi(p(L_w))|_{\mathcal{H}_G} = p(L_w)$, shows that

$$\|\phi(p(L_w))\| = \|p(L_w)\|, \tag{3}$$

for any noncommutative polynomial $p(L_w)$ with $w \in \mathbb{F}^+(G)$. This proves the lemma since this norm estimate is obtained via a reducing subspace decomposition. ∎

Let $\mathfrak{A}$ be a C*-algebra and let $\mathcal{A}$ be a (nonselfadjoint) subalgebra of $\mathfrak{A}$ which generates $\mathfrak{A}$ as a C*-algebra and contains a two-sided contractive approximate unit for $\mathfrak{A}$, i.e., $\mathcal{A}$ is an essential subalgebra for $\mathfrak{A}$. A two-sided ideal $\mathcal{J}$ of $\mathfrak{A}$ is said to be a *boundary ideal* for $\mathcal{A}$ if and only if the quotient map $\pi : \mathfrak{A} \to \mathfrak{A}/\mathcal{J}$ is a complete isometry when restricted to $\mathcal{A}$. It is a result of Hamana [**9**], following the seminal work of Arveson [**1**], that there exists a boundary ideal $\mathcal{J}_S(\mathcal{A})$, the *Shilov boundary ideal*, that contains all other boundary ideals. In that case, the quotient $\mathfrak{A}/\mathcal{J}_S(\mathcal{A})$ is called the C*-*envelope* of $\mathcal{A}$. The C*-envelope is unique in the following sense: Assume that $\phi_1 : \mathcal{A} \to \mathfrak{A}_1$ is a completely isometric isomorphism of $\mathcal{A}$ onto an essential subalgebra of a C*-algebra $\mathfrak{A}_1$ and suppose that the Shilov boundary for $\phi_1(\mathcal{A}) \subseteq \mathfrak{A}_1$ is zero. Then $\mathfrak{A}$ and $\mathfrak{A}_1$ are $*$-isomorphic, via an isomorphism ϕ so that $\phi(\pi(a)) = \phi_1(a)$, for all $a \in \mathfrak{A}$.

Theorem 2.5. *If G is a countable directed graph then the* C*-*envelope of $\mathcal{T}_+(G)$ coincides with the universal Cuntz-Krieger algebra associated with G.*

Proof. Having (completely isometrically) identified $\lambda_G(\mathcal{T}_+(G))$ with an essential subalgebra of $\mathrm{C}^*(G)$, the proof is now similar to that of [**8**, Theorem 5.3.]. Indeed, Theorem 2.2 and Lemmata 2.3 and 2.4 provide a completely isometric isomorphism $\pi \circ \phi$ from $\lambda_G(\mathcal{T}_+(G))$ onto the nonselfadjoint algebra $\mathrm{C}^*_+(G)$ generated by the generators of the universal Cuntz-Krieger algebra $\mathrm{C}^*(G)$. In light of the above discussion, we need to verify that the Shilov boundary ideal $\mathcal{J}_S(\mathrm{C}^*_+(G))$ for $\mathrm{C}^*_+(G)$ inside $\mathrm{C}^*(G)$ is zero. However, the maximality of $\mathcal{J}_S(\mathrm{C}^*_+(G))$ and the invariance of $\mathrm{C}^*_+(G)$ under the gauge action of $\mathbb{T}$ on $\mathrm{C}^*(G)$ imply that $\mathcal{J}_S(\mathrm{C}^*_+(G))$ is a gauge-invariant ideal. Theorem 2.1 in [**2**] shows now that any non-zero gauge-invariant ideal contains at least one of the generating projections P_x, $x \in G^0$, or otherwise the quotient map is an isomorphism. Hence $\mathcal{J}_S(\mathrm{C}^*_+(G)) = \{0\}$, or otherwise the quotient map would not be faithful on $\mathrm{C}^*_+(G)$. ■

The proof above also shows the universal Cuntz-Krieger algebra of G is the C*-envelope of the associated quiver algebra. To obtain this result, we need to use the fact that λ_G is faithful [**6**, Proposition 1.3 and Theorem 4.1]. Our techniques however can give a short proof of this fact.

Theorem 2.6. *Let G be a countable directed graph. Let $\{P'_x\}_{x \in G^0}$ and $\{S'_e\}_{e \in G^1}$ be families of projections and partial isometries acting on a Hilbert space $\mathcal{H}$ which satisfy (†). Then there exists a $*$-epimorphism*

$$\tau : \lambda_G(\mathcal{T}(G)) \longrightarrow \mathrm{C}^*(\{S'_e\}_{e \in G^1})$$

such that $\tau(L_e) = S'_e$, for all $e \in G^1$.

Proof. The Wold decomposition in this setting [**19, 10**] implies the existence of C*-homomorphisms

$$\phi_1 : \mathrm{C}^*(G) \longrightarrow B(\mathcal{H}_1)$$

and

$$\phi_2 : \lambda_G(\mathcal{T}(G)) \longrightarrow B(\mathcal{H}_2)$$

such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and

$$S'_u = (\phi_1 \oplus \phi_2)(S_u \oplus L_u) \quad \text{for} \quad u \in \mathbb{F}^+(G).$$

The desired map is then

$$\tau = (\phi_1 \oplus \phi_2) \circ ((\pi \circ \phi) \oplus id) : \lambda_G(\mathcal{T}(G)) \longrightarrow \mathrm{C}^*(\{S'_e\}_{e \in \mathcal{E}(G)}).$$

Tracing the definitions of these maps shows that $\tau(L_u) = S'_u$ for all $u \in \mathbb{F}^+(G)$. ∎

In [**6**], Fowler and Raeburn obtained a generalization of Coburn's Theorem to arbitrary Toeplitz-Cuntz-Krieger C*-algebras, once again making extensive use of the Hilbert bimodule machinery. For Toeplitz-Cuntz-Krieger C*-algebras, associated with graphs having no sources, the same result was obtained independently in [**12**] by more elementary means. Lemma 2.4 allows us to apply the technique from [**12**] to the general case.

Theorem 2.7. *Let G be a countable directed graph. Let $\{P'_x\}_{x \in G^0}$ and $\{S'_e\}_{e \in G^1}$ be families of projections and partial isometries respectively, acting on a Hilbert space $\mathcal{H}$ and satisfying (†). If,*

$$\sum_{r(e)=x} S'_e (S'_e)^* \neq P'_x \quad \text{for} \quad x \in G^0, \tag{4}$$

then there exists an injective $*$-homorphism $\phi : \lambda_G(\mathcal{T}(G)) \to B(\mathcal{H})$ such that $\phi(L_e) = S'_e$ for all $e \in G^1$.

In particular, the C-algebra $\mathrm{C}^*(\{S'_e\}_{e \in G^1})$, generated by the collection $\{S'_e\}_{e \in G^1}$, is isomorphic to $\lambda_G(\mathcal{T}(G))$.*

For a proof, repeat the arguments in [**12**].

3. Higher rank graphs

In this section we define the natural generalization of the tensor algebra for a higher rank graph and, using a concrete approach, prove its C*-envelope is the corresponding universal Cuntz-Krieger algebra when the graph is row-finite with no sources. This also yields another concrete proof for the case of 1-graphs with no sources.

Definition 3.1. (Kumjian and Pask [**15**]) A k-graph (Λ, d) consists of a countable small category Λ, with range and source maps r and s respectively, together with a functor $d : \Lambda \to \mathbb{Z}_+^k$ satisfying the factorization property: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{Z}_+^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ and $d(\mu) = m$ and $d(\nu) = n$.

By the factorization property we may identify the objects $\mathrm{Obj}(\Lambda)$ of Λ with the subset $\Lambda^0 \equiv d^{-1}(0, \ldots, 0)$. We also write Λ^n for $d^{-1}(n)$, $n \in \mathbb{Z}_+^k$, and put $\Lambda^n(v) = \{\lambda \in \Lambda^n : r(\lambda) = v\}$ for $v \in \Lambda^0$. Following [**15**], if $\Lambda^n(v)$ is nonempty for all choices of $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, we shall say Λ has *no sources*. Further, Λ is said to be *row-finite* if for all $m \in \mathbb{N}^k$ and $v \in \Lambda^0$ the set $\Lambda^m(v)$ is finite. Let us define a grading function $\delta : \Lambda \to \mathbb{N}$ for Λ by $\delta(\lambda) = n_1 + \ldots + n_k$ when $d(\lambda) = (n_1, \ldots, n_k)$.

It is convenient to view a higher rank k-graph as a directed graph with directed edges coloured one of k possible colours and vertices identified with objects. The elements of Λ^1 are the edges of Λ and the factorization property describes how different 'paths' are related. See [**7, 13, 15, 24**] for further introductory discussions on the subject.

Let Λ be a row-finite higher rank graph with no sources. A family $\{S_\lambda : \lambda \in \Lambda\}$ is said to satisfy the Toeplitz-Cuntz-Krieger relations for Λ if the following holds:

$$(\dagger)\left\{\begin{array}{lll} (1) & S_v S_w = 0 & \forall\, v, w \in \Lambda^0,\ v \neq w \\ (2) & S_{\lambda\mu} = S_\lambda S_\mu & \forall\, \lambda, \mu \in \Lambda \\ (3) & S_\lambda^* S_\lambda = S_{s(\lambda)} & \forall\, \lambda \in \Lambda \\ (4) & \sum_{\lambda\in\Lambda^n(v)} S_\lambda S_\lambda^* \leq S_v & \forall\, v \in \Lambda^0 \text{ and } n \in \mathbb{N}^k. \end{array}\right.$$

If equality is satisfied in condition (4) for all v, n, then $\{S_\lambda : \lambda \in \Lambda\}$ is said to satisfy (‡), the analogue of the Cuntz-Krieger relations here. The algebra $\mathrm{C}^*(\Lambda)$ [**15**] is the universal C*-algebra generated by a family of partial isometries satisfying (‡).

The Toeplitz algebra $\mathcal{T}(\Lambda)$ [**23**] is the universal C*-algebra generated by a family of partial isometries satisfying (†). Let us define the following nonselfadjoint algebra as a generalization of the 1-graph case.

Definition 3.2. The norm closed subalgebra of $\mathcal{T}(\Lambda)$ generated by the family $\{S_\lambda : \lambda \in \Lambda\}$ is denoted as $\mathcal{T}_+(\Lambda)$.

We will work with the following faithful representation of $\mathcal{T}_+(\Lambda)$. Given a higher rank graph Λ (row-finite with no sources), let $\mathcal{H}_\Lambda$ be the Fock space Hilbert space with orthonormal basis $\{\xi_\lambda : \lambda \in \Lambda\}$.

Define partial creation operators $L = (L_\lambda)_{\lambda\in\Lambda}$ on $\mathcal{H}_\Lambda$ as follows:

$$L_\lambda \xi_\mu = \begin{cases} \xi_{\lambda\mu} & \text{if } s(\lambda) = r(\mu) \\ 0 & \text{if } s(\lambda) \neq r(\mu). \end{cases}$$

Further define partial isometries $\{R_\lambda : \lambda \in \Lambda\}$ by the corresponding right actions on $\mathcal{H}_\Lambda$. The WOT-closure of this representation is a 'higher rank semigroupoid algebra' of the second author and Power [**13**]. Faithfulness of this 'Fock representation' on $\mathcal{T}(\Lambda)$, and hence on $\mathcal{T}_+(\Lambda)$, was established by Raeburn and Sims [**23**].

In the following discussion let $\Lambda = (\Lambda, d)$ be a row-finite higher rank k-graph with no sources. The no source assumption on Λ allows the following choice of infinite paths. To each vertex $v \in \Lambda^0$ we associate a unique infinite path

$$\mu_v \equiv e_{v,1} e_{v,2} \cdots , \quad e_{v,i} \in \Lambda^1, \quad i = 1, 2, \dots$$

with range v, that is $r(e_{v,1}) = v$. Each path μ_v, $v \in \Lambda^0$, is chosen so that for every $m \in \mathbb{Z}_+^k$, there exists an $i \in \mathbb{N}$ such that $d(\mu_{v,i}) \geq m$ in $\mathbb{N}^k$, where

$$\mu_{v,i} \equiv e_{v,1} e_{v,2} \cdots e_{v,i}.$$

For convenience put $\mu_{v,0} = v$. We will denote the collection of all such paths as

$$\Lambda_{red} = \big\{ \mu_{v,i} : v \in \Lambda^0, \ i = 0, 1, 2, \dots \big\}.$$

We shall construct a faithful representation of $\mathrm{C}^*(\Lambda)$ that will allow us to identify the C^*-envelope of $\mathcal{T}_+(\Lambda)$. First note that if $\lambda \in \Lambda$ then we may define λ^{-1} as the natural equivalence class of products of elements e^{-1}, where $e \in \Lambda^1$, determined by λ. Further, for all $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$ we define $\lambda\mu^{-1}$ to be the equivalence class of reduced products given by applications of the factorization property followed by cancellations of products of the form $ee^{-1} = r(e)$. Let

$$\Gamma = \Lambda\Lambda_{red}^{-1} = \{ \lambda\mu^{-1} : \lambda \in \Lambda, \, \mu \in \Lambda_{red} \, , s(\lambda) = s(\mu) \},$$

where it is understood that $\lambda\mu^{-1}$ corresponds to the equivalence class of reduced products determined by λ, μ as above.

Let $\mathcal{H} = \ell^2(\Gamma)$ be the Hilbert space with orthonormal basis $\{\xi_\gamma : \gamma \in \Gamma\}$. Consider operators $S' = (S'_\lambda)_{\lambda\in\Lambda}$ defined on $\mathcal{H}$ by

$$S'_\lambda \xi_{\lambda_1\mu^{-1}} = \begin{cases} \xi_{\lambda\lambda_1\mu^{-1}} & \text{if } s(\lambda) = r(\lambda_1) \\ 0 & \text{if } s(\lambda) \neq r(\lambda_1). \end{cases}$$

Let τ_Λ be the map that identifies the generators of $\mathrm{C}^*(\Lambda)$ with the operators $S' = (S'_\lambda)_{\lambda\in\Lambda}$, so that $\tau_\Lambda(S_\lambda) = S'_\lambda$ for $\lambda \in \Lambda$.

Proposition 3.3. *The map* $\tau_\Lambda : \mathrm{C}^*(\Lambda) \to \mathcal{B}(\mathcal{H})$ *defines a faithful representation of* $\mathrm{C}^*(\Lambda)$.

Proof. It is evident that $\{S'_\lambda : \lambda \in \Lambda\}$ is a family of partial isometries that satisfy conditions (1), (2) and (3) of the (†) relations above. To see condition (4), let $v \in \Lambda^0$ and let $m \in \mathbb{N}^k$. The range of S'_v is given by $\operatorname{Ran} S'_v = \operatorname{span}\{\xi_\gamma : \gamma = \lambda\mu^{-1} \in \Gamma,\, r(\lambda) = v\}$. Consider an arbitrary element $\gamma = \lambda\mu_{v,i}^{-1} \in \Gamma$. Choose $j > i$ so that

$$d(e_{v,i+1}e_{v,i+2}\cdots e_{v,j}) \geq m.$$

Then $\lambda e_{v,i+1}e_{v,i+2}\cdots e_{v,j}$ belongs to Λ with $d\big(\lambda e_{v,i+1}e_{v,i+2}\cdots e_{v,j}\big) \geq m$ as well. By the factorization property there are $\lambda_1, \lambda_2 \in \Lambda$ such that

$$\lambda e_{v,i+1}e_{v,i+2}\cdots e_{v,j} = \lambda_1\lambda_2 \quad \text{and} \quad d(\lambda_1) = m.$$

Since $s(\lambda_2) = s(e_{v,j})$, we have that $\xi_{\lambda_2\mu_{v,j}^{-1}} \in \mathcal{H}$. Furthermore,

$$\begin{aligned} S_{\lambda_1}(\xi_{\lambda_2\mu_{v,j}^{-1}}) &= \xi_{(\lambda e_{v,i+1}e_{v,i+2}\cdots e_{v,j})\mu_{v,j}^{-1}} \\ &= \xi_{\lambda\mu_{v,i}^{-1}} = \xi_\gamma \end{aligned}$$

and so ξ_γ belongs to $\operatorname{Ran} S_{\lambda_1}$. Finally, the projections $S_\lambda S_\lambda^*$, $\lambda \in \Lambda^m$, are easily seen to have mutually orthogonal ranges by the factorization property. Thus we have verified condition (4) and τ_Λ defines a representation of $\mathrm{C}^*(\Lambda)$ on $\mathcal{H}$.

To show that τ_Λ is faithful, we use the gauge-invariant uniqueness theorem of [**15**]. The universality of $\mathrm{C}^*(\Lambda)$ implies the existence of a canonical action of the k-torus $\mathbb{T}^k$, called the *gauge action*, $\alpha : \mathbb{T}^k \to \operatorname{Aut} \mathrm{C}^*(\Lambda)$ defined for $t = (t_1, t_2, \ldots, t_k) \in \mathbb{T}^k$ and $S_\lambda \in \mathrm{C}^*(\Lambda)$ by

$$\alpha_t(S_\lambda) = t^{d(\lambda)}S_\lambda$$

where $t^m = t_1^{m_1}t_2^{m_2}\cdots t_k^{m_k}$ for $m = (m_1, m_2, \ldots, m_k) \in \mathbb{T}^k$. Let $\tau : \mathrm{C}^*(\Lambda) \to \mathcal{B}$ be a representation of $\mathrm{C}^*(\Lambda)$ so that $\tau(S_v) \neq 0$, for all $v \in \Lambda^0$. The gauge-invariant uniqueness theorem asserts that τ is faithful provided that there exists an action $\beta : \mathbb{T}^k \to \operatorname{Aut} \mathcal{B}$ so that $\tau \circ \alpha_t = \beta_t \circ \tau$, for all $t \in \mathbb{T}^k$.

For each $t \in \mathbb{T}^k$, define a unitary operator $U_t \in \mathcal{B}(\mathcal{H})$ as

$$U_t\, \xi_{\lambda\mu^{-1}} \equiv t^{(d(\mu)-d(\lambda))}\xi_{\lambda\mu^{-1}}, \quad \lambda \in \Lambda,\, \mu \in \Lambda_{\mathrm{red}},$$

and let $\beta_t(A) \equiv U_t^* A U_t$, $A \in B(\mathcal{H})$. It is easy to check that $\tau_\Lambda \circ \alpha_t = \beta_t \circ \tau_\Lambda$, for all $t \in \mathbb{T}^k$, and so τ_Λ is faithful. ■

Lemma 3.4. *Let $\Lambda = (\Lambda, d)$ be a row-finite higher rank graph with no sources and let $\tau_\Lambda : \mathrm{C}^*(\Lambda) \to \mathcal{B}(\mathcal{H})$ be as above. If p is a polynomial in the polynomial ring $\mathcal{P}_\Lambda^+$ generated by Λ then $||p(L)|| = ||\tau_\Lambda(p(S))||$.*

Proof. For $v \in \Lambda^0$ and $\mu \in \Lambda_{red}$ with $r(\mu) = v$, let $\mathcal{H}_{v,\mu}$ be the subspace of $\mathcal{H}$ given by

$$\mathcal{H}_{v,\mu} = \text{span}\{\xi_{\lambda\mu^{-1}} : \lambda \in \Lambda,\, s(\lambda) = s(\mu)\}.$$

Further define for $k \geq 0$ subspaces

$$\mathcal{H}_k = \bigoplus_{v\in\Lambda^0} \mathcal{H}_{v,\mu_{v,k}}.$$

Then $\mathcal{H}_{v,\mu}$ may be naturally identified with the subspace $\mathcal{H}_{s(\mu)} \equiv R_{s(\mu)}\mathcal{H}_\Lambda = \text{span}\{\xi_\lambda : \lambda \in \Lambda,\, s(\lambda) = s(\mu)\}$ of the Fock space $\mathcal{H}_\Lambda$ via the unitary $U_{v,\mu} : \mathcal{H}_{v,\mu} \to \mathcal{H}_{s(\mu)}$ which is defined on standard basis vectors $\xi_{\lambda\mu^{-1}}$ of $\mathcal{H}_{v,\mu}$ by $U_{v,\mu}\,\xi_{\lambda\mu^{-1}} = \xi_\lambda$. The construction of $\Gamma = \Lambda\Lambda_{red}^{-1}$ shows that $U_{v,\mu}$ is well defined and determines a unitary operator. Note that,

$$\tau_\Lambda(S_\lambda)\mid_{\mathcal{H}_{v,\mu}} = U_{v,\mu}^*\big(L_\lambda\mid_{R_{s(\mu)}\mathcal{H}_\Lambda}\big)U_{v,\mu}$$

for any $\lambda \in \Lambda$.

Under these identifications, the restrictions of the $\tau_\Lambda(S_\lambda)$ to the invariant subspace $\mathcal{H}_0 = \oplus_{v\in\Lambda^0}\mathcal{H}_{v,\mu_{v,0}} = \oplus_{v\in\Lambda^0}\mathcal{H}_{v,v}$ are jointly unitarily equivalent to the creation operators L_λ;

$$\tau_\Lambda(S_\lambda)|_{\mathcal{H}_0} \cong L_\lambda \quad \text{for} \quad \lambda \in \Lambda.$$

Thus, if p is a polynomial in the polynomial ring $\mathcal{P}_\Lambda^+$ generated by Λ, then

$$\|p(L)\| = \|p(\tau_\Lambda(S))|_{\mathcal{H}_0}\| \leq \|p(\tau_\Lambda(S))\|$$

On the other hand, the restrictions of the $\tau_\Lambda(S_\lambda)$ to the invariant subspaces $\mathcal{H}_k = \oplus_{v\in\Lambda^0}\mathcal{H}_{v,\mu_{v,k}}$, for $k \geq 0$ are jointly unitarily equivalent to a direct sum of restrictions of L_λ to invariant subspaces of the form $\mathcal{H}_w$, $w \in \Lambda^0$. Therefore,

$$\|p(\tau_\Lambda(S))|_{\mathcal{H}_k}\| \leq \|p(L)\| \quad \text{for} \quad k \geq 0.$$

However, since $\mathcal{H}$ is given by the closed span $\mathcal{H} = \vee_{k\geq 0}\mathcal{H}_k$ of this increasing chain of subspaces, the sequence of compression operators $p(\tau_\Lambda(S))P_{\mathcal{H}_k} = P_{\mathcal{H}_k}p(\tau_\Lambda(S))P_{\mathcal{H}_k}$, $k \geq 0$, converges strongly to $p(\tau_\Lambda(S))$. Hence,

$$\|p(\tau_\Lambda(S))\| \leq \liminf_{k\geq 0} \|p(\tau_\Lambda(S))P_{\mathcal{H}_k}\| \leq \|p(L)\|,$$

and the conclusion follows. ∎

The ampliated version of the above arguments shows that the isometric map

$$\begin{array}{rcl} \mathcal{T}_+(\Lambda) & \rightarrow & \tau_\Lambda(\mathrm{C}^*(\Lambda)) \\ p(L) & \mapsto & \tau_\Lambda(p(S)) \end{array}$$

of Lemma 3.4 is a complete isometry and so we obtain the following.

Corollary 3.5. *There exists a complete isometry* $\phi : \mathcal{T}_+(\Lambda) \to \mathrm{C}^*(\Lambda)$*, which maps generators to generators.*

Proof. Recall that the map τ_Λ is a faithful representation of $\mathrm{C}^*(\Lambda)$. The conclusion now follows from the above. ∎

Theorem 3.6. *If* Λ *is a row-finite higher rank graph with no sources, the* C^**-envelope of* $\mathcal{T}_+(\Lambda)$ *coincides with the universal Cuntz-Krieger algebra* $\mathrm{C}^*(\Lambda)$ *associated with* Λ*.*

Proof. The proof is now identical to that of Theorem 2.5. Indeed, Corollary 3.5 provides a completely isometric isomorphism from $\mathcal{T}_+(\Lambda)$ onto the nonselfadjoint algebra $\mathrm{C}^*_+(\Lambda)$ generated by the generators of the universal Cuntz-Krieger algebra $\mathrm{C}^*(\Lambda)$. In light of the discussion above Theorem 2.5, we need to verify, once again, that the Shilov boundary ideal $\mathcal{J}_S(\mathrm{C}^*_+(\Lambda))$ for $\mathrm{C}^*_+(\Lambda)$ inside $\mathrm{C}^*(\Lambda)$ is zero. However, the maximality of $\mathcal{J}_S(\mathrm{C}^*_+(\Lambda))$ and the invariance of $\mathrm{C}^*_+(G)$ under the gauge action of $\mathbb{T}$ on $\mathrm{C}^*(\Lambda)$ imply that $\mathcal{J}_S(\mathrm{C}^*_+(\Lambda))$ is a gauge-invariant ideal. Theorem 5.2 in **[24]** shows now that any non-zero gauge-invariant ideal contains at least one of the generating projections P_v, $v \in \Lambda^0$. Hence $\mathcal{J}_S(\mathrm{C}^*_+(\Lambda)) = \{0\}$, or otherwise the quotient map would not be faithful on $\mathrm{C}^*_+(\Lambda)$. ∎

Remark 3.7. In light of Theorems 2.6 and 2.7, it is natural to expect that similar results, with similar proofs, are valid in the higher rank context. Indeed, such results have been obtained by Raeburn, et al, making use of the product Hilbert bimodule theory. However, we would expect that a Wold decomposition type approach could yield simpler proofs.

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