C*-ALGEBRAS AND EQUIVALENCES FOR C*-CORRESPONDENCES

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Dedicated to our mentor and friend Aristides Katavolos

Abstract. We study several notions of shift equivalence for C*- correspondences and the effect that these equivalences have on the corresponding Pimsner dilations. Among others, we prove that non- degenerate, regular, full C*-correspondences which are shift equivalent have strong Morita equivalent Pimsner dilations. We also establish that the converse may not be true. These results settle open problems in the literature.

In the context of C*-algebras, we prove that if two non-degenerate, regular, full C*- correspondences are shift equivalent, then their corresponding Cuntz-Pimsner algebras are strong Morita equivalent. This generalizes results of Cuntz and Krieger and Muhly, Tomforde and Pask. As a consequence, if two subshifts of finite type are eventually conjugate, then their Cuntz-Krieger algebras are strong Morita equivalent.

Our results suggest a natural analogue of the Shift Equivalence Problem in the context of C*-correspondences. Even though we do not resolve the general Shift Equivalence Problem, we obtain a positive answer for the class of imprimitivity bimodules.

1. Introduction

In **[19]** Williams introduced three relations for the class of matrices with non-negative integer entries, which are successively weaker. Two matrices E and F with non-negative integer entries are said to be *elementary strong shift equivalent* (symb. $E \overset{\text{s}}{\sim} F$) if there exist matrices R and S with non-negative integer entries such that $E = RS$ and $F = SR$. The transitive closure of the elementary strong equivalence is called *strong shift equivalence* (symb. $\overset{\text{SSE}}{\sim}$) and requires that $E \overset{\text{SSE}}{\sim} F$ if there is a finite sequence of matrices T_i, $i = 0, \dots, n$, such that $E = T_0$, $F = T_n$ and $T_i \overset{\text{SSE}}{\sim} T_{i+1}$. Williams also defined a third more manageable relation: E is *shift equivalent* to F (symb. $E \overset{\text{SE}}{\sim} F$) if there exist matrices R, S with non-negative integer entries such that $E^n = RS$, $F^n = SR$ and $ER = SF$, $FR = SE$ for some $n \in \mathbb{N}$.

The main goal of Williams was to characterize the topological conjugacy of subsifts of finite type using algebraic criteria. In **[19]**, he proved that two

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subsifts σ_E and σ_F of finite type are topologically conjugate if and only if their associated matrices are strong shift equivalent, i.e., $E \overset{\text{SSE}}{\sim} F$. Williams also claimed that the relations $\overset{\text{SSE}}{\sim}$ and $\overset{\text{SE}}{\sim}$ are equivalent, thus providing a more manageable criterion for the conjugacy of shifts. Unfortunately, an error in [**19**] made invalid the proof of that last assertion, and the equivalence of $\overset{\text{SSE}}{\sim}$ and $\overset{\text{SE}}{\sim}$ remained an open problem for over than 20 years, known as *Williams' Conjecture*. The breakthrough came with the work of Kim and Roush [**8**] who proved that the Williams Conjecture for the entire class of non-negative integral matrices is false. Their work reshaped the Williams' Conjecture into what is known today as the *Shift Equivalence Problem*, which for a particular class $\mathcal{S}$ of matrices with entries in a certain ring R asks whether $\overset{\text{SSE}}{\sim}$ is equivalent to $\overset{\text{SE}}{\sim}$ *within* $\mathcal{S}$.

Williams' notions of shift equivalence carry over to the class of C*- correspondences if one replaces in the above definitions the matrices E and F with C*-correspondences and the multiplication of matrices with the internal tensor product. (See Section 4 for the precise definitions). This introduces three notions of relation between C*-correspondences, which will be denoted again as $\overset{\text{s}}{\sim}$, $\overset{\text{SSE}}{\sim}$ and $\overset{\text{SE}}{\sim}$. There exists also a fourth equivalence relation, named *strong Morita equivalence* (symb. $\overset{\text{SME}}{\sim}$), which generalizes the concept of unitary conjugacy for matrices to the realm of C*-correspondences.

The concept of strong Morita equivalence for C*-correspondences was first developed and studied by Abadie, Eilers and Exel [**1**] and Muhly and Solel [**15**]. Among others these authors show that if two C*-correspondences are strong Morita equivalent then the associated Cuntz-Pimsner algebras $\mathcal{O}_E$ and $\mathcal{O}_F$ are (strong) Morita equivalent as well. The notion of elementary and strong shift equivalence for C*-correspondences was first studied by Muhly, Tomforde and Pask [**13**]. These authors also prove that strong shift equivalence of C*-correspondences implies the Morita equivalence of the associated Cuntz-Pimsner algebras, thus extending classical results of Cuntz and Krieger [**3**], Bates [**2**] and Drinen and Sieben [**5**] for graph C*-algebras. In their study of strong shift equivalence [**13**], Muhly, Tomforde and Pask raise two conjectures, which turn out to be important for the further development of the theory [**13**, Remark 5.5].

> **Conjecture 1**. Let E and F be two non-degenerate, regular C*-correspondences and let E_∞ and F_∞ be their associated Pimsner dilations. If $E \overset{\text{SSE}}{\sim} F$, then $E_\infty \overset{\text{SME}}{\sim} F_\infty$.
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> **Conjecture 2**. Let E and F be two non-degenerate, regular C*-correspondences and let E_∞ and F_∞ be their associated Pimsner dilations. If $E_\infty \overset{\text{SME}}{\sim} F_\infty$, then $E \overset{\text{SSE}}{\sim} F$.

The concept of shift equivalence has been studied extensively from both the dynamical and the ring theoretic viewpoint. (See [**18**] for a comprehensive exposition.) In general, shift equivalence has been recognized to be a more manageable invariant than strong shift equivalence, as it is decidable over certain rings [**9**]. Unlike strong shift equivalence, the study of shift equivalence, from the viewpoint of C*-correspondences, has been met with limited success [**12**]. (Other operator theoretic viewpoints however have been quite successful [**11**].)

There are three major objectives that are being met in this work. First we complete the study of strong shift equivalence of Muhly, Tomforde and Pask [**13**] by settling both of their conjectures: with the extra requirement of fullness, Conjecture 1 is settled in the affirmative (Theorem 5.3 and the remarks preceding it), while Conjecture 2 has a negative answer (Theorem 5.13).

A second objective is the detailed study of the shift equivalence for C*-correspondences. First, we raise the analogues of Conjectures 1 and 2 for shift equivalence (instead of strong shift equivalence) and we discover that the answers are the same as in the case of strong shift equivalence. Using that information we prove Theorem 5.10, which states that if two non-degenerate, regular, full C*-correspondences E and F are shift equivalent, then their corresponding Cuntz-Pimsner algebras $\mathcal{O}_E$ and $\mathcal{O}_F$ are (strong) Morita equivalent. This generalizes results of Cuntz and Krieger [**3**], and Muhly, Pask and Tomforde [**13**] and appears to be new even for Cuntz-Krieger algebras, where the two notions of shift and strong shift equivalence are known to be different. Combined with the work of Williams our result says that if two subshifts of finite type are eventually conjugate [**18**], then their Cuntz-Krieger algebras are strong Morita equivalent. In other words, strong Morita equivalence of Cuntz-Krieger C*-algebras is an invariant for shift equivalence.

Our final goal in this paper is the introduction of the Shift Equivalence Problem in the context of C*-correspondences. In light of our previous discussion, it seems natural to ask whether strong shift equivalence and shift equivalence are two different notions of equivalence for C*-correspondences. We coin this problem as the Shift Equivalence Problem for C*- correspondences. The work of Kim and Roush [**8**] shows that the Shift Equivalence Problem has a negative answer within the class of graph correspondences, but it leaves open the option for a positive answer within the whole class of C*-correspondences. In general, we do not know the answer even though the work of Kim and Roush hints that it should be negative. In spite of this, we show that the Shift Equivalence Problem has a positive answer for imprimitivity bimodules: all four notions of "equivalence" described in this paper coincide for imprimitivity bimodules, Theorem 6.1.

There are more things accomplished in this paper, and we describe each of them within the appropriate sections. Most notably, we settle a third conjecture of Muhly, Pask and Tomforde coming from [**13**] by showing that [**13**, Theorem 3.14] is valid without the assumption of non-degeneracy (Theorem 7.1).

The paper is organized as follows. In Section 2 we establish the notation and terminology to be used throughout this paper. In Section 3 we explore the concept of a dilation for a C*-correspondence. In Sections 4, 5 and 6 we present the main results of this paper.

2. Preliminaries

We use **[10]** as a general reference for Hilbert C*-modules and C*- correspondences. An *inner-product right A-module* over a C*-algebra A is a linear space X which is a right A-module together with an A-valued inner product. For $\xi \in X$ we define $\|\xi\|_X := \|\langle \xi, \xi \rangle_A\|_A^{1/2}$. The A-module X will be called a *right Hilbert A-module* if it is complete with respect to the norm $\|\cdot\|_X$. In this case X will be denoted by X_A. It is straightforward to prove that if (a_i) is an approximate unit in A or in the closed ideal $\langle X, X\rangle_X$, then (a_i) is also a right contractive approximate unit (c.a.i.) for X.

Dually we call X a *left Hilbert A-module* if it is complete with respect to the norm induced by a *left A-module inner-product* $[\cdot,\cdot]_X$. The term *Hilbert module* is reserved for the right Hilbert modules, whereas the left case will be clearly stated.

Given a Hilbert A-module X over A, let $X^* = \{\xi^* \in \mathcal{L}(X, A) \mid \xi^*(\zeta) = \langle \xi, \zeta\rangle_X\}$ be the *dual left Hilbert A-module*, with

$$a \cdot \xi^* = (\xi a^*)^* \text{ and } [\xi^*, \zeta^*]_{X^*} = \langle \xi, \zeta \rangle_X \,,$$

for all $\xi, \zeta \in X$ and $a \in A$.

For X, Y Hilbert A-modules let $\mathcal{L}(X, Y)$ be the (closed) linear space of the adjointable maps. For $\xi \in X$ and $y \in Y$, let $\Theta_{y,\xi} \in \mathcal{L}(X, Y)$ such that $\Theta_{y,\xi}(\xi') = y \langle \xi, \xi'\rangle_X$, for all $\xi' \in X$. We denote by $\mathcal{K}(X, Y)$ the closed linear subspace of $\mathcal{L}(X, Y)$ spanned by $\{\Theta_{y,\xi} : \xi \in X, y \in Y\}$. If $X = Y$ then $\mathcal{K}(X, X) \equiv \mathcal{K}(X)$ is a closed ideal of the C*-algebra $\mathcal{L}(X, X) \equiv \mathcal{L}(X)$.

Lemma 2.1. *Let X, Y, Z be Hilbert A-modules. If $\langle X, X\rangle_X$ provides a right c.a.i. (a_i) for Y, then $\overline{\mathcal{K}(X, Y)\mathcal{K}(Z, X)} = \mathcal{K}(Z, Y)$.*

Proof. The existence of the right c.a.i. (a_i) implies that $Y \langle X, X\rangle$ is dense in Y, hence $\overline{\mathcal{K}(X, Y)\mathcal{K}(Z, X)} = \overline{\mathcal{K}(Z, Y \langle X, X\rangle)} = \mathcal{K}(Z, Y)$. ∎

Definition 2.2. An *A-B-correspondence* X is a right Hilbert B-module together with a $*$-homomorphism $\phi_X : A \to \mathcal{L}(X)$. We denote this by ${}_AX_B$. When $A = B$ we refer to X as a *C*-correspondence over A*.

A submodule Y of X is a *subcorrespondence* of ${}_AX_B$, if it is a C-D-correspondence for some C*-subalgebras C and D of A and B, respectively.

A C*-correspondence X is called *non-degenerate* (resp. *strict*) if the closed linear span of $\phi_X(A)X$ is equal to X (resp. complemented in X). We say that X is *full* if $\langle X, X\rangle_X$ is dense in A. Finally, X is called *regular* if both it is *injective*, i.e., ϕ_X is injective, and $\phi_X(A) \subseteq \mathcal{K}(X)$.

Two A-B-correspondences X and Y are called unitarily equivalent (symb. $X \approx Y$) if there is a unitary $u \in \mathcal{L}(X,Y)$ such that $u(\phi_X(a)\xi b) = \phi_Y(a)(u\xi)b$, for all $a \in A, b \in B, \xi \in X$.

Example 2.3. Every Hilbert A-module X is a $\mathcal{K}(X)$-A-correspondence when endowed with the left multiplication $\phi_X \equiv \mathrm{id}_{\mathcal{K}(X)} : \mathcal{K}(X) \to \mathcal{L}(X)$. A left inner product over $\mathcal{K}(X)$ can be defined by $[\xi,\eta]_X = \Theta_{\xi,\eta}$, for all $\xi,\eta \in X$. Also X^* is an A-$\mathcal{K}(X)$-correspondence, when endowed with the following operations

$$\langle \xi^*, \eta^* \rangle_{X^*} = [\xi,\eta]_X\,,\ \xi^* \cdot k = (k^*\xi)^*, \text{ and } \phi_{X^*}(a)\xi^* = a \cdot \xi^* = (\xi \cdot a^*)^*,$$

for all $\xi,\eta \in X$, $k \in \mathcal{K}(X)$ and $a \in A$.

Example 2.4. For Hilbert A-modules X and Y, $\mathcal{L}(X,Y)$ becomes an $\mathcal{L}(Y)$-$\mathcal{L}(X)$-correspondence by defining $\langle s,t\rangle := s^*t$, $t \cdot a := ta$ and $b \cdot t := bt$, for every $s,t \in \mathcal{L}(X,Y), a \in \mathcal{L}(X)$ and $b \in \mathcal{L}(Y)$.

Trivially, $\mathcal{K}(X,Y)$ is a $\mathcal{K}(Y)$-$\mathcal{K}(X)$-subcorrespondence of $\mathcal{L}(X,Y)$. Note that, when $\langle X,X\rangle_X$ provides a right c.a.i. for Y, then $\mathcal{K}(Y)$ acts faithfully on $\mathcal{K}(X,Y)$. When $X=Y$ this is automatically true.

For two C*-correspondences ${}_AX_B$ and ${}_BY_C$, the *interior* or *stabilized tensor product*, denoted by $X \otimes_B Y$ or simply by $X \otimes Y$, is the quotient of the vector space tensor product $X \otimes_{\mathrm{alg}} Y$ by the subspace generated by elements of the form

$$\xi b \otimes y - \xi \otimes \phi(b)y, \text{ for all } \xi \in X, y \in Y, b \in A.$$

It becomes a Hilbert C-module when equipped with

$$(\xi \otimes y)c := \xi \otimes (yc), \qquad (\xi \in X, y \in Y, c \in C),$$
$$\langle \xi_1 \otimes y_1, \xi_2 \otimes y_2\rangle_{X\otimes Y} := \langle y_1, \phi(\langle \xi_1,\xi_2\rangle_X)y_2\rangle_Y\,, \quad (\xi_1,\xi_2 \in X, y_1,y_2 \in Y).$$

For $s \in \mathcal{L}(X)$ we define $s \otimes \mathrm{id}_Y \in \mathcal{L}(X \otimes Y)$ be the map $\xi \otimes y \mapsto (s\xi) \otimes y$. Then $X \otimes_B Y$ becomes an A-C-correspondence by defining $\phi_{X\otimes Y}(a) := \phi_X(a) \otimes \mathrm{id}_Y$. The interior tensor product plays the role of a generalized associative multiplication of C*-correspondences and the following lemmas will be useful in the sequel.

Lemma 2.5. *Let the* C**-correspondences* ${}_AX_B$ *and* ${}_BY_C$. *If* (c_i) *is an approximate identity of* $\langle Y,Y\rangle_Y$, *then* (c_i) *is a right c.a.i. for the interior tensor product* $X \otimes_B Y$.

Proof. The norm on $X \otimes_B Y$ is a submultiplicative tensor norm, i.e., $\|\xi \otimes y\| \le \|y\|\,\|\xi\|$ for all $\xi \in X, y \in Y$. Thus $\lim_i yc_i = y$ implies $\lim_i(\xi \otimes y)c_i = \lim_i \xi \otimes (yc_i) = \xi \otimes y$, for a c.a.i. (c_i) as above. ∎

Lemma 2.6. *Let* ${}_AX_B$ *and* ${}_BY_C$ *be two* C**-correspondences. If* ${}_AX_B$ *is non-degenerate then* $X \otimes_B Y$ *is non-degenerate.*

Proof. Immediate since the tensor norm is submultiplicative. ∎

Lemma 2.7. *Let X, Y be Hilbert A-modules and ${}_AZ_B$ be a regular* C*-*correspondence. Then the mapping*

$$\otimes \mathrm{id}_Z \colon \mathcal{L}(X,Y) \to \mathcal{L}(X \otimes_A Z, Y \otimes_A Z) : t \mapsto t \otimes \mathrm{id}_Z$$

is isometric and maps $\mathcal{K}(X,Y)$ inside $\mathcal{K}(X \otimes_A Z, Y \otimes_A Z)$.

Proof. Set

$$\Psi \equiv \otimes \mathrm{id}_Z \colon \mathcal{L}(X,Y) \to \mathcal{L}(X \otimes_A Z, Y \otimes_A Z),$$
$$\psi \equiv \otimes \mathrm{id}_Z \colon \mathcal{L}(X) \to \mathcal{L}(X \otimes_A Z).$$

The exact analogue of [**10**, Equation 4.6] shows that the mapping Ψ is well defined and contractive. Since

$$\Psi(t_1)^* \Psi(t_2) = \psi(t_1^* t_2), \quad \text{for all } t_1, t_2 \in \mathcal{L}(X,Y),$$

and ψ is isometric [**10**, Proposition 4.7], we obtain that Ψ is also isometric. Finally, let (s_i) be a right approximate unit for $\mathcal{K}(X,Y)$ inside $\mathcal{K}(X)$. Then by the previous, $k \otimes \mathrm{id}_Z = \lim_i (k \otimes \mathrm{id}_Z) \cdot \psi(s_i)$. However, [**10**, Proposition 4.7] shows that $\psi(s_i) \in \mathcal{K}(X \otimes_A Z)$ and the conclusion follows by noting that $(k \otimes \mathrm{id}_Z) \cdot \mathcal{K}(X \otimes_A Z) \subseteq \mathcal{K}(X \otimes_A Z, Y \otimes_A Z)$. ∎

Example 2.8. When a Hilbert A-module X is considered as the $\mathcal{K}(X)$-A-correspondence, then $X \otimes_A X^* \approx \mathcal{K}(X)$ as C*-correspondences over $\mathcal{K}(X)$, via the mapping $u_1 \colon \xi \otimes \zeta^* \mapsto \Theta_{\xi,\zeta}$, and $X^* \otimes_{\mathcal{K}(X)} X \approx \langle X, X \rangle_X$, as C*-correspondences over A, via the mapping $u_2 \colon \xi^* \otimes \zeta \mapsto \langle \xi, \zeta \rangle$. In particular $X^* \otimes_{\mathcal{K}(X)} X \approx A$, when X is full.

Definition 2.9. A *Hilbert A-B-bimodule* is a C*-correspondence ${}_AX_B$ together with a left inner product $[\cdot,\cdot]_X : X \times X \to A$, which satisfies

$$[\xi, \eta]_X \cdot \zeta = \xi \cdot \langle \eta, \zeta \rangle_X , \qquad (\xi, \eta, \zeta \in X).$$

An *A-B-imprimitivity bimodule or equivalence bimodule* is an A-B- bimodule M which is simultaneously a full left and a full right Hilbert A-module, i.e., $[M,M]_M = A$ and $\langle M, M \rangle_M = B$.

An imprimitivity bimodule ${}_AM_B$ is automatically non-degenerate and regular because $A \simeq^{\phi_M} \mathcal{K}(M)$. It is immediate that M is an imprimitivity bimodule if and only if M^* is a B-A-imprimitivity bimodule and Example 2.8 induces the following.

Lemma 2.10. *If M is an A-B-imprimitivity bimodule, then $M \otimes_B M^* \approx A$ and $M^* \otimes_A M \approx B$, where A and B are the trivial* C*-*correspondences over themselves.*

There is a number of ways of considering *a* direct sum of Hilbert modules and *a* direct sum of C*-correspondences; these constructions are sub-correspondences of the *matrix* C*-*correspondence*. Given ${}_AE_A$, ${}_AR_B$, ${}_BS_A$ and ${}_BF_B$, the *matrix* C*-*correspondence* $X = \left[\begin{array}{c|c} E & R \\ S & F \end{array}\right]$ over $A \oplus B$ is the

Hilbert $(A \oplus B)$-module of the linear space of the "matrices" $\left[\begin{array}{c|c} e & r \\ s & f \end{array}\right]$, $e \in E, r \in R, s \in S, f \in F$, with

$$\left[\begin{array}{c|c} e & r \\ s & f \end{array}\right] \cdot (a,b) := \left[\begin{array}{c|c} ea & rb \\ sa & fb \end{array}\right],$$

$$\left\langle \left[\begin{array}{c|c} e_1 & r_1 \\ s_1 & f_1 \end{array}\right], \left[\begin{array}{c|c} e_2 & r_2 \\ s_2 & f_2 \end{array}\right] \right\rangle_X := \left(\langle e_1, e_2\rangle_E + \langle s_1, s_2\rangle_S \, , \langle r_1, r_2\rangle_R + \langle f_1, f_2\rangle_F \right),$$

and the $*$-homomorphism $\phi\colon A \oplus B \to \mathcal{L}\left(\left[\begin{array}{c|c} E & R \\ S & F \end{array}\right]\right)$ is defined by

$$\phi(a,b) \left[\begin{array}{c|c} e & r \\ s & f \end{array}\right] := \left[\begin{array}{c|c} \phi_E(a)e & \phi_R(a)r \\ \phi_S(b)s & \phi_F(b)f \end{array}\right].$$

The E, R, S and F imbed naturally as subcorrespondences in $\left[\begin{array}{c|c} E & R \\ S & F \end{array}\right]$, since the latter is exactly the *exterior direct sum* C*-*correspondence* of the two *interior direct sum* C*-*correspondences* $\left[\begin{array}{c} E \\ S \end{array}\right]$ and $\left[\begin{array}{c} R \\ F \end{array}\right]$.

The following Lemma explains the use of the terminology "matrix C*-correspondence", as tensoring is really "matrix multiplication".

Lemma 2.11. *Let E, F, R, S be* C*-*correspondences as above. Then*

$$\left[\begin{array}{c|c} E & R \\ S & F \end{array}\right] \otimes_{A\oplus B} \left[\begin{array}{c|c} E & R \\ S & F \end{array}\right] \approx \left[\begin{array}{c|c} \left[\begin{array}{c} E \otimes_A E \\ R \otimes_B S \end{array}\right] & \left[\begin{array}{c} E \otimes_A R \\ R \otimes_B F \end{array}\right] \\ & \\ \left[\begin{array}{c} S \otimes_A E \\ F \otimes_B S \end{array}\right] & \left[\begin{array}{c} S \otimes_A R \\ F \otimes_B F \end{array}\right] \end{array}\right].$$

Proof. Note that all entries in the second matrix make sense. It is a matter of routine calculations to show that the mapping

$$\left[\begin{array}{c|c} e_1 & r_1 \\ s_1 & f_1 \end{array}\right] \otimes \left[\begin{array}{c|c} e_2 & r_2 \\ s_2 & f_2 \end{array}\right] \mapsto \left[\begin{array}{c|c} \left[\begin{array}{c} e_1 \otimes e_2 \\ r_1 \otimes s_2 \end{array}\right] & \left[\begin{array}{c} e_1 \otimes r_2 \\ r_1 \otimes f_2 \end{array}\right] \\ & \\ \left[\begin{array}{c} s_1 \otimes e_2 \\ f_1 \otimes s_2 \end{array}\right] & \left[\begin{array}{c} s_1 \otimes r_2 \\ f_1 \otimes f_2 \end{array}\right] \end{array}\right].$$

defines the unitary element that gives the equivalence. ∎

It will be convenient to omit the zero entries, when possible. For example, we write $\mathcal{K}(E,S)$ instead of $\mathcal{K}\left(\left[\begin{array}{c|c} E & 0 \\ 0 & 0 \end{array}\right], \left[\begin{array}{c|c} 0 & 0 \\ S & 0 \end{array}\right]\right)$.

A (*Toeplitz*) *representation* of ${}_AX_A$ into a C*-algebra B, is a pair (π, t), where $\pi\colon A \to B$ is a $*$-homomorphism and $t\colon X \to B$ is a linear map, such that $\pi(a)t(\xi) = t(\phi_X(a)(\xi))$ and $t(\xi)^* t(\eta) = \pi(\langle \xi, \eta\rangle_X)$, for $a \in A$ and $\xi, \eta \in X$. An application of the C*-identity shows that $t(\xi)\pi(a) = t(\xi a)$ is also valid. A representation (π, t) is said to be *injective* if π is injective; in that case t is an isometry.

The C*-algebra generated by a representation (π, t) equals the closed linear span of $t^n(\bar{\xi})t^m(\bar{\eta})^*$, where for simplicity $\bar{\xi} \equiv \xi_1 \otimes \cdots \otimes \xi_n \in X^{\otimes n}$ and $t^n(\bar{\xi}) \equiv t(\xi_1) \dots t(\xi_n)$. For any representation (π, t) there exists a $*$-homomorphism $\psi_t : \mathcal{K}(X) \to B$, such that $\psi_t(\Theta^X_{\xi,\eta}) = t(\xi)t(\eta)^*$.

Let J be an ideal in $\phi_X^{-1}(\mathcal{K}(X))$; we say that a representation (π, t) is J-coisometric if $\psi_t(\phi_X(a)) = \pi(a)$, for any $a \in J$. Following **[7]**, the J_X-coisometric representations (π, t), for

$$J_X = \ker \phi_X^{\perp} \cap \phi_X^{-1}(\mathcal{K}(X)),$$

are called *covariant representations*.

The *Toeplitz-Cuntz-Pimsner algebra* $\mathcal{T}_X$ is the universal C*-algebra for "all" representations of X, and the *Cuntz-Pimsner algebra* $\mathcal{O}_X$ is the universal C*-algebra for "all" covariant representations of X. The *tensor algebra* $\mathcal{T}_X^+$ is the norm-closed algebra generated by the universal copy of A and X in $\mathcal{T}_X$.

If X is an A-B-correspondence, then we may identify X with the $(A \oplus B)$-correspondence $\left[\begin{array}{c|c} 0 & X \\ 0 & 0 \end{array}\right]$ and thus define the Toeplitz-Cuntz-Pimsner, the Cuntz-Pimsner and the tensor algebra of X as the corresponding algebras of the $(A \oplus B)$-correspondence $\left[\begin{array}{c|c} 0 & X \\ 0 & 0 \end{array}\right]$.

If X is a C*-correspondence, then X^* may not be a C*-correspondence (in the usual right-side sense) but it can be described as follows: if (π_u, t_u) is the universal representation of ${}_AX_A$, then X^* is the closed linear span of $t(\xi)^*, \xi \in X$ with the left multiplication and inner product inherited by the correspondence $\mathrm{C}^*(\pi_u, t_u)$. Nevertheless, X^* is an imprimitivity bimodule, whenever X is.

Example 2.12. If $X_{\mathcal{G}}$ is the C*-correspondence coming from a graph $\mathcal{G}$, then $X_{\mathcal{G}}$ is an imprimitivity bimodule if and only if $\mathcal{G}$ is either a cycle or a double-infinite path. In that case, $X_{\mathcal{G}}^*$ is the correspondence coming from the graph having the same edges as $\mathcal{G}$ but its arrows reversed.

If $\alpha \colon A \to B$ is a $*$-homomorphism, then the induced correspondence B_α is an imprimitivity bimodule if and only if α is a $*$-isomorphism. In that case, $B_\alpha^* = A_{\alpha^{-1}}$.

3. Dilations of C*-correspondences

For C*-correspondences ${}_AX_A$ and ${}_BY_B$ we write $X \lesssim Y$ when X is unitarily equivalent to a subcorrespondence of Y. In other words there is a $*$-injective representation $\pi \colon A \to B$, a $\pi(A)$-$\pi(A)$- subcorrespondence Y_0 of Y and a unitary $u \in \mathcal{L}(X, Y_0)$ such that $u : X \to Y_0$, where X is (now) considered a C*-correspondence over $\pi(A)$.

Here we do not impose u to be an isometry in $\mathcal{L}(X, Y)$, i.e., that u has a complemented range in Y [**10**, Theorem 3.2]. What we ask is that u is an isometric map and $X \approx_{(\pi,u)} Y_0 \subseteq Y$, in terms of the representation theory of C*-correspondences.

Definition 3.1. A C*-correspondence ${}_BY_B$ is said to be *a dilation* of the correspondence ${}_AX_A$, if ${}_AX_A \lesssim {}_BY_B$ and the associated Cuntz-Pimsner algebras $\mathcal{O}_X$ and $\mathcal{O}_Y$ are $*$-isomorphic.

Given an injective C*-correspondence X there is a natural way to pass to an injective Hilbert bimodule X_∞, such that X_∞ is a dilation of X. This construction was first introduced by Pimsner in [**16**]. In [**6**, Appendix A] we revisited this construction by using direct limits, as follows.

Define the isometric mapping $\tau \colon X \to \mathcal{L}(X, X^{\otimes 2})$, such that $\tau_\xi(\eta) = \xi \otimes \eta$, and consider the direct limits

$$X \xrightarrow{\tau} \mathcal{L}(X, X^{\otimes 2}) \xrightarrow{\otimes \mathrm{id}_X} \dots \longrightarrow \varinjlim(\mathcal{L}(X^{\otimes n}, X^{\otimes n+1}), \otimes \mathrm{id}_X) =: Y$$

$$A \xrightarrow{\phi_X} \mathcal{L}(X) \xrightarrow{\otimes \mathrm{id}_X} \mathcal{L}(X^{\otimes 2}) \xrightarrow{\otimes \mathrm{id}_X} \dots \longrightarrow \varinjlim(\mathcal{L}(X^{\otimes n}), \otimes \mathrm{id}_X) =: B$$

If $r \in \mathcal{L}(X^{\otimes n})$, $s \in \mathcal{L}(X^{\otimes n}, X^{\otimes n+1})$ and $[r], [s]$ are their equivalence classes in B and Y respectively, then we define $[s] \cdot [r] := [sr]$. From this, it is easy to define a right B-action on Y. Similarly, we may define a B-valued right inner product on Y by setting

$$\big\langle [s'], [s] \big\rangle_Y \equiv [(s')^* s] \in B.$$

for $s, s' \in \mathcal{L}(X^{\otimes n}, X^{\otimes n+1})$, $n \in \mathbb{N}$, and then extending to $Y \times Y$. Finally, we define a $*$-homomorphism $\phi_Y \colon B \to \mathcal{L}(Y)$ by setting

$$\phi_Y([r])([s]) \equiv [rs], \quad r \in \mathcal{L}(X^{\otimes n}), s \in \mathcal{L}(X^{\otimes n-1}, X^{\otimes n}), n \geq 0,$$

and extending to all of B by continuity. We therefore have a left B-action on Y and thus Y becomes a C*-correspondence over B.

Let X_∞ be the Banach subalgebra of $Y = \varinjlim(\mathcal{L}(X^{\otimes n}, X^{\otimes n+1}), \sigma_n)$ generated by the copies of $\mathcal{K}(X^{\otimes n}, X^{\otimes n+1})$, for $n \in \mathbb{Z}_+$ and A_∞ be the C*-subalgebra of $B = \varinjlim(\mathcal{L}(X^{\otimes n}), \rho_n)$ that is generated by the copies of $\mathcal{K}(X^{\otimes n})$, for $n \in \mathbb{Z}_+$. Then X_∞ is an A_∞-subcorrespondence of Y, that contains ${}_AX_A$. In particular, when $\phi_X(A) \subseteq \mathcal{K}(X)$ then $\tau(X) \subseteq \mathcal{K}(X, X^{\otimes 2})$ and Lemma 2.7 implies that

$$X_\infty = \varinjlim(\mathcal{K}(X^{\otimes n}, X^{\otimes n+1}), \sigma_n) \text{ and } A_\infty = \varinjlim(\mathcal{K}(X^{\otimes n}), \rho_n).$$

Theorem 3.2. [**16**, Theorem 2.5], [**6**, Theorem 6.6] *Let X be an injective C*-correspondence and let X_∞ be the A_∞-correspondence constructed above. Then X_∞ is an essential Hilbert bimodule and the Cuntz-Pimsner algebras $\mathcal{O}_{X_\infty}$ and $\mathcal{O}_X$ coincide.*

In the sequel, the A_∞-correspondence X_∞ appearing in the Theorem above will be called the *Pimsner dilation of X*.

Example 3.3. Let (A, α) denote a dynamical system where α is a unital $*$-injective endomorphism of A. We can define the direct limit dynamical system $(A_\infty, \alpha_\infty)$ by

$$\begin{array}{ccccccccc} A & \xrightarrow{\alpha} & A & \xrightarrow{\alpha} & A & \xrightarrow{\alpha} & \cdots & \longrightarrow & A_\infty \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & & & \downarrow{\scriptstyle\alpha_\infty} \\ A & \xrightarrow{\alpha} & A & \xrightarrow{\alpha} & A & \xrightarrow{\alpha} & \cdots & \longrightarrow & A_\infty \end{array}$$

The limit map α_∞ is an automorphism of A_∞ and extends α (note that A imbeds in A_∞ since α is injective). Then the A_∞-A_∞-correspondence X_{α_∞}, is the Pimsner dilation of X_α.

When X is non-degenerate and full, then its Pimsner dilation X_∞ is also full, hence an imprimitivity bimodule. Indeed, it suffices to prove that

$$\overline{\mathcal{K}(X^{\otimes n+1}, X^{\otimes n})\mathcal{K}(X^{\otimes n}, X^{\otimes n+1})} = \mathcal{K}(X^{\otimes n}),$$

for all $n \geq 1$. By Lemma 2.1, it suffices to show that $\left\langle X^{\otimes n+1}, X^{\otimes n+1} \right\rangle$ provides a c.a.i. for $X^{\otimes n}$. For $n = 2$,

$$\left\langle X^{\otimes 2}, X^{\otimes 2} \right\rangle = \langle X, \phi_X(\langle X, X\rangle)X\rangle = \langle X, \phi_X(A)X\rangle = \langle X, X\rangle = A,$$

and an inductive argument completes the claim.

The construction of the Pimsner dilation applies to more general settings. Indeed, let X, Y be Hilbert A-modules and let ${}_AZ_B$ be a regular correspondence. By Lemma 2.7, one can form the following directed systems

$$\begin{gathered} \mathcal{K}(X,Y) \overset{\otimes \mathrm{id}_Z}{\longrightarrow} \mathcal{K}(X \otimes Z, Y \otimes Z) \overset{\otimes \mathrm{id}_Z}{\longrightarrow} \dots \longrightarrow \varinjlim(\mathcal{K}(X \otimes Z^{\otimes n}, Y \otimes Z^{\otimes n}), \otimes \mathrm{id}_Z) \\ \mathcal{K}(X) \overset{\otimes \mathrm{id}_Z}{\longrightarrow} \mathcal{K}(X \otimes Z) \overset{\otimes \mathrm{id}_Z}{\longrightarrow} \dots \longrightarrow \varinjlim(\mathcal{K}(X \otimes Z^{\otimes n}), \otimes \mathrm{id}_Z) \\ \mathcal{K}(Y) \overset{\otimes \mathrm{id}_Z}{\longrightarrow} \mathcal{K}(Y \otimes Z) \overset{\otimes \mathrm{id}_Z}{\longrightarrow} \dots \longrightarrow \varinjlim(\mathcal{K}(Y \otimes Z^{\otimes n}), \otimes \mathrm{id}_Z). \end{gathered}$$

For simplicity, we will write $(\mathcal{K}(X,Y), \mathrm{id}_Z)_\infty$ for the direct limit

$$\varinjlim \left(\mathcal{K}(X \otimes Z^{\otimes n}, Y \otimes Z^{\otimes n}), \mathrm{id}_Z\right).$$

By imitating the proofs in [**6**, Appendix A] and Lemma 2.7, we see that $(\mathcal{K}(X,Y), \mathrm{id}_Z)_\infty$ is a $\mathcal{K}(Y)_\infty$-$\mathcal{K}(X)_\infty$-correspondence, which will be called *the dilation of* $\mathcal{K}(X,Y)$ *by* Z, or more simply the Z*-dilation of* $\mathcal{K}(X,Y)$. If, in addition, $\langle X \otimes Z^{\otimes n}, X \otimes Z^{\otimes n}\rangle$ provides a right c.a.i. for $Y \otimes Z^{\otimes n}$, for every $n \geq 0$, then by Example 2.4 and Lemma 2.1 $(\mathcal{K}(X,Y), \mathrm{id}_Z)_\infty$ is a regular C*-correspondence. In the case of a regular C*-correspondence ${}_AX_A$, the Pimsner dilation X_∞ is simply the X-dilation of $\mathcal{K}(X, X^{\otimes 2})$, which is always regular.

Proposition 3.4. *Let X, Y be Hilbert A-modules and Z, W be regular* C*-*correspondences over A. If $Z \otimes_A W \approx W \otimes_A Z$, then*

$$(\mathcal{K}(X,Y), \mathrm{id}_Z)_\infty \lesssim (\mathcal{K}(X \otimes_A W, Y \otimes_A W), \mathrm{id}_Z)_\infty.$$

Proof. We will identify $Z \otimes W$ with $W \otimes Z$. Since Z commutes with W the diagram

$$\begin{array}{ccccccc}
\mathcal{K}(X,Y) & \xrightarrow{\mathrm{id}_{\mathcal{K}(X,Y)}} & \mathcal{K}(X,Y) & \xrightarrow{\otimes \mathrm{id}_Z} & \mathcal{K}(X\otimes Z, Y\otimes Z) & \xrightarrow{\otimes \mathrm{id}_Z} & \cdots \\
\downarrow{\scriptstyle \mathrm{id}_{\mathcal{K}(X,Y)}} & & \downarrow{\scriptstyle \otimes \mathrm{id}_W} & & \downarrow{\scriptstyle \otimes \mathrm{id}_W} & & \\
\mathcal{K}(X,Y) & \xrightarrow{\otimes \mathrm{id}_W} & \mathcal{K}(X\otimes W, Y\otimes W) & \xrightarrow{\otimes \mathrm{id}_Z} & \mathcal{K}(X\otimes W\otimes Z, Y\otimes W\otimes Z) & \xrightarrow{\otimes \mathrm{id}_Z} & \cdots
\end{array}$$

is commutative and defines a linear map

$$s\colon (\mathcal{K}(X,Y), \mathrm{id}_Z)_\infty \to (\mathcal{K}(X\otimes_A W, Y\otimes_A W), \mathrm{id}_Z)_\infty.$$

That is, if $[k] \in (\mathcal{K}(X,Y), \mathrm{id}_Z)_\infty$, such that $k \in \mathcal{K}(X\otimes Z^{\otimes n}, Y\otimes Z^{\otimes n})$ then

$$k\otimes \mathrm{id}_W \in \mathcal{K}(X\otimes Z^{\otimes n}\otimes W, Y\otimes Z^{\otimes n}\otimes W) = \mathcal{K}(X\otimes W\otimes Z^{\otimes n}, Y\otimes Z^{\otimes n}),$$

and we define $s[k] = [k\otimes \mathrm{id}_W]$. Note that s is defined on $(\mathcal{K}(X), \mathrm{id}_Z)_\infty$ and on $(\mathcal{K}(Y), \mathrm{id}_Z)_\infty$ in the analogous way. For example we get the commutative diagram

$$\begin{array}{ccccccc}
\mathcal{K}(X) & \xrightarrow{\mathrm{id}_{\mathcal{K}(X)}} & \mathcal{K}(X) & \xrightarrow{\otimes \mathrm{id}_Z} & \mathcal{K}(X\otimes Z) & \xrightarrow{\otimes \mathrm{id}_Z} & \cdots \\
\downarrow{\scriptstyle \mathrm{id}_{\mathcal{K}(X)}} & & \downarrow{\scriptstyle \otimes \mathrm{id}_W} & & \downarrow{\scriptstyle \otimes \mathrm{id}_W} & & \\
\mathcal{K}(X) & \xrightarrow{\otimes \mathrm{id}_W} & \mathcal{K}(X\otimes W) & \xrightarrow{\otimes \mathrm{id}_Z} & \mathcal{K}(X\otimes W\otimes Z) & \xrightarrow{\otimes \mathrm{id}_Z} & \cdots
\end{array}$$

It is a matter of routine computations to show that s is a $(\mathcal{K}(Y), \mathrm{id}_Z)_\infty$-$(\mathcal{K}(X), \mathrm{id}_Z)_\infty$-mapping. What is left to show is that s is also a unitary onto its range.

It suffices to prove this at the n-th level. For $k_1, k_2 \in \mathcal{K}(X\otimes Z^{\otimes n}, Y\otimes Z^{\otimes n})$, then

$$\begin{aligned}
\langle s[k_1], s[k_2]\rangle &= \langle [k_1\otimes \mathrm{id}_W], [k_2\otimes \mathrm{id}_W]\rangle = [k_1\otimes \mathrm{id}_W]^*[k_2\otimes \mathrm{id}_W] \\
&= [(k_1^*\circ k_2)\otimes \mathrm{id}_W] = [(k_1^*\circ k_2)] = \langle [k_1], [k_2]\rangle ,
\end{aligned}$$

where we have identified $(\mathcal{K}(X), \mathrm{id}_Z)_\infty$ with its image via s. ∎

For the proof of the following Proposition, recall that there is a major difference between isometric mappings of C*-correspondences and mappings of C*-correspondences that are isometries (of Hilbert modules). However, when an isometric mapping is also onto then it is a unitary between Hilbert modules.

Proposition 3.5. *Let X, Y, W be Hilbert A-modules and Z be a regular* C*-*correspondence over A. If the ideal $\langle X\otimes Z^{\otimes n}, X\otimes Z^{\otimes n}\rangle$ of A provides a right c.a.i. for $Y\otimes Z^{\otimes n}$, for all $n\geq 0$, then*

$$(\mathcal{K}(X,Y), \mathrm{id}_Z)_\infty \otimes_{(\mathcal{K}(X), \mathrm{id}_Z)_\infty} (\mathcal{K}(W,X), \mathrm{id}_Z)_\infty \approx (\mathcal{K}(W,Y), \mathrm{id}_Z)_\infty.$$

Proof. In order to "visualize" the proof, imagine that we multiply "vertically" and term by term the correspondences

$$\mathcal{K}(X,Y) \overset{\otimes \mathrm{id}_Z}{\longrightarrow} \mathcal{K}(X\otimes_A Z, Y\otimes_A Z) \overset{\otimes \mathrm{id}_Z}{\longrightarrow} \ldots \longrightarrow (\mathcal{K}(X,Y), \mathrm{id}_Z)_\infty$$

$$\mathcal{K}(W,X) \overset{\otimes \mathrm{id}_Z}{\longrightarrow} \mathcal{K}(W\otimes_A Z, X\otimes_A Z) \overset{\otimes \mathrm{id}_Z}{\longrightarrow} \ldots \longrightarrow (\mathcal{K}(W,X), \mathrm{id}_Z)_\infty$$

in the order $\mathcal{K}(X \otimes_A Z^{\otimes n}, Y \otimes_A Z^{\otimes n}) \cdot \mathcal{K}(W \otimes_A Z^{\otimes n}, X \otimes_A Z^{\otimes n})$.

We consider $\mathcal{K}(X \otimes Z^{\otimes n})$, $\mathcal{K}(Y \otimes Z^{\otimes n})$ and $\mathcal{K}(W \otimes Z^{\otimes n})$ as C*-subalgebras of $(\mathcal{K}(X), \mathrm{id}_Z)_\infty$, $(\mathcal{K}(Y), \mathrm{id}_Z)_\infty$ and $(\mathcal{K}(W), \mathrm{id}_Z)_\infty$, respectively, for every $n \in \mathbb{Z}_+$. The proof is divided into two parts.

For the first part of the proof note that for every $n \in \mathbb{Z}_+$, the Banach space $\mathcal{K}(X \otimes_A Z^{\otimes n}, Y \otimes_A Z^{\otimes n})$ (resp. $\mathcal{K}(W \otimes_A Z^{\otimes n}, X \otimes_A Z^{\otimes n})$) is an injective $\mathcal{K}(Y \otimes_A Z^{\otimes n})$-$\mathcal{K}(X \otimes_A Z^{\otimes n})$-correspondence (resp. a $\mathcal{K}(X \otimes_A Z^{\otimes n})$-$\mathcal{K}(W \otimes_A Z^{\otimes n})$-correspondence) in the obvious way. Set

$$\mathfrak{A}_n := \mathcal{K}(X \otimes_A Z^{\otimes n}, Y \otimes_A Z^{\otimes n}) \otimes_{\mathcal{K}(X \otimes_A Z^{\otimes n})} \mathcal{K}(W \otimes_A Z^{\otimes n}, X \otimes_A Z^{\otimes n})$$

For every $n \in \mathbb{Z}_+$, the mapping

$$\rho_n \colon \mathfrak{A}_n \to \mathfrak{A}_{n+1} : k \otimes t \mapsto (k \otimes \mathrm{id}_Z) \otimes (t \otimes \mathrm{id}_Z),$$

is isometric on sums of elementary tensors and therefore extends to the closures. Thus $\mathfrak{A}_n \lesssim \mathfrak{A}_{n+1}$, and we can form the direct limit C*-correspondence

$$\mathfrak{A}_0 \xrightarrow{\rho_0} \mathfrak{A}_1 \xrightarrow{\rho_1} \mathfrak{A}_2 \xrightarrow{\rho_2} \dots \longrightarrow \varinjlim(\mathfrak{A}_n, \rho_n),$$

which is a $(\mathcal{K}(Y), \mathrm{id}_Z)_\infty$-$(\mathcal{K}(W), \mathrm{id}_Z)_\infty$-correspondence. Now the mappings

$$\phi_n \colon \mathfrak{A}_n \to (\mathcal{K}(X,Y), \mathrm{id}_Z)_\infty \otimes_{(\mathcal{K}(X), \mathrm{id}_Z)_\infty} (\mathcal{K}(W,X), \mathrm{id}_Z)_\infty : k \otimes t \mapsto [k] \otimes [t]$$

are compatible with the directed system and define a C*-correspondence mapping

$$\phi \colon \varinjlim(\mathfrak{A}_n, \rho_n) \to (\mathcal{K}(X,Y), \mathrm{id}_Z)_\infty \otimes_{(\mathcal{K}(X), \mathrm{id}_Z)_\infty} (\mathcal{K}(W,X), \mathrm{id}_Z)_\infty,$$

which is isometric since every ϕ_{n+1} is. Thus

$$\varinjlim(\mathfrak{A}_n, \rho_n) \lesssim (\mathcal{K}(X,Y), \mathrm{id}_Z)_\infty \otimes_{(\mathcal{K}(X), \mathrm{id}_Z)_\infty} (\mathcal{K}(W,X), \mathrm{id}_Z)_\infty,$$

via ϕ. However $(\mathcal{K}(X,Y), \mathrm{id}_Z)_\infty \otimes_{(\mathcal{K}(X), \mathrm{id}_Z)_\infty} (\mathcal{K}(W,X), \mathrm{id}_Z)_\infty$ is spanned by the elements $[k] \otimes [t]$ for $k \otimes t \in \mathfrak{A}_n$. Therefore ϕ is onto and so

$$\varinjlim(\mathfrak{A}_n, \rho_n) \approx (\mathcal{K}(X,Y), \mathrm{id}_Z)_\infty \otimes_{(\mathcal{K}(X), \mathrm{id}_Z)_\infty} (\mathcal{K}(W,X), \mathrm{id}_Z)_\infty$$

For the second part of the proof we construct an isometric map from $\varinjlim(\mathfrak{A}_n, \rho_n)$ onto $(\mathcal{K}(W,Y), \mathrm{id}_Z)_\infty$. Start by defining the maps

$$u_n \colon \mathfrak{A}_n \to (\mathcal{K}(W,Y), \mathrm{id}_Z)_\infty : k \otimes t \mapsto [kt].$$

These are well defined since $kt \in \mathcal{K}(W \otimes Z^{\otimes n}, Y \otimes Z^{\otimes n})$ and

$$u_n((k \cdot a) \otimes t) = [(k \cdot a)t] = [k(a \cdot t)] = u_n(k \otimes (a \cdot t)),$$

due to the associativity of the multiplication. Also, the maps u_n are isometric and compatible with the direct sequence. Therefore the family $\{u_n\}$ defines an isometric map u from the direct limit into $(\mathcal{K}(W,Y), \mathrm{id}_Z)_\infty$, which extends to a mapping of C*-correspondences. Thus $\mathfrak{A}_n \lesssim (\mathcal{K}(W,Y), \mathrm{id}_Z)_\infty$ and so $\varinjlim(\mathfrak{A}_n, \rho_n) \lesssim (\mathcal{K}(W,Y), \mathrm{id}_Z)_\infty$. Finally note that the assumption that the ideals $\langle X \otimes Z^{\otimes n}, X \otimes Z^{\otimes n} \rangle$ provide a c.a.i. for each $Y \otimes Z^{\otimes n}$, combined with Lemma 2.1, implies that the isometric maps u_n are onto and hence u is onto. ∎

4. Relations associated with C*-correspondences

In [**15**] Muhly and Solel introduced the notion of Morita equivalence of C*-correspondences. This concept generalizes the notion of outer conjugacy for C*-dynamical systems [**15**, Proposition 2.4].

Definition 4.1. The C*-correspondences ${}_AE_A$ and ${}_BF_B$ are called *strong Morita equivalent* if there is an imprimitivity bimodule ${}_AM_B$ such that $E \otimes_A M \approx M \otimes_B F$. In that case we write $E \overset{\text{SME}}{\sim} F$.

Muhly and Solel [**15**] examined this relation under the assumption that the C*-correspondences are both non-degenerate and injective. Nevertheless, non-degeneracy is automatically implied. Indeed, if $E \overset{\text{SME}}{\sim} F$ via M, then

$$E \approx E \otimes_A A \approx E \otimes_A M \otimes_B M^* \approx M \otimes_B F \otimes_B M^* \approx M \otimes_B (F \otimes_B M^*),$$

and Lemma 2.6 implies that E is non-degenerate. A symmetrical argument applies for F.

Remark 4.2. In contrast to non-degeneracy, injectivity is not automatically implied by $\overset{\text{SME}}{\sim}$. For example, pick your favorite non-degenerate and non-injective C*-correspondence ${}_AE_A$ and let $E = F$. Then $E \overset{\text{SME}}{\sim} E$ via the trivial imprimitivity bimodule ${}_AA_A$, but both E and F are not injective.

In [**13**] Muhly, Pask and Tomforde introduced the notion of elementary strong shift equivalence between C*-correspondences that generalizes the corresponding notion for graphs.

Definition 4.3. Let ${}_AE_A$ and ${}_BF_B$ be C*-correspondences. Then E and F will be called *elementary strong shift equivalent* (symb. $E \overset{\text{s}}{\sim} F$) if there are C*-correspondences ${}_AR_B$ and ${}_BS_A$ such that $E \approx R \otimes_B S$ and $F \approx S \otimes_A R$ as C*-correspondences.

Elementary strong shift equivalence is obviously symmetric. Moreover it is reflexive for non-degenerate C*-correspondences; indeed, $E \overset{\text{s}}{\sim} E$ via E and A. But, it may not be transitive as [**19**, Example 2] shows[1]. Nevertheless, we have the following proposition.

Proposition 4.4. *Let ${}_AE_A, {}_BF_B$ and ${}_CG_C$ be C*-correspondences. Assume that $E \overset{s}{\sim} F$ via R, S and $F \overset{s}{\sim} G$ via T, Z. If either Z or R is an imprimitivity bimodule, then $E \overset{s}{\sim} G$.*

Proof. For E, F and G as above we have that

$$E \approx R \otimes_B S\,, F \approx S \otimes_A R\,, F \approx T \otimes_C Z\,, G \approx Z \otimes_B T.$$

[1] Note that [**19**, Example 2] shows that $\overset{\text{s}}{\sim}$ is not transitive when restricted to the class of non-negative integral matrices, which doesn't mean that $\overset{\text{s}}{\sim}$ is not transitive for the whole class of C*-correspondences.

Assume that Z is an imprimitivity bimodule (a symmetric argument can be used if R is an imprimitivity bimodule). Then by Lemma 2.10 $Z^* \otimes_C Z \approx B$ and $Z \otimes_B Z^* \approx C$. Hence,

$$\begin{aligned}(Z \otimes_B S) \otimes_A (R \otimes_B Z^*) &\approx Z \otimes_B (S \otimes_A R) \otimes_B Z^* \approx Z \otimes_B F \otimes_B Z^* \\ &\approx Z \otimes_B T \otimes_C Z \otimes_B Z^* \approx Z \otimes_B T \otimes_C C \\ &\approx Z \otimes_B (T \cdot C) = Z \otimes_B T \approx G.\end{aligned}$$

On the other hand,

$$(R \otimes_B Z^*) \otimes_C (Z \otimes_B S) \approx R \otimes_B B \otimes_B S = (R \cdot B) \otimes_B S = R \otimes_B S \approx E,$$

hence $E \overset{\mathrm{s}}{\sim} G$, which completes the proof. ∎

Following Williams [**19**] we denote by $\overset{\mathrm{SSE}}{\sim}$ the *transitive closure of the relation* $\overset{s}{\sim}$. That is, $E \overset{\mathrm{SSE}}{\sim} F$ if there are n C*-correspondences T_i, $i = 0, \dots, n$ such that $T_0 = E$, $T_n = F$ and $T_i \overset{\mathrm{s}}{\sim} T_{i+1}$.

There is also another relation between C*-correspondences inspired by Williams' work [**19**].

Definition 4.5. Let ${}_AE_A$ and ${}_BF_B$ be C*-correspondences. Then E and F will be called *shift equivalent* (symb. $E \overset{\mathrm{SE}}{\sim} F$) if there are C*-correspondences ${}_AR_B$ and ${}_BS_A$ and a natural number m so that

(i) $E^{\otimes m} \approx R \otimes_A S,\ F^{\otimes m} \approx S \otimes_B R,$

(ii) $S \otimes_A E \approx F \otimes S,\ E \otimes_A R \approx R \otimes_B F.$

The number m is called the *lag* of the equivalence.

If $E \overset{\mathrm{SE}}{\sim} F$ with lag m, then by replacing S with $S \otimes_A E^{\otimes k}$ we obtain another shift equivalence of lag $m + k$, i.e., there is a shift equivalence of lag L for every $L \geq m$.

In contrast to elementary strong shift equivalence, shift equivalence is an equivalence relation for the class of arbitrary C*-correspondences.

Proposition 4.6. *Shift equivalence is an equivalence relation.*

Proof. Note that $E \overset{\mathrm{SE}}{\sim} E$ with lag 2 and $R = S = E$, and it is clear that shift equivalence is symmetric. If $E \overset{\mathrm{SE}}{\sim} F$ with lag m via R, S and $F \overset{\mathrm{SE}}{\sim} G$ with lag n via V, U, then $E \overset{\mathrm{SE}}{\sim} G$ with lag $mn + m$ via

$$\underbrace{R \otimes V \otimes U \otimes \cdots \otimes V}_{V \text{ is repeated } m \text{ times}} \ \text{ and } \ U \otimes S,$$

which shows that $\overset{\mathrm{SE}}{\sim}$ is transitive. ∎

Theorem 4.7. *Let ${}_AE_A$ and ${}_BF_B$ be* C*-*correspondences. Then*

$$E \overset{SME}{\sim} F \Rightarrow E \overset{s}{\sim} F \Rightarrow E \overset{SSE}{\sim} F \Rightarrow E \overset{SE}{\sim} F.$$

Proof. Recall that when $E \overset{\text{SME}}{\sim} F$ then E, F are non-degenerate. Let $R \equiv E \otimes M$ and $S \equiv M^*$. By Lemma 2.11,

$$R \otimes_B S \approx E \otimes_A M \otimes_B M^* \approx E \otimes_A A \approx E,$$

and

$$S \otimes_A R \approx M^* \otimes_A E \otimes_A M \approx M^* \otimes_A M \otimes_B F \approx B \otimes_B F \approx F.$$

Hence $\overset{\text{SME}}{\sim}$ implies $\overset{\text{s}}{\sim}$. Trivially $\overset{\text{s}}{\sim}$ implies $\overset{\text{SSE}}{\sim}$. To complete the proof assume that $E \overset{\text{SSE}}{\sim} F$, i.e., there are T_i, $i = 0, \dots, n$, such that $T_i \overset{\text{s}}{\sim} T_{i+1}$, where $T_0 = E$ and $T_n = F$. Then one can directly verify that $E \overset{\text{SE}}{\sim} F$ with lag n, via $R = R_1 \otimes \cdots \otimes R_n$ and $S = S_n \otimes \cdots \otimes S_1$. ∎

Muhly, Pask and Tomforde [**13**] provide a number of examples to show that strong Morita equivalence differs from elementary strong shift equivalence. In Theorem 4.7 above we prove that in fact it is stronger. Nevertheless, the following result shows that under certain circumstances, the two notions coincide.

Proposition 4.8. *Let ${}_AE_A$ and ${}_BF_B$ be* C**-correspondences and assume that $E \overset{s}{\sim} F$ via R, S. If either R or S is an imprimitivity bimodule, then $E \overset{SME}{\sim} F$.*

Proof. When $E \overset{\text{s}}{\sim} F$ via R, S then $E \otimes_A R \approx R \otimes_B S \otimes_A R \approx R \otimes_B F$ (analogously $S \otimes_B E \approx F \otimes_B S$). ∎

5. Passing to Pimsner Dilations

In this section we show that if $\sim$ is any of the four relations defined in the previous section, then $E \sim F$ implies $E_\infty \sim F_\infty$, under the standing hypothesis that E and F are regular C*-correspondences. Our study is based on the concept of a *bipartite inflation*, an insightful construct originating in the work of Muhly, Pask and Tomforde [**13**].

Assume that ${}_AE_A$ and ${}_BF_B$ are C*-correspondences which are elementary strong shift equivalent via ${}_AR_B$ and ${}_BS_A$. Let $X = \left[\begin{array}{c|c} 0 & R \\ S & 0 \end{array}\right]$ be *the bipartite inflation of S by R*. By Lemma 2.11, we obtain[2]

$$X^{\otimes 2} = \left[\begin{array}{c|c} R \otimes_B S & 0 \\ 0 & S \otimes_A R \end{array}\right] = \left[\begin{array}{c|c} E & 0 \\ 0 & F \end{array}\right].$$

By induction and Lemma 2.11,

$$X^{\otimes 2k} = \left[\begin{array}{c|c} E^{\otimes k} & 0 \\ 0 & F^{\otimes k} \end{array}\right], \quad X^{\otimes 2k+1} = \left[\begin{array}{c|c} 0 & E^{\otimes k} \otimes_A R \\ F^{\otimes k} \otimes_B S & 0 \end{array}\right],$$

[2] In order to ease notation, unitary equivalence of C*-correspondences will be simply denoted as equality in this section.

for all $k \in \mathbb{Z}_+$. In particular, if E and F are regular (resp. non-degenerate) then R and S, and consequently X and $X^{\otimes 2}$, are regular (resp. non-degenerate) as shown in [**13**].

Proposition 5.1. *Let $E \overset{s}{\sim} F$ via R, S. Then, for $k \in \mathbb{Z}_+$,*

(1) *$\left\langle E^{\otimes k} \otimes R, E^{\otimes k} \otimes R \right\rangle$ provides a right c.a.i. for $F^{\otimes k+1}$,*
(2) *$\left\langle F^{\otimes k} \otimes S, F^{\otimes k} \otimes S \right\rangle$ provides a right c.a.i. for $E^{\otimes k+1}$.*

Proof. Let X be the bipartite inflation of S by R and let $k \in \mathbb{Z}_+$. By Lemma 2.5 we have that $\left\langle X^{\otimes 2k+1}, X^{\otimes 2k+1} \right\rangle$ provides a right c.a.i. for $X^{\otimes 2k+2} = X \otimes X^{\otimes 2k+1}$. However,

$$X^{\otimes 2k+2} = \left[\begin{array}{c|c} E^{\otimes k+1} & 0 \\ 0 & F^{\otimes k+1} \end{array}\right]$$

and

$$\begin{aligned} &\left\langle X^{\otimes 2k+1}, X^{\otimes 2k+1} \right\rangle = \\ &= \left\langle \left[\begin{array}{c|c} 0 & E^{\otimes k} \otimes_A R \\ F^{\otimes k} \otimes_B S & 0 \end{array}\right], \left[\begin{array}{c|c} 0 & E^{\otimes k} \otimes_A R \\ F^{\otimes k} \otimes_B S & 0 \end{array}\right] \right\rangle \\ &= \left\langle F^{\otimes k} \otimes S, F^{\otimes k} \otimes S \right\rangle \oplus \left\langle E^{\otimes k} \otimes R, E^{\otimes k} \otimes R \right\rangle . \end{aligned}$$

This completes the proof. ∎

We will make use of the dilation X_∞ of the bipartite inflation of S by R. Let $(A \oplus B)_\infty$ be the direct limit C*-algebra of the following directed system

$$\begin{aligned} &\left[\begin{array}{c|c} A & 0 \\ 0 & B \end{array}\right] \xrightarrow{\rho_0} \mathcal{K}\left(\left[\begin{array}{c|c} 0 & R \\ S & 0 \end{array}\right]\right) \xrightarrow{\rho_1} \mathcal{K}\left(\left[\begin{array}{c|c} E & 0 \\ 0 & F \end{array}\right]\right) \xrightarrow{\rho_2} \\ &\qquad \xrightarrow{\rho_2} \mathcal{K}\left(\left[\begin{array}{c|c} 0 & E \otimes R \\ F \otimes S & 0 \end{array}\right]\right) \xrightarrow{\rho_3} \mathcal{K}\left(\left[\begin{array}{c|c} E^{\otimes 2} & 0 \\ 0 & F^{\otimes 2} \end{array}\right]\right) \xrightarrow{\rho_4} \cdots , \end{aligned}$$

where

$$\begin{aligned} &\rho_0 = \phi_X \colon A = \mathcal{L}(A) \longrightarrow \mathcal{L}(X), \\ &\rho_n = \otimes \mathrm{id}_X \colon \mathcal{L}(X^{\otimes n}) \longrightarrow \mathcal{L}(X^{\otimes n+1}) \colon r \longmapsto r \otimes \mathrm{id}_X,\ n \geq 1. \end{aligned}$$

Note that, since X and $X^{\otimes 2}$ are regular, $(A \oplus B)_\infty$ is $*$-isomorphic to the direct limit C*-algebra of the following directed system

$$\mathcal{K}\left(\left[\begin{array}{c|c} A & 0 \\ 0 & B \end{array}\right]\right) \xrightarrow{\rho_1 \circ \rho_0} \mathcal{K}\left(\left[\begin{array}{c|c} E & 0 \\ 0 & F \end{array}\right]\right) \xrightarrow{\rho_3 \circ \rho_2} \mathcal{K}\left(\left[\begin{array}{c|c} E^{\otimes 2} & 0 \\ 0 & F^{\otimes 2} \end{array}\right]\right) \xrightarrow{\rho_5 \circ \rho_4} \cdots .$$

Since E and F are orthogonal subcorrespondences of $X^{\otimes 2}$ we get that $(A \oplus B)_\infty = A_\infty \oplus B_\infty$ and we can write A_∞ and B_∞ via the following directed systems

$$\begin{aligned} &A_\infty \colon \mathcal{K}\left(\left[\begin{array}{c|c} A & 0 \\ 0 & 0 \end{array}\right]\right) \xrightarrow{\rho_1 \circ \rho_0} \mathcal{K}\left(\left[\begin{array}{c|c} E & 0 \\ 0 & 0 \end{array}\right]\right) \xrightarrow{\rho_3 \circ \rho_2} \mathcal{K}\left(\left[\begin{array}{c|c} E^{\otimes 2} & 0 \\ 0 & 0 \end{array}\right]\right) \xrightarrow{\rho_5 \circ \rho_4} \cdots , \\ &B_\infty \colon \mathcal{K}\left(\left[\begin{array}{c|c} 0 & 0 \\ 0 & B \end{array}\right]\right) \xrightarrow{\rho_1 \circ \rho_0} \mathcal{K}\left(\left[\begin{array}{c|c} 0 & 0 \\ 0 & F \end{array}\right]\right) \xrightarrow{\rho_3 \circ \rho_2} \mathcal{K}\left(\left[\begin{array}{c|c} 0 & 0 \\ 0 & F^{\otimes 2} \end{array}\right]\right) \xrightarrow{\rho_5 \circ \rho_4} \cdots , \end{aligned}$$

The C*-correspondence X_∞ is defined by the directed system

$$\left[\begin{array}{c|c} 0 & R \\ S & 0 \end{array}\right] \xrightarrow{\tau} \mathcal{K}\left(\left[\begin{array}{c|c} 0 & R \\ S & 0 \end{array}\right], \left[\begin{array}{c|c} E & 0 \\ 0 & F \end{array}\right]\right) \xrightarrow{\sigma_1}$$
$$\xrightarrow{\sigma_1} \mathcal{K}\left(\left[\begin{array}{c|c} E & 0 \\ 0 & F \end{array}\right], \left[\begin{array}{c|c} 0 & E\otimes R \\ F\otimes S & 0 \end{array}\right]\right) \xrightarrow{\sigma_2}$$
$$\xrightarrow{\sigma_2} \mathcal{K}\left(\left[\begin{array}{c|c} 0 & E\otimes R \\ F\otimes S & 0 \end{array}\right], \left[\begin{array}{c|c} E^{\otimes 2} & 0 \\ 0 & F^{\otimes 2} \end{array}\right]\right) \xrightarrow{\sigma_3} \cdots$$

where $\tau_\xi(\eta) = \xi \otimes \eta$, and

$$\sigma_n = \otimes \mathrm{id}_X \colon \mathcal{L}(X^{\otimes n}, X^{\otimes n+1}) \to \mathcal{L}(X^{\otimes n+1}, X^{\otimes n+2}) \colon s \mapsto s \otimes \mathrm{id}_X,\ n \geq 1,$$

Let R_∞ be the subcorrespondence of X_∞ that is generated by the copies of

$$R\,, \mathcal{K}\left(\left[\begin{array}{c|c} 0 & 0 \\ 0 & F^{\otimes k} \end{array}\right], \left[\begin{array}{c|c} 0 & E^{\otimes k}\otimes R \\ 0 & 0 \end{array}\right]\right),$$

and

$$\mathcal{K}\left(\left[\begin{array}{c|c} 0 & 0 \\ S & 0 \end{array}\right], \left[\begin{array}{c|c} E & 0 \\ 0 & 0 \end{array}\right]\right), \mathcal{K}\left(\left[\begin{array}{c|c} 0 & 0 \\ F^{\otimes k}\otimes S & 0 \end{array}\right], \left[\begin{array}{c|c} E^{\otimes k+1} & 0 \\ 0 & 0 \end{array}\right]\right),$$

for $k \geq 1$. Because of Lemma 2.7, R_∞ can be written alternatively as a direct limit in the following two forms

$$\text{(i) } R \overset{\tau\circ(\otimes \mathrm{id}_X)}{\longrightarrow} \mathcal{K}(F, E\otimes R) \overset{\otimes \mathrm{id}_{X^{\otimes 2}}}{\longrightarrow} \mathcal{K}\left(F^{\otimes 2}, E^{\otimes 2}\otimes R\right) \overset{\otimes \mathrm{id}_{X^{\otimes 2}}}{\longrightarrow} \cdots,$$
$$\text{(ii) } R \xrightarrow{\tau} \mathcal{K}(S, E) \overset{\otimes \mathrm{id}_{X^{\otimes 2}}}{\longrightarrow} \mathcal{K}\left(F\otimes S, E^{\otimes 2}\right) \overset{\otimes \mathrm{id}_{X^{\otimes 2}}}{\longrightarrow} \cdots,$$

where we omit the zero entries for convenience. The first form of R_∞ shows that it is a Hilbert B_∞-module, whereas the second form shows that the left multiplication by elements of A_∞ defines a left action of A_∞ on R_∞. Hence R_∞ is a A_∞-B_∞-correspondence.

In a dual way we define S_∞ as the subcorrespondence of X_∞ generated by the copies of

$$S\,, \mathcal{K}\left(\left[\begin{array}{c|c} E^{\otimes k} & 0 \\ 0 & 0 \end{array}\right], \left[\begin{array}{c|c} 0 & 0 \\ 0 & F^{\otimes k}\otimes S \end{array}\right]\right)$$

and

$$\mathcal{K}\left(\left[\begin{array}{c|c} 0 & R \\ 0 & 0 \end{array}\right], \left[\begin{array}{c|c} 0 & 0 \\ 0 & F \end{array}\right]\right), \mathcal{K}\left(\left[\begin{array}{c|c} 0 & E^{\otimes k}\otimes R \\ 0 & 0 \end{array}\right], \left[\begin{array}{c|c} 0 & 0 \\ 0 & F^{\otimes k+1} \end{array}\right]\right),$$

for $k \geq 1$. By using Lemma 2.7 and writing S_∞ as a direct limit in the following two forms

$$\text{(i) } S \overset{\tau\circ(\otimes \mathrm{id}_X)}{\longrightarrow} \mathcal{K}(E, F\otimes S) \overset{\otimes \mathrm{id}_{X^{\otimes 2}}}{\longrightarrow} \mathcal{K}\left(E^{\otimes 2}, F^{\otimes 2}\otimes S\right) \overset{\otimes \mathrm{id}_{X^{\otimes 2}}}{\longrightarrow} \cdots,$$
$$\text{(ii) } S \xrightarrow{\tau} \mathcal{K}(R, F) \overset{\otimes \mathrm{id}_{X^{\otimes 2}}}{\longrightarrow} \mathcal{K}\left(E\otimes R, F^{\otimes 2}\right) \overset{\otimes \mathrm{id}_{X^{\otimes 2}}}{\longrightarrow} \cdots,$$

we get that S_∞ becomes a B_∞-A_∞-correspondence.

We must remark here on our use of the subscript ∞. For the correspondences E, F we denote by E_∞, F_∞ their Pimsner dilations, whereas for R, S we denote by R_∞, S_∞ the subcorrespondences in the Pimsner dilation X_∞

of the (injective) bipartite inflation X of S by R. Nevertheless, when the correspondences are non-degenerate, R_∞ is the X-dilation $(\mathcal{K}(B,R),\mathrm{id}_X)_\infty$ and S_∞ is the X-dilation $(\mathcal{K}(A,S),\mathrm{id}_X)_\infty$.

Remark 5.2. Note that R_∞ and S_∞ are both full left Hilbert bimodules (thus injective). In order to prove this for, say S_∞, it is enough (by its second form) to show that

$$\overline{\mathcal{K}(E^{\otimes n}\otimes R, F^{\otimes n+1})\mathcal{K}(E^{\otimes n}\otimes R, F^{\otimes n+1})^*} = \mathcal{K}(F^{\otimes n+1}, F^{\otimes n+1})$$

for all $n \geq 1$. By Lemma 2.1 it suffices to show that $\langle E^{\otimes n}\otimes R, E^{\otimes n}\otimes R\rangle$ contains a right c.a.i. for $F^{\otimes n+1}$, for all $n\geq 1$, a fact established in Proposition 5.1. A symmetrical argument can be used for R_∞.

Moreover, if S is an imprimitivity bimodule, then S_∞ is also right full, hence an imprimitivity bimodule as well. Indeed, by the first form of S_∞ it suffices to show that

$$\overline{\mathcal{K}(E^{\otimes n}, F^{\otimes n}\otimes S)^*\mathcal{K}(E^{\otimes n}, F^{\otimes n}\otimes S)} = \mathcal{K}(E^{\otimes n}, E^{\otimes n}),$$

for all n. By Lemma 2.1, it is enough to show that $\langle F^{\otimes n}\otimes S, F^{\otimes n}\otimes S\rangle$ provides a right c.a.i. for $E^{\otimes n}$. Note that $F\otimes S = S\otimes E$, hence $F^{\otimes n}\otimes S = S\otimes E^{\otimes n}$, thus $E^{\otimes n}$ is non-degenerate. Therefore

$$\begin{aligned}\left\langle F^{\otimes n}\otimes S, F^{\otimes n}\otimes S\right\rangle &= \left\langle S\otimes E^{\otimes n}, S\otimes E^{\otimes n}\right\rangle\\ &= \left\langle E^{\otimes n}, \phi_{E^{\otimes n}}(\langle S,S\rangle)E^{\otimes n}\right\rangle\\ &= \left\langle E^{\otimes n}, \phi_{E^{\otimes n}}(A)E^{\otimes n}\right\rangle = \left\langle E^{\otimes n}, E^{\otimes n}\right\rangle,\end{aligned}$$

and the latter ideal provides a right c.a.i. for $E^{\otimes n}$.

Theorem 5.3. *Let ${}_AE_A, {}_BF_B$ be regular* C*-*correspondences. If* $E \overset{s}{\sim} F$, *then* $E_\infty \overset{s}{\sim} F_\infty$.

Proof. Assume that E,F are elementary strong shift equivalent via R,S. It suffices to prove that the interior tensor product $S_\infty\otimes_{A_\infty} R_\infty$ is (unitarily equivalent to) F_∞. Then, by duality, $R_\infty\otimes_{B_\infty} S_\infty$ is (unitarily equivalent to) E_∞, hence $E_\infty \overset{s}{\sim} F_\infty$. Towards this end we view S_∞ as the $X^{\otimes 2}$-dilation of $\mathcal{K}(E\otimes R, F^{\otimes 2})$, i.e.,

$$\mathcal{K}\left(E\otimes R, F^{\otimes 2}\right) \overset{\otimes\mathrm{id}_{X^{\otimes 2}}}{\longrightarrow} \mathcal{K}\left(E^{\otimes 2}\otimes R, F^{\otimes 3}\right) \overset{\otimes\mathrm{id}_{X^{\otimes 2}}}{\longrightarrow} \cdots,$$

and R_∞ as the $X^{\otimes 2}$-dilation of $\mathcal{K}(F, E\otimes R)$, i.e.,

$$\mathcal{K}\left(F, E\otimes R\right) \overset{\otimes\mathrm{id}_{X^{\otimes 2}}}{\longrightarrow} \mathcal{K}\left(F^{\otimes 2}, E^{\otimes 2}\otimes R\right) \overset{\otimes\mathrm{id}_{X^{\otimes 2}}}{\longrightarrow} \cdots.$$

Note that A_∞ can be written as $(\mathcal{K}(E\otimes R),\mathrm{id}_{X^{\otimes 2}})_\infty$. By Proposition 5.1, the ideal $\left\langle E^{\otimes k}\otimes R, E^{\otimes k}\otimes R\right\rangle$ provides a right c.a.i. for $F^{\otimes k+1}$, for every $k\geq 0$. Therefore, $\left\langle E\otimes R\otimes (X^{\otimes 2})^{\otimes n}, E\otimes R\otimes (X^{\otimes 2})^{\otimes n}\right\rangle$ provides a right c.a.i. for $F^{\otimes 2}\otimes (X^{\otimes 2})^{\otimes n}$, for every $n\geq 0$. Thus, Proposition 3.5 applies, and we obtain that $S_\infty\otimes_{A_\infty} R_\infty$ is $(\mathcal{K}(F,F^{\otimes 2}),\mathrm{id}_{X^{\otimes 2}})_\infty = F_\infty$. ∎

Remark 5.4. In the above proof we have actually shown that

$$(R \otimes_B S)_\infty = E_\infty = R_\infty \otimes_{B_\infty} S_\infty, \text{ and}$$
$$(S \otimes_A R)_\infty = F_\infty = S_\infty \otimes_{A_\infty} R_\infty.$$

An immediate consequence of Theorem 5.3 is the following.

Theorem 5.5. *Let ${}_AE_A, {}_BF_B$ be regular* C*-*correspondences. If* $E \overset{SSE}{\sim} F$, *then* $E_\infty \overset{SSE}{\sim} F_\infty$.

Proof. Assume that $E \overset{\mathrm{SSE}}{\sim} F$ via a sequence of T_i, for $i = 0, \dots, n$. Then $T_i \overset{\mathrm{s}}{\sim} T_{i+1}$, for $i = 0, \dots, n$. By Theorem 5.3, we get that $(T_i)_\infty \overset{\mathrm{s}}{\sim} (T_{i+1})_\infty$ for $i = 0, \dots, n$. Since $T_0 = E$ and $T_n = F$, then $E_\infty \overset{\mathrm{SSE}}{\sim} F_\infty$. ∎

We use Theorem 5.3 to understand the passage to dilations for the strong Morita equivalence.

Theorem 5.6. *Let ${}_AE_A, {}_BF_B$ be regular* C*-*correspondences. If* $E \overset{SME}{\sim} F$, *then* $E_\infty \overset{SME}{\sim} F_\infty$.

Proof. Assume that $E \overset{\mathrm{SME}}{\sim} F$; then E and F are non-degenerate and $E \overset{\mathrm{s}}{\sim} F$ via $E \otimes_A M^*$ and M (see Theorem 4.7). Therefore, by Theorem 5.3, $E_\infty \overset{\mathrm{s}}{\sim} F_\infty$ via $(E \otimes M^*)_\infty$ and M_∞. Since M is assumed an imprimitivity bimodule, then M_∞ is also an imprimitivity bimodule by Remark 5.2. By Proposition 4.8 we conclude that $E_\infty \overset{\mathrm{SME}}{\sim} F_\infty$. ∎

Remark 5.7. An alternative and more direct way of proving Theorem 5.6 would be to show that $E_\infty \overset{\mathrm{SME}}{\sim} F_\infty$ via the imprimitivity bimodule $\mathfrak{X} = A_\infty \otimes_A X \otimes_B B_\infty$. It may take no effort to show that $E \otimes_A A_\infty = A_\infty \otimes_A E$, as sets, and conclude that $E_\infty \otimes_{A_\infty} \mathfrak{X} = \mathfrak{X} \otimes_{B_\infty} F_\infty$. However there is a problem with the definition of $\mathfrak{X}$. It is easy to define a left and right action of A on A_∞ (simply by multiplication), but in order to get the tensor product $A_\infty \otimes_A X$ (and so $\mathfrak{X}$) an inner product of A_∞ taking values in A is needed. The existence of such an inner product is not obvious to us.

We have arrived to the last relation to be examined when passing to Pimsner dilations. Our next Theorem is one of the central results in this paper and will enable us to obtain new information even for concrete classes of operator algebras, i.e., Cuntz-Krieger C*-algebras.

Theorem 5.8. *Let ${}_AE_A, {}_BF_B$ be regular* C*-*correspondences. If* $E \overset{SE}{\sim} F$ *with lag m, then* $E_\infty \overset{SE}{\sim} F_\infty$ *with lag m.*

Proof. Let R, S such that

(i) $E^{\otimes m} = R \otimes_A S$, $F^{\otimes m} = S \otimes_A R$,

(ii) $S \otimes_A E = F \otimes_B S$, $E \otimes_A R = R \otimes_B F$.

Thus $E^{\otimes m} \overset{\text{s}}{\sim} F^{\otimes m}$. Since E is regular, we have

$$E_\infty = (\mathcal{K}(E, E^{\otimes 2}), \mathrm{id}_E)_\infty = (\mathcal{K}(E^{\otimes 2}, E^{\otimes 3}), \mathrm{id}_E)_\infty.$$

Proposition 3.5 implies that

$$\begin{aligned}(E_\infty)^{\otimes 2} &= (\mathcal{K}(E^{\otimes 2}, E^{\otimes 3}), \mathrm{id}_E)_\infty \otimes_{A_\infty} (\mathcal{K}(E, E^{\otimes 2}), \mathrm{id}_E)_\infty \\ &= (\mathcal{K}(E, E^{\otimes 3}), \mathrm{id}_E)_\infty = (E^{\otimes 2})_\infty.\end{aligned}$$

A repetitive use of this argument shows that $(E^{\otimes m})_\infty = (E^{\otimes m-1})_\infty \otimes E_\infty = (E_\infty)^{\otimes m}$. Thus, by the remark following Theorem 5.3, we obtain

$$(E_\infty)^{\otimes m} = (E^{\otimes m})_\infty = (R \otimes_B S)_\infty = R_\infty \otimes_{B_\infty} S_\infty,$$

and in a similar fashion $(F_\infty)^{\otimes m} = S_\infty \otimes_{A_\infty} R_\infty$.

What remains to be proved, in order to complete the proof, is that

$$E_\infty \otimes_{A_\infty} R_\infty = R_\infty \otimes_{B_\infty} F_\infty.$$

Since E is regular, we get that E_∞ coincides with $(\mathcal{K}(E^{\otimes m}, E^{\otimes m+1}), \mathrm{id}_E)_\infty = (\mathcal{K}(E^{\otimes m}, E^{\otimes m+1}), \mathrm{id}_{X^{\otimes 2}})_\infty$, so

$$E_\infty = (\mathcal{K}(E^{\otimes m}, E^{\otimes m+1}), \mathrm{id}_{X^{\otimes 2}})_\infty \,, R_\infty = (\mathcal{K}(S, E^{\otimes m}), \mathrm{id}_{X^{\otimes 2}})_\infty,$$

where X is the bipartite inflation of R and S.

Now $E^{\otimes m+mn} = E^{\otimes m} \otimes (X^{\otimes 2})^n$, $n \geq 0$, and, by Lemma 2.5, the ideal $\langle E^{\otimes m+mn}, E^{\otimes m+mn} \rangle = \left\langle E^{\otimes m} \otimes (X^{\otimes 2})^n, E^{\otimes m} \otimes (X^{\otimes 2})^n \right\rangle$ provides a right c.a.i. for $E \otimes E^{\otimes m+mn} = E^{\otimes 1+m+mn} = E^{\otimes m+1} \otimes (X^{\otimes 2})^n$. Hence Proposition 3.5 implies that

$$E_\infty \otimes_{A_\infty} R_\infty = (\mathcal{K}(S, E^{\otimes m+1}), \mathrm{id}_{X^{\otimes 2}})_\infty.$$

Similarly, we express

$$R_\infty = (\mathcal{K}(F^{\otimes m}, E^{\otimes m} \otimes R), \mathrm{id}_{X^{\otimes 2}})_\infty \,, F_\infty = (\mathcal{K}(F^{\otimes m-1}, F^{\otimes m}), \mathrm{id}_{X^{\otimes 2}})_\infty;$$

then

$$R_\infty \otimes_{B_\infty} F_\infty = (\mathcal{K}(F^{\otimes m-1}, E^{\otimes m} \otimes R), \mathrm{id}_{X^{\otimes 2}})_\infty,$$

since $F^{\otimes m+mn} = F^{\otimes m} \otimes (X^{\otimes 2})^n$ and $\langle F^{\otimes m+mn}, F^{\otimes m+mn} \rangle$ provides a right c.a.i. for $R \otimes F^{\otimes m+mn} = E^{\otimes m} \otimes R \otimes F^{\otimes mn} = E^{\otimes m} \otimes R \otimes (X^{\otimes 2})^n$.

To prove that $E_\infty \otimes R_\infty = R_\infty \otimes F_\infty$ we show that each one is unitarily equivalent to a submodule of the other. First we observe that for $U = \left[\begin{array}{c|c} 0 & E \otimes R \\ F \otimes S & 0 \end{array}\right]$ we get

$$\begin{aligned} X \otimes U &= \left[\begin{array}{c|c} 0 & R \\ S & 0 \end{array}\right] \otimes \left[\begin{array}{c|c} 0 & E \otimes R \\ F \otimes S & 0 \end{array}\right] \\ &= \left[\begin{array}{c|c} R \otimes F \otimes S & 0 \\ 0 & S \otimes E \otimes R \end{array}\right] \\ &= \left[\begin{array}{c|c} E \otimes R \otimes S & 0 \\ 0 & F \otimes S \otimes R \end{array}\right] = U \otimes X, \end{aligned}$$

Thus $X^{\otimes 2} \otimes U = U \otimes X^{\otimes 2}$ and, by Proposition 3.4, we have that the correspondence

$$R_\infty \otimes_{B_\infty} F_\infty = (\mathcal{K}(F^{\otimes m-1}, E^{\otimes m} \otimes R), \mathrm{id}_{X^{\otimes 2}})_\infty$$

is unitarily equivalent, via some unitary u, to a submodule of

$$\begin{aligned}
&(\mathcal{K}(F^{\otimes m-1} \otimes U, E^{\otimes m} \otimes R \otimes U), \mathrm{id}_{X^{\otimes 2}})_\infty = \\
&\qquad = (\mathcal{K}(F^{\otimes m-1} \otimes F \otimes S, E^{\otimes m+1} \otimes R \otimes S), \mathrm{id}_{X^{\otimes 2}})_\infty \\
&\qquad = (\mathcal{K}(S \otimes R \otimes S, E^{\otimes m+1} \otimes R \otimes S), \mathrm{id}_{X^{\otimes 2}})_\infty \\
&\qquad = (\mathcal{K}(S \otimes X^{\otimes 2}, E^{\otimes m+1} \otimes X^{\otimes 2}), \mathrm{id}_{X^{\otimes 2}})_\infty \\
&\qquad = (\mathcal{K}(S, E^{\otimes m+1}), \mathrm{id}_{X^{\otimes 2}})_\infty \\
&\qquad = E_\infty \otimes R_\infty.
\end{aligned}$$

Also define $V = \left[\begin{array}{c|c} 0 & E^{\otimes m-1} \otimes R \\ F^{\otimes m-1} \otimes S & 0 \end{array}\right]$ and verify that $X \otimes V = V \otimes X$. Thus, again by Proposition 3.4, the correspondence

$$E_\infty \otimes R_\infty = (\mathcal{K}(S, E^{\otimes m+1}), \mathrm{id}_{X^{\otimes 2}})_\infty$$

is unitarily equivalent, via some unitary v, to a submodule of

$$\begin{aligned}
&(\mathcal{K}(S \otimes V, E^{\otimes m+1} \otimes V), \mathrm{id}_{X^{\otimes 2}}) = \\
&\qquad = (\mathcal{K}(F^{\otimes 2m-1}, E^{\otimes 2m} \otimes R), \mathrm{id}_{X^{\otimes 2}})_\infty \\
&\qquad = (\mathcal{K}(F^{\otimes m-1} \otimes X^{\otimes 2}, E^{\otimes m} \otimes X^{\otimes 2} \otimes R), \mathrm{id}_{X^{\otimes 2}})_\infty \\
&\qquad = (\mathcal{K}(F^{\otimes m-1} \otimes X^{\otimes 2}, E^{\otimes m} \otimes R \otimes X^{\otimes 2}), \mathrm{id}_{X^{\otimes 2}})_\infty \\
&\qquad = (\mathcal{K}(F^{\otimes m-1}, E^{\otimes m} \otimes R), \mathrm{id}_{X^{\otimes 2}})_\infty \\
&\qquad = R_\infty \otimes F_\infty.
\end{aligned}$$

It remains to show that $uv = \mathrm{id}$ and $vu = \mathrm{id}$. Note that uv maps an element $[q]$ of any compact shift considered above (either it is in a C*-algebra, either it is an element of the module) to $[q \otimes \mathrm{id}_{U \otimes V}] = [q \otimes \mathrm{id}_{X^{\otimes 4}}]$, since

$$\begin{aligned}
U \otimes V &= \left[\begin{array}{c|c} 0 & E \otimes R \\ F \otimes S & 0 \end{array}\right] \otimes \left[\begin{array}{c|c} 0 & E^{\otimes m-1} \otimes R \\ F^{\otimes m-1} \otimes S & 0 \end{array}\right] \\
&= \left[\begin{array}{c|c} E \otimes R \otimes F^{\otimes m-1} \otimes S & 0 \\ 0 & F \otimes S \otimes E^{\otimes m-1} \otimes R \end{array}\right] \\
&= \left[\begin{array}{c|c} E \otimes E^{\otimes m-1} \otimes R \otimes S & 0 \\ 0 & F \otimes F^{\otimes m-1} \otimes S \otimes R \end{array}\right] \\
&= \left[\begin{array}{c|c} E^{\otimes m} \otimes E^{\otimes m} & 0 \\ 0 & F^{\otimes m} \otimes F^{\otimes m} \end{array}\right] = X^{\otimes 4}.
\end{aligned}$$

However, throughout the proof we have used exclusively $X^{\otimes 2}$-dilations and so $[q \otimes \mathrm{id}_{X^{\otimes 4}}] = [q]$. Therefore $uv = \mathrm{id}$. In a similar way $vu = \mathrm{id}$, since $V \otimes U = X^{\otimes 4}$, and the proof is complete. ∎

As mentioned in the Introduction, we were strongly motivated by [**13**, Remark 5.5] and one of our aims was to check whether the alternative way proposed there to prove [**13**, Theorem 3.14] could be achieved, i.e., to show that if $E \overset{\text{s}}{\sim} F$ then $E_\infty \overset{\text{SME}}{\sim} F_\infty$. There is, though, a delicate point in this approach. In [**13**, Remark 5.5] the authors claim that if E is a non-degenerate and regular C*-correspondence, then the dilation E_∞ as introduced by Pimsner [**16**] is an imprimitivity bimodule. This is true, but only in the context of [**16**], because the C*-correspondences there are always assumed full [**16**, Remark 1.2(3)]. This is a consequence of another delicate point in Pimsner's theory, as his version of C*-algebras, unfortunately denoted by the same symbols $\mathcal{T}_E$ and $\mathcal{O}_E$, are not what have eventually become the usual C*-algebras generated by the images of X and A, but they are only generated by the image of X; hence there is no reason to make a distinction between full and non-full correspondences in Pimsner's theory. Note that when X is regular one can recover A in Pimsner's C*-algebra $\mathcal{O}_E$, but this is not the case for $\mathcal{T}_E$.

Of course, if one adds this extra element, then the scheme in [**13**, Remark 5.5] can be implemented, as we are about to show. Note that the previous discussion and the next results settles Conjecture 1 appearing in the Introduction.

Theorem 5.9. *Let ${}_AE_A$ and ${}_BF_B$ be full, non-degenerate and regular* C*-*correspondences. Then the following scheme holds*

$$\begin{array}{ccccccc} E \overset{SME}{\sim} F & \Longrightarrow & E \overset{s}{\sim} F & \Longrightarrow & E \overset{SSE}{\sim} F & \Longrightarrow & E \overset{SE}{\sim} F \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ E_\infty \overset{SME}{\sim} F_\infty & \Longleftrightarrow & E_\infty \overset{s}{\sim} F_\infty & \Longleftrightarrow & E_\infty \overset{SSE}{\sim} F_\infty & \Longleftrightarrow & E_\infty \overset{SE}{\sim} F_\infty \end{array}$$

Proof. E and F are full and non-degenerate, thus E_∞ and F_∞ are imprimitivity bimodules, and Theorem 6.1 (that will follow) applies. ∎

An immediate consequence of Theorem 5.9 and Theorem 3.2 is the following.

Theorem 5.10. *Let ${}_AE_A$ and ${}_BF_B$ be full, non-degenerate and regular* C*-*correspondences. If $E \overset{SE}{\sim} F$, then the corresponding Cuntz-Pimsner algebras are strong Morita equivalent.*

Proof. Suppose that $E \overset{\text{SE}}{\sim} F$. Then by Theorem 5.9 we have that $E_\infty \overset{\text{SME}}{\sim} F_\infty$. Therefore [**15**, Theorem 3.5] implies that $\mathcal{O}_{E_\infty} \overset{\text{SME}}{\sim} \mathcal{O}_{F_\infty}$ and the conclusion follows from Theorem 3.2. ∎

In particular we obtain the following result for Cuntz-Krieger C*-algebras mentioned in the Introduction.

Corollary 5.11. *Let $\mathcal{G}$ and $\mathcal{G}'$ be finite graphs with no sinks or sources and let $A_\mathcal{G}$ and $A_{\mathcal{G}'}$ be their adjacent matrices. If $A_\mathcal{G} \overset{SE}{\sim} A_{\mathcal{G}'}$, in the sense of*

Williams, then the Cuntz-Krieger C^**-algebras* $\mathcal{O}_{\mathcal{G}}$ *and* $\mathcal{O}_{\mathcal{G}'}$ *are strong Morita equivalent.*

There is also an application to unital injective dynamical systems.

Corollary 5.12. *Let* (A,α) *and* (B,β) *be unital injective dynamical systems. If* $X_\alpha \overset{SE}{\sim} X_\beta$*, then* $X_{\alpha_\infty} \overset{SE}{\sim} Y_{\beta_\infty}$ *and the crossed products* $A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}$ *and* $B \rtimes_{\beta_\infty} \mathbb{Z}$ *are strong Morita equivalent.*

We close this section by settling Conjecture 2 of the Introduction. This conjecture asserts that the vertical arrows in Theorem 5.9 are actually equivalences. We will use the following.

Theorem 5.13. *Let* ${}_AE_A$ *be a full, non-degenerate and regular* C^**- correspondence. If* $E \overset{SE}{\sim} E_\infty$ *then* E *is an imprimitivity bimodule.*

Proof. By assumption there exist non-degenerate, regular C^*- correspondences ${}_AR_{A_\infty}, {}_{A_\infty}S_A$ and a positive integer m such that

$$E^{\otimes m} = R \otimes S,\ (E_\infty)^{\otimes m} = (E^{\otimes m})_\infty = S \otimes R$$

and $E \otimes R = R \otimes E_\infty$, $S \otimes E = E_\infty \otimes S$. Then

$$A = \left\langle E^{\otimes m}, E^{\otimes m} \right\rangle = \langle R \otimes S, R \otimes S \rangle = \langle S, \langle R, R \rangle S \rangle \subseteq \langle S, S \rangle \subseteq A,$$

since $E^{\otimes m}$ is also full. Hence S is full. Now, let $k \in \mathcal{K}(S)$. Then $k \otimes \mathrm{id}_R \in \mathcal{K}((E_\infty)^{\otimes m})$, and since $(E_\infty)^{\otimes m}$ is an imprimitivity bimodule, there is an $x \in A_\infty$ such that $k \otimes \mathrm{id}_R = \phi_{(E_\infty)^{\otimes m}}(x) = \phi_S(x) \otimes \mathrm{id}_R$. Thus $\phi_S(x) = k$, since R is regular; therefore $A_\infty \simeq \mathcal{K}(S)$, hence S is an imprimitivity bimodule. Going back to the definition of $E \overset{\mathrm{SE}}{\sim} E_\infty$, we see that $S \otimes E = E_\infty \otimes S$, hence $E = S^* \otimes E_\infty \otimes S$. The fact that S, S^* and E_∞ are imprimitivity bimodules implies that E is also an imprimitivity bimodule. ∎

Let us see now why Conjecture 2 has a negative answer. Let E be any full, non-degenerate and regular C^*-correspondence, which is not an imprimitivity bimodule. If Conjecture 2 was true, then $E \overset{\mathrm{SE}}{\sim} E_\infty$, since both E and E_∞ have unitarily equivalent Pimsner dilations. But then Theorem 5.13 would imply that E is an imprimitivity bimodule, a contradiction.

6. Imprimitivity Bimodules and the Shift Equivalence Problem

Our work on shift equivalences suggests the following generalization of the Williams Problem in the context of C^*-correspondences.

Shift Equivalence Problem for C^*-correspondences. If E, F are non-degenerate, regular C^*-correspondences and $E \overset{\mathrm{SE}}{\sim} F$, then $E \overset{\mathrm{SSE}}{\sim} F$.

One might be tempted to say that the work of Kim and Roush [**8**] readily shows that the answer to the above conjecture is negative. However, it is not known to us whether two graph C^*-correspondences which fail to be strong shift equivalent *via non-negative integral matrices* remain inequivalent if one considers arbitrary C*-correspondences in order to implement the strong shift equivalence. In other words, we do not know the answer to the Williams Problem even for the class of graph C*-correspondences. If it does have a positive answer then it will provide an alternative route for proving our Corollary 5.11, but we conjecture that it doesn't in general.

Nevertheless, in our next result we show that Shift Equivalence Problem has an affirmative answer for the class of imprimitivity bimodules. Recall that this result is essential for the proof of Theorem 5.9.

Theorem 6.1. *Strong Morita equivalence, elementary strong shift equivalence, strong shift equivalence and shift equivalence are equivalent for the class of imprimitivity bimodules.*

Proof. In view of Theorem 4.7, it suffices to show that if ${}_AE_A$ and ${}_BF_B$ are imprimitivity bimodules such that $E \overset{\text{SE}}{\sim} F$, then $E \overset{\text{SME}}{\sim} F$. Assume that $E \overset{\text{SE}}{\sim} F$ via R, S with lag m. Then $E^{\otimes m} = R \otimes S$ and $F^{\otimes m} = S \otimes R$. Since E, F are imprimitivity bimodules, then $E^{\otimes m}$ and $F^{\otimes m}$ are also imprimitivity bimodules. Hence S is regular and, due to $\overset{\text{SE}}{\sim}$, intertwines E and F. It suffices to prove that S is an imprimitivity bimodule. First, it is full right since

$$A = \left\langle E^{\otimes m}, E^{\otimes m} \right\rangle = \langle R \otimes S, R \otimes S \rangle = \langle S, \langle R, R \rangle \cdot S \rangle \subseteq \langle S, S \rangle \subseteq A,$$

thus $\langle S, S \rangle = A$. In order to prove that it is full left, it suffices to prove that $\phi_S \colon B \to \mathcal{L}(S)$ is onto $\mathcal{K}(S)$ (since $\phi_S(B) \subseteq \mathcal{K}(S)$, by regularity of S). To this end, let $k \in \mathcal{K}(S)$. Then, due to regularity of R, we obtain that $k \otimes \mathrm{id}_R \in \mathcal{K}(S \otimes R) = \mathcal{K}(F^{\otimes m})$. Since $F^{\otimes m}$ is an imprimitivity bimodule, there is a $b \in B$ such that $\phi_{F^{\otimes m}}(b) = k \otimes \mathrm{id}_R$; in particular $\phi_S(b) \otimes \mathrm{id}_R = k \otimes \mathrm{id}_R$. Thus $\phi_S(b) = k$, since R is regular. ∎

Combining [**15**, Theorem 3.2, Theorem 3.5] with Theorem 6.1 we obtain the following corollary.

Corollary 6.2. *If ${}_AE_A, {}_BF_B$ are imprimitivity bimodules and $E \overset{SME}{\sim} F, E \overset{s}{\sim} F, E \overset{SSE}{\sim} F$ or $E \overset{SE}{\sim} F$, then the corresponding Toeplitz-Cuntz-Pimsner algebras and Cuntz-Pimsner algebras are strong Morita equivalent as* C**-algebras, and the corresponding tensor algebras are strong Morita equivalent in the sense of* [**4**].

7. Other Applications

7.1. **Extension of [13, Theorem 3.14].** A natural question raised in [**13**] was whether [**13**, Theorem 3.14] is valid without the extra assumption of non-degeneracy. This can be established now with the theory we have developed in Section 5. However, one cannot dispose of regularity, as mentioned explicitly in [**13**] and [**2**, Example 5.4].

Theorem 7.1. *Let ${}_AE_A$ and ${}_BF_B$ be regular* C*-*correspondences. If* $E \overset{SSE}{\sim} F$ *then* $\mathcal{O}_E \overset{SME}{\sim} \mathcal{O}_F$.

Proof. First suppose that $E \overset{\text{s}}{\sim} F$. By Theorem 5.3 we have $E_\infty \overset{\text{s}}{\sim} F_\infty$. But E_∞ and F_∞ are regular Hilbert bimodules hence they are non-degenerate. Therefore [**13**, Theorem 3.14] implies $\mathcal{O}_{E_\infty} \overset{\text{SME}}{\sim} \mathcal{O}_{F_\infty}$ and Theorem 3.2 completes the proof in the case of elementary strong shift equivalence.

Let $E \overset{\text{SSE}}{\sim} F$ via a sequence of T_i, $i = 0, \dots, n$. Then $\mathcal{O}_{T_i} \overset{\text{SME}}{\sim} \mathcal{O}_{T_{i+1}}$, by the previous arguments, for every $i = 0, \dots, n-1$. Strong Morita equivalence of C*-algebras is transitive, hence $\mathcal{O}_E = \mathcal{O}_{T_0} \overset{\text{SME}}{\sim} \mathcal{O}_{T_n} = \mathcal{O}_F$. ∎

7.2. [**13**, Proposition 4.2] **Revisited.** The results in [**13**] and here, concerning strong Morita equivalence of the Cuntz-Pimsner algebras, can be generalized for degenerate correspondences over unital C*- algebras because of [**13**, Proposition 4.2], i.e., if X is a correspondence over a unital C*-algebra A, then $\mathcal{O}_{X_{ess}}$ is a full corner of $\mathcal{O}_X$, where $X_{ess} := \overline{\phi_X(A)X}$. In fact the proofs in [**13**, Proposition 4.2] apply in general for strict correspondences X. The key observation is that, if (a_i) is a c.a.i. in A, then $\phi_X(a_i)$ converges in the s*-topology to a projection, say p, in $\mathcal{L}(X)$. As a consequence $J_X = J_{X_{ess}}$ and, if X is regular, then so is X_{ess}, in the same way as in [**13**, Proposition 4.1].

Theorem 7.2. *Let ${}_AX_A$ be a strict* C*-*correspondence. Then $\mathcal{O}_{X_{ess}}$ is a full corner of $\mathcal{O}_X$.*

Proof. Fix a covariant injective representation (π, t) of X (that admits a gauge action); then $(\pi, t|_{X_{ess}})$ is a covariant injective representation of X_{ess} (that admits a gauge action). Let P be the projection in $\mathcal{M}(\mathcal{O}_X)$ that is defined by $\lim_i \pi(a_i)$ for a c.a.i. (a_i) of A; for example

$$Pt(\xi) := \lim_i \pi(a_i)t(\xi) = t(\lim_i \phi(a_i)\xi) = t(p\xi), \text{ for all } \xi \in X.$$

Then, the rest of the proof goes as the proof of [**13**, Proposition 4.2]. ∎

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