MEET IRREDUCIBLE IDEALS AND REPRESENTATIONS OF LIMIT ALGEBRAS

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Abstract. In this paper we give criteria for an ideal $\mathcal{J}$ of a TAF algebra $\mathcal{A}$ to be meet irreducible. We show that $\mathcal{J}$ is meet irreducible if and only if the $\mathcal{C}^*$-envelope of $\mathcal{A}/\mathcal{J}$ is primitive. In that case, $\mathcal{A}/\mathcal{J}$ admits a faithful nest representation which extends to a $*$-representation of the $\mathcal{C}^*$-envelope for $\mathcal{A}/\mathcal{J}$. We also characterize the meet irreducible ideals as the kernels of bounded nest representations; this settles the question of whether the n-primitive and meet-irreducible ideals coincide.

1. Introduction

Representation theory of operator algebras is still in its infancy. While for $\mathcal{C}^*$-algebras the fundamentals of representation theory have long been known, for nonselfadjoint algebras there are hardly any results of a general nature. For ‘triangular operator algebras’ (a term which we leave undefined), intuition suggests that the fundamental building blocks for representation theory should be nest representations. In the category of $\mathcal{C}^*$-algebras, the nest representations are precisely the irreducible representations.

Recall that a nest representation is a representation for which the closed, invariant subspaces form a nest (i.e., are linearly ordered). In his study of nonselfadjoint crossed products, Lamoureux introduced the notion of n-primitive ideal. An ideal is n-primitive if it is the kernel of a nest representation. Lamoureux has shown that in various contexts in nonselfadjoint algebras the n-primitive ideals play a role analogous to the primitive ideals in $\mathcal{C}^*$-algebras. Thus, one can give the set of n-primitive ideals the hull-kernel topology, and for every (closed, two-sided) ideal $\mathcal{I}$ in the algebra, $\mathcal{I}$ is the intersection of all n-primitive ideals containing $\mathcal{I}$; in other words, $\mathcal{I} = k(h(\mathcal{I}))$. 

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An ideal \( J \) of an algebra \( \mathcal{A} \) is *meet irreducible* if, for any ideals \( I_1 \) and \( I_2 \) containing \( J \), the relation \( I_1 \cap I_2 = J \) implies that either \( I_1 = J \) or \( I_2 = J \). In the case of \( T_n \), the algebra of upper triangular \( n \times n \) matrices, meet irreducible ideals are gotten by ‘cutting a wedge’ from the algebra: let \( 1 \leq i_0 \leq j_0 \leq n \). The ideal

\[
I = \{(a_{ij}) : a_{ij} = 0, \ i_0 \leq i \leq j \leq j_0\}
\]

is meet irreducible, and every meet irreducible ideal of \( T_n \) has this form.

The relationship between meet-irreducible and n-primitive ideals is studied in a variety of examples in [7], and in [3] meet-irreducible ideals in strongly maximal triangular AF-algebras are characterized by sequences of matrix units and also in terms of groupoids. In that paper it is shown that every meet-irreducible ideal is n-primitive. This is done by constructing a nest representation. The converse question, whether every n-primitive ideal is meet-irreducible, was left open.

In a recent work [2], the first two authors examined the \( C^* \) envelope of a quotient \( \mathcal{A}/J \) of a strongly maximal TAF algebra by an ideal \( J \); they showed the \( C^* \) envelope is an AF \( C^* \) algebra, even though the quotient \( \mathcal{A}/J \) is not in general a TAF algebra. It turns out that the \( C^* \) envelope of \( \mathcal{A}/J \) is sensitive enough to detect the meet irreducibility of \( J \). In Theorem 2.3 we show that \( J \) is meet irreducible if and only if \( C^*_{env}(\mathcal{A}/J) \), the \( C^* \) envelope of \( \mathcal{A}/J \), is primitive. The theory of \( C^* \)-envelopes provides the natural framework for studying results of this type. In Theorem 2.4 we show that for a meet irreducible ideal \( J \), there exists a faithful and irreducible \(*\)-representation of \( C^*_{env}(\mathcal{A}/J) \), whose restriction on \( \mathcal{A}/J \) is a nest representation. Since the converse is easily seen to be true, Theorem 2.4 provides a characterization of meet irreducible ideals in terms of the representation theory for \( \mathcal{A} \).

The question of whether the kernel of a nest representation is a meet-irreducible ideal emerged at the Ambelside, U.K. conference in summer, 1997. Subsequently some progress was made. In [4] a partial result was obtained: if the TAF algebra \( \mathcal{A} \) has totally ordered spectrum, or if the nest representation \( \pi \) has the property that the von Neumann algebra generated by \( \pi(\mathcal{A} \cap \mathcal{A}^*) \) contains an atom, then \( \ker(\pi) \) is meet-irreducible. The solution presented in Theorem 2.6 is self-contained and does not make use of the results of [3] or [4]; thus the question is now settled for strongly maximal TAF algebras.

Despite the fact that evidence at hand is limited, it nonetheless seems worthwhile to state the

**Question.** Are there any operator algebras for which the n-primitive ideals, and the meet-irreducible ideals do not coincide?
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2. THE MAIN RESULTS

We begin by recalling a result of Lamoureux [7].

**Lemma 2.1.** Let \( \mathcal{I} \) be a closed, two-sided ideal in a separable C*-algebra \( \mathcal{A} \). Then the following are equivalent:

(i) \( \mathcal{I} \) is \( n \)-primitive
(ii) \( \mathcal{I} \) is primitive
(iii) \( \mathcal{I} \) is prime
(iv) \( \mathcal{I} \) is meet-irreducible

One can actually characterize when an AF C*-algebra is primitive in terms of its Bratteli diagram.

Let \( \mathfrak{A} = \lim_{\to} (\mathfrak{A}_i, \varphi_i) \) be an AF C*-algebra and assume that each \( \mathfrak{A}_i \) decomposes as a direct sum \( \mathfrak{A}_i = \bigoplus_j \mathfrak{A}_{i,j} \) of finite dimensional full matrix algebras \( \mathfrak{A}_{i,j} \). A path \( \Gamma \) for \( \mathfrak{A} = \lim_{\to} (\mathfrak{A}_i, \varphi_i) \) is a sequence \( \mathfrak{A}_{i,j} \) such that for each pair of nodes \( ((i,j),(i+1,j_{i+1})) \) there exist an arrow in the Bratteli diagram for \( \mathfrak{A} = \lim_{\to} (\mathfrak{A}_i, \varphi_i) \) which joins them. It is known that \( \mathfrak{A} \) is primitive iff there is a path \( \Gamma \) for \( \mathfrak{A} = \lim_{\to} (\mathfrak{A}_i, \varphi_i) \) so that each summand of \( \mathfrak{A}_i \) is eventually mapped into a member of \( \Gamma \). We call such a path \( \Gamma \) an essential path for \( \mathfrak{A} \).

Beyond C*-algebras, a meet irreducible ideal need not be primitive. In [3], a description of all meet irreducible ideals was given in terms of matrix unit sequences.

**Definition 2.2.** Let \( \mathcal{A} = \lim_{\to} (\mathcal{A}_i, \varphi_i) \) be a TAF algebra. A sequence \( (e_i)_{i \geq N} \) of matrix units from \( \mathcal{A} \) will be called an \( \text{mi-chain} \) if the following two conditions are satisfied for all \( i \geq N \):

(A) \( e_i \in \mathcal{A}_i \)
(B) \( e_{i+1} \in \text{Id}_{i+1}(e_i) \),

where \( \text{Id}_{i+1}(e_i) \) denotes the ideal generated by \( e_i \) in \( \mathcal{A}_{i+1} \).

If \( (e_i)_{i \geq N} \) is an mi-chain for \( \mathcal{A} = \lim_{\to} (\mathcal{A}_i, \varphi_i) \), let \( \mathcal{J} \) be the join of all ideals which do not contain any matrix unit \( e_i \) from the chain. In [3, Theorem 1.2] it is shown that for a TAF algebra \( \mathcal{A} = \lim_{\to} (\mathcal{A}_i, \varphi_i) \), given an mi-chain \( (e_i)_{i \geq N} \), the ideal \( \mathcal{J} \) associated with \( (e_i)_{i \geq N} \) is meet irreducible. Conversely, every proper meet irreducible ideal in \( \mathcal{A} = \lim_{\to} (\mathcal{A}_i, \varphi_i) \) is induced by some mi-chain, chosen from some contraction of this representation.
In this paper we give a characterization of the meet irreducible ideals of TAF algebras in terms of C*-envelopes of quotient algebras. We need to recall the notation and machinery from [2].

Let \( A = \lim_{\rightarrow} (A_i, \varphi_i) \) be the enveloping C*-algebra for a TAF algebra \( A = \lim_{\rightarrow} (A_i, \varphi_i) \) and let \( J \subseteq A \) be a closed ideal; let \( J_i := J \cap A_i \). For each \( i \geq 1 \), \( S_i \) denotes the collection of all diagonal projections \( p \) which are semi-invariant for \( A_i \), are supported on a single summand of \( A_i \) and satisfy \( (p A_i p) \cap J = \{0\} \). We form finite dimensional C*-algebras \( B_i := \sum_{p \in S_i} B(\text{Ran } p) \) where \( B(\text{Ran } p) \) denotes the bounded operators on \( \text{Ran } p \); of course, \( B(\text{Ran } p) \) is isomorphic to \( M_{\text{rank } p} \). Let \( \sigma_i \) be the map from \( A_i \) into \( B_i \) given by \( \sigma_i(a) = \sum_{p \in S_i} p a p|_{\text{Ran } p} \). The map \( \sigma_i|_{A_i} \) factors as \( \rho_i q_i \) where \( q_i \) is the quotient map of \( A_i \) onto \( A_i / J_i \) and \( \rho_i \) is a completely isometric homomorphism of \( A_i / J_i \) into \( B_i \). Notice that \( B_i \) equals the C*-algebra generated by \( \rho_i(A_i / J_i) \).

We then consider unital embeddings \( \pi_i \) of \( B_i \) into \( B_{i+1} \) defined as follows. For each \( q \in S_{i+1} \) we choose projections \( p \in S_i \) which maximally embed into \( q \) under the action of \( \varphi_i \). This way, we determine multiplicity one embeddings of \( B(\text{Ran } p) \) into \( B(\text{Ran } q) \). Taking into account all such possible embeddings, we determine the embedding \( \pi_i \) of \( B_i \) into \( B_{i+1} \).

Finally we form the subsystem of the directed limit \( \mathcal{B} = \lim_{\rightarrow} (\mathcal{B}_i, \pi_i) \) corresponding to all summands which are never mapped into a summand \( B(\text{Ran } p) \) where \( p \) is a maximal element of some \( S_i \). Evidently this system is directed upwards. It is also hereditary in the sense that if every image of a summand lies in one of the selected blocks, then it clearly does not map into a maximal summand and thus already lies in our system. By [1, Theorem III.4.2], this system determines an ideal \( J \) of \( \mathcal{B} \). The quotient \( \mathcal{B}' = \mathcal{B} / J \) is the AF algebra corresponding to the remaining summands and the remaining embeddings; it can be expressed as a direct limit \( \mathcal{B}' = \lim_{\rightarrow} (\mathcal{B}_i', \pi_i') \), with the understanding that \( \mathcal{B}_i' = \oplus_j \mathcal{B}_{i,j} \) for these remaining summands \( \mathcal{B}_{i,j} \) of \( \mathcal{B}_i \). It can be seen that the quotient map is isometric on \( A / J \) and that \( \mathcal{B}' \) is the C*-envelope of \( A / J \).

**Theorem 2.3.** Let \( A \) be a TAF algebra and let \( J \subseteq A \) be an ideal. Then \( J \) is meet irreducible if and only if the algebra \( C^*_{\text{env}}(A / J) \) is primitive.
Proof. Assume that \( \mathcal{B}' = C^*_{\text{env}}(\mathcal{A}/\mathcal{J}) \) is primitive and let \( \Gamma = (\mathcal{B}_{i,j})_{i=1}^{\infty} \) an essential path for \( \mathcal{B}' \). Let \( e_i \) for \( \mathcal{B}_{i,j} \) be the characteristic matrix units for \( \mathcal{B}_{i,j} \), i.e., the ones on the top right corner of \( \mathcal{B}_{i,j} \).

Assume that there exist ideals \( I_1 \) and \( I_2 \), properly containing \( \mathcal{J} \). Since \( I_1 \) and \( I_2 \) properly contain \( \mathcal{J} \), there exist matrix units \( f_k \in I_k \) with \( f_k \notin \mathcal{J} \), \( k = 1, 2 \). So the images of the \( f_k \) appear in the presentation for the \( C^* \)-envelope in perhaps different summands. However, the existence of an essential path \( \Gamma \) implies that some subordinates for the \( f_k \) will appear in some member of \( \Gamma \), say \( \mathcal{B}_{i,j} \), for \( i \) sufficiently large, and so \( e_i \in I_1 \cap I_2 \). However, \( e_i \notin \mathcal{J} \) and so \( \mathcal{J} \) is properly contained in \( I_1 \cap I_2 \). It follows \( \mathcal{J} \) is meet irreducible.

Conversely, assume that \( \mathcal{J} \) is meet irreducible. In light of Lemma 2.1 and the subsequent comments, it suffices to show that the trivial ideal \( \{0\} \) is meet irreducible in the \( C^* \)-envelope \( \mathcal{B}' \).

By way of contradiction assume that there are non-trivial ideals \( I_1 \) and \( I_2 \) of \( \mathcal{B}' \) so that \( I_1 \cap I_2 = \{0\} \). We claim that \( (\mathcal{A}/\mathcal{J}) \cap I_k \neq \{0\} \), \( k = 1, 2 \). Indeed, any non-trivial summand of \( I_k \) will eventually be mapped into a direct summand \( \mathcal{B}_{i,j} \) of \( \mathcal{B}' \) corresponding to some maximal element of \( S_i \). Hence all matrix units in \( \mathcal{B}_{i,j} \) belong to \( I_k \), including the characteristic one. This one however also belongs to \( \mathcal{A}/\mathcal{J} \) and therefore in the intersection \( (\mathcal{A}/\mathcal{J}) \cap I_k \).

The claim shows that the zero ideal is not meet irreducible in \( \mathcal{A}/\mathcal{J} \). By considering the pullback, this implies that \( \mathcal{J} \) is not meet irreducible in \( \mathcal{A} \), which is the desired contradiction.

Notice that the sequence \((e_i)_{i=1}^{\infty}\) associated with the path \( \Gamma \) in the proof above satisfies the Conditions (A) and (B) of the Definition 2.2 and is therefore an mi-chain for the ideal \( \mathcal{J} \).

Theorem 2.4. If \( \mathcal{A} \) is a TAF algebra and \( \mathcal{J} \) an ideal of \( \mathcal{A} \), then the following are equivalent:

(i) There exists a faithful representation \( \tau : C^*_{\text{env}}(\mathcal{A}/\mathcal{J}) \longrightarrow \mathcal{B}(\mathcal{H}) \) so that \( \tau(\mathcal{A}/\mathcal{J}) \) is weakly dense in some nest algebra.

(ii) \( \mathcal{J} \) is meet irreducible.

Proof. Assume that (i) is valid and let \( \tau : C^*_{\text{env}}(\mathcal{A}/\mathcal{J}) \longrightarrow \mathcal{B}(\mathcal{H}) \) be a faithful representation so that \( \tau(\mathcal{A}/\mathcal{J}) \) is weakly dense in some nest algebra \( \text{Alg} \mathcal{N} \). By way of contradiction assume that \( \mathcal{J} \) is not meet irreducible. Theorem 2.3 and Lemma 2.1 imply the existence of nonzero closed ideals \( I_1 \) and \( I_2 \) in \( C^*_{\text{env}}(\mathcal{A}/\mathcal{J}) \) so that \( I_1 I_2 = \{0\} \). Consider the mutually orthogonal subspaces \( [\tau(I_i)\mathcal{H}] \) These are both invariant under \( \tau(\mathcal{A}/\mathcal{J}) \), and therefore belong to \( \mathcal{N} \), a contradiction.
Conversely, assume that (ii) is valid and so, by Theorem 2.3, $C^*_\text{env}(\mathcal{A}/\mathcal{J})$ is primitive. Retain the notation established in the paragraphs preceding Theorem 2.3. Hence

$$C^*_\text{env}(\mathcal{A}/\mathcal{J}) = \mathcal{B}' = \lim\downarrow (\mathcal{B}'_i, \pi'_i)$$

where $\mathcal{B}'_i = \oplus_j \mathcal{B}_{ij}$ for the remaining summands $\mathcal{B}_{ij}$ of $\mathcal{B}_i$. Let $\Gamma = (\mathcal{B}_{ij}, i \in \mathbb{N})$ be the essential path for $\mathcal{B}'$.

Each $\mathcal{B}_{ij}$ is a full matrix algebra and therefore contains the algebra $\mathcal{B}_{ij}$ of upper triangular matrices. Form the finite dimensional algebras $\mathcal{B}'_i = \oplus_j \mathcal{B}_{ij}$ and consider the direct limit algebra

$$\mathcal{B}' = \lim\downarrow (\mathcal{B}'_i, \pi'_i),$$

where $\pi'_i$ is as earlier. Clearly, $\mathcal{B}'$ is a TAF algebra whose enveloping $C^*$-algebra is $\mathcal{B}'$. Moreover, $\mathcal{B}'$ contains $\mathcal{A}/\mathcal{J}$.

We define a state $\omega$ on $\mathcal{B}'$ as follows. Let $(p_i)_{i=1}^\infty$ be a sequence of diagonal projections with $p_i \in \mathcal{B}_{ij}$, so that $p_{i+1}$ is a subordinate of $p_i$, $i \in \mathbb{N}$. We define $\omega_i : \mathcal{B}'_i \to \mathbb{C}$ to be the compression on $p_i$ and we let $\omega$ to be the direct limit $\omega = \lim\downarrow \omega_i$. Consider the GNS triple $(\tau, \mathcal{H}, g)$ associated with the state $\omega$, i.e., $\tau$ is a representation of $\mathcal{B}'$ on $\mathcal{H}$ and $g \in \mathcal{H}$ so that $\omega(a) = \langle \tau(a)g, g \rangle$, $a \in \mathcal{B}'$. Since $\omega$ is pure, $\tau$ is irreducible. Moreover, $p_i \in \mathcal{B}_{ij}$, $i \in \mathbb{N}$ and so $\tau$ is also faithful.

An alternative presentation for $(\tau, \mathcal{H}, g)$ was given in [8, Proposition II.2.2]. Since $\omega$ is multiplicative on the diagonal $\mathcal{B}' \cap (\mathcal{B}')^*$, one considers $\mathcal{H}$ to be $L^2(\mathcal{X}, \mu)$, where $\mathcal{X}$ is the Gelfand spectrum of $\mathcal{B}' \cap (\mathcal{B}')^*$ and $\mu$ the counting measure on the orbit of $\omega$ in $\mathcal{X}$. With these identifications, given any matrix unit $e$, $\tau(e)$ is the translation operator on $\mathcal{X}$ defined in the paragraphs preceding [8, Theorem II.1.1].

In [8, Proposition II.2.2] it is shown that $\tau$ maps $\mathcal{B}'$ in a weakly dense subset of some nest algebra. The proof of the theorem will follow if we show that the weak closure of $\tau(\mathcal{A}/\mathcal{J})$ contains $\tau(\mathcal{B}')$.

A moment’s reflection shows that given any contraction $a \in \mathcal{B}'_{ij}$ and matrix units $e_1, e_2, \ldots, e_n$ and $f_1, f_2, \ldots, f_n$ in $\mathcal{B}'$, there exists a contraction $\hat{a} \in \mathcal{A}/\mathcal{J}$ so that

$$\omega(f_k^* \hat{a} e_k) = \omega(f_k^* a e_k)$$

and therefore

$$\langle \tau(\hat{a}) \tau(e_k)g, \tau(f_k)g \rangle = \langle \tau(a) \tau(e_k)g, \tau(f_k)g \rangle$$

for all $k = 1, 2, \ldots, n$. However the collection of all vectors of the form $\tau(e)g$, where $e$ ranges over all matrix units of $\mathcal{B}'$, forms a dense subset of $\mathcal{H}$ and so the desired density follows.
Remark 2.5. 1. The implication (i) $\implies$ (ii) also follows from Theorem 2.6.
2. Note that there exists a faithful representation $\tau : C^*_\text{env}(A/J) \to B(H)$ so that $\tau(A/J)$ is weakly dense in a nest algebra if and only if there is a faithful irreducible representation $\tau : C^*_\text{env}(A/J) \to B(H)$ so that $\tau(A/J)$ is weakly dense in a nest algebra.

Theorem 2.6. Let $A$ be a strongly maximal TAF algebra, $\pi : A \to B(H)$ a bounded nest representation on a Hilbert space $H$. Then $\ker(\pi)$ is a meet-irreducible ideal.

Proof. Since a bounded representation of the diagonal masa $A \cap A^*$ is completely bounded ([9, Theorem 8.7]), and a completely bounded representation is similar to a completely contractive representation [9, Theorem 8.1], we may assume that the restriction of $\pi$ to the diagonal masa is completely contractive. It follows that the restriction of $\pi$ to the diagonal masa is a star representation. Let $J = \ker(\pi)$, and $J_1, J_2$ be ideals in $A$ properly containing $J$. We are to show that $J_1 \cap J_2$ properly contains $J$.

Since $\pi$ is a nest representation, we have (after possibly interchanging $J_1, J_2$),

$$(0) \neq [\pi(J_1)H] \subseteq [\pi(J_2)H]$$

where $[X]$ denotes the closed subspace generated by $X \subset H$. Let $u$ be a matrix unit in $J_1 \cap A_n, n \in \mathbb{N}$, and $h \in H$ be such that $\|\pi(u)h\| = 1$. There exist $m, N \in \mathbb{N}, m \geq n$, matrix units $v_t \in J_2 \cap A_m, h_t \in H, 1 \leq t \leq N$ such that

$$\|\pi(u)h - \sum_{t=1}^{N} \pi(v_t)h_t\| < \frac{1}{4}.$$ 

We may assume that $\pi(v_t) \neq 0$ for all $t$. Since the diagonal matrix units $e_t = v_t v_t^*$ need not be distinct, we set $E = \sum \pi(e_t)$, where the sum is taken over the distinct elements $e_t$. As $\pi$ is a star representation on the diagonal, $E$ is a projection in $B(H)$. Since for all $s, t$,

$$\pi(e_s)\pi(v_t) = \pi(v_s v_s^* v_t) = \begin{cases} \pi(v_t) & \text{if } v_s v_s^* = v_t v_t^* \\ 0 & \text{otherwise} \end{cases}$$

we have $E \sum \pi(v_t)h_t = \sum_{s,t} \pi(e_s)\pi(v_t)h_t = \sum \pi(v_t)h_t$.

Now

$$\|E(\pi(u)h - \sum_{t=1}^{N} \pi(v_t)h_t)\| \leq \|\pi(u)h - \sum_{t=1}^{N} \pi(v_t)h_t\| < \frac{1}{4},$$

which completes the proof.
Hence
\[
\|E(\pi(u)h - \sum_{t=1}^{N} \pi(v_t)h_t)\| = \|E\pi(u)h - \sum_{t=1}^{N} \pi(v_t)h_t\| < \frac{1}{4}.
\]

We now claim that there exists at least one matrix unit \(e_t, 1 \leq t \leq N\), which satisfies \(\pi(e_t)\pi(u) \neq 0\).

Indeed, assume on the contrary that \(\pi(e_t)\pi(u) = 0\), for all \(1 \leq t \leq N\). Then \(E\pi(u)h = 0\) and so
\[
\|\sum_{t=1}^{N} \pi(v_t)h_t\| < \frac{1}{4}.
\]
However
\[
\|\sum_{t=1}^{N} \pi(v_t)h_t\| \geq \|\pi(u)h\| - \|\pi(u)h - \sum_{t=1}^{N} \pi(v_t)h_t\| > 1 - \frac{1}{4} = \frac{3}{4},
\]
which is a contradiction, proving the claim.

Embed \(u \in \mathcal{A}_n \hookrightarrow \mathcal{A}_m\) and decompose it as a sum \(u = \sum u_s\) of matrix units in \(\mathcal{A}_m\). Then \(e_tu = u_s\), for some \(s\), and so by the above claim \(\pi(u_s) \neq 0\), i.e., \(u_s \notin \mathcal{J}\). Thus we have identified matrix units \(u_s \in \mathcal{J}_1 \setminus \mathcal{J}\) and \(v_t \in \mathcal{J}_2 \setminus \mathcal{J}\) of \(\mathcal{A}_m\) with the same final projection. Say \(u_s = e_{ij}^{(m,r)}\) and \(v_t = e_{ik}^{(m,r)}\). We now distinguish three cases:

If \(j = k\), then \(u_s = v_t \in \mathcal{J}_1 \cap \mathcal{J}_2 \setminus \mathcal{J}\);

If \(j < k\), then \(v_t = u_se_{jk}^{(m,r)} \in \mathcal{J}_1 \cap \mathcal{J}_2 \setminus \mathcal{J}\);

If \(j > k\), then \(u_s = v_te_{kj}^{(m,r)} \in \mathcal{J}_1 \cap \mathcal{J}_2 \setminus \mathcal{J}\).

It follows that in all three cases \(\mathcal{J}_1 \cap \mathcal{J}_2\) properly contains \(\mathcal{J}\). Thus \(\mathcal{J}\) is meet-irreducible.

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