PRIMITIVE LIMIT ALGEBRAS AND C*-ENVELOPES

KENNETH R. DAVIDSON AND ELIAS KATSOULIS

Abstract. In this paper we study irreducible representations of regular limit subalgebras of AF-algebras. The main result is twofold: every closed prime ideal of a limit of direct sums of nest algebras (NSAF) is primitive, and every prime regular limit algebra is primitive. A key step is that the quotient of an NSAF algebra by any closed ideal has an AF C*-envelope, and this algebra is exhibited as a quotient of a concretely represented AF algebra. When the ideal is prime, the C*-envelope is primitive. The GNS construction is used to produce algebraically irreducible (in fact n-transitive for all $n \geq 1$) representations for quotients of NSAF algebras by closed prime ideals. Thus the closed prime ideals of NSAF algebras coincide with the primitive ideals. Moreover these representations extend to $*$-representations of the C*-envelope of the quotient, so that a fortiori these algebras are also operator primitive. The same holds true for arbitrary limit algebras and the $\{0\}$ ideal.

The regular limit algebras form a class of operator algebras that has enjoyed a lot of attention in recent years. These are non-selfadjoint subalgebras of AF C*-algebras modeled upon diagraph algebras and constructed via a direct limit process that satisfies certain regularity conditions (see below for a precise definition). One of their attractive features is the availability of abstract schemes for their classification, parallel to the ones in the selfadjoint theory. Another one is the rich and tractable ideal structure that allows for considerations which are redundant in the selfadjoint case.

The purpose of this paper is to construct algebraically irreducible (primitive) representations of quotients of regular limit algebras. In a recent article [**16**], Hudson and the second author produced a variety of primitive TUHF algebras. Elaborating on a representation of Orr and Peters [**22**], it was shown that if the standard embedding appears infinitely many times in the presentation of a TUHF algebra $\mathcal{A}$, then $\mathcal{A}$ is primitive. This led to a complete classification of the primitive ideals

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of Power's lexicographic algebras, with applications to their epimorphic theory.

The question of when a TUHF, or more generally a TAF algebra, is primitive was implicit in [**16**] and was raised explicitly in several conferences. Similar questions had also been raised and investigated in [**21**] in the broader context of subalgebras of groupoid C*-algebras algebras. A well-known necessary condition for an ideal to be primitive is closed and prime. However, the converse is generally false for Banach algebras. In this paper, we establish the converse for NSAF algebras, which includes the strongly maximal TAF algebras. Moreover the representations constructed extend to $*$-representations of the C*-envelope of the quotient, and in particular are completely isometric representations as operator algebras. While this general result for ideals remains elusive for the larger class of regular limit algebras, we are able to deal with the $\{0\}$ ideal and show that if $\mathcal{A}$ is prime, then it is also primitive.

We adopt a C*-algebraic approach. It is a consequence of the Blecher–Ruan–Sinclair characterization [**5**] of operator algebras that the quotient of an operator algebra by a closed ideal is also an operator algebra. Moreover due the work of Arveson [**2, 3**] and Hamana [**14**], every operator algebra can be completely isometrically embedded in a unique minimal C*-algebra known as its C*-envelope. In general it is very difficult to identify this algebra. In our case, the algebra $\mathcal{A}$ comes with a presentation as a subalgebra of an AF C*-algebra. Because of the explicit nature of this presentation, we are able to describe an explicit presentation of each quotient as an NSAF algebra. The C*-envelope is always a quotient of the C*-algebra constructed, and thus will also be AF.

Hudson [**15**] characterized closed prime ideals in TAF algebras by a modification of Donsig's characterization [**12**] of semisimple TAF algebras. In the case when the ideal is prime, this allows a more precise description of the C*-envelope of the quotient sufficient to show that it is always primitive.

The next step is to use the GNS construction to obtain concrete representations of this C*-algebra in such a way that the image of the nonself-adjoint algebra is WOT-dense in $\mathcal{B}(\mathcal{H})$. This is a new approach to non-selfadjoint problems of this nature and we expect it to have additional applications. It is important that these are completely isometric representations of the quotient as operator algebras, not just representations on some arbitrary Banach space. It follows that the notion of operator primitive ideal and primitive ideal coincide for these algebras.

1. Preliminaries

We briefly recall the framework for studying limit algebras and list some necessary results. For more details see the monograph [**26**].

Let $\mathfrak{A}$ be an AF C*-algebra. For subalgebras $\mathcal{X}$, $\mathcal{Y}$ of $\mathfrak{A}$, let $\mathrm{PI}(\mathcal{X})$ denote the set of partial isometries in $\mathcal{X}$. The normalizer of $\mathcal{Y}$ in $\mathcal{X}$ is

$$\mathrm{N}_{\mathcal{Y}}(\mathcal{X}) = \{x \in \mathrm{PI}(\mathcal{X}) : xyx^*, x^*yx \in \mathcal{Y} \text{ for all } y \in \mathcal{Y}\}.$$

A maximal abelian selfadjoint subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ is called a *canonical masa* in $\mathfrak{A}$ if there is a nested sequence $(\mathfrak{A}_i)_i$ of finite dimensional subalgebras of $\mathfrak{A}$ so that $\mathfrak{A} = \overline{\bigcup_i \mathfrak{A}_i}$ and $\mathfrak{C}_i = \mathfrak{A}_i \cap \mathfrak{C}$ is a masa in $\mathfrak{A}_i$ for each $i \geq 1$ and $\mathrm{N}_{\mathfrak{C}_i}(\mathfrak{A}_i) \subseteq \mathrm{N}_{\mathfrak{C}_{i+1}}(\mathfrak{A}_{i+1})$. It follows that $\mathfrak{C} = \overline{\bigcup_i \mathfrak{C}_i}$ Let φ_i denote the $*$-monomorphism injecting $\mathfrak{A}_i$ into $\mathfrak{A}_{i+1}$ for this particular sequence.

A *regular limit algebra* $\mathcal{A}$ is a norm-closed subalgebra of an AF C*-algebra which contains a canonical masa $\mathfrak{C}$. With the choices of the previous paragraph, we define $\mathcal{A}_i = \mathcal{A} \cap \mathfrak{A}_i$. By a slight abuse of notation, we denote the injection of $\mathcal{A}_i$ into $\mathcal{A}_{i+1}$ by φ_i as well. As an $\mathfrak{C}$-bimodule, it follows [**26**] that $\mathcal{A}$ is inductive in the sense that $\mathcal{A} = \overline{\bigcup_i \mathcal{A}_i}$.

More abstractly, a norm closed subalgebra $\mathcal{A}$ of $\mathfrak{A}$ is a limit algebra if and only if it is the limit $\varinjlim(\mathcal{A}_i, \varphi_i)$ of a directed system

$$\mathcal{A}_1 \xrightarrow{\varphi_1} \mathcal{A}_2 \xrightarrow{\varphi_2} \mathcal{A}_3 \xrightarrow{\varphi_3} \mathcal{A}_4 \xrightarrow{\varphi_4} \cdots$$

where for each $i \geq 1$, $\mathcal{A}_i := \mathcal{A} \cap \mathfrak{A}_i$ satisfy

(i) $\mathfrak{C}_i \subseteq \mathcal{A}_i \subseteq \mathfrak{A}_i$,
(ii) φ_i extends to a $*$-monomorphism from $\mathfrak{A}_i$ into $\mathfrak{A}_{i+1}$, and
(iii) the extension of φ_i maps $\mathrm{N}_{\mathfrak{C}_i}(\mathfrak{A}_i)$ into $\mathrm{N}_{\mathfrak{C}_{i+1}}(\mathfrak{A}_{i+1})$.

The algebra $\mathfrak{A}_i$ may be decomposed as a sum $\sum_{j=1}^{\oplus k_i} \mathfrak{A}_{i,j}$ where $\mathfrak{A}_{i,j}$ is isomorphic to a full matrix algebra $\mathfrak{M}_{n_{i,j}}$ of all $n_{i,j} \times n_{i,j}$ matrices. One may choose a basis for each matrix algebra which will diagonalize the masa $\mathfrak{C}_i$. The matrix units for $\mathfrak{A}_i$ will be denoted by $e^{i,j}_{kl}$ for $1 \leq j \leq k_i$ and $1 \leq k, l \leq n_{i,j}$. The diagonal matrix units $e^{i,j}_{kk}$ span $\mathfrak{C}_i$. Let $p_{i,j} = \sum_{k=1}^{n_{i,j}} e^{i,j}_{kk}$ denote the projection onto the jth summand $\mathfrak{A}_{i,j}$.

The normalizing partial isometries in $\mathfrak{A}_i$ are just orthogonal sums of modulus one multiples of standard matrix units. So condition (iii) implies that each matrix unit is sent to a sum of modulus one multiples of matrix units with pairwise orthogonal domains and ranges. It is a routine exercise to recursively replace the basis vectors at each level by an appropriate modulus one multiple so that the scalars involved are all 1's, and each matrix unit is sent to a sum of matrix units.

Now condition (ii) means that each summand $\mathfrak{A}_{i,j}$ is embedded into $\mathfrak{A}_{i+1,k}$ with a certain multiplicity $m(i,j,k)$ by $\varphi_{i,j,k} = p_{i+1,k}\varphi_i|_{\mathfrak{A}_{i,j}}$. The normalizing condition means that this embedding decomposes as a direct sum of multiplicity one embeddings, meaning that there are $m(i,j,k)$ orthogonal projections $q_s = q_{i,j,k,s}$ in $\mathfrak{C}_{i+1}$ of rank $n_{i,j}$ such that $\varphi_{i,j,k}(\cdot) = \sum_{s=1}^{m(i,j,k)} q_s\varphi_{i,j,k}(\cdot)q_s$, and each $\varphi_{i,j,k,s}(\cdot) = q_s\varphi_{i,j,k}(\cdot)q_s$ is a unitary equivalence obtained by matching the basis for $\mathfrak{C}_i$ with the basis for $q_s\mathfrak{C}_{i+1}$.

The limit algebra $\mathcal{A}$ is called *triangular* AF (TAF) if $\mathcal{A}\cap\mathcal{A}^*$ equals a canonical masa $\mathfrak{C}$ in $\mathfrak{A}$. In this case, the basis for each matrix algebra may be ordered so that each algebra $\mathcal{A}_i\cap\mathfrak{A}_{i,j}$ is identified with a subalgebra of the upper triangular matrix algebra $\mathcal{T}_{n_{i,j}}$ of $\mathfrak{M}_{n_{i,j}}$.

If $\mathcal{A}$ is a TAF algebra and $\mathcal{A}+\mathcal{A}^*$ is dense in $\mathfrak{A}$, then $\mathcal{A}$ is called *strongly maximal triangular*. If $\mathcal{A}$ is a strongly maximal TAF algebra, then we can write $\mathcal{A} = \varinjlim(\mathcal{A}_i,\varphi_i)$, where each $\mathcal{A}_i$ is a direct sum of the full upper triangular algebras $\mathcal{T}_{n_{i,j}}$. Once this is done and φ_i is decomposed into a direct sum of multiplicity one embeddings, each of these embeddings will respect this order because each triangular algebra is mapped into another triangular algebra. More generally, we shall consider *nest subalgebras of AF-algebras* (NSAF) where each $\mathcal{A}_i$ is the direct sum of nest subalgebras of $\mathfrak{M}_{n_{i,j}}$. The standard basis will be ordered so that $\mathcal{A}_i$ contains $\mathcal{T}_{n_{i,j}}$ in each summand.

An element e of an algebra $\mathcal{A}$ is said to have a link in $\mathcal{A}$ if $e\mathcal{A}e \neq \{0\}$. A homomorphism φ between two algebras $\mathcal{A}$ and $\mathcal{B}$ is said to be *mixing* if for any element e in $\mathcal{A}$, $\varphi(e)$ has a link. For $*$-extendible homomorphisms between direct sums of upper triangular algebras preserving the normalizers, mixing simply means that the image of any matrix unit has a link in $\mathcal{B}$. A directed system $\mathcal{A} = \varinjlim(\mathcal{A}_i,\varphi_i)$ is called mixing if all the maps φ_i are mixing.

If $\mathcal{A} = \varinjlim(\mathcal{A}_i,\varphi_i)$ is a presentation for a limit algebra $\mathcal{A}$, then a *compression* of this presentation is a direct system of the form $\varinjlim(\mathcal{A}_{i_k},\psi_k)$ for some subsequence (i_k) of $\mathbb{N}$ and maps $\psi_k = \varphi_{i_{k+1}-1}\circ\cdots\varphi_{i_k+1}\circ\varphi_{i_k}$. It is easy to construct examples of non-mixing embeddings whose compositions are mixing.

Unlike their building blocks, the upper triangular matrices, strongly maximal triangular limit algebras may be semisimple. More generally semisimple TAF algebras are characterized by Donsig [**12**]. An inspection of the proof shows that this result is valid for all regular limit algebras.

Theorem 1.1 (Donsig)**.** *Let $\mathcal{A} = \varinjlim(\mathcal{A}_i,\varphi_i)$ be a regular limit algebra. Then the following are equivalent:*

(i) *$\mathcal{A}$ is semisimple*
(ii) *every element $e \in \mathrm{N}_{\mathfrak{C}}(\mathcal{A})$ has a link*
(iii) *there is a compression $\varinjlim(\mathcal{A}_{i_k}, \psi_k)$ of this presentation which is mixing.*

An ideal $\mathcal{I}$ of an algebra $\mathcal{A}$ is said to be *prime* if given any two ideals $\mathcal{J}_1, \mathcal{J}_2$, so that $\mathcal{J}_1\mathcal{J}_2 \subseteq \mathcal{I}$, then either $\mathcal{J}_1 \subseteq \mathcal{I}$ or $\mathcal{J}_2 \subseteq \mathcal{I}$. If $\mathcal{A}$ is a Banach algebra and $\mathcal{I}$ is closed, then it is easily seen that the ideals $\mathcal{J}_1$ and $\mathcal{J}_2$ in the above criterion can be taken to be closed. Hudson [**15**] characterized the closed prime ideals of a TAF algebra by a criterion based on Donsig's Theorem. Again the proof only requires $\mathcal{A}$ to be a regular limit algebra.

Theorem 1.2 (Hudson)**.** *A closed ideal $\mathcal{P}$ of a regular limit algebra $\mathcal{A}$ is prime if and only if for every pair $e, f \in \mathrm{N}_{\mathfrak{C}}(\mathcal{A})$ with $e, f \notin \mathcal{P}$, there is a link $s \in \mathrm{N}_{\mathfrak{C}}(\mathcal{A})$ such that $esf \notin \mathcal{P}$.*

An ideal $\mathcal{I}$ of an algebra $\mathcal{A}$ is said to be *primitive* if it is the kernel of an algebraically irreducible representation π of $\mathcal{A}$ on some vector space $\mathcal{X}$, meaning that the algebra $\pi(\mathcal{A})$ has no invariant manifolds. If $\{0\}$ is a primitive ideal, then $\mathcal{A}$ is said to be primitive. When $\mathcal{A}$ is a Banach algebra, the primitive ideals are closed and prime. The converse is not generally true. In the case of limit algebras, [**15**] and [**16**] show that for a variety of significant algebras the converse is valid. For AF-algebras, the equivalence is fairly elementary [**6**, Theorem 3.8]. For more general separable C*-algebras, one has an old result of Dixmier [**11**]:

Theorem 1.3 (Dixmier)**.** *Let $\mathfrak{A}$ be a separable* C*-*algebra. Then an ideal of $\mathfrak{A}$ is primitive if and only if it is closed and prime.*

A representation is topologically irreducible if it has no proper closed invariant subspaces. The famous Kadison Transitivity Theorem [**18**] (see [**9**, Theorem I.9.4]) shows that every topologically irreducible representation π of a C*-algebra is not only algebraically irreducible as well, it is in fact n-transitive for every $n \geq 1$. That is, given any finite dimensional subspace $\mathcal{M}$ of $\mathcal{H}_\pi$ and any operator T, there is an element $a \in \mathcal{A}$ so that $\pi(a)|\mathcal{M} = T|\mathcal{M}$. For more general operator algebras, this is no longer true. However a useful substitute is the following result of Hudson and the second author [**16**]. This result only establishes algebraic irreducibility, but by applying an old result of Rickart and Yood [**27**, Theorem 8.4], one concludes that the algebra is strictly dense, meaning that it is strictly n-transitive for all $n \geq 1$.

Lemma 1.4 (Hudson–Katsoulis)**.** *Let $\mathcal{A}$ be a closed subalgebra of $\mathcal{B}(\mathcal{H})$. If for some finite r, the r-ball of $\mathcal{A}$ is* WOT-*dense in the unit ball of $\mathcal{B}(\mathcal{H})$, then $\mathcal{A}$ is strictly dense in $\mathcal{B}(\mathcal{H})$.*

In the category of operator algebras, it is natural to define an operator algebra to be *operator primitive* if it has a completely isometric representation on a Hilbert space which is algebraically irreducible. When the operator algebra is given as a subalgebra of a C*-algebra, one can ask for a possibly stronger property that the representation extend to a $*$-representation of the enveloping C*-algebra. We shall call an algebra with this latter property *C*-primitive*. Dixmier's Theorem shows that separable C*-algebras are operator primitive (or equivalently C*-primitive) if they are primitive. So the representations constructed are in the right category.

In his seminal work on operator algebras and completely bounded maps, Arveson [**2, 3**] introduced the notion of the *C*-envelope* of an operator algebra. The C*-envelope $\mathfrak{A}$ of a unital operator algebra $\mathcal{A}$ is actually a pair $(\mathfrak{A}, i)$ consisting of the C*-algebra $\mathfrak{A}$ together with a completely isometric unital homomorphism i of $\mathcal{A}$ into $\mathfrak{A}$ with the following universal properties: $\mathfrak{A} = \mathrm{C}^*(i(\mathcal{A}))$ and if j is any completely isometric unital homomorphism j of $\mathcal{A}$ into another C*-algebra $\mathfrak{B} = \mathrm{C}^*(j(\mathcal{A}))$, then there is a (unique) $*$-homomorphism π of $\mathfrak{B}$ onto $\mathfrak{A}$ such that $i = \pi j$. The existence of this object was established by Hamana [**14**].

In general, it is very difficult to compute the C*-envelope of an abstractly presented operator algebra. In our case however, the limit algebras come with a presentation as a subalgebra of an AF C*-algebra. Thus the C*-envelope is a quotient of this C*-algebra, and thus is also AF. Laurie and Power [**20**] were able to show that starting with a direct limit of digraph algebras with (completely) contractive regular maps, the C*-envelope is always AF. Here we are concerned with quotients of NSAF algebras. Unfortunately there is no known relationship between the C*-envelope of an operator algebra and of one of its quotients. Because of the nice structure of the ideals of the upper triangular matrices, we are able to explicitly represent the quotient of a NSAF algebra by any closed ideal completely isometrically as a subalgebra of an AF-algebra. Thus the C*-envelope is also AF. When the ideal $\mathcal{P}$ is prime, an application of Hudson's Theorem will allow us to embed $\mathcal{A}/\mathcal{P}$ completely isometrically in a primitive AF algebra.

As an appetizer, we provide the following easy result for arbitrary regular limit algebras. The result mentioned above for quotients lies deeper.

Proposition 1.5. *Let $\mathcal{A} = \varinjlim(\mathcal{A}_i, \varphi_i)$ be a regular limit algebra. If $\mathcal{A}$ has a faithful topologically irreducible representation π on a Banach space $\mathcal{X}$, then $\mathrm{C}^*(\mathcal{A})$ is primitive.*

Proof. Let $\mathfrak{J}$ and $\mathfrak{K}$ be two non-zero ideals in $\mathfrak{A} = \mathrm{C}^*(\mathcal{A}) = \varinjlim(\mathfrak{A}_i, \varphi_i)$. The inductivity of closed ideals [**29, 26**] implies the existence of an integer i and non-zero projections $e \in \mathfrak{J} \cap \mathfrak{C}_i$ and $f \in \mathfrak{K} \cap \mathfrak{C}_i$. Since π is faithful and topologically irreducible, $\pi(\mathcal{A}f)\mathcal{X}$ is dense in $\mathcal{X}$ and thus $\pi(e\mathcal{A}f) \neq \{0\}$. A fortiori, $\mathfrak{J}\mathfrak{K} \supset e\mathfrak{A}f \neq \{0\}$. This shows that $\{0\}$ is a prime ideal, i.e. $\mathfrak{A}$ is prime. Hence $\mathfrak{A}$ is primitive by [**6**, Theorem 3.8] or by Dixmier's Theorem. ∎

2. C*-Envelopes of Quotients

Let us fix a closed ideal $\mathcal{I}$ in a NSAF algebra $\mathcal{A}$. If the C*-algebras $\mathfrak{A}$ and $\mathfrak{C}$ have inductive sequences $\mathfrak{A}_i$ and $\mathfrak{C}_i$, then $\mathcal{I}$ is the inductive limit of $\mathcal{I}_i := \mathfrak{A}_i \cap \mathcal{I} = \mathcal{A}_i \cap \mathcal{I}$ (see [**26**]). It is easy to see that $\mathcal{I}_i$ is an ideal of $\mathcal{A}_i$. By assumption, $\mathcal{A}_i$ is the direct sum of nest algebras $\mathcal{A}_{i,j}$ containing the triangular algebra $\mathcal{T}_{n_{i,j}}$ for $1 \le j \le k_i$. Since this contains a canonical masa, the ideal $\mathcal{I}_i$ is determined by the matrix units it contains. It is evident that one condition needed for a set of matrix units in $\mathcal{A}_{i,j}$ to span an ideal is that $e^{i,j}_{k,l} \in \mathcal{I}_i$ implies that $e^{i,j}_{k',l'} \in \mathcal{I}_i$ for all $k' \le k$ and $l' \ge l$. A necessary and sufficient in terms of the atoms of the nest is easy to describe, but will not be explicitly needed.

Let q denote the quotient map of $\mathcal{A}$ onto $\mathcal{A}/\mathcal{I}$, and let q_i denote the quotient maps of $\mathcal{A}_i$ onto $\mathcal{A}_i/\mathcal{I}_i$. Then there are induced maps $\bar{\varphi}_i$ from $\mathcal{A}_i/\mathcal{I}_i$ into $\mathcal{A}_{i+1}/\mathcal{I}_{i+1}$. These maps are completely isometric (see Corollary 2.5).

For each $i \ge 1$, let S_i denote the collection of all projections p in $\mathfrak{C}_i$ which are semi-invariant for $\mathcal{A}_i$, are supported on a single summand of $\mathfrak{A}_i$ and satisfy $(p\mathcal{A}_i p) \cap \mathcal{I}_i = \{0\}$. For notational convenience, we denote the typical projection in S_i by $p^{i,j}_{k,l} = \sum_{s=k}^{l} e^{i,j}_{s,s}$. It is easy to see that $p = p^{i,j}_{k,l}$ belongs to S_i if and only if p is an interval of the nest and $e^{i,j}_{k,l}$ is not in $\mathcal{I}_i$.

Form finite dimensional C*-algebras

$$\mathfrak{B}_i := \sum_{p \in S_i} \oplus\, \mathcal{B}(\operatorname{Ran} p)$$

where $\mathcal{B}(\operatorname{Ran} p)$ denotes the bounded operators on $\operatorname{Ran} p$; of course, $\mathcal{B}(\operatorname{Ran} p)$ is isomorphic to $\mathfrak{M}_{\operatorname{rank} p}$. Let σ_i be the completely positive map from $\mathfrak{A}_i$ into $\mathfrak{B}_i$ given by $\sigma_i(a) = \sum^{\oplus}_{p \in S_i} pap|_{\operatorname{Ran} p}$. The first lemma identifies $\mathcal{A}_i/\mathcal{I}_i$ via a completely isometric isomorphism with a subalgebra of $\mathfrak{B}_i$.

Lemma 2.1. *The map $\sigma_i|_{\mathcal{A}_i}$ from $\mathcal{A}_i$ into $\mathfrak{B}_i$ factors as $\rho_i q_i$ where q_i is the quotient map of $\mathcal{A}_i$ onto $\mathcal{A}_i/\mathcal{I}_i$ and ρ_i is a completely isometric homomorphism of $\mathcal{A}_i/\mathcal{I}_i$ into $\mathfrak{B}_i$. Moreover* $\mathrm{C}^*(\sigma_i(\mathcal{A}_i)) = \mathfrak{B}_i$.

Proof. Since each $p \in S_i$ is semi-invariant for $\mathcal{A}_i$, the compression map taking $a \in \mathcal{A}_i$ to pap is a homomorphism; and thus $\sigma_i|_{\mathcal{A}_i}$ is multiplicative. It is clear from the construction that $p\mathcal{I}_i p = \{0\}$ for every $p \in S_i$, and thus $\mathcal{I}_i$ is contained in $\ker \sigma_i|_{\mathcal{A}_i}$. Thus there is a uniquely defined homomorphism ρ_i of $\mathcal{A}_i/\mathcal{I}_i$ into $\mathfrak{B}_i$ so that $\rho_i q_i = \sigma_i|_{\mathcal{A}_i}$. It remains to show that ρ_i is completely isometric.

Let $A = \big[a_{st}\big]$ belong to $\mathfrak{M}_m(\mathcal{A}_i)$. We must show that

$$\big\|\sigma_i^{(m)}(A)\big\| = \operatorname{dist}(A, \mathfrak{M}_m(\mathcal{I}_i)).$$

Computing the right hand side is a matrix completion problem. Note first that as $\mathcal{A}_i$ is a direct sum of algebras $\mathcal{A}_{i,j}$ and $\mathcal{I}_i$ decomposes in a compatible way, that it suffices to assume that $\mathcal{A}_i = \mathcal{T}$ is a nest subalgebra of $\mathfrak{M}_n$ and $\mathcal{I}_i$ is an ideal $\mathcal{I}$ of $\mathcal{T}$. The distance of A to $\mathcal{I}$ is given by

$$\operatorname{dist}(A, \mathfrak{M}_m(\mathcal{I})) = \inf \big\| \big[b_{st}\big] \big\|$$

where $b_{s,t} = a_{s,t}$ if $e_{s,t} \notin \mathcal{I}$. The coefficients $b_{s,t}$ for $e_{s,t} \in \mathcal{I}$ are arbitrary.

Thus the entries are specified for the lower triangular part of the picture and may be completed arbitrarily in the upper triangular part corresponding to $\mathcal{I}$. This distance may be computed using the Arveson distance formula [**4**] (see also [**8**]) as

$$\operatorname{dist}(A, \mathfrak{M}_m(\mathcal{I})) = \sup \big\|q^{(m)} A r^{(m)}\big\|$$

where (q, r) runs over the set of all pairs of diagonal projections such that $q\mathcal{I}r = \{0\}$.

It is enough to consider maximal pairs. Because $\mathcal{I}$ is upper triangular, r may be enlarged to the smallest invariant projection dominating it, namely the range projection $\bar{r}$ of $\mathcal{T}r$; and likewise, q may be enlarged to the range projection $\bar{q}$ of $\mathcal{T}^*q$. For indeed

$$\{0\} = q\mathcal{I}r = q\mathcal{T}\mathcal{I}\mathcal{T}r = q\mathcal{T}(\bar{q}\mathcal{I}\bar{r})\mathcal{T}r$$

from which it follows that $\bar{q}\mathcal{I}\bar{r} = \{0\}$. Then $p = \bar{q}\bar{r}$ is a semi-invariant projection for $\mathcal{T}$ and $\bar{q}\mathcal{T}\bar{r} = p\mathcal{T}p$. In particular, $p\mathcal{I}p = \{0\}$, so that $p \in S_i$. Thus

$$\operatorname{dist}(A, \mathfrak{M}_m(\mathcal{I})) = \sup_{p \in S_i} \big\|p^{(m)} A p^{(m)}\big\| = \big\|\sigma_i^{(m)}(A)\big\|.$$

To evaluate $\mathrm{C}^*(\sigma_i(\mathcal{A}_i))$, it clearly suffices to consider the case of a single summand, say $\mathcal{A}_i \simeq \mathcal{T}$. Consider any maximal projection $p_{k,l}$ in S_i. If $q \in S_i$, the matrix unit $e_{k,l}$ is annihilated by the compression

$qe_{k,l}q$ except when $q = p_{k',l'}$ with $k' \le k$ and $l' \ge l$. Thus the maximality of p ensures that $\sigma_i(e_{k,l})$ is zero on every summand of $\mathfrak{B}_i$ except in $\mathcal{B}(\operatorname{Ran} p_{k,l})$. Hence if $k \le k', l' \le l$, then $\sigma_i(e_{k,k'})^*\sigma_i(e_{k,l})\sigma_i(e_{l',l})^*$ determines the matrix unit $e_{k',l'}$ in $\mathcal{B}(\operatorname{Ran} p_{k,l})$ and it vanishes in all other summands. By induction, one shows that $\mathrm{C}^*(\sigma_i(\mathcal{A}_i)) = \mathfrak{B}_i$. ∎

Remark 2.2. The curious reader should ask why we have to restrict ourselves to nest subalgebras of $\mathfrak{M}_n$ for these arguments. The reason is that the distance formula for the distance to an ideal in terms of compressions to intervals which are disjoint from the ideal is special property which is not valid in general. Consider the digraph algebra $\mathcal{A}$ of all 5×5 matrices of the form

$$\begin{bmatrix} * & 0 & a & b & x \\ 0 & * & c & y & d \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & * \end{bmatrix}$$

where the $*$'s, a, b, c, d, x, y are arbitrary entries. The ideal $\mathcal{I}$ consists of arbitrary entries for x and y.

Even considering elements with zero diagonal, the compressions to semi-invariant subspaces will yield the matrices

$$\begin{bmatrix} 0 & 0 & a \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & c & d \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

By [**10**, Theorem 2.1], the distance to $\mathcal{I}$ is not equal to the maximum of the norms of these three matrices, but rather a constant of $\sqrt{9/8}$ is required.

This does not mean that there is not a more clever way to embed these quotient algebras into matrix algebras. In particular, we raise as a test question: what is $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A}/\mathcal{I})$ for this example.

Now we describe the embedding of $\mathfrak{B}_i$ into $\mathfrak{B}_{i+1}$. Consider a projection $q \in S_{i+1}$. This is supported on some summand $\mathfrak{A}_{i+1,k}$. Recall that the identity $p_{i+1,k}$ of this summand decomposes as a sum of certain diagonal projections $q_{i,j,k,s}$ corresponding to multiplicity one embeddings of $\mathfrak{A}_{i,j}$ into $\mathfrak{A}_{i+1,k}$. Because q is an interval and the maps $\varphi_{i,j,k,s}$ preserve the order of the bases, it follows that $p = \varphi_{i,j,k,s}^{-1}(q)p_{i,j}$ is an interval of $\mathfrak{A}_{i,j}$. Now φ_i maps $\mathcal{I}_i$ into $\mathcal{I}_{i+1}$, and maps $p\mathcal{A}_{i,j}p$ isometrically into $q\mathcal{A}_{i+1,k}q$. Since $q \in S_{i+1}$, we have $q\mathcal{I}_{i+1}q = \{0\}$. Consequently $p\mathcal{I}_ip = \{0\}$, and thus $p \in S_i$. Each pair (p, q) obtained in this way determines a multiplicity one embedding of $\mathcal{B}(\operatorname{Ran} p)$ into $\mathcal{B}(\operatorname{Ran} q)$.

Clearly this completely defines a unital embedding π_i of $\mathfrak{B}_i$ into $\mathfrak{B}_{i+1}$. A moment's thought reveals that we have chosen those p which maximally embed into q in the sense that for each choice of a multiplicity one embedding of $\mathfrak{A}_{i,j}$ into $\mathfrak{A}_{i+1,k}$, one has chosen the unique largest interval projection p so that the natural map into $q\mathfrak{A}_{i+1,k}q$ is faithful on $p\mathfrak{A}_{i,j}p$.

Let $\mathfrak{B} = \varinjlim(\mathfrak{B}_i, \pi_i)$ be the AF algebra determined by this sequence.

Lemma 2.3. *With notation as above, $\sigma_{i+1}\varphi_i = \pi_i\sigma_i$.*

Proof. Let $a \in \mathfrak{A}_i$. Then $\sigma_i(a) = \sum^{\oplus}_{p\in S_i} pap|_{\operatorname{Ran} p}$ and thus $\pi_i\sigma_i(a)$ is a direct sum over $q \in S_{i+1}$ of the images of the various pap which are embedded faithfully into $\mathcal{B}(\operatorname{Ran} q) = q\mathfrak{A}_{i+1,k}q$ and are maximal with respect to this property. On the other hand,

$$\sigma_{i+1}\varphi_i(a) = \sum_{q\in S_{i+1}} \oplus\, q\varphi_i(a)q|_{\operatorname{Ran} q}.$$

Thus the same summands of a are mapped by φ_i into $\mathfrak{A}_{i+1,k}$ as a sum of multiplicity one embeddings, and then are mapped by σ_{i+1} to a direct sum of terms $q\varphi_i(a)q|_{\operatorname{Ran} q}$ which decompose as direct sums of compressions of the various multiplicity embeddings. One needs to observe what the resulting compression does on each multiplicity one piece. A moments reflection shows this to be the compressions to the same intervals p determining the map π_i, establishing the identity. ■

Theorem 2.4. *Let $\mathcal{A} = \varinjlim(\mathcal{A}_i, \varphi_i)$ be an NSAF algebra of the AF-algebra $\mathfrak{A} = \varinjlim(\mathfrak{A}_i, \varphi_i)$, and let $\mathcal{I}$ be a closed ideal. Then $\sigma = \varinjlim \sigma_i$ of $\mathfrak{A}$ into the AF C*-algebra $\mathfrak{B}$ is a unital completely positive map such that the restriction to $\mathcal{A}$ is a homomorphism which factors as $\sigma|_{\mathcal{A}} = \rho q$ where q is the quotient onto $\mathcal{A}/\mathcal{I}$ and ρ is a completely isometric homomorphism. Moreover $\mathrm{C}^*(\sigma(\mathcal{A})) = \mathfrak{B}$.*

Proof. The previous lemma shows that the following diagram commutes

$$\begin{array}{ccccccc}
\mathfrak{A}_1 & \xrightarrow{\varphi_1} & \mathfrak{A}_2 & \xrightarrow{\varphi_2} & \mathfrak{A}_3 & \xrightarrow{\varphi_3} & \cdots \\
\sigma_1\downarrow & & \sigma_2\downarrow & & \sigma_3\downarrow & & \\
\mathfrak{B}_1 & \xrightarrow{\pi_1} & \mathfrak{B}_2 & \xrightarrow{\pi_2} & \mathfrak{B}_3 & \xrightarrow{\pi_3} & \cdots
\end{array}$$

All the maps are completely positive and unital, and thus $\sigma = \varinjlim \sigma_i$ is unital and completely positive. By Lemma 2.1, the restriction of σ to each $\mathcal{A}_i$ is a homomorphism, and thus σ is a homomorphism. Moreover the kernel of $\sigma_i|_{\mathcal{A}_i}$ is precisely $\mathcal{I}_i$, and thus σ vanishes on $\mathcal{I}$. Hence $\sigma|_{\mathcal{A}}$ factors as ρq for a unique unital homomorphism ρ of $\mathcal{A}/\mathcal{I}$ into $\mathfrak{B}$. Again by Lemma 2.1 and Lemma 2.3, it follows that $\rho|_{\mathcal{A}_i} = \pi_i\rho_i$ factors as the completely isometric map ρ_i followed by the $*$-monomorphism π_i of $\mathfrak{B}_i$ into $\mathfrak{B}$. Hence ρ is completely isometric on each $\mathcal{A}_i$ and thus also on all of $\mathcal{A}$.

Finally, $\mathrm{C}^*(\sigma(\mathcal{A}))$ contains (the image of) $\mathrm{C}^*(\sigma_i(\mathcal{A}_i))$ which equals $\mathfrak{B}_i$ by Lemma 2.1. As the union of the $\mathfrak{B}_i$'s is dense in $\mathfrak{B}$, we obtain $\mathrm{C}^*(\sigma(\mathcal{A})) = \mathfrak{B}$. ∎

Corollary 2.5. *The injection of $\mathcal{A}_i/\mathcal{I}_i$ into $\mathcal{A}_{i+1}/\mathcal{I}_{i+1}$ is a complete isometry.*

Proof. This map is injective and $*$-extendible, hence completely isometric. ∎

Corollary 2.6. *Let $\mathcal{A} = \varinjlim(\mathcal{A}_i, \varphi_i)$ be an NSAF algebra of the AF-algebra $\mathfrak{A} = \varinjlim(\mathfrak{A}_i, \varphi_i)$, and let $\mathcal{I}$ be a closed ideal. Then the C*-envelope of $\mathcal{A}/\mathcal{I}$ is AF.*

Proof. This is immediate from Theorem 2.4 and Hamana's Theorem [**14**] because the C*-envelope is a quotient of $\mathrm{C}^*(\rho(\mathcal{A})) = \mathfrak{B}$, which is AF because $\mathfrak{B}$ is AF. ∎

Now we wish to specialize to prime ideals. The key additional ingredient is Hudson's Theorem. This allows us to explicitly exhibit the C*-envelope as a direct system.

Theorem 2.7. *Let $\mathcal{A} = \varinjlim(\mathcal{A}_i, \varphi_i)$ be a NSAF algebra of the AF-algebra $\mathfrak{A} = \varinjlim(\mathfrak{A}_i, \varphi_i)$, and let $\mathcal{P}$ be a closed prime ideal. Then the C*-envelope of $\mathcal{A}/\mathcal{P}$ is prime and hence primitive; and it can be expressed explicitly as a direct system.*

Proof. Hudson's Theorem 1.2 shows that if e, f are matrix units in $\mathcal{A}_i \setminus \mathcal{P}_i$, then there is a link s so that esf is not in $\mathcal{P}$. By dropping to a compression sequence, we may assume that every matrix unit in $\mathcal{A}_i \setminus \mathcal{P}_i$ has such a link in $\mathcal{A}_{i+1}$.

Consider two summands of $\mathfrak{B}_i$ corresponding to *maximal* elements of S_i, say $p = p^{i,j}_{k,l}$ and $p' = p^{i,j'}_{k',l'}$. Let $e = e^{i,j}_{kl}$ and $f = e^{i,j'}_{k'l'}$. There is a matrix unit s in $\mathcal{A}_{i+1}$ such that $x = \varphi_i(e)s\varphi_i(f)$ is not in $\mathcal{P}$. Let q be any maximal element of S_{i+1} such that $qxq = x$. Then in $\mathcal{B}(\operatorname{Ran} q)$, the element x factors in this way. Thus there is some compression of an image of $\mathfrak{A}_{i,j}$ containing a non-zero image of e in $\mathcal{B}(\operatorname{Ran} q)$. But by the maximality of p, there is no other projection r in S_i so that $\mathcal{B}(\operatorname{Ran} r)$ contains an image of e (see the proof of $\mathrm{C}^*(\sigma_i(\mathcal{A}_i)) = \mathfrak{B}_i$ in Lemma 2.1). Thus this means that $\mathcal{B}(\operatorname{Ran} p)$ is mapped into $\mathcal{B}(\operatorname{Ran} q)$ with non-zero multiplicity. Similarly, it follows that $\mathcal{B}(\operatorname{Ran} p')$ is mapped into the same summand.

Consider a subsystem of the directed limit $\varinjlim(\mathfrak{B}_i, \pi_i)$ corresponding to all summands which are *never* mapped into a summand $\mathcal{B}(\operatorname{Ran} p)$ where p is a maximal element of some S_i. Evidently this system is directed upwards. It is also hereditary in the sense that if every image of a summand lies in one of our selected blocks, then it clearly does not map into a maximal summand and thus already lies in our system. By [**6**] (see [**9**, Theorem III.4.2]), this system determines an ideal $\mathfrak{I}$ of $\mathfrak{B}$. The quotient $\mathfrak{B}' = \mathfrak{B}/\mathfrak{I}$ is the AF algebra corresponding to the remaining summands and the remaining embeddings. Let κ denote this quotient map.

In the quotient algebra $\mathfrak{B}'$, we see by construction that any two summands at level i are mapped into maximal summands at some higher level. Then by the arguments of the second paragraph, they are then mapped into two maximal summands on the same level and thence into a common maximal summand one level higher. Consequently for each i, there is a single summand at some higher level so that every summand at level i is mapped into this common summand at the higher level. Note for future reference that the argument of the second paragraph shows moreover that any pair of matrix units in $\mathcal{A}/\mathcal{P}$ has a link in $\mathfrak{B}'$ at some higher level. We may now drop to a compression and relabel if necessary so that every summand at level i is mapped into the first summand of level $i+1$ in such a way that every pair of matrix units in $\mathcal{A}_i/\mathcal{P}_i$ has a link in $\mathcal{A}_{i+1}$ supported in this first block $\mathfrak{B}'_{i+1,1}$ of level $i+1$. In particular, these embeddings are mixing.

Now suppose that $\mathfrak{J}$ and $\mathfrak{K}$ are two non-zero ideals in $\mathfrak{B}'$. Then at some level i, both have a non-empty intersection with $\mathfrak{B}'_i$. By the previous paragraph, they then both contain the first summand at the next level. Consequently $\mathfrak{J}\mathfrak{K} \neq \{0\}$. This shows that $\mathfrak{B}'$ is prime. By [**6**, Theorem 3.8] or by Dixmier's Theorem, it follows that $\mathfrak{B}'$ is primitive.

Consider the canonical map $\rho' = \kappa\rho$ of $\mathcal{A}/\mathcal{P}$ into $\mathfrak{B}'$. By Lemma 2.1, each coset $\bar{a}$ for an element $a \in \mathcal{A}$ has norm $\|\bar{a}\| = \sup_{p\in S_i} \|pap\|$. Moreover it is clear that if one only sups over the maximal elements of S_i that one obtains the same norm. Thus the map ρ' is isometric on $\mathcal{A}/\mathcal{P}$. Similarly one shows that ρ' is completely isometric. As in Theorem 2.4, we have $\mathfrak{B}' = \mathrm{C}^*(\rho(\mathcal{A}/\mathcal{P}))$.

It remains to show that $\mathfrak{B}'$ is the C*-envelope of $\mathcal{A}/\mathcal{P}$. This only requires that we show that the quotient by any ideal of $\mathfrak{B}'$ fails to be isometric on the image of $\mathcal{A}/\mathcal{P}$. Let $\mathfrak{J}$ be an ideal of $\mathfrak{B}'$ and let

$\mathfrak{J}_i = \mathfrak{J} \cap \mathfrak{B}'_i$. Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "q"] & \mathcal{A}/\mathcal{I} \arrow[r, "\rho"] & \mathfrak{B}' \arrow[r, "\kappa"] & \mathfrak{B}'/\mathfrak{J} \\
\mathcal{A}_{i+1} \arrow[u] \arrow[r, "q_{i+1}"] & \mathcal{A}_{i+1}/\mathcal{I}_{i+1} \arrow[u] \arrow[r, "\rho_{i+1}"] & \mathfrak{B}'_{i+1} \arrow[u] \arrow[r, "\kappa_{i+1}"] & \mathfrak{B}'_{i+1}/\mathfrak{J}_{i+1} \arrow[u] \\
\mathcal{A}_i \arrow[u, "\varphi_i"] \arrow[r, "q_i"] & \mathcal{A}_i/\mathcal{I}_i \arrow[u] \arrow[r, "\rho_i"] & \mathfrak{B}'_i \arrow[u, "\pi_i"] \arrow[r, "\kappa_i"] & \mathfrak{B}'_i/\mathfrak{J}_i \arrow[u]
\end{tikzcd}$$

If $\mathfrak{J}$ is non-zero, then it contains a non-zero summand at some level, and thus contains the first (maximal) summand $\mathfrak{B}'_{i,1}$ at the next level. This summand is determined by the maximal element $p^{i,1}_{k,l}$ of S_i. However the element $\rho_i q_i(e^{i,1}_{kl})$ is annihilated in every summand of $\mathfrak{B}'_i$ except for the summand $\mathfrak{B}'_{i,1}$. Consequently in the quotient by $\mathfrak{J}_i$, we see that $\kappa_i \rho_i q_i(e^{i,1}_{k,l}) = 0$. As the diagram commutes, it follows that $\kappa\rho q(e^{i,1}_{k,l}) = 0$, which shows that κ is not isometric on $\mathcal{A}/\mathcal{P}$. Thus $\mathfrak{B}'$ is the C*-envelope of $\mathcal{A}/\mathcal{P}$. ■

We record for future use the additional structure, most of which was obtained in the proof of the last theorem.

Corollary 2.8. *Let $\mathcal{A} = \varinjlim(\mathcal{A}_i, \varphi_i)$ be an NSAF algebra of the AF-algebra $\mathfrak{A} = \varinjlim(\mathfrak{A}_i, \varphi_i)$, and let $\mathcal{P}$ be a closed prime ideal. Then the C*-envelope $\mathrm{C}^*_{env}(\mathcal{A}/\mathcal{P})$ is an AF-algebra which has a presentation $\mathfrak{B}' = \varinjlim(\mathfrak{B}'_i, \psi_i)$ as a direct limit of finite dimensional C*-subalgebras $\mathfrak{B}'_i$ with the following properties:*

- (i) *There is a sequence of subalgebras $\mathcal{A}'_i$ of $\mathfrak{B}'_i$ which are completely isometrically isomorphic to $\mathcal{A}_{k_i}/\mathcal{P}_{k_i}$ such that $\mathcal{A}/\mathcal{P} = \varinjlim(\mathcal{A}'_i, \psi_i)$.*
- (ii) *The maps ψ_i map every summand of $\mathfrak{B}'_i$ into the first block $\mathfrak{B}'_{i+1,1}$.*
- (iii) *The embedding of $\mathcal{A}'_i$ into the compression $\mathcal{A}'_{i+1,1}$ of $\mathcal{A}'_{i+1}$ to this first block is mixing.*
- (iv) *The compression $\mathcal{A}'_{i,j}$ of $\mathcal{A}'_i$ to the summand $\mathfrak{B}'_{i,j}$ of $\mathfrak{B}'_i$ is a nest algebra containing the upper triangular matrices (with respect to an appropriate order on the basis of $\mathfrak{C}'_i$). The matrix units of $\mathcal{A}'_i$ are partial isometries which are rank 0 or 1 in each summand.*
- (v) *$\mathfrak{B}'$ contains a canonical masa $\mathfrak{C}' = \varinjlim(\mathfrak{C}'_i, \psi_i)$ where each $\mathfrak{C}'_i$ is a masa in $\mathfrak{B}'_i$ and $\psi_i(\mathrm{N}_{\mathfrak{C}'_i}(\mathfrak{B}'_i)) \subset \mathrm{N}_{\mathfrak{C}'_{i+1}}(\mathfrak{B}'_{i+1})$.*

Proof. Parts (i), (ii) and (iii) are contained in the proof above.

Since $\mathcal{A}$ is NSAF, each $\mathcal{A}_i$ is the direct sum of nest algebras. It is evident that the compression by any projection $p \in S_i$ maps the nest algebra onto a nest algebra in $\mathcal{B}(\operatorname{Ran} p)$. Since the quotient by $\mathfrak{J}$ merely deletes certain summands, this property remains in the quotient. Also it is clear that each matrix unit of $\mathcal{A}_i/\mathcal{P}_i$ is sent by σ_i to a direct sum of a matrix unit or 0 in each summand.

The masa $\mathfrak{C}'$ is defined in the natural way by setting $\mathfrak{C}'_i$ to be the unique masa generated by $\mathfrak{C}_{k_i}/(\mathfrak{C}_{k_i} \cap \mathcal{P}_{k_i})$ and the centre $Z(\mathfrak{B}'_i)$. With the canonical basis thereby obtained for each $\mathcal{B}(\operatorname{Ran} p)$ by keeping the basis vectors dominated by p, the algebra $\mathfrak{C}'_i$ is seen to consist of the diagonal operators. The definition of ψ_i as the amalgamation of the compressions of the maps ψ_i to the various summands $\mathcal{B}(\operatorname{Ran} p)$ shows that matrix units are taken to sums of matrix units, and thus this map is regular with respect to $\mathfrak{C}'$. ∎

Remark 2.9. When the original algebra is TUHF, additional hypotheses [**16**] such as lexicographic ordering allow one to show that prime ideals intersect each matrix algebra $\mathcal{T}_{k_n}$ in a prime ideal of this subalgebra. The prime ideals of $\mathcal{T}_n$ are of the form $\mathcal{P}_n = \{A : PAP = 0\}$ for some interval P, and thus $\mathcal{T}_n/\mathcal{P}_n$ is completely isometrically isomorphic to $P\mathcal{T}_n|_{P\mathcal{H}} \simeq \mathcal{T}_m$ where $m = \operatorname{rank} P$. In this case, the C*-envelope construction is seen to have a single block at each level and a UHF algebra is obtained.

In general however, there are prime ideals in TUHF algebras which do not intersect any $\mathcal{T}_n$ in a prime ideal, and the enveloping AF-algebra is not simple (and hence not UHF), as the following example shows.

The construction is a bit messy, so we only sketch the ideas. Start with the ideal $\mathcal{P}_1 = \mathbb{C}e_{13}$ of $\mathcal{T}_3$. Suppose that we have constructed a chain of ideals $\mathcal{P}_k$ in $\mathcal{T}_{n_k}$ and $*$-embeddings of $\mathfrak{M}_{n_{k-1}}$ into $\mathfrak{M}_{n_k}$ taking $\mathcal{T}_{n_{k-1}}$ into $\mathcal{T}_{n_k}$ and so that two important things occur. The first is that $\mathcal{P}_n$ is the ideal generated by the image of $\mathcal{P}_{n-1}$; and secondly, every pair of matrix units in $\mathcal{T}_{n_{k-1}} \setminus \mathcal{P}_{n-1}$ has a link with product in $\mathcal{T}_{n_k} \setminus \mathcal{P}_n$. The first condition makes the ideals $\mathcal{P}_n$ nested. The latter condition guarantees that the limit ideal $\mathcal{P}$ is prime in the limit algebra $\mathcal{A}$ by Hudson's Theorem. Moreover it guarantees that no matrix unit of $\mathcal{T}_{n_k}$ lies in any $\mathcal{P}_l$ for any $l > k$. Thus $\mathcal{P} \cap \mathcal{T}_{n_k} = \mathcal{P}_k$.

Given the ideal $\mathcal{P}_k$ of $\mathcal{T}_{n_k}$, let $I_{k,i}$ for $1 \le i \le s_k$ denote the maximal intervals of $\mathcal{T}_{n_k}$ with compressions which annihilate $\mathcal{P}_k$. (Note that there are s_k maximal intervals, but $s_k - 1$ is the minimal number of matrix units needed to generate $\mathcal{P}_k$.) We will define a multiplicity s_k embedding of $\mathfrak{M}_{n_k}$ into $\mathfrak{M}_{n_{k+1}}$ where $n_{k+1} = s_k n_k$ as the direct sum of s_k multiplicity one embeddings. For each $2 \le i \le s_k$, let $I^-_{k,i}$ be the invariant interval so that $I^-_{k,i} \cup I_{k,i}$ is also invariant. Similarly let $I^+_{k,i}$ be the remaining coinvariant interval for $1 \le i \le s_k - 1$. Let the image intervals under the ith multiplicity one embedding of these intervals be denoted by $J^{\pm}_{k,i}$. The embedding is accomplished by ordering these intervals as follows:

$$J^-_{k,s_k} J^-_{k,s_k-1} \dots J^-_{k,2} \quad J_{k,s_k} J_{k,s_k-1} \dots J_{k,1} \quad J^+_{k,s_k-1} J^+_{k,s_k-2} \dots J^+_{k,1}$$

Notice that in the middle of this sequence, the maximal blocks are represented in reverse order. This ensures that they are all contained in a maximal interval of $\mathcal{P}_{k+1}$, and thus all the desired links exist. The new ideal $\mathcal{P}_{k+1}$ can be seen to have $s_{k+1} - 1 = s_k(s_k - 1)$ generating matrix units. In particular, $\mathcal{P}_k$ is not prime, and $s_{k+1} = s_k^2 - s_k + 1$. The AF-algebra constructed in Theorem 2.4 has at least s_k summands at the kth level because the maximal intervals all need to occur. Moreover, in the quotient to the C*-envelope, these maximal blocks are always retained. However the first maximal block of one level is never mapped into the last maximal block at a higher level. Thus this AF-algebra is not simple, and in particular is not UHF.

Remark 2.10. It is important to note that the algebra $\mathcal{A}/\mathcal{P}$, as a subalgebra of $\mathfrak{B}'$, need not contain the masa $\mathfrak{C}'$. So this is not a limit algebra. This is a stumbling block, but we are able to work around it. We are unable to determine if this is just a limitation of our proof, or if there really are examples for which $\mathcal{A}/\mathcal{P}$ is not a limit algebra.

3. Irreducible Representations

In this section, we proceed to construct irreducible representations. We will need the following lemma which provides additional information about mixing homomorphisms. Let ξ^1_i, $1 \le i \le n$ and ξ^2_j, $1 \le j \le m$ be the standard bases for $\mathbb{C}^n$ and $\mathbb{C}^m$, and let e_{ij} and f_{ij} be the corresponding matrix units for $\mathfrak{M}_n$ and $\mathfrak{M}_m$ respectively.

Lemma 3.1. *Let φ be an (not necessarily unital) regular embedding of $\mathfrak{M}_n$ into $\mathfrak{M}_m$ such that φ is a mixing embedding of $\mathcal{T}_n$ into $\mathcal{T}_m$. Then there are two regular isometries u, v of $\mathbb{C}^n$ into $\mathbb{C}^m$ such that $f = ue_{n1}v^*$ is a matrix unit in $\mathcal{T}_m$ and*

$$\varphi(a) = uau^* + vav^* + (I - uu^* - vv^*)\varphi(a)(I - uu^* - vv^*).$$

for all $a \in \mathfrak{M}_n$. Moreover $\varphi(e_{in}) f \varphi(e_{1j}) = ue_{ij}v^*$ for $1 \le i, j \le n$.

Proof. Since φ is mixing, there is a matrix unit $f = f_{pq} \in \mathcal{T}_m$ such that $\varphi(e_{1n}) f_{pq} \varphi(e_{1n}) \ne 0$. In particular, $\zeta_1 = \varphi(e_{1n})\xi^2_p$ and $\eta_n = \varphi(e_{1n})^*\xi^2_q$ are non-zero. As $\varphi(e_{ij})$ is a sum of matrix units, its action on a standard basis vector is to send it to another basis vector or to 0. Thus there are integers k_1 and l_n so that $\zeta_1 = \xi^2_{k_1}$ and $\eta_n = \xi^2_{l_n}$.

Define $\zeta_i = \varphi(e_{i,n})\xi^2_p$ and $\eta_i = \varphi(e_{i1})\xi^2_q$ for $1 \le i \le n$. Observe that from the definitions,

$$\begin{aligned} \varphi(e_{1i})\zeta_i &= \zeta_1 \quad \text{for} \quad 1 \le i \le n \\ \varphi(e_{in})^*\eta_i &= \eta_n \quad \text{for} \quad 1 \le i \le n \\ f_{pq}\eta_1 &= \zeta_n \end{aligned}$$

It follows that there are integers k_i and l_i so that $\zeta_i = \xi^2_{k_i}$ and $\eta_i = \xi^2_{l_i}$ for $1 \le i \le n$. Moreover since $\varphi(e_{1i})$, $\varphi(e_{in})$ and f_{pq} are strictly upper triangular, it follows that

$$k_1 < k_2 < \cdots < k_n = p < q = l_1 < l_2 < \cdots < l_n.$$

It is evident from the relations of the matrix units that $\mathcal{M}_1 = \operatorname{span}\{\xi^2_{k_i} : 1 \le i \le n\}$ and $\mathcal{M}_2 = \operatorname{span}\{\xi^2_{l_i} : 1 \le i \le n\}$ are reducing subspaces for $\varphi(\mathfrak{M}_n)$. Now define two isometries by $u\xi^1_i = \xi^2_{k_i}$ and $v\xi^1_i = \xi^2_{l_i}$ for $1 \le i \le n$, and let $Q = (P_{\mathcal{M}_1} + P_{\mathcal{M}_2})^\perp$. It is evident that $\varphi(a) = uau^* + vav^* + Q\varphi(a)Q$. Also

$$ue_{n1}v^*\eta_j = ue_{n1}\xi^1_l = \delta_{1j}u\xi^1_n = \delta_{1j}\zeta_n = f\eta_j$$

As both f and $ue_{n1}v^*$ vanish on the complement of $\operatorname{Ran} v$, it follows that $f = ue_{n1}v^*$. Now $\varphi(a)$ commutes with uu^* and vv^*, so

$$\varphi(e_{in})f\varphi(e_{1j}) = \varphi(e_{in})ue_{n1}v^*\varphi(e_{1j}) = ue_{in}e_{n1}e_{1j}v^* = ue_{ij}v^*. \qquad \blacksquare$$

Theorem 3.2. *Let $\mathcal{A} = \varinjlim(\mathcal{A}_i, \varphi_i)$ be an NSAF algebra. Then every closed prime ideal $\mathcal{P}$ of $\mathcal{A}$ is C*-primitive. Moreover the primitive representation constructed is n-transitive for all $n \ge 1$.*

Proof. The goal will be to use the GNS construction to obtain a faithful irreducible representation π of $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A}/\mathcal{P})$ in such a way that the 2-ball of $\pi(\mathcal{A}/\mathcal{P})$ contains the unit ball of $\mathcal{B}(\mathcal{H})$ in its WOT-closure. Then an application of Lemma 1.4 will show that $\pi(\mathcal{A}/\mathcal{P})$ has no invariant manifolds, and thus π is a primitive representation. Finally, if one wishes to consider $\mathcal{A}/\mathcal{P}$ (completely isometrically embedded) as a subalgebra of some other C*-algebra $\mathfrak{B}$ via an embedding j, then by definition of the C*-envelope, there is a $*$-homomorphism ρ of $\mathrm{C}^*(j(\mathcal{A}/\mathcal{P}))$ onto $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A}/\mathcal{P})$ which is the 'identity' on $\mathcal{A}/\mathcal{P}$ in the sense that $i = \rho j$. Composing this with π yields an irreducible $*$-representation of $\mathfrak{B}$ which is primitive and faithful on $\mathcal{A}/\mathcal{P}$.

We will make use of the structure of $\mathfrak{B} = \mathrm{C}^*_{\mathrm{env}}(\mathcal{A}/\mathcal{P})$ obtained in Corollary 2.8. In particular, we assume that $\mathfrak{B} = \varinjlim(\mathfrak{B}_i, \psi_i)$ and $\mathcal{A}/\mathcal{P} = \varinjlim(\mathcal{A}_i, \psi_i)$. Moreover ψ_i maps $\mathfrak{B}_i$ isometrically into the first summand $\mathfrak{B}_{i+1,1}$ of $\mathfrak{B}_{i+1}$ and this map is mixing for the injection of $\mathcal{A}_i$ into the first block of $\mathcal{A}_{i+1}$. Also the compression of $\mathcal{A}_i$ to the summand $\mathfrak{B}_{i,1}$ is a nest algebra containing upper triangular matrices.

At this stage, we apply Lemma 3.1 to the compression $\psi_{i,11}$ of ψ_i to the injection of matrix algebra $\mathfrak{B}_{i,1} \simeq \mathfrak{M}_{n_i}$ into $\mathfrak{B}_{i+1,1} \simeq \mathfrak{M}_{n_{i+1}}$. The triangular subalgebra is contained in the compression $\mathcal{A}_{i,1}$ of $\mathcal{A}_i$ to $\mathfrak{B}_{i,1}$, and similarly for $\mathcal{A}_{i+1,1}$ in $\mathfrak{B}_{i+1,1}$. Consider $\mathfrak{B}_{i,1}$ as acting on a Hilbert space $\mathcal{H}_i \simeq \mathbb{C}^{n_i}$ with standard matrix unit system $\{e^{i,1}_{kl}\}$. We obtain isometries u_i and v_i of $\mathcal{H}_i$ into $\mathcal{H}_{i+1}$ such that $f = u_i e^{i,1}_{n_i 1} v_i^*$ is a matrix unit in $\mathcal{A}_{i+1,1}$ and for all $a \in \mathfrak{B}_{i,1}$,

$$\psi_{i,11}(a) = u_i a u_i^* + v_i a v_i^* + q_i \psi_i(a) q_i$$

where $q_i = 1 - u_i u_i^* - v_i v_i^*$. Let f_{i+1} be the matrix unit in $\mathcal{A}_{i+1}$ whose compression to $\mathfrak{B}_{i+1,1}$ is f. Then $u_i^* f_{i+1} v_i = e^{i,1}_{n_i 1}$. In addition we have

$$\psi_i(e^i_{kn_i}) f_{i+1} \psi_i(e^i_{1l}) = \psi_i(e^{i,1}_{kn_i}) f_{i+1} \psi_i(e^{i,1}_{1l}) = u_i e^{i,1}_{kl} v_i^*$$

for all $1 \leq i, j \leq n_i$.

Now we define a sequence of unit vectors ζ_i in $\mathcal{H}_i$ as follows. Choose ζ_1 to be any unit vector in $\mathcal{H}_1$. Then recursively define

$$\zeta_{i+1} = \tfrac{1}{\sqrt{2}} u_i \zeta_i + \tfrac{1}{\sqrt{2}} v_i \zeta_i \quad \text{for} \quad i \geq 1.$$

Let $\omega_i(a) = \omega_{\zeta_i}(a) = \langle a\zeta_i, \zeta_i \rangle$ be the corresponding vector state on $\mathfrak{B}_i$.

Notice that if $a \in \mathfrak{B}_{i,1}$, then

$$\begin{aligned} \omega_{i+1}\psi_i(a) &= \tfrac{1}{2}\big\langle (u_i a u_i^* + v_i a v_i^* + q_i \psi_i(a) q_i)(u_i\zeta_i + v_i\zeta_i), (u_i\zeta_i + v_i\zeta_i) \big\rangle \\ &= \tfrac{1}{2}\langle a\zeta_i, \zeta_i\rangle + \tfrac{1}{2}\langle a\zeta_i, \zeta_i \rangle = \omega_i(a). \end{aligned}$$

Since ψ_i maps $\mathfrak{B}_i$ faithfully into $\mathfrak{B}_{i+1,1}$, it follows that

$$\omega_{i+2}\psi_{i+1}\psi_i = \omega_{i+1}\psi_i \quad \text{for all} \quad i \geq 1.$$

Consequently, we may define a state ω on $\mathfrak{B}$ by $\omega(a) = \varinjlim \omega_i$, and it has the property that $\omega(a) = \omega_{i+1}\psi_i(a)$ for all $a \in \mathfrak{B}_i$.

Let $\pi = (\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ denote the cyclic representation obtained from ω by the GNS construction. We wish to observe that π is a faithful irreducible representation of $\mathfrak{B}$. Indeed, each ω_i is a pure state on $\mathfrak{B}_i$. It follows that ω is pure. (This is the easy direction of [**6**, Corollary 4.16].) Thus the representation π is irreducible c.f. [**9**, Theorem I.9.8]. Moreover this representation is non-trivial on the simple summand $\mathfrak{B}_{i+1,1}$. Therefore it is injective on $\mathfrak{B}_{i+1,1}$. However ψ_i injects $\mathfrak{B}_i$ faithfully into $\mathfrak{B}_{i+1,1}$. Hence π is faithful on each $\mathfrak{B}_i$ for all $i \geq 1$, and thus is faithful.

Our next goal is to show that $\pi(\mathcal{A}/\mathcal{P})$ is WOT-dense in $\pi(\mathfrak{B})$. Let a be an arbitrary element in $\mathfrak{B}_{i,1}$, and write it as $a = \sum_{k,l=1}^{n_i} \alpha_{kl} e^{i,1}_{kl}$. From

Corollary 2.8(iv), we know that if $k \le l$, then $e^{i,1}_{kl}$ is the compression to $\mathfrak{B}_{i,1}$ of a partial isometry e^i_{kl} in $\mathcal{A}_i$. Define an element $\hat{a}$ by

$$\hat{a} = u_i a v_i^* = \sum_{k,l=1}^{n_i} \alpha_{kl} u_i e^{i,1}_{kl} v_i^* = \sum_{k,l=1}^{n_i} \alpha_{kl} \psi_i(e^i_{kn_i}) f_{i+1} \psi_i(e^i_{1l})$$

These formulae show that $\hat{a} \in \mathcal{A}_{i+1}$ and $\|\hat{a}\| = \|a\|$.

Observe that

$$\begin{aligned}\omega\big(\psi_i(e^i_{kn_i}) f_{i+1} \psi_i(e^i_{1l})\big) &= \omega\big(\psi_i(e^{i,1}_{kn_i}) f_{i+1} \psi_i(e^{i,1}_{1l})\big) \\ &= \langle u_i e^{i,1}_{kl} v_i^* \zeta_{i+1}, \zeta_{i+1}\rangle = \tfrac{1}{2}\langle e^{i,1}_{kl}\zeta_i, \zeta_i\rangle = \tfrac{1}{2}\omega(e^{i,1}_{kl}).\end{aligned}$$

Therefore for any two matrix units $e^{i,1}_{pq}$ and $e^{i,1}_{rs}$ in $\mathfrak{B}_{i,1}$,

$$\begin{aligned}\big\langle \pi(a)\pi(e^{i,1}_{pq})\xi_\omega, \pi(e^{i,1}_{rs})\xi_\omega\big\rangle &= \omega(e^{i,1\,*}_{rs} a e^{i,1}_{pq}) = \omega(\alpha_{rp} e^{i,1}_{rq}) \\ &= 2\alpha_{rp}\omega\big(\psi_i(e^i_{rn_i}) f_{i+1} \psi_i(e^i_{1q})\big) \\ &= 2\alpha_{rp}\omega\big(\psi_i(e^i_{rs})\psi_i(e^1_{sn_i}) f_{i+1} \psi_i(e^i_{1p})\psi_i(e^i_{pq})\big) \\ &= 2\omega\big(\psi_i(e^i_{rs})\hat{a}\psi_i(e^i_{pq})\big) \\ &= 2\big\langle \pi(\hat{a})\pi(e^i_{pq})\xi_\omega, \pi(e^i_{rs})\xi_\omega\big\rangle \\ &= 2\big\langle \pi(\hat{a})\pi(e^{i,1}_{pq})\xi_\omega, \pi(e^{i,1}_{rs})\xi_\omega\big\rangle\end{aligned}$$

The set of vectors of the form $\pi(e^{j,1}_{rs})\xi_\omega$ for $j \ge 1$ and $1 \le r,s \le n_j$ are dense in $\mathcal{H}_\omega$. Thus if a lies in the unit ball of $\mathfrak{B}_i$, let a_j denote the component of $\psi_j\psi_{j-1}\dots\psi_i(a)$ supported on $\mathfrak{B}_{j,1}$ for all $j > i$. Then $\omega(a) = \omega(a_j)$ for all $j > i$. The construction above now shows that there is an element $\hat{a}_j$ in the unit ball of $\mathcal{A}_{j+1}$ such that

$$\begin{aligned}\big\langle \pi(a)\pi(e^{j,1}_{pq})\xi_\omega, \pi(e^{j,1}_{rs})\xi_\omega\big\rangle &= \big\langle \pi(a_j)\pi(e^{j,1}_{pq})\xi_\omega, \pi(e^{j,1}_{rs})\xi_\omega\big\rangle \\ &= \big\langle \pi(2\widehat{a_j})\pi(e^{j,1}_{pq})\xi_\omega, \pi(e^{j,1}_{rs})\xi_\omega\big\rangle.\end{aligned}$$

Consequently the 2-ball of $\pi(\mathcal{A}/\mathcal{P})$ is WOT-dense in the unit ball of $\overline{\pi(\mathfrak{B})}^{\text{WOT}} = \mathcal{B}(\mathcal{H}_\omega)$. Now an application of Lemma 1.4 shows that $\pi(\mathcal{A})$ is n-transitive for all $n \ge 1$, and in particular is algebraically irreducible. So we have constructed a faithful $*$-representation of $\mathrm{C}^*_{\text{env}}(\mathcal{A}/\mathcal{P})$ which is a primitive representation of $\mathcal{A}/\mathcal{P}$. Hence $\mathcal{P}$ is a C*-primitive ideal. ■

There is a great deal of freedom in defining the state ω. At the first step (which could start at any $\mathcal{A}_{k_1}$), we begin with an arbitrary vector. At each stage, the choice is determined by the link f_{i+1}. However it is easy to arrange for there to be two (or more) distinct links intertwining different pairs of multiplicity one embeddings of $\mathcal{A}_i$ into $\mathcal{A}_{i+1}$, say $f^{\varepsilon_i}_{i+1}$

for $\varepsilon_i \in \{1, 2\}$. In this way, for each choice of an infinite sequence (ε_i), one obtains a different pure state at a limit of vector states. Any two distinct sequences differ at some point k. At this point of the construction, the vectors ζ_k constructed by the two methods become orthogonal. They remain orthogonal in all subsequent steps.

Each choice yields an irreducible representation of $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A}/\mathcal{P})$. With a little more care, we can show that this yields a large collection of inequivalent representations.

Corollary 3.3. *Let $\mathcal{A}$ be a primitive NSAF algebra and let $\mathcal{P}$ be a closed prime ideal. Then $\mathcal{A}/\mathcal{P}$ admits uncountably many inequivalent $\mathrm{C}^*_{env}(\mathcal{A}/\mathcal{P})$-extendible algebraically irreducible representation on a separable Hilbert space $\mathcal{H}$.*

Proof. Assume that ω_1 and ω_2 are two pure states of an AF algebra $\mathfrak{B} = \varinjlim(\mathfrak{B}_i, \psi_i)$ that produce equivalent irreducible representations π_1 and π_2. Then by [**19**, Theorem 10.2.6], there exists a unitary $u \in \mathfrak{A}$ so that $\omega_1 = \omega_2 \operatorname{ad}_u$, where $\operatorname{ad}_u(a) = uau^*$. Every unitary in an AF algebra is a limit of unitaries in its finite dimensional subalgebras. Hence there is a unitary v in some $\mathfrak{B}_i$ so that $\|\omega_1 - \omega_2 \operatorname{ad}_v\| < 1$. However if our two sequences differ beyond the ith level, then ω_1 and $\omega_2 \operatorname{ad}_v$ correspond to a limit of vector states which are orthogonal and thus these states are distance 1 apart. Consequently any two sequences which are not tail equivalent yield inequivalent representations. Since any equivalence class for tail equivalence contains countably many sequences (ϵ_i), it is evident that there are uncountably many equivalence classes. ■

It is well-known for any Banach algebra that the kernel of a topologically irreducible representation is closed and prime. Thus the following five conditions are successively weaker than the next. So combining this fact with the previous theorem, we obtain:

Corollary 3.4. *For a closed ideal $\mathcal{I}$ of an NSAF algebra, the following are equivalent:*

(i) *$\mathcal{I}$ is C*-primitive.*
(ii) *$\mathcal{I}$ is operator primitive.*
(iii) *$\mathcal{I}$ is primitive.*
(iv) *$\mathcal{I}$ is the kernel of a topologically irreducible representation.*
(v) *$\mathcal{I}$ is closed and prime.*

4. Primitive Limit Algebras

In this section, we show how to modify the technique of the previous section to deal with arbitrary regular limit algebras at the expense of having to limit ourselves to the algebra itself, rather than quotients. The reason for avoiding quotients by an ideal is that we do not have the analogue of Theorem 2.4 for these quotients because of Remark 2.2. However the algebra itself comes with a nice embedding into an AF-algebra provided. As the proof is essentially the same as for Theorem 3.2, we only outline the method.

Theorem 4.1. *Let $\mathcal{A}$ be a regular limit algebra. If $\mathcal{A}$ is prime, then it is C*-primitive.*

Proof. By Proposition 1.5, the C*-algebra $\mathfrak{A}$ is primitive. Thus by dropping to a compression of the original sequence, we may suppose that each summand of $\mathfrak{A}_i$ is mapped into the first block $\mathfrak{A}_{i+1,1}$ of $\mathfrak{A}_i$.

The key again is Hudson's Theorem 1.2. Since $\{0\}$ is a prime ideal, any two non-zero elements in $\mathcal{A}$ have a link. Let $e_1, \dots, e_{k_i}$ be the (diagonal) matrix units of $\mathfrak{C}_{i,1}$. We inductively find matrix units x_j and y_j living in some algebra $\mathcal{A}_{i'}$ so that

$$e_1 x_1 e_2 x_2 \dots x_{k_i-1} e_{k_i} x_{k_i} e_1 y_1 e_2 y_2 \dots y_{k_i-1} e_{k_i} y_{k_i} e_1 \neq 0.$$

It is now easy to further compress our sequence so that these elements all live in $\mathcal{A}_{i+1,1}$. Moreover it is a routine matter to refine these elements so that each x_j and y_j is a matrix unit in $\mathcal{A}_{i+1,1}$. Set $z_{ij} = e_i x_i \dots e_{k_i} x_{k_i} e_1 y_1 \dots e_{j-1} y_{j-1} e_j$ for $1 \le i, j \le k_i$.

Let $\mathcal{H}_i = \mathbb{C}^{k_i}$ be the space on which $\mathcal{A}_{i,1}$ acts, with a standard basis $\{\xi^i_j : 1 \le j \le k_i\}$. Define regular unitaries u_i and v_i from $\mathcal{H}_i$ into $\mathcal{H}_{i+1}$ by setting $u\xi_j$ to be the range vector of x_j and similarly $v\xi^i_j$ is the range vector of y_j. Then as in the proof of Theorem 3.2, we show that that for any $a = \sum_{k,l=1}^{k_i} \alpha_{kl} e^{i,1}_{kl}$ in $\mathfrak{A}_{i,1}$, the element

$$\hat{a} = uav^* = \sum_{k,l=1}^{k_i} \alpha_{kl} e_k z_{kl} e_l$$

is an element of $\mathcal{A}_{i+1,1}$ and $\|\hat{a}\| = \|a\|$.

A sequence of unit vectors ζ_i is defined exactly as before. The argument that the state $\omega = \varinjlim \omega_{\zeta_i}$ yields a faithful irreducible representation of $\mathfrak{A}$ is exactly the same. Next we repeat the long calculation of Theorem 3.2 which shows that

$$\big\langle \pi(a)\pi(e^{i,1}_{pq})\xi_\omega, \pi(e^{i,1}_{rs})\xi_\omega \big\rangle = 2\big\langle \pi(\hat{a})\pi(e^{i,1}_{pq})\xi_\omega, \pi(e^{i,1}_{rs})\xi_\omega \big\rangle$$

for all $1 \le p, q, r, s \le k_i$. The argument showing that the 2-ball of $\pi(\mathcal{A})$ is WOT-dense in the unit ball of $\mathcal{B}(\mathcal{H})$ is the same, and we conclude that this representation is C*-primitive. ■

Mimicking Corollary 3.4, we obtain

Corollary 4.2. *For a regular limit algebra $\mathcal{A}$, the following are equivalent:*

(i) *$\mathcal{A}$ is C*-primitive.*
(ii) *$\mathcal{A}$ is operator primitive.*
(iii) *$\mathcal{A}$ is primitive.*
(iv) *$\mathcal{A}$ has a faithful topologically irreducible representation.*
(v) *$\mathcal{A}$ is prime.*

In [**22**], Orr and Peters considered the problem: given a strongly maximal TAF algebra $\mathcal{A}$, determine which WOT-closed subalgebras of $\mathcal{B}(\mathcal{H})$ containing a masa are the WOT-closure of a faithful representation of $\mathcal{A}$. This corollary solves this problem completely when the algebra is $\mathcal{B}(\mathcal{H})$, thus considerably improving on [**22**, Proposition III.3.1].

It is instructive to observe that the although the standard algebra admits uncountably many inequivalent C*-extendible irreducible representations on a separable Hilbert space $\mathcal{H}$, it admits only one class of $*$-extendible representations which is dense in Alg($\mathbb{N}$), the nest algebra of a nest ordered as $\mathbb{N}$ with one dimensional atoms. This representation π_s due to R. Smith, was introduced in [**22**]; for the reader's convenience we describe it for the case of the 2^∞ standard algebra.

Let $\mathcal{A} = \varinjlim(\mathcal{A}_n, \varphi_n)$ be the 2^∞ standard algebra, i.e., $\mathcal{A} = \overline{\cup_n \mathcal{A}_n}$ where each $\mathcal{A}_n$ is isomorphic to the $2^n \times 2^n$ upper triangular matrices and for each matrix unit e^n_{ij}, we have $e^n_{ij} = e^{n+1}_{ij} + e^{n+1}_{2^n+i\,2^n+j}$. Let $\mathcal{N}$ be a multiplicity-free atomic nest, whose atoms are ordered like $\mathbb{N}$ and let $\{P_l\}_{l=1}^\infty$ denote the atoms of $\mathcal{N}$ in order. Define

$$\pi_s(e^n_{ij})P_l = \begin{cases} P_{i+k2^n} & \text{if } l = j + k2^n \\ 0 & \text{otherwise} \end{cases}$$

for any $k \in \mathbb{N}$. The map π_s extends by linearity to a representation of $\mathcal{A}_n$, $n \in \mathbb{N}$, and by a completeness argument to a representation of $\mathcal{A}$.

Theorem 4.3. *Let $\mathcal{A} = \varinjlim(\mathcal{A}_n, \varphi_n)$ be a standard limit algebra and let $\mathcal{N}$ be a multiplicity-free atomic nest, whose atoms are ordered like $\mathbb{N}$. Suppose that π be a C*-extendible representation of $\mathcal{A}$ so that $\pi(\mathcal{A})$ is* WOT*-dense in* Alg $\mathcal{N}$*. Then π is unitarily equivalent to the Smith representation π_s.*

Proof. Let $\{P_l\}_{l=1}^\infty$ denote the atoms of $\mathcal{N}$ in order. For convenience we assume that $\mathcal{A}$ is the 2^∞ standard algebra.

Claim: $\pi(e^n_{ii})P_i = P_i$ for $1 \leq i \leq 2^n$.

The map $\rho_i(a) = P_i\pi(a)P_i$ is a multiplicative linear form on $\mathcal{A}$. Thus $\rho_i(e^n_{jj})$ is 0 or 1, and (since $\mathcal{A}_n$ is unital) there is a unique j taking the value 1; and this satisfies $\pi(e^n_{jj})P_i = P_i$. Each $\pi(e^n_{jj})$ is a projection in the diagonal $\operatorname{Alg}\mathcal{N} \cap \operatorname{Alg}\mathcal{N}^*$, and thus is a sum of those minimal diagonal projections P_i for which $\rho_i(e^n_{jj}) = 1$.

First we show that $\pi(e^n_{jj})P_k = P_k$ implies that $j \le k$. The partial isometries $\pi(e^n_{ij})P_k$ for $1 \le i \le j$ have rank one and their ranges are orthogonal minimal projections P_{k_i} for distinct $k_i < k$. Thus $j \le k$.

The proof of the claim follows by induction on i. If $i = 1$, then since $\pi(e^n_{jj})P_1 = P_1$ for some $j \le 1$, we must have $j = 1$. Assume that the claim is true for all $i < k$ and find j so that $\pi(e^n_{jj})P_k = P_k$. Then $j \le k$. Suppose that $j < k$. The two multiplicative forms ρ_j and ρ_k on $\mathcal{A}$ are distinct because $\pi(\mathcal{A})$ is dense in $\operatorname{Alg}\mathcal{N}$. Thus for some $m > n$ there must be distinct diagonal elements e^m_{rr} and e^m_{ss} so that $\rho_j(e^m_{rr}) = 1$ and $\rho_k(e^m_{ss}) = 1$. Note that $\rho(e^n_{jj}e^m_{rr}) = 1$ and thus $e^m_{rr} < e^n_{jj}$. Similarly $e^m_{ss} < e^n_{jj}$. Therefore one has $r \equiv s \equiv j \pmod{2^n}$. So one of them is greater than 2^n (and $s > r$). But then $\pi(e^m_{ss})P_k = P_k$ and $k < s$ which contradicts the previous paragraph. We conclude that $j = k$ as claimed.

Repeated use of the claim now shows that $\pi(e^n_{ii}) = \pi_s(e^n_{ii})$. Indeed, let $k \in \mathbb{N}$ and choose $m \ge i$ so large that $k \le 2^m$. Then

$$\pi(e^n_{ii})P_k = \sum_{r \equiv i \mod 2^n} \pi(e^m_{rr})P_k = \begin{cases} P_k & \text{if} \quad k \equiv i \pmod{2^n} \\ 0 & \text{otherwise} \end{cases}$$

It is precisely the same for π_s.

Let $\{E_{ij} : 1 \le i \le j\}$ be the matrix units for $\operatorname{Alg}\mathcal{N}$. A moments thought shows that $P_i\pi(e^n_{ij})P_j = \lambda^{(n)}_{ij}E_{ij}$ for some scalar $\lambda^{(n)}_{ij}$ of modulus one. Since this map is a homomorphism, a standard argument shows that, given $n \in \mathbb{N}$, there is a diagonal unitary U_n such that $P_iU_n\pi(e^n_{ij})U_n^*P_j = E_{ij}$ for all $1 \le i \le j \le 2^n$. Let U be any weak limit of the sequence $\{U_n\}_{n\in\mathbb{N}}$. Then U is a diagonal unitary and satisfies $P_iU\pi(e^n_{ij})U^*P_j = E_{ij}$ for all $1 \le i \le j \le 2^n$ and $n \in \mathbb{N}$. It is now evident that $\operatorname{ad} U\, \pi = \pi_s$, which proves the theorem. ∎

5. Derivations

If $\mathcal{A}$ is a Banach algebra and $a \in \mathcal{A}$, then let $\delta(a, \mathcal{A})$ denote the inner derivation of $\mathcal{A}$ induced by a, and let $\operatorname{dist}(a, Z(\mathcal{A}))$ denote the distance from a to $Z(\mathcal{A})$, the centre of $\mathcal{A}$. A simple calculation shows that

$$\|\delta(a, \mathcal{A})\| \le 2 \operatorname{dist}(a, Z(\mathcal{A})).$$

Let $\kappa(\mathcal{A})$ be the smallest number in $[\,0, \infty\,]$ such that

$$\operatorname{dist}(a, Z(\mathcal{A})) \le \kappa(\mathcal{A})\|\delta(a, \mathcal{A})\| \quad \text{for all} \quad a \in \mathcal{A}.$$

Note that $\kappa(\mathcal{A}) \ge 1/2$ unless $\mathcal{A}$ is abelian.

The constant $\kappa(\mathcal{A})$ was introduced, in the context of C*-algebras by Archbold [**1**]. In [**17**], this constant was studied in the context of TAF algebras. It turns out that for any TAF algebra $\mathcal{A}$, the constant $\kappa(\mathcal{A})$ is intimately related to its ideal structure and $\overline{\mathcal{A}+\mathcal{A}^*}$. Moreover, for triangular subalgebras of primitive AF algebras, the seminorms $\operatorname{dist}(a, Z(\mathcal{A}))$ and $\delta(a, \mathcal{A})$ are equivalent. However, no specific calculations of $\kappa(\mathcal{A})$ were made in [**17**] and it was raised as an open problem to calculate for the familiar examples of TAF algebras.

Proposition 5.1. *Let $\mathcal{A}$ be a primitive NSAF algebra. Then $\kappa(\mathcal{A}) = \frac{1}{2}$. More generally, if $\mathcal{A}$ is a triangular subalgebra of a primitive AF-algebra, then $\frac{1}{2} \leq \kappa(\mathcal{A}) \leq 2$.*

Proof. By Theorem 3.2, $\mathcal{A}$ has an isometric representation π on a separable Hilbert space so that $\pi(\mathcal{A})$ is WOT-dense in $\mathcal{B}(\mathcal{H})$. Thus $Z(\mathcal{A}) = \mathbb{C}$ and

$$\frac{1}{2} \leq \kappa(\mathcal{A}) = \kappa(\pi(\mathcal{A})) \leq \kappa(\mathcal{B}(\mathcal{H})) = \frac{1}{2}$$

by a theorem of Stampfli [**28**].

If $\mathcal{A}$ is a triangular subalgebra of a primitive AF C*-algebra, then by Proposition II.2.2 of [**22**], there is a representation π of $\mathcal{A}$ which is a dense subalgebra of a nest algebra $\operatorname{Alg}\mathcal{N}$. Hence $\|\delta(a, \mathcal{A})\| = \|\delta(\pi(a), \operatorname{Alg}\mathcal{N})\|$. Now a result of Christensen [**7**] shows that

$$\operatorname{dist}(\pi(a), \mathbb{C}I) \leq 2\|\delta(\pi(a), \operatorname{Alg}\mathcal{N})\|$$

and the conclusion folows. ∎

In particular, the standard, alternation and the lexicographic algebras $\mathbb{A}(\mathbb{Q}, \nu)$ all have constant κ equal to $\frac{1}{2}$. The calculation $\frac{1}{2} \leq \kappa(\mathcal{A}) \leq 2$ was done in [**17**] without the use of representation theory.

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Pure Math. Dept., U. Waterloo, Waterloo, ON N2L–3G1, CANADA
E-mail address: `krdavids@uwaterloo.ca`

Math. Dept., East Carolina University, Greenville, NC 27858, USA
E-mail address: `KatsoulisE@mail.ecu.edu`