TENSOR ALGEBRAS OF PRODUCT SYSTEMS AND THEIR C*-ENVELOPES

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Abstract. Let (G, P) be an abelian, lattice ordered group and let X be a compactly aligned, $\tilde{\phi}$-injective product system over P. We show that the C*-envelope of the Nica tensor algebra $\mathcal{NT}_X^+$ is the Cuntz-Nica-Pimsner algebra $\mathcal{NO}_X$ as defined by Sims and Yeend. We give several applications of this result. First we resolve a problem posed by Skalski and Zacharias on dilating isometric representations of product systems to unitary representations. As a second application we characterize the C*-envelope of the tensor algebra of a finitely aligned higher-rank graph. An analogous result holds for the C*-envelope of the tensor algebra of a compactly aligned topological k-graph. As a third application we show that the Hao-Ng isomorphism problem for generalized gauge actions of discrete groups on C*-algebras of product systems has an affirmative answer in many cases, generalizing recent results of Bedos, Quigg, Kaliszewski and Robertson and of the second author. As a final application, we show the existence of a co-universal C*-algebra for injective, gauge compatible, Nica-covariant representations of a compactly aligned product system over an abelian, lattice ordered group. This is done without the assumption of $\tilde{\phi}$-injectivity and we therefore resolve in that case a problem that was left open in the work of Carlsen, Larsen, Sims and Vittadello.

1. Introduction

Since its inception by Arveson [**3**] in the late 60's, the concept of the C*-envelope, or non-commutative Shilov boundary, has played an important role in operator algebra theory with its importance recently increasing considerably. Indeed, it is through the ideas of non-commutative boundaries that the work of Kennedy and Kalantar on C*-simplicity took flight [**30**], and more recently, in the work of Davidson and Kennedy [**15**], a conjecture of Arveson [**6**] related to approximation theory was solved for the commutative case. In addition, in [**39**] it is shown that the Arveson-Douglas conjecture can be phrased in terms of the C*-envelope of non-self-adjoint operator algebras arising from homogeneous polynomial relations.

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The central result of this paper, Theorem 3.2, characterizes the C*-envelope of the tensor algebra of a product system X over P, where (G, P) is an abelian, lattice ordered group and X satisfies certain mild hypotheses. This is a new development even in the case of the abelian ordered groups $(\mathbb{Z}^d, \mathbb{N}^d)$ for $d \in \mathbb{N}$. A result in the spirit of Theorem 3.2 has been sought after ever since the special case of the ordered group $(\mathbb{Z}, \mathbb{N})$ was established in **[33]**, over a decade ago. Concrete cases of it have already appeared in the literature with the proofs sometimes requiring considerable effort. Our approach incorporates many of these special cases by providing a unified treatment and furthermore, it gives answers in new cases which were previously inaccessible by other means.

In **[53]**, Pimsner generalized many constructions of operator algebras by associating them to C*-correspondences. Pimsner associates two C*-algebras $\mathcal{T}_X$ and $\mathcal{O}_X$ to a C*-correspondence X, where the algebra $\mathcal{O}_X$ generalizes Cuntz-Krieger algebras **[12]** arising from directed graphs and crossed products by $\mathbb{Z}$, while $\mathcal{T}_X$ generalizes their Toeplitz extensions. In a sequence of papers **[35, 36, 37]**, Katsura drew insight from these specific examples and expanded Pimsner's construction. Katsura removed any assumptions on the C*-correspondence while simplifying and expanding many of Pimsner's results.

Many examples of (non-self-adjoint) operator algebras arise as tensor algebras of C*-correspondences, i.e., subalgebras of $\mathcal{T}_X$ generated as closed algebras by a copy of the C*-correspondence and the coefficient algebra. Following Arveson's programme on the C*-envelope, it makes sense to ask for a characterization of the C*-envelope of a tensor algebra $\mathcal{T}_X^+$. Muhly and Solel **[49]** and Fowler, Muhly and Raeburn **[25]** established that the C*-envelope of $\mathcal{T}_X^+$ is Pimsner's Cuntz-Pimsner algebra $\mathcal{O}_X$, under various restrictions on the C*-correspondence X. The problem was finally settled by Katsoulis and Kribs **[33]** who removed all restrictions and showed that the C*-envelope of any tensor algebra $\mathcal{T}_X^+$ is Katsura's Cuntz-Pimsner algebra $\mathcal{O}_X$. The proof required an intricate tail adding technique, as developed by Muhly and Tomforde in **[50]**. Although successful attempts have been made for C*-dynamical systems over $\mathbb{N}^d$ **[16, 29]**, this tail-adding technique is difficult to generalize beyond single C*-correspondences. This has been an impediment for the continued development of the non-selfadjoint theory.

The concept of a product system over a partially ordered discrete group generalizes that of a C*-correspondence. In order to obtain a satisfactory theory one needs to add the extra requirement that the product system is compactly aligned over a quasi-lattice ordered semigroup. Given a compactly aligned product system X over a quasi-lattice ordered group, one builds a Nica-Toeplitz algebra $\mathcal{NT}_X$ as a universal object for a suitable class of representations called *Nica-covariant* representations. The development of this theory owes to the work of many hands **[20, 45, 51, 26, 23]**. The theory of the Toeplitz-Nica-Pimsner algebras $\mathcal{NT}_X$ associated with product system X reached its current state with the work of Fowler **[24]**, who successfully refined all previous ideas and provided a uniqueness theorem for it (see Theorem 2.2). However, it was unclear what the right analogue of the Cuntz-Pimsner algebra of a non-injective product system X should be and this was open for quite some time. Nevertheless this has now been settled in many cases by Sims and Yeend [**58**], who gave a definition for the Cuntz-Nica-Pimsner algebra $\mathcal{NO}_X$ and by Carlsen, Larson, Sims and Vittadello [**13**], who provided the gauge invariant uniqueness theorem for $\mathcal{NO}_X$. (See Theorem 2.3).

Non-selfadjoint operator algebras can also often be described as tensor algebras of compactly aligned product systems over lattice ordered semigroups (G, P) with G abelian. The list is long and includes examples that do not materialize as tensor algebras of a single C*-correspondence. These examples include the tensor algebras of a k-graphs, studied by Kribs and Power [**42**] and Davidson, Power and Yang [**18**] and the recent Nica semicrossed products of C*-dynamical systems over $\mathbb{N}^d$, studied by Davidson, Fuller and Kakariadis [**16**] (see [**17**] for a survey). In these cases, the C*-envelopes have been calculated successfully, sometimes with considerable effort. Motivated by the special case of the tensor algebra of a C*-correspondence, it is tempting to ask whether the aforementioned result of Katsoulis and Kribs [**33**] holds in the greater generality of product system, i.e., whether the C*-envelope of the Nica tensor algebra of a product system is isomorphic to the Cuntz-Nica-Pimsner algebra of the system. Our Theorem 3.2 answers this question by showing that if X is a compactly aligned product system over an abelian, lattice ordered semigroup (G, P), then indeed the C*-envelope of the Nica tensor algebra $\mathcal{NT}_X^+$ is the Cuntz-Nica-Pimsner algebra $\mathcal{NO}_X$ of Sims and Yeend. This result generalizes the earlier result of Katsoulis and Kribs [**33**], without the use of any tail adding technique in its proof (such a technique is currently unavailable for arbitrary product systems). It further shows that the view of $\mathcal{NO}_X$ as a co-universal object can be fully materialized within Arveson's programme for the C*-envelope, without any reference to gauge actions. This is something that was desired (but not attained) in [**13**] and conjectured indirectly in [**17**]. In fact, in the last section we show in Theorem 7.2 that by using the C*-envelope picture, one can go beyond $\tilde{\phi}$-injective cases of [**13**], to obtain a co-universal object in the sense of Carlsen, Larsen, Sims and Vitadello, for any compactly aligned product systems over abelian, lattice ordered groups.

We give several applications of our main result. In the context of a sourceless row-finite higher rank graph Λ, it was shown in [**32**, Theorem 3.6] (see also [**18**, Theorem 3.5]) that the C*-envelope of the tensor algebra $\mathcal{T}_+(\Lambda)$ is the higher rank graph C*-algebra $\mathrm{C}^*(\Lambda)$ of Kumjian and Pask [**43**]. In [**57**] a more general C*-algebra was associated to a finitely aligned higher rank graph Λ, and a gauge invariant uniqueness theorem was proven for it. A tensor algebra $\mathcal{T}_+(\Lambda)$ can still be defined, and our main theorem (Theorem 2.3) is used to easily extend [**32**, Theorem 3.6] to finitely aligned higher rank graphs (see Theorem 5.1).

As another application of our results, we are able to resolve a question of Skalski and Zacharias from the end of [**59**]. They ask the following question: given a product system X over $(\mathbb{Z}^d, \mathbb{N}^d)$, and a representation of X in the sense that $(\sigma, T_1, ..., T_d)$ is a $(d+1)$-tuple where each (σ, T_i) is an isometric representation of X_i with some commutation condition, is there a common dilation $(\pi, U_1, ..., U_d)$ to a representation of X where each (π, U_i) is both isometric *and* fully-coisometric. In Corollary 4.2 we show that there is a positive answer to their question when X is regular, and the representation $(\sigma, T_1, ..., T_d)$ is *doubly-commuting* (which is equivalent to Nica covariance in this case). We also construct in Example 5.2 a higher rank graph and an isometric representation for its associated product system that has no isometric and fully coisometric dilation. This shows that Nica-covariance is necessary, and together with earlier observations in [**59**], this shows that our result is optimal.

We also give another application to C*-algebra theory; we use our main result to obtain analogues of the Hao-Ng isomorphism theorem in the context of product systems over more general semigroups. Recall that the Hao-Ng isomorphism problem, as popularized in [**8**], asks for the validity of the isomorphism

$$\mathcal{O}_X \rtimes_\alpha^r \mathcal{G} \cong \mathcal{O}_{X \rtimes_\alpha^r \mathcal{G}} \tag{1.1}$$

in the case where X is a single C*-correspondence and $\mathcal{G}$ a locally compact group acting on X. Even though this problem is still open in general, two important cases have been worked out: when $\mathcal{G}$ is amenable, by Hao and Ng in their original work [**28**], and more recently when $\mathcal{G}$ is discrete, by the second author [**31**]. In this paper we investigate the obvious generalization of the Hao-Ng isomorphism for product systems and verify its validity for discrete groups acting on finitely aligned product systems. More specifically, in Theorem 6.10 we show that if (G, P) is an abelian, lattice ordered group, X is a finitely aligned and $\tilde{\phi}$-injective product system over P and $\mathcal{G}$ is a discrete group acting on X, then the analogue of equation (1.1) holds in the sense that $\mathcal{NO}_X \rtimes_\alpha^r \mathcal{G} \cong \mathcal{NO}_{X \rtimes_\alpha^r \mathcal{G}}$. This generalizes results from [**8, 28, 31, 38**].

In the final section of the paper we consider a problem that was left open in the work of Carlsen, Larsen, Sims and Vittadello [**13**]. In [**13**], the authors study the concept of a co-universal C*-algebra for a compactly aligned product system X over a quasi-lattice ordered group (G, P). This C*-algebra, denoted as $\mathcal{NO}_X^r$, is co-universal with respect to all injective, gauge compatible representations of X and its existence is not automatically guaranteed. The main result of [**13**], when applied to *abelian*, lattice ordered groups, shows that $\mathcal{NO}_X^r$ always exists, provided that the system X is $\tilde{\phi}$-injective. (See [**13**, Theorem 4.1].) Therefore the work in [**13**] leaves open the case for various product systems, including those in [**58**, Example 3.16] which are not $\tilde{\phi}$-injective. As it turns out, the non-selfadjoint techniques of our paper can be used to tackle this problem as well. In Theorem 7.2 we show that for *any* compactly aligned product system X over an abelian, lattice ordered group (G, P), the co-universal C*-algebra $\mathcal{NO}_X^r$ always exists and coincides with $\mathrm{C}_e^*(\mathcal{NT}_X^+)$ in a canonical way. This further necessitates the identification of $\mathrm{C}_e^*(\mathcal{NT}_X^+)$ beyond $\tilde{\phi}$-injective product systems, a problem that we plan to pursue in subsequent work.

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2. Preliminaries

2.1. Operator algebras and C*-envelopes.

We will survey the theory of non-commutative boundaries for unital operator algebras, and we refer the reader to [**3, 4, 7, 10**] for a more in-depth treatment of the theory. A brief exposition of the results described below, including proofs, can be found in [**9**, Section 2.5].

Let $\mathcal{A}$ be an operator algebra. We say that the pair $(\mathcal{B}, \iota)$ is a *C*-cover* for $\mathcal{A}$, if $\iota : \mathcal{A} \to \mathcal{B}$ is a completely isometric homomorphism, and $\mathrm{C}^*(\iota(\mathcal{A})) = \mathcal{B}$.

There is always a unique, smallest C*-cover for an operator algebra $\mathcal{A}$. This C*-cover $(C_e^*(\mathcal{A}), \kappa)$ is called the *C*-envelope* of $\mathcal{A}$ and it satisfies the following universal property: given any other C*-cover $(\mathcal{B}, \iota)$ for $\mathcal{A}$, there exists a (necessarily unique and surjective) $*$-homomorphism $\pi : \mathcal{B} \to C_e^*(\mathcal{A})$, such that $\pi \circ \iota = \kappa$. We will sometimes identify $\mathcal{A}$ with its image $\iota(\mathcal{A})$ under a given C*-cover $(\mathcal{B}, \iota)$ for $\mathcal{A}$.

The existence of the C*-envelope for a *unital* operator algebra was first established in the seventies by Hamana [**27**], following the pioneering work of Arveson [**3**]. (See [**52**, Theorem 15.16] for a proof in the spirit of Hamana's original work.) For a non-unital operator algebra, the existence of the C*-envelope was established much later via unitization, which we now describe.

If $\mathcal{A} \subseteq B(\mathcal{H})$ is a non-unital operator algebra generating a C*-algebra $\mathcal{B} = \mathrm{C}^*(\mathcal{A})$, a theorem of Meyer [**47**, Section 3] (see also [**10**, Corollary 2.1.15]) states that every representation $\varphi : \mathcal{A} \to B(\mathcal{K})$ extends to a unital representation φ^1 on the *unitization* $\mathcal{A}^1 = \mathcal{A} \oplus \mathbb{C} I_{\mathcal{H}}$ of $\mathcal{A}$ by specifying $\varphi^1(a + \lambda I_{\mathcal{H}}) = \varphi(a) + \lambda I_{\mathcal{K}}$. Meyer's theorem shows that $\mathcal{A}$ has a *unique* (one-point) unitization, in the sense that if $(\mathcal{B}, \iota)$ is a C*-cover for the operator algebra $\mathcal{A}$, and $\mathcal{B} \subseteq B(\mathcal{H})$ is some faithful representation of $\mathcal{B}$, then the operator-algebraic structure on $\mathcal{A}^1 \cong \iota(\mathcal{A}) + \mathbb{C}1_{\mathcal{H}}$ is independent of the C*-cover and the faithful representation of $\mathcal{B}$.

The C*-envelope of a non-unital operator algebra can be computed from the C*-envelope of its unitization. More precisely, as the pair $(\mathrm{C}_e^*(\mathcal{A}), \iota)$ where $\mathrm{C}_e^*(\mathcal{A})$ is the C*-subalgebra generated by $\iota(\mathcal{A})$ inside the C*-envelope

$(\mathrm{C}^*_e(\mathcal{A}^1), \iota)$ of the (unique) unitization $\mathcal{A}^1$ of $\mathcal{A}$. By the proof of [**10**, Proposition 4.3.5] this C*-envelope of an operator algebra $\mathcal{A}$ has the desired (co-)universal property, that for any C*-cover $(\iota', \mathcal{B}')$ of $\mathcal{A}$, there exists a (necessarily unique and surjective) $*$-homomorphism $\pi : \mathcal{B}' \to \mathrm{C}^*_e(\mathcal{A})$, such that $\pi \circ \iota' = \iota$.

For an operator algebra $\mathcal{A}$ generating a C*-algebra $\mathcal{B}$, an ideal $\mathcal{J}$ of $\mathcal{B}$ is called *a boundary ideal* for $\mathcal{A}$ if the quotient map $\mathcal{B} \to \mathcal{B}/\mathcal{J}$ is a complete isometry on $\mathcal{A}$. The existence of a C*-envelope for $\mathcal{A}$ implies the existence of a largest boundary ideal $\mathcal{J}_{\mathcal{A}}$ of $\mathcal{A}$ in $\mathcal{B}$, which is called *the Shilov ideal* of $\mathcal{A}$ in $\mathcal{B}$. Its importance in our context is that it gives a way of computing the C*-envelope. Namely, the C*-envelope of $\mathcal{A}$ is always isomorphic to $\mathcal{B}/\mathcal{J}_{\mathcal{A}}$.

2.2. **Maximal maps and the unique extension property.** The results of the previous subsection form the necessary prerequisites from abstract operator algebra theory for the proof of our main result, which is Theorem 3.2. The applications of Theorem 3.2 however require more from that theory. We survey these additional results from abstract operator algebra theory in this subsection.

Suppose $\mathcal{A}$ is a *unital* operator algebra generating a C*-algebra $\mathcal{B}$. We say that a unital complete contraction $\rho : \mathcal{A} \to B(\mathcal{H})$ has the *unique extension property* if the only unital completely positive extension to $\mathcal{B}$ is a $*$-representation.

When $\mathcal{A}$ is a unital operator algebra, any unital completely contractive map ρ can be extended to a unital completely positive map on the operator system $\mathcal{A}^* + \mathcal{A}$. Hence, when $\mathcal{A}$ is a unital operator algebra, by Arveson's extension theorem any unital complete contraction $\rho : \mathcal{A} \to B(\mathcal{H})$ has *some* unital completely positive extension $\phi : \mathcal{B} \to B(\mathcal{H})$. When ρ has the unique extension property, it must be automatically multiplicative.

When $\pi : \mathcal{B} \to B(\mathcal{H})$ is a $*$-representation such that $\pi|_{\mathcal{A}}$ has the unique extension property, then any boundary ideal of $\mathcal{A}$ in $\mathcal{B}$ is annihilated by π. We will call such π a boundary representation if it is also irreducible. The boundary theorem of Arveson, Davidson and Kennedy (See [**19**] and [**5**]) then shows that the Shilov ideal is the intersection of all kernels of boundary representations, providing another way of computing the C*-envelope.

For a unital operator algebra $\mathcal{A}$ and a unital complete contraction $\varphi : \mathcal{A} \to B(\mathcal{H})$, a unital complete contraction $\psi : \mathcal{A} \to B(\mathcal{K})$ is said to *dilate* φ if there is an isometry $V : \mathcal{H} \to \mathcal{K}$ such that for all $a \in \mathcal{A}$ we have $\varphi(a) = V^*\psi(a)V$. Since V is an isometry, we can identify $\mathcal{H} \cong V(\mathcal{H})$ as a subspace of $\mathcal{K}$, so that ψ dilates φ if and only if there is a larger Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ such that for all $a \in \mathcal{A}$ we have that $\varphi(a) = P_{\mathcal{H}}\psi(a)|_{\mathcal{H}}$ where $P_{\mathcal{H}}$ is the projection onto $\mathcal{H}$. We say that a unital complete contraction $\rho : \mathcal{A} \to B(K)$ is *maximal* if whenever π is a unital complete contraction dilating ρ, then in fact $\pi = \rho \oplus \psi$ for some unital complete contraction ψ.

Based on ideas of Agler [**1**] Muhly and Solel from [**48**], Dritschel and McCullough [**21**, Theorem 1.1] (See also [**7**]) showed that a unital complete contraction $\rho : \mathcal{A} \to B(\mathcal{K})$ is maximal if and only if it has the unique extension property. Dritschel and McCullough [**21**, Theorem 1.2] then used this to show that every unital complete contraction ρ on $\mathcal{A}$ can be dilated to a *maximal* unital complete contraction π on $\mathcal{A}$.

2.3. **Quasi-lattice ordered semigroups.** Let P be a subsemigroup of a group G such that $P \cap P^{-1} = \{e\}$, where e is the identity element of G. Then P induces a partial order on G by defining $p \leq q$ iff $p^{-1}q \in P$ which is left-invariant in the sense that if $p \leq q$ then $rp \leq rq$ for $p, q, r \in P$. Following Nica [**51**], we say that (G, P) is a *quasi-lattice ordered group* if every finite subset of G with a common upper bound in P, has a (necessarily unique) least upper bound in P. When $p, q \in P$ have a common upper bound, we will denote their least upper bound by $p \vee q$; and when they do not, we write $p \vee q = \infty$. We note that by [**14**, Lemma 7], the pair (G, P) is quasi-lattice ordered if and only every finite subset of G with a common upper bound in G has a least upper bound in G. When any finite set in G has both a least upper bound and greatest lower bound, we will say that (G, P) is lattice ordered.

An *Ore* semigroup is a (left and right) cancellative semigroup P such that $Ps \cap Pt \neq \emptyset$ for all $s, t \in P$. It is easy to see from this, that if (G, P) is a quasi-lattice ordered semigroup, then P is Ore if and only if P is *directed* in the sense that any two elements $p, q \in P$ have a common upper bound. When (G, P) is abelian, then xy is an upper bound for any two elements x and y in P, and we may define $x \wedge y = (x^{-1} \vee y^{-1})^{-1}$. With the operation $\wedge$, the pair (G, P) becomes an abelian, lattice ordered group, and is automatically directed.

For a quasi-lattice ordered group (G, P), a representation $V : P \to B(\mathcal{H})$ of P by isometries is said to be *Nica-covariant* if it satisfies

$$V_p V_p^* V_q V_q^* = V_{p \vee q} V_{p \vee q}^*.$$

By a theorem of Laca from [**44**] (See also [**16**, Proposition 2.4.8]), every isometric representation $V : P \to B(\mathcal{H})$ of an *Ore* semigroup has a dilation to a unitary representation $U : P \to B(\mathcal{H})$. In this case, it is easy to verify that U is automatically Nica-covariant. This theorem is the semigroup analogue of the classical theorem of Itô from dilation theory, that every d-tuple of commuting isometries have a power dilation to a d-tuple of commuting unitaries. In particular, we see that every Nica-covariant isometric representation of an abelian, lattice ordered group (G, P) has a unitary dilation.

2.4. **C*-correspondences.** Here we will give an overview of Hilbert C*-correspondences. For further details and material, we recommend [**46**].

Let $\mathcal{A}$ be a C*-algebra. A right inner product $\mathcal{A}$-module is a complex vector space X equipped with a right action of $\mathcal{A}$ and an $\mathcal{A}$-valued map $\langle \cdot, \cdot \rangle : X \times X \to \mathcal{A}$ which is $\mathcal{A}$-linear in the second argument, such that for $x, y \in X$ and $a \in \mathcal{A}$ we have

(i) $\langle x, x \rangle \geq 0$

(ii) $\langle x, x\rangle = 0$ if and only if $x = 0$
(iii) $\langle x, y\rangle = \langle y, x\rangle^*$

When X is complete with respect to the norm given by $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ we say that X is a Hilbert $\mathcal{A}$-module.

Let X be a Hilbert $\mathcal{A}$ module. We say that a map $T : X \to X$ is adjointable if there's a map $T^* : X \to X$ such that $\langle Tx, y\rangle = \langle xT^*y\rangle$ for every $x, y \in X$. Every adjointable operator is automatically $\mathcal{A}$-linear and continuous. We denote by $\mathcal{L}(X)$ the C*-algebra of adjointable operators equipped with the operator norm. For $x, y \in X$ there is a special adjointable operator $\theta_{x,y} \in \mathcal{L}(X)$ given by $\theta_{x,y}(z) = x \cdot \langle y, z\rangle$. We will denote by $\mathcal{K}(X) \lhd \mathcal{L}(X)$ the ideal of generalized compact operators generated by $\theta_{x,y}$ with $x, y \in X$.

A $\mathcal{B}$-$\mathcal{A}$ C*-correspondence is then just a (right) Hilbert $\mathcal{A}$-module X along with a $*$-homomorphism $\phi : \mathcal{B} \to \mathcal{L}(X)$ which is non-degenerate, i.e., $[\phi(B)X] = X$ (this is sometimes called essential). If X is an $\mathcal{A}$-$\mathcal{A}$ C*-correspondence we will just call X an $\mathcal{A}$-correspondence. We think of ϕ as implementing a left action of $\mathcal{B}$ on X, and we often write $b \cdot x$ for $\phi(b)x$.

When X is a $\mathcal{C}$-$\mathcal{B}$-correspondences and Y a $\mathcal{B}$-$\mathcal{A}$-correspondence, we may form the interior tensor product $X \otimes_{\mathcal{B}} Y$. Indeed, let $X \odot_{\mathcal{B}} Y$ be the algebraic $\mathcal{B}$-balanced tensor product. Then the formula

$$\langle x \odot y, w \odot z\rangle := \langle y, \langle x, w\rangle \cdot z\rangle,$$

determines an $\mathcal{A}$-valued sesquilinear form on $X \odot_{\mathcal{B}} Y$, whose Hausdorff completion $X \otimes_{\mathcal{B}} Y$ is a (right) Hilbert $\mathcal{A}$-module. There is then a left $\mathcal{C}$ action $\mathcal{C} \to \mathcal{L}(X \otimes_{\mathcal{B}} Y)$ given by $c \cdot (x \odot y) = (c \cdot x) \odot y$ for each $x \in X$, $y \in Y$ and $c \in \mathcal{C}$.

An important example of an $\mathcal{A}$-correspondence is $\mathcal{A}$ itself, equipped with the left and right actions coming from $\mathcal{A}$, and the inner product $\langle a, b\rangle = a^*b$. Then $\mathcal{K}(\mathcal{A})$ is isomorphic to $\mathcal{A}$ via $\theta_{a,b} \mapsto ab^*$, while $\mathcal{L}(\mathcal{A})$ is isomorphic to the multiplier algebra of $\mathcal{A}$.

2.5. **Product systems over quasi-lattice ordered semigroups.** Let $\mathcal{A}$ be a C*-algebra and P a semigroup with identity e. A product system over P with coefficients $\mathcal{A}$ is a semigroup of $\mathcal{A}$-correspondences $X = (X_p)_{p \in P}$ such that

(i) X_e is $\mathcal{A}$ as an $\mathcal{A}$-correspondence.
(ii) For $p, q \in P$, there exists a unitary $\mathcal{A}$-linear isomorphism $M_{p,q} : X_p \otimes X_q \to X_{pq}$
(iii) The left and right multiplication on each X_p are given via $M_{e,p}$ and $M_{p,e}$ for each $p \in P$ and we also have associativity in the sense that for $p, q, r \in P$,

$$M_{p,qr}(I_{X_p} \otimes M_{q,r}) = M_{pq,r}(M_{p,q} \otimes I_{X_r})$$

Note that our definition forces each X_p to be non-degenerate since $X_e \otimes X_p$ is isomorphic to X_p via the left action implemented by $M_{e,p}$. We will denote

$M_{p,q}(x \otimes y) = xy \in X_{pq}$ for every $x \in X_p$ and $y \in X_q$. We will also denote by $\phi_p : \mathcal{A} \to \mathcal{L}(X_p)$ the left action on X_p for each $p \in P$. In particular, $\phi_{pq}(a)(xy) = (\phi_p(a)x)y$ for all $p, q \in P$, $a \in A$ and $x \in X_p$, $y \in X_q$.

Given $p \in P \setminus \{e\}$ and $q \in P$, the unitary $\mathcal{A}$-linear map $M_{p,q} : X_p \otimes X_q \to X_{pq}$ induces a $*$-homomorphism $\iota_p^{pq} : \mathcal{L}(X_p) \to \mathcal{L}(X_{pq})$ via

$$\iota_p^{pq} := M_{p,q} \circ (S \otimes id_{X_p}) M_{p,q}^{-1}$$

for each $S \in \mathcal{L}(X_p)$. Alternatively, we have that the $*$-homomorphism ι_p^{pq} is given by the formula $\iota_p^{pq}(S)(xy) = (Sx)y$ for each $S \in \mathcal{L}(X_p)$, $x \in X_p$ and $y \in X_q$. For ι_e^q, we first define on $\mathcal{A} \cong \mathcal{K}(\mathcal{A})$ via $\iota_e^p(a) = \phi_p(a)$, and then extend uniquely to $\mathcal{L}(\mathcal{A})$ via [**46**, Proposition 2.5] to obtain a map $\iota_e^q : \mathcal{L}(\mathcal{A}) \to \mathcal{L}(X_q)$. Finally, for notational purposes, we will define ι_p^r to be zero whenever $r \neq pq$ for all $q \in P$.

When $X = (X_p)_{p \in P}$ is a product system over a quasi-lattice ordered semigroup (G, P), we will say that X is *compactly aligned* if whenever $S \in \mathcal{K}(X_p)$ and $T \in \mathcal{K}(X_q)$ for some $p, q \in P$ with $p \vee q < \infty$, then $S \vee T := \iota_p^{p\vee q}(S)\iota_q^{p \vee q}(T) \in \mathcal{K}(X_{p \vee q})$.

2.6. **Nica-Toeplitz representations.** We begin with an important definition.

Definition 2.1. Suppose (G, P) is a quasi-lattice ordered group, and X a compactly aligned product system over P with coefficients in $\mathcal{A}$. An *isometric representation* of X into a C*-algebra $\mathcal{B}$ is a map $\psi : X \to \mathcal{B}$ comprised of linear maps $\psi_p : X_p \to \mathcal{B}$ for each $p \in P$ such that

(i) ψ_e is a $*$-homomorphism from $\mathcal{A} = X_e$ into $\mathcal{B}$.
(ii) $\psi_p(x)\psi_q(y) = \psi_{pq}(xy)$ for all $p, q \in P$ and $x \in X_p$, $y \in X_q$.
(iii) $\psi_p(x)^*\psi_p(y) = \psi_e(\langle x, y\rangle)$ for all $p \in P$ and $x, y \in X_p$.

It is standard to show that each ψ_p is contractive, and is isometric precisely when ψ_e is injective. We will say that ψ is non-degenerate provided that $\mathcal{B} \subseteq B(\mathcal{H})$ and ψ_e is non-degenerate. For each $p \in P$ we obtain a $*$-homomorphism $\psi^{(p)} : \mathcal{K}(X_p) \to \mathcal{B}$ given by $\psi^{(p)}(\theta_{x,y}) = \psi_p(x)\psi_p(y)^*$ for all $x, y \in X_p$.

We will say that an isometric representation ψ of a compactly aligned product system X is *Nica-covariant* if for any $p, q \in P$ and $S \in \mathcal{K}(X_p)$, $T \in \mathcal{K}(X_q)$ we have that

$$\psi^{(p)}(S)\psi^{(q)}(T) = \begin{cases} \psi^{(p\vee q)}(S \vee T) & \text{if } p \vee q < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

In the case where $\mathcal{B} \subseteq B(\mathcal{H})$ is a non-degenerate representation of $\mathcal{B}$, the above definition simplifies when we consider ψ into $B(\mathcal{H})$. Indeed by [**46**, Proposition 2.5] the $*$-homomorphism ψ admits a unique strict-SOT continuous extension on $\mathcal{L}(X_p)$, which will be still denoted as $\psi^{(p)}$. Then

$\psi : X \to B(\mathcal{H})$ is Nica-covariant if and only if for any $p, q \in P$ we have

$$\psi^{(p)}(I)\psi^{(q)}(I) = \begin{cases} \psi^{(p\vee q)}(I) & \text{if } p \vee q < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Actually the above condition is the one appearing in the original definition of Nica-covariance, as it appeared in [**26**]. It has the drawback that it applies only to representations into concrete Hilbert spaces, but has the advantage that it works for arbitrary product systems which are not necessarily compactly aligned. With the advent of compactly aligned systems, condition (2.2) was replaced by (2.1). Both conditions are equivalent for concrete representations by [**24**, Proposition 5.6].

Each product system X has a natural Nica-covariant isometric representation on Fock space which we now describe. We denote by $\mathcal{F}_X := \oplus_{p\in P} X_p$ the direct sum of sequences, which is naturally an $\mathcal{A}$-correspondence in its own. We then define $l : X \to \mathcal{L}(\mathcal{F}_X)$ given by $l_p(x)(y_q)_{q\in P} = (xy_q)_{q\in P}$ for each $p \in P$, $x \in X_p$, and $(y_q)_{q\in P} \in \mathcal{F}_X$. We call l the Fock representation, which is an isometric Nica-covariant representation of X by [**24**, Lemma 5.3].

We denote by $\mathcal{NT}_X$ the universal C*-algebra generated by a Nica-covariant representation for X, which exists due to [**24**, Theorem 6.3]. Put in other words, there is an isometric Nica-covariant representation $i_X : X \to \mathcal{NT}_X$ such that $\mathcal{NT}_X$ is generated by the image of i_X and for any other isometric Nica-covariant representation $\psi : X \to B(\mathcal{H})$ there exists a $*$-homomorphism $\psi_* : \mathcal{NT}_X \to B(\mathcal{H})$ such that $\psi_* \circ i_{X,p} = \psi_p$, for every $p \in P$. We remark that in the pertinent selfadjoint literature $\mathcal{NT}_X$ is usually denoted as $\mathcal{T}_{cov}(X)$.

By [**13**, Proposition 3.5] (See also [**24**, Proposition 4.7]) there is a canonical gauge coaction $\delta_X : \mathcal{NT}_X \to \mathcal{NT}_X \otimes \mathrm{C}^*(G)$ given by $\delta_X(i_{X,p}(x)) = i_{X,p}(x) \otimes i_G(p)$ for each $p \in P$ and $x \in X_p$. We will say that a representation $\psi : X \to B(\mathcal{H})$ is *gauge-compatible* if there is a (full) gauge coaction β of G on $\mathrm{C}^*(\{\psi_p(X_p)\}_{p\in P})$ such that $\beta(\psi_p(x)) = \psi_p(x) \otimes i_G(p)$ for all $x \in X_p$ and $p \in P$. When ψ is gauge compatible, the induced $*$-homomorphism $\psi_* : \mathcal{NT}_X \to \mathrm{C}^*(\{\psi_p(X_p)\}_{p\in P})$ is equivariant in the sense that $(\psi_* \otimes \mathrm{id}_{\mathrm{C}^*(G)}) \circ \delta_X = \beta \circ \psi_*$.

In [**24**, Theorem 7.2] Fowler provides a condition on an isometric Nica-covariant representation ψ to induce a faithful $*$-representation ψ_* of $\mathcal{NT}_X$. We state it here for the case where (G, P) is abelian, lattice ordered.

As we saw earlier, for an isometric Nica-covariant representation ψ we have the projections $\{\psi^{(p)}(I)\}_{p\in P}$. Each $\psi^{(p)}(I)$ is the SOT limit of (any) contractive approximate identity for $\psi^{(p)}(\mathcal{K}(X_p))$. Furthermore, since each X_p is essential, $\psi^{(p)}(I)$ projects onto $\psi_p(X_p)\mathcal{H}$.

Theorem 2.2 (Fowler)**.** *Suppose (G, P) is an abelian, lattice ordered group, X is a compactly aligned product system over P with coefficients in $\mathcal{A}$, and $\psi : X \to B(\mathcal{H})$ an isometric Nica-covariant representation. Suppose that* *for any finite set $F \subseteq P \setminus \{e\}$ the representation $A \to B(\mathcal{H})$ given by*

$$a \mapsto \psi_e(a) \prod_{p \in F} \big(I - \psi^{(p)}(I)\big)$$

is faithful. Then the induced $*$-representation $\psi_* : \mathcal{NT}_X \to B(\mathcal{H})$ is faithful.

As an easy application of this theorem, we see that the $*$-homomorphism $l_* : \mathcal{NT}_X \to \mathcal{L}(\mathcal{F}_X)$ induced from the Fock representation l is faithful. To show this, suppose that $\iota : \mathcal{A} \to B(\mathcal{H})$ is a faithful non-degenerate representation, so that it induces a representation $\psi := l \otimes \iota : X \to B(\mathcal{F}_X \otimes_{\mathcal{A}} \mathcal{H})$ on a *Hilbert space*. It is clear that l_* is injective if and only if ψ_* is injective, since the induced map from $\mathcal{L}(\mathcal{F}_X) \to \mathcal{L}(\mathcal{F}_X \otimes \mathcal{H})$ given by $T \mapsto T \otimes I_{\mathcal{H}}$ is an injective $*$-homomorphism. Hence, we will show instead that ψ_* is injective. For that purpose, note that $\psi_e : \mathcal{A} \to \mathcal{L}(\mathcal{F}_X \otimes \mathcal{H})$ is faithful because it is faithful on $\mathcal{A}$ after restricting $\psi_e(a)$ to the reducing subspace $X_e \otimes \mathcal{H} = \mathcal{A} \otimes \mathcal{H}$ for every $a \in \mathcal{A}$. Next, since $(I - \psi^{(p)}(I))|_{X_e \otimes \mathcal{H}} = P_{X_e \otimes \mathcal{H}}$, we see that the map $a \mapsto \psi_e(a) \prod_{p \in F} \big(I - \psi^{(p)}(I)\big)$ is faithful on $\mathcal{A}$. Thus, by the theorem above, we are done.

2.7. **Cuntz-Nica-Pimsner representations.** In order to define Cuntz-Nica-Pimsner representations, we need to recall some definitions from [**58**]. Suppose (G, P) is quasi-lattice ordered and X is a compactly aligned product system over P with coefficients in $\mathcal{A}$. We denote $I_e(X) = \mathcal{A}$, and for each $q \in P \setminus \{e\}$ write $I_q(X) := \cap_{e < p \leq q} \mathrm{Ker}(\phi_p)$. We then denote

$$\tilde{X}_q := \oplus_{p \leq q} X_p \cdot I_{p^{-1}q}(X).$$

The homomorphism implementing the left action on the $\mathcal{A}$-correspondence is denoted by $\tilde{\phi}_q$, and we say that X is $\tilde{\phi}$-injective if $\tilde{\phi}_q$ is injective for all $q \in P$. By [**58**, Lemma 3.15], if every bounded subset of P has a maximal element, every product system over P is automatically $\tilde{\phi}$-injective. This holds, for instance, when $P = \mathbb{N}^d$.

Recalling the definitions of the maps ι_p^q, we obtain homomorphisms $\tilde{\iota}_p^q : \mathcal{L}(X_p) \to \mathcal{L}(\tilde{X}_q)$ for all $p, q \in P$ with $p \neq e$ defined by $\tilde{\iota}_p^q(T) = \oplus_{p \leq r \leq q} \iota_p^r(T)$ for all $p, q \in P$ with $p \neq e$. When $p = e$, similarly to the above we obtain a homomorphism $\tilde{\iota}_e^q : \mathcal{K}(X_e) \to \mathcal{L}(\tilde{X}_q)$.

Suppose now that X is $\tilde{\phi}$-injective. We say that a Nica-covariant isometric representation ψ of X is *Cuntz-Nica-Pimsner* covariant if for every $s \in P$ there exists $r \geq s$ such that for every $q \geq r$ and every finite set $F \subseteq P$ for which $\sum_{p \in F} \tilde{\iota}_p^q(T_p) = 0$, we have $\sum_{p \in F} \psi^{(p)}(T_p) = 0$ when $T_p \in \mathcal{K}(X_p)$ for $p \in F$.

When X is $\tilde{\phi}$-injective, by [**58**, Proposition 3.12] there is a universal C*-algebra $\mathcal{NO}_X$ generated by a (universal) CNP covariant representation $j_X : X \to \mathcal{NO}_X$ called the *Cuntz-Nica-Pimsner algebra* of X. One of the main results [**58**, Theorem 4.1] in the paper of Sims and Yeend is that the representation $j_{X,p} := (j_X)_p$ is isometric on each X_p. By [**13**, Remark 4.5], the gauge coaction δ_X on $\mathcal{NT}_X$ then yields a natural gauge coaction ν_X on

$\mathcal{NO}_X$ which makes j_X into a gauge compatible representation. We combine [**13**, Corollary 4.12] and [**13**, Definition 4.10] to obtain the following gauge invariant uniqueness theorem.

Theorem 2.3 (Carlsen, Larsen, Sims & Vittadello)**.** *Let (G,P) be an abelian, lattice ordered group and let X be a compactly aligned product system over P with coefficients in $\mathcal{A}$. Assume that X is $\tilde{\phi}$-injective. If $\psi : X \to B(\mathcal{H})$ is a Cuntz-Nica-Pimsner covariant gauge-compatible isometric representation such that ψ_e is injective on $\mathcal{A}$, then the induced $*$-homomorphism $\psi_* : \mathcal{NO}_X \to \mathrm{C}^*(\{\psi_p(X_p)\}_{p\in P})$ is a $*$-isomorphism.*

Since we are assuming in this paper that G is abelian, the concept of gauge compatibility reduces to a more familiar one and the statement of the theorem above simplifies considerably.

Proposition 2.4. *Let (G,P) be an abelian, lattice ordered group, and X a compactly aligned product system over P with coefficients in $\mathcal{A}$. Then a representation $\psi : X \to B(\mathcal{H})$ is gauge-compatible if and only if there exists a gauge action*

$$\hat{\alpha}_X\colon \hat{G} \longrightarrow \mathrm{Aut}\left(\mathrm{C}^*(\{\psi_p(X_p)\}_{p\in P})\right)$$

such that $\hat{\alpha}_{X,\gamma}(\psi_p(x)) = \gamma(p)\psi_p(x)$ for all $x \in X_p$, $p \in P$ and $\gamma \in \hat{G}$, the Pontryagin dual of G.

Proof. For notational simplicity, let $\mathcal{B} \equiv \mathrm{C}^*(\{\psi_p(X_p)\}_{p\in P})$. Assume first that ψ is gauge-compatible via a coaction δ of G. Let

$$F\colon \mathrm{C}^*(G) \longrightarrow \mathrm{C}(\hat{G})$$

be the Fourier transform. Then for each $\gamma \in \hat{G}$, the desired $\hat{\alpha}_{X,\gamma}\colon \mathcal{B} \to \mathcal{B}$ is given by the composition

$$\mathcal{B} \xrightarrow{\delta} \mathcal{B} \otimes \mathrm{C}^*(G) \xrightarrow{\mathrm{id}\otimes F} \mathcal{B} \otimes \mathrm{C}(\hat{G}) \xrightarrow{\mathrm{id}\otimes e_\gamma} \mathcal{B},$$

where $e_\gamma\colon \mathrm{C}(\hat{G}) \to \mathbb{C}$ denotes the evaluation at $\gamma \in \hat{G}$.

Conversely, assume that ψ admits a gauge action $\hat{\alpha}_X\colon \hat{G} \longrightarrow \mathrm{Aut}\,\mathcal{B}$ with $\hat{\alpha}_{X,\gamma}(\psi_p(x)) = \gamma(p)\psi_p(x)$ for all $x \in X_p$, $p \in P$ and $\gamma \in \hat{G}$. Then the desired coaction δ comes from the composition

$$\mathcal{B} \xrightarrow{R} \mathcal{B} \otimes \mathrm{C}(\hat{G}) \xrightarrow{\mathrm{id}\otimes F^{-1}} \mathcal{B} \otimes \mathrm{C}^*(G),$$

where $R\colon \mathcal{B} \to \mathcal{B} \otimes \mathrm{C}(\hat{G})$ is defined by $R(b)(\gamma) \equiv \hat{\alpha}_{X,\gamma}(b)$ for $b \in \mathcal{B}$, $\gamma \in \hat{G}$, where we identify $\mathcal{B} \otimes \mathrm{C}(\hat{G})$ with $\mathrm{C}(\hat{G}, \mathcal{B})$. ∎

3. The C*-envelope of a Nica tensor algebra

In this section we characterize the C*-envelope of the Nica tensor algebra $\mathcal{NT}_X^+$ of a product system X over a (discrete) abelian, lattice ordered group (G,P).

The Nica tensor algebra of the product system X is given by

$$\mathcal{NT}_X^+ := \overline{\mathrm{alg}}^{\|\cdot\|}\{i_p^X(X_p)\}.$$

Alternatively, $\mathcal{NT}_X^+$ is completely isometrically isomorphic to the non-selfadjoint operator algebra generated by the image of the Fock representation $l\colon X \to \mathcal{L}(\mathcal{F}_X)$. It is naturally a subalgebra of $\mathcal{NT}_X$, and is also the universal norm-closed operator algebra generated by a Nica-covariant isometric representation of X.

In the special case where (G,P) is the ordered group $(\mathbb{Z},\mathbb{N})$, the algebra $\mathcal{NT}_X^+$ is just the tensor algebra of a C*-correspondence of Muhly and Solel [**49**]. Such algebras are at the forefront of non-selfadjoint operator algebra research. See [**9**, Chapter 2] and the references therein for more details.

Let (G,P) be a quasi-lattice ordered group. If $\psi : X \to B(\mathcal{H})$ and $V : P \to B(\mathcal{K})$ are isometric representations, then we define a representation

$$\psi \otimes V : X \longrightarrow B(\mathcal{H}\otimes\mathcal{K});\ X_p \ni x \longmapsto \psi_p(x)\otimes V_p, \quad p \in P. \tag{3.1}$$

Clearly $\psi\otimes V$ is an isometric representation of X.

Lemma 3.1. *Let (G,P) be a quasi-lattice ordered group and let X be a compactly aligned product system over P with coefficients in $\mathcal{A}$. Assume that $\psi : X \to B(\mathcal{H})$ and $V : P \to B(\mathcal{K})$ are isometric representations. Then*

(i) *If both ψ and V are Nica-covariant then $\psi\otimes V$ is also Nica-covariant.*

(ii) *If ψ is a CNP representation and U is a unitary representation of P, then $\psi\otimes U$ is also a CNP representation of X.*

Proof. (i) According to (2.2), we need to verify that

$$(\psi\otimes V)^{(p)}(I)(\psi\otimes V)^{(q)}(I) = \begin{cases} (\psi\otimes V)^{(p\vee q)}(I) & \text{if } p\vee q<\infty \\ 0 & \text{otherwise.}\end{cases}$$

Now notice that

$$\begin{aligned}(\psi\otimes V)^{(p)}(I)(\mathcal{H}\otimes\mathcal{K}) &= (\psi\otimes V)_p(X)(\mathcal{H}\otimes\mathcal{K})\\ &= \big(\psi_p(X)\otimes V_p\big)(\mathcal{H}\otimes\mathcal{K})\\ &= \psi_p(X)(\mathcal{H})\otimes V_p(\mathcal{K})\\ &= \big(\psi^{(p)}(I)\otimes V_pV_p^*\big)(\mathcal{H}\otimes\mathcal{K}).\end{aligned}$$

Hence we have $(\psi\otimes V)^{(p)}(I) = \psi^{(p)}(I)\otimes V_pV_p^*$ and similarly $(\psi\otimes V)^{(q)}(I) = \psi^{(q)}(I)\otimes V_qV_q^*$. Therefore, if $p\vee q<\infty$, we have

$$\begin{aligned}(\psi\otimes V)^{(p)}(I)(\psi\otimes V)^{(q)}(I) &= \psi^{(p)}(I)\psi^{(q)}(I)\otimes V_pV_p^*V_qV_q^*\\ &= \psi^{(p\vee q)}(I)\otimes V_{p\vee q}V_{p\vee q}^*\\ &= (\psi\otimes V)^{(p\vee q)}(I)\end{aligned}$$

and similarly if p and q have no upper bound, $(\psi\otimes V)^{(p)}(I)(\psi\otimes V)^{(q)}(I) = 0$, as desired.

(ii) Since $(\psi \otimes U)^{(p)} = \psi^{(p)} \otimes I_{\mathcal{K}}$, the conclusion follows directly from the definition of Cuntz-Nica-Pimsner covariance. ∎

The following is the central result of the paper.

Theorem 3.2. *Let (G, P) be an abelian, lattice ordered group. Let X be a compactly aligned $\tilde{\phi}$-injective product system over P with coefficients in $\mathcal{A}$. Then*

$$\mathrm{C}^*_e(\mathcal{NT}_X^+) \simeq \mathcal{NO}_X$$

via a map that sends generators to generators.

Proof. Let $j_X : X \to \mathcal{NO}_X \subseteq B(\mathcal{H})$ be the universal CNP representation of X where $\mathcal{NO}_X \subseteq B(\mathcal{H})$ is non-degenerate, and let $V : P \to B(\mathcal{K})$ be the universal Nica-covariant isometric representation of P. We define the representation

$$j_X \otimes V : X \longrightarrow B(\mathcal{H} \otimes \mathcal{K});\ X_p \ni x \longmapsto j_X(x) \otimes V_p, \quad p \in P.$$

Since both representations j_X and V are Nica-covariant, Lemma 3.1(i) implies that $j_X \otimes V$ is also Nica-covariant.

Now V contains the left-regular representation of P as a direct summand and so there is some $0 \neq \xi \in \mathcal{K}$ such that $V_t V_t^* \xi = 0$ for all $t \in P \setminus \{e\}$. The subspace

$$\mathcal{H}_\xi := \mathcal{H} \otimes \operatorname{span}\{V_t \xi\}_{t \in P}$$

is reducing for $j_X \otimes V$ and for all $p \in P \setminus \{e\}$ we have

$$\big(I - (j_X \otimes V)^{(p)}(I)\big)|_{\mathcal{H}_\xi} = I \otimes P_\xi,$$

where P_ξ is the projection onto $\mathbb{C}\xi$. Hence, for each finite set $F \subseteq P \setminus \{e\}$, the map

$$\mathcal{A} \longrightarrow B(\mathcal{H} \otimes \mathcal{K});\ a \longmapsto (j_X \otimes V)_e(a) \prod_{p \in F} \big(I - (j_X \otimes V)^{(p)}(I)\big)$$

is injective when restricted to the reducing subspace $\mathcal{H}_\xi$. Hence, Theorem 2.2 implies that the induced $*$-homomorphism

$$(j_X \otimes V)_* : \mathcal{NT}_X \longrightarrow B(\mathcal{H} \otimes \mathcal{K})$$

is injective. In particular, it is completely isometric on $\mathcal{NT}_X^+$.

Laca's theorem [**44**, Theorem 1.4] now implies the existence of a unitary dilation

$$U : P \longrightarrow B(\mathcal{K}')$$

of V on some Hilbert space $\mathcal{K}' \supseteq \mathcal{K}$, and by Lemma 3.1(ii) we have that $j_X \otimes U$ is a CNP representation of X. Since $j_X \otimes U$ dilates $j_X \otimes V$, the induced $*$-homomorphism

$$(j_X \otimes U)_* : \mathcal{NT}_X \longrightarrow B(\mathcal{H} \otimes \mathcal{K}')$$

is completely isometric on $\mathcal{NT}_X^+$.

On the other hand, the representation $j_X \otimes U$ also induces a $*$-homomorphism

$$(j_X \otimes U)_{**} : \mathcal{NO}_X \longrightarrow B(\mathcal{H} \otimes \mathcal{K}').$$

which is faithful by Theorem 2.3. Indeed, using Proposition 2.4, the C*-algebra $(j_X \otimes U)_{**}(\mathcal{NO}_X)$ admits a gauge coaction coming from the gauge action of $\hat{G}$ given by $\hat{\alpha}_X \otimes \mathrm{id}$ on $(j \otimes U)_*(\mathcal{NT}_X) \subseteq \mathcal{NO}_X \otimes \mathrm{C}^*(G)$.

Since

$$j_{X*} \mid_{\mathcal{NT}_X^+} = (j_X \otimes U)_{**}^{-1} \circ (j_X \otimes U)_* \mid_{\mathcal{NT}_X^+},$$

the last two paragraphs imply that $j_{X*} \mid_{\mathcal{NT}_X^+}$ is a complete isometry and so $(\mathcal{NO}_X, j_{X*} \mid_{\mathcal{NT}_X^+})$ is a C*-cover for $\mathcal{NT}_X^+$.

To finish the proof we need to verify that $(\mathcal{NO}_X, j_{X*} \mid_{\mathcal{NT}_X^+})$ is actually the C*-envelope of $\mathcal{NT}_X^+$. Let $\mathcal{J}_X$ be the Shilov ideal of $\mathcal{NT}_X^+ \subseteq \mathcal{NO}_X$. We will show that $\mathcal{J}_X = \{0\}$ and so $\mathrm{C}_e^*(\mathcal{NT}_X^+) \simeq \mathcal{NO}_X$ via a map that sends generators to generators.

By way of contradiction, assume that $\mathcal{J}_X \neq \{0\}$. Let

$$\hat{\alpha}_X : \hat{G} \longrightarrow \mathrm{Aut}(\mathcal{NO}_X)$$

be the canonical gauge action of Proposition 2.4. Clearly each $\hat{\alpha}_{X,\gamma}$ leaves invariant $\mathcal{NT}_X^+ \subseteq \mathcal{NO}_X$ and so it will leave invariant the *largest* boundary ideal of $\mathcal{NT}_X^+$, i.e., $\hat{\alpha}_{X,\gamma}(\mathcal{J}_X) = \mathcal{J}_X$, for all $\gamma \in \hat{G}$. Therefore $\mathcal{NO}_X/\mathcal{J}_X$ inherits a canonical gauge action

$$\breve{\alpha}_X : \hat{G} \longrightarrow \mathrm{Aut}(\mathcal{NO}_X/\mathcal{J}_X)$$

so that $\breve{\alpha}_{X,\gamma} \circ \pi = \pi \circ \hat{\alpha}_{X,\gamma}$, for all $\gamma \in \hat{G}$, where

$$\pi : \mathcal{NO}_X \to \mathcal{NO}_X/\mathcal{J}_X$$

is the natural quotient map. By Proposition 2.4, the quotient map π is gauge compatible, as a representation of X. Since π is not faithful, Theorem 2.3 implies that $\pi_e = \pi \mid_{\mathcal{A}}$ is not faithful. But since $\mathcal{A} \subseteq \mathcal{NT}_X^+$, this implies that π is not faitfhful on $\mathcal{NT}_X^+$ and so $\mathcal{J}_X$ is not a boundary ideal, which is a contradiction. Hence $\mathcal{J}_X = \{0\}$ and so $\mathrm{C}_e^*(\mathcal{NT}_X^+) = \mathcal{NO}_X$ as desired. ∎

Remark 3.3. (i) It is shown in **[13]** that $\mathcal{NO}_X$ is co-universal in the sense that it is the unique C*-algebra which: (1) is generated by an injective Nica-covariant representation of X; (2) carries a gauge coaction that is equivariant with the canonical gauge coaction on Fowler's $\mathcal{NT}_X$; and (3) has the property that, given any other C*-algebra B satisfying (1) and (2), there is a canonical $*$-homomorphism $\phi \colon B \to \mathcal{NO}_X$.

The above properties do not a priori imply that $\mathcal{NO}_X \simeq \mathrm{C}_e^*(\mathcal{NT}_X^+)$ even though they seemingly come close to doing that. The reason is the universality property (3). Indeed, to be the C*-envelope of $\mathcal{NT}_X$, $\mathcal{NO}_X$ must have the property that given any other C*-algebra B generated by a completely isometric representation ψ of $\mathcal{NT}_X^+$, there is a canonical *-homomorphism

$\phi\colon B \to \mathcal{NO}_X$. It is easy to construct examples of completely isometric representations ψ of $\mathcal{NT}_X^+$, whose restrictions on X do not produce isometric representations, in the sense of Definition 2.1. Hence property (3) does not give the universal property for the C*-envelope.

(ii) In the special case where X is an injective product system over $(\mathbb{Z}^d, \mathbb{N}^d)$ and satisfies an additional condition (strong compact alignment), Theorem 3.2 has been obtained independently by Evgenios Kakariadis using different methods.

4. Dilations of Nica-Toeplitz representations

In this section we answer a question of Skalski and Zacharias, which appears at the end of [**59**]. Suppose that X is a compactly aligned product system with coefficients in $\mathcal{A}$ over a quasi-lattice ordered semigroup (G, P). We will say that a Nica-covariant isometric representation ψ of X is *fully-coisometric* if $\psi_p(X_p)\mathcal{H} = \psi_e(\mathcal{A})\mathcal{H}$ for every $p \in P$.

When $(G, P) = (\mathbb{Z}^d, \mathbb{N}^d)$, we have a standard set of generators $\{e_i\}_{i=1}^d$ for $\mathbb{N}^d$, so that an isometric representation ψ can be considered as $d+1$-tuples $(\sigma^\psi, T_1^\psi, ..., T_d^\psi)$ where $\sigma^\psi := \psi_e$ is a representation of the coefficient algebra $\mathcal{A}$, and $T_i^\psi := \psi_{e_i}$ is a representation of X_{e_i}. In this case, the $d+1$-tuple $(\sigma^\psi, T_1^\psi, ..., T_d^\psi)$ becomes an isometric representation of X in the sense of [**59**, Definition 1.2]. Nica covariance of ψ is equivalent to the assumption that $(T_1^\psi, ..., T_d^\psi)$ are doubly commuting in the sense of [**59**, Definition 2.1], and when ψ is non-degenerate, it is fully-coisometric if and only if $(\sigma^\psi, T_1^\psi, ..., T_d^\psi)$ is fully-coisometric as in [**59**, Definition 1.1]. In fact, the above forms a bijection between Nica-covariant isometric representations ψ of X, and doubly-commuting isometric representations $(\sigma, T_1, ..., T_d)$ as in definition [**59**, Definition 1.1].

In the classical context of a unitary dilation of isometries, a theorem of Itô shows that every commuting d-tuple of isometries dilate to a commuting d-tuple of unitaries. In the hope of generalizing Itô's theorem, Skalski and Zacharias (See [**59**, Section 5]) ask when an isometric representation $(\sigma, T_1, ..., T_d)$ of a product system X over $\mathbb{N}^d$ has an isometric and fully-coisometric dilation $(\pi, U_1, ..., U_d)$. Laca's theorem [**44**] generalizes Itô's theorem to show that every isometric representation $V : P \to B(\mathcal{H})$ of an Ore semigroup (G, P) dilates to a unitary representation.

Recall that a product system X over a quasi-lattice ordered semigroup (G, P) is said to be *regular* when the left action ϕ_p of each X_p is injective with image in $\mathcal{K}(X_p)$. By [**58**, Corollary 5.2] we know that for regular product systems X with P directed, an isometric representation ψ of X is CNP if and only if $\psi^{(p)} \circ \phi_p = \psi_e$ for all $p \in P$. Hence, when X is regular, since the projection $\psi^{(p)}(I)$ onto $\psi_p(X_p)\mathcal{H}$ is the SOT limit of (any) contractive approximate identity for $\psi^{(p)}(\mathcal{K}(X_p))$, we have that ψ is CNP if and only if each ψ_p is fully-coisometric. The following yields an answer to Skalski and Zacharias' question in the positive when X is regular and ψ is Nica-covariant, and is a generalization of [**59**, Theorem 5.4] to the semigroup context.

Theorem 4.1. *Let X be a* regular *product system over an abelian, lattice ordered semigroup (G, P) and let ψ be a* Nica-covariant *isometric representation of X on a Hilbert space $\mathcal{H}$. Then there is an isometric and* fully-coisometric *representation $\tilde{\psi}$ of X on $\mathcal{K}$ that dilates ψ in the sense that $\mathcal{H} \subseteq \mathcal{K}$ and $\psi_p(x_p) = P_{\mathcal{H}}\tilde{\psi}_p(x_p)|_{\mathcal{H}}$ for every $x_p \in X_p$.*

Proof. Assume that $\mathcal{NT}_X^+$ is not unital. Since $\mathcal{NT}_X^+$ has an approximate unit, by [**10**, Remark 2.1.8] no C*-cover of $\mathcal{NT}_X^+$ is unital.

Let ψ be a Nica-covariant isometric representation of X. This induces a representation of $\mathcal{NT}_X$; in particular, a completely contractive representation $\psi_* : \mathcal{NT}_X^+ \to B(\mathcal{H})$ is induced. By Meyer's Theorem, we have a completely contractive unitization

$$\psi_*^1 : (\mathcal{NT}_X^+)^1 \longrightarrow B(\mathcal{H})$$

By the Dritschel-McCullough theorem [**21**, Theorem 1.2], we have that ψ_*^1 dilates to a *maximal* representation

$$\tilde{\psi}_*^1 : (\mathcal{NT}_X^+)^1 \longrightarrow B(\mathcal{H}),$$

which by [**21**, Theorem 1.1] has the unique extension property. By Theorem 3.2 and Meyer's Theorem, $(\mathcal{NO}_X)^1$ becomes a C*-cover of $(\mathcal{NT}_X^+)^1$, via a map that sends generators to generators canonically. We may therefore extend $\tilde{\psi}_*^1$ to a $*$-representation of $(\mathcal{NO}_X)^1$, which we denote with the same symbol. We now define

$$\tilde{\psi} : X \longrightarrow B(\mathcal{K});\, \xi_p = \tilde{\psi}_*^1(i_{X,p}(\xi_p)), \quad p \in P,$$

to obtain an isometric Nica-covariant representation $\tilde{\psi} : X \to B(\mathcal{K})$ that dilates ψ. Since

$$j_X : X \longrightarrow \mathcal{NO}_X$$

in a CNP representation [**58**, Proposition 3.12], we have that $\tilde{\psi} = \tilde{\psi}_*^1 \circ j_X$ is also a CNP representation. By the remarks preceding the theorem, $\tilde{\psi}$ is an isometric and fully-coisometric representation that dilates ψ. This completes the proof in the non-unital case.

If $\mathcal{NT}_X^+$ is unital, then a repetition of the above arguments without resorting to unitizations, suffices for the proof. ∎

The following is the positive answer to the original question of Skalski and Zacharias, under the assumption of Nica-covariance.

Corollary 4.2. *Let X be a* regular *product system over $\mathbb{N}^d$, and suppose that $(\sigma, T_1, ..., T_d)$ is a* doubly-commuting *isometric representation of X on a Hilbert space $\mathcal{H}$. Then there is an isometric and* fully-coisometric *representation $(\pi, U_1, ..., U_d)$ of X that dilates ψ.*

As another corollary of [**58**, Corollary 5.2] and Theorem 3.2, we obtain

Corollary 4.3. *Let X be a* regular *product system over an abelian, lattice ordered semigroup (G, P). Then $\mathrm{C}^*_e(\mathcal{NT}_X^+)$ is the universal C^*-algebra generated by the image of representations ψ_* where ψ is a representation of X such that $\psi^{(p)} \circ \phi_p = \psi_e$ for all $p \in P$.*

5. Examples and comparisons

5.1. **Higher rank graphs.** Here we collect some terminology on higher rank graphs and their product systems. More details can be found in [**55**], [**56**] and [**57**]. Let $G = (V, E, r, s)$ be directed graph, and partition $E = E_1 \cup ... \cup E_d$ such that each edge carries a unique color from a selection of d colors. Denote by $E^\bullet$ the collection of all paths in G. We may then define a multi-degree function $d : E^\bullet \to \mathbb{Z}^d_+$ by $d(\lambda) = (n_1, ..., n_d)$, where n_i is the number of edges in λ from E_i.

A higher rank structure on G is an equivalence relation $\sim$ on $E^\bullet$ such that for all $\lambda \in E^\bullet$ and $p, q \in \mathbb{Z}^d_+$ with $d(\lambda) = p + q$, there exist *unique* $\mu, \nu \in E^\bullet$ with $s(\lambda) = s(\nu)$, $r(\lambda) = r(\mu)$, such that $d(\mu) = p$ and $d(\nu) = q$ and $\lambda \sim \mu\nu$. We denote $\Lambda = E^\bullet / \sim$ and keep denoting $d : \Lambda \to \mathbb{Z}^d_+$ the multi-degree map. We call the pair (Λ, d) a higher rank graph, so that this way, it becomes a higher rank graph as in [**56**, Definition 2.1]. We will keep denoting elements in Λ by λ with the understanding that they may be represented in various ways. For each $p \in \mathbb{Z}^d_+$ we denote $\Lambda^p := \{\lambda \in \Lambda \mid d(\lambda) = p\}$, and when $\lambda \in \Lambda$ and $F \subseteq \Lambda$, we denote $\lambda F := \{\lambda\mu \mid \mu \in F \text{ with } s(\lambda) = r(\mu)\}$ and $F\lambda := \{\mu\lambda \mid \mu \in F \text{ with } s(\mu) = r(\lambda)\}$. Wealso write $\Lambda^1 := \Lambda^0 \cup \bigcup_{i=1}^d \Lambda^{e_i}$.

For $\mu, \nu \in \Lambda$ let

$$\Lambda^{\min}(\mu, \nu) := \{(\alpha, \beta) : \mu\alpha = \nu\beta, d(\mu\alpha) = d(\mu) \vee d(\nu)\}$$

be the set of minimal common extensions of μ and ν. We will say that (Λ, d) is finitely aligned if $|\Lambda^{\min}(\mu, \nu)| < \infty$. Given a vertex $v \in \Lambda^0$, we say that a subset $F \subseteq v\Lambda$ is *exhaustive* if for every $\mu \in v\Lambda$ there is $\nu \in F$ such that $\Lambda^{\min}(\mu, \nu) \neq \emptyset$. Given a finitely aligned higher rank graph (Λ, d), a set of partial isometries $\{S_\lambda\}_{\lambda \in \Lambda}$ is called a *Toeplitz-Cuntz-Krieger* Λ-family if

(P) $\{S_v\}_{v \in \Lambda^0}$ is a collection of pairwise orthogonal projections;
(C) $S_\mu S_\nu = S_{\mu\nu}$ when $s(\mu) = r(\nu)$;
(NC) $S^*_\mu S_\nu = \sum_{(\alpha,\beta) \in \Lambda^{\min}(\mu,\nu)} S_\alpha S^*_\beta$.

It is called a *Cuntz-Krieger* Λ-family if it additionally satisfies

(CK) $\prod_{\lambda \in F}(S_v - S_\lambda S^*_\lambda) = 0$ for every $v \in \Lambda^0$ and all non-empty finite exhaustive sets $F \subseteq v\Lambda$.

We will denote the universal C^*-algebra generated by a Cuntz-Krieger Λ-family by $\mathrm{C}^*(\Lambda)$.

Every higher rank graph (Λ, d) has a natural product system $X(\Lambda)$ associated to it as in [**55**]. More precisely, for each $p \in \mathbb{Z}^d_+$ we put a pre-Hilbert

$c_0(\Lambda^0)$-bimodule structure on $c_0(\Lambda^p)$ via the formula

$$\langle \xi, \eta \rangle(v) := \sum_{s(\lambda)=v} \overline{\xi(\lambda)}\eta(\lambda) \ , \text{ and } (a \cdot \xi \cdot b)(\lambda) := a(r(\lambda))\xi(\lambda)b(s(\lambda))$$

The completion $X(\Lambda)_p$ of this pre-Hilbert bimodule then becomes a product system where the identification $X(\Lambda)_p \otimes X(\Lambda)_q \cong X(\Lambda)_{p+q}$ is given by $\delta_\mu \otimes \delta_\nu = \delta_{\mu\nu}$ when $s(\mu) = r(\nu)$ (and is 0 otherwise in both sides if $s(\mu) \neq r(\nu)$). It was shown in [**55**, Theorem 5.4] that $X(\Lambda)$ is compactly aligned if and only if Λ is finitely aligned. Furthermore, $X(\Lambda)$ is regular if and only if Λ is row-finite and sourceless (in each color separately).

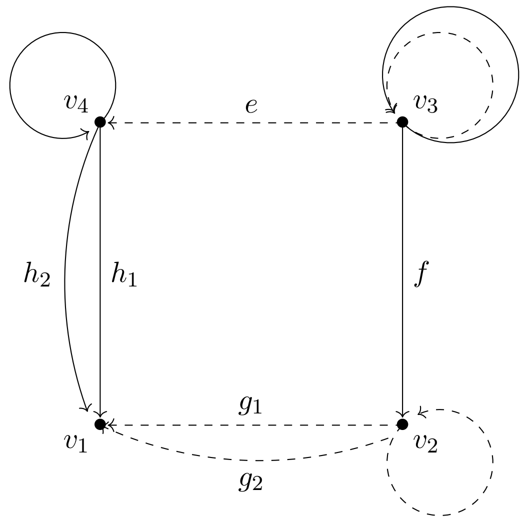
Given a representation ψ of $X(\Lambda)$, the family $\{ \ t_\lambda \mid \lambda \in \Lambda \ \}$ given by $t_\lambda = \psi_p(\delta_\lambda)$ becomes a Λ-family that satisfies conditions (P) and (C). the representation ψ is then Nica-covariant if and only if $\{t_\lambda\}_{\lambda\in\Lambda}$ satisfies condition (NC). In fact, Nica-covariant representations ψ of $X(\Lambda)$ are in bijection with Toeplitz-Cuntz-Krieger Λ families. We will denote by $\mathcal{T}(\Lambda) = \mathcal{NT}_{X(\Lambda)}$ the universal C*-algebra generated by a Toeplitz-Cuntz-Krieger family. By [**58**, Theorem 5.4] we see that the Cuntz-Nica-Pimsner algebra $\mathcal{NO}_{X(\Lambda)}$ coincides with the universal C*-algebra $\mathrm{C}^*(\Lambda)$ generated by a Cuntz-Krieger Λ-family, and that CNP representations of $X(\Lambda)$ are in bijection with Cuntz-Krieger Λ-families. Denote by $\mathcal{T}_+(\Lambda)$ the norm closed generated by a universal Toeplitz-Cuntz-Krieger Λ-family, which coincides with $\mathcal{NT}^+_{X(\Lambda)}$. As a corollary of Theorem 3.2 we obtain the following generalization of [**32**, Theorem 3.6] to the finitely-aligned case.

Corollary 5.1. *Let Λ be a finitely aligned higher rank graph. The C*-envelope of $\mathcal{T}_+(\Lambda)$ coincides with the universal Cuntz-Krieger algebra* $\mathrm{C}^*(\Lambda)$ *associated to Λ.*

In Theorem 4.1, the assumption of faithfulness of left actions ϕ_p for a product system X over (G, P) is easily seen to be a necessary assumption for every isometric Nica-covariant representation to have an isometric and fully-coisometric dilation. Less clear is the assumption that $\phi_p(\mathcal{A}) \subseteq \mathcal{K}(X_p)$. However, in [**59**, Example 5.5] it is shown that an isometric representation of a product system over $\mathbb{N}$ fails to have an isometric and fully-coisometric dilation, even with faithful left actions. Our next goal is to show that in the multivariable context, the assumption of double commutation in Corollary 4.2 cannot be dropped completely.

Example 5.2. There exists an isometric representation ψ of a regular product system $X(\Lambda)$ coming from a finite sourceless graph Λ for which there is an isometric representation with no isometric and fully coisometric dilation (and in particular this representation is not Nica-covariant).

Indeed, we begin by describing a colored graph $G = (V, E)$ on four vertices $V := \{v_1, v_2, v_3, v_4\}$. v_3 has two loops $\nu_3^{(1)}$ and $\nu_3^{(2)}$ of distinct colors 1 and 2 respectively, v_4 has one loop $\nu_4^{(1)}$ of color 1 and v_2 has one loop $\nu_2^{(2)}$ of color 2. There are two edges h_1, h_2 from v_4 to v_1 with color 1, two edges g_1, g_2



from v_2 to v_1 of color 2, an edge f from v_3 to v_2 of color 1 and an edge e from v_3 to v_4 of color 2. This defines a 2-colored graph as in the drawing. We obtain a higher-rank graph Λ on it by specifying the commutation of any pair of concatenating edges of distinct colors

$$h_j e = g_j f, \quad \nu_2^{(2)} f = f \nu_3^{(2)}, \quad \nu_4^{(1)} e = e \nu_3^{(1)}, \quad \nu_3^{(1)} \nu_3^{(2)} = \nu_3^{(2)} \nu_3^{(1)}.$$

For each $1 \leq i \leq 4$ define a different copy $\mathcal{H}_i$ of the Hilbert space $\mathcal{M} := \mathcal{H}_1^{(c)} \oplus \mathcal{H}_2^{(c)} \oplus \mathcal{H}_1^{(s)} \oplus \mathcal{H}_2^{(s)}$, identified via a unitary operator $J_i : \mathcal{H}_i \to \mathcal{M}$, where each summand is $\aleph_0$ dimensional and $\mathcal{H}_j^{(c)}$ and $\mathcal{H}_j^{(s)}$ have orthonormal bases $\{\xi_{c,j}^{(n)}\}_{n \in \mathbb{N}}$ and $\{\xi_{s,j}^{(n)}\}_{n \in \mathbb{N}}$ respectively.

To define a Λ-family $\{S_\lambda\}_{\lambda \in \Lambda}$ that satisfy (P) and (C), it will suffice to specify isometries $T_\lambda : \mathcal{M} \to \mathcal{M}$ where the source $s(\lambda) = v_i$ and range $r(\lambda) = v_{i'}$ and then set $S_\lambda = J_{i'}^{-1} T_\lambda J_i$.

Fix an isometry $V : \mathcal{M} \to \mathcal{H}_1^{(c)} \oplus \mathcal{H}_2^{(c)}$, and take two unitaries $T_1 : \mathcal{M} \to \mathcal{H}_1^{(c)} \oplus \mathcal{H}_1^{(s)}$ and $T_2 : \mathcal{M} \to \mathcal{H}_2^{(c)} \oplus \mathcal{H}_2^{(s)}$. Next, define a switching operator $W : \mathcal{M} \to \mathcal{M}$ given by

$$W(\xi_{c,j}^{(n)}) = \xi_{c,j}^{(n)}, \quad W(\xi_{s,1}^{(n)}) = \xi_{s,2}^{(n)} \text{ and } W(\xi_{s,2}^{(n)}) = \xi_{s,1}^{(n)}.$$

We then set

$$T_e = V = T_f, \quad T_{h_1} = T_1, \quad T_{h_2} = T_2 \quad T_{g_1} = WT_1 \text{ and } T_{g_2} = WT_2.$$

By the previous paragraph, we obtain $S_e, S_f, S_{h_1}, S_{h_2}, S_{g_1}$ and S_{g_2}. We then set each S_{v_i} to be the projection onto the i-th copy of $\mathcal{H}_i$, the loops $S_{\nu_3^{(1)}} = S_{\nu_3^{(2)}}$ to be the projection onto $\mathcal{H}_3$, together with $S_{\nu_2^{(2)}}$ and $S_{\nu_4^{(1)}}$ as the projections onto $\mathcal{H}_2$ and $\mathcal{H}_4$ respectively.

Then, we see that $S_{g_i} S_f = S_{h_i} S_e$ so that $S = \{S_\lambda\}_{\lambda \in E \cup V}$ clearly satisfies conditions (P) and (C), and can hence be extended to a Λ-family $\{S_\lambda\}_{\lambda \in \Lambda}$.

However, $S_{g_1}^* S_{h_2}$ is identified as an operator on $\mathcal{M}$ with $T_1^* W^* T_2 \neq 0$, so we get that $S_{g_1}^* S_{h_2} \neq 0$ while $\Lambda^{\min}(g_1, h_2) = \emptyset$. The former holds since for $\xi = T_2^{-1}(\xi_{s,2}^{(n)}) \in \mathcal{M}$ we have $T_1^* W T_2(\xi) = T_1^* W(\xi_{s,2}^{(n)}) = T_1^*(\xi_{s,1}^{(n)}) \neq 0$, and the latter holds since if it didn't then there would be $\alpha, \beta \in \Lambda$ such that $g_1\alpha = h_2\beta$, forcing $\alpha = e$ and $\beta = f$. This will contradict the unique factorization property since we already have $g_1\alpha = h_1\beta$. Hence, $S = \{S_\lambda\}$ fails (NC).

We next show that S cannot have a dilation that satisfies condition (NC). Assume towards contradiction that $W = \{W_\lambda\}_{\lambda\in\Lambda}$ is a TCK Λ-family that dilates S. It would have to have the form

$$W_\lambda = \begin{bmatrix} * & X_\lambda & * \\ 0 & S_\lambda & * \\ 0 & 0 & * \end{bmatrix}$$

for every $\lambda \in \Lambda$. Since $W_{g_1} W_{g_1}^* = W_{h_2} W_{h_2}^* = W_{v_1}$, we get that

$$X_{g_1}^* X_{g_1} + S_{g_1}^* S_{g_1} = S_{v_1} = X_{h_2}^* X_{h_2} + S_{h_1}^* S_{h_1},$$

and since $S_{g_1}^* S_{g_1} = S_{v_1} = S_{h_1}^* S_{h_1}$ we see that $X_{g_1} = X_{h_2} = 0$. Since W satisfies condition (NC), we must have that $W_{g_1}^* W_{h_2} = 0$, where the $(2,2)$ corner of this equation yields

$$X_{g_1}^* X_{h_2} + S_{g_1}^* S_{h_2} = S_{g_1}^* S_{h_2} \neq 0$$

in contradiction. Hence, if ψ is the representation of X associate to the Λ-family S, we see that ψ is fails to have an isometric and fully-coisometric dilation. Indeed, such a dilation would be Cuntz-Pimsner covariant in the sense that $\psi^{(p)} \circ \phi_p = \psi_e$ so that by [**24**, Proposition 5.4] it would automatically be Nica-covariant.

5.2. **Topological higher rank graphs.** There is a class of product systems that generalizes those arising from higher-rank graphs. To describe this class, one needs to introduce the class of topological higher-rank graphs. We will not do this here but instead we direct the reader to the papers of Yeend [**60, 61**] for the pertinent definitions and additional details.

Corollary 5.3. *Let Λ be a compactly aligned topological k-graph. Let $\mathcal{G}_\Lambda$ be Yeend's boundary path groupoid* [**61**, Definition 4.8] *and let $X(\Lambda)$ be the product system associated with Λ as in* [**13**, Proposition 5.9]*. Then the C*-envelope of $\mathcal{T}_{X(\Lambda)}^+$ coincides with the groupoid* C* *algebra* $\mathrm{C}^*(\mathcal{G}_\Lambda)$.

Proof. In [**13**, Theorem 5.20] it is shown that $\mathcal{NO}(X(\Lambda)) \simeq \mathrm{C}^*(\mathcal{G}_\Lambda)$. The conclusion now follows from Theorem 3.2. ∎

Remark 5.4. In the work of Davidson, Fuller and Kakariadis [**16**], noncommutative C*-dynamical systems $\alpha : \mathbb{Z}_+^N \to \mathrm{End}(A)$ and associated Nica-Toeplitz $\mathcal{NT}(A,\alpha)$ and Cuntz-Nica-Toeplitz $\mathcal{NO}(A,\alpha)$ crossed product algebras were considered. The majority of [**16**, Subsection 4.3] is devoted to obtain a multivariable tail-adding technique that allows them to prove a strong Morita equivalence between $\mathcal{NO}(A,\alpha)$ and $\mathcal{NO}(B,\beta)$ where $\beta : \mathbb{Z}_+^N \to \mathrm{End}(B)$ is *injective*. This yields several consequences for non-injective systems, among which are a gauge-invariant uniqueness theorem [**16**, Theorem 4.3.17] and a characterization of the C*-envelope [**16**, Corollary 4.3.18]. In an upcoming paper of the first author with Kakariadis, we recover [**16**, Theorem 4.3.17] in an alternative way, and the results in this current paper can then be used to obtain [**16**, Corollary 4.3.18], for non-degenerate systems, as an immediate consequence. This makes for a substantial simplification of the characterization of the C*-envelope, as it no longer requires the use of a multivariable tail-adding technique.

6. An application to C*-algebra theory: the Hao-Ng isomorphism for product systems

In this section we examine the Hao-Ng isomorphism problem in the context of product systems. In what follows, we assume some familiarity with the theory of reduced crossed product of (not-necessarily-self-adjoint) operator algebras, as it appears in [**34**].

Let (G,P) be a quasi-lattice ordered group and $X = (X_p)_{p\in P}$ product system over P with coefficients in $\mathcal{A}$. Consider an action $\alpha : \mathcal{G} \to \mathrm{Aut}\,\mathcal{NT}_X$ so that $\alpha_g(X_p) = X_p$, for all $g \in \mathcal{G}$ and $p \in P$. We call such an action α a *generalized gauge action* and we say that the group $\mathcal{G}$ *acts on* X. (Note that we do not insist that the automorphisms α_g fix X_p, $p \in P$, element-wise but instead that they fix them only as sets.) Clearly the action α restricts to a generalized gauge action $\alpha : \mathcal{G} \to \mathrm{Aut}\,\mathcal{NT}_X^+$. This generalized gauge action extends to a generalized gauge action on $\mathrm{C}_e^*(\mathcal{NT}_X^+)$ so that when G is abelian and X is $\tilde{\phi}$-injective, we have a generalized gauge action on $\mathrm{C}_e^*(\mathcal{NT}_X^+) \cong \mathcal{NO}_X$ by Theorem 3.2. The crossed product of $\mathcal{NO}_X$ by such actions play an important role in C*-algebra theory: in the case of a Cuntz or a Cuntz-Krieger C*-algebra, examples of such actions are the so-called *quasi-free* actions whose crossed products have been studied extensively [**22, 40, 41**].

For each $p \in P$ let $X_p \rtimes_\alpha^r \mathcal{G}$ be the closed subspace of $\mathcal{NT}_X \rtimes_\alpha^r \mathcal{G}$ generated by $C_c(\mathcal{G}, X_p) \subseteq \mathcal{NT}_X \rtimes_\alpha^r \mathcal{G}$, i.e., all finite sums of the form $\sum_s x_s U_s$, $x_s \in X_p$, $s \in \mathcal{G}$. Just as in [**34**, Lemma 7.11], one can verify that for every $p \in P$, $X_p \rtimes_\alpha^r \mathcal{G}$ is an $\mathcal{A} \rtimes_\alpha^r \mathcal{G}$-correspondence with inner product defined by $\langle x, y\rangle = x^*y$, $x, y \in X_p \rtimes_\alpha^r \mathcal{G} \subseteq \mathcal{NT}_X \rtimes_\alpha^r \mathcal{G}$. Furthermore, it is easily seen that $(X_p \rtimes_\alpha^r \mathcal{G})(X_q \rtimes_\alpha^r \mathcal{G}) \subseteq X_{pq} \rtimes_\alpha^r \mathcal{G}$ is dense for any $p, q \in P$. Therefore $(X_p \rtimes_\alpha^r \mathcal{G})_{p\in P}$ forms a product system that we denote as $X \rtimes_\alpha^r \mathcal{G}$.

The C*-algebra $\mathcal{NT}_X \rtimes_\alpha^r \mathcal{G}$ also contains a non-selfadjoint crossed product algebra which we denote as $\mathcal{NT}_X^+ \rtimes_\alpha^r \mathcal{G}$. This is the norm closed algebra generated by a completely isometric copy of $\mathcal{NT}_X^+ \subseteq \mathcal{NT}_X \rtimes_\alpha^r \mathcal{G}$ and $\mathrm{C}_r^*(\mathcal{G})$. In particular have the inclusions

$$X \rtimes_\alpha^r \mathcal{G} \subseteq \mathcal{NT}_X^+ \rtimes_\alpha^r \mathcal{G} \subseteq \mathcal{NT}_X \rtimes_\alpha^r \mathcal{G}.$$

Furthermore, $\mathcal{NT}_X^+ \rtimes_\alpha^r \mathcal{G}$ is generated as a closed algebra by $X \rtimes_\alpha^r \mathcal{G}$ and so $\mathcal{NT}_X \rtimes_\alpha^r \mathcal{G}$ is a C*-cover for $\mathcal{NT}_X^+ \rtimes_\alpha^r \mathcal{G}$.

Remark 6.1. (i) The reader familiar with the theory of non-selfadjoint crossed products (as developed in [**34**]) recognizes that $\mathcal{NT}_X^+ \rtimes_\alpha^r \mathcal{G}$ coincides with the reduced crossed product of the dynamical system $(\mathcal{NT}_X^+, \mathcal{G}, \alpha)$, as defined in [**34**, Definition 3.16]. Indeed this follows from an immediate application of [**34**, Corollary 3.15].

(ii) When G is abelian, and X is $\tilde{\phi}$-injective, we see that $\mathcal{NT}_X^+ \rtimes_\alpha^r \mathcal{G}$ is completely isometrically isomorphic to the natural subalgebra of $\mathcal{NO}_X \rtimes_\alpha^r \mathcal{G}$ generated by a copy of X and $C_r^*(G)$. This follows from Theorem 3.2 and [**34**, Corollary 3.16]. Hence, we also obtain an injective copy of $X \rtimes_\alpha^r \mathcal{G}$ sitting naturally inside $\mathcal{NO}_X \rtimes_\alpha^r \mathcal{G}$.

In order to apply the theory of Section 3 we need to verify that certain properties of a product system are being preserved when passing to crossed products by discrete groups. First we investigate compact alignment. The following definition was inspired by the theory of higher-rank graph C*-algebras.

Definition 6.2. Let (G, P) be a quasi-lattice ordered group and X a product system over P. We say that X is *finitely aligned* if there exists a family $\mathcal{S}_p \subseteq X_p$, $p \in P$, of total subsets so that whenever $p, q \in P$ with $p \vee q < \infty$, then $\theta_{x_1,x_2} \vee \theta_{y_1,y_2} \in \mathcal{L}(X_{p\vee q})$ is a generalized finite rank operator for any choice of $x_1, x_2 \in \mathcal{S}_p$ and $y_1, y_2 \in \mathcal{S}_q$.

If (Λ, d) is a finitely-aligned higher-rank graph, then the natural product system $X(\Lambda)$ associated with (Λ, d) is finitely aligned. Indeed in that case simply take

$$\mathcal{S}_p := \{\delta_\mu \mid \mu \in \Lambda^p\}, \quad p \in P$$

and then follow the proof of [**55**, Theorem 5.4]. Therefore the class of finitely aligned product systems includes a very natural and broad class of examples. But there are many more

Example 6.3. (i) If (G, P) is totally ordered and X is a product system over P, then X is finitely aligned.

Indeed in this case we simply take $\mathcal{S}_p = X_p$ and we note that if $p, q \in P$, then $p \vee q$ is equal to either p or q, say p. But then $\iota_p^{p\vee q}(\theta_{x_1,x_2}) = \theta_{x_1,x_2}$ and so

$$\theta_{x_1,x_2} \vee \theta_{y_1,y_2} = \iota_p^{p\vee q}(\theta_{x_1,x_2})\iota_q^{p\vee q}(\theta_{y_1,y_2}) = \theta_{x_1,x_2}\iota_q^{p\vee q}(\theta_{y_1,y_2})$$

is a generalized rank one operator.

(ii) Let (G, P) be a quasi-lattice ordered group and X is a product system over P with coefficients in $\mathcal{A}$. If $\mathcal{A}$ acts on each fiber X_p by finite rank operators, then X finitely aligned.

Indeed it is enough to show that for any $p, q \in P$ and $x, y \in X_p$, the operator $\theta_{x,y} \otimes I \in \mathcal{L}(X_{p\vee q})$ is of finite rank. To see that express $x = x'a$

for a suitable $a \in X_e$ and then note that

$$\theta_{x,y} \otimes I = \theta_{x'} \phi(a) \theta_y^*,$$

where

$$\theta_{x'} \colon X_{p^{-1}(p\vee q)} \longrightarrow X_{p\vee q};\, X_{p^{-1}(p\vee q)} \ni \xi \longmapsto x'\xi$$

and similarly for θ_y. (See the proof of [**46**, Proposition 4.7]).

We would have liked to show that the property of compact alignment is preserved when taking crossed products. Even though we cannot do that, we obtain a result which is sufficiently close for our purposes.

Proposition 6.4. *Let (G, P) be a quasi-lattice ordered group and X a product system over P. Let $\alpha : \mathcal{G} \to \operatorname{Aut} \mathcal{NT}_X$ be a generalized gauge action by a discrete group $\mathcal{G}$. If X is finitely aligned, then $X \rtimes_\alpha^r \mathcal{G}$ is compactly aligned.*

Proof. Let $\mathcal{S}_p$, $p \in P$, be as in Definition 6.2 and let $p, q \in P$ with $p\vee q < \infty$. To show that $X \rtimes_\alpha^r \mathcal{G}$ is compactly aligned, it suffices to verify that $S \vee T$ is always a generalized finite rank operator, whenever S and T range over suitable total subsets of $\mathcal{K}(X_p \rtimes_\alpha^r \mathcal{G})$ and $\mathcal{K}(X_q \rtimes_\alpha^r \mathcal{G})$ respectively.

Let

$$S = \theta_{x_1 U_{s_1}, x_2 U_{s_2}} \quad \text{and} \quad T = \theta_{y_1 U_{t_1}, y_2 U_{t_2}}, \tag{6.1}$$

where $s_1, s_2, t_1, t_2 \in \mathcal{G}$ and

$$x_1 \in \alpha_s(\mathcal{S}_p), \quad x_2 \in \alpha_{t_1 t_2^{-1}}(\mathcal{S}_p), \quad y_1 \in \alpha_{t_1 t_2^{-1}}(\mathcal{S}_q), \quad y_2 \in \mathcal{S}_q,$$

with $s := s_1 s_2^{-1} t_1 t_2^{-1}$. Then for any $z \in X_{p\vee q}$ and $r \in \mathcal{G}$, we have

$$\begin{aligned} (S \vee T) z U_r &= (x_1 U_{s_1} U_{s_2}^* x_2^*)(y_1 U_{t_1} U_{t_2}^* y_2^*) z U_r \\ &= U_s x_3 x_4^* y_3 y_2^* z U_r \\ &= U_s \big((\theta_{x_3,x_4} \vee \theta_{y_3,y_2}) z \big) U_r, \end{aligned} \tag{6.2}$$

where

$$x_3 = \alpha_{s^{-1}}(x_1), \quad x_4 = \alpha_{t_2 t_1^{-1}}(x_2) \quad \text{and} \quad y_3 = \alpha_{t_2 t_1^{-1}}(y_1).$$

Note that $x_3, x_4 \in \mathcal{S}_p$ and $y_3, y_2 \in \mathcal{S}_q$ and so the hypothesis of X being finitely aligned implies the existence of $z_i, z_i' \in X_{p\vee q}$, $i = 1, 2, \dots n$, so that

$$\theta_{x_3,x_4} \vee \theta_{y_3,y_2} = \sum_{i=1}^n \theta_{z_i, z_i'}.$$

Hence (6.2) implies that $S \vee T = \sum_{i=1}^n \theta_{\alpha_s(z_i) U_s, z_i'}$ is a generalized finite rank operator. Since $\alpha_s(\mathcal{S}_p), \alpha_{t_1 t_2^{-1}}(\mathcal{S}_p)$ and $\alpha_{t_1 t_2^{-1}}(\mathcal{S}_q), \mathcal{S}_q$ are total subsets of X_p and X_q respectively, the conclusion follows. ∎

Now we need a good supply of Nica-covariant representations for the crossed product system $X \rtimes_\alpha^r \mathcal{G}$.

Lemma 6.5. *Let (G, P) be a quasi-lattice ordered group and X a product system over P. Let $\alpha : \mathcal{G} \to \operatorname{Aut}\mathcal{NT}_X$ be a generalized gauge action by a discrete group $\mathcal{G}$. If $\pi \colon \mathcal{NT}_X \to B(\mathcal{H})$ is a $*$-representation, then the restriction of the regular representation*

$$\operatorname{Ind}_\pi \colon \mathcal{NT}_X \rtimes_\alpha^r \mathcal{G} \to B(\mathcal{H} \otimes \ell^2(\mathcal{G}))$$

on the product system $X \rtimes_\alpha^r \mathcal{G} \subseteq \mathcal{NT}_X \rtimes_\alpha^r \mathcal{G}$ forms a Nica-covariant representation.

Proof. Let $p \in P$. Since the correspondence $X \rtimes_\alpha^r \mathcal{G}$ is non-degenerate (essential), we have that $\operatorname{Ind}_\pi^{(p)}(I)$ projects on the subspace

$$\begin{aligned} \operatorname{Ind}_\pi\big((X \rtimes_\alpha^r \mathcal{G})_p\big)(\mathcal{H} \otimes \ell^2(\mathcal{G})) &= \operatorname{Ind}_\pi(X_p \rtimes_\alpha^r \mathcal{G})(\mathcal{H} \otimes \ell^2(\mathcal{G})) \\ &= \pi(X_p)(\mathcal{H}) \otimes \ell^2(\mathcal{G}) \end{aligned}$$

and so $\operatorname{Ind}_\pi^{(p)}(I) = \pi^{(p)}(I) \otimes I$. This suffices to prove the result. ∎

Finally we wish to assure that $\tilde{\phi}$-injectivity is preserved under discrete crossed products, so as to get the relations of Sims and Yeend. We use the following result from [**58**].

Lemma 6.6. ([**58**, Lemma 3.2]) *Let (G, P) be a quasi-lattice ordered group and X a product system over P with coefficients in $\mathcal{A}$. Let $p, q \in P$ with $p \leq q$ and let $x \in X_p$. Then $x \in X_p I_{p^{-1}q}(X)$ if and only if $xy = 0$ for all $y \in X_r$ with $e < r \leq p^{-1}q$.*

Corollary 6.7. *Let (G, P) be a quasi-lattice ordered group and X a product system over P. Let $\alpha : \mathcal{G} \to \operatorname{Aut}\mathcal{NT}_X$ be a generalized gauge action by a discrete group $\mathcal{G}$. Fix $p, q \in P$ with $p \leq q$.*

(i) *If $x \in X_p I_{p^{-1}q}(X)$, then $\alpha_s(x) \in X_p I_{p^{-1}q}(X)$, for any $s \in \mathcal{G}$.*
(ii) *If $x \in X_p I_{p^{-1}q}(X)$, then $x = xU_e \in (X \rtimes_\alpha^r \mathcal{G})_p I_{p^{-1}q}(X \rtimes_\alpha^r \mathcal{G})$.*

Proof. To prove (i) note that if $y \in X_r$ with $e < r \leq p^{-1}q$, then $\alpha_{s^{-1}}(y) \in X_r$ and so

$$\alpha_{s^{-1}}\big(\alpha_s(x)y\big) = x\alpha_{s^{-1}}(y) = 0.$$

Hence $\alpha_s(x)y = 0$. Since this is true for all $y \in X_r$ with $e < r \leq p^{-1}q$, Lemma 6.6 implies that $\alpha_s(x) \in X_p I_{p^{-1}q}(X)$.

The proof of (ii) is an immediate application of Lemma 6.6. ∎

In order to establish the permanence of $\tilde{\phi}$-injectivity under reduced crossed products, we also need a basic fact regarding crossed products of C*-algebras. Let $\mathcal{A}$ be a C*-algebra and let $\alpha : \mathcal{G} \to \operatorname{Aut}\mathcal{A}$. It is well-known that the map

$$\mathbb{E} \colon C_c(\mathcal{G}, \mathcal{A}) \longrightarrow \mathcal{A};\, C_c(\mathcal{G}, \mathcal{A}) \ni \sum a_g U_g \longmapsto a_e$$

extends to a faithful expectation of $\mathcal{A} \rtimes_\alpha^r \mathcal{G}$. Similarly, we have maps $\mathbb{E}_s \colon C_c(\mathcal{G}, \mathcal{A}) \to \mathcal{A}$ given by $\mathbb{E}_s(A) = \mathbb{E}(AU_{s^{-1}})$, $A \in \mathcal{A} \rtimes_\alpha^r \mathcal{G}$. The maps $\mathbb{E}_s$ allow for a useful "Fourier series" viewpoint for elements of $\mathcal{A} \rtimes_\alpha^r \mathcal{G}$. We write $A \sim \sum_{s \in \mathcal{G}} \mathbb{E}_s(A) U_s$, with the understanding that if $\mathbb{E}_s(A) = 0$ for every $s \in \mathcal{G}$, then $A = 0$. See [**54**, Proposition 6.1] for a proof of this fact.

Proposition 6.8. *Let (G, P) be a quasi-lattice ordered group and X a product system over P with coefficients in $\mathcal{A}$. Let $\alpha : \mathcal{G} \to \operatorname{Aut} \mathcal{NT}_X$ be a generalized gauge action by a discrete group $\mathcal{G}$. If X is $\tilde{\phi}$-injective, then $X \rtimes_\alpha^r \mathcal{G}$ is also $\tilde{\phi}$-injective.*

Proof. We need to verify that for all $q \in P$ if the action of $A \in \mathcal{A} \rtimes_\alpha^r \mathcal{G}$ by left multiplication on $(X \rtimes_\alpha^r \mathcal{G})_q$ is null then $A = 0$.

Let $0 \neq A \in \mathcal{A} \rtimes_\alpha^r \mathcal{G}$ and $q \in P$. If $A \sim \sum_{s \in \mathcal{G}} a_s U_s$ is the Fourier expansion of $A \in \mathcal{A} \rtimes_\alpha^r \mathcal{G}$, then by the discussion above there exists $t \in \mathcal{G}$ so that $a_t \neq 0$. Since X is $\tilde{\phi}$-injective, there exists $p \leq q$ and $x \in X_p I_{p^{-1}q}(X)$ so that $a_t x \neq 0$. Corollary 6.7 implies now that $\alpha_t(x) U_e \in (X \rtimes_\alpha^r \mathcal{G})_p I_{p^{-1}p}(X \rtimes_\alpha^r \mathcal{G})$ for all $t \in \mathcal{G}$. Furthermore

$$A \alpha_{t^{-1}}(x) U_e \sim \sum_{s \in \mathcal{G}} a_s \alpha_{s^{-1}t}(x) U_s$$

Since the t-term of the series above is $a_t \alpha_{t^{-1}t}(x) U_t = a_t x U_t \neq 0$, we have that $A \alpha_{t^{-1}}(x) U_e \neq 0$. Hence A acts non trivially on $(X \rtimes_\alpha^r \mathcal{G})_q$ and the conclusion follows. ∎

Theorem 6.9. *Let (G, P) be an abelian, lattice ordered group and X a finitely aligned product system over P with coefficients in $\mathcal{A}$. Let $\alpha : \mathcal{G} \to \operatorname{Aut} \mathcal{NT}_X$ be a generalized gauge action by a discrete group $\mathcal{G}$. Then*

$$\mathcal{NT}_X^+ \rtimes_\alpha^r \mathcal{G} \simeq \mathcal{NT}^+_{X \rtimes_\alpha^r \mathcal{G}}.$$

Therefore, if X is $\tilde{\phi}$-injective, then we have

$$\mathrm{C}^*_e\big(\mathcal{NT}_X^+ \rtimes_\alpha^r \mathcal{G}\big) \simeq \mathcal{NO}_{X \rtimes_\alpha^r \mathcal{G}}.$$

Proof. Let $i = i_X \colon X \to B(\mathcal{H})$ be the universal Nica-covariant representation of X and let

$$V \colon P \longrightarrow B(\ell^2(P)); p \longmapsto V_p$$

be the left regular representation of P on $\ell^2(P)$. Let $\pi \equiv i \otimes V$ be defined as in (3.1). By Lemma 3.1, π is a Nica-covariant representation of X. It is easily seen to satisfy the requirements of Fowler's Theorem (Theorem 2.2) and therefore it induces a faithful representation $\pi_* : \mathcal{NT}_X \to B(\mathcal{H} \otimes \ell^2(P))$.

Consider the regular representation

$$\operatorname{Ind}_{\pi_*} \colon \mathcal{NT}_X \rtimes_\alpha^r \mathcal{G} \longrightarrow B\big(\mathcal{H} \otimes \ell^2(P) \otimes \ell^2(\mathcal{G})\big)$$

and let

$$\psi \colon X \rtimes_\alpha^r \mathcal{G} \longrightarrow B\big(\mathcal{H} \otimes \ell^2(P) \otimes \ell^2(\mathcal{G})\big)$$

be the restriction of $\operatorname{Ind}_{\pi_*}$ on $X \rtimes_\alpha^r \mathcal{G} \subseteq \mathcal{NT} \rtimes_\alpha^r \mathcal{G}$. By Lemma 6.5, ψ is Nica-covariant.

We claim that ψ satisfies the requirements of Fowler's Theorem. Indeed let $Q_p \in \ell^2(P)$ be the projection on the one dimensional subspace corresponding to the characteristic function of $p \in P$ and let

$$\hat{Q}_p \equiv (I \otimes Q_p) \otimes I \in B\big(\mathcal{H} \otimes \ell^2(P) \otimes \ell^2(\mathcal{G})\big).$$

If $f \in C_c(\mathcal{G}, X_p)$, $p \in P$, is a finite sum in $X_p \rtimes_\alpha^r \mathcal{G}$ and $h \in \mathcal{H} \otimes \ell^2(P) \otimes \ell^2(\mathcal{G})$, then

$$\begin{aligned}\big(\hat{Q}_e \psi_p(f) h\big)(s) &= \hat{Q}_e(\mathrm{Ind}_{\pi_*} f) h(s)\\ &= \sum_{s\in\mathcal{G}} (I \otimes Q_e)\pi_*\big(\alpha_t^{-1}(f(s))\big) h(s^{-1}t)\\ &= \sum_{s\in\mathcal{G}} (I \otimes Q_e)\big(i_p(\alpha_t^{-1}(f(s))) \otimes V_p\big) h(s^{-1}t)\\ &= \sum_{s\in\mathcal{G}} \big(i_p(\alpha_t^{-1}(f(s))) \otimes Q_e V_p\big) h(s^{-1}t) = 0\end{aligned}$$

Hence $\hat{Q}_e \psi^{(p)}(I) = 0$ and so each projection $I - \psi^{(p)}(I)$, $p \in P$, dominates $\hat{Q}_e$. Therefore if $F \subseteq P\backslash\{e\}$ is any finite set, then the product $\prod_{p\in F}\big(I - \psi^{(p)}(I)\big)$ also dominates $\hat{Q}_e$. Hence

$$\Big\|\psi_e(f) \prod_{p\in F}\big(I - \psi^{(p)}(I)\big)\Big\| \geq \big\|\psi_e(f)\hat{Q}_e\big\| = \big\|\mathrm{Ind}_\pi(f)\hat{Q}_e\big\|, \tag{6.3}$$

for any $f \in \mathcal{C}_c(\mathcal{G}, \mathcal{A})$. On the other hand

$$\pi_* \mid_{\mathcal{A}} \simeq \big(\oplus_{p\in P}(I \otimes Q_p)\pi_*\big)\mid_{\mathcal{A}} \simeq \big(\oplus(I\otimes Q_e)\pi_*\big)\mid_{\mathcal{A}},$$

i.e., the restriction of π_* on $\mathcal{A}$ is unitarily equivalent to a direct sum indexed by P of copies of $(I \otimes Q_e)\pi_*$ restricted to $\mathcal{A}$. From this we obtain,

$$\psi_e = \mathrm{Ind}_{\pi_*}\mid_{\mathcal{A}\rtimes_\alpha^r\mathcal{G}} \simeq \oplus \mathrm{Ind}_{Q_e\pi_*}\mid_{\mathcal{A}\rtimes_\alpha^r\mathcal{G}} \simeq \oplus \hat{Q}_e \mathrm{Ind}_{\pi_*}\mid_{\mathcal{A}\rtimes_\alpha^r\mathcal{G}}.$$

Combining the above with (6.3) we now obtain

$$\Big\|\psi_e(f)\prod_{p\in F}\big(I - \psi^{(p)}(I)\big)\Big\| \geq \big\|\mathrm{Ind}_{\pi_*}(f)\hat{Q}_e\big\| = \|\psi_e(f)\|$$

for any $f \in \mathcal{C}_c(\mathcal{G}, \mathcal{A})$, which establishes the claim.

From the claim we obtain that the induced representation ψ_* is a faithful $*$-representation of $\mathcal{NT}_{X\rtimes_\alpha^r\mathcal{G}}$. In addition, note that $\psi_*(\mathcal{NT}^+_{X\rtimes_\alpha^r\mathcal{G}}) \simeq \mathcal{NT}^+_{X\rtimes_\alpha^r\mathcal{G}}$ is equal to the closed linear span of

$$\psi_*(X \rtimes_\alpha^r \mathcal{G}) = \bigcup_{p\in P} \psi_p(X_p \rtimes_\alpha^r \mathcal{G}) = \bigcup_{p\in P} \overline{\mathrm{Ind}_{\pi_*} C_c(\mathcal{G}, X_p)}.$$

However, $\mathcal{NT}_X^+ \rtimes_\alpha^r \mathcal{G}$ is also completely isometrically isomorphic to the closed linear span of

$$\bigcup_{p\in P} \overline{\mathrm{Ind}_{\pi_*} C_c(\mathcal{G}, X_p)}.$$

Hence, $\mathcal{NT}_X^+ \rtimes_\alpha^r \mathcal{G} \simeq \mathcal{NT}^+_{X \rtimes_\alpha^r \mathcal{G}}$.

Finally if (G, P) is directed and X is $\tilde{\phi}$-injective, then Proposition 6.8 implies that $X \rtimes_\alpha^r \mathcal{G}$ is $\tilde{\phi}$-injective and so

$$\mathrm{C}^*_e\big(\mathcal{NT}_X^+ \rtimes_\alpha^r \mathcal{G}\big) \simeq \mathrm{C}^*_e(\mathcal{NT}^+_{X \rtimes_\alpha^r \mathcal{G}}) \simeq \mathcal{NO}_{X \rtimes_\alpha^r \mathcal{G}},$$

with the last identification following from Theorem 3.2. ∎

We now use the above to obtain our extension of the Hao-Ng isomorphism.

Theorem 6.10. *Let (G, P) be an abelian, lattice ordered group and let X be a finitely aligned and $\tilde{\phi}$-injective product system over P. Let $\alpha : \mathcal{G} \to \operatorname{Aut} \mathcal{NT}_X$ be a generalized gauge action by a discrete group $\mathcal{G}$. Then*

$$\mathcal{NO}_X \rtimes_\alpha^r \mathcal{G} \simeq \mathcal{NO}_{X \rtimes_\alpha^r \mathcal{G}}.$$

Proof. By Theorem 6.9 we have

$$\mathrm{C}^*_e\big(\mathcal{NT}_X^+ \rtimes_\alpha^r \mathcal{G}\big) \simeq \mathcal{NO}_{X \rtimes_\alpha^r \mathcal{G}}.$$

On the other hand [**31**, Theorem 2.5] implies that

$$\mathrm{C}^*_e\big(\mathcal{NT}_X^+ \rtimes_\alpha^r \mathcal{G}\big) \simeq \mathrm{C}^*_e(\mathcal{NT}_X^+) \rtimes_\alpha^r \mathcal{G}.$$

Hence $\mathrm{C}^*_e(\mathcal{NT}_X^+) \rtimes_\alpha^r \mathcal{G} \simeq \mathcal{NO}_{X \rtimes_\alpha^r \mathcal{G}}$. Now an application of Theorem 3.2 shows that $\mathrm{C}^*_e(\mathcal{NT}_X^+) \simeq \mathcal{NO}_X$ via a $\mathcal{G}$-equivariant map that intertwines the corresponding generalized gauge actions and the conclusion follows. ∎

Let us indicate the utility of Theorem 6.10 with a quick application. Let (Λ, d) be a finitely aligned k-graph and let $\mathcal{G}$ be a discrete group acting on $X(\Lambda)$ via $\alpha : \mathcal{G} \to \operatorname{Aut} \mathrm{C}^*(\Lambda)$, e.g., let $\mathcal{G} = \mathbb{F}^k$ be the free group on k-generators acting on $\mathrm{C}^*(\Lambda)$ by twisting the generators with unimodular scalars. As we discussed in Remark 6.1(ii), $X(\Lambda) \rtimes_\alpha^r \mathcal{G}$ is just the closed subspace of $\mathrm{C}^*(\Lambda) \rtimes_\alpha^r \mathcal{G}$ generated by all monomials of the form xU_s, with $x \in X(\Lambda)$ and $s \in \mathcal{G}$. Theorem 6.10 implies that the Cuntz-Nica-Pimsner C*-algebra of $X(\Lambda) \rtimes_\alpha^r \mathcal{G}$ is isomorphic to $\mathrm{C}^*(\Lambda) \rtimes_\alpha^r \mathcal{G}$. Even in very special cases, it is quite intricate to verify this result directly from the definition of the Cuntz-Nica-Pimsner C*-algebra.

7. Beyond $\tilde{\phi}$-injectivity

In this section we exhibit the existence of a co-universal C*-algebra with respect to gauge compatible isometric Nica-covariant representations of product systems over abelian, lattice ordered groups. Our approach goes through the C*-envelope and does not assume $\tilde{\phi}$-injectivity as in [**58**]. In fact, this is what allows for more complicated abelian semigroup pairs such as $(\mathbb{Q}^d, \mathbb{Q}^d_+)$ for which product systems are not easily shown to be $\tilde{\phi}$-injective, and those appearing in [**13**, Example 3.16] for which there are product systems that fail to be $\tilde{\phi}$-injective.

Lemma 7.1. *Let (G, P) be an abelian, lattice ordered group. Let X be a compactly aligned product system over P. Suppose $\psi : X \to B(\mathcal{H})$ is an isometric Nica-covariant representation of X such that*

(i) *ψ_e is faithful, and*
(ii) *ψ is gauge compatible.*

Then ψ_* is completely isometric on $\mathcal{NT}_X^+$.

Proof. Suppose ψ is a representation as above. Since ψ is gauge compatible, there is a $*$-homomorphism

$$\beta : \mathrm{C}^*(\{\psi(X_p)\}_{p \in P}) \to \mathrm{C}^*(\{\psi(X_p)\}_{p \in P}) \otimes \mathrm{C}^*(G)$$

such that $\beta(\psi_p(x)) = \psi_p(x) \otimes U_p$, $p \in P$, where $g \mapsto U_g$ is the left regular representation of G. Hence, we have the commutative diagram

$$\begin{tikzcd} \mathcal{NT}_X \arrow[r, "\delta_X"] \arrow[d, "\psi_*"] & \mathcal{NT}_X \otimes \mathrm{C}^*(G) \arrow[d, "\psi_* \otimes \mathrm{id}"] \\ \mathrm{C}^*(\{\psi(X_p)\}_{p \in P}) \arrow[r, "\beta"'] & \mathrm{C}^*(\{\psi(X_p)\}_{p \in P}) \otimes \mathrm{C}^*(G)) \end{tikzcd}$$

In order to prove that ψ_* is completely isometric on $\mathcal{NT}_X^+$, it is therefore sufficient to show that $(\psi_* \otimes \mathrm{id}) \circ \delta$ is completely isometric on $\mathcal{NT}_X^+$.

Consider the isometric representation

$$\psi \otimes V \colon X \longrightarrow B\big(\mathcal{H} \otimes \ell^2(P)\big); X_p \ni x \longmapsto \psi_p(x) \otimes V_p, \quad p \in P,$$

where $p \mapsto V_p$ is the left regular representation of P. We claim that φ satisfies the conditions of Theorem 2.2 (Fowler's Theorem). Indeed, let P_e be the projection onto the characteristic function of $e \in P$ in $\ell^2(P)$. Then for any $p \in P \backslash \{e\}$ we have

$$(\psi \otimes V)^{(p)}(I)(I \otimes P_e) = (I \otimes P_e)(\psi \otimes V)^{(p)}(I) = 0$$

and for all $p \in P \setminus \{e\}$ we have $\big(I - (\psi \otimes V)^{(p)}(I)\big)(I \otimes P_e) = I \otimes P_e$. Hence, for each finite set $F \subseteq P \setminus \{e\}$, the map

$$X_e \longrightarrow B(\mathcal{H} \otimes \ell^2(P));\ a \longmapsto (\psi \otimes V)_e(a) \prod_{p \in F} \big(I - (\psi \otimes V)^{(p)}(I)\big)$$

is injective when restricted to the reducing subspace $\mathcal{H}_\xi$. Hence, Fowler's Theorem implies that the induced $*$-homomorphism

$$(\psi \otimes V)_* : \mathcal{NT}_X \longrightarrow B(\mathcal{H} \otimes \ell^2(P))$$

is injective. In particular, it is completely isometric on $\mathcal{NT}_X^+$.

Let Q be the projection from $\ell^2(G)$ to $\ell^2(P)$. Then

$$(I \otimes Q)(\psi \otimes U)(x_p)(I \otimes Q) = (\psi \otimes l)(x_p),$$

for all $x_p \in X_p$, $p \in Q$. Since $I \otimes Q$ is semi-invariant for $(\psi \otimes U)_*(X)$, the above equation promotes to

$$(I \otimes Q)(\psi \otimes U)_*(x)(I \otimes Q) = (\psi \otimes V)_*(x), \tag{7.1}$$

for all $x \in \mathcal{NT}_X^+$. As we saw earlier, $(\psi \otimes V)_*$ is completely isometric on $\mathcal{NT}_X^+$. Therefore (7.1) implies the same for $(\psi \otimes U)_*$. However

$$(\psi \otimes U)_* = (\psi_* \otimes \mathrm{id})\delta$$

and so $(\psi_* \otimes \mathrm{id}) \circ \delta$ is isometric on $\mathcal{NT}_X^+$, as desired. ∎

The main result of this section now follows.

Theorem 7.2 (Co-universality)**.** *Let (G, P) be an abelian, lattice ordered group and X a compactly aligned product system over P. Then $\mathrm{C}^*_e(\mathcal{NT}_X^+)$ is a co-universal C^*-algebra for isometric, Nica-covariant, gauge compatible representations of X in the following sense: the representation $j : X \to \mathrm{C}^*_e(\mathcal{NT}_X^+)$ that induces the canonical quotient $j_* : \mathcal{NT}_X \to \mathrm{C}^*_e(\mathcal{NT}_X^+)$ satisfies*

(i) *j_e is faithful,*
(ii) *j_* is gauge compatible, and*
(iii) *for any gauge-compatible Nica-covariant isometric representation $\psi : X \to B(\mathcal{H})$ for which ψ_e is faithful, there is a surjective $*$-homomorphism $q : \mathrm{C}^*(\{\psi_p(X_p)\}_{p\in P}) \to \mathrm{C}^*_e(\mathcal{NT}_X^+)$ such that*

$$q \circ \psi(x) = j(x), \quad \text{for all } \xi \in X.$$

Proof. The validity of (ii) follows as in the proof of Theorem 3.2. In order to establish (iii) notice that Lemma 7.1 implies that if such a ψ exists then ψ_* is faithful on $\mathcal{NT}_X^+$. The existence of the map $q : \mathrm{C}^*(\{\psi_p(X_p)\}_{p\in P}) \to \mathrm{C}^*_e(\mathcal{NT}_X^+)$ now follows from the defining properties of the C^*-envelope. ∎

The universality conditions of Theorem 7.2 are exactly the defining properties of the C^*-algebra $\mathcal{NO}_X^r$ of Carlsen, Larsen, Sims and Vittadello. Since this algebra is canonically isomorphic to any other C^*-algebra satisfying the same universality conditions (see Theorem 4.1 in **[13]**), we obtain that $\mathcal{NO}_X^r$ exists for any compactly aligned product system over an abelian, lattice ordered group. This improves [**13**, Theorem 4.1] for this case, since we do not require that the system be $\tilde{\phi}$-injective.

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