SEMICROSSED PRODUCTS OF THE DISK ALGEBRA

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Abstract. If \( \alpha \) is the endomorphism of the disk algebra, \( A(D) \), induced by composition with a finite Blaschke product \( b \), then the semicrossed product \( A(D) \times_{\alpha} \mathbb{Z}^+ \) imbeds canonically, completely isometrically into \( C(T) \times_{\alpha} \mathbb{Z}^+ \). Hence in the case of a non-constant Blaschke product \( b \), the C*-envelope has the form \( C(S_b) \times_s \mathbb{Z} \), where \( (S_b, s) \) is the solenoid system for \( (T, b) \). In the case where \( b \) is a constant, then the C*-envelope of \( A(D) \times_{\alpha} \mathbb{Z}^+ \) is strongly Morita equivalent to a crossed product of the form \( C(S_e) \times_s \mathbb{Z} \), where \( e: T \times \mathbb{N} \to T \times \mathbb{N} \) is a suitable map and \( (S_e, s) \) is the solenoid system for \( (T \times \mathbb{N}, e) \).

1. Introduction

If \( \mathcal{A} \) is a unital operator algebra and \( \alpha \) is a completely contractive endomorphism, the semicrossed product is an operator algebra \( \mathcal{A} \times_{\alpha} \mathbb{Z}^+ \) which encodes the covariant representations of \( (\mathcal{A}, \alpha) \): namely completely contractive unital representations \( \rho: \mathcal{A} \to B(\mathcal{H}) \) and contractions \( T \) satisfying

\[
\rho(a)T = T \rho(\alpha(a)) \quad \text{for all } a \in \mathcal{A}.
\]

Such algebras were defined by Peters [9] when \( \mathcal{A} \) is a C*-algebra.

One can readily extend Peter’s definition [9] of the semicrossed product of a C*-algebra by a \( * \)-endomorphism to unital operator algebras and unital completely contractive endomorphisms. One forms the polynomial algebra \( \mathcal{P}(\mathcal{A}, t) \) of formal polynomials of the form \( p = \sum_{i=0}^{n} t^i a_i \), where \( a_i \in \mathcal{A} \), with multiplication determined by the covariance relation \( at = t \alpha(a) \) and the norm

\[
\|p\| = \sup_{(\rho,T) \text{covariant}} \left\| \sum_{i=0}^{n} T^i \rho(a_i) \right\|.
\]

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This supremum is clearly dominated by $\sum_{i=0}^{n} \|a_i\|$; so this norm is well defined. The completion is the semicrossed product $\mathcal{A} \times_\alpha \mathbb{Z}_+$. Since this is the supremum of operator algebra norms, it is also an operator algebra norm. By construction, for each covariant representation $(\rho, T)$, there is a unique completely contractive representation $\rho \times T$ of $\mathcal{A} \times_\alpha \mathbb{Z}_+$ into $\mathcal{B}(\mathcal{H})$ given by

$$\rho \times T(p) = \sum_{i=0}^{n} T^i \rho(a_i).$$

This is the defining property of the semicrossed product.

In this note, we examine semicrossed products of the disk algebra by an endomorphism which extends to a $*$-endomorphism of $C(T)$. In the case where the endomorphism is injective, these have the form $\alpha(f) = f \circ b$ where $b$ is a non-constant Blaschke product. We show that every covariant representation of $(A(D), \alpha)$ dilates to a covariant representation of $(C(T), \alpha)$. This is readily dilated to a covariant representation $(\sigma, V)$, where $\sigma$ is an $*$-representation of $C(T)$ (so $\sigma(z)$ is unitary) and $V$ is an isometry. To go further, we use the recent work of Kakariadis and Katsoulis [6] to show that $C(T) \times_\alpha \mathbb{Z}_+^+$ imbeds completely isometrically into a $C^*$-crossed product $C(S_b) \times_\alpha \mathbb{Z}$. In fact, $C^*_e(C(T) \times_\alpha \mathbb{Z}_+^+) = C(S_b) \times_\alpha \mathbb{Z}$ and as a consequence, we obtain that $(\rho, T)$ dilates to a covariant representation $(\tau, W)$, where $\tau$ is a $*$-representation of $C(T)$ (so $\sigma(z)$ is unitary) and $W$ is a unitary.

In contrast, if $\alpha$ is induced by a constant Blaschke product, we can no longer identify $C^*_e(C(T) \times_\alpha \mathbb{Z}_+^+)$ up to isomorphism. In that case, $\alpha$ is evaluation at a boundary point. Even though every covariant representation of $(A(D), \alpha)$ dilates to a covariant representation of $(C(T), \alpha)$, the theory of [6] is not directly applicable since $\alpha$ is not injective. Instead, we use the process of “adding tails to $C^*$-correspondences” [8], as modified in [3, 7] and we identify $C^*_e(C(T) \times_\alpha \mathbb{Z}_+^+)$ up to strong Morita equivalence as a crossed product. In Theorem 2.6 we show that $C^*_e(C(T) \times_\alpha \mathbb{Z}_+^+)$ is strongly Morita equivalent to a $C^*$-algebra of the form $C(S_e) \times_s \mathbb{Z}$, where $e: T \times N \to T \times N$ is a suitable map and $(S_e, s)$ is the solenoid system for $(T \times N, e)$.

Semi-crossed products of the disc algebra were introduced and first studied by Buske and Peters in [1], following relevant work of Hoover, Peters and Wogen [5]. The algebras $A(D) \times_\alpha \mathbb{Z}_+$, where $\alpha$ is an arbitrary endomorphism, where classified up to algebraic endomorphism in [2]. Results associated with their $C^*$-envelope can be found in [1, Proposition III.13] and [10, Theorem 2]. The results of the present paper subsume and extend these earlier results.
2. The Disk Algebra

The C*-envelope of the disk algebra $A(D)$ is $C(T)$, the space of continuous functions on the unit circle. Suppose that $\alpha$ is an endomorphism of $C(T)$ which leaves $A(D)$ invariant. We refer to the restriction of $\alpha$ to $A(D)$ as $\alpha$ as well. Then $b = \alpha(z) \in A(D)$; and has spectrum $\sigma_{A(D)}(b) \subset \sigma_{A(D)}(z) = D$ and $\sigma_{C(T)}(b) \subset \sigma_{C(T)}(z) = T$.

Thus $\|b\| = 1$ and $b(T) \subset T$. It follows that $b$ is a finite Blaschke product. Therefore $\alpha(f) = f \circ b$ for all $f \in C(T)$. When $b$ is not constant, $\alpha$ is completely isometric.

A (completely) contractive representation $\rho$ of $A(D)$ is determined by $\rho(z) = A$, which must be a contraction. The converse follows from the matrix von Neumann inequality; and shows that $\rho(f) = f(A)$ is a complete contraction. A covariant representation of $(A(D), \alpha)$ is thus determined by a pair of contractions $(A, T)$ such that $AT = Tb(A)$.

The representation of $A(D) \times_\alpha \mathbb{Z}^+$ is given by

$$\rho \times T \left( \sum_{i=0}^{n} t^i f_i \right) = \sum_{i=0}^{n} T^i f_i(A),$$

which extends to a completely contractive representation of the semicrossed product by the universal property.

A contractive representation $\sigma$ of $C(T)$ is a *-representation, and is likewise determined by $U = \sigma(z)$, which must be unitary; and all unitary operators yield such a representation by the functional calculus. A covariant representation of $(C(T), \alpha)$ is given by a pair $(U, T)$ where $U$ is unitary and $T$ is a contraction satisfying $UT = Tb(U)$. To see this, multiply on the left by $U^*$ and on the right by $b(U)^*$ to obtain the identity

$$U^* T = Tb(U)^* = T \bar{b}(U) = T \alpha(z)(U).$$

The set of functions $\{f \in C(T) : f(U)T = T\alpha(f)(U)\}$ is easily seen to be a norm closed algebra. Since it contains $z$ and $\bar{z}$, it is all of $C(T)$. So the covariance relation holds.

**Theorem 2.1.** Let $b$ be a finite Blaschke product, and let $\alpha(f) = f \circ b$. Then $A(D) \times_\alpha \mathbb{Z}^+$ is (canonically completely isometrically isomorphic to) a subalgebra of $C(T) \times_\alpha \mathbb{Z}^+$.

**Proof.** To establish that $A(D) \times_\alpha \mathbb{Z}^+$ is completely isometric to a subalgebra of $C(T) \times_\alpha \mathbb{Z}^+$, it suffices to show that each $(A, T)$ with $AT = Tb(A)$ has a dilation to a pair $(U, S)$ with $U$ unitary and $S$ a contraction such that $US = Sb(U)$ and $P_H S^n U^m |_H = T^n A^m$ for all
This latter condition is equivalent to $\mathcal{H}$ being semi-invariant for the algebra generated by $U$ and $S$.

The covariance relation can be restated as

$$\begin{bmatrix} A & 0 \\ 0 & b(A) \end{bmatrix} \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & b(A) \end{bmatrix}$$

Dilate $A$ to a unitary $U$ which leaves $\mathcal{H}$ semi-invariant. Then $\begin{bmatrix} A & 0 \\ 0 & b(A) \end{bmatrix}$ dilates to $\begin{bmatrix} U & 0 \\ 0 & b(U) \end{bmatrix}$. By the Sz.Nagy-Foiaş Commutant Lifting Theorem, we may dilate $\begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}$ to a contraction of the form $\begin{bmatrix} \ast & S \\ \ast & \ast \end{bmatrix}$ which commutes with $\begin{bmatrix} U & 0 \\ 0 & \alpha(U) \end{bmatrix}$ and has $\mathcal{H} \oplus \mathcal{H}$ as a common semi-invariant subspace. Clearly, we may take the $\ast$ entries to all equal 0 without changing things. So $(U, S)$ satisfies the same covariance relations $US = Sb(U)$. Therefore we have obtained a dilation to the covariance relations for $(C(T), \alpha)$.

Once we have a covariance relation for $(C(T), \alpha)$, we can try to dilate further. Extending $S$ to an isometry $V$ follows a well-known path. Observe that

$$b(U)S^*S = S^*US = S^*Sb(U).$$

Thus $D = (I - S^*S)^{1/2}$ commutes with $b(U)$. Write $b^{(n)}$ for the composition of $b$ with itself $n$ times, Hence we can now use the standard Schaeffer dilation of $S$ to an isometry $V$ and simultaneously dilate $U$ to $U_1$ as follows:

$$V = \begin{bmatrix} S & 0 & 0 & 0 & \ldots \\ D & 0 & 0 & 0 & \ldots \\ 0 & I & 0 & 0 & \ldots \\ 0 & 0 & I & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad U_1 = \begin{bmatrix} U & 0 & 0 & 0 & \ldots \\ 0 & b(U_1) & 0 & 0 & \ldots \\ 0 & 0 & b^{(2)}(U_1) & 0 & \ldots \\ 0 & 0 & 0 & b^{(3)}(U_1) & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$ 

A simple calculation shows that $U_1V = Vb(U_1)$. So as above, $(U, V)$ satisfies the covariance relations for $(C(T), \alpha)$.

We would like to make $V$ a unitary as well. This is possible in the case where $b$ is non-constant, but the explicit construction is not obvious. Instead, we use the theory of C*-envelopes and maximal dilations. First we need the following.

**Lemma 2.2.** Let $b$ be a finite Blaschke product, and let $\alpha(f) = f \circ b$. Then

$$C^*_e(A(\mathbb{D}) \times_\alpha \mathbb{Z}^+) \simeq C^*_e(C(T) \times_\alpha \mathbb{Z}^+).$$
Proof. The previous Theorem identifies $A(D) \times_\alpha \mathbb{Z}^+$ completely isometrically as a subalgebra of $C(T) \times_\alpha \mathbb{Z}^+$. The C*-envelope $\mathcal{C}$ of $C(T) \times_\alpha \mathbb{Z}^+$ is a Cuntz-Pimsner algebra containing a copy of $C(T)$ which is invariant under gauge actions. Now $\mathcal{C}$ is a C*-cover of $C(T) \times_\alpha \mathbb{Z}^+$, so it is easy to see that it is also a C*-cover of $A(D) \times_\alpha \mathbb{Z}^+$. Since $A(D) \times_\alpha \mathbb{Z}^+$ is invariant under the same gauge actions, its Shilov ideal $\mathcal{J} \subseteq \mathcal{C}$ will be invariant by these actions as well. If $\mathcal{J} \neq 0$ then by gauge invariance $\mathcal{J} \cap C(T) \neq 0$. Since the quotient map

$$A(D) \longrightarrow C(T)/(\mathcal{J} \cap C(T))$$

is completely isometric, we obtain a contradiction. Hence $\mathcal{J} = 0$ and the conclusion follows.

We now recall some of the theory of semicrossed products of C*-algebras. When $\mathfrak{A}$ is a C*-algebra, the completely isometric endomorphisms are the faithful $^*$-endomorphisms. In this case, Peters shows [9, Prop.I.8] that there is a unique C*-algebra $\mathfrak{B}$, a $^*$-automorphism $\beta$ of $\mathfrak{B}$ and an injection $j$ of $\mathfrak{A}$ into $\mathfrak{B}$ so that $\beta \circ j = j \alpha$ and $\mathfrak{B}$ is the closure of $\bigcup_{n \geq 0} \beta^{-n}(j(\mathfrak{A}))$. It follows [9, Prop.II.4] that $A(D) \times_\alpha \mathbb{Z}^+$ is completely isometrically isomorphic to the subalgebra of the crossed product algebra $\mathfrak{B} \times_\beta \mathbb{Z}$ generated as a non-self-adjoint algebra by an isomorphic copy $j(\mathfrak{A})$ of $\mathfrak{A}$ and the unitary $u$ implementing $\beta$ in the crossed product. Actually, Kakariadis and the second author [6, Thm.2.5] show that $\mathfrak{B} \times_\beta \mathbb{Z}$ is the C*-envelope of $\mathfrak{A} \times_\alpha \mathbb{Z}^+$.

In the case where $\mathfrak{A} = C(X)$ is commutative and $\alpha$ is induced by an injective self-map of $X$, the pair $(\mathfrak{B}, \beta)$ has an alternative description.

Definition 2.3. Let $X$ be a topological space and $\varphi$ a surjective self-map of $X$. We define the solenoid system of $(X, \varphi)$ to be the pair $(S_\varphi, s)$, where

$$S_\varphi = \{(x_n)_{n \geq 1} : x_n = \varphi(x_{n+1}), x_n \in X, n \geq 1\}$$

equipped with the relative topology inherited from the product topology on $\prod_{i=1}^{\infty} X_i$, $X_i = X$, $i = 1, 2, \ldots$, and $s$ is the backward shift on $S_\varphi$.

It is easy to see that in the case where $\mathfrak{A} = C(X)$ and $\alpha$ is induced by an injective self-map $\varphi$ of $X$, the pair $(\mathfrak{B}, \beta)$ for $(\mathfrak{A}, \alpha)$ described above, is conjugate to the solenoid system $(S_\varphi, s)$. Therefore, we obtain

Corollary 2.4. Let $b$ be a non-constant finite Blaschke product, and let $\alpha(f) = f \circ b$ on $C(T)$. Then

$$C^*_e(A(D) \times_\alpha \mathbb{Z}^+) = C^*_e(C(S_b) \times_s \mathbb{Z})$$

where $(S_b, s)$ is the solenoid system of $(T, b)$. 
It is worth restating this theorem as a dilation result.

**Corollary 2.5.** Let $\alpha$ be an endomorphism of $A(D)$ induced by a non-constant finite Blaschke product and let $A, T \in B(H)$ be contractions satisfying $AT = T\alpha(A)$. Then there exist unitary operators $U$ and $W$ on a Hilbert space $K \supset H$ which simultaneously dilate $A$ and $T$, in the sense that $P_H W^m U^n \big|_H = T^m A^n$ for all $m, n \geq 0$, so that $UW = W\alpha(U)$.

**Proof.** Every covariant representation $(A, T)$ of $(A(D), \alpha)$ dilates to a covariant representation $(U_1, V)$ of $(C(T), \alpha)$. This in turn dilates to a maximal dilation $\tau$ of $C(T) \times_{\alpha} Z^+$, in the sense of Dritschel and McCullough [4]. The maximal dilations extend to $\ast$-representations of the $C^*$-envelope. Then $A$ is dilated to $\tau(j(z)) = U$ is unitary and $T$ dilates to the unitary $W$ which implements the automorphism $\beta$ on $B$, and restricts to the action of $\alpha$ on $C(T)$.

The situation changes when we move to non-injective endomorphisms $\alpha$ of $A(D)$. Indeed, let $\lambda \in T$ and consider the endomorphism $\alpha_\lambda$ of $A(D)$ induced by evaluation on $\lambda$, i.e., $\alpha_\lambda(f)(z) = f(\lambda)$, $\forall z \in D$. (Thus $\alpha_\lambda$ is the endomorphism of $A(D)$ corresponding to a constant Blaschke product.) If two contractions $A, T$ satisfy $AT = T\alpha_\lambda(A) = \lambda T$, then the existence of unitary operators $U, W$, dilating $A$ and $T$ respectively, implies that $A = \lambda I$. It is easy to construct a pair $A, T$ satisfying $AT = \lambda T$ and yet $A \neq \lambda I$. This shows that the analogue Corollary 2.5 fails for $\alpha = \alpha_\lambda$ and therefore one does not expect $C^\ast_e(A(D) \times_{\alpha_\lambda} Z^+)$ to be isomorphic to the crossed product of a commutative $C^*$-algebra, at least under canonical identifications. However as we have seen, a weakening of Corollary 2.5 is valid for $\alpha = \alpha_\lambda$ if one allows $W$ to be an isometry instead of a unitary operator. In addition, we can identify $C^\ast_e(A(D) \times_{\alpha_\lambda} Z^+)$ as being strongly Morita equivalent to a crossed product $C^*$-algebra. Indeed, if $e: \mathbb{T} \times \mathbb{N} \to \mathbb{T} \times \mathbb{N}$ is defined as

$$e(z, n) = \begin{cases} (1, 1) & \text{if } n = 1 \\ (z, n - 1) & \text{otherwise,} \end{cases}$$

then

**Theorem 2.6.** Let $\alpha = \alpha_\lambda$ be an endomorphism of $A(D)$ induced by evaluation at a point $\lambda \in \mathbb{T}$. Then $C^\ast_e(A(D) \times_{\alpha_\lambda} Z^+)$ is strongly Morita equivalent to $C(S_e) \times_s Z$, where $e: \mathbb{T} \times \mathbb{N} \to \mathbb{T} \times \mathbb{N}$ is defined above and $(S_e, s)$ is the solenoid system of $(\mathbb{T} \times \mathbb{N}, e)$. 
Proof. In light of Lemma 2.2, it suffices to identify the C*-envelope of $C(T) \times_\alpha \mathbb{Z}^+$. As $\alpha$ is no longer an injective endomorphism of $C(T)$, we invoke the process of adding tails to C*-correspondences [8], as modified in [3, 7]. Indeed, [7, Example 4.3] implies that the C*-envelope of the tensor algebra associated with the dynamical system $(C(T), \alpha)$ is strongly Morita equivalent to the Cuntz-Pimsner algebra associated with the injective dynamical system $(T \times \mathbb{N}, e)$ defined above. Therefore by invoking the solenoid system of $(T \times \mathbb{N}, e)$, the conclusion follows from the discussion following Lemma 2.2.

References


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