OPERATOR ALGEBRAS OF HIGHER RANK NUMERICAL SEMIGROUPS

EVGENIOS T.A. KAKARIADIS, ELIAS G. KATSOULIS, AND XIN LI

ABSTRACT. A higher rank numerical semigroup is a positive cone whose seminormalization is isomorphic to the free abelian semigroup. We show that the induced nonselfadjoint semigroup algebras form a complete isomorphic invariant for this class of semigroups.

1. INTRODUCTION

Semigroup C*-algebras, i.e., C*-algebras generated by left regular representations of left-cancellative semigroups, form a natural class of C*-algebras which generalize (reduced) group C*-algebras. They have been studied for various classes of semigroups, for instance positive cones in totally ordered groups **[2, 14, 22]**, examples coming from group theory **[4, 5, 21]** or examples of number-theoretic origin **[6, 7, 8]**. We refer the reader to **[9]** and the references therein.

Nonselfadjoint semigroup operator algebras are formed by considering the non-involutive part of the left regular representation, or through families of representations. They are trivially examples of semicrossed products, a construct introduced by Arveson **[1]** and formalized by Peters **[23]**, that enraptures the properties of semigroup actions on C*-algebras. Actions over $\mathbb{Z}_+^d$ and $\mathbb{F}_+^d$ (the free semigroup on d generators) are by now well studied, with a comprehensive list of pertinent papers being impossiblele to present here. We direct the interested reader to the surveys **[11, 12, 19]** for more information. However less is known for other semigroups, even at the level of the semigroup algebras, with only recent dilation results obtained for lattice-ordered semigroups **[10, 20]**.

In this paper we focus on a particular class, which we call higher rank numerical semigroups as they generalize classical numerical semigroups in a natural way. These are positive cones in $\mathbb{Z}^d$ whose seminormalization is isomorphic to $\mathbb{Z}_+^d$. Our main result (Theorem 3.8) says that two higher rank numerical semigroups are isomorphic if and only if their nonselfadjoint operator algebras are isomorphic. In fact we show that being completely isometric isomorphic coincides with being algebraic isomorphic.

This rigidity of nonselfadjoint operator algebras is another example of stark contrast with the C*-algebra setting. The difference is particularly striking for classical numerical semigroups, i.e., subsemigroups of the natural numbers (with respect to addition) which have finite complement. For such semigroups, it is straightforward to check that they all have isomorphic semigroup C*-algebras. However, the nonselfadjoint operator algebra remembers the numerical semigroup completely.

The realm of semigroups is too vast to be treated in one stroke and it is appropriate to reflect on a case-by-case study. The main tool we are using here is the canonical embedding of a semigroup into $\mathbb{Z}_+^d$, and its effect on the character spaces. We show that this embedding is a homeomorphism exactly for the class of higher rank numerical semigroups. The metamathematical conclusion is that this is the widest class this tool tackles.

2. POSITIVE CONES

We give some preliminary results for positive cones. We will write $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. The reader should be familiar with the general theory of nonselfadjoint

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operator algebras and dilations of their representations, that we will avoid repeating here. We just recall that the C*-envelope $\mathrm{C}^*_{\text{env}}(\mathcal{A})$ of a nonselfadjoint operator algebra $\mathcal{A}$ is the co-universal C*-algebra in the sense that: (i) there exists a completely isometric homomorphism $i\colon \mathcal{A} \to \mathrm{C}^*_{\text{env}}(\mathcal{A}) = \mathrm{C}^*(i(\mathcal{A}))$; and (ii) for any other completely isometric homomorphism $j\colon \mathcal{A} \to \mathrm{C}^*(j(\mathcal{A}))$ there exists a unique $*$-epimorphism $\Phi\colon \mathrm{C}^*(j(\mathcal{A})) \to \mathrm{C}^*_{\text{env}}(\mathcal{A})$ such that $\Phi \circ j = i$. It follows by **[15]** that $\mathrm{C}^*_{\text{env}}(\mathcal{A}) = \mathrm{C}^*(\rho(\mathcal{A}))$ for any completely isometric representation of $\mathcal{A}$ that does not admit non-trivial contractive dilations.

Recall that a *positive cone* $\mathcal{S}$ of an abelian group $\mathcal{G}$ is a unital sub-semigroup of $\mathcal{G}$ such that: (i) $\mathcal{S} \cap (-\mathcal{S}) = (0)$; and (ii) for every $g \in \mathcal{G}$ there exist $s, t \in \mathcal{S}$ such that $g = s - t$. The *Fock representation* $V: \mathcal{S} \to \mathcal{B}(\ell^2(\mathcal{S}))$ is given by

$$V_s e_t = e_{s+t}.$$

We define

$$\mathcal{A}(\mathcal{S}) := \overline{\mathrm{alg}}\{V_s \mid s \in \mathcal{S}\} \quad \text{and} \quad \mathrm{C}^*(\mathcal{S}) := \mathrm{C}^*(V_s \mid s \in \mathcal{S}).$$

Since $V_s V_t = V_{s+t}$ we get that $\mathcal{A}(\mathcal{S})$ is in fact densely spanned by the monomials V_s. Notice that the Fock representation does not use any property of the positive cone and can be defined for general semigroups, including groups of course. To allow comparisons however we reserve the notation $U\colon \mathcal{G} \to \mathcal{B}(\ell^2(\mathcal{G}))$ for the left regular group representation of an abelian group $\mathcal{G}$. In this case $\mathrm{C}^*(\mathcal{G})$ is the usual group C*-algebra.

The following proposition can be derived as an application of dilation results for C*-dynamics that have appeared for example in **[10, 17]** when applied for trivial dynamical systems. In the absence of the dynamics, a simpler proof can be given, included here.

Proposition 2.1. *Let $\mathcal{S}$ be a positive cone of an abelian group $\mathcal{G}$. Then the mapping $V_s \mapsto U_s$ extends to a completely isometric map $\rho\colon \mathcal{A}(\mathcal{S}) \to \mathrm{C}^*(\mathcal{G})$.*

Proof. It suffices to show that the unique linear extension is isometric on polynomials, i.e. we will show that

$$\| \sum_{s \in F} \lambda_s V_s \| = \| \sum_{s \in F} \lambda_s U_s \| \quad \text{for all finite } F \subset \mathcal{S}.$$

Similar arguments at any matrix level yield that this mapping ρ is completely isometric. By identifying $\ell^2(\mathcal{S})$ with the obvious subspace inside $\ell^2(\mathcal{G})$ we get that

$$\| \sum_{s \in F} \lambda_s V_s \| = \| P_{\ell^2(\mathcal{S})} \left(\sum_{s \in F} \lambda_s U_s \right) |_{\ell^2(\mathcal{S})} \| \leq \| \sum_{s \in \mathcal{S}} \lambda_s U_s \|.$$

For the reverse inequality fix $\varepsilon > 0$. Let $\xi = \sum_{i=1}^n k_i e_{g_i}$ in the unit ball of $\ell^2(\mathcal{G})$ such that

$$\| \sum_{s \in F} \lambda_s U_s \| - \varepsilon \leq \| \sum_{s \in F} \lambda_s U_s \xi \|_{\ell^2(\mathcal{G})}.$$

Since $\mathcal{S}$ is a positive cone we have that there are $s_i, t_i \in \mathcal{S}$ such that $g_i = s_i - t_i$ for all $i = 1, \dots, n$. Set $t := \sum_{i=1}^n t_i \in \mathcal{S}$ so that $t + g_i + F \subset \mathcal{S}$ for all $i = 1, \dots, n$. Then the vector

$$\xi' := U_t \xi = \sum_{i=1}^n k_i e_{t+g_i}$$

is in the unit ball of $\ell^2(\mathcal{S})$. Therefore we obtain

$$\begin{aligned} \| \sum_{s \in F} \lambda_s U_s \| - \varepsilon &\leq \| U_t \sum_{s \in F} \lambda_s U_s \xi \|_{\ell^2(\mathcal{G})} = \| \sum_{s \in F} \lambda_s U_s U_t \xi \|_{\ell^2(\mathcal{G})} \\ &= \| \sum_{s \in F} \lambda_s U_s \xi' \|_{\ell^2(\mathcal{G})} = \| \sum_{s \in F} \lambda_s V_s \xi' \|_{\ell^2(\mathcal{S})} \leq \| \sum_{s \in F} \lambda_s V_s \|. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary we have equality of the norms. ∎

Corollary 2.2. *Let $\mathcal{S}$ be a positive cone of an abelian group $\mathcal{G}$. Then $\mathrm{C}^*(\mathcal{G})$ is the C*-envelope of the semigroup algebra $\mathcal{A}(\mathcal{S})$.*

Proof. By Proposition 2.1 we have that $\mathcal{A}(\mathcal{S}) \hookrightarrow \mathrm{C}^*(\mathcal{G})$ completely isometrically. As the copy of $\mathcal{A}(\mathcal{S})$ contains the generators of $\mathrm{C}^*(\mathcal{G})$ we get that $\mathrm{C}^*(\mathcal{G})$ is a C*-cover of $\mathcal{A}(\mathcal{G})$ and it is generated by unitaries. Since a contractive dilation of a unitary is trivial we get that this is a maximal representation of $\mathcal{A}(\mathcal{S})$ and thus $\mathrm{C}^*(\mathcal{G})$ is the C*-envelope **[15]**. ∎

The completely contractive representations of $\mathcal{A}(\mathcal{S})$ are characterized in the following theorem.

Theorem 2.3. *Let $\mathcal{S}$ be a positive cone of an abelian group $\mathcal{G}$. A representation $\rho\colon \mathcal{A}(\mathcal{S}) \to \mathcal{B}(\mathcal{H})$ is completely contractive if and only if there is a unitary representation $U\colon \mathcal{G} \to \mathcal{B}(\mathcal{K})$ for $\mathcal{K} \supseteq \mathcal{H}$ such that $\rho(V_s) = P_{\mathcal{H}} U_s|_{\mathcal{H}}$ for all $s \in \mathcal{S}$.*

Proof. By Proposition 2.1 we have that $\mathcal{A}(\mathcal{S}) \subset \mathrm{C}^*(\mathcal{G})$. If ρ is a completely contractive representation of $\mathcal{A}(\mathcal{S})$ then it extends to a completely contractive map of $\mathrm{C}^*(\mathcal{G})$ and Stinespring's Theorem produces the required unitary representation. Conversely, if we have such a unitary representation then its compression to $\mathcal{H}$ is a completely contractive map of $\mathrm{C}^*(\mathcal{G})$. Hence the restriction ρ to $\mathcal{A}(\mathcal{S})$ is a completely contractive map. ∎

Remark 2.4. There is a small subtlety as the above does not imply that commuting contractions induce a completely contractive representation even when $\mathcal{S} \subset \mathbb{Z}$. One would expect that starting with a contraction T we can build a representation of $\mathcal{A}(\mathcal{S})$ by assigning $V_n \mapsto T^n$. However, in **[16]** it is shown that there is a contraction that fails to induce a completely contractive representation for the semigroup $\mathcal{S} = \{0, 2, 3, \dots\}$. The example given fails to be 2-contractive.

Henceforth we will restrict our attention to $\mathcal{G} = \mathbb{Z}^d$ for some finite d. For a positive cone $\mathcal{S} \subset \mathbb{Z}_+^d$ of $\mathbb{Z}^d$ we make the following identifications

$$\mathcal{S} \ni s \equiv V_s \equiv z^s \in \mathbb{A}(\mathbb{D}^d).$$

By Corollary 2.2 we have a canonical identification $\mathcal{A}(\mathcal{S}) \subset \mathcal{A}(\mathbb{Z}_+^d) \subset \mathrm{C}^*(\mathbb{Z}^d)$. Hence we can use on $\mathcal{A}(\mathcal{S})$ the Fourier transform inherited from $\mathrm{C}^*(\mathbb{Z}^d)$. A straightforward application gives the following corollary.

Corollary 2.5. *Let $\mathcal{S} \subset \mathbb{Z}_+^d$ be a positive cone in $\mathbb{Z}^d$ and let $f \in \mathcal{A}(\mathbb{Z}_+^d)$. Then $f \in \mathcal{A}(\mathcal{S})$ if and only if $\{s \in S \mid f^{(s)}(0) \neq 0\} \subset \mathcal{S}$. Therefore, $s \in \mathcal{S}$ if and only if there exists an $f \in \mathcal{A}(\mathcal{S})$ such that $f^{(s)}(0) \neq 0$.*

Corollary 2.6. *Let $\mathcal{S}_1 \subset \mathbb{Z}_+^{d_1}$ and $\mathcal{S}_2 \subset \mathbb{Z}_+^{d_2}$ be positive coves in $\mathbb{Z}^{d_1}$ and $\mathbb{Z}^{d_2}$ respectively. If $\mathcal{A}(\mathcal{S}_1)$ and $\mathcal{A}(S_2)$ are completely isometrically isomorphic then $d_1 = d_2$.*

Proof. Being completely isometrically isomorphic yields that the associated C*-envelopes $\mathrm{C}^*(\mathbb{Z}^{d_1})$ and $\mathrm{C}^*(\mathbb{Z}^{d_2})$ are $*$-isomorphic. ∎

If $\rho\colon \mathcal{A} \to \mathcal{B}$ is an algebraic epimorphism for the Banach algebras $\mathcal{A}$ and $\mathcal{B}$, then the discontinuity of ρ is quantified by the ideal

$$\mathfrak{S}(\rho) := \{b \in \mathcal{B} \mid \exists (a_n) \subset \mathcal{A} \text{ such that } a_n \to 0 \text{ and } \rho(a_n) \to b\}.$$

By the closed graph theorem ρ is continuous if and only if $\mathfrak{S}(\rho) = (0)$. Due to a result of Sinclair **[26]**, for any sequence (b_n) in $\mathcal{B}$ there exists an $N \in \mathbb{N}$ such that

$$\overline{b_1 \cdots b_N \mathfrak{S}(\rho)} = \overline{b_1 \cdots b_n \mathfrak{S}(\rho)} \text{ and } \overline{\mathfrak{S}(\rho) b_n \cdots b_1} = \overline{\mathfrak{S}(\rho) b_N \cdots b_1}, \text{ for all } n \geq N.$$

Proposition 2.7. *Let $\mathcal{S} \subset \mathbb{Z}_+^d$ be a positive cone in $\mathbb{Z}^d$. Then any algebraic epimorphism $\rho\colon \mathcal{A} \to \mathcal{A}(\mathcal{S})$ for any Banach algebra $\mathcal{A}$ is automatically continuous.*

Proof. Fix $s \in \mathcal{S}$. The Fourier transform yields $\bigcap_{n=0}^\infty V_s^n \mathcal{I} = (0)$ for any ideal $\mathcal{I} \subset \mathcal{A}(\mathcal{S})$. Then by using the isometries $b_n = V_s^n$ we get that $\mathfrak{S}(\rho) = \bigcap_{n=0}^\infty V_s^n \mathfrak{S}(\rho) = (0)$. ∎

3. Numerical semigroups

Recall that a positive cone $\mathcal{S}$ of a group $\mathcal{G}$ is called *seminormal* if whenever $3s = 2t$ for $s, t \in \mathcal{S}$ then there exists a (necessarily unique) $p \in \mathcal{S}$ such that $2p = s$ and $3p = t$ **[3**, Definition 1.7**]**; equivalently if $p = t - s$ is in $\mathcal{S}$. Every positive cone admits a *seminormalization* which by **[3**, Example 1.12**]** can be expressed as

$$\mathcal{S}_{\mathrm{sn}} := \{g \in \mathcal{G} \mid ng \in \mathcal{S} \text{ eventually for all } n \in \mathbb{N}\}.$$

Alternatively $\mathcal{S}_{\mathrm{sn}}$ is the universal seminormal monoid that contains an injective copy of $\mathcal{S}$ **[3**, Lemma 1.7**]**. The seminormalization of a positive cone is itself a positive cone (for the same generating group).

Remark 3.1. Positive cones in $\mathbb{Z}$ are also known as numerical semigroups and have several equivalent characterizations. For example $\mathcal{S}$ is a numerical semigroup, if and only if $\gcd(\mathcal{S}) = 1$ if and only if there is an $N \in \mathcal{S}$ such that $n \in S$ for all $n > N$, if and only $\mathcal{S}_{\mathrm{sn}} = \mathbb{Z}_+$. We will consider their higher rank analogue.

Definition 3.2. A positive cone $\mathcal{S}$ of a group $\mathcal{G}$ is called a *higher rank numerical semigroup* if $\mathcal{S}_{\mathrm{sn}} \simeq \mathbb{Z}_+^d$. If $d = 1$ then $\mathcal{S}$ is called simply a *numerical semigroup*.

Remark 3.3. The above definition implies several items. First of all it is not hard to see that an isomorphism $\sigma \colon \mathcal{S}_1 \to \mathcal{S}_2$ between two positive cones induces an isomorphism $\widetilde{\sigma} \colon \mathcal{G}_1 \to \mathcal{G}_2$ of their generating groups given by $\widetilde{\sigma}(s - t) = \sigma(s) - \sigma(t)$. Moreover $\widetilde{\sigma}$ restrics to an isomorphism of the seminormalizations. Therefore if $\mathcal{S} \subset \mathcal{G}$ is a higher rank numerical semigroup then $\mathcal{S} \hookrightarrow \mathbb{Z}_+^d$ and $\mathcal{G} \simeq \mathbb{Z}^d$.

We will thus restrict to positive cones of $\mathbb{Z}^d$ with seminormalization *equal* to $\mathbb{Z}_+^d$. For notational purposes we write $\{\mathbf{1}, \dots, \mathbf{d}\}$ for the usual generators in $\mathbb{Z}^d$. We will also use the multivariable notation $z^s = z^{s_1} \cdots z^{s_d}$ for $z = (z_1, \dots, z_d) \in \mathbb{C}^d$ and $s = (s_1, \dots, s_d) \in \mathbb{Z}^d$.

Let $\mathcal{S} \subset \mathbb{Z}_+^d$ be a positive cone of $\mathbb{Z}^d$. By Corollary 2.2 the algebra $\mathcal{A}(\mathbb{Z}_+^d)$ contains $\mathcal{A}(\mathcal{S})$ and so there is a continuous map between their character spaces $\overline{\mathbb{D}}^d$ and $\mathfrak{M}_{\mathcal{S}}$; namely

$$\iota^* \colon \overline{\mathbb{D}}^d \to \mathfrak{M}_{\mathcal{S}} : \zeta \mapsto \mathrm{ev}_\zeta|_{\mathcal{A}(\mathcal{S})}.$$

The next proposition shows that this map is injective exactly when $\mathcal{S}_{\mathrm{sn}} = \mathbb{Z}_+^d$.

Proposition 3.4. *Let $\mathcal{S} \subset \mathbb{Z}_+^d$ be a positive cone of $\mathbb{Z}^d$. Let $\iota^* \colon \overline{\mathbb{D}}^d \to \mathfrak{M}_{\mathcal{S}}$ be the continuous map induced by the embedding $\mathcal{A}(\mathcal{S}) \hookrightarrow \mathcal{A}(\mathbb{Z}_+^d)$. Then the following are equivalent:*

(i) $\mathcal{S}_{\mathrm{sn}} = \mathbb{Z}_+^d$;
(ii) *the intersection of $\mathcal{S}$ with all axes is a non-trivial positive cone of $\mathbb{Z}$;*
(iii) *ι^* is injective.*

In particular, ι^* is a homeomorphism when it is injective.

Proof. [(i) $\Leftrightarrow$ (ii)]: For simplicity let us write $\mathcal{S}(i) := \mathcal{S} \cap \{n\mathbf{i} \mid n \in \mathbb{Z}_+\}$. If $\mathbb{Z}_+^d = \mathcal{S}_{\mathrm{sn}}$ then $n\mathbf{i} \in \mathcal{S}$ eventually for every $n \in \mathbb{N}$. Hence $\gcd(\mathcal{S}(i)) = 1$ giving that $\mathcal{S}(i)$ is a positive cone in $\mathbb{Z}$. Conversely if $\mathcal{S}(i)$ is a positive cone then

$$\mathbf{i} \in \mathbb{N}\mathbf{i} \subset (\mathcal{S}(i))_{\mathrm{sn}} \subset \mathcal{S}_{\mathrm{sn}},$$

and thus $\mathbb{Z}_+^d = \mathcal{S}_{\mathrm{sn}}$.

[(ii) $\Leftrightarrow$ (iii)]: Suppose that ι^* is injective. First we show that $\mathcal{S}$ intersects with all axes. Assume without loss of generality that $\mathcal{S}(1) = \{0\}$. Then for $\lambda \neq 0$ we would have that

$$\mathrm{ev}_{(\lambda,0,\dots,0)}(z^s) = 0^{s_2} \cdots 0^{s_d} = \mathrm{ev}_{(0,\dots,0)}(z^s)$$

for all $s \in \mathcal{S}$ as $s_1 = 0$. Hence $\mathrm{ev}_{(\lambda,0,\dots,0)} = \mathrm{ev}_{(0,\dots,0)}$ which contradicts injectivity of ι^*. Secondly we show that every $\mathcal{S}(i)$ is a positive cone in $\mathbb{Z}$. Without loss of generality assume that $\mathcal{S}(1)$ is not such and set $k := \gcd(S(1)) \neq 1$. Let λ, μ be two distinct non-trivial k-th roots of the unit. If $s \in \mathcal{S}(1)$ then

$$\mathrm{ev}_{(\lambda,0,\dots,0)}(z^s) = \lambda^{s_1} = \mu^{s_1} = \mathrm{ev}_{(\mu,0,\dots,0)}(z^s).$$

If $s \in \mathcal{S} \setminus \mathcal{S}(1)$ then there is at least one $j \in \{2, \dots, d\}$ such that $s_j \neq 0$ and so

$$\mathrm{ev}_{(\lambda,0,\dots,0)}(z^s) = 0 = \mathrm{ev}_{(\mu,0,\dots,0)}(z^s).$$

Therefore $\mathrm{ev}_{(\lambda,0,\dots,0)} = \mathrm{ev}_{(\mu,0,\dots,0)}$ which again contradicts injectivity of ι^*.

For the converse recall that a semicharacter on $\mathcal{S}$ is a semigroup homomorphism $h \colon \mathcal{S} \to \overline{\mathbb{D}}$. By **[18**, Theorem 4.2.1**]** the character space of $\mathcal{A}(\mathcal{S})$ is homeomorphic to the semicharacter space of $\mathcal{S}$. We will show that every semicharacter of $\mathcal{S}$ extends uniquely to a semicharacter of $\mathcal{S}_{\mathrm{sn}}$. This will give that the character spaces of $\mathcal{A}(\mathcal{S})$ and $\mathcal{A}(\mathcal{S}_{\mathrm{sn}})$ are homeomorphic, and so if $\mathcal{S}_{\mathrm{sn}} = \mathbb{Z}_+^d$ then ι^* is a homeomorphism. To this end for $h \colon \mathcal{S} \to \overline{\mathbb{D}}$ we define $\widetilde{h} \colon \mathcal{S}_{\mathrm{sn}} \to \overline{\mathbb{D}}$ by

$$\widetilde{h}(t) := \begin{cases} h((n+1)t)/h(nt) & \text{if } (n+1)t, nt \in \mathcal{S} \text{ and } h(nt) \neq 0 \text{ for some } n \in \mathbb{N}, \\ 0 & \text{if } h(nt) = 0 \text{ for every } n \in \mathbb{N}. \end{cases}$$

To see that $\widetilde{h}$ is well defined first suppose that $h(nt) = 0$ for some $n \in \mathbb{N}$. Then for every $m \in \mathbb{N}$ with $mt \in \mathcal{S}$ we have that

$$h(mt)^n = h(mnt) = h(nt)^m = 0.$$

Thus $h(mt) = 0$ for every $m \in \mathbb{N}$. Moreover if there are $n, m \in \mathbb{N}$ such that $(n+1)t, nt \in \mathcal{S}$ and $(m+1)t, mt \in \mathcal{S}$ then we have

$$h((n+1)t)h(mt) = h((n+m+1)t) = h((m+1)t)h(nt)$$

showing that $\widetilde{h}(t)$ does not depend on the choice of n. ∎

Remark 3.5. Contrary to **[18**, Proposition 3.5.6**]**, we use that semicharacters of $\mathcal{S}$ extend uniquely to the seminormalization of $\mathcal{S}$ rather than to the *normalization* $\mathcal{S}_{\mathrm{n}} := \{g \in \mathcal{G} \mid \exists n \in \mathbb{N} \text{ such that } ng \in \mathcal{S}\}$.

Remark 3.6. It is worth noticing that the equivalence of items (ii) and (iii) of Proposition 3.4 follows by the universal property of seminormalizations, by applying **[3**, Lemma 1.11**]** for the pointed monoid $\overline{\mathbb{D}}$. Therein ι^* follows by applying a Zorn's Lemma. However it is the analytic form of ι^* that we will be requiring and wish to make explicit here.

Suppose that $\mathcal{S} \subset \mathbb{Z}_+^d$ is a positive cone with $\mathcal{S}_{\mathrm{sn}} = \mathbb{Z}_+^d$. For every $\mathbf{i} \in \mathbb{Z}_+^d$ let $n_i \in \mathbb{N}$ such that both $(n_i + 1)\mathbf{i}$ and $n_i\mathbf{i}$ are in $\mathcal{S}$. Proposition 3.4 then asserts that if $\chi \in \mathfrak{M}_{\mathcal{S}}$ with $\chi = \mathrm{ev}_\zeta |_{\mathcal{A}(\mathcal{S})}$ then ζ is uniquely given by

$$\zeta_i = \begin{cases} \chi(z_i^{n_i+1})/\chi(z_i^{n_i}) & \text{if } \chi(z_i^{n_i}) \neq 0, \\ 0 & \text{if } \chi(z_i^{n_i}) = 0. \end{cases} \tag{3.1}$$

Recall that if $\mathcal{S}_1 \simeq \mathcal{S}_2$ and are both positive cones of $\mathbb{Z}$ then $\mathcal{S}_1 = \mathcal{S}_2$. This property passes also to higher ranks.

Proposition 3.7. *Let $\mathcal{S}_1 \subset \mathbb{Z}^{d_1}$ and $\mathcal{S}_2 \subset \mathbb{Z}^{d_2}$ be positive cones such that $(\mathcal{S}_1)_{\mathrm{sn}} = \mathbb{Z}_+^{d_1}$ and $(\mathcal{S}_2)_{\mathrm{sn}} = \mathbb{Z}_+^{d_2}$. Then $\mathcal{S}_1 \simeq \mathcal{S}_2$ if and only if $d_1 = d_2$ and $\mathcal{S}_1 = \mathcal{S}_2$ up to a permutation of the co-ordinates.*

Proof. Let $\sigma \colon \mathcal{S}_1 \to \mathcal{S}_2$ be a semigroup isomorphism. Since it defines an isomorphism between their generating groups we get that $d_1 = d_2$, which we name as d from now on. Moreover the induced group isomorphism is given by a unitary map, say $U \in \mathrm{GL}_d(\mathbb{Z})$. For every $i \in \{1, \dots, d\}$, choose $n_i \in \mathbb{N}$ so that $n_i\mathbf{i} \in \mathcal{S}_1(i)$. Then the i-th column $\sigma(n_i\mathbf{i})$ of n_iU is in $\mathcal{S}_2 \subset \mathbb{Z}_+^d$ and so it has non-negative entries. Hence all entries of U are non-negative. As the same holds for U^{-1} we get that U is a permutation matrix. ∎

We now have arrived to the main rigidity result. We will be using an idea of **[13]** for rotating isomorphisms to vacuum preserving isomorphisms. From the proof it will become apparent that rotations are the only vacuum preserving isomorphisms (up to a permutation of the co-ordinates) for this class of semigroup algebras.

Theorem 3.8. *Let $\mathcal{S}_1 \subset \mathcal{G}_1$ and $\mathcal{S}_2 \subset \mathcal{G}_2$ be higher rank numerical semigroups. Then the following are equivalent:*

(i) $\mathcal{S}_1 \simeq \mathcal{S}_2$*;*
(ii) $\mathcal{A}(\mathcal{S}_1) \simeq \mathcal{A}(\mathcal{S}_2)$ *by a completely isometric isomorphism;*
(iii) $\mathcal{A}(\mathcal{S}_1) \simeq \mathcal{A}(\mathcal{S}_2)$ *by an algebraic isomorphism.*

Proof. First we remark that semigroup isomorphisms induce completely isometric isomorphisms. Indeed for an isomorphism $\sigma \colon \mathcal{S}_1 \to \mathcal{S}_2$ we can define $U \colon \ell^2(\mathcal{S}_1) \to \ell^2(\mathcal{S}_2)$ to be the permutation unitary $Ue_t = e_{\sigma(t)}$. It then follows that $UV_sU^* = V_{\sigma(s)}$ for all $s \in \mathcal{S}_1$. Therefore it suffices to show that item (iii) implies item (i). In particular the isomorphism in item (iii) is automatically bounded by Proposition 2.7.

Combining the above with Remark 3.3, without loss of generality we may assume that $\mathcal{S}_1 \subset \mathbb{Z}_+^{d_1}$ with $(\mathcal{S}_1)_{\rm sn} = \mathbb{Z}_+^{d_1}$, and likewise for $\mathcal{S}_2$. Thus by Proposition 3.7 it suffices to show that item (iii) implies that $d_1 = d_2$ and that $\mathcal{S}_1 = \mathcal{S}_2$ up to a permutation of the variables.

An algebraic isomorphism ρ between $\mathcal{A}(\mathcal{S}_1)$ and $\mathcal{A}(S_2)$ implements a homeomorphism ρ^* of their character spaces $\overline{\mathbb{D}}^{d_1}$ and $\overline{\mathbb{D}}^{d_2}$. Therefore $d_1 = d_2$, which we name as d henceforth. For convenience we will treat the cases $d = 1$ and $d > 1$ separately.

The one-variable case. For $d = 1$ we have $\mathfrak{M}_{\mathcal{S}_1} \simeq \mathfrak{M}_{\mathcal{S}_2} \simeq \overline{\mathbb{D}}$. We will employ an idea from **[13]** to rotate the isomorphism to one that matches the zeroes of the character space. To this end let the maps

$$\rho_\vartheta = \operatorname{ad}_{u_\vartheta}, \vartheta \in [0, 2\pi] \quad \text{for } u_\vartheta \colon \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+) : e_n \mapsto e^{i\vartheta n} e_n.$$

It is immediate that ρ_ϑ gives an automorphism of $\mathcal{A}(\mathbb{Z}_+)$ that sends every generator to a scalar multiple of the same generator. Hence the restriction to $\mathcal{A}(\mathcal{S}_1)$ and to $\mathcal{A}(\mathcal{S}_2)$ gives complete isometric automorphisms such that

$$(\rho_\vartheta)^*(\zeta) = e^{i\vartheta}\zeta \quad \text{for all } \zeta \in \mathbb{D}.$$

Suppose that $\rho^*(0) = \zeta \neq 0$ and so $(\rho^{-1})^*(0) = \eta \neq 0$. Then $\rho_\vartheta^*(\eta)$ defines a circle of radius $|\eta|$ around the origin in $\mathfrak{M}_{\mathcal{S}_2}$. By applying ρ^* we can implement a closed curve $(\rho_\vartheta \circ \rho)^*(\eta)$ that has ζ in its interior and passes through $0 \in \mathfrak{M}_{S_1}$. Likewise we can form a circle $\rho_\vartheta^*(\zeta)$ of radius $|\zeta|$ around the origin in $\mathfrak{M}_{\mathcal{S}_1}$ and move it to the closed curve $(\rho_\vartheta \circ \rho^{-1})^*(\zeta)$ that passes through $0 \in \mathfrak{M}_{\mathcal{S}_2}$ with η in its interior. By construction $\rho_\vartheta^*(\eta)$ intersects with $(\rho_\vartheta \circ \rho^{-1})^*(\zeta)$ at least in one point, say η'. By applying ρ^* we thus derive that $(\rho_\vartheta \circ \rho)^*(\eta)$ intersects $\rho_\vartheta^*(\zeta)$ at $\zeta' = \rho^*(\eta')$. Choose ϑ_1 that rotates ζ to ζ' and ϑ_2 that rotates η' to η. Then we can define the isomorphism

$$\pi := \rho \circ \rho_{\vartheta_1} \circ \rho^{-1} \circ \rho_{\vartheta_2} \circ \rho \colon \mathcal{A}(\mathcal{S}_1) \to \mathcal{A}(\mathcal{S}_2)$$

for which

$$\pi^*(0) = \rho^* \circ \rho_{\vartheta_2}^* \circ (\rho^{-1})^* \circ \rho_{\vartheta_1}^* \circ \rho^*(0) = \rho^* \circ \rho_{\vartheta_2}^*(\eta') = \rho^*(\eta) = 0.$$

This transformation is depicted in the figure of the next page. Hence without loss of generality we may assume that $\mathcal{A}(\mathcal{S}_1) \simeq \mathcal{A}(\mathcal{S}_2)$ by an isomorphism ρ such that $\rho^*(0) = 0$.

Now we use the explicit construction of equation (3.1). Recall here that we identify elements in $\mathcal{A}(\mathcal{S})$ with their corresponding holomorphic functions. Fix $0 \neq n \in \mathcal{S}_1$ such that $n + 1 \in \mathcal{S}_1$ and set

$$f := \rho(z^{n+1}) \quad \text{and} \quad g := \rho(z^n).$$

Then $\rho^*(\zeta) = f(\zeta)/g(\zeta)$ whenever $g(\zeta) \neq 0$. However f/g is holomorphic in $\mathbb{D} \setminus g^{-1}(\{0\})$ and continuously extendable at any $w \in g^{-1}(\{0\})$ by $\rho^*(w)$. By Riemann's Theorem on removable singularities f/g is holomorphically extendable to $\mathbb{D}$, and thus its extension ρ^* is holomorphic on $\mathbb{D}$.

Clearly ρ^* is not constant. Thus by the open mapping theorem for holomorphic functions we have that $\rho^*(\mathbb{D}) \subset \mathbb{D}$. By symmetry we have the same for its inverse. Hence ρ^* is a biholomorphism of $\mathbb{D}$ with $\rho^*(0) = 0$. Thus by Schwarz Lemma it follows that $\rho^*(\zeta) = e^{i\vartheta}\zeta$ for $\vartheta \in [0, 2\pi]$. Hence we get $(\rho_{-\vartheta} \circ \rho)^* = \mathrm{id}$. As rotations are automorphisms we may work with $\rho_{-\vartheta} \circ \rho$ instead of ρ. Thus without loss of generality we may assume that $\rho^* = \mathrm{id}$ on $\mathbb{D}$; and hence on $\overline{\mathbb{D}}$.

To finish the first part, let $s \in \mathcal{S}_1$ and write $\rho(z^s) = h(z)$. Then for every $\overline{\mathbb{D}} \ni \zeta \equiv \mathrm{ev}_\zeta \in \mathfrak{M}_{\mathcal{S}_2}$ we have that $\rho^*(\mathrm{ev}_\zeta) = \mathrm{ev}_\zeta \in \mathfrak{M}_{\mathcal{S}_1}$ and so

$$\zeta^s = \mathrm{ev}_\zeta(z^s) = \rho^*(\mathrm{ev}_\zeta)(z^s) = \mathrm{ev}_\zeta(\rho(z^s)) = h(\zeta). \tag{3.2}$$

As this holds for all $\zeta \in \overline{\mathbb{D}}$ we derive that $z^s = h(z) = \rho(z^s)$. Since $s \in \mathcal{S}_1$ was arbitrary we get that $\rho = \mathrm{id}|_{\mathcal{S}_1}$ giving that $\mathcal{S}_1 \subset \mathcal{S}_2$. By symmetry on ρ^{-1} we have equality.

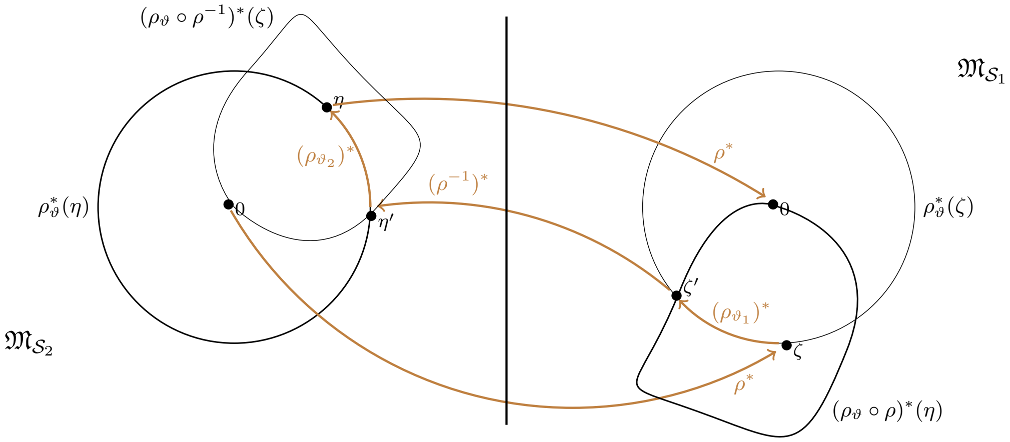

Figure. Matching zeroes of the character spaces.

The multi-variable case. Let the continuous functions $\rho_i^* \colon \overline{\mathbb{D}}^d \to \overline{\mathbb{D}}$ so that the homeomorphism $\rho^* \colon \mathfrak{M}_{\mathcal{S}_2} \to \mathfrak{M}_{\mathcal{S}_1}$ is written as

$$\rho^*(\zeta) = (\rho_1^*(\zeta), \dots, \rho_d^*(\zeta)).$$

First we show that every ρ_i^* is holomorphic on $\mathbb{D}^d$. Fix $n_i \in \mathbb{N}$ so that $n_i\mathbf{i}$ and $(n_i+1)\mathbf{i}$ are both in $\mathcal{S}_1(i)$. Set $f_i := \rho(z_i^{n_i+1})$ and $g_i := \rho(z_i^{n_i})$. By using equation (3.1) we can write

$$\rho_i^*(\zeta) = \frac{f_i(\zeta)}{g_i(\zeta)} \text{ for all } \zeta \in \mathbb{D}^d \setminus g_i^{-1}(\{0\}).$$

Zero sets of analytic functions are thin sets, and by **[24**, Theorem 3.4**]** the set $g_i^{-1}(\{0\})$ can be removed so that ρ_i^* is holomorphic on $\mathbb{D}^d$.

Now we have that ρ_1^* cannot be constant as in that case we would have the contradiction $\overline{\mathbb{D}}^d = \rho^*(\overline{\mathbb{D}}^d) \subset \{\rho_1^*(0, \dots, 0)\} \times \overline{\mathbb{D}}^{d-1}$. By applying the open mapping theorem for holomorphic functions on several variables, e.g. **[24**, Theorem 1.21**]**, we get that $\rho_1^*(\mathbb{D}^d) \subset \mathbb{D}$. Likewise it follows that

$$\rho_i^*(\mathbb{D}^d) \subset \mathbb{D} \text{ for all } i = 1, \dots, d.$$

Hence we have that $\rho^*(\mathbb{D}^d) \subset \mathbb{D}^d$, and thus by symmetry on $(\rho^*)^{-1}$, we conclude that $\rho^*(\mathbb{D}^d) = \mathbb{D}^d$. Therefore ρ^* restricts to a biholomorphism of the polydisc. Recall that

$$\mathrm{Aut}(\mathbb{D}^d) \simeq (\times_{i=1}^d \mathrm{Aut}(\mathbb{D})) \rtimes \mathcal{S}_d,$$

e.g. **[25**, Theorem 2, pp. 48**]**. Hence ρ^* is the product of automorphisms of $\mathbb{D}$ up to a permutation of the variables, say σ. As every permutation on the variables implies a completely isometric isomorphism, without loss of generality, we may substitute S_2 by $\sigma(S_2)$ so that the ρ_i^* depends only on ζ_i.

Notice that rotating co-ordinatewise on $\mathcal{A}(\mathbb{Z}_+^d)$ restricts to automorphisms on $\mathcal{A}(\mathcal{S}_1)$ and on $\mathcal{A}(\mathcal{S}_2)$. Therefore we rotate ρ appropriately so that $\rho_i^*(0) = 0$ for all $i = 1, \dots, d$. That is, every ρ_i^* restricts to a biholomorphism of $\mathbb{D}$ fixing the zero. By using the one-variable arguments we derive that every $\rho_i^*|_\mathbb{D}$ is a rotation. Hence without loss of generality $\rho^* = \mathrm{id}$ on $\mathbb{D}^d$ and thus on $\overline{\mathbb{D}}^d$. A computation as in equation (3.2) shows that $\rho = \mathrm{id}$ and the proof is complete. ∎

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SCHOOL OF MATHEMATICS, STATISTICS AND PHYSICS, NEWCASTLE UNIVERSITY, NEWCASTLE UPON TYNE, NE1 7RU, UK
E-mail address: evgenios.kakariadis@ncl.ac.uk

DEPARTMENT OF MATHEMATICS, EAST CAROLINA UNIVERSITY, GREENVILLE, NC 27858, USA
E-mail address: katsoulise@ecu.edu

SCHOOL OF MATHEMATICAL SCIENCES, QUEEN MARY UNIVERSITY OF LONDON, MILE END ROAD, LONDON, E1 4NS, UNITED KINGDOM
E-mail address: xin.li@qmul.ac.uk