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JACQUET MODULES OF p-ADIC GENERAL LINEAR GROUPS

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Abstract. In this paper, we study Jacquet modules for p-adic general linear groups. More precisely, we have results—formulas and algorithms—aimed at addressing the following question: Given the Langlands data for an irreducible representation, can we determine its (semisimplified) Jacquet module? We use our results to answer this question in a number of cases, as well as to recover some familiar results as relatively easy consequences.

1. Introductory material

1.1. **Introduction.** In this paper, we begin a study of Jacquet modules for representations of p-adic general linear groups. We show how to calculate those subquotients of Jacquet modules satisfying certain properties, as well as giving some applications of these results.

We remark that our interest in such questions was originally spurred by the work of Tadić (cf. [Tad1], [Tad2], [Tad3]) on the use of Jacquet modules in analyzing induced representations. Section 3.1 of this paper contains such an analysis, though most of this paper is focused on the calculation of the Jacquet modules themselves.

First, we note that by a result of Zelevinsky (cf. section 1.2 of this paper), it suffices to study Jacquet modules for irreducible representations π having supercuspidal support contained in a set of the form $\{\nu^z\rho\}_{z\in\mathbb{Z}}$, with $\nu = |\det|$ and ρ an irreducible, unitary supercuspidal representation. Let ρ be an irreducible unitary supercuspidal representation of $GL(n_\rho, F)$ and π an irreducible representation of $GL(n_\pi, F)$ whose supercuspidal support is contained in $\{\nu^z\rho\}_{z\in\mathbb{Z}}$. Write π in terms of its Langlands data (more precisely, we reorder the Langlands data to correspond to the lowest lexicographic term in $r_{min}\pi$; cf. section 1.2):

$$\pi = L(\delta([\nu^{b_1}\rho, \nu^{a_1}\rho]), \delta([\nu^{b_2}\rho, \nu^{a_1}\rho]), \dots, \delta([\nu^{b_k}\rho, \nu^{a_k}\rho])),$$

with $\delta([\nu^b\rho, \nu^a\rho])$ the generalized Steinberg representation, which is the unique irreducible subrepresentation of $\mathrm{Ind}_P^G(\nu^a\rho \otimes \nu^{a-1}\rho \otimes \cdots \otimes \nu^b\rho)$. Note that when we refer to calculating subquotients of Jacquet modules, we mean that they are to be identified by their Langlands data.

Sections 1.2 and 1.3 constitute the remainder of the introductory material, with section 1.2 discussing notation and background material, and section 1.3 giving a number of lemmas which are used later in the paper.

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The second chapter constitutes the technical heart of the paper. Let $r_{min}\pi$ denote the (semisimplified) minimal (nonzero) Jacquet module for π (i.e., with respect to the smallest parabolic subgroup having nonzero Jacquet module, namely the standard parabolic subgroup having Levi factor $GL(n_\rho, F) \times GL(n_\rho, F) \times \cdots \times GL(n_\rho, F)$). In Proposition 2.1.4 (also, cf. Corollary 2.1.5), we determine the initial frequency $f_\pi(a)$, the largest value of f such that $r_{min}\pi$ contains a term of the form $\underbrace{\nu^a\rho \otimes \nu^a\rho \otimes \cdots \otimes \nu^a\rho}_{f} \otimes \ldots$. Furthermore, up to multiplicity, there is a unique term of the form $\underbrace{\nu^a\rho \otimes \cdots \otimes \nu^a\rho}_{f} \otimes \theta$ (θ irreducible) in $r_{M,G}\pi$, the Jacquet module of π with respect to the standard parabolic subgroup having Levi factor $M = \underbrace{GL(n_\rho, F) \times \cdots \times GL(n_\rho, F)}_{f} \otimes GL(n_\pi - fn_\rho, F)$ (cf. Lemma 2.1.2). Theorem 2.2.1 determines this θ.

This is generalized in section 2.3. There, the single value a is replaced by a subset X of the supercuspidal support. $f_\pi(X)$ is the largest value of f such that $r_{min}\pi$ contains a term of the form $\underbrace{\nu^{x_1}\rho \otimes \nu^{x_2}\rho \otimes \cdots \otimes \nu^{x_f}\rho}_{f} \otimes \ldots$ with $x_1, x_2, \ldots, x_f \in X$. There is a unique irreducible $\tau \otimes \theta \leq r_{M,G}\pi$ (where $P = MU$ is the standard parabolic subgroup having Levi factor $M = GL(n_\tau, F) \otimes GL(n_\pi - n_\tau, F)$) such that $\underbrace{\nu^{x_1}\rho \otimes \nu^{x_2}\rho \otimes \cdots \otimes \nu^{x_f}\rho}_{f} \otimes \ldots$ lies in $r_{min}(\tau \otimes \theta)$ (cf. Lemma 2.1.2). An algorithm for calculating this τ and θ is also given. Dual results, focusing on the rightmost terms in the tensor product rather than the leftmost terms, are discussed in section 2.4

We now discuss the applications of these results, which are given in section 3. In section 3.1, we look at the situation where π is a regular irreducible representation. In this case, our results may be used to write π as an alternating sum of induced representations, similar to the formula for the Steinberg representation (cf. Proposition 3.1.3). We remark that such a formula is given in 9.13 of [Zel], though our approach is a bit different.

In section 3.2, we relate the Langlands classification to the Zelevsinky classification (cf. Proposition 3.2.4; also Remark 3.2.5). In [Zel], Zelevinsky gives a classification for irreducible representations of general linear groups which is essentially dual to the Langlands classification. This is used in [Zel] and a number of papers which build on [Zel]. In this section, we show how the two classifications are related and discuss how to convert between the Langlands data and the Zelevinsky data. This conversion is discussed modulo the ability to calculate the dual of a representation, dual here being in the sense of [Aub], [S-S] (i.e., the Zelevinsky involution), which is discussed in section 3.3.

In section 3.3, we discuss how our results may be used to calculate the dual to π. In particular, we obtain an algorithm for calculating the dual of π from π. This has a similar flavor to the algorithm of [M-W], though again from a different point of view. (There is also a more combinatorial description given in [K-Z].)

One of the interests in starting this project was the goal of being able to calculate Jacquet modules for π. More precisely, the goal is to calculate $r_{M,G}\pi$, where $P = MU$ is the standard parabolic subgroup having Levi factor $M = GL(n_\rho, F) \times$

$GL(n_\pi - n_\rho, F)$. The ability to this in general would imply the ability to calculate $r_{min}\pi$ (by iteration). This is a difficult problem with a number of interesting consequences, which we discuss momentarily. In section 3.4, we give a couple of cases where this can be done (under the assumption that Jacquet modules for lower rank groups are known). Of course, if $f_\pi(a) \leq 1$ for all a, $r_{M,G}\pi$ may be read off from Theorem 2.2.1. Thus our concern is the cases where $f_\pi(a) > 1$.

We discuss two situations where this may be done. The first situation (cf. Proposition 3.4.3) is when there is a b_j such that (1) $r_{min}\pi$ contains a term of the form $\cdots \otimes \nu^{b_j}\rho$, (2) $a_i \neq b_j, b_j - 1$ for all i, and (3) $b_i \neq b_{j+1}$ for all i. (Note that if the lowest value of b_j which appears is less than a_1, conditions (2) and (3) will automatically be satisfied.) The second situation is when the supercuspidal support of π is $\{\nu\rho, \nu^2\rho, \nu^3\rho\}$.

As mentioned above, the ability to calculate Jacquet modules in general has a couple of noteworthy consequences. The first is that it would allow one to decompose arbitrarily induced representations. From a result of Zelevinsky (cf. section 1.2 of this paper), to decompose a general induced representation, it suffices to be able to decompose induced representations of the form $\mathrm{Ind}_P^G(\pi_1 \otimes \pi_2)$ with π_i an irreducible representation of $GL(n_{\pi_i}, F)$ supported on $\{\nu^z\rho\}_{z\in\mathbb{Z}}$ and P the standard parabolic subgroup of $GL(n_{\pi_1}+n_{\pi_2}, F)$ having Levi factor $GL(n_{\pi_1}, F)\times GL(n_{\pi_2}, F)$. Since we can calculate $r_{min}\pi_1$ and $r_{min}\pi_2$, we can calculate $r_{min}(\mathrm{Ind}_P^G(\pi_1 \otimes \pi_2))$ (using Lemma 2.12 of [B-Z] or the results from section 6 of [Cas1]). Consider the lowest term in $\mathrm{Ind}_P^G(\pi_1 \otimes \pi_2)$ with respect to the lexicographic ordering (cf. section 1.2). From this term, we can read off the Langlands data of one irreducible subquotient of $\mathrm{Ind}_P^G(\pi_1 \otimes \pi_2)$ (cf. section 1.2); call it λ_1. We then calculate $r_{min}\lambda_1$ and subtract the result from $r_{min}(\mathrm{Ind}_P^G(\pi_1 \otimes \pi_2))$. From the lowest lexicographic term in $r_{min}(\mathrm{Ind}_P^G(\pi_1 \otimes \pi_2)) - r_{min}\lambda_1$, we can read off the Langlands data of another irreducible subquotient; call it λ_2. We iterate, looking at $r_{min}(\mathrm{Ind}_P^G(\pi_1 \otimes \pi_2)) - r_{min}\lambda_1 - r_{min}\lambda_2$, etc., until we have accounted for all of $r_{min}(\mathrm{Ind}_P^G(\pi_1 \otimes \pi_2))$.

We note that the solution to this problem has another consequence. If the calculations of Jacquet modules do not depend on the particular ρ in the supercuspidal support, which is the case with everything done in this paper, then the multiplicities of the irreducible subquotients of induced representations also do not depend on the particular ρ in the supercuspidal support. Consequently, the Hopf subalgebras $R(\rho)$ are all isomorphic (cf. section 1.2). This was conjectured by Zelevinsky and essentially follows from the Hecke algebra results of chapter 7 of [B-K]. However, this would provide a proof which does not depend on the classification of supercuspidal representations, hence would have a better chance of generalizing to classical groups (at least at this point in time).

1.2. **Notation and preliminaries.** In this section, we introduce notation and recall some results that will be needed in the rest of the paper.

Let F be a p-adic field with $\mathrm{char}\, F = 0$. Let $|\cdot|$ denote the absolute value on F, normalized so that $|\varpi| = q^{-1}$, ϖ a uniformizer. As in [B-Z], we let $\nu = |det|$ on $GL(n, F)$ (with the value of n clear from context). Define $\times$ for general linear groups as in [B-Z]: if $\rho_1, \ldots, \rho_k$ are representations of $GL(n_1, F), \ldots, GL(n_k, F)$, let $\rho_1 \times \cdots \times \rho_k$ denote the representation of $GL(n_1 + \cdots + n_k, F)$ obtained by inducing $\rho_1 \otimes \cdots \otimes \rho_k$ from the standard parabolic subgroup of $GL(n_1 + \cdots + n_k, F)$ with Levi factor $GL(n_1, F) \times \cdots \times GL(n_k, F)$.

Frequently, we work in the Grothendieck group setting. That is, we work with the semisimplified representation. So, for any representation π and irreducible representation ρ, let $m(\rho,\pi)$ denote the multiplicity of ρ in π. We write $\pi = \pi_1+\cdots+\pi_k$ if $m(\rho,\pi)=m(\rho,\pi_1)+\cdots+m(\rho,\pi_k)$ for every irreducible ρ. Similarly, we write $\pi \geq \pi_0$ if $m(\rho,\pi)\geq m(\rho,\pi_0)$ for every such ρ. For clarity, we use $=$ when defining something or working in the Grothendieck group; $\cong$ is used to denote an actual equivalence.

We recall some notation of Bernstein and Zelevinsky ([B-Z]). If $P=MU$ is a standard parabolic subgroup of G and ξ a representation of M, we let $i_{G,M}(\xi)$ denote the representation obtained by (normalized) parabolic induction. Similarly, if π is a representation of G, we let $r_{M,G}(\pi)$ denote the (normalized) Jacquet module of π with respect to P.

As we work with representations supported on $\{\nu^z\rho\}_{z\in\mathbb{Z}}$, we modify the notation of Tadić a bit. Suppose ρ is a representation of $GL(r,F)$ (defining r) and π a representation of $GL(mr,F)$ supported on $\{\nu^z\rho\}_{z\in\mathbb{Z}}$. If $(\alpha)=(m_1,\dots,m_t)$ is a partition of $k\leq m$, we let $r_{(\alpha)}\pi$ denote the Jacquet module $r_{M,G}\pi$, where M is the standard Levi of the form $GL(m_1r,F)\times\cdots\times GL(m_tr,F)\times GL((m-k)r,F)$. For convenience, we write $r_{min}\pi$ for $r_{\underbrace{(1,1,\dots,1)}_{m-1}}\pi$; M_{min} is the smallest standard Levi such that $r_{M_{min},G}\pi\neq 0$. Also, on occasions when it is awkward to describe the standard Levi factor M, we may simply write $r_{app}\pi$ for $r_{M,G}\pi$ (i.e., the Jacquet module taken with respect to the appropriate standard parabolic subgroup). Such notation will only be used in cases where a subquotient of $r_{M,G}\pi$ is explicitly written out, in which case the appropriate M is clear from context.

It will also be useful to have the following shorthand. If τ is an irreducible representation of $GL(m_1r,F)\times\cdots\times GL(m_sr,F)$, we define $r_\tau\pi$ to be the sum of everything in $r_{(m_1,\dots,m_s)}\pi$ which has the form $\tau\otimes\theta$.

We recall some structures which will be useful later (cf. section 1 of [Zel]). Let $R(GL(n,F))$ denote the Grothendieck group of the category of all smooth finite-length $GL(n,F)$-modules. Set $R=\bigoplus_{n\geq 0}R(GL(n,F))$. Then $\times$ lifts naturally to a multiplication

$$\times : R\otimes R\longrightarrow R.$$

If π is a representation of $GL(n,F)$, set

$$m^*(\tau)=\sum_{i=0}^{n} r_{(i)}(\tau).$$

Observe that we may lift m^* to a map $m^*:R\longrightarrow R\otimes R$. With multiplication given by $\times$ and comultiplication given by m^* (and antipode given by the Zelevinsky involution; cf. section 3.3), R has the structure of a Hopf algebra.

We now review some results on induced representations for $GL(n,F)$. This is based on the work of Zelevinsky ([Zel]).

First, if ρ is an irreducible supercuspidal representation of $GL(r,F)$ and $m\equiv n \bmod 1$, we define the segment

$$[\nu^m\rho,\nu^n\rho]=\nu^m\rho,\nu^{m+1}\rho,\dots\nu^n\rho.$$

We note that the induced representation $\nu^m\rho\times\nu^{m+1}\rho\times\cdots\times\nu^n\rho$ has a unique irreducible subrepresentation, which we denote by $\zeta([\nu^m\rho,\nu^n\rho])$, and a unique irreducible quotient, which we denote by $\delta([\nu^m\rho,\nu^n\rho])$.

Lemma 1.2.1. *Let $(\rho_1, \alpha_1), \ldots, (\rho_s, \alpha_s)$ be distinct pairs with $\rho_1, \ldots, \rho_s$ irreducible unitary supercuspidal representations of $GL(r_1, F), \ldots, GL(r_s, F)$ and $\alpha_1, \ldots, \alpha_s \in \mathbb{R}$ with $0 \leq \alpha_1, \ldots, \alpha_s < 1$. Let $\tau(\rho_i, \alpha_i)$ be an irreducible representation of a general linear group supported on $\{\nu^{\alpha_i+z}\rho_i\}_{z\in\mathbb{Z}}$. Let M be the standard Levi subgroup of $G = GL(n, F)$ which admits $\tau(\rho_1, \alpha_1) \otimes \cdots \otimes \tau(\rho_s, \alpha_s)$ as a representation. Then,*

(1) *$\tau(\rho_1, \alpha_1) \times \cdots \times \tau(\rho_s, \alpha_s)$ is irreducible.*

(2)
$$mult(\tau(\rho_1, \alpha_1) \otimes \cdots \otimes \tau(\rho_s, \alpha_s), r_{M,G}(\tau(\rho_1, \alpha_1) \times \cdots \times \tau(\rho_s, \alpha_s))) = 1.$$

Furthermore, if $\tau'(\rho_i, \alpha_i)$ is an irreducible representation of a general linear group supported on $\{\nu^{\alpha}\rho_i\}_{\alpha\in\alpha_i+\mathbb{Z}}$, then
$$mult(\tau'(\rho_1, \alpha_1) \otimes \cdots \otimes \tau'(\rho_s, \alpha_s), r_{M,G}(\tau(\rho_1, \alpha_1) \times \cdots \times \tau(\rho_s, \alpha_s))) = 0$$
unless $\tau'(\rho_i, \alpha_i) \cong \tau(\rho_i, \alpha_i)$ for all i.

(3) *If π is an irreducible representation of $GL(n, F)$ and $r_{M,G}(\pi) \geq \tau(\rho_1, \alpha_1) \otimes \cdots \otimes \tau(\rho_s, \alpha_s)$, then*
$$\pi = \tau(\rho_1, \alpha_1) \times \cdots \times \tau(\rho_s, \alpha_s).$$

Proof. The first claim is an immediate consequence of Proposition 8.5 of [Zel]. Claims (2) and (3) follow fairly easily; see Corollary 5.6 and section 10 of [Jan2] for details. □

Let $\rho_1, \ldots, \rho_s$ be irreducible unitary supercuspidal representations and $\alpha_1, \ldots, \alpha_s \in \mathbb{R}$ with $0 \leq \alpha_1, \ldots, \alpha_s < 1$. We assume the pairs (ρ_i, α_i) are distinct, i.e., if $\rho_i \cong \rho_j$ for $i \neq j$, then $\alpha_i \neq \alpha_j$. We let $R\left((\rho_1, \alpha_1), \ldots, (\rho_s, \alpha_s)\right)$ denote the subalgebra of R generated by representations supported on $\{\nu^{\alpha_1+z}\rho_1\}_{z\in\mathbb{Z}} \cup \cdots \cup \{\nu^{\alpha_s+z}\rho_s\}_{z\in\mathbb{Z}}$. It follows from the work of Zelevinsky that
$$R\left((\rho_1, \alpha_1), \ldots, (\rho_s, \alpha_s)\right) \cong R\left((\rho_1, \alpha_1)\right) \otimes \cdots \otimes R\left((\rho_s, \alpha_s)\right)$$
as Hopf algebras. In particular, in one direction the isomorphism is determined by
$$\tau(\rho_1, \alpha_1) \otimes \cdots \otimes \tau(\rho_s, \alpha_s) \longmapsto \tau(\rho_1, \alpha_1) \times \cdots \times \tau(\rho_s, \alpha_s)$$
for irreducible $\tau(\rho_i, \alpha_i)$. In the other direction, one may use Jacquet modules to determine the image of an irreducible representation in $R\left((\rho_1, \alpha_1), \ldots, (\rho_s, \alpha_s)\right)$. Lemma 1.2.1 above provides the necessary technical results. When $\alpha = 0$, we write $R(\rho)$ rather than $R\left((\rho, 0)\right)$. We note that the map $\pi \mapsto \nu^{\alpha}\pi$ gives an isomorphism $R(\rho) \cong R\left((\rho, \alpha)\right)$. Thus we focus on the subalgebras of the form $R(\rho)$.

Let us briefly review the Langlands classification (cf. [Sil], [B-W], [Kon]) for general linear groups. First, if δ is an essentially square-integrable representation of $GL(n, F)$, then there is an $\varepsilon(\delta) \in \mathbb{R}$ such that $\nu^{-\varepsilon(\delta)}\delta$ is unitarizable. Suppose $\delta_1, \ldots, \delta_k$ are irreducible, essentially square-integrable representations of $GL(n_1, F), \ldots, GL(n_k, F)$ with $\varepsilon(\delta_1) \leq \cdots \leq \varepsilon(\delta_k)$. (We allow weak inequalities since we are assuming δ_i is essentially square-integrable; if we allowed δ_i essentially tempered, we would have strict inequalities. The formulations are equivalent.) Then, $\delta_1 \times \cdots \times \delta_k$ has a unique irreducible subrepresentation (Langlands subrepresentation), and this representation appears with multiplicity one in $\delta_1 \times \cdots \times \delta_k$. Furthermore, any irreducible representation of a general linear group may be realized in this way. We favor the subrepresentation version of the Langlands classification over the quotient version since $\pi \hookrightarrow \delta_1 \times \cdots \times \delta_k$ tells us that $\delta_1 \otimes \cdots \otimes \delta_k$ appears in the (appropriate) Jacquet module for π.

Let ρ be an irreducible unitary supercuspidal representation of $GL(r,F)$. As in [Jan4], we use the notation $\succeq$ for the lexicographic ordering on tensor products of terms of the form $\nu^s\rho$. That is, if $\chi_1=\nu^{x_1}\rho\otimes\cdots\otimes\nu^{x_s}\rho$ and $\chi_2=\nu^{y_1}\rho\otimes\cdots\otimes\nu^{y_s}\rho$, we have $\chi_1\succeq\chi_2$ if $x_1>y_1$, or $x_1=y_1$ and $x_2>y_2$, etc. Note that this is a total ordering. Let π be a representation of $GL(nr,F)$ of finite length supported on $\{\nu^\alpha\rho\}_{\alpha\in\mathbb{Z}}$. Then, let $\chi_0(\pi)$ denote the lowest element of $r_{min}(\pi)$ with respect to the lexicographic order. $\chi_0(\pi)$ is unique up to multiplicity.

Lemma 1.2.2. *$\chi_0(\pi)$ has the form*

$$\chi_0(\pi)=(\nu^{a_1}\rho\otimes\nu^{a_1-1}\rho\otimes\cdots\otimes\nu^{b_1})\otimes\cdots\otimes(\nu^{a_k}\rho\otimes\nu^{a_k-1}\rho\otimes\cdots\otimes\nu^{b_k}\rho),$$

with $a_1\leq a_2\leq\cdots\leq a_k$ and $a_i\in\mathbb{Z}$ for all i.

Proof. This is Lemma 2.2.2 of [Jan4]. □

With notation as above, if

$$\chi_0(\pi)=(\nu^{a_1}\rho\otimes\nu^{a_1-1}\rho\otimes\cdots\otimes\nu^{b_1}\rho)\otimes\cdots\otimes(\nu^{a_k}\rho\otimes\nu^{a_k-1}\rho\otimes\cdots\otimes\nu^{b_k}\rho),$$

set

$$\delta_0(\pi)=\delta([\nu^{b_1}\rho,\nu^{a_1}\rho])\otimes\cdots\otimes\delta([\nu^{b_k}\rho,\nu^{a_k}\rho]).$$

Then, by Corollary 2.2.4 of [Jan4],

$$\pi\hookrightarrow i_{G,M}(\delta_0(\pi)).$$

For convenience, we write $\chi_1\succ\chi_2$ if $\chi_1\succeq\chi_2$ and $\chi_1\neq\chi_2$. As a minor abuse of notation, we also apply this notation to δ_0.

Suppose τ_i,τ_i', $i=1,\ldots,k$, are representations of $GL(n_i,F),GL(n_i',F)$, resp. Let us say $\tau_1\otimes\cdots\otimes\tau_k$ and $\tau_1'\otimes\cdots\otimes\tau_k'$ are inductively equivalent if $\tau_1\times\cdots\times\tau_k\cong\tau_1'\times\cdots\times\tau_k'$. Then, as discussed in section 2.4 of [Jan4], for an irreducible representation π supported on $\{\nu^\alpha\rho\}_{\alpha\in\mathbb{Z}}$, we have $\delta_0(\pi)$ inductively equivalent to the Langlands data for π. For this reason, we shall freely move between $\delta_0(\pi)$ and the Langlands data for π, referring to both as the Langlands classification and freely using both in the notation for the Langlands subrepresentation. In fact, we extend this convention to anything inductively equivalent to the Langlands data. We remark that the reason for also using the δ_0 point of view is that the underlying ordering is a total ordering, unlike that of the Langlands classification. If $\delta_1\otimes\cdots\otimes\delta_k$ is Langlands data, we write $L(\delta_1,\ldots,\delta_k)$ for the unique irreducible subrepresentation of $\delta_1\times\cdots\times\delta_k$ (the Langlands subrepresentation).

In general, if one considers the standard representation obtained by inducing Langlands data, the Langlands subrepresentation is minimal with respect to a partial order defined by Langlands (cf. [B-W]). We may strengthen this somewhat for general linear groups by using the total ordering above:

Proposition 1.2.3. *Let $\pi\in R(\rho)$ be an irreducible representation. Suppose $\pi'\leq Ind_P^G(\delta_0(\pi))$ is an irreducible subquotient. Then*

$$\chi_0(\pi)\preceq\chi_0(\pi'),$$

with equality if and only if $\pi\cong\pi'$. Furthermore, we note that π occurs with multiplicity one in $Ind_P^G(\delta_0(\pi))$.

Proof. The first claim follows from Lemma 2.4.2 of [Jan4]. Multiplicity one is part of the Langlands classification. □

1.3. **Assorted lemmas.**

Lemma 1.3.1. *Let* $\pi \leq \delta([\nu^{b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{b_k}\rho, \nu^{a_k}\rho])$ *be an irreducible representation, where* $a_1 \leq \cdots \leq a_k$. *Then,*

$$\pi = L(\delta([\nu^{b'_1}\rho, \nu^{a_1}\rho]), \delta([\nu^{b'_2}\rho, \nu^{a_2}\rho]), \ldots, \delta([\nu^{b'_k}\rho, \nu^{a_k}\rho]))$$

for some permutation $b'_1, b'_2, \ldots, b'_k$ *of* $b_1, b_2, \ldots, b_k$.

Proof. This follows from [Zel] or an argument like that in Proposition 5.3.2 of [Jan4] (but this case is simpler). □

Corollary 1.3.2. *Let*

$$\pi = L(\delta([\nu^{b_1}\rho, \nu^{a_1}\rho]), \ldots, \delta([\nu^{b_i}\rho, \nu^{a_i}\rho]), \delta([\nu^{b_{i+1}}\rho, \nu^{a}\rho]), \ldots, \delta([\nu^{b_j}\rho, \nu^{a}\rho]), \quad \delta([\nu^{b_{j+1}}\rho, \nu^{a_{j+1}}\rho]), \ldots, \delta([\nu^{b_k}\rho, \nu^{a_k}\rho]))$$

with

$$a_1 \leq a_2 \leq \cdots \leq a_i < a < a_{j+1} \leq a_{j+2} \leq \cdots \leq a_k.$$

If $\nu^a\rho \otimes \theta \leq r_{\nu^a\rho}\pi$ *with* θ *irreducible, then*

$$\theta = L(\delta([\nu^{b'_1}\rho, \nu^{a_1}\rho]), \ldots, \delta([\nu^{b'_i}\rho, \nu^{a_i}\rho]), \delta([\nu^{b'_{i+1}}\rho, \nu^{a-1}\rho]), \delta([\nu^{b'_{i+2}}\rho, \nu^{a}\rho]), \ldots, \delta([\nu^{b'_j}\rho, \nu^{a}\rho]), \delta([\nu^{b'_{j+1}}\rho, \nu^{a_{j+1}}\rho]), \ldots, \delta([\nu^{b'_k}\rho, \nu^{a_k}\rho]))$$

for some permutation $b'_1, b'_2, \ldots, b'_k$ *of* $b_1, b_2, \ldots, b_k$.

Proof. We have

$$\pi \leq \delta([\nu^{b_1}\rho, \nu^{a_1}\rho]) \times \cdots \times \delta([\nu^{b_i}\rho, \nu^{a_i}\rho]) \times \delta([\nu^{b_{i+1}}\rho, \nu^{a}\rho]) \times \cdots \times \delta([\nu^{b_j}\rho, \nu^{a}\rho]) \times \delta([\nu^{b_{j+1}}\rho, \nu^{a_{j+1}}\rho]) \times \cdots \times \delta([\nu^{b_k}\rho, \nu^{a_k}\rho])$$

and

$$r_{\nu^a\rho}\delta([\nu^{b_\ell}\rho, \nu^{a_\ell}\rho]) = \begin{cases} \nu^a\rho \otimes \delta([\nu^{b_\ell}\rho, \nu^{a-1}\rho]) & \text{if } i+1 \leq \ell \leq j, \\ 0 & \text{if not.} \end{cases}$$

The corollary now follows from Lemma 1.3.1 and m^* considerations. □

Lemma 1.3.3. *Suppose* $\delta([\nu^b\rho, \nu^a\rho]) \times \delta([\nu^{b_i}\rho, \nu^{a_i}\rho])$ *is irreducible for* $i = 1, \ldots, k$. *Then* $\delta([\nu^b\rho, \nu^a\rho]) \times L(\delta([\nu^{b_1}\rho, \nu^{a_1}\rho]), \ldots, \delta([\nu^{b_k}\rho, \nu^{a_k}\rho]))$ *is also irreducible.*

Proof. The proof of the lemma is based on that of Theorem 2.6 of [Jan3].

Let i be such that $\frac{a_i+b_i}{2} \leq \frac{a+b}{2} < \frac{a_{i+1}+b_{i+1}}{2}$, i.e., $\delta([\nu^b\rho, \nu^a\rho])$ occurs between $\delta([\nu^{b_i}\rho, \nu^{a_i}\rho])$ and $\delta([\nu^{b_{i+1}}\rho, \nu^{a_{i+1}}\rho])$ in the ordering for the Langlands classification. Let

$$\pi = L(\delta([\nu^{b_1}\rho, \nu^{a_1}\rho]), \ldots, \delta([\nu^{b_i}\rho, \nu^{a_i}\rho]), \delta([\nu^{b}\rho, \nu^{a}\rho]), \delta([\nu^{b_{i+1}}\rho, \nu^{a_{i+1}}\rho]), \ldots, \delta([\nu^{b_k}\rho, \nu^{a_k}\rho])).$$

We first claim that π is the unique irreducible subrepresentation of $\delta([\nu^b\rho,\nu^a\rho])\times L(\delta([\nu^{b_1}\rho,\nu^{a_1}\rho]),\dots,\delta([\nu^{b_k}\rho,\nu^{a_k}\rho]))$. To this end, observe that

$$\begin{aligned}
&\delta([\nu^b\rho,\nu^a\rho])\times L(\delta([\nu^{b_1}\rho,\nu^{a_1}\rho]),\dots,\delta([\nu^{b_k}\rho,\nu^{a_k}\rho]))\\
&\quad\hookrightarrow \delta([\nu^b\rho,\nu^a\rho])\times\delta([\nu^{b_1}\rho,\nu^{a_1}\rho])\times\cdots\times\delta([\nu^{b_i}\rho,\nu^{a_i}\rho])\\
&\qquad\qquad\qquad\times\delta([\nu^{b_{i+1}}\rho,\nu^{a_{i+1}}\rho])\times\cdots\times\delta([\nu^{b_k}\rho,\nu^{a_k}\rho])\\
&\quad\cong \delta([\nu^{b_1}\rho,\nu^{a_1}\rho])\times\delta([\nu^b\rho,\nu^a\rho])\times\cdots\times\delta([\nu^{b_i}\rho,\nu^{a_i}\rho])\\
&\qquad\qquad\qquad\times\delta([\nu^{b_{i+1}}\rho,\nu^{a_{i+1}}\rho])\times\cdots\times\delta([\nu^{b_k}\rho,\nu^{a_k}\rho])\\
&\quad\vdots\\
&\quad\cong \delta([\nu^{b_1}\rho,\nu^{a_1}\rho])\times\cdots\times\delta([\nu^b\rho,\nu^a\rho])\times\delta([\nu^{b_i}\rho,\nu^{a_i}\rho])\\
&\qquad\qquad\qquad\times\delta([\nu^{b_{i+1}}\rho,\nu^{a_{i+1}}\rho])\times\cdots\times\delta([\nu^{b_k}\rho,\nu^{a_k}\rho])\\
&\quad\cong \delta([\nu^{b_1}\rho,\nu^{a_1}\rho])\times\cdots\times\delta([\nu^{b_i}\rho,\nu^{a_i}\rho])\times\delta([\nu^b\rho,\nu^a\rho])\\
&\qquad\qquad\qquad\times\delta([\nu^{b_{i+1}}\rho,\nu^{a_{i+1}}\rho])\times\cdots\times\delta([\nu^{b_k}\rho,\nu^{a_k}\rho]),
\end{aligned}$$

noting that we may "commute" $\delta([\nu^b\rho,\nu^a\rho])$ to the right since $\delta([\nu^b\rho,\nu^a\rho])\times\delta([\nu^{b_j}\rho,\nu^{a_j}\rho])\cong\delta([\nu^{b_j}\rho,\nu^{a_j}\rho])\times\delta([\nu^b\rho,\nu^a\rho])$ (by the irreducibility hypothesis for $j=1,\dots,i$). By the Langlands classification, π is the unique irreducible subrepresentation of $\delta([\nu^{b_1}\rho,\nu^{a_1}\rho])\times\cdots\times\delta([\nu^{b_i}\rho,\nu^{a_i}\rho])\times\delta([\nu^b\rho,\nu^a\rho])\times\delta([\nu^{b_{i+1}}\rho,\nu^{a_{i+1}}\rho])\times\cdots\times\delta([\nu^{b_k}\rho,\nu^{a_k}\rho])$. The claim follows. Therefore, $\tilde{\pi}$ is the unique irreducible quotient of $\delta(\widetilde{[\nu^b\rho,\nu^a\rho]})\times L(\delta(\widetilde{[\nu^{b_1}\rho,\nu^{a_1}\rho]),\dots,}\delta([\nu^{b_k}\rho,\nu^{a_k}\rho]))$. By Lemma 1.1 of [Jan3],

$$\begin{aligned}
\tilde{\pi}=L(\delta([\nu^{-a_k}\tilde{\rho},\nu^{-b_k}\tilde{\rho}]),\dots,\delta([\nu^{-a_{i+1}}\tilde{\rho},\nu^{-b_{i+1}}\tilde{\rho}]),\delta([\nu^{-a}\tilde{\rho},\nu^{-b}\tilde{\rho}]),&\\
\delta([\nu^{-a_i}\tilde{\rho},\nu^{-b_i}\tilde{\rho}]),\dots,\delta([\nu^{-a_1}\tilde{\rho},\nu^{-b_1}\tilde{\rho}])).&
\end{aligned}$$

and

$$\begin{aligned}
&\delta(\widetilde{[\nu^b\rho,\nu^a\rho]})\times L(\delta(\widetilde{[\nu^{b_1}\rho,\nu^{a_1}\rho]),\dots,}\delta([\nu^{b_k}\rho,\nu^{a_k}\rho]))\\
&\qquad\cong\delta([\nu^{-a}\tilde{\rho},\nu^{-b}\tilde{\rho}])\times L(\delta([\nu^{-a_k}\tilde{\rho},\nu^{-b_k}\tilde{\rho}]),\dots,\delta([\nu^{-a_1}\tilde{\rho},\nu^{-b_1}\tilde{\rho}])).
\end{aligned}$$

Next, we claim $\tilde{\pi}$ is the unique irreducible subrepresentation of $\delta([\nu^{-a}\tilde{\rho},\nu^{-b}\tilde{\rho}])\times L(\delta([\nu^{-a_k}\tilde{\rho},\nu^{-b_k}\tilde{\rho}]),\dots,\delta([\nu^{-a_1}\tilde{\rho},\nu^{-b_1}\tilde{\rho}]))$. Observe that

$$\begin{aligned}
&\delta([\nu^{-a}\tilde{\rho},\nu^{-b}\tilde{\rho}])\times L(\delta([\nu^{-a_k}\tilde{\rho},\nu^{-b_k}\tilde{\rho}]),\dots,\delta([\nu^{-a_1}\tilde{\rho},\nu^{-b_1}\tilde{\rho}]))\\
&\quad\hookrightarrow\delta([\nu^{-a}\tilde{\rho},\nu^{-b}\tilde{\rho}])\times\delta([\nu^{-a_k}\tilde{\rho},\nu^{-b_k}\tilde{\rho}])\times\cdots\times\delta([\nu^{-a_{i+1}}\tilde{\rho},\nu^{-b_{i+1}}\tilde{\rho}])\\
&\qquad\qquad\qquad\times\delta([\nu^{-a_i}\tilde{\rho},\nu^{-b_i}\tilde{\rho}])\times\cdots\times\delta([\nu^{-a_1}\tilde{\rho},\nu^{-b_1}\tilde{\rho}])\\
&\quad\cong\delta([\nu^{-a_k}\tilde{\rho},\nu^{-b_k}\tilde{\rho}])\times\cdots\times\delta([\nu^{-a_{i+1}}\tilde{\rho},\nu^{-b_{i+1}}\tilde{\rho}])\times\delta([\nu^{-a}\tilde{\rho},\nu^{-b}\tilde{\rho}])\\
&\qquad\qquad\qquad\times\delta([\nu^{-a_i}\tilde{\rho},\nu^{-b_i}\tilde{\rho}])\times\cdots\times\delta([\nu^{-a_1}\tilde{\rho},\nu^{-b_1}\tilde{\rho}]),
\end{aligned}$$

noting that here we may commute $\delta([\nu^{-a}\tilde{\rho},\nu^{-b}\tilde{\rho}])$ to the right since $\delta([\nu^{-a}\tilde{\rho},\nu^{-b}\tilde{\rho}])\times\delta([\nu^{-a_j}\tilde{\rho},\nu^{-b_j}\tilde{\rho}])\cong\delta([\nu^{-a_j}\tilde{\rho},\nu^{-b_j}\tilde{\rho}])\times\delta([\nu^{-a}\tilde{\rho},\nu^{-b}\tilde{\rho}])$ (by contragredience and the irreducibility hypothesis for $j=i+1,\dots,k$). Again, by the Langlands classification, $\tilde{\pi}$ is the unique irreducible subrepresentation of $\delta([\nu^{-a_k}\tilde{\rho},\nu^{-b_k}\tilde{\rho}])\times\cdots\times\delta([\nu^{-a_{i+1}}\tilde{\rho},\nu^{-b_{i+1}}\tilde{\rho}])\times\delta([\nu^{-a}\tilde{\rho},\nu^{-b}\tilde{\rho}])\times\delta([\nu^{-a_i}\tilde{\rho},\nu^{-b_i}\tilde{\rho}])\times\cdots\times\delta([\nu^{-a_1}\tilde{\rho},\nu^{-b_1}\tilde{\rho}])$, hence of $\delta([\nu^{-a}\tilde{\rho},\nu^{-b}\tilde{\rho}])\times L(\delta([\nu^{-a_k}\tilde{\rho},\nu^{-b_k}\tilde{\rho}]),\dots,\delta([\nu^{-a_1}\tilde{\rho},\nu^{-b_1}\tilde{\rho}]))$, as claimed.

We now have $\tilde{\pi}$ as both the unique irreducible subrepresentation and unique irreducible quotient of $\delta([\nu^{-a}\tilde{\rho}, \nu^{-b}\tilde{\rho}]) \times L(\delta([\nu^{-a_k}\tilde{\rho}, \nu^{-b_k}\tilde{\rho}]), \ldots, \delta([\nu^{-a_1}\tilde{\rho}, \nu^{-b_1}\tilde{\rho}]))$. Since it appears with multiplicity one, this implies that $\delta([\nu^{-a}\tilde{\rho}, \nu^{-b}\tilde{\rho}]) \times L(\delta([\nu^{-a_k}\tilde{\rho}, \nu^{-b_k}\tilde{\rho}]), \ldots, \delta([\nu^{-a_1}\tilde{\rho}, \nu^{-b_1}\tilde{\rho}]))$ is irreducible. The lemma follows from this by contragredience. □

We note the similarity of the following lemma to Lemmes II.8 and II.10.1 of [M-W]. (We generalize part (1) of this lemma in Lemma 2.1.3.)

Lemma 1.3.4. (1) *Suppose $b_1 < b_2 \le b_3$. Then,*

$$\begin{aligned} &L(\delta([\nu^{b_1}\rho, \nu^{a-1}\rho]), \delta([\nu^{b_2}\rho, \nu^{a}\rho]), \delta([\nu^{b_3}\rho, \nu^{a}\rho])) \\ &\qquad = L(\delta([\nu^{b_1}\rho, \nu^{a-1}\rho]), \delta([\nu^{b_2}\rho, \nu^{a}\rho])) \times \delta([\nu^{b_3}\rho, \nu^{a}\rho]). \end{aligned}$$

(2) *Suppose $b_1 \le b_2 < b_3$. Then,*

$$\begin{aligned} &L(\delta([\nu^{b_1}\rho, \nu^{a-1}\rho]), \delta([\nu^{b_2}\rho, \nu^{a-1}\rho]), \delta([\nu^{b_3}\rho, \nu^{a}\rho])) \\ &\qquad = \delta([\nu^{b_1}\rho, \nu^{a-1}\rho]) \times L(\delta([\nu^{b_2}\rho, \nu^{a-1}\rho]), \delta([\nu^{b_3}\rho, \nu^{a}\rho])). \end{aligned}$$

Proof. We focus on (2); (1) is similar.

Let $\pi_I = \delta([\nu^{b_1}\rho, \nu^{a-1}\rho]) \times L(\delta([\nu^{b_2}\rho, \nu^{a-1}\rho]), \delta([\nu^{b_3}\rho, \nu^{a}\rho]))$. Since $\pi_I \le \delta([\nu^{b_1}\rho, \nu^{a-1}\rho]) \times \delta([\nu^{b_2}\rho, \nu^{a-1}\rho]) \times \delta([\nu^{b_3}\rho, \nu^{a}\rho])$, it follows from Lemma 1.3.1 that the only possible components of π_I are

$$\begin{aligned} \pi_1 &= L(\delta([\nu^{b_1}\rho, \nu^{a-1}\rho]), \delta([\nu^{b_2}\rho, \nu^{a-1}\rho]), \delta([\nu^{b_3}\rho, \nu^{a}\rho])), \\ \pi_2 &= L(\delta([\nu^{b_1}\rho, \nu^{a-1}\rho]), \delta([\nu^{b_3}\rho, \nu^{a-1}\rho]), \delta([\nu^{b_2}\rho, \nu^{a}\rho])), \\ \pi_3 &= L(\delta([\nu^{b_2}\rho, \nu^{a-1}\rho]), \delta([\nu^{b_3}\rho, \nu^{a-1}\rho]), \delta([\nu^{b_1}\rho, \nu^{a}\rho])) \end{aligned}$$

(noting that if $b_1 = b_2$, then $\pi_2 = \pi_3$). It follows from the Langlands classification that π_1 appears with multiplicity one in π_I. Thus, it remains to show that π_2, π_3 do not appear.

That π_3 does not appear follows from the observations that $r_{\nu^a\rho}(\pi_3) \ne 0$ (as $\pi_3 = L(\delta([\nu^{b_2}\rho, \nu^{a-1}\rho]), \delta([\nu^{b_3}\rho, \nu^{a-1}\rho])) \times \delta([\nu^{b_1}\rho, \nu^{a}\rho])$ by Lemma 1.3.3) and $r_{\nu^a\rho}(\pi_I) = 0$ (as $r_{\nu^a\rho}L(\delta([\nu^{b_2}\rho, \nu^{a-1}\rho]), \delta([\nu^{b_3}\rho, \nu^{a}\rho])) = 0$, which follows from [Zel]). Since $\pi_2 = \pi_3$ if $b_1 = b_2$, we may now assume $b_1 < b_2$. To see that π_2 does not appear in π_I, first observe that by Lemma 1.3.3,

$$\begin{gathered} \pi_2 = \delta([\nu^{b_3}\rho, \nu^{a-1}\rho]) \times L(\delta([\nu^{b_1}\rho, \nu^{a-1}\rho]), \delta([\nu^{b_2}\rho, \nu^{a}\rho])) \\ \Downarrow \\ r_{\delta([\nu^{b_1}\rho, \nu^{a-1}\rho]) \otimes \delta([\nu^{b_2}\rho, \nu^{a}\rho])}(\pi_2) \ne 0. \end{gathered}$$

On the other hand,

$$\begin{gathered} r_{\delta([\nu^{b_1}\rho, \nu^{a-1}\rho])}(\pi_I) = \delta([\nu^{b_1}\rho, \nu^{a-1}\rho]) \otimes L(\delta([\nu^{b_2}\rho, \nu^{a-1}\rho]), \delta([\nu^{b_3}\rho, \nu^{a}\rho])) \\ \Downarrow \\ r_{\delta([\nu^{b_1}\rho, \nu^{a-1}\rho]) \otimes \delta([\nu^{b_2}\rho, \nu^{a}\rho])}(\pi_I) = 0. \end{gathered}$$

Thus, we see that π_2 does not appear in π_I, finishing the proof. □

Lemma 1.3.5. *Suppose π is an irreducible representation with supercuspidal support contained in $\{\nu^{a-1}\rho, \nu^a\rho\}$. Then, π has the form*

$$\pi = L(\underbrace{\nu^{a-1}\rho, \ldots \nu^{a-1}\rho}_{\ell_1}, \underbrace{\delta([\nu^{a-1}\rho, \nu^{a}\rho]), \ldots, \delta([\nu^{a-1}\rho, \nu^{a}\rho])}_{\ell_2}, \underbrace{\nu^{a}\rho, \ldots, \nu^{a}\rho}_{\ell_3}).$$

Furthermore,

(1) *If* $\ell_3 \geq \ell_1$*, then*

$$\pi = \underbrace{L(\nu^{a-1}\rho, \nu^a\rho), \times \cdots \times L(\nu^{a-1}\rho, \nu^a\rho)}_{\ell_1} \times \underbrace{\delta([\nu^{a-1}\rho, \nu^a\rho]) \times \cdots \times \delta([\nu^{a-1}\rho, \nu^a\rho])}_{\ell_2} \times \underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{\ell_3-\ell_1}.$$

(2) *If* $\ell_3 \leq \ell_1$*, then*

$$\pi = \underbrace{\nu^{a-1}\rho \times \cdots \times \nu^{a-1}\rho}_{\ell_1-\ell_3} \times \underbrace{L(\nu^{a-1}\rho, \nu^a\rho), \times \cdots \times L(\nu^{a-1}\rho, \nu^a\rho)}_{\ell_3} \times \underbrace{\delta([\nu^{a-1}\rho, \nu^a\rho]) \times \cdots \times \delta([\nu^{a-1}\rho, \nu^a\rho])}_{\ell_2}.$$

Proof. That π has the given form is clear. The irreducibility of the induced representations in (1) and (2) follows easily from Lemma 1.3.3 and [Zel]. It is then straightforward to check that the induced representations in (1) and (2) actually correspond to π; e.g., in (1)

$$\begin{aligned}
&\underbrace{L(\nu^{a-1}\rho, \nu^a\rho), \times \cdots \times L(\nu^{a-1}\rho, \nu^a\rho)}_{\ell_1} \times \underbrace{\delta([\nu^{a-1}\rho, \nu^a\rho]) \times \cdots \times \delta([\nu^{a-1}\rho, \nu^a\rho])}_{\ell_2} \times \underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{\ell_3-\ell_1} \\
&\hookrightarrow \underbrace{\nu^{a-1}\rho \times \cdots \times \nu^{a-1}\rho}_{\ell_1} \times \underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{\ell_1} \times \underbrace{\delta([\nu^{a-1}\rho, \nu^a\rho]) \times \cdots \times \delta([\nu^{a-1}\rho, \nu^a\rho])}_{\ell_2} \times \underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{\ell_3-\ell_1} \\
&\cong \underbrace{\nu^{a-1}\rho \times \cdots \times \nu^{a-1}\rho}_{\ell_1} \times \underbrace{\delta([\nu^{a-1}\rho, \nu^a\rho]) \times \cdots \times \delta([\nu^{a-1}\rho, \nu^a\rho])}_{\ell_2} \times \underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{\ell_3}
\end{aligned}$$

(since $\nu^a\rho \times \delta([\nu^{a-1}\rho, \nu^a\rho]) \cong \delta([\nu^{a-1}\rho, \nu^a\rho]) \times \nu^a\rho$ by irreducibility). By the Langlands classification, this has π as a unique irreducible subrepresentation. (1) follows. (2) is similar. □

2. Terms of maximal initial frequency

2.1. **The definition of $f_\pi(a)$.**

Definition 2.1.1. Let $\pi \in R(\rho)$ be an irreducible representation, $X \subset \mathbb{Z}$. We define $f_\pi(X)$ to be the largest value of f such that $r_{min}\pi$ contains a term of the form $\nu^{x_1}\rho \otimes \cdots \otimes \nu^{x_f}\rho \otimes \ldots$ with $x_1, \ldots, x_f \in X$. If $X = \{a\}$, we write $f_\pi(a)$ rather than $f_\pi(\{a\})$. We let $m^*_X\pi$ be the sum of everything in $m^*\pi$ of the form $\tau \otimes \theta$ with τ, θ irreducible and $r_{min}\tau$ containing a term of the form $\nu^{x_1}\rho \otimes \cdots \otimes \nu^{x_f}\rho$ with $f = f_\pi(X)$ and $x_1, \ldots, x_f \in X$.

Lemma 2.1.2. *Suppose $\pi \in R(\rho)$ is an irreducible representation, $X \subset \mathbb{Z}$. Then, there are unique irreducible representations τ, θ and unique f such that the following are all satisfied:*

(1) $\pi \hookrightarrow \tau \times \theta$.
(2) *If* $\nu^{x_1}\rho \otimes \cdots \otimes \nu^{x_f}\rho \leq r_{min}\tau$, *then* $x_1, \ldots, x_f \in X$.
(3) $r_{\nu^x\rho}\theta = 0$ *for all* $x \in X$.

Furthermore, $f = f_\pi(X)$ *and* $m^*_X(\pi) = \tau \otimes \theta$. *We write* $\tau_\pi(X) \otimes \theta_\pi(X)$ *for this* $\tau \otimes \theta$.

In fact, $\pi \hookrightarrow \tau_\pi(X) \times \theta_\pi(X)$ *as the unique irreducible subrepresentation. In particular, if* π' *is an irreducible representation with* $m^*_X\pi = m^*_X\pi'$, *then* $\pi = \pi'$.

Proof. We begin with existence. Let $f = f_\pi(X)$. Then there is a $\nu^{x_1}\rho \otimes \cdots \otimes \nu^{x_f}\rho \otimes \cdots \leq r_{min}\pi$ with $x_1, \ldots, x_f \in X$. By central character considerations,

$$\begin{gathered}\pi \hookrightarrow (\nu^{x_1}\rho \times \cdots \times \nu^{x_f}\rho) \times (\nu^{x_{f+1}}\rho \times \cdots \times \nu^{x_n}\rho) \\ \Downarrow \\ \pi \hookrightarrow \tau \times \theta\end{gathered}$$

for some irreducible $\tau \leq \nu^{x_1}\rho \times \cdots \times \nu^{x_f}\rho$ and irreducible $\theta \leq \nu^{x_{f+1}}\rho \times \cdots \times \nu^{x_n}\rho$ (cf. Lemma 5.5 of [Jan2]). Observe that (1) and (2) clearly hold. For (3), observe that by Frobenius reciprocity, $m^*\pi \geq \tau \otimes \theta$. If $r_{\nu^x\rho}\theta \neq 0$ for some $x \in X$, then $r_{min}\pi \geq \nu^{x_1} \otimes \cdots \otimes \nu^{x_f}\rho \otimes \nu^x\rho \otimes \ldots$ with $x_1, \ldots, x_f, x \in X$, contradicting the definition of $f = f_\pi(X)$. Thus, (3) also holds.

We now address uniqueness. Suppose uniqueness failed. Let τ, θ, f and τ', θ', f' satisfy (1)–(3). Then,

$$\begin{aligned}&\pi \hookrightarrow \tau \otimes \theta \\ &\qquad \Downarrow \text{ (Frobenius reciprocity)} \\ &r_\tau\pi \neq 0 \\ &\qquad \Downarrow \text{ (since } \pi \hookrightarrow \tau' \times \theta' \text{)} \\ &r_\tau(\tau' \times \theta') \neq 0.\end{aligned}$$

Since $r_{\nu^x\rho}\theta' = 0$ for all $x \in X$, we have $r_\tau(\tau' \times \theta') \neq 0 \Rightarrow r_\tau\tau' \neq 0$. Reversing roles, we also have $r_{\tau'}\tau \neq 0$. Therefore $f = f'$. Furthermore, since τ, τ' are irreducible, this also implies $\tau \cong \tau'$. Now, $r_\tau\pi \leq r_\tau(\tau \times \theta) = \tau \otimes \theta$ and $r_\tau\pi \leq r_\tau(\tau \times \theta') = \tau \otimes \theta'$ give $\theta \cong \theta'$, as needed.

That $f = f_\pi(X)$ and $m^*_X\pi = \tau \otimes \theta$ follow immediately from the preceding arguments.

We now show that $\pi \hookrightarrow \tau_\pi(X) \times \theta_\pi(X)$ as the unique irreducible subrepresentation. We assume that both $\tau_\pi(X)$ and $\theta_\pi(X)$ are nontrivial (if not, the result is trivial). We now observe that $m^*(\tau_\pi(X) \times \theta_\pi(X))$ contains $\tau_\pi(X) \otimes \theta_\pi(X)$ with multiplicity one (since the supercuspidal support of $\tau_\pi(X)$ is contained in X but $r_{\nu^x\rho}\theta_\pi(X) = 0$ for all $x \in X$). Therefore, by Frobenius reciprocity, $\tau_\pi(X) \times \theta_\pi(X)$ has a unique irreducible subrepresentation, as needed. □

In the remainder of this section, we focus on the case $X = \{a\}$; more general X will be considered in section 2.3.

First, suppose $\pi = L(\delta([\nu^{b_1}\rho, \nu^{a-1}\rho]), \ldots, \delta([\nu^{b_i}\rho, \nu^{a-1}\rho]), \delta([\nu^{b_{i+1}}\rho, \nu^a\rho]), \ldots, \delta([\nu^{b_k}\rho, \nu^a\rho]))$. Let $\delta([\nu^{b'_1}\rho, \nu^{a'_1}\rho]), \ldots, \delta([\nu^{b'_k}\rho, \nu^{a'_k}\rho])$ be the permutation of $\delta([\nu^{b_1}\rho, \nu^{a-1}\rho]), \ldots, \delta([\nu^{b_i}\rho, \nu^{a-1}\rho]), \delta([\nu^{b_{i+1}}\rho, \nu^a\rho]), \ldots, \delta([\nu^{b_k}\rho, \nu^a\rho])$ satisfying the following: $b'_j \leq b'_{j+1}$ for all j, and if $b'_j = b'_{j+1}$, then $a'_j \geq a'_{j+1}$. We note that this is an inductively equivalent shuffle, so we may also write $\pi = L(\delta([\nu^{b'_1}\rho, \nu^{a'_1}\rho]), \ldots, \delta([\nu^{b'_k}\rho, \nu^{a'_k}\rho]))$. For $1 \leq j \leq k$, we define

$$n_a(j) = |\{x \leq j \mid a'_x = a\}|;$$

by convention, $n_a(0) = 0$ (which will be useful in the statement of Proposition 2.1.4 below). We have the following:

Lemma 2.1.3. *Suppose $\pi = L(\delta([\nu^{b'_1}\rho, \nu^{a'_1}\rho]), \ldots, \delta([\nu^{b'_k}\rho, \nu^{a'_k}\rho]))$ as above. Furthermore, suppose $n_a(k) - n_{a-1}(k) = 1$ but $n_a(j) \leq n_{a-1}(j)$ for all $j < k$. Then, $a'_k = a$ and*

$$\pi \cong \delta([\nu^{b'_k}\rho, \nu^{a'_k}\rho]) \times L(\delta([\nu^{b'_1}\rho, \nu^{a'_1}\rho]), \ldots, \delta([\nu^{b'_{k-1}}\rho, \nu^{a'_{k-1}}\rho]))$$

(with the induced representation irreducible).

Proof. That $a'_k = a$ is obvious.

We first show that $\pi \hookrightarrow \delta([\nu^{b'_k}\rho, \nu^{a'_k}\rho]) \times L(\delta([\nu^{b'_1}\rho, \nu^{a'_1}\rho]), \ldots, \delta([\nu^{b'_{k-1}}\rho, \nu^{a'_{k-1}}\rho]))$ as unique irreducible subrepresentation.

By the Langlands classification and the observation that $\delta([\nu^{b'_1}\rho, \nu^{a'_1}\rho]), \ldots, \delta([\nu^{b'_k}\rho, \nu^{a'_k}\rho])$ is inductively equivalent to $\delta([\nu^{b_1}\rho, \nu^{a-1}\rho]), \ldots, \delta([\nu^{b_i}\rho, \nu^{a-1}\rho]), \delta([\nu^{b_{i+1}}\rho, \nu^{a}\rho]), \ldots, \delta([\nu^{b_k}\rho, \nu^{a}\rho])$, we have

$$\pi \hookrightarrow \delta([\nu^{b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{b'_k}\rho, \nu^{a'_k}\rho])$$

as a unique irreducible subrepresentation.

Since $n_a(k) - n_{a-1}(k) = 1$, we see that $k = 2m+1$ (where $m = n_{a-1}(k)$ and $m+1 = n_a(k)$). Let $a'_{x_1}, \ldots, a'_{x_m}$ be those elements of $a'_1, \ldots, a'_k$ which are equal to $a-1$; without loss of generality, $x_1 < \cdots < x_m$. We note that if no such x_1 exists, the lemma holds trivially. We also note that $x_m < k-1$; if $a'_{k-1} = a-1$, then (noting $a'_k = a$), we would have $n_a(k-2) - n_{a-1}(k-2) = 1$, a contradiction.

Now, we claim

$$\begin{aligned}
\pi \hookrightarrow\ & \delta([\nu^{b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{b'_{x_m-1}}\rho, \nu^{a'_{x_m-1}}\rho]) \\
& \times \delta([\nu^{b'_{x_m}}\rho, \nu^{a-1}\rho]) \times \delta([\nu^{b'_{x_m+1}}\rho, \nu^{a}\rho]) \\
& \times \delta([\nu^{b'_{x_m+2}}\rho, \nu^{a}\rho]) \times \cdots \times \delta([\nu^{b'_k}\rho, \nu^{a}\rho]) \\
& \Downarrow \\
\pi \hookrightarrow\ & \delta([\nu^{b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{b'_{x_m-1}}\rho, \nu^{a'_{x_m-1}}\rho]) \\
& \times L(\delta([\nu^{b'_{x_m}}\rho, \nu^{a-1}\rho]), \delta([\nu^{b'_{x_m+1}}\rho, \nu^{a}\rho])) \\
& \times \delta([\nu^{b'_{x_m+2}}\rho, \nu^{a}\rho]) \times \cdots \times \delta([\nu^{b'_k}\rho, \nu^{a}\rho]),
\end{aligned}$$

as a unique irreducible subrepresentation. This follows immediately from the observation that

$$\begin{aligned}
& \delta([\nu^{b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{b'_{x_m-1}}\rho, \nu^{a'_{x_m-1}}\rho]) \\
& \quad \times L(\delta([\nu^{b'_{x_m}}\rho, \nu^{a-1}\rho]), \delta([\nu^{b'_{x_m+1}}\rho, \nu^{a}\rho])) \\
& \quad \times \delta([\nu^{b'_{x_m+2}}\rho, \nu^{a}\rho]) \times \cdots \times \delta([\nu^{b'_k}\rho, \nu^{a}\rho]) \\
& \quad \hookrightarrow \delta([\nu^{b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{b'_k}\rho, \nu^{a'_k}\rho])
\end{aligned}$$

which has π as a unique irreducible subrepresentation. When $x \geq x'_{m+2}$, we have $\delta([\nu^{b'_x}\rho, \nu^a\rho]) \times L(\delta([\nu^{b'_{x_m}}\rho, \nu^{a-1}\rho]), \delta([\nu^{b'_{x_{m+1}}}\rho, \nu^a\rho]))$ irreducible (by Lemma 1.3.4), so

$$\begin{aligned}\pi \hookrightarrow\ & \delta([\nu^{b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{b'_{x_m-1}}\rho, \nu^{a'_{x_m-1}}\rho]) \\ & \times \delta([\nu^{b'_{x_m+2}}\rho, \nu^a\rho]) \times \cdots \times \delta([\nu^{b'_k}\rho, \nu^a\rho]) \\ & \times L(\delta([\nu^{b'_{x_m}}\rho, \nu^{a-1}\rho]), \delta([\nu^{b'_{x_m+1}}\rho, \nu^a\rho]))\end{aligned}$$

as a unique irreducible subrepresentation. We let $y_m = x_{m+1}$.

We now apply the same argument starting with x_{m-1}. Observe that

$$\begin{aligned}\pi \hookrightarrow\ & \delta([\nu^{b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{b'_{x_{m-1}-1}}\rho, \nu^{a'_{x_{m-1}-1}}\rho]) \\ & \times \delta([\nu^{b'_{x_{m-1}}}\rho, \nu^{a-1}\rho]) \times \delta([\nu^{b'_{x_{m-1}+1}}\rho, \nu^a\rho]) \\ & \times \cdots \times \delta([\nu^{b'_{x_m-1}}\rho, \nu^a\rho]) \times \delta([\nu^{b'_{x_m+2}}\rho, \nu^a\rho]) \times \cdots \times \delta([\nu^{b'_k}\rho, \nu^a\rho]) \\ & \times L(\delta([\nu^{b'_{x_m}}\rho, \nu^{a-1}\rho]), \delta([\nu^{b'_{x_m+1}}\rho, \nu^a\rho]))\end{aligned}$$

has at least one of $\delta([\nu^{b'_{x_{m-1}+1}}\rho, \nu^a\rho]), \delta([\nu^{b'_{x_m+2}}\rho, \nu^a\rho])$ actually occuring (or else $n_a(x_{m-1}) - n_{a-1}(x_{m-1}) = 1$ since only $\delta([\nu^{b'_{x_m}}\rho, \nu^{a-1}\rho]), \delta([\nu^{b'_{y_m}}\rho, \nu^a\rho])$ are missing). Let

$$y_{m-1} = \begin{cases} x_{m-1}+1 & \text{if } \delta([\nu^{b'_{x_{m-1}+1}}\rho, \nu^a\rho]) \text{ occurs,} \\ x_{m+2} & \text{if not.}\end{cases}$$

Then, the same argument as above tells us that

$$\begin{aligned}\pi \hookrightarrow\ & \delta([\nu^{b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{b'_{x_{m-1}-1}}\rho, \nu^{a'_{x_{m-1}-1}}\rho]) \\ & \times L(\delta([\nu^{b'_{x_{m-1}}}\rho, \nu^{a-1}\rho]), \delta([\nu^{b'_{y_{m-1}}}\rho, \nu^a\rho])) \\ & \times \delta([\nu^{b'_{z_1}}\rho, \nu^a\rho]) \times \cdots \times \delta([\nu^{b'_{z_{k-x_{m-1}-4}}}\rho, \nu^a\rho]) \\ & \times L(\delta([\nu^{b'_{x_m}}\rho, \nu^{a-1}\rho]), \delta([\nu^{b'_{y_m}}\rho, \nu^a\rho])) \\ \cong\ & \delta([\nu^{b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{b'_{x_{m-1}-1}}\rho, \nu^{a'_{x_{m-1}-1}}\rho]) \\ & \times \delta([\nu^{b'_{z_1}}\rho, \nu^a\rho]) \times \cdots \times \delta([\nu^{b'_{z_{k-x_{m-1}-4}}}\rho, \nu^a\rho]) \\ & \times L(\delta([\nu^{b'_{x_{m-1}}}\rho, \nu^{a-1}\rho]), \delta([\nu^{b'_{y_{m-1}}}\rho, \nu^a\rho])) \\ & \times L(\delta([\nu^{b'_{x_m}}\rho, \nu^{a-1}\rho]), \delta([\nu^{b'_{y_m}}\rho, \nu^a\rho])),\end{aligned}$$

where $\delta([\nu^{b'_{z_1}}\rho, \nu^a\rho]), \ldots, \delta([\nu^{b'_{z_{k-x_{m-1}-4}}}\rho, \nu^a\rho])$ are just $\delta([\nu^{b'_{x_{m-1}+2}}\rho, \nu^a\rho]), \ldots,$ $\delta([\nu^{b'_{x_m-1}}\rho, \nu^a\rho]), \delta([\nu^{b'_{x_m+2}}\rho, \nu^a\rho]), \ldots, \delta([\nu^{b'_k}\rho, \nu^a\rho])$ with $\delta([\nu^{b'_{y_{m-1}}}\rho, \nu^a\rho])$ removed. Iterating this argument, we eventually obtain

$$\begin{aligned}\pi \hookrightarrow\ & \delta([\nu^{b'_k}\rho, \nu^a\rho]) \times L(\delta([\nu^{b'_{x_1}}\rho, \nu^{a-1}\rho]), \delta([\nu^{b'_{y_1}}\rho, \nu^a\rho])) \times \ldots \\ & \times L(\delta([\nu^{b'_{x_m}}\rho, \nu^{a-1}\rho]), \delta([\nu^{b'_{y_m}}\rho, \nu^a\rho]))\end{aligned} \tag{1}$$

as a unique irreducible subrepresentation.

By Lemma 5.5 of [Jan2],

$$\begin{aligned}\pi &\hookrightarrow \delta([\nu^{b'_k}\rho,\nu^a\rho]) \times L(\delta([\nu^{b'_{x_1}}\rho,\nu^{a-1}\rho]),\delta([\nu^{b'_{y_1}}\rho,\nu^a\rho])) \times \dots \\ &\quad \times L(\delta([\nu^{b'_{x_m}}\rho,\nu^{a-1}\rho]),\delta([\nu^{b'_{y_m}}\rho,\nu^a\rho])) \\ &\qquad\qquad\Downarrow \\ \pi &\hookrightarrow \delta([\nu^{b'_k}\rho,\nu^a\rho]) \times \theta\end{aligned}$$

for some irreducible

$$\theta \le L(\delta([\nu^{b'_{x_1}}\rho,\nu^{a-1}\rho]),\delta([\nu^{b'_{y_1}}\rho,\nu^a\rho]))\times\cdots\times L(\delta([\nu^{b'_{x_m}}\rho,\nu^{a-1}\rho]),\delta([\nu^{b'_{y_m}}\rho,\nu^a\rho])).$$

Let $\theta_0 = L(\delta([\nu^{b'_1}\rho,\nu^{a'_1}\rho]),\delta([\nu^{b'_2}\rho,\nu^{a'_2}\rho]),\dots,\delta([\nu^{b'_{k-1}}\rho,\nu^{a'_{k-1}}\rho]))$. We claim $\theta = \theta_0$. To show this, we show $\chi_0(\theta)=\chi_0(\theta_0)$. Since

$$\begin{aligned}\theta &\le L(\delta([\nu^{b'_{x_1}}\rho,\nu^{a-1}\rho]),\delta([\nu^{b'_{y_1}}\rho,\nu^a\rho])) \times\cdots\times L(\delta([\nu^{b'_{x_m}}\rho,\nu^{a-1}\rho]),\delta([\nu^{b'_{y_m}}\rho,\nu^a\rho])) \\ &\le \delta([\nu^{b'_1}\rho,\nu^{a'_1}\rho])\times\delta([\nu^{b'_2}\rho,\nu^{a'_2}\rho])\times\cdots\times\delta([\nu^{b'_{k-1}}\rho,\nu^{a'_{k-1}}\rho]),\end{aligned}$$

it follows from Proposition 1.2.3 that $\chi_0(\theta_0)\preceq\chi_0(\theta)$. On the other hand, suppose $\chi_0(\theta_0)\ne\chi_0(\theta)$. Then, by Lemma 1.3.1, we have

$$\delta_0(\theta)=\delta([\nu^{d_1}\rho,\nu^{c_1}\rho])\otimes\delta([\nu^{d_2}\rho,\nu^{c_2}\rho])\otimes\cdots\otimes\delta([\nu^{d_{k-1}}\rho,\nu^{c_{k-1}}\rho])$$

with $d_1,d_2,\dots,d_{k-1}$ (resp., $c_1,c_2,\dots,c_{k-1}$) a permutation of $b'_1,b'_2,\dots,b'_{k-1}$ (resp., $a'_1,a'_2,\dots,a'_{k-1}$). Since $b'_k\ge d_j$ for all j and $a'_k=a\ge c_j$ for all j, we see that $\delta([\nu^{d_1}\rho,\nu^{c_1}\rho])\otimes\cdots\otimes\delta([\nu^{d_{k-1}}\rho,\nu^{c_{k-1}}\rho])\otimes\delta([\nu^{b'_k}\rho,\nu^a\rho])$ satisfies the conditions for Langlands data (δ_0 version). Therefore, by Proposition 1.2.3, any irreducible

$$\pi'\le\theta\times\delta([\nu^{b'_k}\rho,\nu^a\rho])\hookrightarrow\delta([\nu^{d_1}\rho,\nu^{c_1}\rho])\times\cdots\times\delta([\nu^{d_{k-1}}\rho,\nu^{c_{k-1}}\rho])\times\delta([\nu^{b'_k}\rho,\nu^a\rho])$$

has

$$\begin{aligned}\delta_0(\pi') &\succeq \delta([\nu^{d_1}\rho,\nu^{c_1}\rho])\otimes\cdots\otimes\delta([\nu^{d_{k-1}}\rho,\nu^{c_{k-1}}\rho])\otimes\delta([\nu^{b'_k}\rho,\nu^a\rho]) \\ &= \delta_0(\theta)\otimes\delta([\nu^{b'_k}\rho,\nu^a\rho]).\end{aligned}$$

In particular, $\delta_0(\pi)\succeq\delta_0(\theta)\otimes\delta([\nu^{b'_k}\rho,\nu^a\rho])$. However, this contradicts $\delta_0(\pi)=\delta_0(\theta_0)\otimes\delta([\nu^{b'_k}\rho,\nu^a\rho])\prec\delta_0(\theta)\otimes\delta([\nu^{b'_k}\rho,\nu^a\rho])$. Thus, we have $\chi_0(\theta)=\chi_0(\theta_0)$, as needed.

We now check irreducibility. We remark that the argument is similar to that of Lemma 1.3.3. First, we note that

$$\tilde\pi = L(\delta([\nu^{-a'_k}\tilde\rho,\nu^{-b'_k}\tilde\rho]),\delta([\nu^{-a'_{k-1}}\tilde\rho,\nu^{-b'_{k-1}}\tilde\rho]),\dots,\delta([\nu^{-a'_1}\tilde\rho,\nu^{-b'_1}\tilde\rho]))$$

(cf. Lemma 1.1 of [Jan3]). Since

$$\pi\hookrightarrow\delta([\nu^{b'_k}\rho,\nu^{a'_k}\rho])\times L(\delta([\nu^{b'_1}\rho,\nu^{a'_1}\rho]),\dots,\delta([\nu^{b'_{k-1}}\rho,\nu^{a'_{k-1}}\rho]))$$

as a unique irreducible subrepresentation, $\tilde\pi$ is the unique irreducible quotient of

$$\begin{aligned}&\Big(\delta([\nu^{b'_k}\rho,\nu^{a'_k}\rho])\times L(\delta([\nu^{b'_1}\rho,\nu^{a'_1}\rho]),\dots,\delta([\nu^{b'_{k-1}}\rho,\nu^{a'_{k-1}}\rho]))\Big)^{\sim} \\ &\cong \delta(\widetilde{[\nu^{b'_k}\rho,\nu^{a'_k}}\rho])\times L(\delta([\nu^{b'_1}\rho,\nu^{a'_1}\rho]),\widetilde{\dots,}\,\delta([\nu^{b'_{k-1}}\rho,\nu^{a'_{k-1}}\rho])) \\ &\cong \delta([\nu^{-a'_k}\tilde\rho,\nu^{-b'_k}\tilde\rho])\times L(\delta([\nu^{-a'_{k-1}}\tilde\rho,\nu^{-b'_{k-1}}\tilde\rho]),\dots,\delta([\nu^{-a'_1}\tilde\rho,\nu^{-b'_1}\tilde\rho])).\end{aligned}$$

On the other hand, it follows from the Langlands classification that

$$\tilde{\pi} \hookrightarrow \delta([\nu^{-a'_k}\tilde{\rho}, \nu^{-b'_k}\tilde{\rho}]) \times L(\delta([\nu^{-a'_{k-1}}\tilde{\rho}, \nu^{-b'_{k-1}}\tilde{\rho}]), \ldots, \delta([\nu^{-a'_1}\tilde{\rho}, \nu^{-b'_1}\tilde{\rho}]))$$

as a unique irreducible subrepresentation. Since $\tilde{\pi}$ appears with multiplicity one in $\delta([\nu^{-a'_k}\tilde{\rho}, \nu^{-b'_k}\tilde{\rho}]) \times L(\delta([\nu^{-a'_{k-1}}\tilde{\rho}, \nu^{-b'_{k-1}}\tilde{\rho}]), \ldots, \delta([\nu^{-a'_1}\tilde{\rho}, \nu^{-b'_1}\tilde{\rho}]))$ (also by the Langlands classification), the only possibility is that

$$\tilde{\pi} \cong \delta([\nu^{-a'_k}\tilde{\rho}, \nu^{-b'_k}\tilde{\rho}]) \times L(\delta([\nu^{-a'_{k-1}}\tilde{\rho}, \nu^{-b'_{k-1}}\tilde{\rho}]), \ldots, \delta([\nu^{-a'_1}\tilde{\rho}, \nu^{-b'_1}\tilde{\rho}])),$$

with the induced representation being irreducible. The lemma follows. □

We are now ready to characterize $f_\pi(a)$ for general (irreducible) π. For notational convenience, write

$$\begin{aligned}\pi = L(&\delta([\nu^{d_1}\rho, \nu^{c_1}\rho]), \ldots, \delta([\nu^{d_r}\rho, \nu^{c_r}\rho]), \delta([\nu^{b_1}\rho, \nu^{a-1}\rho]), \ldots, \delta([\nu^{b_i}\rho, \nu^{a-1}\rho]),\\ &\delta([\nu^{b_{i+1}}\rho, \nu^{a}\rho]), \ldots, \delta([\nu^{b_k}\rho, \nu^{a}\rho]), \delta([\nu^{d_{r+1}}\rho, \nu^{c_{r+1}}\rho]), \ldots, \delta([\nu^{d_s}\rho, \nu^{c_s}\rho]))\end{aligned}$$

with $c_r < a-1$ and $c_{r+1} > a$.

Proposition 2.1.4. $f_\pi(a) = max_{j\geq 0}\{n_a(j) - n_{a-1}(j)\}$.

Proof. Let $f = \max_{j\geq 0}\{n_a(j) - n_{a-1}(j)\}$. Let us suppose $f \geq 1$ for the general argument below; the special case $f = 0$ is much easier (and only requires the bound $f_\pi(a) \leq f$ done at the end). For $1 \leq m \leq f$, let

$$j_m = \text{ smallest } j \text{ having } n_a(j) - n_{a-1}(j) = m.$$

Now, let $L_1 = L(\delta([\nu^{d_1}\rho, \nu^{c_1}\rho]), \ldots, \delta([\nu^{d_r}\rho, \nu^{c_r}\rho]))$ and $L_2 = L(\delta([\nu^{d_{r+1}}\rho, \nu^{c_{r+1}}\rho]), \ldots, \delta([\nu^{d_s}\rho, \nu^{c_s}\rho]))$. Then,

$$\begin{aligned}L(&\delta([\nu^{b_1}\rho, \nu^{a-1}\rho]), \ldots, \delta([\nu^{b_i}\rho, \nu^{a-1}\rho]), \delta([\nu^{b_{i+1}}\rho, \nu^{a}\rho]), \ldots, \delta([\nu^{b_k}\rho, \nu^{a_k}\rho]))\\ &= L(\delta([\nu^{b'_1}\rho, \nu^{a'_1}\rho]), \ldots, \delta([\nu^{b'_k}\rho, \nu^{a'_k}\rho])),\end{aligned}$$

so we have

$$\begin{aligned}\pi &\hookrightarrow L_1 \times L(\delta([\nu^{b'_1}\rho, \nu^{a'_1}\rho]), \ldots, \delta([\nu^{b'_k}\rho, \nu^{a'_k}\rho])) \times L_2\\ &\hookrightarrow L_1 \times L(\delta([\nu^{b'_1}\rho, \nu^{a'_1}\rho]), \ldots, \delta([\nu^{b'_{j_1-1}}\rho, \nu^{a'_{j_1-1}}\rho]), \delta([\nu^{b'_{j_1}}\rho, \nu^{a}\rho]))\\ &\quad \times \cdots \times L(\delta([\nu^{b'_{j_{f-1}+1}}, \nu^{a'_{j_{f-1}+1}}\rho]), \ldots, \delta([\nu^{b'_{j_f-1}}\rho, \nu^{a'_{j_f-1}}\rho]), \delta([\nu^{b'_{j_f}}\rho, \nu^{a}\rho]))\\ &\quad \times L(\delta([\nu^{b'_{j_f+1}}, \nu^{a'_{j_f+1}}\rho]), \ldots, \delta([\nu^{b'_k}\rho, \nu^{a'_k}\rho])) \times L_2.\end{aligned}$$

Applying the previous lemma to each $L(\delta([\nu^{b'_{j_{i-1}+1}}\rho,\nu^{a'_i}\rho]),\dots,\delta([\nu^{b'_{j_i}}\rho,\nu^{a}\rho]))$ for $1\leq i\leq f$,

$$\begin{aligned}
\pi &\hookrightarrow L_1\times\delta([\nu^{b'_{j_1}}\rho,\nu^{a}\rho])\times L(\delta([\nu^{b'_1}\rho,\nu^{a'_1}\rho]),\dots,\delta([\nu^{b'_{j_1-1}}\rho,\nu^{a'_{j_1-1}}\rho]))\\
&\quad\times\cdots\times\delta([\nu^{b'_{j_f}}\rho,\nu^{a}\rho])\times L(\delta([\nu^{b'_{j_{f-1}+1}},\nu^{a'_{j_{f-1}+1}}\rho]),\dots,\delta([\nu^{b'_{j_f-1}}\rho,\nu^{a'_{j_f-1}}\rho]))\\
&\quad\times L(\delta([\nu^{b'_{j_f+1}},\nu^{a'_{j_f+1}}\rho]),\dots,\delta([\nu^{b'_k}\rho,\nu^{a'_k}\rho]))\times L_2\\
&\cong L_1\times\delta([\nu^{b'_{j_1}}\rho,\nu^{a}\rho])\cdots\times\delta([\nu^{b'_{j_f}}\rho,\nu^{a}\rho])\\
&\quad\times L(\delta([\nu^{b'_1}\rho,\nu^{a'_1}\rho]),\dots,\delta([\nu^{b'_{j_1-1}}\rho,\nu^{a'_{j_1-1}}\rho]))\\
&\quad\times\cdots\times L(\delta([\nu^{b'_{j_{f-1}+1}},\nu^{a'_{j_{f-1}+1}}\rho]),\dots,\delta([\nu^{b'_{j_f-1}}\rho,\nu^{a'_{j_f-1}}\rho]))\\
&\quad\times L(\delta([\nu^{b'_{j_f+1}},\nu^{a'_{j_f+1}}\rho]),\dots,\delta([\nu^{b'_k}\rho,\nu^{a'_k}\rho]))\times L_2
\end{aligned}$$

by a commuting argument using Lemma 2.1.3. Since

$$\begin{aligned}
\delta([\nu^{b'_{j_1}}\rho,\nu^{a}\rho])\times\cdots\times\delta([\nu^{b'_{j_f}}\rho,\nu^{a}\rho])\hookrightarrow \underbrace{\nu^a\rho\times\cdots\times\nu^a\rho}_{f}\\
\times\delta([\nu^{b'_{j_1}}\rho,\nu^{a-1}\rho])\times\cdots\times\delta([\nu^{b'_{j_f}}\rho,\nu^{a-1}\rho]),
\end{aligned}$$

we have (by Lemma 1.3.3)

$$\begin{aligned}
\pi &\hookrightarrow L_1\times\underbrace{\nu^a\rho\times\cdots\times\nu^a\rho}_{f}\times\delta([\nu^{b'_{j_1}}\rho,\nu^{a-1}\rho])\times\cdots\times\delta([\nu^{b'_{j_f}}\rho,\nu^{a-1}\rho])\\
&\quad\times L(\delta([\nu^{b'_1}\rho,\nu^{a'_1}\rho]),\dots,\delta([\nu^{b'_{j_1-1}}\rho,\nu^{a'_{j_1-1}}\rho]))\\
&\quad\times\cdots\times L(\delta([\nu^{b'_{j_{f-1}+1}},\nu^{a'_{j_{f-1}+1}}\rho]),\dots,\delta([\nu^{b'_{j_f-1}}\rho,\nu^{a'_{j_f-1}}\rho]))\\
&\quad\times L(\delta([\nu^{b'_{j_f+1}},\nu^{a'_{j_f+1}}\rho]),\dots,\delta([\nu^{b'_k}\rho,\nu^{a'_k}\rho]))\times L_2\\
&\cong\underbrace{\nu^a\rho\times\cdots\times\nu^a\rho}_{f}\times L_1\times\delta([\nu^{b'_{j_1}}\rho,\nu^{a-1}\rho])\times\cdots\times\delta([\nu^{b'_{j_f}}\rho,\nu^{a-1}\rho])\\
&\quad\times L(\delta([\nu^{b'_1}\rho,\nu^{a'_1}\rho]),\dots,\delta([\nu^{b'_{j_1-1}}\rho,\nu^{a'_{j_1-1}}\rho]))\\
&\quad\times\cdots\times L(\delta([\nu^{b'_{j_{f-1}+1}},\nu^{a'_{j_{f-1}+1}}\rho]),\dots,\delta([\nu^{b'_{j_f-1}}\rho,\nu^{a'_{j_f-1}}\rho]))\\
&\quad\times L(\delta([\nu^{b'_{j_f+1}},\nu^{a'_{j_f+1}}\rho]),\dots,\delta([\nu^{b'_k}\rho,\nu^{a'_k}\rho]))\times L_2.
\end{aligned}\tag{2}$$

By Frobenius reciprocity, it follows immediately that $f_\pi(a)\geq f$.

To show $f_\pi(a)\leq f$, we show that $r_{\nu^a\rho}=0$ for all the terms except the $\nu^a\rho$ terms which appear in the right-hand side of (2). Certainly, $r_{\nu^a\rho}\delta([\nu^{b'_{j_m}}\rho,\nu^{a-1}\rho])=0$ for $m=1,\dots,f$. Also, $r_{\nu^a\rho}L_1=0$ and $r_{\nu^a\rho}L_2=0$ (since, e.g., $L_1\hookrightarrow\delta([\nu^{d_1}\rho,\nu^{c_1}\rho])\times\cdots\times\delta([\nu^{d_r}\rho,\nu^{c_r}\rho])$ and $r_{\nu^a\rho}\delta([\nu^{c_i}\rho,\nu^{d_i}\rho])=0$ for all i). For $L(\delta([\nu^{b'_1}\rho,\nu^{a'}\rho]),\dots,\delta([\nu^{b'_{j_1-1}}\rho,\nu^{a'_{j_1-1}}\rho]))$, the argument leading to (1) tells us that

$$\begin{aligned}
&L(\delta([\nu^{b'_1}\rho,\nu^{a'_1}\rho]),\dots,\delta([\nu^{b'_{j_1-1}}\rho,\nu^{a'_{j_1-1}}\rho]))\\
&\quad\hookrightarrow L(\delta([\nu^{x'_1}\rho,\nu^{a-1}\rho]),\delta([\nu^{y'_1}\rho,\nu^{a}\rho]))\times\dots\\
&\quad\times L(\delta([\nu^{x'_{m_1}}\rho,\nu^{a-1}\rho]),\delta([\nu^{y'_{m_1}}\rho,\nu^{a}\rho]))
\end{aligned}$$

and all terms in the product in the right-hand side have $r_{\nu^a\rho} = 0$, so

$$r_{\nu^a\rho} L(\delta([\nu^{b'_1}\rho, \nu^{a'}\rho]), \dots, \delta([\nu^{b'_{j_1-1}}\rho, \nu^{a'_{j_1-1}}\rho])) = 0.$$

The same argument applies to $L(\delta([\nu^{b'_{j_1+1}}\rho, \nu^{a'_{j_1+1}}\rho]), \dots, \delta([\nu^{b'_{j_2-1}}\rho, \nu^{a'_{j_2-1}}\rho]))$, $\dots, L(\delta([\nu^{b'_{j_{f-1}+1}}\rho, \nu^{a'_{j_{f-1}+1}}\rho]), \dots, \delta([\nu^{b'_{j_f-1}}\rho, \nu^{a'_{j_f-1}}\rho]))$. The argument for $L(\delta([\nu^{b'_{j_f+1}}\rho, \nu^{a'_{j_f+1}}\rho]), \dots, \delta([\nu^{b'_k}\rho, \nu^{a'_k}\rho]))$ is similar. The only difference is that in this case, the embedding has the form

$$\begin{aligned} &L(\delta([\nu^{b'_{j_f+1}}\rho, \nu^{a'_{j_f+1}}\rho]), \dots, \delta([\nu^{b'_k}\rho, \nu^{a'_k}\rho])) \\ &\quad \hookrightarrow \delta([\nu^{x_1}\rho, \nu^{a-1}\rho]) \times \dots \times \delta([\nu^{x_n}\rho, \nu^{a-1}\rho]) \\ &\qquad \times L(\delta([\nu^{x_{n+1}}\rho, \nu^{a-1}\rho]), \delta([\nu^{y_{n+1}}\rho, \nu^{a}\rho])) \\ &\qquad\quad \times \dots \times L(\delta([\nu^{x_m}\rho, \nu^{a-1}\rho]), \delta([\nu^{y_m}\rho, \nu^{a}\rho])). \end{aligned}$$

Thus, we see $f_\pi(a) \leq f$, implying $f_\pi(a) = f$, as needed. □

The above characterization of $f_\pi(a)$ may be made more explicit. For convenience, let us shift things (essentially working with $\nu^x\pi$ in place of π) so that the supercuspidal support of π is contained in $\{\nu\rho, \nu^2\rho, \dots, \nu^n\rho\}$ (with both $\nu\rho$ and $\nu^n\rho$ occurring). Let $\ell_{i,j}$ be the number of times $\delta([\nu^i\rho, \nu^j\rho])$ appears in the Langlands data for π (i.e., in $\delta_0(\pi)$).

Corollary 2.1.5. *We have the following:*

$$f_\pi(1) = \ell_{1,1},$$

$$f_\pi(2) = \ell_{1,2} + max\{\ell_{2,2} - \ell_{1,1}, 0\},$$

$$f_\pi(3) = \ell_{1,3} + max\{\ell_{2,3} - \ell_{1,2} + max\{\ell_{3,3} - \ell_{2,2}, 0\}, 0\},$$

and in general, for $1 \leq k \leq n$,

$$\begin{aligned} f_\pi(k) = \ell_{1,k} &+ max\{\ell_{2,k} - \ell_{1,k-1} + max\{\ell_{3,k} - \ell_{2,k-1} + \dots \\ &+ max\{\ell_{k,k} - \ell_{k-1,k-1}, 0\}, 0\} \dots, 0\}. \end{aligned}$$

2.2. **Calculation of terms of maximum initial frequency.**

Theorem 2.2.1. *With notation as in 2.1, let*

$$\begin{aligned} \theta_\pi(a) = L(\delta([\nu^{d_1}\rho, \nu^{c_1}\rho]), \dots, \delta([\nu^{d_r}\rho, \nu^{c_r}\rho]), \delta([\nu^{b'_1}\rho, \nu^{a''_1}\rho]), \dots, \delta([\nu^{b'_k}\rho, \nu^{a''_k}\rho]), \\ \delta([\nu^{d_{r+1}}\rho, \nu^{c_{r+1}}\rho]), \dots, \delta([\nu^{d_s}\rho, \nu^{c_s}\rho])), \end{aligned}$$

where

$$a''_j = \begin{cases} a'_j & \text{if } j \neq j_m, \\ a'_j - 1 & \text{if } j = j_m \text{ for some } m. \end{cases}$$

Then,

$$r_{\underbrace{\nu^a\rho \times \dots \times \nu^a\rho}_{f_\pi(a)}}\pi = (\underbrace{\nu^a\rho \times \dots \times \nu^a\rho}_{f_\pi(a)}) \otimes \theta_\pi(a).$$

Proof. From (2) in the proof of Proposition 2.1.4, we know that

$$\begin{aligned}
\pi \hookrightarrow\ & \underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{f} \times L_1 \times \delta([\nu^{b'_{j_1}}\rho, \nu^{a-1}\rho]) \times \cdots \times \delta([\nu^{b'_{j_f}}\rho, \nu^{a-1}\rho]) \\
& \times L(\delta([\nu^{b'_1}\rho, \nu^{a'_1}\rho]), \ldots, \delta([\nu^{b'_{j_1-1}}\rho, \nu^{a'_{j_1-1}}\rho])) \\
& \times \cdots \times L(\delta([\nu^{b'_{j_{f-1}+1}}, \nu^{a'_{j_{f-1}+1}}\rho]), \ldots, \delta([\nu^{b'_{j_f-1}}\rho, \nu^{a'_{j_f-1}}\rho])) \\
& \times L(\delta([\nu^{b'_{j_f+1}}, \nu^{a'_{j_f+1}}\rho]), \ldots, \delta([\nu^{b'_k}\rho, \nu^{a'_k}\rho])) \times L_2 \\
\hookrightarrow\ & \underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{f} \times \delta([\nu^{d_1}\rho, \nu^{c_1}\rho]) \times \cdots \times \delta([\nu^{d_r}\rho, \nu^{c_r}\rho]) \times \delta([\nu^{b'_{j_1}}\rho, \nu^{a-1}\rho]) \\
& \times \cdots \times \delta([\nu^{b'_{j_f}}\rho, \nu^{a-1}\rho]) \times \delta([\nu^{b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \delta([\nu^{b'_{j_1-1}}\rho, \nu^{a'_{j_1-1}}\rho]) \\
& \times \cdots \times \delta([\nu^{b'_{j_{f-1}+1}}, \nu^{a'_{j_{f-1}+1}}\rho]) \times \cdots \times \delta([\nu^{b'_{j_f-1}}\rho, \nu^{a'_{j_f-1}}\rho]) \\
& \times \delta([\nu^{b'_{j_f+1}}, \nu^{a'_{j_f+1}}\rho]) \times \cdots \times \delta([\nu^{b'_k}\rho, \nu^{a'_k}\rho]) \\
& \times \delta([\nu^{d_{r+1}}\rho, \nu^{c_{r+1}}\rho]) \times \cdots \times \delta([\nu^{d_s}\rho, \nu^{c_s}\rho]).
\end{aligned}$$

Since

$$\begin{aligned}
\delta([\nu^{b'_1}\rho, \nu^{a'}\rho]), \ldots, \delta([\nu^{b'_{j_1-1}}\rho, \nu^{a'_{j_1-1}}\rho]), \ldots, \delta([\nu^{b'_{j_{f-1}+1}}, \nu^{a'_{j_{f-1}+1}}\rho]), \\
\ldots, \delta([\nu^{b'_{j_f-1}}\rho, \nu^{a'_{j_f-1}}\rho]), \delta([\nu^{b'_{j_f+1}}, \nu^{a'_{j_f+1}}\rho]), \ldots, \delta([\nu^{b'_k}\rho, \nu^{a'_k}\rho])
\end{aligned}$$

may be shuffled back into lexicographic order (Langlands classification order) via an inductively equivalent shuffle, and then commuted past any segment of the form $\delta([\nu^b\rho, \nu^{a-1}\rho])$ into lexicographic position, we have

$$\pi \hookrightarrow \underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{f} \times \mathrm{Ind}(\delta_0(\theta_\pi(a))).$$

On the one hand, by Lemma 5.5 of [Jan2], this tells us that

$$\pi \hookrightarrow \underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{f} \times \theta$$

for some irreducible $\theta \leq \mathrm{Ind}(\delta_0(\theta_\pi(a)))$. By Proposition 1.2.3, we have $\delta_0(\theta) \succeq \delta_0(\theta_\pi(a))$. On the other hand, taking Jacquet modules in stages tells us there is an irreducible θ' such that $m^*\pi \geq \underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{f} \otimes \theta'$ and $r_{app}\theta' \geq \delta_0(\theta_\pi(a))$. Then, $\delta_0(\theta') \preceq \delta_0(\theta_\pi(a))$. However, by Lemma 2.1.2, we have $\theta = \theta'$, which implies $\delta_0(\theta) = \delta_0(\theta') = \delta_0(\theta_\pi(a))$, so $\theta = \theta' = \theta_\pi(a)$. The theorem follows. □

Remark 2.2.2. Suppose $\underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{f} \otimes \tau$ has τ irreducible. Then there is a unique irreducible representation π such that $m^*_{\{a\}}\pi = \underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{f} \otimes \tau$. Moreover, using the preceding theorem, it is a straightforward matter to recover (the Langlands data for) π from (the Langlands data for) τ. In particular, the only possibilities have f generalized Steinberg representations $\delta([\nu^{b_i}\rho, \nu^{a-1}\rho])$ in the Langlands data for τ (possibly with $b_i = a$) replaced by $\delta([\nu^{b_i}\rho, \nu^a\rho])$. One can simply use the theorem to check all such possibilities to see which works. (However, in practice it is probably easier to use a repeated application of Remark 2.2.4 below.)

The same sort of considerations used above give the following:

Corollary 2.2.3. *With π as above, there is a unique irreducible θ' such that $r_{\nu^a\rho}\pi \geq \nu^a\rho \otimes \theta'$ and $f_\pi(a) = f_{\theta'}(a) + 1$. Furthermore, $\nu^a\rho \otimes \theta'$ appears with multiplicity $f_\pi(a)$ in $r_{\nu^a\rho}\pi$. We may describe θ' explicitly*

$$\theta' = L(\delta([\nu^{d_1}\rho, \nu^{c_1}\rho]), \dots, \delta([\nu^{d_r}\rho, \nu^{c_r}\rho]), \delta([\nu^{b'_1}\rho, \nu^{a''_1}\rho]), \dots \delta([\nu^{b'_k}\rho, \nu^{a''_k}\rho]), \delta([\nu^{d_{r+1}}\rho, \nu^{c_{r+1}}\rho]), \dots, \delta([\nu^{d_s}\rho, \nu^{c_s}\rho])),$$

where

$$a''_j = \begin{cases} a'_j & \text{if } j \neq j_{f_\pi(a)}, \\ a'_j - 1 & \text{if } j = j_{f_\pi(a)}. \end{cases}$$

We note that $\pi \hookrightarrow \nu^a\rho \times \theta'$ as a unique irreducible subrepresentation.

Proof. Observe that by Lemma 2.1.2,

$$\pi \hookrightarrow \underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{f_\pi(a)} \times \theta_\pi(a)$$
$$\Downarrow$$
$$\pi \hookrightarrow \nu^a\rho \times L$$

for some irreducible $L \leq \underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{f_\pi(a)-1} \times \theta_\pi(a)$ (cf. Lemma 5.5 of [Jan2]). It follows from m^* considerations that $f_\pi(a) \leq f_L(a) + 1$. On the other hand, by Frobenius reciprocity we must also have $f_L(a) \leq f_\pi(a) - 1$. Thus,

$$f_L(a) = f_\pi(a) - 1.$$

In particular, we have (Lemma 2.1.2)

$$m^*_{\{a\}}L = \underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{f_\pi(a)-1} \times \theta_\pi(a).$$

Since the θ' given above satisfies (cf. Theorem 2.2.1)

$$m^*_{\{a\}}\theta' = \underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{f_\pi(a)-1} \times \theta_\pi(a),$$

it follows from Lemma 2.1.2 that $\theta' = L$.

Observe that since $\nu^a\rho \times \theta' \hookrightarrow \underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{f_\pi(a)} \times \theta_\pi(a)$ and π is the unique irreducible subrepresentation of $\underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{f_\pi(a)} \times \theta_\pi(a)$, we have π as the unique irreducible subrepresentation of $\nu^a\rho \times \theta'$. That $\nu^a\rho \otimes \theta'$ appears with multiplicity $f_\pi(a)$ in $r_{(1)}\pi$ follows from the observation that $\nu^a\rho \otimes \underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{f_\pi(a)-1} \otimes \theta_\pi(a)$ has multiplicity $f_\pi(a)$ in $r_{(1,f_\pi(a)-1)}\pi$ and multiplicity one in $r_{(1,f_\pi(a)-1)}\nu^a\rho \otimes \theta'$. □

Remark 2.2.4. (1) Suppose $\nu^a\rho \otimes \tau$ has τ irreducible. Then there is a unique irreducible representation π such that $r_{\nu^a\rho}\pi \geq \nu^a\rho \otimes \tau$ and $f_\pi(a) = f_\tau(a) + 1$. Moreover, using the preceding corollary, it is a straightforward matter to recover (the Langlands data for) π from (the Langlands data for) τ. In particular, the only possibilities have a generalized Steinberg representation $\delta([\nu^{b_i}\rho, \nu^{a-1}\rho])$ in the Langlands data for τ (possibly with $b_i = a$) replaced by $\delta([\nu^{b_i}\rho, \nu^{a}\rho])$. One can simply use the corollary to check all such possibilities to see which works.

(2) Fix $k \in \mathbb{N}$. One could ask whether (1) holds if $r_{\nu^a\rho}\pi \geq \nu^a\rho \otimes \tau$ has $f_\pi(a) = f_\tau(a)+k$ when $k > 1$ (i.e., whether such a π is unique). The following example shows this is not the case: let $\pi_1 = L(\delta([\nu\rho, \nu^2\rho]), \nu^2\rho, \nu^3\rho, \delta([\nu^2\rho, \nu^4\rho]))$ and $\pi_2 = L(\delta([\nu\rho, \nu^2\rho]), \nu^2\rho, \delta([\nu^2\rho, \nu^3\rho]), \nu^3\rho, \nu^4\rho)$. Then,

$$\begin{aligned} r_{(1)}\pi_1 = {} & \nu^2\rho \otimes L(\nu\rho, \nu^2\rho, \nu^3\rho, \delta([\nu^2\rho, \nu^4\rho])) \\ & + 2\nu^2\rho \otimes L(\delta([\nu\rho, \nu^2\rho]), \nu^3\rho, \delta([\nu^2\rho, \nu^4\rho])) \\ & + \nu^2\rho \otimes L(\delta([\nu\rho, \nu^3\rho]), \delta([\nu^2\rho, \nu^4\rho])) \\ & + \nu^4\rho \otimes L(\delta([\nu\rho, \nu^2\rho]), \nu^2\rho, \delta([\nu^2\rho, \nu^3\rho]), \nu^3\rho) \end{aligned}$$

and

$$\begin{aligned} r_{(1)}\pi_2 = {} & \nu^2\rho \otimes L(\delta(\nu\rho, \nu^2\rho, \delta([\nu^2\rho, \nu^3\rho]), \nu^3\rho, \nu^4\rho) \\ & + 2\nu^2\rho \otimes L(\delta([\nu\rho, \nu^2\rho]), \delta([\nu^2\rho, \nu^3\rho]), \nu^3\rho, \nu^4\rho) \\ & + \nu^2\rho \otimes L(\delta([\nu\rho, \nu^3\rho]), \delta([\nu^2\rho, \nu^4\rho])). \end{aligned}$$

In particular, $\nu^2\rho \otimes L(\delta([\nu\rho, \nu^3\rho]), \delta([\nu^2\rho, \nu^4\rho])) \leq r_{(1)}\pi_i$ for $i = 1, 2$, and has $f_{\pi_i}(2) = 2 = f_{L(\delta([\nu\rho,\nu^3\rho]),\delta([\nu^2\rho,\nu^4\rho]))}(2) + 2$.

2.3. **Calculation of $m^*_X\pi$.** In this section, we take up the question of how to calculate $m^*_X\pi$ for general X. We note that the case $|X| = 1$ is covered by the preceding section and is used in what follows.

First, choose $x_1 \in X$ such that $f_\pi(x_1) \neq 0$. Let $f_1 = f_\pi(x_1)$ and $\theta_1 = \theta_\pi(x_1)$ (cf. Proposition 2.1.4 and Theorem 2.2.1), so that

$$\pi \hookrightarrow \underbrace{\nu^{x_1}\rho \times \cdots \times \nu^{x_1}\rho}_{f_1} \times \theta_1.$$

Next, choose $x_2 \in X$ such that $f_{\theta_1}(x_2) \neq 0$. Let $f_2 = f_{\theta_1}(x_2)$ and $\theta_2 = \theta_{\theta_1}(x_2)$. Then,

$$\pi \hookrightarrow \underbrace{\nu^{x_1}\rho \times \cdots \times \nu^{x_1}\rho}_{f_1} \times \underbrace{\nu^{x_2}\rho \times \cdots \times \nu^{x_2}\rho}_{f_2} \times \theta_2.$$

We continue this process: at the jth step we have

$$\pi \hookrightarrow \underbrace{\nu^{x_1}\rho \times \cdots \times \nu^{x_1}\rho}_{f_1} \times \cdots \times \underbrace{\nu^{x_{j-1}}\rho \times \cdots \times \nu^{x_{j-1}}\rho}_{f_{j-1}} \times \theta_{j-1},$$

and choose $x_j \in X$ such that $f_{\theta_{j-1}}(x_j) \neq 0$. We let $f_j = f_{\theta_{j-1}}(x_j)$ and $\theta_j = \theta_{\theta_{j-1}}(x_j)$. Then,

$$\pi \hookrightarrow \underbrace{\nu^{x_1}\rho \times \cdots \times \nu^{x_1}\rho}_{f_1} \times \cdots \times \underbrace{\nu^{x_{j-1}}\rho \times \cdots \times \nu^{x_{j-1}}\rho}_{f_{j-1}} \times \underbrace{\nu^{x_j}\rho \times \cdots \times \nu^{x_j}\rho}_{f_j} \times \theta_j.$$

We continue this process until we reach θ_k such that $f_{\theta_k}(x) = 0$ for all $x \in X$, at which point we stop. We then have

$$\pi \hookrightarrow \underbrace{\nu^{x_1}\rho \times \cdots \times \nu^{x_1}\rho}_{f_1} \times \cdots \times \underbrace{\nu^{x_{k-1}}\rho \times \cdots \times \nu^{x_{k-1}}\rho}_{f_{k-1}} \times \underbrace{\nu^{x_k}\rho \times \cdots \times \nu^{x_k}\rho}_{f_k} \times \theta_k.$$

By Lemma 5.5 of [Jan2], this implies that there is an irreducible subquotient

$$\tau' \leq \underbrace{\nu^{x_1}\rho \times \cdots \times \nu^{x_1}\rho}_{f_1)} \times \cdots \times \underbrace{\nu^{x_k}\rho \times \cdots \times \nu^{x_k}\rho}_{f_k}$$

such that $\pi \hookrightarrow \tau' \otimes \theta_k$. In particular, by Lemma 2.1.2, we have $\theta_k = \theta_\pi(X)$ (and $\tau' = \tau_\pi(X)$).

We now describe how to recover τ. Let $\tau_k = \underbrace{\nu^{x_k}\rho \times \cdots \times \nu^{x_k}\rho}_{f_k}$. By Remark 2.2.2 there is a unique irreducible representation τ_{k-1} satisfying $f_{\tau_{k-1}}(x_{k-1}) = f_{k-1}$ and $m^*_{\{x_{k-1}\}}\tau_{k-1} = \underbrace{\nu^{x_{k-1}}\rho \times \cdots \times \nu^{x_{k-1}}\rho}_{f_{k-1}} \otimes \tau_k$, which may be determined. Next, by Remark 2.2.2, we may calculate the unique irreducible representation τ_{k-2} satisfying $f_{\tau_{k-2}}(x_{k-2}) = f_{k-2}$ and $m^*_{\{x_{k-2}\}}\tau_{k-2} = \underbrace{\nu^{x_{k-2}}\rho \times \cdots \times \nu^{x_{k-2}}\rho}_{f_{k-2}} \otimes \tau_{k-1}$. We continue this process. If we have τ_j, we obtain τ_{j-1} as follows: by Remark 2.2.2, we calculate the unique irreducible representation τ_{j-1} satisfying $f_{\tau_{j-1}}(x_{j-1}) = f_{j-1}$ and $m^*_{\{x_{j-1}\}}\tau_{j-1} = \underbrace{\nu^{x_{j-1}}\rho \times \cdots \times \nu^{x_{j-1}}\rho}_{f_{j-1}} \otimes \tau_j$. The process stops at τ_1. We claim that $\tau_1 = \tau$.

To see this, consider the general step. We have

$$r_{app}\pi \geq \cdots \otimes \underbrace{\nu^{x_{j-1}}\rho \times \cdots \times \nu^{x_{j-1}}\rho}_{f_{j-1}} \otimes \tau_j \otimes \theta,$$

so there is some irreducible τ'_{j-1} having $m^*\tau'_{j-1} \geq \underbrace{\nu^{x_{j-1}}\rho \times \cdots \times \nu^{x_{j-1}}\rho}_{f_{j-1}} \otimes \tau_j$ and $r_{app}\pi \geq \cdots \otimes \underbrace{\nu^{x_{j-2}}\rho \times \cdots \times \nu^{x_{j-2}}\rho}_{f_{j-2}} \otimes \tau'_{j-1} \otimes \theta$. Clearly, $f_{\tau'_{j-1}}(x_{j-1}) \geq f_{j-1}$. On the other hand, we claim that $f_{\tau'_{j-1}}(x_{j-1}) \leq f_{j-1}$. Since

$$\begin{aligned} r_{(\underbrace{\nu^{x_1}\rho \times \cdots \times \nu^{x_1}\rho}_{f_1}) \otimes \cdots \otimes (\underbrace{\nu^{x_{j-2}}\rho \times \cdots \times \nu^{x_{j-2}}\rho}_{f_{j-2}})}\pi = (\underbrace{\nu^{x_1}\rho \times \cdots \times \nu^{x_1}\rho}_{f_1}) \\ \otimes \cdots \otimes (\underbrace{\nu^{x_{j-2}}\rho \times \cdots \times \nu^{x_{j-2}}\rho}_{f_{j-2}}) \otimes \theta_{j-2} \end{aligned}$$

and $f_{j-1} = f_{\theta_{j-2}}(x_{j-1})$, we see that

$$r_{(\underbrace{\nu^{x_1}\rho \times \cdots \times \nu^{x_1}\rho}_{f_1}) \otimes \cdots \otimes (\underbrace{\nu^{x_{j-2}}\rho \times \cdots \times \nu^{x_{j-2}}\rho}_{f_{j-2}}) \otimes (\underbrace{\nu^{x_{j-1}}\rho \times \cdots \times \nu^{x_{j-1}}\rho}_{f_{j-1}+1})}\pi = 0.$$

However, if we had $f_{\tau'_{j-1}}(x_{j-1}) > f_{j-1}$, this would imply

$$r_{(\underbrace{\nu^{x_1}\rho \times \cdots \times \nu^{x_1}\rho}_{f_1}) \otimes \cdots \otimes (\underbrace{\nu^{x_{j-2}}\rho \times \cdots \times \nu^{x_{j-2}}\rho}_{f_{j-2}}) \otimes (\underbrace{\nu^{x_{j-1}}\rho \times \cdots \times \nu^{x_{j-1}}\rho}_{f_{j-1}+1})}\pi \neq 0,$$

a contradiction. Thus $f_{\tau'_{j-1}}(x_{j-1}) \leq f_{j-1}$, so $f_{\tau'_{j-1}}(x_{j-1}) = f_{j-1}$. In particular, the only possibility for τ'_{j-1} is then obtained by Remark 2.2.2, i.e., τ_{j-1}.

Since $\tau_1 \otimes \theta \leq m^*\pi$ and $r_{min}\tau_1$ contains a term of the form $\nu^{x_1}\rho \otimes \cdots \otimes \nu^{x_{f_\pi(X)}}\rho$ with $x_1, \ldots, x_{f_\pi(X)} \in X$, it follows from (the uniqueness in) Lemma 2.1.2 that $\tau_1 = \tau$, as needed.

Remark 2.3.1. (1) We note that a tuple $(x_1, \ldots, x_k)$ as above can contain a given value of x more than once. Furthermore, not all tuples which can be used to calculate $m^*_X\pi$ need to have the same number of entries (i.e., same k).
(2) One can show (using Lemma 1.3.5) that if $X = \{a-1, a\}$, we have $k \leq 3$.

Since the algorithm for calculating $m^*_X\pi$ allows one to start with any $x_1 \in X$ having $f_\pi(x_1) \neq 0$, we observe the following:

Corollary 2.3.2. *Suppose $a \in X$. If π, τ are irreducible representations with $\nu^a\rho \otimes \tau \leq r_{\nu^a\rho}\pi$ and $f_\tau(X) = f_\pi(X) - 1$, then $f_\tau(a) = f_\pi(a) - 1$.*

In the spirit of Remark 2.2.4, and for future use, we consider the following question: How can we recover an irreducible representation π from $m^*_X\pi$? More precisely, suppose $\tau(X) \otimes \theta(X)$ is irreducible and satisfies (1) the supercuspidal support of $\tau(X)$ is contained in X and (2) $f_{\theta(X)}(x) = 0$ for all $x \in X$. We argue that there is a unique irreducible representation π having $m^*_X\pi = \tau(X) \otimes \theta(X)$, and we give an algorithm for recovering π from $\tau(X) \otimes \theta(X)$. Note that the uniqueness follows from Lemma 2.1.2; existence follows from the algorithm below.

The algorithm for recovering π from $\tau(X) \otimes \theta(X)$ is very much like that used in calculating $m^*_X\pi$ (especially, the calculation of $\tau_\pi(X)$). Let $\tau_1 = \tau(X)$ and choose x_1 such that $f_1 = f_{\tau_1}(x_1) \neq 0$. Write

$$r_{\underbrace{\nu^{x_1}\rho \times \cdots \times \nu^{x_1}\rho}_{f_1}}\tau(X) = \underbrace{\nu^{x_1}\rho \times \cdots \times \nu^{x_1}\rho}_{f_1} \otimes \tau_2.$$

Next, choose x_2 such that $f_2 = f_{\tau_2}(x_2) \neq 0$ and write

$$r_{\underbrace{\nu^{x_1}\rho \times \cdots \times \nu^{x_1}\rho}_{f_1} \otimes \underbrace{\nu^{x_2}\rho \times \cdots \times \nu^{x_2}\rho}_{f_2}}\tau(X) = \underbrace{\nu^{x_1}\rho \times \cdots \times \nu^{x_1}\rho}_{f_1} \otimes \underbrace{\nu^{x_2}\rho \times \cdots \times \nu^{x_2}\rho}_{f_2} \otimes \tau_3.$$

We continue until we have $\tau_{k+1} = 1$ and

$$r_{app}\tau(X) \geq \underbrace{\nu^{x_1}\rho \times \cdots \times \nu^{x_1}\rho}_{f_1} \otimes \cdots \otimes \underbrace{\nu^{x_k}\rho \times \cdots \times \nu^{x_k}\rho}_{f_k}.$$

If $m^*_X\pi = \tau(X) \otimes \theta(X)$, we then have

$$r_{app}\tau(X) \geq \underbrace{\nu^{x_1}\rho \times \cdots \times \nu^{x_1}\rho}_{f_1} \otimes \cdots \otimes \underbrace{\nu^{x_k}\rho \times \cdots \times \nu^{x_k}\rho}_{f_k} \otimes \theta(X).$$

We now recover π in much the same way we obtained $\tau_\pi(X)$ in the calculation of $m^*_X\pi$. By repeated application of Remark 2.2.2, there is a unique irreducible representation θ_k satisfying $f_{\theta_k}(x_k) = f_k$ and $m^*_{\{x_k\}}\theta_k = \underbrace{\nu^{x_k}\rho \times \cdots \times \nu^{x_k}\rho}_{f_k} \otimes \theta(X)$, which may be calculated. Next, there is a unique irreducible θ_{k-1} satisfying $f_{\theta_{k-1}}(x_{k-1}) = f_{k-1}$ and $m^*_{\{x_{k-1}\}}\theta_{k-1} = \underbrace{\nu^{x_{k-1}}\rho \times \cdots \times \nu^{x_{k-1}}\rho}_{f_{k-1}} \otimes \theta_k$, which again may be calculated. Continuing this, we eventually reach θ_1. We claim $\pi = \theta_1$. The argument is the same as that used above (in showing that $\tau_\pi(X) = \tau_1$ in the algorithm for calculating $m^*_X\pi$), so we do not repeat it here.

2.4. **Dual results.** In this section, we formulate some results dual to those in sections 2.1–2.3. By dual, we mean formulated looking at the Jacquet modules from right to left rather than left to right (i.e., focusing on the last terms in the tensor products rather than the first terms). The proofs are mirror images of those in sections 2.1-2.3, so we are content to simply state the appropriate definitions and results. We note that these results bear a close connection with duality in the sense of [Aub], [S-S]; cf. Remark 2.4.6.

We need notation dual to r_τ. Suppose ξ is an irreducible representation of $GL(m_1 r, F) \times \cdots \times GL(m_k r, F)$ and π a representation of $GL(mr, F)$, where $m \geq m_1 + \cdots + m_k$ and both representations are supported on $\{\nu^z \rho\}_{z \in \mathbb{Z}}$. Write $m_0 = m - (m_1 + \cdots + m_k)$. We let $s_\xi \pi$ consist of everything in $r_{(m_0, m_1, \ldots, m_{k-1})}\pi$ of the form $\eta \otimes \xi$.

The following are the counterparts to Definition 2.1.1 and Lemma 2.1.2.

Definition 2.4.1. Let $\pi \in R(\rho)$ be an irreducible representation, $Y \subset \mathbb{Z}$. We define $g_\pi(Y)$ to be the largest value of g such that $r_{min}\pi$ contains a term of the form $\cdots \otimes \nu^{y_g}\rho \otimes \nu^{y_{g-1}}\rho \otimes \cdots \otimes \nu^{y_1}\rho \otimes$ with $y_1, \ldots, y_g \in Y$. If $Y = \{b\}$, we write $g_\pi(b)$ rather than $g_\pi(\{b\})$. We let ${}_Y m^* \pi$ be the sum of everything in $m^*\pi$ of the form $\eta \otimes \xi$ with η, ξ irreducible and $r_{min}\xi$ containing a term of the form $\nu^{y_g}\rho \otimes \cdots \otimes \nu^{y_1}\rho$ with $g = g_\pi(Y)$ and $y_1, \ldots, y_g \in Y$.

Lemma 2.4.2. *Suppose $\pi \in R(\rho)$ is an irreducible representation, $Y \subset \mathbb{Z}$. Then, there are unique irreducible representations η, ξ and unique g such that the following are all satisfied:*

1. $\pi \hookrightarrow \eta \times \xi$.
2. *If $\nu^{y_g}\rho \otimes \nu^{y_{g-1}}\rho \otimes \cdots \otimes \nu^{y_1}\rho \leq r_{min}\xi$, then $y_1, \ldots, y_g \in Y$.*
3. $s_{\nu^y \rho}\eta = 0$ *for all $y \in Y$.*

Furthermore, $g = g_\pi(Y)$ and ${}_Y m^*(\pi) = \eta \otimes \xi$. We write $\eta(Y) \otimes \xi(Y)$ for this $\eta \otimes \xi$.

Next, we take up the dual versions of Proposition 2.1.4 and Corollary 2.1.5. Using an inductively equivalent shuffle, write

$$\pi = L(\delta([\nu^{b''_1}\rho, \nu^{a_1}\rho]), \ldots, \delta([\nu^{b''_k}\rho, \nu^{a_k}\rho])),$$

where $a_1 \leq \cdots \leq a_k$ and if $a_i = a_{i+1}$, then $b''_i \geq b''_{i+1}$ (it is this second condition which is different than $\delta_0(\pi)$). We define m_b by

$$m_b(i) = |\{j \geq k+1-i \,|\, b''_j = b\}|;$$

by convention, $m_b(0) = 0$. Then, we have the following:

Proposition 2.4.3. $g_b(\pi) = max_{i \geq 0}\{m_b(i) - m_{b+1}(i)\}$

The dual to Corollary 2.1.5 is then the following:

Corollary 2.4.4. *With notation as in Corollary 2.1.5, we have the following:*

$$g_\pi(n) = \ell_{n,n},$$

$$g_\pi(n-1) = \ell_{n-1,n} + max\{\ell_{n-1,n-1} - \ell_{n,n}, 0\},$$

$$g_\pi(n-2) = \ell_{n-2,n} + max\{\ell_{n-2,n-1} - \ell_{n-1,n} + max\{\ell_{n-2,n-2} - \ell_{n-1,n-1}, 0\}, 0\},$$

and in general, for $1 \leq k \leq n$,

$$g_\pi(k) = \ell_{k,n} + max\{\ell_{k,n-1} - \ell_{k+1,n} + max\{\ell_{k,n-2} - \ell_{k+1,n-1} + \dots + max\{\ell_{k,k} - \ell_{k+1,k+1}, 0\}, 0\} \dots, 0\}.$$

Suppose $g_\pi(b) > 0$. For each $1 \leq n \leq g_\pi(b)$, we let

$$i_n = \text{smallest } i \text{ such that } m_b(i) - m_{b+1}(i) = n.$$

Let $\eta_\pi(b)$ be defined by

$$\eta_\pi(b) = L(\delta([\nu^{b_1'''}\rho, \nu^{a_1}\rho]), \dots, \delta([\nu^{b_k'''}\rho, \nu^{a_k}\rho])),$$

where

$$b_i''' = \begin{cases} b_i'' & \text{if } i \neq k+1-i_n \text{ for any } n, \\ b_i''+1 & \text{if } i = k+1-i_n \text{ for some } n. \end{cases}$$

We note that the definition of $\eta_\pi(b)$ makes sense; the data which appears is inductively equivalent to Langlands data. The dual to Theorem 2.2.1 is then the following:

Theorem 2.4.5. *With notation as above,*

$$s_{\underbrace{\nu^b\rho\times\cdots\times\nu^b\rho}_{g_\pi(b)}}\pi = \eta_\pi(b) \otimes \underbrace{\nu^b\rho \times \cdots \times \nu^b\rho}_{g_\pi(b)}.$$

As with Remark 2.2.2, we can also reverse this process. It is a straightforward matter to obtain the duals to Corollary 2.2.3 and Remark 2.2.4 (1).

The calculation of ${}_Ym^*$ is a straightforward analogue of the calculation of m_X^* given in section 2.3. The details are left to the reader. Similarly, the algorithm for recovering π from ${}_Ym^*\pi$ is also a straightforward analogue of that given in section 2.3.

Remark 2.4.6. The dual results discussed above are also related to the results of sections 2.1-2.3 through the duality of [Aub], [S-S]. Let $\hat{\pi}$ denote the dual to π in the sense of [Aub], [S-S]. Observe that by Théorème 1.7 (2) of [Aub], we have $\tau_1 \otimes \tau_2 \leq m^*\pi$ if and only if $\hat{\tau_2} \otimes \hat{\tau_1} \leq m^*\hat{\pi}$. Thus, e.g., we have $\underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{f} \otimes \theta \leq m^*\pi$ if and only if $\hat{\theta} \otimes \underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{f} \leq m^*\hat{\pi}$. Therefore, $f_\pi(a) = g_{\hat{\pi}}(a)$. Furthermore, writing $f = f_\pi(a) = g_{\hat{\pi}}(a)$, if $r_{\underbrace{\nu^a\rho\times\cdots\times\nu^a\rho}_{f}}\pi = \underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{f} \otimes \theta$, then $s_{\underbrace{\nu^a\rho\times\cdots\times\nu^a\rho}_{f}} = \hat{\theta} \otimes \underbrace{\nu^a\rho \times \cdots \times \nu^a\rho}_{f}.$

3. Some applications

3.1. **Example: regular representations.** In this section, we show how the results of the previous section may be applied to regular representations. In particular, we obtain an alternating sum formula analogous to that for the Steinberg representation (cf. [Cas2]). We note that a similar result appears in 9.13 of [Zel], but with a different approach and formulated in terms of the Zelevinsky classification.

As our goal is to write a regular irreducible representation π as a sum of induced representations, the reductions discussed in section 2 allow us to assume that $\pi \in \mathrm{Irr}(\rho)$. Such a π then necessarily has the form

$$\pi = L(\delta([\nu^{b_1}\rho, \nu^{a_1}\rho]), \delta([\nu^{b_2}\rho, \nu^{a_2}\rho]), \dots, \delta([\nu^{b_k}\rho, \nu^{a_k}\rho])),$$

with $a_i < b_{i+1}$ for $1 \le i \le k-1$. The following note reduces us to the case where $a_i = b_{i+1} - 1$ for $1 \le i \le k-1$, i.e.,

$$\pi = L(\delta([\nu^{a_0+1}\rho, \nu^{a_1}\rho]), \delta([\nu^{a_1+1}\rho, \nu^{a_2}\rho]), \dots, \delta([\nu^{a_{k-1}+1}\rho, \nu^{a_k}\rho]))$$

(writing $a_0 + 1$ for b_1).

Note 3.1.1. If $a_i < b_{i+1} - 1$, we have

$$\pi = L(\delta([\nu^{b_1}\rho, \nu^{a_1}\rho]), \dots, \delta([\nu^{b_i}\rho, \nu^{a_i}\rho])) \times L(\delta([\nu^{b_{i+1}}\rho, \nu^{a_{i+1}}\rho]), \dots, \delta([\nu^{b_k}\rho, \nu^{a_k}\rho])).$$

In the case where ρ is trivial, this follows from Theorem 3.1.2 of [Jan1] or Theorem 2.6 of [Jan3]; the general case may be argued similarly.

To analyze such representations, we begin with the following lemma:

Lemma 3.1.2. *For $k \ge 2$,*

$$\begin{aligned}
&\delta([\nu^{a_0+1}\rho, \nu^{a_1}\rho]) \times L(\delta([\nu^{a_1+1}\rho, \nu^{a_2}\rho]), \delta([\nu^{a_2+1}\rho, \nu^{a_3}\rho]), \dots, \delta([\nu^{a_{k-1}+1}\rho, \nu^{a_k}\rho])) \\
&= L(\delta([\nu^{a_0+1}\rho, \nu^{a_1}\rho]), \delta([\nu^{a_1+1}\rho, \nu^{a_2}\rho]), \delta([\nu^{a_2+1}\rho, \nu^{a_3}\rho]), \dots, \delta([\nu^{a_{k-1}+1}\rho, \nu^{a_k}\rho])) \\
&+ L(\delta([\nu^{a_0+1}\rho, \nu^{a_2}\rho]), \delta([\nu^{a_2+1}\rho, \nu^{a_3}\rho]), \delta([\nu^{a_3+1}\rho, \nu^{a_4}\rho]), \dots, \delta([\nu^{a_{k-1}+1}\rho, \nu^{a_k}\rho])).
\end{aligned}$$

Proof. The proof is by induction on the parabolic rank of the supercuspidal support, i.e., $a_k - a_0$. The case $a_k - a_0 = 2$ is trivial.

Now, we assume the result holds for parabolic rank of the supercuspidal support $< a_k - a_0$. Using the m^* structure and Theorem 2.2.1, we get

$$\begin{aligned}
&r_{(1)}\delta([\nu^{a_0+1}\rho, \nu^{a_1}\rho]) \times L(\delta([\nu^{a_1+1}\rho, \nu^{a_2}\rho]), \delta([\nu^{a_2+1}\rho, \nu^{a_3}\rho]), \dots, \delta([\nu^{a_{k-1}+1}\rho, \nu^{a_k}\rho])) \\
&\quad = \nu^{a_1}\rho \otimes \delta([\nu^{a_0+1}\rho, \nu^{a_1-1}\rho]) \times L(\delta([\nu^{a_1+1}\rho, \nu^{a_2}\rho]), \delta([\nu^{a_2+1}\rho, \nu^{a_3}\rho]), \\
&\qquad\qquad \dots, \delta([\nu^{a_{k-1}+1}\rho, \nu^{a_k}\rho])) \\
&\quad + \sum_{i=2}^{k} m_i \nu^{a_i}\rho \otimes \delta([\nu^{a_0+1}\rho, \nu^{a_1}\rho]) \times L(\delta([\nu^{a_1+1}\rho, \nu^{a_2}\rho]), \delta([\nu^{a_2+1}\rho, \nu^{a_3}\rho]), \\
&\qquad \dots, \delta([\nu^{a_{i-1}+1}\rho, \nu^{a_i-1}\rho]), \delta([\nu^{a_i+1}\rho, \nu^{a_{i+1}}\rho]), \delta([\nu^{a_{i+1}+1}\rho, \nu^{a_{i+2}}\rho]), \\
&\qquad\qquad \dots, \delta([\nu^{a_{k-1}+1}\rho, \nu^{a_k}\rho])) \\
&\quad = \nu^{a_1}\rho \otimes \delta([\nu^{a_0+1}\rho, \nu^{a_1-1}\rho]) \times L(\delta([\nu^{a_1+1}\rho, \nu^{a_2}\rho]), \delta([\nu^{a_2+1}\rho, \nu^{a_3}\rho]), \\
&\qquad\qquad \dots, \delta([\nu^{a_{k-1}+1}\rho, \nu^{a_k}\rho])) \\
&\quad + \sum_{i=2}^{k} m_i \nu^{a_i}\rho \otimes \delta([\nu^{a_0+1}\rho, \nu^{a_1}\rho]) \times L(\delta([\nu^{a_1+1}\rho, \nu^{a_2}\rho]), \delta([\nu^{a_2+1}\rho, \nu^{a_3}\rho]), \\
&\qquad \dots, \delta([\nu^{a_{i-1}+1}\rho, \nu^{a_i-1}\rho])) \times L(\delta([\nu^{a_i+1}\rho, \nu^{a_{i+1}}\rho]), \delta([\nu^{a_{i+1}+1}\rho, \nu^{a_{i+2}}\rho]), \\
&\qquad\qquad \dots, \delta([\nu^{a_{k-1}+1}\rho, \nu^{a_k}\rho])),
\end{aligned}$$

by Note 3.1.1, where the multiplicity

$$m_i = \begin{cases} 1 & \text{if } a_i > a_{i-1} + 1 \text{ or } i = 2, \\ 0 & \text{if } a_i = a_{i-1} + 1 \text{ and } i > 2. \end{cases}$$

Now, applying the inductive hypothesis and Note 3.1.1, we have

$$
\begin{aligned}
&r_{(1)}\delta([\nu^{a_0+1}\rho,\nu^{a_1}\rho])\times L(\delta([\nu^{a_1+1}\rho,\nu^{a_2}\rho]),\delta([\nu^{a_2+1}\rho,\nu^{a_3}\rho]),\dots,\delta([\nu^{a_{k-1}+1}\rho,\nu^{a_k}\rho]))\\
&\quad=\nu^{a_1}\rho\otimes L(\delta([\nu^{a_0+1}\rho,\nu^{a_1-1}\rho]),\delta([\nu^{a_1+1}\rho,\nu^{a_2}\rho]),\delta([\nu^{a_2+1}\rho,\nu^{a_3}\rho]),\\
&\qquad\qquad\dots,\delta([\nu^{a_{k-1}+1}\rho,\nu^{a_k}\rho]))\\
&\quad+\sum_{i=2}^{k}m_iL(\delta([\nu^{a_0+1}\rho,\nu^{a_1}\rho]),\delta([\nu^{a_1+1}\rho,\nu^{a_2}\rho]),\dots,\delta([\nu^{a_{i-2}+1}\rho,\nu^{a_{i-1}}\rho]),\\
&\qquad\delta([\nu^{a_{i-1}+1}\rho,\nu^{a_i-1}\rho]),\delta([\nu^{a_i+1}\rho,\nu^{a_{i+1}}\rho]),\delta([\nu^{a_{i+1}+1}\rho,\nu^{a_{i+2}}\rho]),\\
&\qquad\qquad\dots,\delta([\nu^{a_{k-1}+1}\rho,\nu^{a_k}\rho]))\\
&\quad+\sum_{i=2}^{k}m_i\nu^{a_i}\rho\otimes L(\delta([\nu^{a_0+1}\rho,\nu^{a_2}\rho]),\delta([\nu^{a_2+1}\rho,\nu^{a_3}\rho]),\dots,\delta([\nu^{a_{i-2}+1}\rho,\nu^{a_{i-1}}\rho]),\\
&\quad\delta([\nu^{a_{i-1}+1}\rho,\nu^{a_i-1}\rho]),\delta([\nu^{a_i+1}\rho,\nu^{a_{i+1}}\rho]),\delta([\nu^{a_{i+1}+1}\rho,\nu^{a_{i+2}}\rho]),\\
&\qquad\qquad\dots,\delta([\nu^{a_{k-1}+1}\rho,\nu^{a_k}\rho]))
\end{aligned}
$$

By looking at the minimal lexicographic term (cf. section 1.2), we see that one component is

$$\pi_1=L(\delta([\nu^{a_0+1}\rho,\nu^{a_1}\rho]),\delta([\nu^{a_1+1}\rho,\nu^{a_2}\rho]),\dots,\delta([\nu^{a_{k-1}+1}\rho,\nu^{a_k}\rho])).$$

By Theorem 2.2.1, we see that $r_{(1)}\pi_1$ accounts for the $\nu^{a_1}\rho\otimes$ term and those terms appearing in the first sum. Now, by looking at the minimal lexicographic term in the second sum (the $\nu^{a_2}\rho\otimes$ term, which is nonzero), we see that another component is

$$
\begin{aligned}
\pi_2=L(&\delta([\nu^{a_0+1}\rho,\nu^{a_2}\rho]),\delta([\nu^{a_2+1}\rho,\nu^{a_3}\rho]),\delta([\nu^{a_3+1}\rho,\nu^{a_4}\rho]),\\
&\dots,\delta([\nu^{a_{k-1}+1}\rho,\nu^{a_k}\rho])).
\end{aligned}
$$

Theorem 2.2.1 tells us that $r_{(1)}\pi_2$ accounts for all the remaining terms. Thus π_1 and π_2 are the only components, and both occur. The lemma now follows from induction. □

Proposition 3.1.3. *Let*

$$\pi=L(\delta([\nu^{a_0+1}\rho,\nu^{a_1}\rho]),\delta([\nu^{a_1+1}\rho,\nu^{a_2}\rho]),\dots,\delta([\nu^{a_{k-1}+1}\rho,\nu^{a_k}\rho])).$$

Then,

$$
\begin{aligned}
\pi=\sum_{\substack{S=\{s_1,\dots,s_\ell\}\\ S\subset\{1,\dots,k-1\}}}&(-1)^{k+1-|S|}\delta([\nu^{a_0+1}\rho,\nu^{a_{s_1}}\rho])\\
&\times\delta([\nu^{a_{s_1}+1}\rho,\nu^{a_{s_2}}\rho])\times\cdots\times\delta([\nu^{a_{s_\ell}+1}\rho,\nu^{a_k}\rho]),
\end{aligned}
$$

where the sum is over all subsets of $\{1,2,\dots,k-1\}$ *and* $s_1<s_2<\cdots<s_\ell$.

Proof. The proof is by induction on k. The case $k=1$ is trivial; the case $k=2$ follows from [Zel].

By Lemma 3.1.2,

$$\begin{aligned}
\pi = {} & \delta([\nu^{a_0+1}\rho, \nu^{a_1}\rho]) \times L(\delta([\nu^{a_1+1}\rho, \nu^{a_2}\rho]), \delta([\nu^{a_2+1}\rho, \nu^{a_3}\rho]), \ldots, \delta([\nu^{a_{k-1}+1}\rho, \nu^{a_k}\rho])) \\
& - L(\delta([\nu^{a_0+1}\rho, \nu^{a_2}\rho]), \delta([\nu^{a_2+1}\rho, \nu^{a_3}\rho]), \delta([\nu^{a_3+1}\rho, \nu^{a_4}\rho]), \ldots, \delta([\nu^{a_{k-1}+1}\rho, \nu^{a_k}\rho])) \\
= {} & \sum_{\substack{S=\{s_1,\ldots,s_\ell\} \\ S\subset\{2,\ldots,k-1\}}} (-1)^{k-|S|}\delta([\nu^{a_0+1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{a_1+1}\rho, \nu^{a_{s_1}}\rho]) \\
& \times \delta([\nu^{a_{s_1}+1}\rho, \nu^{a_{s_2}}\rho]) \times \cdots \times \delta([\nu^{a_{s_\ell}+1}\rho, \nu^{a_k}\rho]) \\
& - \sum_{\substack{S=\{s_1,\ldots,s_\ell\} \\ S\subset\{2,\ldots,k-1\}}} (-1)^{k-|S|}\delta([\nu^{a_0+1}\rho, \nu^{a_{s_1}}\rho]) \\
& \times \delta([\nu^{a_{s_1}+1}\rho, \nu^{a_{s_2}}\rho]) \times \cdots \times \delta([\nu^{a_{s_\ell}+1}\rho, \nu^{a_k}\rho])
\end{aligned}$$

by inductive hypothesis. We may rewrite this as follows (using $\{t_1, t_2, \ldots, t_\ell\}$ for $\{1, s_1, s_2, \ldots, s_\ell\}$ and $\{s_1, s_2, \ldots, s_\ell\}$, resp.):

$$\begin{aligned}
\pi = {} & \sum_{\substack{T=\{t_1,\ldots,t_\ell\} \\ T\subset\{1,\ldots,k-1\} \\ with\ t_1=1}} (-1)^{k-(|T|-1)}\delta([\nu^{a_0+1}\rho, \nu^{a_{t_1}}\rho]) \\
& \times \delta([\nu^{a_{t_1}+1}\rho, \nu^{a_{t_2}}\rho]) \times \delta([\nu^{a_{t_2}+1}\rho, \nu^{a_{t_3}}\rho]) \times \cdots \times \delta([\nu^{a_{t_\ell}+1}\rho, \nu^{a_k}\rho]) \\
& + \sum_{\substack{T=\{t_1,\ldots,t_\ell\} \\ T\subset\{1,\ldots,k-1\} \\ with\ t_1>1}} (-1)\cdot(-1)^{k-|T|}\delta([\nu^{a_0+1}\rho, \nu^{a_{t_1}}\rho]) \\
& \times \delta([\nu^{a_{t_1}+1}\rho, \nu^{a_{t_2}}\rho]) \times \cdots \times \delta([\nu^{a_{t_\ell}+1}\rho, \nu^{a_k}\rho]),
\end{aligned}$$

which easily reduces to the desired result. □

3.2. **The Langlands classification and Zelevsinky classification.** In Theorem 6.1 of [Zel], Zelevinsky gives a classification of the irreducible non-supercuspidal representations of general linear groups. In this section, we show how to relate the "Zelevinsky data" for such a representation to its Langlands data.

We begin with a definition.

Definition 3.2.1. For a representation $\pi \in R(\rho)$, we let $\chi_M(\pi)$ denote the highest element of $r_{min}\pi$ with respect to the lexicographic order (unique up to multiplicity).

A proof analogous to that of Lemma 2.2.2 of [Jan4] shows the following:

Lemma 3.2.2. *$\chi_M(\pi)$ has the form*

$$\chi_M(\pi) = (\nu^{x_1}\rho \otimes \nu^{x_1+1}\rho \otimes \cdots \otimes \nu^{y_1}\rho) \otimes \cdots \otimes (\nu^{x_\ell}\rho \otimes \nu^{x_\ell+1}\rho \otimes \cdots \otimes \nu^{y_\ell}\rho)$$

with $x_1 \geq x_2 \geq \cdots \geq x_\ell$ and $y_i \geq y_{i+1}$ if $x_i = x_{i+1}$.

We note that the analogue of the last condition is not stated in Lemma 2.2.2 of [Jan4] but can easily be deduced from Corollary 2.2.4 of [Jan4] (or argued directly).

We let $\zeta([\nu^x\rho, \nu^y\rho])$ denote the unique irreducible subrepresentation of $\nu^x\rho \times \nu^{x+1}\rho \times \cdots \times \nu^y\rho$. With χ_M as above, we define ζ_M by

$$\zeta_M(\pi) = \zeta([\nu^{x_1}\rho, \nu^{y_1}\rho]) \otimes \cdots \otimes \zeta([\nu^{x_\ell}\rho, \nu^{y_\ell}\rho]).$$

An argument like that of Corollary 2.2.4 of [Jan4] then gives the following:

Corollary 3.2.3. *If π is irreducible, then*

$$\pi \hookrightarrow i_{G,M_{app}} \zeta_M(\pi),$$

where M_{app} is the appropriate parabolic subgroup.

We now observe that if π is irreducible and has

$$\zeta_M(\pi) = \zeta([\nu^{x_1}\rho, \nu^{y_1}\rho]) \otimes \cdots \otimes \zeta([\nu^{x_\ell}\rho, \nu^{y_\ell}\rho]),$$

the segments $[\nu^{x_1}\rho, \nu^{y_1}\rho], \ldots, [\nu^{x_\ell}\rho, \nu^{y_\ell}\rho]$ satisfy the requirements of Theorem 6.1 of [Zel]. Furthermore, any permutation $[\nu^{x'_1}\rho, \nu^{y'_1}\rho], \ldots, [\nu^{x'_\ell}\rho, \nu^{y'_\ell}\rho]$ of $[\nu^{x_1}\rho, \nu^{y_1}\rho], \ldots, [\nu^{x_\ell}\rho, \nu^{y_\ell}\rho]$ which satisfies the hypotheses of Theorem 6.1 of [Zel] has $\zeta([\nu^{x'_1}\rho, \nu^{y'_1}\rho]) \otimes \cdots \otimes \zeta([\nu^{x'_\ell}\rho, \nu^{y'_\ell}\rho])$ inductively equivalent to $\zeta([\nu^{x_1}\rho, \nu^{y_1}\rho]) \otimes \cdots \otimes \zeta([\nu^{x_\ell}\rho, \nu^{y_\ell}\rho])$. Thus, we call $\zeta_M(\pi)$ the Zelevinsky data for π.

We now relate the Langlands classification and Zelevinsky classification through duality (cf. [Aub], [S-S] or section 9 of [Zel]).

Proposition 3.2.4. *Suppose $\pi = L(\delta([\nu^{b_1}\rho, \nu^{a_1}\rho]), \ldots, \delta([\nu^{b_k}\rho, \nu^{a_k}\rho]))$. Then, the Zelevinsky data for $\hat{\pi}$ is $\zeta([\nu^{b_k}\rho, \nu^{a_k}\rho]) \otimes \cdots \otimes \zeta([\nu^{b_1}\rho, \nu^{a_1}\rho])$ up to inductive equivalence.*

Proof. We have

$$\chi_0(\pi) = (\nu^{a_1}\rho \otimes \nu^{a_1-1}\rho \otimes \cdots \otimes \nu^{b_1+1}\rho \otimes \nu^{b_1}\rho) \otimes \cdots \otimes (\nu^{a_k}\rho \otimes \nu^{a_k-1}\rho \otimes \cdots \otimes \nu^{b_k+1}\rho \otimes \nu^{b_k}\rho) \leq r_{min}\pi.$$

By Théorème 1.7 of [Aub], we have

$$(\nu^{b_k}\rho \otimes \nu^{b_k+1}\rho \otimes \cdots \otimes \nu^{a_k-1}\rho \otimes \nu^{a_k}\rho) \otimes \cdots \otimes (\nu^{b_1}\rho \otimes \nu^{b_1+1}\rho \otimes \cdots \otimes \nu^{a_1-1}\rho \otimes \nu^{a_1}\rho) \leq r_{min}\hat{\pi}$$

In fact, this is the minimal term in $r_{min}\hat{\pi}$ with respect to the right-to-left lexicographic order. An argument like that in Lemma 2.2.2 of [Jan4] shows that this implies

$$\hat{\pi} \hookrightarrow \zeta([\nu^{b_k}\rho, \nu^{a_k}\rho]) \times \cdots \times \zeta([\nu^{b_1}\rho, \nu^{a_1}\rho]).$$

It remains to show that this is inductively equivalent to the Zelevinsky data. Observe that if $i < j$ has $b_i < b_j$, then $\zeta([\nu^{b_i}\rho, \nu^{a_i}\rho]) \times \zeta([\nu^{b_j}\rho, \nu^{a_j}\rho]) \cong \zeta([\nu^{b_j}\rho, \nu^{a_j}\rho]) \times \zeta([\nu^{b_i}\rho, \nu^{a_i}\rho])$ by irreducibility. Thus by a sequence of such transpositions, the $\zeta([\nu^{b_i}\rho, \nu^{a_i}\rho])$ may be rearranged to get an inductively equivalent shuffle $\zeta([\nu^{b'_1}\rho, \nu^{a'_1}\rho]) \times \cdots \times \zeta([\nu^{b'_k}\rho, \nu^{a'_k}\rho])$ satisfying $b'_1 \geq \cdots \geq b'_k$ and if $b'_i = b'_{i+1}$, then $a'_i \geq a'_{i+1}$. □

Remark 3.2.5. The above proposition is essentially a precise version of the claim that the Zelevinsky classification is dual to the Langlands classification. Note that if π is an irreducible representation, the Zelevinsky data for π may be easily be obtained if the Langlands data for $\hat{\pi}$ is known. A method for obtaining the Langlands data for $\hat{\pi}$ from that for π is discussed in the next section. Similarly, if the Zelevinsky data for π is known, one may obtain the Langlands data for $\hat{\pi}$, from which one can then obtain the Langlands data for π.

3.3. **The Zelevinsky involution.** In this section, we give an algorithm for calculating the Zelevinsky involution of an irreducible representation. More precisely, let π be an irreducible representation supported on $\{\nu^z\rho\}_{z\in\mathbb{Z}}$ and $\hat{\pi}$ its dual (cf. [Aub],[S-S])—the Zelevinsky involution of π. (In the case where ρ is trivial, this also corresponds to the Iwahori-Matsumoto involution of π; cf. [Kat].) We show how to calculate the Langlands data for $\hat{\pi}$ from that for π. A similar result is given in [M-W], but their approach is different and is done in the setting of the Zelevinsky classification. (However, we note the similarity of our lexicographic ordering on Jacquet modules and Lemma 1.3.4 to their ordering on multisegments and Lemmes II.8 and II.10.1.) We first describe the algorithm and show that it works; we close with an example to illustrate this approach.

We now describe the algorithm.

(1) Set $\tau_1 = \pi$ and let ψ_1 be the trivial representation of $GL(0,F)$. Starting with $i=1$, we iterate the following procedure:

(a) Choose x_i such that $g_{\tau_i}(x_i) \neq 0$. Let $g_i = g_{\tau_i}(x_i)$ and write (cf. Theorem 2.4.5)

$$s_{\underbrace{\nu^{x_i}\rho\times\cdots\times\nu^{x_i}\rho}_{g_i}}\tau_i = \tau_{i+1}\otimes\underbrace{\nu^{x_i}\rho\times\cdots\times\nu^{x_i}\rho}_{g_i}.$$

(b) Set

$$\psi_{i+1} = \psi_i\otimes\underbrace{\nu^{x_i}\rho\times\cdots\times\nu^{x_i}\rho}_{g_i}.$$

This procedure stops after ℓ steps if $\tau_{\ell+1}$ is the trivial representation of $GL(0,F)$ (so that $g_{\tau_{\ell+1}}(x)=0$ for all x). Then

$$\psi_\ell = \underbrace{\nu^{x_1}\rho\times\cdots\times\nu^{x_1}\rho}_{g_1}\otimes\underbrace{\nu^{x_2}\rho\times\cdots\times\nu^{x_2}\rho}_{g_2}\otimes\cdots\otimes\underbrace{\nu^{x_\ell}\rho\times\cdots\times\nu^{x_\ell}\rho}_{g_\ell}.$$

(2) Let $L_{\ell+1}$ be the trivial representation of $GL(0,F)$. Starting with $i=\ell+1$ and working down to $i=1$, consider $\underbrace{\nu^{x_i}\rho\times\cdots\times\nu^{x_i}\rho}_{g_i}\otimes L_{i+1}$. By Remark 2.2.2 and Theorem 2.2.1, we can determine the unique irreducible L_i such that $r_{\underbrace{\nu^{x_i}\rho\times\cdots\times\nu^{x_i}\rho}_{g_i}}L_i = \underbrace{\nu^{x_i}\rho\times\cdots\times\nu^{x_i}\rho}_{g_i}\otimes L_{i+1}$ (n.b., $f_{L_i}(x_i) = g_i + f_{L_{i+1}}(x_i)$).

We have $\hat{\pi} = L_1$.

Proof of the algorithm. To see that the algorithm works, it is enough to show the following: If

$$r_{app}\hat{\pi} \geq \underbrace{\nu^{x_1}\rho\times\cdots\times\nu^{x_1}\rho}_{g_1}\otimes\cdots\otimes\underbrace{\nu^{x_{i-1}}\rho\times\cdots\times\nu^{x_{i-1}}\rho}_{g_{i-1}}\otimes\theta_i$$

has

$$m^*\theta_i \geq \underbrace{\nu^{x_i}\rho\times\cdots\times\nu^{x_i}\rho}_{g_i}\otimes L_{i+1},$$

then $f_{\theta_i}(x_i) = f_{L_{i+1}}(x_i) + g_i$, making θ_i the unique irreducible representation with this property, i.e., L_i. Suppose this is not the case. Since we clearly have $f_{\theta_i}(x_i) \geq$

$f_{L_{i+1}}(x_i)+g_i$, this would mean $f_{\theta_i}(x_i)>f_{L_{i+1}}(x_i)+g_i$. In this case,

$$r_{app}\hat{\pi} \geq \underbrace{\nu^{x_1}\rho\times\cdots\times\nu^{x_1}\rho}_{g_1}\otimes\cdots\otimes\underbrace{\nu^{x_{i-1}}\rho\times\cdots\times\nu^{x_{i-1}}\rho}_{g_{i-1}}\otimes\underbrace{\nu^{x_i}\rho\times\cdots\times\nu^{x_i}\rho}_{g_i'}\otimes\dots$$

with $g_i'>g_i$. By Théorème 1.7 (2) of [Aub], this implies

$$r_{app}\pi \geq \cdots\otimes\underbrace{\nu^{x_i}\rho\times\cdots\times\nu^{x_i}\rho}_{g_i'}\otimes\underbrace{\nu^{x_{i-1}}\rho\times\cdots\times\nu^{x_{i-1}}\rho}_{g_{i-1}}\otimes\cdots\otimes\underbrace{\nu^{x_1}\rho\times\cdots\times\nu^{x_1}\rho}_{g_1}.$$

However, we know that

$$\begin{aligned}&s_{\underbrace{\nu^{x_{i-1}}\rho\times\cdots\times\nu^{x_{i-1}}\rho}_{g_{i-1}}\otimes\cdots\otimes\underbrace{\nu^{x_1}\rho\times\cdots\times\nu^{x_1}\rho}_{g_1}}\pi\\ &\qquad=\tau_i\otimes\underbrace{\nu^{x_{i-1}}\rho\times\cdots\times\nu^{x_{i-1}}\rho}_{g_{i-1}}\otimes\cdots\otimes\underbrace{\nu^{x_1}\rho\times\cdots\times\nu^{x_1}\rho}_{g_1}.\end{aligned}$$

This implies that $s_{\underbrace{\nu^{x_i}\rho\times\cdots\times\nu^{x_i}\rho}_{g_i'}}\tau_i\neq 0$, contradicting $g_{\tau_i}(x_i)=g_i$. □

Example 3.3.1. $\pi=L(\delta([\nu\rho,\nu^2\rho]),\nu^2\rho,\delta([\nu^2\rho,\nu^3\rho]),\nu^3\rho)$

For (1), we calculate:

$$\begin{gathered}L(\delta([\nu\rho,\nu^2\rho]),\nu^2\rho,\delta([\nu^2\rho,\nu^3\rho]),\nu^3\rho)\\ \downarrow\ \text{taking } x_1=3\text{, so } g_1=1\\ L(\delta([\nu\rho,\nu^2\rho]),\nu^2\rho,\delta([\nu^2\rho,\nu^3,\rho]))\otimes\nu^3\rho\\ \downarrow\text{ taking } x_2=2\text{, so } g_2=2\\ L(\delta([\nu\rho,\nu^2\rho]),\nu^3\rho)\otimes\nu^2\rho\times\nu^2\rho\otimes\nu^3\rho\\ \downarrow\text{ taking } x_3=3\text{, so } g_3=1\\ \delta([\nu\rho,\nu^2\rho])\rho\otimes\nu^3\rho\otimes\nu^2\rho\times\nu^2\rho\otimes\nu^3\rho\\ \downarrow\text{ taking } x_4=1\text{, so } g_4=1\\ \nu^2\rho\otimes\nu\rho\otimes\nu^3\rho\otimes\nu^2\rho\times\nu^2\rho\otimes\nu^3\rho.\end{gathered}$$

Thus, $\ell=5$ and

$$\psi_5=\nu^3\rho\otimes\nu^2\rho\times\nu^2\rho\otimes\nu^3\rho\otimes\nu\rho\otimes\nu^2\rho.$$

For (2), we calculate:

$$\begin{gathered}\nu^3\rho\otimes\nu^2\rho\times\nu^2\rho\otimes\nu^3\rho\otimes\nu\rho\otimes\nu^2\rho\\ \downarrow\\ \nu^3\rho\otimes\nu^2\rho\times\nu^2\rho\otimes\nu^3\rho\otimes L(\nu\rho,\nu^2\rho)\\ \downarrow\\ \nu^3\rho\otimes\nu^2\rho\times\nu^2\rho\otimes L(\nu\rho,\delta([\nu^2\rho,\nu^3\rho]))\\ \downarrow\\ \nu^3\rho\otimes L(\delta([\nu\rho,\nu^2\rho]),\nu^2\rho,\delta([\nu^2\rho,\nu^3\rho]))\\ \downarrow\\ L(\delta([\nu\rho,\nu^2\rho]),\delta([\nu^2\rho,\nu^3\rho]),\delta([\nu^2\rho,\nu^3\rho])).\end{gathered}$$

Thus, $\hat{\pi}=L(\delta([\nu\rho,\nu^2\rho]),\delta([\nu^2\rho,\nu^3\rho]),\delta([\nu^2\rho,\nu^3\rho]))$.

3.4. **Toward an algorithm when $f_\pi(a) \geq 2$.** In this section, we give a couple of cases where we have algorithms for calculating $r_{(1)}\pi$, $\pi = L(\delta([\nu^{b_1}\rho, \nu^{a_1}\rho]), \ldots, \delta([\nu^{b_k}\rho, \nu^{a_k}\rho]))$, subject to certain restrictions on a_i, b_i. These algorithms are inductive in nature, building up from lower rank general linear groups. We note that the first algorithm is not closed; if π satisfies the constraints on a_i, b_i, the same may not be true of all the representations of the lower rank groups which are used by the algorithm. Thus, it may not be possible to actually calculate $r_{(1)}(\pi)$ using the first algorithm, since some of the embedded calculations may not be possible. (It is perhaps better to view the first algorithm as a special case of a general algorithm, rather than an algorithm for a special case.)

Definition 3.4.1. Let $\pi \in R(\rho)$ be an irreducible representation and $X, Y \subset \mathbb{Z}$. For M_α a standard Levi subgroup, we let $r_{(\alpha)}^{\mathcal{P}_f(X)}\pi$ (resp., $r_{(\alpha)}^{\mathcal{P}_g(Y)}\pi$) denote the sum of all $\lambda \leq r_{(\alpha)}\pi$ satisfying $f_\lambda(X) = f_\pi(X)$ (resp., $g_\lambda(Y) = g_\pi(Y)$).

We now describe how to calculate $r_{(1)}^{\mathcal{P}_f(X)}\pi$ and $r_{(1)}^{\mathcal{P}_g(Y)}\pi$ (under the inductive assumption that lower rank Jacquet modules are known) when X, Y are proper subsets of the supercuspidal support. We start with $r_{(1)}^{\mathcal{P}_f(X)}\pi$. By the results of section 2.3, we may calculate $m_X^*\pi = \tau_\pi(X) \otimes \theta_\pi(X)$. By inductive assumption, write
$$r_{(1)}\tau_\pi(X) = \sum_i m_i \nu^{a_i}\rho \otimes \tau_i,$$
with τ_i irreducible and m_i the multiplicity. Then, for $f = f_\pi(X)$, we have
$$r_{(1,f-1)}^{\mathcal{P}_f(X)}\pi = \sum_i m_i \nu^{a_i}\rho \otimes \tau_i \otimes \theta_\pi(X).$$
We note the following:

Lemma 3.4.2. *Suppose $\nu^a\rho \otimes L \leq r_{(1)}^{\mathcal{P}_f(X)}\pi$, and L irreducible. Then there is a unique i with $a_i = a$ and*
$$m_X^* L = \tau_i \otimes \theta_\pi(X).$$
Furthermore, the multiplicity of $\nu^a\rho \otimes L$ in $r_{(1)}^{\mathcal{P}_f(X)}\pi$ is m_i.

Proof. Certainly, there is some i with $a_i = a$ and $\tau_i \otimes \theta_\pi(X) \leq m^*L$. This tells us $f_L(X) \geq f_\pi(X) - 1$; as $f_L(X) \leq f_\pi(X) - 1$ must hold, we have $f_L(X) = f_\pi(X) - 1$. It then follows from Lemma 2.1.2 that $m_X^* L = \tau_i \otimes \theta_\pi(X)$. The rest of the lemma now follows. □

In particular, there is a bijective correspondence between the $\nu^{a_i}\rho \otimes \tau_i \otimes \theta_\pi(X) \leq r_{(1,f-1)}^{\mathcal{P}_f(X)}\pi$ and the $\nu^a\rho \otimes L \leq r_{(1)}^{\mathcal{P}_f(X)}\pi$ (including multiplicities). For $\tau_i \otimes \theta_\pi(X)$, we then calculate the unique irreducible L_i having $m_X^* L_i = \tau_i \otimes \theta_\pi(X)$ (cf. section 2.3). We have
$$r_{(1)}^{\mathcal{P}_f(X)}\pi = \sum_i m_i \nu^{a_i}\rho \otimes L_i.$$

We now describe how to calculate $r_{(1)}^{\mathcal{P}_g}(Y)\pi$. By the results of section 2.4, we may calculate ${}_Y m^*\pi = \eta_\pi(Y) \otimes \xi_\pi(Y)$. This time, by inductive hypothesis (letting $g = g_\pi(Y)$), write
$$r_{(1)}\eta_\pi(Y) = \sum_i n_i \nu^{a_i}\rho \otimes \eta_i.$$

Therefore,

$$r^{\mathcal{P}_g(Y)}_{(1,n-g-1)}\pi = \sum_i n_i \nu^{a_i}\rho \otimes \eta_i \otimes \xi_\pi(X).$$

For $\eta_i \otimes \xi_\pi(Y)$, there is a unique irreducible L_i such that ${}_Y m^* L_i = \eta_i \otimes \xi_\pi(Y)$, and we may determine this L_i (cf. section 2.4). As with $m^*_X\pi$ above (cf. Lemma 3.4.2 et seq.), we then see that

$$r^{\mathcal{P}_g(Y)}_{(1)}\pi = \sum_i n_i \nu^{a_i}\rho \otimes L_i.$$

Dual calculations allow one to calculate $r^{\mathcal{P}_f(X)}_{(n-1)}\pi$ and $r^{\mathcal{P}_g(Y)}_{(n-1)}\pi$ (where ρ is a representation of $GL(r,F)$ and π is a representation of $GL(nr,F)$). Given the similarity to the above calculations, we forgo the details.

We note that in some cases, the above are enough to produce all of $r_{(1)}\pi$. One such situation is the following (which then constitutes the first algorithm referred to at the start of this section):

Proposition 3.4.3. *Suppose there is a b such that $g_\pi(b) \neq 0$ and the following hold: (i) $a_i \neq b, b-1$ for all i, and (ii) $b_i \neq b+1$ for all i. Then, $r_{(1)}\pi = r^{\mathcal{P}_g(b)}_{(1)}\pi$.*

Proof. Consider $\nu^x\rho \otimes \theta \leq r_{(1)}\pi$ (θ irreducible). We need to show that $g_\theta(b) = g_\pi(b)$.

First, observe that Corollary 1.3.2 and (i) ensure $n_\theta(b) = n_\pi(b)$, where $n(b) = |\{i \,|\, b_i = b\}|$ (the subscript indicating the representation considered). By Proposition 2.4.3, condition (ii) is now enough to ensure that $g_\theta(b) = n_\theta(b)$ and $g_\pi(b) = n_\pi(b)$, finishing the lemma. □

We now give a different situation where the above calculations may be used to calculate $r_{(1)}\pi$. For $M_{(\alpha)}$, a standard Levi subgroup, let $r^{max}_{(\alpha)}\pi$ consist of all $\tau \leq r_{(\alpha)}\pi$ satisfying $f_\tau(X) = f_\pi(X)$ or $g_\tau(Y) = g_\pi(Y)$ for some proper subset X or Y of the supercuspidal support. That is,

$$r^{max}_{(\alpha)}\pi = \left(\bigcup_X r^{\mathcal{P}_f(X)}_{(\alpha)}\pi\right) \bigcup \left(\bigcup_Y r^{\mathcal{P}_g(Y)}_{(\alpha)}\pi\right),$$

where the union is in the multiset sense (i.e., the multiplicity of x in $A \cup B$ is the maximum of the multiplicity in A and the multiplicity in B) and X, Y run over the proper subsets of the supercuspidal support. (We remark that by Corollary 2.3.2 and its counterpart for ${}_Y m^*\pi$, it is enough to consider X, Y maximal proper subsets, i.e., missing just one element of the supercuspidal support.) By the preceding discussion and inductive assumption, we may calculate $r^{max}_{(1)}\pi$ and $r^{max}_{(n-1)}\pi$.

In general, we may normalize matters as at the end of section 2.1, so that the supercuspidal support of π is contained in a set of the form $\{\nu\rho, \nu^2\rho, \ldots, \nu^k\rho\}$ (and assuming both $\nu\rho$ and $\nu^k\rho$ actually occur). The case $k=1$ is trivial; the case $k=2$ is covered by Lemma 1.3.5. In what follows, we show how to calculate $r_{(1)}\pi$ in the case where $k=3$ (but noting that some of the results apply to general k). In particular, we show that at least one of $r_{(1)}\pi = r^{max}_{(1)}\pi$ or $r_{(n-1)}\pi = r^{max}_{(n-1)}\pi$ must hold (and which are). If $r_{(1)}\pi = r^{max}_{(1)}\pi$, we are done; if $r_{(n-1)}\pi = r^{max}_{(n-1)}\pi$, we are done by Note 3.4.10 below. We close with some additional observations which may be obtained using the sort of arguments in this section.

Lemma 3.4.4. *Suppose $\pi \in R(\rho)$ is an irreducible representation with $\nu^a\rho$ in its supercuspidal support. Then $m^*\pi$ contains a term of the form $\eta \otimes L(\delta([\nu^{a-i}\rho, \nu^a\rho]), \nu^{a+1}\rho, \dots, \nu^{a+j}\rho)$ for some $i, j \geq 0$.*

Proof. This follows from arguments like those used in section 2.2 of [Jan4]. In particular, one considers the "last appearance" of $\nu^a\rho$ in $r_{min}\pi$, i.e., term(s) of the form $\nu^{x_1}\rho \otimes \cdots \otimes \nu^{x_n}\rho \leq r_{min}\pi$ having $x_h = a$ for the largest possible h. Using an argument like that in the proof of Lemma 2.2.2 of [Jan4], one shows that such a last appearance occurs in a term of the form $\cdots \otimes \nu^a\rho \otimes \nu^{a-1}\rho \otimes \cdots \otimes \nu^{a-i}\rho \otimes \nu^{a+1}\rho \otimes \nu^{a+2}\rho \otimes \cdots \otimes \nu^{a+j}\rho$. Then, an argument like that used in the proof of Corollary 2.2.4 of [Jan4] implies $\pi \hookrightarrow \eta \times L(\delta([\nu^{a-i}\rho, \nu^a\rho]), \nu^{a+1}\rho, \dots, \nu^{a+j}\rho)$ for some irreducible η, from which the conclusion follows. $\square$

Lemma 3.4.5. *Suppose $\pi \in R(\rho)$ is irreducible and $\nu^a\rho \otimes L \leq r_{(1)}\pi$ but $\nu^a\rho \otimes L \not\leq r_{(1)}^{max}\pi$. Then for proper subsets $X, Y \subset \{1, \dots, k\}$, the following must hold:*

(1) (a) *If $a \in X$, then*

$$f_L(X) < f_\pi(X) - 1.$$

(b) *If $a, a-1, a+1 \notin X$, then*

$$f_L(X) \leq f_\pi(X).$$

(c) *If $a \notin X$ but $a-1 \in X$ or $a+1 \in X$, let $\alpha, \beta \geq 0$ be the largest values such that $\{a-\alpha, a-\alpha+1, \dots, a-1\} \subset X$ and $\{a+1, a+2, \dots, a+\beta\} \subset X$. Then,*

$$f_L(X) \leq f_\pi(X) + \alpha + \beta.$$

(2) *$g_L(Y) \leq g_\pi(Y)$ with equality possible only if both are 0.*

Proof. (1)(a) and (2) follow directly from the definition. For (1)(b), observe that if $f_L(X) > f_\pi(X) = f$, then $r_{min}\pi \geq r_{min}(\nu^a\rho \otimes L)$ tells us that

$$r_{min}\pi \geq \nu^a\rho \otimes \nu^{x_1}\rho \otimes \cdots \otimes \nu^{x_{f+1}}\rho \otimes \dots$$

with $x_1, \dots, x_{f+1} \in X$. A commuting argument (cf. proof of Lemma 1.3.3) then implies that

$$r_{min}\pi \geq \nu^{x_1}\rho \otimes \cdots \otimes \nu^{x_{f+1}}\rho \otimes \nu^a\rho \otimes \dots,$$

a contradiction. For (1)(c), we have

$$\begin{gathered} \nu^a\rho \otimes \tau_L(X) \otimes \theta_L(X) \leq r_{app}\pi \\ \Downarrow \\ \tau' \otimes \theta_L(X) \leq m^*\pi \end{gathered}$$

for some irreducible τ' with $r_{(1)}\tau' \geq \nu^a\rho \otimes \tau_L(X)$. Lemma 3.4.4 applied to τ' gives

$$\begin{gathered} m^*\tau' \geq \eta \otimes L(\delta([\nu^{a-i}\rho, \nu^a\rho]), \nu^{a+1}\rho, \dots, \nu^{a+j}\rho) \\ \Downarrow \\ r_{app}\pi \geq \eta \otimes L(\delta([\nu^{a-i}\rho, \nu^a\rho]), \nu^{a+1}\rho, \dots, \nu^{a+j}\rho) \otimes \theta_L(X) \\ \Downarrow \\ f_\pi(X) \geq f_\eta(X) = f_L(X) - i - j. \end{gathered}$$

The result now follows from the observation that the largest possible values of i, j in that lemma are α, β, respectively. $\square$

The technical heart of the analysis for $k = 3$ is the following:

Proposition 3.4.6. *Suppose that π is irreducible with supercuspidal support in $\{\nu\rho, \nu^2\rho, \nu^3\rho\}$ (with $\nu\rho$ and $\nu^3\rho$ actually occurring) and $\nu^a\rho \otimes L \leq r_{(1)}\pi$ with $\nu^a\rho \otimes L$ irreducible.*

1. *If $a = 1$, then $\nu^a\rho \otimes L \leq r_{(1)}^{max}\pi$.*
2. *If $a = 2$, then $\nu^a\rho \otimes L \leq r_{(1)}^{max}\pi$ unless the following conditions on π hold: $\ell_{3,3} \geq \ell_{2,2} > \ell_{1,1}$ and $\ell_{2,3} = 0$. Furthermore, L must satisfy $\ell'_{1,1} = \ell_{1,1} + 1$, $\ell'_{1,3} = \ell_{1,3} + 1$, $\ell'_{3,3} = \ell_{3,3} - 1$, where $\ell'_{i,j}$ denotes the number of times $\delta([\nu^i\rho, \nu^j\rho])$ appears in the Langlands data for L (i.e., in $\delta_0(L)$).*
3. *If $a = 3$, then $\nu^a\rho \otimes L \leq r_{(1)}^{max}\pi$.*

Proof. We first consider the case $a = 2$. Suppose $\nu^2\rho \otimes L \leq r_{(1)}\pi$ with $\nu^2\rho \otimes L \not\leq r_{(1)}^{max}\pi$. By counting the number of times $\nu\rho$ (resp., $\nu^2\rho, \nu^3\rho$) appear in the supercuspidal support for $\nu^2\rho \otimes L$ and π, we see that

$$\begin{aligned} \ell'_{1,1} + \ell'_{1,2} + \ell'_{1,3}, &= \ell_{1,1} + \ell_{1,2} + \ell_{1,3}, \\ 1 + \ell'_{1,2} + \ell'_{2,2} + \ell'_{1,3} + \ell'_{2,3} &= \ell_{1,2} + \ell_{2,2} + \ell_{1,3} + \ell_{2,3}, \\ \ell'_{1,3} + \ell'_{2,3} + \ell'_{3,3} &= \ell_{1,3} + \ell_{2,3} + \ell_{3,3}. \end{aligned}$$

Next, write $\ell'_{1,1} = \ell_{1,1} - a$ and $\ell'_{3,3} = \ell_{3,3} - c$; by Lemma 3.4.5, $a \geq -1$ and $c \geq 0$. Also, we claim that $\ell'_{1,3} \geq \ell_{1,3}$. By Lemma 1.3.3, we have $\pi = \underbrace{\delta([\nu\rho, \nu^3\rho]) \times \cdots \times \delta([\nu\rho, \nu^3\rho])}_{\ell_{1,3}} \times \pi'$, where π' has the same Langlands data as π except that the $\delta([\nu\rho, \nu^3\rho])$'s are removed. If $r_{\nu^2\rho}\pi' = \sum \nu^2\rho \otimes \lambda_i$, then an m^* argument tells us $r_{\nu^2\rho}\pi' = \sum \nu^2\rho \otimes \underbrace{\delta([\nu\rho, \nu^3\rho]) \times \cdots \times \delta([\nu\rho, \nu^3\rho])}_{\ell_{1,3}} \times \lambda_i$. Again, Lemma 1.3.3 tells us that $\underbrace{\delta([\nu\rho, \nu^3\rho]) \times \cdots \times \delta([\nu\rho, \nu^3\rho])}_{\ell_{1,3}} \times \lambda_i$ is irreducible, so L must have this form. It then follows that $\ell'_{1,3} \geq \ell_{1,3}$, as claimed. With this, we may write $\ell'_{1,3} = \ell_{1,3} + b$, where $b \geq 0$. Using (3), we may now rewrite the $\ell'_{i,j}$ in terms of the $\ell_{i,j}$:

$$\begin{aligned} \ell'_{1,1} &= \ell_{1,1} - a, & \ell'_{1,3} &= \ell_{1,3} + b, \\ \ell'_{1,2} &= \ell_{1,2} + a - b, & \ell'_{2,3} &= \ell_{2,3} + c - b, \\ \ell'_{2,2} &= \ell_{2,2} - 1 + b - a - c, & \ell'_{3,3} &= \ell_{3,3} - c, \end{aligned} \tag{3}$$

with $a \geq -1$ and $b, c \geq 0$.

We consider two cases (showing the first does not actually occur).

Case 1: $\ell_{1,1} \geq \ell_{2,2}$. In this case, Corollary 2.1.5 tells us $f_\pi(2) = \ell_{1,2}$. By Lemma 3.4.5, we have

$$\begin{gathered} \ell'_{1,2} \leq \ell'_{1,2} + max\{\ell'_{2,2} - \ell'_{1,1}, 0\} = f_L(2) < f_\pi(2) - 1 = \ell_{1,2} - 1 \\ \Downarrow \text{ by (4)} \\ \ell_{1,2} + a - b < \ell_{1,2} - 1 \\ \Downarrow \\ a + 1 < b. \end{gathered}$$

We now consider $g_\pi(1)$. By Corollary 2.4.4, we have

$$\begin{aligned}&\ell'_{1,3} + max\{\ell'_{1,1} + \ell'_{1,2} - \ell'_{2,2} - \ell'_{2,3}, 0\}\\ &\quad \leq \ell'_{1,3} + max\{\ell'_{1,2} - \ell'_{2,3} + max\{\ell'_{1,1} - \ell'_{2,2}, 0\}, 0\} = g_L(1)\\ &\quad \leq g_\pi(1) = \ell_{1,3} + max\{\ell_{1,1} + \ell_{1,2} - \ell_{2,2} - \ell_{2,3}, 0\},\end{aligned}$$

with equality only if $g_\pi(1) = 0$. By (4), this reduces to

$$\ell_{1,3}+b+max\{\ell_{1,1}+\ell_{1,2}-\ell_{2,2}-\ell_{2,3}+a-b+1, 0\} \leq \ell_{1,3}+max\{\ell_{1,1}+\ell_{1,2}-\ell_{2,2}-\ell_{2,3}, 0\},$$

with equality only if both are 0. First, suppose the right-hand side is positive. Then, $g_\pi(1) > 0$, so the inequality becomes strict. Since $b \geq 0$, this forces $\ell_{1,1} + \ell_{1,2} - \ell_{2,2} - \ell_{2,3} > 0$. We then have

$$\begin{gathered}\ell_{1,3} + b + \ell_{1,1} + \ell_{1,2} - \ell_{2,2} - \ell_{2,3} + a - b + 1\\ \leq g_L(1) < g_\pi(1) = \ell_{1,3} + \ell_{1,1} + \ell_{1,2} - \ell_{2,2} - \ell_{2,3}\\ \Downarrow\\ a + 1 < 0,\end{gathered}$$

contradicting $a \geq -1$. Thus, we must have the right-hand side equal to 0. However, this forces $b = 0$, contradicting $b > a + 1$ as observed above. Thus Case 1 cannot occur.

<u>Case 2:</u> $\ell_{1,1} < \ell_{2,2}$. In this case, Corollary 2.1.5 tells us $f_\pi(2) = \ell_{1,2} + \ell_{2,2} - \ell_{1,1}$. By Lemma 3.4.5, we have

$$\begin{gathered}\ell'_{1,2} + \ell'_{2,2} - \ell'_{1,1} \leq f_L(2) < f_\pi(2) - 1 = \ell_{1,2} + \ell_{2,2} - \ell_{1,1} - 1\\ \Downarrow\\ \ell_{1,2} + a - b + \ell_{2,2} - 1 + b - a - c - \ell_{1,1} + a < \ell_{1,2} + \ell_{2,2} - \ell_{1,1} - 1\\ \Downarrow\\ a < c.\end{gathered}$$

We now consider $g_\pi(2)$. By Corollary 2.4.4, we have $g_\pi(2) = \ell_{2,3} + max\{\ell_{2,2} - \ell_{3,3}, 0\}$. First, we claim $\ell_{3,3} \geq \ell_{2,2}$. Suppose not, then $\ell_{2,2} > \ell_{3,3}$ and $g_\pi(2) > 0$. Therefore, by Lemma 3.4.5 (n.b.–the assumption that the supercuspidal support actually contains both $\nu\rho$ and $\nu^3\rho$ ensures that $g_{\nu^2\rho\otimes L}(2) = g_L(2)$),

$$\begin{gathered}\ell'_{2,3} + \ell'_{2,2} - \ell'_{3,3} \leq g_L(2) < g_\pi(2) = \ell_{2,3} + \ell_{2,2} - \ell_{3,3}\\ \Downarrow\\ \ell_{2,3} + c - b + \ell_{2,2} - 1 + b - a - c - \ell_{3,3} + c < \ell_{2,3} + \ell_{2,2} - \ell_{3,3}\\ \Downarrow\\ c < a + 1.\end{gathered}$$

However, we cannot have $a < c < a + 1$ for a, c integers. Therefore, $\ell_{3,3} \geq \ell_{2,2}$, as claimed. Then,

$$\begin{gathered}\ell'_{2,3} \leq g_L(2) \leq g_\pi(2) = \ell_{2,3}\\ \Downarrow\\ c \leq b,\end{gathered}$$

with equality only if $g_\pi(2) = g_L(2) = 0$ (which requires $\ell_{2,3} = \ell'_{2,3} = 0$). Note that we have now established the key inequality $\ell_{3,3} \geq \ell_{2,2} > \ell_{1,1}$ which is sufficient for the applications that follow (cf. Theorem 3.4.9). We further add that this forces $c > 0$: if $c = 0$, then $g_L(3) = \ell'_{3,3} = \ell_{3,3} = g_\pi(3)$, so both must be 0 by Lemma 3.4.5. This contradicts $\ell_{3,3} \geq \ell_{2,2} > \ell_{1,1} \geq 0$, so we must have

$$b \geq c \geq 1.$$

Next, we consider $g_\pi(1)$. Since $\ell_{2,2} > \ell_{1,1}$, Corollary 2.4.4 gives $g_\pi(1) = \ell_{1,3} + max\{\ell_{1,2} - \ell_{2,3}, 0\}$. Thus,

$$\begin{gathered}\ell'_{1,3} + max\{\ell'_{1,2} - \ell'_{2,3}, 0\} \leq g_L(1) \leq g_\pi(1) = \ell_{1,3} + max\{\ell_{1,2} - \ell_{2,3}, 0\} \\ \Downarrow \\ \ell_{1,3} + b + max\{\ell_{1,2} - \ell_{2,3} + a - c, 0\} \leq \ell_{1,3} + max\{\ell_{1,2} - \ell_{2,3}, 0\}.\end{gathered}$$

First, we note that $g_\pi(1) > 0$: if not, we must have $b = 0$, contradicting $b \geq c \geq 1$. Therefore, by Lemma 3.4.5, we have $g_L(1) < g_\pi(1)$, so

$$\begin{gathered}\ell_{1,3} + b + \ell_{1,2} - \ell_{2,3} + a - c < \ell_{1,3} + \ell_{1,2} - \ell_{2,3} \\ \Downarrow \\ a + b < c.\end{gathered}$$

Since $b \geq c$ and $a \geq -1$, we must have $b = c$ and $a = -1$. As noted above, $b = c$ requires $\ell_{2,3} = \ell'_{2,3} = 0$. Observe that if we show $b = c = 1$, we have finished the proof of (2) from the statement of the proposition.

We consider $f_\pi(3)$. By Corollary 2.1.5 (since $\ell_{3,3} \geq \ell_{2,2}$), $f_\pi(3) = \ell_{1,3} + max\{\ell_{2,3} + \ell_{3,3} - \ell_{1,2} - \ell_{2,2}, 0\}$. Therefore, by Lemma 3.4.5,

$$\begin{aligned}&\ell'_{1,3} + max\{\ell'_{2,3} + \ell'_{3,3} - \ell'_{1,2} - \ell'_{2,2}, 0\} \leq f_{\nu^2\rho\otimes L}(3) \leq f_\pi(3) + 1 \\ &\qquad = 1 + \ell_{1,3} + max\{\ell_{2,3} + \ell_{3,3} - \ell_{1,2} - \ell_{2,2}, 0\} \\ &\qquad\qquad\qquad\qquad \Downarrow \\ &\ell_{1,3} + b + max\{\ell_{2,3} + \ell_{3,3} - \ell_{1,2} - \ell_{2,2} + 1 - b + c, 0\} \\ &\qquad \leq \ell_{1,3} + max\{\ell_{2,3} + \ell_{3,3} - \ell_{1,2} - \ell_{2,2}, 0\}.\end{aligned}$$

Since $b \geq 1$, we have $\ell_{1,3} + b \geq 1 + \ell_{1,3}$; since $b = c$, $\ell_{2,3} + \ell_{3,3} - \ell_{1,2} - \ell_{2,2} + 1 - b + c = \ell_{2,3} + \ell_{3,3} - \ell_{1,2} - \ell_{2,2} + 1$. Thus, the only way the above inequality can hold is if $b = 1$ (and $\ell_{2,3} + \ell_{3,3} - \ell_{1,2} - \ell_{2,2} \leq 0$). This finishes the proof of (2).

The proofs of (1) and (3) are similar. Note that for (3), one uses the formula

$$\begin{aligned}f_\pi(X) = \ell_{1,2} + 2\ell_{1,3} + \ell_{2,3} &+ max\{\ell_{2,2} - \ell_{1,1}, 0\} + max\{\ell_{3,3} - min\{\ell_{1,1}, \ell_{2,2}\}, 0\} \\ &+ max\{\ell_{2,3} + min\{\ell_{1,1}, \ell_{2,2}\} - \ell_{1,1} - \ell_{1,2}, 0\}\end{aligned}$$

for $X = \{1, 2\}$. We note that this formula may be obtained by writing

$$m^*_{\{\nu^2\rho\}}\pi = \underbrace{\nu^2\rho \times \cdots \times \nu^2\rho}_{f_1} \otimes \theta_1, \mu^*_{\{\nu^3\rho\}}\theta_1 = \underbrace{\nu^3\rho \times \cdots \times \nu^3\rho}_{f_2} \otimes \theta_2,$$

and

$$m^*_{\{\nu^2\rho\}}\theta_2 = \underbrace{\nu^2\rho \times \cdots \times \nu^2\rho}_{f_3} \otimes \theta_3$$

as in section 2.3 (also, cf. Remark 2.3.1). □

Example 3.4.7. Let $\pi = L(\delta([\nu\rho, \nu^2\rho]), \delta([\nu\rho, \nu^2\rho]), \nu^2\rho, \nu^3\rho, \nu^3\rho)$. Then

$$\begin{aligned}r_{(1)}\pi = \nu^2\rho \otimes L(\nu\rho, \nu^2\rho, \delta([\nu\rho, \nu^3\rho]), \nu^3\rho) &+ 2\nu^2\rho \otimes L(\delta([\nu\rho, \nu^2\rho]), \delta([\nu\rho, \nu^3\rho]), \nu^3\rho) \\ &+ 2\nu^2\rho \otimes L(\nu\rho, \delta([\nu\rho, \nu^2\rho]), \nu^2\rho, \nu^3\rho, \nu^3\rho) \\ &+ 3\nu^2\rho \otimes L(\delta([\nu\rho, \nu^2\rho]), \delta([\nu\rho, \nu^2\rho]), \nu^3\rho, \nu^3\rho)\end{aligned}$$

and

$$\begin{aligned} r_{(1)}^{max}\pi = &2\nu^2\rho \otimes L(\delta([\nu\rho,\nu^2\rho]),\delta([\nu\rho,\nu^3\rho]),\nu^3\rho)\\ &+ 2\nu^2\rho \otimes L(\nu\rho,\delta([\nu\rho,\nu^2\rho]),\nu^2\rho,\nu^2\rho,\nu^3\rho)\\ &+ 3\nu^2\rho \otimes L(\delta([\nu\rho,\nu^2\rho]),\delta([\nu\rho,\nu^2\rho]),\nu^3\rho,\nu^3\rho). \end{aligned}$$

In particular, there are examples for (2) in Proposition 3.4.6 where $\nu^2\rho\otimes L \not\leq r_{(1)}^{max}\pi$.

Corollary 3.4.8. *Suppose that π is irreducible with supercuspidal support in $\{\nu\rho,\nu^2\rho,\nu^3\rho\}$ (with $\nu\rho$ and $\nu^3\rho$ actually occurring) and $L\otimes\nu^b\rho \leq r_{(n-1)}\pi$ with $L\otimes\nu^b\rho$ irreducible.*

1. *If $b=1$, then $L\otimes\nu^b\rho\leq r_{(n-1)}^{max}\pi$.*
2. *If $b=2$, then $L\otimes\nu^b\rho\leq r_{(n-1)}^{max}\pi$ unless the following conditions on π hold: $\ell_{1,1}\geq\ell_{2,2}>\ell_{3,3}$ and $\ell_{1,2}=0$. Furthermore, L must satisfy $\ell'_{3,3}=\ell_{3,3}+1$, $\ell'_{1,3}=\ell_{1,3}+1$, $\ell'_{1,1}=\ell_{1,1}-1$.*
3. *If $b=3$, then $L\otimes\nu^a\rho\leq r_{(n-1)}^{max}\pi$.*

Proof. These results are just the duals to Proposition 3.4.6. □

The conditions required to have $r_{(1)}^{max}\pi\neq r_{(1)}\pi$ and $r_{(n-1)}^{max}\pi\neq r_{(n-1)}\pi$ cannot be satisfied simultaneously. In particular, we have the following:

Theorem 3.4.9. *If π is an irreducible representation with supercuspidal support $\{\nu\rho,\nu^2\rho,\nu^3\rho\}$, then we have either $r_{(1)}\pi=r_{(1)}^{max}\pi$ or $r_{(n-1)}\pi=r_{(n-1)}^{max}\pi$ (or both). Furthermore, unless $\ell_{3,3}\geq\ell_{2,2}>\ell_{1,1}$, we must have $r_{(1)}\pi=r_{(1)}^{max}\pi$; unless $\ell_{1,1}\geq\ell_{2,2}>\ell_{3,3}$, we must have $r_{(n-1)}\pi=r_{(n-1)}^{max}\pi$.*

In the case where we have $r_{(n-1)}\pi=r_{(n-1)}^{max}\pi$ but not $r_{(1)}\pi=r_{(1)}^{max}\pi$, the following allows us to recover $r_{(1)}\pi$:

Note 3.4.10. We take a brief look at the question of recovering $r_{(1)}\pi$ from $r_{(n-1)}\pi$. As the algorithm can be applied more generally, we work in a more general setting. As in Definition 3.4.1, for a property $\mathcal{P}$, we let $r_{(\alpha)}^{\mathcal{P}}\pi$ denote the sum of all $\lambda\leq r_{(\alpha)}\pi$ which satisfy $\mathcal{P}$. Suppose the property $\mathcal{P}$ satisfies the following:

1. $r_{(1,n-2)}^{\mathcal{P}}\pi\leq r_{(1,n-2)}\circ r_{(1)}^{\mathcal{P}}\pi$ and $r_{(1,n-2)}^{\mathcal{P}}\pi\leq r_{(1,n-2)}\circ r_{(n-1)}^{\mathcal{P}}\pi$.
2. If $\nu^a\rho\otimes L$ appears in $r_{(1)}^{\mathcal{P}}\pi$ with multiplicity m and $\Lambda\otimes\nu^b\rho\leq r_{(n-2)}L$ has $g_L(b)=g_\Lambda(b)+1$, then $\nu^a\rho\otimes\Lambda\otimes\nu^b\rho$ appears in $r_{(1,n-2)}^{\mathcal{P}}\pi$ with multiplicity m.

We note that the property of being in the Jacquet module clearly satisfies (1) and (2).

To recover $r_{(1)}^{\mathcal{P}}\pi$ from $r_{(n-1)}^{\mathcal{P}}\pi$, we first calculate $r_{(1,n-2)}^{\mathcal{P}}\pi$. By inductive hypothesis, we may calculate $r_{(1,n-2)}\circ r_{(n-1)}^{\mathcal{P}}\pi$; by (1), we may then obtain $r_{(1,n-2)}^{\mathcal{P}}\pi$ by removing any terms which fail to satisfy $\mathcal{P}$.

At the first step, set $r_{(1,n-2)}^{\mathcal{P},1}\pi=r_{(1,n-2)}^{\mathcal{P}}\pi$ and $r_{(1)}^{\mathcal{P},1}\pi=0$. At the ith step, choose $\nu^a\rho\otimes\Lambda\otimes\nu^b\rho\leq r_{(1,n-2)}^{\mathcal{P},i-1}\pi$ having $g_\Lambda(b)$ maximal. Using the dual to Remark 2.2.4 (cf. section 2.4), calculate the unique irreducible L having $r_{(n-2)}L\geq\Lambda\otimes\nu^b\rho$ and $g_L(b)=g_\Lambda(b)+1$. We set $r_{(1)}^{\mathcal{P},i}\pi=r_{(1)}^{\mathcal{P},i-1}\pi+\nu^a\rho\otimes L$ and $r_{(1,n-2)}^{\mathcal{P},i}\pi=r_{(1,n-2)}^{\mathcal{P},i-1}\pi-r_{(1,n-2)}^{\mathcal{P}}(\nu^a\rho\otimes L)$, noting that the latter term may be calculated by inductive assumption. We now check that $\nu^a\rho\otimes L\leq r_{(1)}^{\mathcal{P}}\pi$. By (1), there is some

$\nu^a\rho \otimes L' \leq r^{\mathcal{P}}_{(1)}\pi$ having $\nu^a\rho \otimes \Lambda \otimes \nu^b\rho \leq r_{(1,n-2)}(\nu^a\rho \otimes L')$. By (2), we have $g_\Lambda(b)+1 = g_{L'}(b)$, where $g_\Lambda(b)+1 < g_{L'}(b)$, $\nu^a\rho \otimes \Lambda \otimes \nu^b\rho$ would have been removed from $r^{\mathcal{P},j}_{(1,n-2)}\pi$ at an earlier stage. By the dual to Corollary 2.2.3 (cf. section 2.4), we have $L'=L$. Of course, the process stops when $r^{\mathcal{P},i}_{(1,n-2)}\pi$ reaches 0.

Remark 3.4.11. Arguments like those used in the proof of Proposition 3.4.6 may also be used to show the following:

(1) π is determined by the values of $f_\pi(X)$, $g_\pi(Y)$ for $X, Y \subset \{1,2,3\}$. That is, if $f_{\pi'}(X) = f_\pi(X)$, $g_{\pi'}(Y) = g_\pi(Y)$ for all $X, Y \subset \{1,2,3\}$, then $\pi' = \pi$.

(2) We have

$$\pi = \left(\bigcap_X \tau_\pi(X) \times \theta_\pi(X)\right) \bigcap \left(\bigcap_Y \eta_\pi(Y) \times \xi_\pi(Y)\right),$$

where X, Y run over proper subsets of $\{1,2,3\}$; intersections are interpreted in the obvious multiset sense (i.e., the multiplicity with which an object appears in $A \cap B$ is the minimum of the multiplicity in A and the multiplicity in B), and $\tau_\pi(X) \otimes \theta_\pi(X)$ and $\eta_\pi(Y) \otimes \xi_\pi(Y)$ are as in Lemmas 2.1.2 and 2.4.2, respectively. (We note that this fails if the supercuspidal support is just $\{\nu\rho, \nu^2\rho\}$.)

We remark that while we do not expect $r^{max}_{(1)}\pi = r_{(1)}\pi$ or $r^{max}_{(n-1)}\pi = r_{(n-1)}\pi$ to have to hold in general, these may still serve as a starting point for a more general algorithm.

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