DEGENERATE PRINCIPAL SERIES FOR SYMPLECTIC AND ODD-ORTHOGONAL GROUPS

CHRIS JANTZEN

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Contents

Abstract

Let F be a p-adic field and $G = SO_{2n+1}(F)$ (resp. $Sp_{2n}(F)$). A maximal parabolic subgroup of G has the form $P = MU$, with Levi factor $M \cong GL_k(F) \times SO_{2(n-k)+1}(F)$ (resp. $M \cong GL_k(F) \times Sp_{2(n-k)}(F)$). A one-dimensional representation of M has the form $\chi \circ det_k \otimes triv_{(n-k)}$, with χ a one-dimensional representation of $F^{\times}$; this may be extended trivially to get a representation of P. We consider representations of the form $\mathrm{Ind}_P^G(\chi \circ det_k \otimes triv_{(n-k)}) \otimes 1$. (More generally, we allow Zelevinsky segment representations for the inducing representation.)

In this paper, we study the reducibility of such representations. We determine the reducibility points, give Langlands data and Jacquet modules for each of the irreducible composition factors, and describe how they are arranged into composition series. (Note: it turns out that the composition series has length ≤ 4.) Our approach is based on Jacquet module techniques developed by M. Tadić.

key words and phrases: p-adic field, symplectic group, orthogonal group, induced representation, Jacquet module, Langlands classification.

1. Introduction

Let S_n denote either $Sp_{2n}(F)$ or $SO_{2n+1}(F)$, F p-adic, char$F$=0. A degenerate principal series for S_n is a representation obtained by inducing a one-dimensional representation from a maximal parabolic subgroup in S_n. In this paper, we determine the composition series for such representations, specifying the components (irreducible composition factors) by giving their Langlands data.

First, we note that a maximal parabolic subgroup of S_n has Levi factor $M \cong GL_m(F) \times S_{n-m}$; n possible maximal parabolic subgroups ($1 \leq m \leq n$). So, a typical degenerate principal series representation is $\pi = i_{GM}(\chi \circ det_m \otimes tr_{n-m})$ (i_{GM} denotes induction from the parabolic subgroup with Levi factor M). Such representations have been studied in [Gus], [Jan1], [Jan2], [K-R]; also [Tad3]. [Gus] uses Hecke algebra methods to determine composition series in the case where $m = n$ and χ is unramified (for $Sp_{2n}(F)$). [K-R] is the companion to [Gus], using intertwining operators to determine composition series when $m = n$ and χ is ramified (for $Sp_{2n}(F)$). [Jan1] uses Hecke algebra methods to determine composition series for the case $m = 1$ and any χ (for $Sp_{2n}(F)$). Also, the reducibility points for $n \leq 3$ (any m) are determined using the Jacquet module methods of Tadić. A general reducibility condition, subject to a regularity hypothesis is also given. In [Jan2], Jacquet module methods are used to determine the components (irreducible composition factors) for $n \leq 3$ (any m) and in general for the regular case (for $SO_{2n+1}(F)$). [Tad3] uses the structure theory from [Tad2], which simplifies the calculation of Jacquet modules, to recover many of the results above (among other things). In this paper, we use the sort of approach used in [Tad3] to determine composition series in general (any m, n, χ; no restriction on regularity). We give Langlands data and Jacquet modules for each component, and specify where they lie in the composition series.

We follow the lead of [Tad3] and work in a slightly more general setting. Let ν denote $|det|$ on GL. Let $\rho_1, \ldots, \rho_k$ be representations of $GL_{p_1}(F), \ldots, GL_{p_k}(F)$, respectively, and τ a representation of S_m. Let $\rho_1 \times \rho_2 \ldots \times \rho_k$ denote the representation obtained by inducing the representation $\rho_1 \otimes \rho_2 \otimes \ldots \otimes \rho_k$ from the appropriate subgroup of $GL_{p_1+\cdots+p_k}(F)$. Similarly, let $\rho_1 \times \ldots \times \rho_k \rtimes \tau$ denote the representation of $S_{p_1+\cdots+p_k+m}$ obtained by inducing the representation $\rho_1 \otimes \ldots \otimes \rho_k \otimes \tau$ from the appropriate parabolic subgroup of $S_{p_1+\cdots+p_k+m}$. If $m = n$, we write $\rho_1 \times \ldots \times \rho_k \rtimes 1_{S_0}$, using the $\rtimes 1_{S_0}$ to distinguish this from induction in $GL_n(F)$. (See the next section for more details on notation.)

Now, if ρ_0 is an irreducible unitarizable supercuspidal representation of $GL_{p_0}(F)$, then $\nu^{\frac{-k+1}{2}}\rho_0 \times \nu^{\frac{-k+1}{2}+1}\rho_0 \times \ldots \times \nu^{\frac{k-1}{2}}\rho_0$ has a unique irreducible subrepresentation $\zeta(\rho_0, k)$. Similarly, suppose that ρ is an irreducible unitarizable supercuspidal representation of $GL_p(F)$ and σ an irreducible supercuspidal representation of S_m such that $\nu^{-\frac{1}{2}}\rho \rtimes \sigma$ (resp. $\nu^{-1}\rho \rtimes \sigma$) is reducible and $\nu^{\beta}\rho \rtimes \sigma$ is irreducible for all $\beta \in \mathbb{R}$ with $|\beta| \neq \frac{1}{2}$ (resp. $|\beta| \neq 1$). Then $\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-\ell+\frac{3}{2}}\rho \times \ldots \times \nu^{-\frac{1}{2}}\rho \rtimes \sigma$ (resp. $\nu^{-\ell}\rho \times \nu^{-\ell+1}\rho \times \ldots \times \nu^{-1}\rho \rtimes \sigma$) contains a unique irreducible subrepresentation which we denote $\zeta(\rho, \ell; \sigma)$ (in either case). In this paper, we look

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at representations of the form $\nu^\alpha\zeta(\rho_0,k)\rtimes\zeta(\rho,\ell;\sigma)$ for $\alpha\in\mathbb{R}$.

Let σ be an irreducible supercuspidal representation of S_m. Let us say ρ satisfies (C1) (resp. C(1/2), C(0)) if ρ is an irreducible unitarizable supercuspidal representation of some $GL_p(F)$ satisfying

(C1) $\nu^{-1}\rho\rtimes\sigma$ is reducible and $\nu^\beta\rho\rtimes\sigma$ is irreducible for all $\beta\in\mathbb{R}$ with $|\beta|\neq 1$.

(C1/2) $\nu^{-\frac{1}{2}}\rho\rtimes\sigma$ is reducible and $\nu^\beta\rho\rtimes\sigma$ is irreducible for all $\beta\in\mathbb{R}$ with $|\beta|\neq\frac{1}{2}$.

(C0) $\rho\rtimes\sigma$ is reducible and $\nu^\beta\rho\rtimes\sigma$ is irreducible for all $\beta\in\mathbb{R}$ with $\beta\neq 0$.

Note that any of these conditions (or more generally, $\nu^\alpha\rho\rtimes\sigma$ reducible for some $\alpha\in\mathbb{R}$) implies $\tilde{\rho}\cong\rho$. Now, $\nu^\alpha\zeta(\rho_0,k)\rtimes\zeta(\rho,\ell;\sigma)$ generalizes degenerate principal series for $SO_{2n+1}(F)$ as follows. Consider $\pi=\chi\circ\det_m\rtimes tr_{n-m}$. Write $\chi=|\cdot|^\alpha\chi_0$ with $\alpha\in\mathbb{R}$. Let $1_{SO(1)}$ denote the trivial representation of $SO_1(F)$. Note that for $\sigma=1_{SO(1)}$, the trivial representation, $\rho=1$, of $GL_1(F)$ satisfies (C1/2). Thus we have $\pi=\nu^\alpha\zeta(\chi_0,m)\rtimes\zeta(1,n-m;1_{SO(1)})$. The only difference for degenerate principal series for $Sp_{2n}(F)$ is that when σ is the trivial representation of $Sp_0(F)$, $\rho=1$ satisfies (C1) instead. However, we still have $\chi\circ det_m\rtimes tr_{n-m}=\nu^\alpha\zeta(\chi_0,m)\rtimes\zeta(1,n-m;1_{Sp(0)})$.

Let us now describe the rest of this paper, section-by-section. In the next section, we introduce notation and background results which will be needed in the rest of the paper. In section 3, we focus on certain special cases, namely $\nu^\alpha\rho\rtimes\zeta(\rho,\ell;\sigma)$ (ρ satisfying (C1/2) or (C1)) and $\nu^\alpha\zeta(\rho,n)\rtimes\sigma$ (ρ satisfying (C0), (C1/2), or (C1)). For these special cases, we identify the components by Langlands data; Jacquet module information is also provided. We deal with $\nu^\alpha\rho\rtimes\zeta(\rho,\ell;\sigma)$ separately, in part because the results are already known. $\nu^\alpha\zeta(\rho,n)\rtimes\sigma$ is handled separately because the results will be needed in section 5. This will also save time later, as these would have to be dealt with as separate cases when doing the general results.

The fourth section contains reducibility results for $\pi=\nu^\alpha\zeta(\rho_0,k)\rtimes\zeta(\rho,\ell;\sigma)$ for both ρ satisfying (C1/2) (Theorem 4.1) and ρ satisfying (C1) (Theorem 4.3). We note that the proof for $\rho_0\cong\rho$ is done there. When $\rho_0\ncong\rho$, the approach to the study of π is a bit different, so although we include the results for $\rho_0\ncong\rho$ there, we do not include a proof. The reducibility results when $\rho_0\ncong\rho$ are corollaries of the results in section 5.

In section 5, we determine the components of $\nu^\alpha\zeta(\rho_0,k)\rtimes\zeta(\rho,\ell;\sigma)$ with $\rho_0\ncong\rho$, $\alpha\in\mathbb{R}$. The main result in this regard is Proposition 5.3, which relates the components of $\nu^\alpha\zeta(\rho_0,k)\rtimes\zeta(\rho,\ell;\sigma)$ to those of $\nu^\alpha\zeta(\rho_0,k)\rtimes\sigma$. (Note that this gives the results on the reducibility of $\nu^\alpha\zeta(\rho_0,k)\rtimes\zeta(\rho,\ell;\sigma)$ which were given in section 4.) As corollaries, we explicitly write out components and their Jacquet modules when ρ and ρ_0 both satisfy (C1/2) (Corollary 5.7) or ρ satisfies (C1) and ρ_0 satisfies (C0) (Corollary 5.8). We single out these particular combinations of conditions on ρ, ρ_0 because they generalize degenerate principal series of the form $(|\cdot|^\alpha sgn\circ det_k)\rtimes tr_\ell$, where sgn denotes a (nontrivial) character of order 2 (sgn satisfies (C1/2) for $\sigma=1_{SO(1)}$; (C0) for $\sigma=1_{Sp(0)}$). One other consequence of Proposition 5.3 is that if $\tilde{\rho}_0\ncong\rho_0$, then $\nu^{\alpha}\zeta(\rho_0,k)\rtimes\zeta(\rho,\ell;\sigma)$ is irreducible for all $\alpha\in\mathbb{R}$. This means that the degenerate principal series $|\cdot|^{\alpha}\chi_0\circ det_k\rtimes tr_{\ell}$, $\alpha\in\mathbb{R}$, with $\chi_0^2\neq 1$ is always irreducible.

In section 6, we determine the components of $\pi=\nu^{\alpha}\zeta(\rho,k)\rtimes\zeta(\rho,\ell;\sigma)$ when ρ is (C1/2) or (C1). Again, the components are identified by Langlands data, with Jacquet module information also given. The main results, Theorems 6.1 and 6.2 ((C1/2) and (C1), resp.) hold for all k, ℓ. Jacquet modules are given for $\ell\geq 1$, $k\geq 2$; when $\ell=0$ or $k=1$, the Jacquet modules are covered by the results in section 3.

In section 7, we determine composition series for the representations from sections 5 and 6. That is, we identify which components occurs as subrepresentations, quotients, etc. We use the results from sections 5 and 6 in the following way–e.g., if π has four components, there are other (generalized) degenerate principal series representations π' and π'' which have components in common with π. This allows us to compare Jacquet modules for π, π', π'' to see which components contain certain key Jacquet module components. Then we can use Frobenius reciprocity and other arguments to determine where the different components lie in the composition series.

We now give a summary of where the results are located. Let $\pi=\nu^{\alpha}\zeta(\rho_0,k)\rtimes\zeta(\rho,\ell;\sigma)$.

(1) ρ is (C1/2): reducibility points in Theorem 4.1

$\rho_0\cong\rho$:	components in Theorem 6.1	
	composition series in Theorem 7.1	
	Jacquet modules	Proposition 3.1 ($k=1$)
		Proposition 3.6 ($\ell=0$)
		Theorem 6.1 ($k\geq 2$, $\ell\geq 1$) (tabulated in proof)
$\rho_0\not\cong\rho$ with ρ_0 (C1/2):	components in Corollary 5.7	
	composition series in Theorem 7.1	
	Jacquet modules in Corollary 5.7 (also cf. Remark 5.6)	

(2) ρ is (C1): reducibility points in Theorem 4.3

$\rho_0\cong\rho$:	components in Theorem 6.2	
	composition series in Theorem 7.2	
	Jacquet modules	Proposition 3.9 ($k=1$)
		Proposition 3.10 ($\ell=0$)
		Theorem 6.2 ($k\geq 2$, $\ell\geq 1$) (tabulated in proof)
$\rho_0\not\cong\rho$ with ρ_0 (C0):	components in Corollary 5.8	
	composition series in Theorem 7.2	
	Jacquet modules in Corollary 5.8 (also cf. Remark 5.6)	

We remark that Proposition 5.3 and Corollary 5.5 coupled with the results of section 3 may be used to cover cases with $\rho_0\not\cong\rho$ other than those mentioned above.

Before closing this introduction, there are a few people I would like to thank. Part of this work was done at the SFB 170 in Göttingen; I would like to take this opportunity to thank them for their hospitality. In addition, I would like to thank Marko Tadić for many valuable contributions to this work. Conversations with Dihua Jiang and Paul Sally were also helpful, and I take this opportunity to thank them as well. Finally, I would like to thank the referee.

2. Notation and preliminaries

In this section, we introduce notation and recall some results that will be needed in the rest of the paper. Much of this, though not all, follows the setup used in [Tad3].

Let F be a p-adic field with char$F$=0. Let $|\cdot|$ denote the absolute value on F, normalized so that $|\varpi| = q^{-1}$, ϖ a uniformizer.

In most of this paper, we work with the components (irreducible composition factors) of a representation rather than with the actual composition series. That is, we usually work with the semisimplified representation (even in chapter 7, where we determine composition series, most of the argument uses semisimplified Jacquet modules). So, for any representation π and irreducible representation ρ, let $m(\pi,\rho)$ denote the multiplicity of ρ in π. We write $\pi = \pi_1 + \cdots + \pi_k$ if $m(\pi,\rho) = m(\pi_1,\rho) + \cdots + m(\pi_k,\rho)$ for every irreducible ρ. Similarly, we write $\pi \geq \pi_0$ if $m(\pi,\rho) \geq m(\pi_0,\rho)$ for every such ρ. We write $\pi \cong \pi_0$ if we mean that they are actually equivalent.

We now turn to symplectic and odd-orthogonal groups. Let

$$J_n = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}$$

denote the $n \times n$ antidiagonal matrix above. Then,

$$SO_{2n+1}(F) = \{X \in SL_{2n+1}(F) | {}^T X J_{2n+1} X = J_{2n+1}\}$$

$$Sp_{2n}(F) = \left\{ X \in GL_{2n}(F) | {}^T X \begin{pmatrix} & -J \\ J & \end{pmatrix} X = \begin{pmatrix} & -J \\ J & \end{pmatrix} \right\}.$$

We use S_n to denote either $SO_{2n+1}(F)$ or $Sp_{2n}(F)$. In either case, the Weyl group is $W =\{$ permutations and sign changes on n letters $\}$.

We take as minimal parabolic subgroup in S_n the subgroup P_{min} consisting of upper triangular matrices. Let $\alpha = (n_1, \ldots, n_k)$ be an ordered partition of a nonnegative integer $m \leq n$

into positive integers. Let $M_\alpha \subset S_n$ be the subgroup

$$M_\alpha = \left\{ \left(\begin{array}{ccccccc} X_1 & & & & & & \\ & \ddots & & & & & \\ & & X_k & & & & \\ & & & X & & & \\ & & & & {}^\tau X_k & & \\ & & & & & \ddots & \\ & & & & & & {}^\tau X_1 \end{array} \right) \middle| \, X_i \in GL_{n_i}(F),\ X \in S_{n-m} \right\},$$

where ${}^\tau X = J^T X^{-1} J$. Then $P_\alpha = M_\alpha P_{min}$ is a parabolic subgroup of S_n and every parabolic subgroup is of this form (up to conjugation). For $\alpha = (n_1, \ldots, n_k)$, let $\rho_1, \ldots, \rho_k$ be representations of $GL_{n_1}(F), \ldots, GL_{n_k}(F)$, respectively, and τ a representation of S_{n-m}. Let $\rho_1 \times \ldots \times \rho_k \rtimes \tau$ denote the representation of S_n obtained by inducing the representation $\rho_1 \otimes \ldots \otimes \rho_k \otimes \tau$ of M_α (extended trivially to P_α). If $m = n$, we write $\rho_1 \times \ldots \times \rho_k \rtimes 1_{S_0}$, where 1_{S_0} denotes the trivial representation of S_0.

We now give the Langlands classification for S_n (cf. [Tad1] or [Tad2]). As in [Zel], let $\nu = |det|$ on $GL_n(F)$ (with the value of n clear from context). Suppose that δ is an irreducible essentially square integrable representation of $GL_n(F)$. Then, there is an $\varepsilon(\delta) \in \mathbb{R}$ such that $\nu^{-\varepsilon(\delta)}\delta$ is unitarizable. Let $\delta_1, \ldots, \delta_k$ be irreducible essentially square integrable representations satisfying $\varepsilon(\delta_1) \leq \cdots \leq \varepsilon(\delta_k) < 0$ and τ a tempered representation of S_{n-m}. Then, $\delta_1 \times \ldots \times \delta_k \rtimes \tau$ has a unique irreducible subrepresentation which we denote by $L(\delta_1, \ldots, \delta_k; \tau)$. At times, it will be convenient not to have to worry about listing $\delta_1, \ldots, \delta_k$ in increasing order. So, if $\delta_1, \ldots, \delta_k$ satisfy $\varepsilon(\delta_i) < 0$, then there is some permutation $\delta_{\sigma_1}, \ldots, \delta_{\sigma_k}$ which satisfies $\varepsilon(\delta_{\sigma_1}) \leq \cdots \leq \varepsilon(\delta_{\sigma_k}) < 0$. Then, by $L(\delta_1, \ldots, \delta_k; \tau)$ we mean $L(\delta_{\sigma_1}, \ldots, \delta_{\sigma_k}; \tau)$. At times, it will also be convenient to use $i(\delta_1, \ldots, \delta_k; \tau) = \delta_{\sigma_1} \otimes \ldots \otimes \delta_{\sigma_k} \otimes \tau$ and $I(\delta_1, \ldots, \delta_k; \tau) = \delta_{\sigma_1} \times \ldots \times \delta_{\sigma_k} \rtimes \tau$ (so $I(\delta_1, \ldots, \delta_k; \tau)$ has $L(\delta_1, \ldots, \delta_k; \tau)$ as its unique irreducible subrepresentation). Note that we use Langlands classification in the subrepresentation setting rather than the quotient setting for the following reason: in the subrepresentation setting, $\delta_1 \otimes \ldots \otimes \delta_k \otimes \tau$ will lie in the appropriate Jacquet module of $L(\delta_1, \ldots, \delta_k; \tau)$ (by Frobenius reciprocity, cf. Theorem 2.2).

At this point, we introduce a little shorthand. Let ρ be a unitarizable supercuspidal representation of $GL_p(F)$. Then, $\nu^{\frac{-k+1}{2}}\rho \times \nu^{\frac{-k+1}{2}+1}\rho \times \ldots \times \nu^{\frac{k-1}{2}}\rho$ has a unique irreducible subrepresentation which we denote $\zeta(\rho, k)$ and a unique irreducible quotient which we denote by $\delta(\rho, k)$ (n.b. $\delta(\rho, k)$ is square-integrable). Similarly, suppose that σ is a supercuspidal representation of S_m and $\nu^\alpha \rho \rtimes \sigma$ reduces for some $\alpha < 0$ (note that this implies $\tilde{\rho} \cong \rho$, where $\tilde{\rho}$ denotes the contragredient of ρ). Then,

$$\nu^{-\ell+1+\alpha}\rho \times \nu^{-\ell+2+\alpha}\rho \times \ldots \times \nu^{\alpha}\rho \rtimes \sigma$$

has a unique irreducible subrepresentation which we denote $\zeta(\rho, \ell; \sigma)$ and a unique irreducible quotient which we denote $\delta(\rho, \ell; \sigma)$ (n.b. $\delta(\rho, \ell; \sigma)$ is square-integrable). We also use the segment notation of Zelevinsky [Zel]; let

$$[\nu^\beta \rho, \nu^{\beta+m}\rho] = \nu^\beta \rho, \nu^{\beta+1}\rho, \ldots, \nu^{\beta+m}\rho.$$

Then, $\zeta(\rho,k;\sigma)=L([\nu^{-k+\alpha}\rho,\nu^{\alpha}\rho];\sigma)$. For example, in S_n, $tr_n=\zeta(1,n;1_{S_0})$ (for $\rho=1$ and $\sigma=1_{SO(1)}$, $\alpha=-\frac{1}{2}$; for $\rho=1$ and $\sigma=1_{Sp(0)}$, $\alpha=-1$).

The following facts about induced representations for $GL_n(F)$ will be needed later.

THEOREM **2.1** (Zelevinsky)**.** *Let ρ, ρ_0, ρ_1 be irreducible unitarizable supercuspidal representations of $GL_p(F)$, $GL_{p_0}(F)$, $GL_{p_1}(F)$, respectively, and $\alpha,\beta,\gamma\in\mathbb{R}$.*

(1) *$\nu^{\alpha}\zeta(\rho,m)\times\nu^{\beta}\zeta(\rho_0,n)$ is reducible if and only if $\rho_0\cong\rho$ and $[\nu^{\alpha+\frac{-m+1}{2}}\rho,\nu^{\alpha+\frac{m-1}{2}}\rho]\cup[\nu^{\beta+\frac{-n+1}{2}}\rho,\nu^{\beta+\frac{n-1}{2}}\rho]$ is also a segment and strictly contains both $[\nu^{\alpha+\frac{-m+1}{2}}\rho,\nu^{\alpha+\frac{m-1}{2}}\rho]$ and $[\nu^{\beta+\frac{-n+1}{2}}\rho,\nu^{\beta+\frac{n-1}{2}}\rho]$.*

(2) *$\nu^{\alpha}\zeta(\rho,m)\times\nu^{\beta}\zeta(\rho_0,n)\times\nu^{\gamma}\zeta(\rho_1,r)$ is reducible if and only if one (or more) of $\nu^{\alpha}\zeta(\rho,m)\times\nu^{\beta}\zeta(\rho_0,n)$, $\nu^{\alpha}\zeta(\rho,m)\times\nu^{\gamma}\zeta(\rho_1,r)$, or $\nu^{\beta}\zeta(\rho_0,n)\times\nu^{\gamma}\zeta(\rho_1,r)$ reduces.*

Proof. See Theorem 4.2 of [Zel]. $\square$

Let σ be an irreducible supercuspidal representation of S_m. Let us say ρ satisfies (C1) (resp. C(1/2), C(0)) if ρ is an irreducible unitarizable supercuspidal representation of some $GL_p(F)$ satisfying

(C1) $\nu^{-1}\rho\rtimes\sigma$ is reducible and $\nu^{\beta}\rho\rtimes\sigma$ is irreducible for all $\beta\in\mathbb{R}$ with $|\beta|\neq 1$.

(C1/2) $\nu^{-\frac{1}{2}}\rho\rtimes\sigma$ is reducible and $\nu^{\beta}\rho\rtimes\sigma$ is irreducible for all $\beta\in\mathbb{R}$ with $|\beta|\neq\frac{1}{2}$.

(C0) $\rho\rtimes\sigma$ is reducible and $\nu^{\beta}\rho\rtimes\sigma$ is irreducible for all $\beta\in\mathbb{R}$ with $\beta\neq 0$.

Next, we introduce some notation for Jacquet modules. If π is a representation of some S_n and α is a partition of $m\leq n$, let $s_{\alpha}\pi$ denote the Jacquet module with respect to M_{α}. Further, for $\pi=\nu^{\alpha}\zeta(\rho,k)\rtimes\zeta(\rho,\ell;\sigma)$, it makes sense to define $s_{min}\pi=s_{(p,p,\ldots,p)}\pi$ (with $k+\ell$ copies of p in the subscript) and $s_{GL}\pi=s_{((k+\ell)p)}\pi$. Note that, by abuse of notation, we also allow s_{α} and s_{min} to be applied to representations of compatible M_{β}'s. We will occasionally use similar notation for representations of $GL_n(F)$. If $\alpha=(n_1,\ldots,n_k)$ is a partition of $m\leq n$, $GL_n(F)$ has a standard parabolic subgroup with Levi factor $L_{\alpha}\cong GL_{n_1}(F)\times\ldots\times GL_{n_k}(F)\times GL_{n-m}(F)$ (L_{α} consists of block-diagonal matrices; the corresponding parabolic subgroup of block upper triangular matrices). If π is a representation of $GL_n(F)$, we let $r_{\alpha}\pi$ denote the Jacquet module of π with respect to L_{α}. Similarly, for representations such as $\nu^{\alpha_1}\zeta(\rho,k_1)\times\nu^{\alpha_2}\zeta(\rho,k_2)$, it makes sense to define $r_{min}=r_{(p,\ldots,p)}$ (with k_1+k_2 copies of p).

We now give two theorems on Jacquet modules.

THEOREM **2.2** (Frobenius reciprocity)**.** *Let G be a connected reductive p-adic group, $P=MU$ a parabolic subgroup, ρ an (admissible) representation of M, π an (admissible) representation of G. Then*

$$Hom_M(r_{MG}\pi,\rho)\cong Hom_G(\pi,i_{GM}\rho).$$

THEOREM **2.3** (Bernstein-Zelevinsky/Casselman)**.** *Let G be a connected reductive p-adic group, MU and NV standard parabolic subgroups. Let ρ be an (admissible) representation of M.*

Then, $r_{NG} \circ i_{GM}\rho$ has a composition series with factors

$$i_{NN'} \circ w \circ r_{M'M}\rho$$

where $M' = M \cap w^{-1}(N)$, $N' = w(M) \cap N$, and $W^{MN} = \{w \in W | w(P_{min} \cap M) \subset P_{min},\ w^{-1}(P_{min} \cap N) \subset P_{min}\}$.

Proof. See [B-Z] or [Cas]. □

Suppose ρ_0, ρ are irreducible unitarizable supercuspidal representations of $GL_p(F)$ (i.e., $p_0 = p$; though see Remark 5.6) and σ an irreducible supercuspidal representation of S_m. Further, suppose ρ satisfies (C1/2). Let $\pi = \nu^{\alpha}\zeta(\rho_0, k) \rtimes \zeta(\rho, \ell; \sigma)$. Then, set

$$\begin{aligned}\chi_0 &= s_{min}\nu^{\alpha}\zeta(\rho_0, k) \otimes \zeta(\rho, \ell; \sigma)\\ &= (\nu^{\alpha+\frac{-k+1}{2}}\rho_0 \otimes \nu^{\alpha+\frac{-k+1}{2}+1}\rho_0 \otimes \ldots \otimes \nu^{\alpha+\frac{k-1}{2}}\rho_0)\\ &\qquad \otimes(\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{3}{2}}\rho \otimes \ldots \otimes \nu^{-\frac{1}{2}}\rho) \otimes \sigma.\end{aligned}$$

For $0 \leq r \leq k$, set

$$\begin{aligned}\chi_r = \;& (\nu^{\alpha+\frac{-k+1}{2}}\rho_0 \otimes \nu^{\alpha+\frac{-k+1}{2}+1}\rho_0 \otimes \ldots \otimes \nu^{\alpha+\frac{-k+1}{2}+(r-1)}\rho_0)\\ & \otimes(\nu^{-\alpha+\frac{-k+1}{2}}\tilde{\rho}_0 \otimes \nu^{-\alpha+\frac{-k+1}{2}+1}\tilde{\rho}_0 \otimes \ldots \otimes \nu^{-\alpha+\frac{-k+1}{2}+(k-r-1)}\tilde{\rho}_0)\\ & \otimes(\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{3}{2}}\rho \otimes \ldots \otimes \nu^{-\frac{1}{2}}\rho) \otimes \sigma.\end{aligned}$$

By analogy with [K-R], let us call a shuffle of χ_r a permutation on χ_r satisfying

1. $\nu^{\alpha+\frac{-k+1}{2}}\rho_0, \ldots, \nu^{\alpha+\frac{-k+1}{2}+(r-1)}\rho_0$ appear in that order
2. $\nu^{-\alpha+\frac{-k+1}{2}}\tilde{\rho}_0, \ldots, \nu^{-\alpha+\frac{-k+1}{2}+(k-r-1)}\tilde{\rho}_0$ appear in that order
3. $\nu^{-\ell+\frac{1}{2}}\rho, \ldots, \nu^{-\frac{1}{2}}\rho$ appear in that order.

That is, the relative orders in the three parenthesized pieces remain intact. Then,

$$s_{min}\pi = \sum_{r=0}^{k} (\text{all shuffles of } \chi_r)$$

(cf. Lemma 4.4, [Tad2]). A similar description holds if ρ satisfies (C1).

We now recall some structure theory related to Jacquet modules.

DEFINITION **2.4.** (1) *If τ is a representation of $GL_n(F)$, set*

$$m^*\tau = \sum_{i=0}^{n} r_{(i)}\tau$$

(2) *If π is a representation of S_n, set*

$$\mu^*\pi = \sum_{i=0}^{n} s_{(i)}\pi.$$

If τ_1 and τ_2 are representations of $GL_{n_1}(F), GL_{n_2}(F)$, respectively, let $s(\tau_1 \otimes \tau_2) = \tau_2 \otimes \tau_1$ and $m(\tau_1 \otimes \tau_2) = \tau_1 \times \tau_2$. If τ is a representation of $GL_n(F)$ and ϑ is a representation of S_m, define $\hat{\rtimes}$ by $(\tau_1 \otimes \tau_2) \hat{\rtimes} (\tau \otimes \vartheta) = (\tau_1 \times \tau) \otimes (\tau_2 \rtimes \vartheta)$. Set $M_S^* = (m \otimes 1) \circ (\,\tilde{}\, \otimes m^*) \circ s \circ m^*$ ($\tilde{}$ denotes contragredient).

Theorem **2.5** (Tadić)**.** *If τ is a representation of $GL_n(F)$ and ϑ a representation of S_m, then*

$$\mu^*(\tau \rtimes \vartheta) = M_S^*(\tau) \hat{\rtimes} \mu^*(\vartheta).$$

Proof. See [Tad2]. $\square$

We mention that this has a counterpart for general linear groups. If we define $\hat{\times}$ by $(\tau_1 \otimes \tau_2) \hat{\times} (\tau_1' \otimes \tau_2') = (\tau_1 \times \tau_1') \otimes (\tau_2 \times \tau_2')$, then $m^*(\pi_1 \times \pi_2) = m^*(\pi_1) \hat{\times} m^*(\pi_2)$. See section 1.7 of [Zel].

We now give two corollaries of this. We give their complete statements for the half-integral case and simply indicate the few changes required for the integral case.

Corollary **2.6.** *Let ρ_0,ρ be irreducible unitarizable supercuspidal representations of $GL_{p_0}(F)$, $GL_p(F)$, respectively, and σ an irreducible supercuspidal representation of S_m. Suppose ρ satisfies (C1/2). Set $\pi = \nu^\alpha \zeta(\rho_0, k) \rtimes \zeta(\rho, \ell; \sigma)$. Then,*

$$\mu^*\pi = \sum_{i=0}^{k} \sum_{t=0}^{i} \sum_{j=0}^{\ell} \{[\nu^{-\alpha - \frac{i}{2}} \zeta(\tilde{\rho}_0, k-i) \times \nu^{\alpha + \frac{-k+t}{2}} \zeta(\rho_0, t) \times \nu^{-\ell + \frac{j}{2}} \zeta(\rho, j)] \otimes [\nu^{\alpha + \frac{-k+i+t}{2}} \zeta(\rho_0, i-t) \rtimes \zeta(\rho, \ell - j; \sigma)]\}$$

Proof. This follows from Theorem 2.5 and

$$m^*(\nu^\alpha \zeta(\rho_0, k)) = \sum_{i=0}^{k} \nu^{\alpha + \frac{-k+i}{2}} \zeta(\rho_0, i) \otimes \nu^{\alpha + \frac{i}{2}} \zeta(\rho_0, k-i)$$

$$\mu^*(\zeta(\rho, \ell; \sigma)) = \sum_{j=0}^{\ell} \nu^{-\ell + \frac{j}{2}} \zeta(\rho, j) \otimes \zeta(\rho, \ell - j; \sigma)$$

(cf. Lemma 2.9 for properties of $\tilde{}$). $\square$

We note that the case when ρ is (C1) is very similar–the only change necessary is to replace $\nu^{-\ell + \frac{j}{2}} \zeta(\rho, j)$ with $\nu^{-\ell + \frac{j-1}{2}} \zeta(\rho, j)$ in the formula. (In the proof, the same change is needed for $\mu^*(\zeta(\rho, \ell; \sigma))$.)

Corollary **2.7.** *Suppose σ is an irreducible supercuspidal representation of S_m and ρ is a representation of $GL_p(F)$ satisfying (C1/2). Let $\pi = \nu^\alpha \zeta(\rho, k) \rtimes \zeta(\rho, \ell; \sigma)$ (i.e., $\rho_0 = \rho \cong \tilde{\rho}$ above). Then*

$$\begin{aligned} s_{(p)}\pi &= \nu^{\alpha + \frac{-k+1}{2}} \rho \otimes (\nu^{\alpha + \frac{1}{2}} \zeta(\rho, k-1) \rtimes \zeta(\rho, \ell; \sigma)) \\ &+ \nu^{-\alpha + \frac{-k+1}{2}} \rho \otimes (\nu^{\alpha - \frac{1}{2}} \zeta(\rho, k-1) \rtimes \zeta(\rho, \ell; \sigma)) \\ &+ \nu^{-\ell + \frac{1}{2}} \rho \otimes (\nu^\alpha \zeta(\rho, k) \rtimes \zeta(\rho, \ell - 1; \sigma)) \end{aligned}$$

(note that the third term is missing if $\ell = 0$)

$$s_{GL}\pi = \sum_{i=0}^{k}[\nu^{-\alpha-\frac{i}{2}}\zeta(\rho,k-i)\times\nu^{\alpha+\frac{-k+i}{2}}\zeta(\rho,i)\times\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)]\otimes\sigma$$

$$s_{((k+\ell-1)p)}\pi = \sum_{i=0}^{k}[\nu^{-\alpha-\frac{i}{2}}\zeta(\rho,k-i)\times\nu^{\alpha+\frac{-k+i}{2}}\zeta(\rho,i)\times\nu^{\frac{-\ell-1}{2}}\zeta(\rho,\ell-1)]\otimes\zeta(\rho,1;\sigma)$$

$$+\sum_{i=1}^{k}[\nu^{-\alpha-\frac{i}{2}}\zeta(\rho,k-i)\times\nu^{\alpha+\frac{-k+i-1}{2}}\zeta(\rho,i-1)\times\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)]\otimes(\nu^{\alpha+\frac{-k+2i-1}{2}}\rho\rtimes\sigma)$$

(note that the first sum is missing if $\ell = 0$).

Proof. In the preceding corollary, take $\rho_0 = \rho$. Then, e.g., for a term in $\mu^*\pi$ to be in $s_{GL}\pi = s_{((k+\ell)p)}\pi$, one needs $j = \ell$ and $t = i$ in the sum for $\mu^*\pi$. □

Again, the results for the case when ρ is (C1) are similar. The formula for $s_{(p)}\pi$ requires replacing $\nu^{-\ell+\frac{1}{2}}\rho$ with $\nu^{-\ell}\rho$ in the third term (the first and second terms remain the same). For $s_{GL}\pi$, replace $\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)$ with $\nu^{\frac{-\ell-1}{2}}\zeta(\rho,\ell)$. Finally, for $s_{((k+\ell-1)p)}\pi$, replace $\nu^{\frac{-\ell-1}{2}}\zeta(\rho,\ell-1)$ with $\nu^{\frac{-\ell-2}{2}}\zeta(\rho,\ell-1)$ in the first sum and replace $\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)$ with $\nu^{\frac{-\ell-1}{2}}\zeta(\rho,\ell)$ in the second.

LEMMA **2.8.** *Let δ_i be an irreducible essentially square-integrable representation of $GL_{p_i}(F)$ for $i = 1,\dots,k$, and τ an irreducible tempered representation of S_m. Suppose $\varepsilon(\delta_1) \leq \dots \leq \varepsilon(\delta_k) < 0$, and set $\pi = L(\delta_1,\dots,\delta_k;\tau)$. Then, for any $0 \leq j \leq k$,*

$$s_{(p_1,\dots,p_j)}\pi \geq \delta_1\otimes\dots\otimes\delta_j\otimes L(\delta_{j+1},\dots,\delta_k;\tau).$$

Proof. By exactness and induction in stages,

$$\delta_1\times\dots\times\delta_k\rtimes L(\delta_{j+1},\dots,\delta_k;\tau)\hookrightarrow\delta_1\times\dots\times\delta_j\rtimes(\delta_{j+1}\times\dots\delta_k\rtimes\tau).$$

However, the right-hand side has $\pi = L(\delta_1,\dots,\delta_k;\tau)$ as its unique irreducible subrepresentation. Therefore,

$$\pi\hookrightarrow\delta_1\times\dots\times\delta_j\rtimes L(\delta_{j+1},\dots\delta_k;\tau).$$

The claim is then immediate from Frobenius reciprocity. □

The next lemma gives a few properties of the contragredient representation.

LEMMA **2.9.** (1) *$\pi\rtimes\vartheta$ and $\tilde{\pi}\rtimes\vartheta$ have the same components.*

(2) *The functor $\pi\mapsto\tilde{\pi}$ is an exact contravariant functor. Further, with respect to induction, we have $(\delta_1\times\dots\delta_k\rtimes\tau)\tilde{} \cong \tilde{\delta_1}\times\dots\tilde{\delta_k}\rtimes\tilde{\tau}$.*

(3) *The contragredient of $L(\delta_1,\dots,\delta_k;\tau)$ is $L(\delta_1,\dots,\delta_k;\tilde{\tau})$.*

(4) *The contragredient of $\nu^{\alpha}\zeta(\rho_0,k)\rtimes\zeta(\rho,\ell;\sigma)$ is $\nu^{-\alpha}\zeta(\tilde{\rho_0},k)\rtimes\zeta(\rho,\ell;\tilde{\sigma})$.*

Proof. (1) and (2) are standard facts. (3) is done in chapter 6 of [Tad1] (in the quotient setting for Langlands classification). The same argument works in this setting. Finally, (4) follows from (2) once we have that $(\zeta(\rho,\ell;\sigma))\tilde{} \cong \zeta(\rho,\ell;\tilde{\sigma})$ and $(\nu^{\alpha}\zeta(\rho_0,k))\tilde{} \cong \nu^{-\alpha}\zeta(\rho_0,k)$. The first of these is an immediate consequence of (3); the second follows from the GL-analogue of (3)

(For $GL_n(F)$, $\delta_1,\ldots,\delta_k$ constitute Langlands data if $\varepsilon(\delta_1)\leq\cdots\leq\varepsilon(\delta_k)$. The analogue to (3) is then $(\mathcal{L}(\delta_1,\ldots,\delta_k))\ \cong \mathcal{L}(\tilde{\delta_k},\ldots,\tilde{\delta_1})$.) $\square$

We close with the following observation from section 2 of [Tad3]. Suppose ρ is an irreducible unitarizable supercuspidal representation of $GL_p(F)$ and σ is an irreducible supercuspidal representation of S_m. Recall that if $\rho\not\cong\tilde{\rho}$, then $\nu^{\alpha}\rho\rtimes\sigma$ is irreducible for all $\alpha\in\mathbb{R}$. Suppose $\tilde{\rho}\cong\rho$. Then, conjectures in the ninth section in [Sha1] and [Sha2] imply that for any such ρ and σ, ρ must satisfy one of (C0), (C1/2), (C1). This helps explain why we focus on the conditions (C0), (C1/2), and (C1).

3. Components: useful special cases

This section focuses on a couple of special cases. The components and Jacquet modules of $\nu^{\alpha}\rho\rtimes\zeta(\rho,\ell;\sigma)$ are given in Proposition 3.1 (for ρ satisfying (C1/2)) and Proposition 3.9 (for ρ satisfying (C1)). Also, the components and Jacquet modules for $\nu^{\alpha}\zeta(\rho,n)\rtimes\sigma$ are given in Proposition 3.6 (for ρ satisfying (C1/2)), Proposition 3.10 (for ρ satisfying (C1)), and Proposition 3.11 (for ρ satisfying (C0)). There are certain advantages to dealing with these separately. First, Propositions 3.1, 3.6, 3.9, 3.10 would have to be dealt with as special cases in the proofs of later theorems, anyway. Also, Propositions 3.6 and 3.11 will be important in section 5 as well. In addition, a couple of the lemmas here will also be useful later (keeping the already too-long section 6 from being even longer).

Proposition **3.1.** *Let σ be an irreducible supercuspidal representation of S_m and suppose ρ is a representation of $GL_p(F)$ satisfying (C1/2). Let $\pi=\nu^{\alpha}\rho\rtimes\zeta(\rho,\ell;\sigma)$ with $\alpha\in\mathbb{R}$, $\ell\geq 1$. Then, π is reducible if and only if $\alpha\in\{\pm\frac{1}{2},\pm(\ell+\frac{1}{2})\}$. Suppose π is reducible. By Lemma 2.9, we may without loss of generality assume $\alpha\leq 0$.*

(1) $\alpha=-\frac{1}{2}$

$\pi=\pi_1+\pi_2$ *with*

$$\pi_1=L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\nu^{-\frac{1}{2}}\rho;\sigma)\quad \pi_2=L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{3}{2}}\rho];\mathcal{T}),$$

where $\mathcal{T}$ is the unique (irreducible) common component of $\nu^{-\frac{1}{2}}\rho\rtimes\zeta(\rho,1;\sigma)$ and $\delta(\rho,2)\rtimes\sigma$.

(a) $\ell=1$

$$s_{(p)}\pi_1=2\nu^{-\frac{1}{2}}\rho\otimes L(\nu^{-\frac{1}{2}}\rho;\sigma)+\nu^{-\frac{1}{2}}\rho\otimes\delta(\nu^{-\frac{1}{2}}\rho;\sigma)$$

$$s_{(p)}\pi_2=\nu^{\frac{1}{2}}\rho\otimes L(\nu^{-\frac{1}{2}}\rho;\sigma)$$

(b) $\ell\geq 2$

$$\begin{aligned} s_{(p)}\pi_1 &=\nu^{-\ell+\frac{1}{2}}\rho\otimes L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\nu^{-\frac{1}{2}}\rho;\sigma)\\ &+\nu^{-\frac{1}{2}}\rho\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\end{aligned}$$

$$\begin{aligned} s_{(p)}\pi_2 &=\nu^{-\ell+\frac{1}{2}}\rho\otimes L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{3}{2}}\rho];\mathcal{T})\\ &+\nu^{\frac{1}{2}}\rho\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\end{aligned}$$

(2) $\alpha=-\ell-\frac{1}{2}$

$$\pi = L([\nu^{-\ell-\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)+L(\nu^{-\ell}\delta(\rho,2),[\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)$$

$$s_{(p)}L([\nu^{-\ell-\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma) = \nu^{-\ell-\frac{1}{2}}\rho\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)$$

$$\begin{aligned} s_{(p)}L(\nu^{-\ell}\delta(\rho,2),[\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma) &= \nu^{\ell+\frac{1}{2}}\rho\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\ &+\nu^{-\ell+\frac{1}{2}}\rho\otimes L(\nu^{-\ell-\frac{1}{2}}\rho,[\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\end{aligned}$$

Proof. Theorem 6.1 of [Tad3] gives the Langlands (quotient) data for π. It is not difficult to determine $s_{(p)}\pi_i$; enough similar arguments will be done later to justify omitting it here. □

In this section, we also will deal with $\nu^{\alpha}\zeta(\rho,k)\rtimes\sigma$. We first give a few useful lemmas. As we will also need results like these later, we do them in more useful generality–that is, we work with $\nu^{\alpha}\zeta(\rho,k)\rtimes\zeta(\rho,\ell;\sigma)$.

LEMMA **3.2.** *Suppose (π,M,V) is an admissible representation of a reductive p-adic group M. Let Z denote the center of M. If λ is a one-dimensional representation of Z, let*

$$V_{\lambda}=\{v\in V|\ \text{there is an } r\in\mathbb{N} \text{ such that } [\pi(z)-\lambda(z)]^{r}v=0\ \forall z\in Z\}.$$

Then, $V=\bigoplus_{\lambda}V_{\lambda}$ is a direct sum of M-invariant subspaces.

Proof. This is Lemma 8.2 in [Gus]. □

LEMMA **3.3.** *Let σ be an irreducible supercuspidal representation of S_m and suppose ρ is a representation of $GL_p(F)$ satisfying (C1/2). Let $\pi=\nu^{\alpha}\zeta(\rho,k)\rtimes\zeta(\rho,\ell;\sigma)$, with $\alpha\in\mathbb{R}$, $\alpha\leq 0$, $\alpha\equiv\frac{k}{2} \bmod 1$. Then, π has a component π_1 whose Langlands data comes from the smallest possible parabolic subgroup. More precisely, we have the following:*

(1) $\alpha+\frac{k-1}{2}<0$

$$\pi_1=L([\nu^{\alpha+\frac{-k+1}{2}}\rho,\nu^{\alpha+\frac{k-1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)$$

Furthermore, if $\alpha+\frac{k-1}{2}=-\frac{1}{2}$

$$\begin{aligned} s_{(p)}\pi_1 &\geq \nu^{\alpha+\frac{-k+1}{2}}\rho\otimes L([\nu^{\alpha+\frac{-k+1}{2}+1}\rho,\nu^{\alpha+\frac{k-1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\ &+\nu^{-\ell+\frac{1}{2}}\otimes L([\nu^{\alpha+\frac{-k+1}{2}}\rho,\nu^{\alpha+\frac{k-1}{2}}\rho],[\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\end{aligned}$$

(for $\alpha+\frac{k-1}{2}<-\frac{1}{2}$, we are only guaranteed of getting the first term in the inequality above).

(2) $\alpha+\frac{k-1}{2}>0$

$$\pi_1=L([\nu^{\alpha+\frac{-k+1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\alpha+\frac{-k+1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma).$$

Furthermore,

$$\begin{aligned} s_{(p)}\pi_1 &\geq \nu^{\alpha+\frac{-k+1}{2}}\rho\otimes L([\nu^{\alpha+\frac{-k+1}{2}+1}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\alpha+\frac{-k+1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\ &+\nu^{-\alpha+\frac{-k+1}{2}}\rho\otimes L([\nu^{\alpha+\frac{-k+1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\alpha+\frac{-k+1}{2}+1}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\ &+\nu^{-\ell+\frac{1}{2}}\rho\otimes L([\nu^{\alpha+\frac{-k+1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\alpha+\frac{-k+1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma).\end{aligned}$$

Note that if $\ell=0$, there is no $\nu^{-\ell+\frac{1}{2}}\rho$–term in the $s_{(p)}$ inequalities above.

Proof. We do the second case, assuming $\ell>0$. The others are similar.

Let $i(\Delta)$ denote a permutation of

$$\nu^{\alpha+\frac{-k+1}{2}}\rho\otimes\ldots\otimes\nu^{-\frac{1}{2}}\rho\otimes\nu^{-\alpha+\frac{-k+1}{2}}\rho\otimes\ldots\otimes\nu^{-\frac{1}{2}}\rho\otimes\nu^{-\ell+\frac{1}{2}}\rho\otimes\ldots\nu^{-\frac{1}{2}}\rho$$

which has the form $\nu^{\beta_1}\otimes\ldots\otimes\nu^{\beta_n}\otimes\sigma$ with $\beta_i\leq\beta_j$ for $i\leq j$. In particular, π_1 is the Langlands subrepresentation of the representation obtained by inducing $i(\Delta)$. We write $L(\Delta)$ for π_1 and $I(\Delta)$ for the representation obtained by inducing $i(\Delta)$.

We begin by showing that π_1 is a component of π. First, observe that $i(\Delta)<s_{min}\pi$ (cf. Lemma 2.3 et seq.). Let π' be a component of π with $i(\Delta)<s_{min}\pi'$. By Lemma 3.2 applied to the representation $s_{min}\pi'$ of M, we see that $\mathrm{Hom}_M(s_{min}\pi',i(\Delta))\neq 0$. Therefore, by Frobenius reciprocity, $\mathrm{Hom}_G(\pi',I(\Delta))\neq 0$, i.e., π' is an irreducible subrepresentation of $I(\Delta)$. Since $I(\Delta)$ has $\pi_1=L(\Delta)$ as unique irreducible subrepresentation, we have $\pi'=\pi_1$ is a component of π.

Next, recall that

$$\begin{aligned}s_{(p)}\pi &= \nu^{\alpha+\frac{-k+1}{2}}\rho\otimes\nu^{\alpha+\frac{1}{2}}\zeta(\rho,k-1)\rtimes\zeta(\rho,\ell;\sigma)\\ &+\nu^{-\alpha+\frac{-k+1}{2}}\rho\otimes\nu^{\alpha-\frac{1}{2}}\zeta(\rho,k-1)\rtimes\zeta(\rho,\ell;\sigma)\\ &+\nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{\alpha}\zeta(\rho,k)\rtimes\zeta(\rho,\ell-1;\sigma)\\ \\ &=\tau'+\tau''+\tau'''\end{aligned}$$

with τ's in the order listed. Using the identification of π_1 done above, we have

$$\begin{aligned}\tau_1' &= \nu^{\alpha+\frac{-k+1}{2}}\rho\otimes L([\nu^{\alpha+\frac{-k+1}{2}+1}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\alpha+\frac{-k+1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\ \tau_1'' &= \nu^{-\alpha+\frac{-k+1}{2}}\rho\otimes L([\nu^{\alpha+\frac{-k+1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\alpha+\frac{-k+1}{2}+1}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\ \tau_1''' &= \nu^{-\ell+\frac{1}{2}}\rho\otimes L([\nu^{\alpha+\frac{-k+1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\alpha+\frac{-k+1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma).\end{aligned}$$

Also, recall that

$$s_{GL}\pi=\sum_{i=0}^{k}\nu^{-\alpha-\frac{i}{2}}\zeta(\rho,k-i)\times\nu^{\alpha+\frac{-k+i}{2}}\zeta(\rho,i)\times\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\otimes\sigma.$$

The $i=-\alpha+\frac{k}{2}$ term (necessarily, this is an integer) is

$$\kappa=\nu^{-\frac{\alpha}{2}-\frac{k}{4}}\zeta(\rho,\frac{k}{2}+\alpha)\times\nu^{\frac{\alpha}{2}-\frac{k}{4}}\zeta(\rho,\frac{k}{2}-\alpha)\times\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\otimes\sigma.$$

Observe that κ is irreducible (cf. Theorem 2.1) and that $s_{min}\kappa$ consists of terms of the form $\nu^{\beta_1}\otimes\ldots\nu^{\beta_n}\otimes\sigma$ with $\beta_j<0$ for $j=1,\ldots,n$. Furthermore, every term in $s_{min}\pi$ of the form $\nu^{\beta_1}\otimes\ldots\nu^{\beta_n}\otimes\sigma$ with $\beta_j<0$ for $j=1,\ldots,n$ comes from $s_{min}\kappa$ (i.e., $s_{min}\kappa$ contains every such "all-negative" term). Since $s_{min}\pi_1\geq i(\Delta)$ and $i(\Delta)$ is such an all-negative term, we must have $\kappa\leq s_{GL}\pi_1$. Therefore, $s_{min}\pi_1$ contains all the all-negative terms. Now, observe that $s_{min}\tau_1'$

(resp. $s_{min}\tau_1''$, $s_{min}\tau_1'''$) also has such an all-negative term. Therefore, we must have $s_{(p)}\pi_1 \geq \tau_1'$ (resp. $s_{(p)}\pi_1 \geq \tau_1''$, $s_{(p)}\pi_1 \geq \tau_1'''$). $\square$

REMARK **3.4.** *More generally, if $\alpha \not\equiv \frac{k}{2}+\frac{1}{2} \bmod 1$, one has a similar representation π_1 having the form $\pi_1 = L(\nu^{\beta_1}\rho, \ldots, \nu^{\beta_n}\rho; \sigma)$ with $\beta_1 \leq \cdots \leq \beta_n < 0$ (i.e., having minimal support). The same basic argument shows that π_1 is a component of π. If $\alpha \equiv \frac{k}{2}+\frac{1}{2} \bmod 1$, one has π_1 of the form $L(\nu^{\beta_1}\rho, \ldots, \nu^{\beta_{n-1}}\rho; \rho \rtimes \sigma)$ (i.e., $\beta_n = 0$) by similar considerations. It turns out that in these cases π is irreducible (cf. Theorem 4.1), so $\pi_1 = \pi$. This allows us to get Langlands data for π in these situations.*

LEMMA **3.5.** *Let σ be an irreducible supercuspidal representation of S_m and suppose ρ is a representation of $GL_p(F)$ satisfying (C1/2). Let $\pi = \nu^{\alpha}\zeta(\rho,k) \rtimes \zeta(\rho,\ell;\sigma)$ with $\alpha \in \mathbb{R}$, $\alpha \leq 0$.*

1. *Suppose that $\alpha+\frac{-k+1}{2} \neq -\alpha+\frac{-k+1}{2},\ -\ell+\frac{1}{2}$. Then, $\nu^{\alpha+\frac{-k+1}{2}}\rho \otimes \nu^{\alpha+\frac{1}{2}}\zeta(\rho,k-1) \rtimes \zeta(\rho,\ell;\sigma)$ reducible implies π reducible. Write*
$$\nu^{\alpha+\frac{1}{2}}\zeta(\rho,k-1) \rtimes \zeta(\rho,\ell;\sigma) = \sum_i L(\Lambda_i),$$
where Λ_i is Langlands data. Then, no component of π contains more than one term of the form $\nu^{\alpha+\frac{-k+1}{2}}\rho \otimes L(\Lambda_i)$ in its $s_{(p)}$.
2. *Suppose in addition that $\alpha+\frac{-k+1}{2} < -\alpha+\frac{-k+1}{2},\ -\ell+\frac{1}{2}$. Then,*
$$\pi \geq \sum_i L(\nu^{\alpha+\frac{-k+1}{2}}\rho, \Lambda_i).$$
3. *Suppose further that $\alpha+\frac{-k+1}{2}+1 < -\ell+\frac{1}{2},\ -\alpha+\frac{-k+1}{2}$ (so that $\nu^{\alpha+\frac{-k+1}{2}}\rho \times \nu^{-\ell+\frac{1}{2}}\rho$ and $\nu^{\alpha+\frac{-k+1}{2}}\rho \times \nu^{-\alpha+\frac{-k+1}{2}}\rho$ are both irreducible). Then,*
$$\pi = \sum_i L(\nu^{\alpha+\frac{-k+1}{2}}\rho, \Lambda_i).$$

In the case where $\ell = 0$, the conditions above involving ℓ may be ignored (though they automatically hold in (1) and (2), anyway).

If $-\alpha+\frac{-k+1}{2} \neq \alpha+\frac{-k+1}{2}, -\ell+\frac{1}{2}$, then the analogue to (1) holds with $\nu^{-\alpha+\frac{-k+1}{2}}\rho \otimes \nu^{\alpha-\frac{1}{2}}\zeta(\rho,k-1) \rtimes \zeta(\rho,\ell;\sigma)$ replacing $\nu^{\alpha+\frac{-k+1}{2}}\rho \otimes \nu^{\alpha+\frac{1}{2}}\zeta(\rho,k-1) \rtimes \zeta(\rho,\ell;\sigma)$. (Note that $-\alpha+\frac{-k+1}{2} \geq \alpha+\frac{-k+1}{2}$, so that there are no counterparts to (2) and (3)) If $\ell \geq 1$, there are analogues to (1), (2), (3) with $\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{\alpha}\zeta(\rho,k) \rtimes \zeta(\rho,\ell-1;\sigma)$ replacing $\nu^{\alpha+\frac{-k+1}{2}}\rho \otimes \nu^{\alpha+\frac{1}{2}}\zeta(\rho,k-1) \rtimes \zeta(\rho,\ell;\sigma)$. They require $-\ell+\frac{1}{2} \neq \alpha+\frac{-k+1}{2}, -\alpha+\frac{-k+1}{2}$ for (1), $-\ell+\frac{1}{2} < \alpha+\frac{-k+1}{2}, -\alpha+\frac{-k+1}{2}$ for (2), and $-\ell+\frac{1}{2}+1 < \alpha+\frac{-k+1}{2}, -\alpha+\frac{-k+1}{2}$ for (3).

Proof. We work with the case explicitly described, i.e., $\nu^{\alpha+\frac{-k+1}{2}}\rho \otimes \nu^{\alpha+\frac{1}{2}}\zeta(\rho,k-1) \rtimes \zeta(\rho,\ell;\sigma)$. The other cases are similar.

Part 1: Choose a component π_1 of π such that $s_{(p)}\pi_1$ contains a term of the form $\nu^{\alpha+\frac{-k+1}{2}}\rho\otimes L(\Lambda_i)$ (this is not necessarily the π_1 from Lemma 3.3). By Lemma 3.2 and Frobenius reciprocity, we must have $\pi_1 \hookrightarrow \nu^{\alpha+\frac{-k+1}{2}}\rho \rtimes L(\Lambda_i)$ for some i; without loss of generality take $i=1$.

Now, consider $\nu^{\alpha+\frac{-k+1}{2}}\rho \rtimes L(\Lambda_1)$. Suppose

$$s_{(p)}L(\Lambda_1)=\sum_i \nu^{\alpha_i}\rho\otimes M_i.$$

Then,

$$s_{(p)}\nu^{\alpha+\frac{-k+1}{2}}\rho \rtimes L(\Lambda_1)=\nu^{\alpha+\frac{-k+1}{2}}\rho\otimes L(\Lambda_1)+\nu^{-\alpha+\frac{k-1}{2}}\rho\otimes L(\Lambda_1)+\sum_i \nu^{\alpha_i}\rho\otimes\nu^{\alpha+\frac{-k+1}{2}}\rho\rtimes M_i.$$

Observe that since $L(\Lambda_1)$ is a component of $\nu^{\alpha+\frac{1}{2}}\zeta(\rho,k-1)\rtimes\zeta(\rho,\ell;\sigma)$, looking at $s_{(p)}\nu^{\alpha+\frac{1}{2}}\zeta(\rho,k-1)\rtimes\zeta(\rho,\ell;\sigma)$ tells us that $\nu^{\alpha_i}\rho$ must be one of $\nu^{\alpha+\frac{-k+1}{2}+1}\rho$, $\nu^{-\alpha+\frac{-k+1}{2}}\rho$, or $\nu^{-\ell+\frac{1}{2}}\rho$. In particular, π_1 contains only one term of the form $\nu^{\alpha+\frac{-k+1}{2}}\rho\otimes L(\Lambda_i)$ in its $s_{(p)}$. This verifies the first part.

Part 2: Now, suppose $\alpha+\frac{-k+1}{2}<-\alpha+\frac{-k+1}{2}$, $-\ell+\frac{1}{2}$. Let π_i denote the component of π with $s_{(p)}\pi_i\geq\nu^{\alpha+\frac{-k+1}{2}}\rho\otimes L(\Lambda_i)$ (if $\Lambda_i=\Lambda_j$ for some $i\neq j$, choose a π_i and π_j with this property). Since this is the only term of the form $\nu^{\alpha+\frac{-k+1}{2}}\rho\otimes\dots$ in $s_{(p)}\pi_i$, by Lemma 3.2 and Frobenius reciprocity,

$$\pi_i\hookrightarrow\nu^{\alpha+\frac{-k+1}{2}}\rho\rtimes L(\Lambda_i).$$

Suppose that $\Lambda_i=\nu^{\alpha_1}\delta_1\otimes\dots\otimes\nu^{\alpha_r}\delta_r\otimes\tau$ with δ_i a square-integrable representation of some GL_{m_i}, τ a tempered representation of some appropriate S_m and $\alpha_1\leq\dots\leq\alpha_r<0$. Then, $\alpha+\frac{-k+1}{2}<\alpha_1$ since $\alpha+\frac{-k+1}{2}$ is lower than any exponent in $s_{min}\nu^{\alpha+\frac{1}{2}}\zeta(\rho,k-1)\rtimes\zeta(\rho,\ell;\sigma)$ by assumption. Thus, we have

$$\pi_i\hookrightarrow\nu^{\alpha+\frac{-k+1}{2}}\rho\rtimes L(\Lambda_i)\hookrightarrow\nu^{\alpha+\frac{-k+1}{2}}\rho\times\nu^{\alpha_1}\delta_1\times\dots\times\nu^{\alpha_r}\delta_r\rtimes\tau,$$

which has unique irreducible subrepresentation $L(\nu^{\alpha+\frac{-k+1}{2}}\rho,\Lambda_i)$. So, $\pi_i=L(\nu^{\alpha+\frac{-k+1}{2}}\rho,\Lambda_i)$. We can then conclude

$$\pi\geq\sum_i L(\nu^{\alpha+\frac{-k+1}{2}}\rho,\Lambda_i)$$

as claimed.

Part 3: From Part 2, we have

$$\pi\geq\sum_i L(\nu^{\alpha+\frac{-k+1}{2}}\rho,\Lambda_i)=\sum_i\pi_i$$

with $s_{(p)}\pi_i\geq\nu^{\alpha+\frac{-k+1}{2}}\rho\otimes L(\Lambda_i)$. Suppose that π had another component–call it π_0. Then, $s_{(p)}\pi_0\not\geq\nu^{\alpha+\frac{-k+1}{2}}\rho\otimes L(\Lambda_i)$ for any i. Since $s_{(p)}\pi_0\neq 0$, suppose that $s_{(p)}\pi_0\geq\nu^{-\ell+\frac{1}{2}}\rho\otimes L(\Lambda)$.

Now, $L(\Lambda) \leq \nu^{\alpha}\zeta(\rho,k) \rtimes \zeta(\rho,\ell-1;\sigma)$. We claim $s_{(p)}L(\Lambda) \geq \nu^{\alpha+\frac{-k+1}{2}}\rho \otimes L(\Lambda')$ for some Λ'. Since $\alpha + \frac{-k+1}{2} + 1 < -(\ell-1)+\frac{1}{2},\ -\alpha+\frac{-k+1}{2}$, this follows by induction on $n = \ell + k$. Therefore, we have the following:

$$\begin{gathered}
s_{(p)}\pi_0 \geq \nu^{-\ell+\frac{1}{2}}\rho \otimes L(\Lambda) \\
\Downarrow \\
s_{(p,p)}\pi_0 \geq \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{\alpha+\frac{-k+1}{2}}\rho \otimes L(\Lambda') \\
\Downarrow \\
s_{(2p)}\pi_0 \geq \nu^{-\ell+\frac{1}{2}}\rho \times \nu^{\alpha+\frac{-k+1}{2}}\rho \otimes L(\Lambda') \quad (\text{since } \nu^{-\ell+\frac{1}{2}}\rho \times \nu^{\alpha+\frac{-k+1}{2}}\rho \text{ is irreducible}). \\
\Downarrow \\
s_{(p,p)}\pi_0 \geq \nu^{\alpha+\frac{-k+1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes L(\Lambda') \\
\Downarrow \\
s_{(p)}\pi_0 \geq \nu^{\alpha+\frac{-k+1}{2}}\rho \otimes L(\Lambda_i)
\end{gathered}$$

for some i, a contradiction. Thus,

$$\pi = \sum_i \pi_i$$

as claimed. □

Proposition 3.6. *Suppose σ is an irreducible supercuspidal representation of S_m and ρ is a representation of $GL_p(F)$ satisfying (C1/2). Let $\pi = \nu^{\alpha}\zeta(\rho,n) \rtimes \sigma$ with $\alpha \in \mathbb{R}$, $n \geq 2$. Then π is reducible if and only if $\alpha \in \{-\frac{n}{2}, -\frac{n}{2}+1, \ldots, \frac{n}{2}\}$. Suppose π is reducible. By Lemma 2.9, we may without loss of generality assume that $\alpha \leq 0$. Write $\alpha = -\frac{n}{2}+j$, $0 \leq j \leq \frac{n}{2}$.*

(1) $j = 0$

$\pi = \pi_1 + \pi_2$ with

$$\pi_1 = L([\nu^{-n+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho];\sigma)$$

$$\pi_2 = L([\nu^{-n+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho];\delta(\nu^{-\frac{1}{2}}\rho;\sigma))$$

$$s_{(p)}\pi_1 = \nu^{-n+\frac{1}{2}}\rho \otimes L([\nu^{-n+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho];\sigma)$$

$$s_{(p)}\pi_2 = \nu^{-n+\frac{1}{2}}\rho \otimes L([\nu^{-n+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho];\delta(\nu^{-\frac{1}{2}}\rho;\sigma)) + \nu^{\frac{1}{2}}\rho \otimes L([\nu^{-n+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho];\sigma)$$

(2) $1 \leq j < \frac{n}{2}$

$\pi = \pi_1 + \pi_2 + \pi_3$ with

$$\pi_1 = L([\nu^{-n+j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho];\sigma)$$

$$\pi_2 = L([\nu^{-n+j+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho];\delta(\nu^{-\frac{1}{2}}\rho;\sigma))$$

$$\pi_3 = L([\nu^{-n+j+\frac{1}{2}}\rho, \nu^{-j-\frac{3}{2}}\rho], \nu^{-j}\delta(\rho,2), \nu^{-j+1}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma)).$$

(a) $j=1=\frac{n-1}{2}$ *(so $n=3$)*

$$\begin{aligned} s_{(p)}\pi_1 &= \nu^{-\frac{3}{2}}\rho \otimes L(\nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho;\sigma) + \nu^{-\frac{1}{2}}\rho \otimes L(\nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho;\sigma) \\ s_{(p)}\pi_2 &= \nu^{-\frac{3}{2}}\rho \otimes L(\nu^{-\frac{1}{2}}\rho;\delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\ s_{(p)}\pi_3 &= \nu^{-\frac{1}{2}}\rho \otimes L(\nu^{-\frac{3}{2}}\rho;\delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \end{aligned}$$

(b) $j=1$, $n>3$

$$\begin{aligned} s_{(p)}\pi_1 &= \nu^{-n+\frac{3}{2}}\rho \otimes L([\nu^{-n+\frac{5}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\frac{1}{2}}\rho;\sigma) \\ &\quad + \nu^{-\frac{1}{2}}\rho \otimes L([\nu^{-n+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho];\sigma) \\ s_{(p)}\pi_2 &= \nu^{-n+\frac{3}{2}}\rho \otimes L([\nu^{-n+\frac{5}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\frac{1}{2}}\rho;\delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\ s_{(p)}\pi_3 &= \nu^{-n+\frac{3}{2}}\rho \otimes L([\nu^{-n+\frac{5}{2}}\rho, \nu^{-\frac{5}{2}}\rho], \nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\ &\quad + \nu^{-\frac{1}{2}}\rho \otimes L([\nu^{-n+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho];\delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \end{aligned}$$

(c) $j=\frac{n-1}{2}$, $n>3$ *(n odd)*

$$\begin{aligned} s_{(p)}\pi_1 &= \nu^{-\frac{n}{2}}\rho \otimes L([\nu^{-\frac{n}{2}+1}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\frac{n}{2}+1}\rho, \nu^{-\frac{1}{2}}\rho];\sigma) \\ &\quad + \nu^{-\frac{n}{2}+1}\rho \otimes L([\nu^{-\frac{n}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\frac{n}{2}+2}\rho, \nu^{-\frac{1}{2}}\rho];\sigma) \\ s_{(p)}\pi_2 &= \nu^{-\frac{n}{2}} \otimes L([\nu^{-\frac{n}{2}+1}\rho, \nu^{-\frac{3}{2}}\rho], [\nu^{-\frac{n}{2}+1}\rho, \nu^{-\frac{1}{2}}\rho];\delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\ &\quad + \nu^{-\frac{n}{2}+1} \otimes L([\nu^{-\frac{n}{2}}\rho, \nu^{-\frac{3}{2}}\rho], [\nu^{-\frac{n}{2}+2}\rho, \nu^{-\frac{1}{2}}\rho];\delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\ s_{(p)}\pi_3 &= \nu^{-\frac{n}{2}+1}\rho \otimes L(\nu^{-\frac{n}{2}}\rho, \nu^{-\frac{n}{2}+\frac{3}{2}}\delta(\rho,2), \nu^{-\frac{n}{2}+\frac{5}{2}}\delta(\rho,2), \\ &\qquad \ldots, \nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \end{aligned}$$

(d) $1<j<\frac{n-1}{2}$

$$\begin{aligned} s_{(p)}\pi_1 &= \nu^{-n+j+\frac{1}{2}}\rho \otimes L([\nu^{-n+j+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho];\sigma) \\ &\quad + \nu^{-j+\frac{1}{2}}\rho \otimes L([\nu^{-n+j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-j+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho];\sigma) \\ s_{(p)}\pi_2 &= \nu^{-n+j+\frac{1}{2}}\rho \otimes L([\nu^{-n+j+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho];\delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\ &\quad + \nu^{-j+\frac{1}{2}}\rho \otimes L([\nu^{-n+j+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], [\nu^{-j+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho];\delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\ s_{(p)}\pi_3 &= \nu^{-n+j+\frac{1}{2}}\rho \otimes L([\nu^{-n+j+\frac{3}{2}}\rho, \nu^{-j-\frac{3}{2}}\rho], \nu^{-j}\delta(\rho,2), \nu^{-j+1}\delta(\rho,2), \\ &\qquad \ldots, \nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\ &\quad + \nu^{-j+\frac{1}{2}}\rho \otimes L([\nu^{-n+j+\frac{1}{2}}\rho, \nu^{-j-\frac{1}{2}}\rho], \nu^{-j+1}\delta(\rho,2), \nu^{-j+2}\delta(\rho,2), \\ &\qquad \ldots, \nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \end{aligned}$$

(3) $j=\frac{n}{2}$ *(n even)*

$\pi = \pi_1 + \pi_2$ *with*

$$\pi_1 = L([\nu^{\frac{-n+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\frac{-n+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$$

$$\pi_2 = L([\nu^{\frac{-n+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\frac{-n+1}{2}}\rho, \nu^{-\frac{3}{2}}\rho]; \delta(\nu^{-\frac{1}{2}}\rho; \sigma))$$

(a) $n = 2$

$$s_{(p)}\pi_1 = 2\nu^{-\frac{1}{2}}\rho \otimes L(\nu^{-\frac{1}{2}}\rho; \sigma) + \nu^{-\frac{1}{2}}\rho \otimes \delta(\nu^{-\frac{1}{2}}\rho; \sigma)$$

$$s_{(p)}\pi_2 = \nu^{-\frac{1}{2}}\rho \otimes \delta(\nu^{-\frac{1}{2}}\rho; \sigma)$$

(b) $n > 2$

$$\begin{aligned} s_{(p)}\pi_1 \;&= 2\nu^{\frac{-n+1}{2}}\rho \otimes L([\nu^{\frac{-n+3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\frac{-n+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\ &+\nu^{\frac{-n+1}{2}}\rho \otimes L(\nu^{-\frac{n}{2}+1}\delta(\rho, 2), \nu^{-\frac{n}{2}+2}\delta(\rho, 2), \ldots, \nu^{-1}\delta(\rho, 2); \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \\ s_{(p)}\pi_2 \;&= 2\nu^{\frac{-n+1}{2}}\rho \otimes L([\nu^{\frac{-n+3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\frac{-n+1}{2}}\rho, \nu^{-\frac{3}{2}}\rho]; \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \\ &+\nu^{\frac{-n+1}{2}}\rho \otimes L(\nu^{-\frac{n}{2}+1}\delta(\rho, 2), \nu^{-\frac{n}{2}+2}\delta(\rho, 2), \ldots, \nu^{-1}\delta(\rho, 2); \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \end{aligned}$$

Proof. The reducibility points are given in Theorem 7.2 (ii) of [Tad3].

We now turn to the task of showing that the components and Jacquet modules (for reducibility points) are as claimed above. The proof is by induction on n. We go through it case-by-case. Recall that

$$s_{(p)}\pi = \nu^{-n+j+\frac{1}{2}}\rho \otimes \nu^{-\frac{n}{2}+j+\frac{1}{2}}\zeta(\rho, n-1) \rtimes \sigma + \nu^{-j+\frac{1}{2}}\rho \otimes \nu^{-\frac{n}{2}+j-\frac{1}{2}}\zeta(\rho, n-1) \rtimes \sigma.$$

<u>Case 1:</u> By inductive hypothesis, we have

$$\begin{aligned} s_{(p)}\pi \;&= \nu^{-n+\frac{1}{2}}\rho \otimes \nu^{-\frac{n}{2}+\frac{1}{2}}\zeta(\rho, n-1) \rtimes \sigma + \nu^{\frac{1}{2}}\rho \otimes \nu^{-\frac{n}{2}-\frac{1}{2}}\zeta(\rho, n-1) \rtimes \sigma \\ &= \nu^{-n+\frac{1}{2}}\rho \otimes L([\nu^{-n+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) + \nu^{-n+\frac{1}{2}}\rho \otimes L([\nu^{-n+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho]; \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \\ &+\nu^{\frac{1}{2}}\rho \otimes L([\nu^{-n+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho]; \sigma) \end{aligned}$$

By Lemma 3.5 3, we have $\pi = \pi_1 + \pi_2$ with π_1 and π_2 as given. Since $\pi_1 = \zeta(\rho, n; \sigma)$, we have $s_{(p)}\pi_1 = \nu^{-n+\frac{1}{2}}\rho \otimes \zeta(\rho, n-1; \sigma) = \nu^{-n+\frac{1}{2}}\rho \otimes L([\nu^{-n+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$, as claimed. Necessarily, $s_{(p)}\pi_2$ consists of everything else in $s_{(p)}\pi$.

<u>Case 2a:</u> The case $\rho = 1$, $\sigma = 1$ is in Theorem 4.5 of [Jan2]. The same basic proof works here.

<u>Case 2b:</u> The proof is like that for case 2d below.

Case 2c: By inductive hypothesis, we have

$$\begin{aligned}
s_{(p)}\pi &= \nu^{-\frac{n}{2}}\rho \otimes \zeta(\rho, n-1) \rtimes \sigma + \nu^{-\frac{n}{2}+1}\rho \otimes \nu^{-1}\zeta(\rho, n-1) \rtimes \sigma \\
&= \nu^{-\frac{n}{2}}\rho \otimes L([\nu^{-\frac{n}{2}+1}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\frac{n}{2}+1}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\
&+\nu^{-\frac{n}{2}}\rho \otimes L([\nu^{-\frac{n}{2}+1}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\frac{n}{2}+1}\rho, \nu^{-\frac{3}{2}}\rho]; \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \\
&+\nu^{-\frac{n}{2}+1}\rho \otimes L([\nu^{-\frac{n}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\frac{n}{2}+2}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\
&+\nu^{-\frac{n}{2}+1}\rho \otimes L([\nu^{-\frac{n}{2}}\rho, \nu^{-\frac{3}{2}}\rho], [\nu^{-\frac{n}{2}+2}\rho, \nu^{-\frac{1}{2}}\rho]; \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \\
&+\nu^{-\frac{n}{2}+1}\rho \otimes L(\nu^{-\frac{n}{2}}\rho, \nu^{-\frac{n}{2}+\frac{3}{2}}\delta(\rho,2), \nu^{-\frac{n}{2}+\frac{5}{2}}\delta(\rho,2) \ldots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \\
&= \tau_1' + \tau_2' + \tau_1'' + \tau_2'' + \tau_3'' \text{ (in the order listed)}
\end{aligned}$$

First, by Lemma 3.3, $\pi_1 = L([\nu^{-\frac{n}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\frac{n}{2}+1}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$ is a component of π and has $s_{(p)}\pi_1 \geq \tau_1' + \tau_1''$. By Lemma 3.5 1, we also have that τ_2', τ_2'', $\tau_3'' \not\leq s_{(p)}\pi_1$. Thus, $s_{(p)}\pi_1 = \tau_1' + \tau_1''$.

Next, by Lemma 3.5 2, we have $\pi_2 = L([\nu^{-\frac{n}{2}}\rho, \nu^{-\frac{3}{2}}\rho], [\nu^{-\frac{n}{2}+1}\rho, \nu^{-\frac{1}{2}}\rho]; \delta(\nu^{-\frac{1}{2}}\rho; \sigma))$ is a component of π. By Lemma 2.8, $s_{(p)}\pi_2 \geq \tau_2'$. Now, recall that

$$s_{GL}\pi = \sum_{i=0}^{n} \nu^{\frac{-i+1}{2}}\zeta(\rho, n-i) \times \nu^{\frac{-n+i-1}{2}}\zeta(\rho, i) \otimes \sigma.$$

We claim that $s_{GL}\pi$ contains $\nu^{\frac{-n-1}{4}}\zeta(\rho, \frac{n-3}{2}) \times \nu^{\frac{-n+1}{4}}\zeta(\rho, \frac{n+3}{2}) \otimes \sigma$ with multiplicity 2. In particular, it is the (irreducible) $i = \frac{n+3}{2}$ term and a component of the (reducible) $i = \frac{n-1}{2}$ term (a quick look at s_{min} of the terms with $i \neq \frac{n-1}{2}, \frac{n+3}{2}$ shows that these are the only copies). Next, let

$$\begin{aligned}
\psi = \ & \nu^{-\frac{n}{2}+1}\rho \otimes \nu^{-\frac{n}{2}}\rho \otimes \nu^{-\frac{n}{2}+1}\rho \otimes (\nu^{-\frac{n}{2}+2}\rho \times \nu^{-\frac{n}{2}+2}\rho) \otimes (\nu^{-\frac{n}{2}+3}\rho \times \nu^{-\frac{n}{2}+3}\rho) \\
& \otimes \ldots \otimes (\nu^{-1}\rho \times \nu^{-1}\rho) \otimes \nu^{-\frac{1}{2}}\rho \otimes \nu^{\frac{1}{2}}\rho \otimes \sigma.
\end{aligned}$$

By the description of $s_{min}\pi$ (cf. Theorem 2.3 et seq.), we see that ψ occurs in $s_{(p,p,p,2p,2p,\ldots,2p,p,p)}\pi$ with multiplicity 2. Further, $\psi \leq s_{(p,p,p,2p,2p,\ldots,2p,p,p)}\nu^{\frac{-n-1}{4}}\zeta(\rho, \frac{n-3}{2}) \times \nu^{\frac{-n+1}{4}}\zeta(\rho, \frac{n+3}{2}) \otimes \sigma$ (just look at $s_{min}\nu^{\frac{-n-1}{4}}\zeta(\rho, \frac{n-3}{2}) \times \nu^{\frac{-n+1}{4}}\zeta(\rho, \frac{n+3}{2}) \otimes \sigma$) and $\psi \leq s_{(p,p,p,2p,2p,\ldots,2p,p,p)}\tau_2''$ (it comes from the Langlands data for τ_2''). Therefore, if π' denotes the component of π with $s_{(p)}\pi' \geq \tau_2''$, $s_{(p,p,p,2p,2p,\ldots,2p,p,p)}\pi' \geq \psi$. Since both copies of ψ come from a copy of $\nu^{\frac{-n-1}{4}}\zeta(\rho, \frac{n-3}{2}) \times \nu^{\frac{-n+1}{4}}\zeta(\rho, \frac{n+3}{2}) \otimes \sigma$, we have $s_{GL}\pi' \geq \nu^{\frac{-n-1}{4}}\zeta(\rho, \frac{n-3}{2}) \times \nu^{\frac{-n+1}{4}}\zeta(\rho, \frac{n+3}{2}) \otimes \sigma$. This implies $s_{min}\pi'$ has terms of the form $\nu^{-\frac{n}{2}}\rho \otimes \ldots$. Therefore, $s_{(p)}\pi'$ has a term of the form $\nu^{-\frac{n}{2}}\rho \otimes \ldots$, i.e., τ_1' or τ_2'. Since $\pi' \neq \pi_1$ (since $\tau_2'' \leq s_{(p)}\pi'$), we have $s_{(p)}\pi' \geq \tau_2'$. Thus, $\pi' = \pi_2$ and we have $s_{(p)}\pi_2 \geq \tau_2' + \tau_2''$. By Lemma 3.5 1, we have $\tau_3'' \not\leq s_{(p)}\pi_2$. Thus $s_{(p)}\pi_2 = \tau_2' + \tau_2''$ and $s_{(p)}\pi_3 = \tau_3''$, where π_3 denotes the remaining component (which has yet to be identified).

The final step is to show $\pi_3 = L(\nu^{-\frac{n}{2}+\frac{1}{2}}\delta(\rho,2),\nu^{-\frac{n}{2}+\frac{3}{2}}\delta(\rho,2)\ldots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma))$. Since $s_{(p)}\pi_3 = \tau_3''$, by Lemma 2.8 we have

$$s_{(p,p)}\pi_3 \geq \nu^{-\frac{n}{2}+1}\rho\otimes\nu^{-\frac{n}{2}}\rho\otimes L(\nu^{-\frac{n}{2}+\frac{3}{2}}\delta(\rho,2),\nu^{-\frac{n}{2}+\frac{5}{2}}\delta(\rho,2)\ldots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma)).$$

Further, by Lemma 3.5 1 applied to $\nu^{-\frac{n}{2}+1}\rho\otimes\nu^{-1}\zeta(\rho,n-1)\rtimes\sigma$, it is the only component of $s_{(p,p)}\pi_3$ of the form $\nu^{-\frac{n}{2}+1}\rho\otimes\nu^{-\frac{n}{2}}\rho\otimes\ldots$. This means

$$s_{(2p)}\pi_3 \geq \nu^{-\frac{n}{2}+\frac{1}{2}}\delta(\rho,2)\otimes L(\nu^{-\frac{n}{2}+\frac{3}{2}}\delta(\rho,2),\nu^{-\frac{n}{2}+\frac{5}{2}}\delta(\rho,2),\ldots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma))$$

and this is the only component of $s_{(2p)}\pi_3$ with this central character. If we let

$$\lambda = \nu^{-\frac{n}{2}+\frac{1}{2}}\delta(\rho,2)\otimes L(\nu^{-\frac{n}{2}+\frac{3}{2}}\delta(\rho,2),\nu^{-\frac{n}{2}+\frac{5}{2}}\delta(\rho,2),\ldots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma)),$$

then Lemma 3.2 implies $\mathrm{Hom}(s_{(2p)}\pi_3,\lambda)\neq 0$. By Frobenius reciprocity,

$$\pi_3 \hookrightarrow i_{GM}\lambda \hookrightarrow \nu^{-\frac{n}{2}+\frac{1}{2}}\delta(\rho,2)\times\nu^{-\frac{n}{2}+\frac{3}{2}}\delta(\rho,2)\times\ldots\times\nu^{-1}\delta(\rho,2)\rtimes\delta(\nu^{-\frac{1}{2}}\rho;\sigma),$$

which has $L(\nu^{-\frac{n}{2}+\frac{1}{2}}\delta(\rho,2),\nu^{-\frac{n}{2}+\frac{3}{2}}\delta(\rho,2),\ldots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma))$ as unique irreducible subrepresentation. Thus, $\pi_3 = L(\nu^{-\frac{n}{2}+\frac{1}{2}}\delta(\rho,2),\ldots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma))$, as claimed. This finishes 2c.

<u>Case 2d:</u> By induction,

$$\begin{aligned}
s_{(p)}\pi &= \nu^{-n+j+\frac{1}{2}}\rho\otimes\nu^{-\frac{n}{2}+j+\frac{1}{2}}\zeta(\rho,n-1)\rtimes\sigma+\nu^{-j+\frac{1}{2}}\rho\otimes\nu^{-\frac{n}{2}+j-\frac{1}{2}}\zeta(\rho,n-1)\rtimes\sigma\\
&= \nu^{-n+j+\frac{1}{2}}\rho\otimes L([\nu^{-n+j+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-j+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\
&+\nu^{-n+j+\frac{1}{2}}\rho\otimes L([\nu^{-n+j+\frac{3}{2}}\rho,\nu^{-\frac{3}{2}}\rho],[\nu^{-j+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\delta(\nu^{-\frac{1}{2}}\rho;\sigma))\\
&+\nu^{-n+j+\frac{1}{2}}\rho\otimes L([\nu^{-n+j+\frac{3}{2}}\rho,\nu^{-j-\frac{3}{2}}\rho],\nu^{-j}\delta(\rho,2),\ldots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma))\\
&+\nu^{-j+\frac{1}{2}}\rho\otimes L([\nu^{-n+j+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-j+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\
&+\nu^{-j+\frac{1}{2}}\rho\otimes L([\nu^{-n+j+\frac{1}{2}}\rho,\nu^{-\frac{3}{2}}\rho],[\nu^{-j+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\delta(\nu^{-\frac{1}{2}}\rho;\sigma))\\
&+\nu^{-j+\frac{1}{2}}\rho\otimes L([\nu^{-n+j+\frac{1}{2}}\rho,\nu^{-j-\frac{1}{2}}\rho],\nu^{-j+1}\delta(\rho,2),\ldots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma))\\
&= \tau_1'+\tau_2'+\tau_3'+\tau_1''+\tau_2''+\tau_3'' \text{ (in the order listed).}
\end{aligned}$$

First, observe that we may apply Lemma 3.5 3 (to the $\nu^{-n+j+\frac{1}{2}}\rho$-terms) to get $\pi=\pi_1+\pi_2+\pi_3$, with π_1, π_2, π_3 as given. We need to identify $s_{(p)}\pi_i$, $i=1,2,3$.

By Lemma 3.3, $s_{(p)}\pi_1 \geq \tau_1'+\tau_1''$. By Lemma 3.5 1, $\tau_2',\tau_3',\tau_2'',\tau_3'' \not\leq s_{(p)}\pi_1$. Thus, $s_{(p)}\pi_1 = \tau_1'+\tau_1''$. Also, we necessarily have $s_{(p)}\pi_2 \geq \tau_2'$ and $s_{(p)}\pi_3 \geq \tau_3'$ (cf. Lemma 2.8). Thus, all that remains is to show $s_{(p)}\pi_2 \geq \tau_2''$ and $s_{(p)}\pi_3 \geq \tau_3''$.

To identify Jacquet modules, we work with $s_{(p,p)}\pi$. Observe that

$$\begin{aligned} s_{(p,p)}\pi &= \nu^{-n+j+\frac{1}{2}}\rho \otimes \nu^{-n+j+\frac{3}{2}}\rho \otimes \nu^{-\frac{n}{2}+j+1}\zeta(\rho, n-2) \rtimes \sigma \\ &+\nu^{-n+j+\frac{1}{2}}\rho \otimes \nu^{-j+\frac{1}{2}}\rho \otimes \nu^{-\frac{n}{2}+j}\zeta(\rho, n-2) \rtimes \sigma \\ &+\nu^{-j+\frac{1}{2}}\rho \otimes \nu^{-n+j+\frac{1}{2}}\rho \otimes \nu^{-\frac{n}{2}+j}\zeta(\rho, n-2) \rtimes \sigma \\ &+\nu^{-j+\frac{1}{2}}\rho \otimes \nu^{-j+\frac{3}{2}}\rho \otimes \nu^{-\frac{n}{2}+j-1}\zeta(\rho, n-2) \rtimes \sigma. \end{aligned}$$

For our purposes, it is the second and third terms which are of interest. We note that $\nu^{-\frac{n}{2}+j}\zeta(\rho, n-2) \rtimes \sigma$ has three distinct components (it is a "case 2" representation: it has $\hat{n} = n-2$, $\hat{j} = j-1$ so that $0 < \hat{j} < \frac{\hat{n}-1}{2}$). In particular, the components of $\nu^{-n+j+\frac{1}{2}}\rho \otimes \nu^{-j+\frac{1}{2}}\rho \otimes \nu^{-\frac{n}{2}+j}\zeta(\rho, n-2) \rtimes \sigma$ and $\nu^{-j+\frac{1}{2}}\rho \otimes \nu^{-n+j+\frac{1}{2}}\rho \otimes \nu^{-\frac{n}{2}+j}\zeta(\rho, n-2) \rtimes \sigma$ appear in $s_{(p,p)}\pi$ with multiplicity one. Furthermore, we claim:

$$\begin{aligned} s_{(p,p)}\tau_2' &\geq \nu^{-n+j+\frac{1}{2}}\rho \otimes \nu^{-j+\frac{1}{2}}\rho \otimes L([\nu^{-n+j+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], [\nu^{-j+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \\ s_{(p,p)}\tau_3' &\geq \nu^{-n+j+\frac{1}{2}}\rho \otimes \nu^{-j+\frac{1}{2}}\rho \otimes L([\nu^{-n+j+\frac{3}{2}}\rho, \nu^{-j-\frac{1}{2}}\rho], \nu^{-j+1}\delta(\rho, 2), \\ &\qquad \ldots, \nu^{-1}\delta(\rho, 2); \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \\ s_{(p,p)}\tau_2'' &\geq \nu^{-j+\frac{1}{2}}\rho \otimes \nu^{-n+j+\frac{1}{2}}\rho \otimes L([\nu^{-n+j+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], [\nu^{-j+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \\ s_{(p,p)}\tau_3'' &\geq \nu^{-j+\frac{1}{2}}\rho \otimes \nu^{-n+j+\frac{1}{2}}\rho \otimes L([\nu^{-n+j+\frac{3}{2}}\rho, \nu^{-j-\frac{1}{2}}\rho], \nu^{-j+1}\delta(\rho, 2), \\ &\qquad \ldots, \nu^{-1}\delta(\rho, 2); \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \end{aligned}$$

This follows easily from induction: $\tau' = \nu^{-n+j+\frac{1}{2}}\rho \otimes \nu^{-\frac{n}{2}+j+\frac{1}{2}}\zeta(\rho, n-1) \rtimes \sigma$ (resp. $\tau'' = \nu^{-j+\frac{1}{2}}\rho \otimes \nu^{-\frac{n}{2}+j-\frac{1}{2}}\zeta(\rho, n-1) \rtimes \sigma$) has induced part with $n' = n-1$, $j' = j$ (resp. $n'' = n-1$, $j'' = j-1$). Thus, the Jacquet modules for τ' are governed by case 2c or 2d (resp. τ'' governed by case 2b or 2d). In either case, the claim follows.

Using the above, we may argue as follows (using the irreducibility of $\nu^{-n+j+\frac{1}{2}}\rho \times \nu^{-j+\frac{1}{2}}\rho$ for the second implication):

$$\begin{gathered} s_{(p)}\pi_2 \geq \tau_2' \\ \Downarrow \\ s_{(p,p)}\pi_2 \geq \nu^{-n+j+\frac{1}{2}}\rho \otimes \nu^{-j+\frac{1}{2}}\rho \otimes L([\nu^{-n+j+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], [\nu^{-j+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \\ \Downarrow \\ s_{(2p)}\pi_2 \geq (\nu^{-n+j+\frac{1}{2}}\rho \times \nu^{-j+\frac{1}{2}}\rho) \otimes L([\nu^{-n+j+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], [\nu^{-j+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \\ \Downarrow \\ s_{(p,p)}\pi_2 \geq \nu^{-j+\frac{1}{2}}\rho \otimes \nu^{-n+j+\frac{1}{2}}\rho \otimes L([\nu^{-n+j+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], [\nu^{-j+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \\ \Downarrow \\ s_{(p)}\pi_2 \geq \tau_2''. \end{gathered}$$

Similarly, $s_{(p)}\pi_3 \geq \tau_3''$. Thus, we have $s_{(p)}\pi_i = \tau_i' + \tau_i''$ for $i = 1, 2, 3$, as needed.

Case 3a: Case 3a is similar to case 3b below, but somewhat simpler.

Case 3b: By induction, we have (noting that $\nu^{\frac{1}{2}}\zeta(\rho, n-1) \rtimes \sigma$ and $\nu^{-\frac{1}{2}}\zeta(\rho, n-1) \rtimes \sigma$ have the same components)

$$\begin{aligned}
s_{(p)}\pi &= \nu^{\frac{-n+1}{2}}\rho \otimes \nu^{\frac{1}{2}}\zeta(\rho, n-1) \rtimes \sigma + \nu^{\frac{-n+1}{2}}\rho \otimes \nu^{-\frac{1}{2}}\zeta(\rho, n-1) \rtimes \sigma \\
&= 2\nu^{\frac{-n+1}{2}}\rho \otimes L([\nu^{\frac{-n+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\frac{-n+3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\
&+2\nu^{\frac{-n+1}{2}}\rho \otimes L([\nu^{\frac{-n+1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], [\nu^{\frac{-n+3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \\
&+2\nu^{\frac{-n+1}{2}}\rho \otimes L(\nu^{-\frac{n}{2}+1}\delta(\rho, 2), \nu^{-\frac{n}{2}+2}\delta(\rho, 2), \dots, \nu^{-1}\delta(\rho, 2); \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \\
&= 2\tau_1 + 2\tau_2 + 2\tau_3 \text{ (in order listed).}
\end{aligned}$$

First, we show that π has exactly two components. Since π is known to be reducible, there are at least two components. On the other hand, observe that

$$\theta_0 = \nu^{-\frac{n}{2}+\frac{1}{2}}\rho \otimes \nu^{-\frac{n}{2}+\frac{3}{2}}\rho \otimes \dots \otimes \nu^{\frac{n}{2}-\frac{1}{2}}\rho \otimes \sigma = s_{min}(\zeta(\rho, n) \otimes \sigma)$$

occurs in $s_{min}\pi$ with multiplicity two (cf. Lemma 2.3 et seq.). Thus, $\mathrm{Hom}_{M_{min}}(s_{min}\pi, \theta_0)$ has dimension≤ 2. Therefore, by Frobenius reciprocity, π has at most two components.

By Lemma 3.3,

$$\pi_1 = L([\nu^{\frac{-n+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\frac{-n+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$$

is a component of π and $s_{(p)}\pi_1 \geq 2\tau_1$. Let π_2 denote the other component of π. We claim

$$\pi_2 = L([\nu^{\frac{-n+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\frac{-n+1}{2}}\rho, \nu^{-\frac{3}{2}}\rho]; \delta(\nu^{-\frac{1}{2}}\rho; \sigma)).$$

To see this, consider

$$\theta_2 = \nu^{\frac{-n+1}{2}}\rho \otimes \nu^{\frac{-n+1}{2}}\rho \otimes \nu^{\frac{-n+3}{2}}\rho \otimes \nu^{\frac{-n+3}{2}}\rho \otimes \dots \nu^{-\frac{3}{2}}\rho \otimes \nu^{-\frac{3}{2}}\rho \otimes \nu^{-\frac{1}{2}}\rho \otimes \nu^{\frac{1}{2}}\rho \otimes \sigma.$$

Clearly, $\theta_2 \leq s_{min}L([\nu^{\frac{-n+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\frac{-n+1}{2}}\rho, \nu^{-\frac{3}{2}}\rho]; \delta(\nu^{-\frac{1}{2}}\rho; \sigma))$ (cf. Lemma 2.8). Also, by the description of $s_{min}\pi$ (Lemma 2.3 et seq.), we see that θ_2 has the same multiplicity in π as in $i_{GM_{min}}\theta_2$ (namely $2^{\frac{n}{2}}$). Therefore, $L([\nu^{\frac{-n+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\frac{-n+1}{2}}\rho, \nu^{-\frac{3}{2}}\rho]; \delta(\nu^{-\frac{1}{2}}\rho; \sigma))$ must be a component of π; necessarily π_2, as claimed. Of course, $s_{(p)}\pi_2 \geq \tau_2$.

Next, observe that for $n>4$, $s_{(p)}\nu^{-\frac{1}{2}}\zeta(\rho,n-1)\rtimes\sigma$ is given by 2c (by 2a for $n=4$). Thus, for $n>4$

$$\begin{aligned} s_{(p,p)}\tau_1 &= \nu^{\frac{-n+1}{2}}\rho\otimes\nu^{\frac{-n+1}{2}}\rho\otimes L([\nu^{\frac{-n+3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{\frac{-n+3}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\ &+\nu^{\frac{-n+1}{2}}\rho\otimes\nu^{\frac{-n+3}{2}}\rho\otimes L([\nu^{\frac{-n+1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{\frac{-n+5}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\ \\ s_{(p,p)}\tau_2 &= \nu^{\frac{-n+1}{2}}\rho\otimes\nu^{\frac{-n+1}{2}}\rho\otimes L([\nu^{\frac{-n+3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{\frac{-n+3}{2}}\rho,\nu^{-\frac{3}{2}}\rho];\delta(\nu^{-\frac{1}{2}}\rho;\sigma))\\ &+\nu^{\frac{-n+1}{2}}\rho\otimes\nu^{\frac{-n+3}{2}}\rho\otimes L([\nu^{\frac{-n+5}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{\frac{-n+1}{2}}\rho,\nu^{-\frac{3}{2}}\rho];\delta(\nu^{-\frac{1}{2}}\rho;\sigma))\\ \\ s_{(p,p)}\tau_3 &= \nu^{\frac{-n+1}{2}}\rho\otimes\nu^{\frac{-n+3}{2}}\rho\otimes L(\nu^{\frac{-n+1}{2}}\rho,\nu^{-\frac{n}{2}+2}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \end{aligned}$$

For $n=4$, the second term in $s_{(p,p)}\tau_2$ is missing. Therefore, we may argue as follows (using the irreducibility of $\nu^{\frac{-n+1}{2}}\rho\times\nu^{\frac{-n+1}{2}}\rho$ for the second implication):

$$\begin{gathered} s_{(p)}\pi_2\geq\tau_2\\ \Downarrow\\ s_{(p,p)}\pi_2\geq\nu^{\frac{-n+1}{2}}\rho\otimes\nu^{\frac{-n+1}{2}}\rho\otimes L([\nu^{\frac{-n+3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{\frac{-n+3}{2}}\rho,\nu^{-\frac{3}{2}}\rho];\delta(\nu^{-\frac{1}{2}}\rho;\sigma))\\ \Downarrow\\ s_{(2p)}\pi_2\geq(\nu^{\frac{-n+1}{2}}\rho\times\nu^{\frac{-n+1}{2}}\rho)\otimes L([\nu^{\frac{-n+3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{\frac{-n+3}{2}}\rho,\nu^{-\frac{3}{2}}\rho];\delta(\nu^{-\frac{1}{2}}\rho;\sigma))\\ \Downarrow\\ s_{(p,p)}\pi_2\geq 2\nu^{\frac{-n+1}{2}}\rho\otimes\nu^{\frac{-n+1}{2}}\rho\otimes L([\nu^{\frac{-n+3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{\frac{-n+3}{2}}\rho,\nu^{-\frac{3}{2}}\rho];\delta(\nu^{-\frac{1}{2}}\rho;\sigma))\\ \Downarrow\\ s_{(p)}\pi_2\geq 2\tau_2 \end{gathered}$$

Finally, by Frobenius reciprocity, $\theta_0\leq s_{min}\pi_i$ for $i=1,2$. Therefore, $\theta_0\not\leq s_{min}\tau_1$, $\theta_0\not\leq s_{min}\tau_2$. Thus, $\theta_0\leq s_{min}\tau_3$ and therefore $\tau_3\leq s_{(p)}\pi_i$ for $i=1,2$. Hence $s_{(p)}\pi_1=2\tau_1+\tau_3$, $s_p\pi_2=2\tau_2+\tau_3$. $\square$

Note **3.7.** *If we had done the preceding proposition in conjunction with Theorem 6.1, we could shorten the proof of case 3 considerably. In particular, we could use the fact that π_1 is also a component of $\nu^{-\frac{n}{4}}\zeta(\rho,\frac{n}{2})\rtimes\zeta(\rho,\frac{n}{2};\sigma)$, and the components of $s_{(p)}\nu^{-\frac{n}{4}}\zeta(\rho,\frac{n}{2})\rtimes\zeta(\rho,\frac{n}{2};\sigma)$ would be known (by inductive hypothesis). A comparison then implies $s_{(p)}\pi_1\leq 2\tau_1+\tau_3$. Since π has only two components, $s_{(p)}\pi_2\geq 2\tau_2+\tau_3$. Again, by Frobenius reciprocity, $\theta_0\leq s_{min}\pi_i$ for $i=1,2$, implying that $\theta_0\leq s_{min}\tau_3$. Therefore $s_{(p)}\pi_1=2\tau_1+\tau_3$, $s_{(p)}\pi_2=2\tau_2+\tau_3$.*

We now give the analogues to Propositions 3.1 and 3.6 when ρ satisfies (C1). We note that the analogue to Lemma 3.5 holds in this case. Lemma 3.3 has an analogue for this case; as it is slightly different, we state it explicitly.

Lemma **3.8.** *Suppose that σ is an irreducible supercuspidal representation of S_m and ρ is a representation of $GL_p(F)$ satisfying (C1). Let $\pi=\nu^{\alpha}\zeta(\rho,k)\rtimes\zeta(\rho,\ell;\sigma)$ with $\alpha\in\mathbb{R}$, $\alpha\leq 0$, $\alpha\equiv\frac{k-1}{2}$ mod 1. Then, π has a component π_1 whose Langlands data comes from the smallest possible parabolic subgroup. More precisely, we have the following:*

(1) $\alpha + \frac{k-1}{2} < 0$

$$\pi_1 = L([\nu^{\alpha+\frac{-k+1}{2}}\rho, \nu^{\alpha+\frac{k-1}{2}}\rho], [\nu^{-\ell}\rho, \nu^{-1}\rho]; \sigma).$$

Furthermore, if $\alpha + \frac{k-1}{2} = -1$,

$$\begin{aligned} s_{(p)}\pi_1 &\geq \nu^{\alpha+\frac{-k+1}{2}}\rho \otimes L([\nu^{\alpha+\frac{-k+1}{2}+1}\rho, \nu^{\alpha+\frac{k-1}{2}}\rho], [\nu^{-\ell}\rho, \nu^{-1}\rho]; \sigma) \\ &+\nu^{-\ell}\rho \otimes L([\nu^{\alpha+\frac{-k+1}{2}}\rho, \nu^{\alpha+\frac{k-1}{2}}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho]; \sigma) \end{aligned}$$

(for $\alpha + \frac{k-1}{2} < -1$*, we are only guaranteed of getting the first term in the inequality above).*

(2) $\alpha + \frac{k-1}{2} \geq 0$

$$\pi_1 = L([\nu^{\alpha+\frac{-k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{-\alpha+\frac{-k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{-\ell}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma)$$

Furthermore,

$$\begin{aligned} s_{(p)}\pi_1 &\geq \nu^{\alpha+\frac{-k+1}{2}}\rho \otimes L([\nu^{\alpha+\frac{-k+1}{2}+1}\rho, \nu^{-1}\rho], [\nu^{-\alpha+\frac{-k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{-\ell}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma) \\ &+\nu^{-\alpha+\frac{-k+1}{2}}\rho \otimes L([\nu^{\alpha+\frac{-k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{-\alpha+\frac{-k+1}{2}+1}\rho, \nu^{-1}\rho], [\nu^{-\ell}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma) \\ &+\nu^{-\ell}\rho \otimes L([\nu^{\alpha+\frac{-k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{-\alpha+\frac{-k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma). \end{aligned}$$

Note that if $\ell = 0$*, there is no* $\nu^{-\ell}\rho$*–term in the* $s_{(p)}$ *inequalities above.*

Proof. The proof is like that of Lemma 3.3. However, for the second part, the key Jacquet module representation is

$$\nu^{-\frac{\alpha}{2}+\frac{-k-1}{4}}\zeta(\rho, \alpha + \frac{k-1}{2}) \times \nu^{\frac{\alpha}{2}+\frac{-k-1}{4}}\zeta(\rho, -\alpha + \frac{k-1}{2}) \times \nu^{\frac{-\ell-1}{2}}\zeta(\rho, \ell) \otimes (\rho \rtimes \sigma) < s_{((k+\ell-1)p)}\pi,$$

which is irreducible. □

Proposition **3.9.** *Suppose that* σ *is an irreducible supercuspidal representation of* S_m *and* ρ *is a representation of* $GL_p(F)$ *satisfying (C1). Let* $\pi = \nu^\alpha\rho \rtimes \zeta(\rho, \ell; \sigma)$ *with* $\alpha \in \mathbb{R}$, $\ell \geq 1$*. Then,* π *is reducible if and only if* $\alpha \in \{0, \pm(\ell+1)\}$*. Suppose* π *is reducible. By Lemma 2.9, we may without loss of generality assume that* $\alpha \leq 0$*.*

(1) $\alpha = 0$

$\pi = \pi_1 + \pi_2$ *with*

$$\pi_1 = L([\nu^{-\ell}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma)$$

$$\pi_2 = L([\nu^{-\ell}\rho, \nu^{-2}\rho], \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma)$$

(a) $\ell = 1$

$$s_{(p)}\pi_1 = \nu^{-1}\rho \otimes (\rho \rtimes \sigma) + \rho \otimes L(\nu^{-1}\rho; \sigma)$$

$$s_{(p)}\pi_2 = \rho \otimes L(\nu^{-1}\rho; \sigma)$$

(b) $\ell \geq 2$

$$s_{(p)}\pi_1 = \nu^{-\ell}\rho \otimes L([\nu^{-\ell+1}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma) + \rho \otimes L([\nu^{-\ell}\rho, \nu^{-1}\rho]; \sigma)$$

$$s_{(p)}\pi_2 = \nu^{-\ell}\rho \otimes L([\nu^{-\ell+1}\rho, \nu^{-2}\rho], \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma) + \rho \otimes L([\nu^{-\ell}\rho, \nu^{-1}\rho]; \sigma)$$

(2) $\alpha = -\ell - 1$

$\pi = \pi_1 + \pi_2$ *with*

$$\pi_1 = L([\nu^{-\ell-1}\rho, \nu^{-1}\rho]; \sigma)$$

$$\pi_2 = L(\nu^{-\ell-\frac{1}{2}}\delta(\rho, 2), [\nu^{-\ell+1}\rho, \nu^{-1}\rho]; \sigma)$$

$$s_{(p)}\pi_1 = \nu^{-\ell-1}\rho \otimes L([\nu^{-\ell}\rho, \nu^{-1}\rho]; \sigma)$$

$$s_{(p)}\pi_2 = \nu^{-\ell}\rho \otimes L(\nu^{-\ell-1}\rho, [\nu^{-\ell+1}\rho, \nu^{-1}\rho]; \sigma) + \nu^{\ell+1}\rho \otimes L([\nu^{-\ell}\rho, \nu^{-1}\rho]; \sigma)$$

Proof. Theorem 5.2 of [Tad3] gives the Langlands (quotient) data for π. It is not difficult to determine $s_{(p)}\pi_i$–enough similar (but harder) arguments will be done to justify omitting it here. □

PROPOSITION **3.10.** *Suppose that σ is an irreducible supercuspidal representation of S_m and ρ is a representation of $GL_p(F)$ satisfying (C1). Let $\pi = \nu^{\alpha}\zeta(\rho, n) \rtimes \sigma$ with $\alpha \in \mathbb{R}$, $n \geq 2$. Then, π is reducible if and only if $\alpha \in \{\frac{-n-1}{2}, \frac{-n+1}{2}, \ldots, \frac{n+1}{2}\}$. Suppose π is reducible. By Lemma 2.9, we may without loss of generality assume that $\alpha \leq 0$. When $\alpha \geq \frac{-n+1}{2}$ (cases 2,3,4 below), write $\alpha = \frac{-n+1}{2} + j$, with $0 \leq j \leq \frac{n-1}{2}$.*

(1) $\alpha = \frac{-n-1}{2}$

$\pi = \pi_1 + \pi_2$ *with* $\pi_1 = L([\nu^{-n}\rho, \nu^{-1}\rho]; \sigma)$ *and* $\pi_2 = L([\nu^{-n}\rho, \nu^{-2}\rho]; \delta(\nu^{-1}\rho; \sigma))$.

$$s_{(p)}\pi_1 = \nu^{-n}\rho \otimes L([\nu^{-n+1}\rho, \nu^{-1}\rho]; \sigma)$$

$$s_{(p)}\pi_2 = \nu^{-n}\rho \otimes L([\nu^{-n+1}\rho, \nu^{-2}\rho]; \delta(\nu^{-1}\rho; \sigma)) + \nu\rho \otimes L([\nu^{-n}\rho, \nu^{-2}\rho]; \sigma).$$

(2) $j = 0$

$\pi = \pi_1 + \pi_2$ *with*

$$\pi_1 = L([\nu^{-n+1}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma)$$

$$\pi_2 = L([\nu^{-n+1}\rho, \nu^{-2}\rho]; \mathcal{T}),$$

where $\mathcal{T}$ denotes the unique (irreducible) common component of $\nu^{-\frac{1}{2}}\zeta(\rho, 2) \rtimes \sigma$ and $\rho \rtimes \delta(\nu^{-1}\rho; \sigma)$.

(a) $n = 2$

$$s_{(p)}\pi_1 = \nu^{-1}\rho \otimes (\rho \rtimes \sigma) + \rho \otimes L(\nu^{-1}\rho; \sigma)$$

$$s_{(p)}\pi_2 = \rho \otimes \delta(\nu^{-1}\rho; \sigma)$$

(b) $n > 2$

$$s_{(p)}\pi_1 = \nu^{-n+1}\rho \otimes L([\nu^{-n+2}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma) + \rho \otimes L([\nu^{-n+1}\rho, \nu^{-1}\rho]; \sigma)$$

$$s_{(p)}\pi_2 = \nu^{-n+1}\rho \otimes L([\nu^{-n+2}\rho, \nu^{-2}\rho]; \mathcal{T}) + \rho \otimes L([\nu^{-n+1}\rho, \nu^{-2}\rho]; \delta(\nu^{-1}\rho; \sigma))$$

(3) $1 \le j \le \frac{n-1}{2}$

$\pi = \pi_1 + \pi_2 + \pi_2$ *with*

$$\pi_1 = L([\nu^{-n+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma)$$

$$\pi_2 = L([\nu^{-n+j+1}\rho, \nu^{-2}\rho], [\nu^{-j}\rho, \nu^{-1}\rho]; \mathcal{T})$$

$$\pi_3 = L([\nu^{-n+j+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho, 2), \nu^{-j+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{3}{2}}\delta(\rho, 2); \mathcal{T}),$$

where $\mathcal{T}$ *is as above.*

(a) $j = 1 = \frac{n-2}{2}$ *(*$n = 4$*)*

$$s_{(p)}\pi_1 = \nu^{-2}\rho \otimes L(\nu^{-1}\rho, \nu^{-1}\rho; \rho \rtimes \sigma) + \nu^{-1}\rho \otimes L(\nu^{-2}\rho, \nu^{-1}\rho; \rho \rtimes \sigma)$$

$$s_{(p)}\pi_2 = \nu^{-2}\rho \otimes L(\nu^{-1}\rho; \mathcal{T})$$

$$s_{(p)}\pi_3 = \nu^{-1}\rho \otimes L(\nu^{-2}\rho; \mathcal{T})$$

(b) $j = 1$, $n > 4$

$$\begin{aligned} s_{(p)}\pi_1 \quad &= \nu^{-n+2}\rho \otimes L([\nu^{-n+3}\rho, \nu^{-1}\rho], \nu^{-1}\rho; \rho \rtimes \sigma) \\ &+ \nu^{-1}\rho \otimes L([\nu^{-n+2}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma) \end{aligned}$$

$$s_{(p)}\pi_2 \quad = \nu^{-n+2}\rho \otimes L[\nu^{-n+3}\rho, \nu^{-2}\rho], \nu^{-1}\rho; \mathcal{T})$$

$$\begin{aligned} s_{(p)}\pi_3 \quad &= \nu^{-n+2}\rho \otimes L([\nu^{-n+3}\rho, \nu^{-3}\rho], \nu^{-\frac{3}{2}}\delta(\rho, 2); \mathcal{T}) \\ &+ \nu^{-1}\rho \otimes L([\nu^{-n+2}\rho, \nu^{-2}\rho]; \mathcal{T}) \end{aligned}$$

(c) $j = \frac{n-2}{2}$, $n > 4$ *(*n *even)*

$$\begin{aligned} s_{(p)}\pi_1 \quad &= \nu^{-\frac{n}{2}} \otimes L([\nu^{-\frac{n}{2}+1}\rho, \nu^{-1}\rho], [\nu^{-\frac{n}{2}+1}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma) \\ &+ \nu^{-\frac{n}{2}+1}\rho \otimes L([\nu^{-\frac{n}{2}}\rho, \nu^{-1}\rho], [\nu^{-\frac{n}{2}+2}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma) \end{aligned}$$

$$\begin{aligned} s_{(p)}\pi_2 \quad &= \nu^{-\frac{n}{2}}\rho \otimes L([\nu^{-\frac{n}{2}+1}\rho, \nu^{-2}\rho], [\nu^{-\frac{n}{2}+1}\rho, \nu^{-1}\rho]; \mathcal{T}) \\ &+ \nu^{-\frac{n}{2}+1}\rho \otimes L([\nu^{-\frac{n}{2}}\rho, \nu^{-2}\rho], [\nu^{-\frac{n}{2}+2}\rho, \nu^{-1}\rho]; \mathcal{T}) \end{aligned}$$

$$s_{(p)}\pi_3 \quad = \nu^{-\frac{n}{2}+1}\rho \otimes L(\nu^{-\frac{n}{2}}\rho, \nu^{-\frac{n}{2}+\frac{3}{2}}\delta(\rho, 2), \nu^{-\frac{n}{2}+\frac{5}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{3}{2}}\delta(\rho, 2); \mathcal{T})$$

(d) $2 \leq j < \frac{n-2}{2}$

$$\begin{aligned} s_{(p)}\pi_1 &= \nu^{-n+j+1}\rho \otimes L([\nu^{-n+j+2}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma) \\ &+\nu^{-j}\rho \otimes L([\nu^{-n+j+1}\rho, \nu^{-1}\rho], [\nu^{-j+1}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma) \end{aligned}$$

$$\begin{aligned} s_{(p)}\pi_2 &= \nu^{-n+j+1}\rho \otimes L([\nu^{-n+j+2}\rho, \nu^{-2}\rho], [\nu^{-j}\rho, \nu^{-1}\rho]; \mathcal{T}) \\ &+\nu^{-j}\rho \otimes L([\nu^{-n+j+1}\rho, \nu^{-2}\rho], [\nu^{-j+1}\rho, \nu^{-1}\rho]; \mathcal{T}) \end{aligned}$$

$$\begin{aligned} s_{(p)}\pi_3 &= \nu^{-n+j+1}\rho \otimes L([\nu^{-n+j+2}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{3}{2}}\delta(\rho,2); \mathcal{T}) \\ &+\nu^{-j}\rho \otimes L([\nu^{-n+j+1}\rho, \nu^{-j-1}\rho], \nu^{-j+\frac{1}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{3}{2}}\delta(\rho,2); \mathcal{T}) \end{aligned}$$

(4) $j = \frac{n-1}{2}$ *(n odd)*

$\pi = \pi_1 + \pi_2$ *with*

$$\pi_1 = L([\nu^{\frac{-n+1}{2}}\rho, \nu^{-1}\rho], [\nu^{\frac{-n+1}{2}}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma)$$

$$\pi_2 = L([\nu^{\frac{-n+1}{2}}\rho, \nu^{-2}\rho], [\nu^{\frac{-n+1}{2}}\rho, \nu^{-1}\rho]; \mathcal{T})$$

(a) $n = 3$

$$s_{(p)}\pi_1 = 2\nu^{-1}\rho \otimes L(\nu^{-1}\rho; \rho \rtimes \sigma) + \nu^{-1}\rho \otimes \mathcal{T}$$

$$s_{(p)}\pi_2 = \nu^{-1}\rho \otimes \mathcal{T}$$

(b) $n > 3$

$$\begin{aligned} s_{(p)}\pi_1 &= 2\nu^{\frac{-n+1}{2}}\rho \otimes L([\nu^{\frac{-n+1}{2}}\rho, \nu^{-1}\rho], [\nu^{\frac{-n+3}{2}}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma) \\ &+\nu^{\frac{-n+1}{2}}\rho \otimes L(\nu^{-\frac{n}{2}+1}\delta(\rho,2), \ldots, \nu^{-\frac{3}{2}}\delta(\rho,2); \mathcal{T}) \end{aligned}$$

$$\begin{aligned} s_{(p)}\pi_2 &= 2\nu^{\frac{-n+1}{2}}\rho \otimes L([\nu^{\frac{-n+1}{2}}\rho, \nu^{-2}\rho], [\nu^{\frac{-n+3}{2}}\rho, \nu^{-1}\rho]; \mathcal{T}) \\ &+\nu^{\frac{-n+1}{2}}\rho \otimes L(\nu^{-\frac{n}{2}+1}\delta(\rho,2), \ldots, \nu^{-\frac{3}{2}}\delta(\rho,2); \mathcal{T}) \end{aligned}$$

Proof. The reducibility points are given in Theorem 7.2 (iv) of [Tad3]. The identification of components and Jacquet modules is similar to that in Proposition 3.6. □

Proposition **3.11.** *Suppose that σ is an irreducible supercuspidal representation of S_m and ρ is a representation of $GL_p(F)$ satisfying (C0). Let $\rho \rtimes \sigma = T_1 + T_2$. Let $\pi = \nu^{\alpha}\zeta(\rho, n) \rtimes \sigma$ with $\alpha \in \mathbb{R}$, $n \geq 2$. Then π is reducible if and only if $\alpha \in \{\frac{-n+1}{2}, \frac{-n+3}{2}, \ldots, \frac{n-1}{2}\}$. Suppose π is reducible. By Lemma 2.9, we may without loss of generality assume that $\alpha \leq 0$. Write $\alpha = \frac{-n+1}{2} + j$ with $0 \leq j \leq \frac{n-1}{2}$.*

(1) $j = \frac{n-1}{2}$

$\pi = \pi_1 + \pi_2$ *with*

$$\pi_i = L([\nu^{\frac{-n+1}{2}}\rho, \nu^{-1}\rho], [\nu^{\frac{-n+1}{2}}\rho, \nu^{-1}\rho]; T_i)$$

for $i = 1, 2$. *Further,*

$$\begin{aligned} s_{(p)}\pi_i &= 2\nu^{\frac{-n+1}{2}}\rho \otimes L([\nu^{\frac{-n+1}{2}}\rho, \nu^{-1}\rho], [\nu^{\frac{-n+3}{2}}\rho, \nu^{-1}\rho]; T_i) \\ &+\nu^{\frac{-n+1}{2}}\rho \otimes L(\nu^{-\frac{n}{2}+1}\delta(\rho,2), \nu^{-\frac{n}{2}+2}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma) \end{aligned}$$

for $i = 1, 2$.

(2) $0 \leq j < \frac{n-1}{2}$

$\pi = \pi_1 + \pi_2 + \pi_3$ *with*

$$\pi_i = L([\nu^{-n+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho]; T_i)$$

for $i = 1, 2$ *and*

$$\pi_3 = L([\nu^{-n+j+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho,2), \nu^{-j+\frac{1}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma).$$

(a) $j = 0 = \frac{n-2}{2}$ *(*$n = 2$*)*

$$s_{(p)}\pi_i = \nu^{-1}\rho \otimes T_i$$

for $i = 1, 2$.

$$s_{(p)}\pi_3 = \rho \otimes L(\nu^{-1}\rho; \sigma).$$

(b) $j = 0$, $n > 2$

$$s_{(p)}\pi_i = \nu^{-n+1}\rho \otimes L([\nu^{-n+2}\rho, \nu^{-1}\rho]; T_i)$$

for $i = 1, 2$.

$$\begin{aligned} s_{(p)}\pi_3 &= \nu^{-n+1}\rho \otimes L([\nu^{-n+2}\rho, \nu^{-2}\rho], \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma) \\ &+\rho \otimes L([\nu^{-n+1}\rho, \nu^{-1}\rho]; \sigma). \end{aligned}$$

(c) $j = \frac{n-2}{2}$, $n \geq 4$ *(*n *even)*

$$\begin{aligned} s_{(p)}\pi_i &= \nu^{-\frac{n}{2}}\rho \otimes L([\nu^{-\frac{n}{2}+1}\rho, \nu^{-1}\rho], [\nu^{-\frac{n}{2}+1}\rho, \nu^{-1}\rho]; T_i) \\ &+\nu^{-\frac{n}{2}+1}\rho \otimes L([\nu^{-\frac{n}{2}}\rho, \nu^{-1}\rho], [\nu^{-\frac{n}{2}+2}\rho, \nu^{-1}\rho]; T_i) \end{aligned}$$

for $i = 1, 2$.

$$s_{(p)}\pi_3 = \nu^{-\frac{n}{2}+1}\rho \otimes L(\nu^{-\frac{n}{2}}\rho, \nu^{\frac{-n+3}{2}}\delta(\rho,2), \ldots, \nu^{-\frac{1}{2}}\delta(\rho,2); \sigma).$$

(d) $0 < j < \frac{n-2}{2}$

$$\begin{aligned} s_{(p)}\pi_i &= \nu^{-n+j+1}\rho \otimes L([\nu^{-n+j+2}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho]; T_i) \\ &+\nu^{-j}\rho \otimes L([\nu^{-n+j+1}\rho, \nu^{-1}\rho], [\nu^{-j+1}\rho, \nu^{-1}\rho]; T_i) \end{aligned}$$

for $i=1,2$.

$$\begin{aligned} s_{(p)}\pi_3 &= \nu^{-n+j+1}\rho\otimes L([\nu^{-n+j+2}\rho,\nu^{-j-2}\rho],\nu^{-j-\frac{1}{2}}\delta(\rho,2),\dots,\nu^{-\frac{1}{2}}\delta(\rho,2);\sigma) \\ &+\nu^{-j}\rho\otimes L([\nu^{-n+j+1}\rho,\nu^{-j-1}\rho],\nu^{-j+\frac{1}{2}}\delta(\rho,2),\dots,\nu^{-\frac{1}{2}}\delta(\rho,2);\sigma). \end{aligned}$$

Proof. The reducibility points are given in Theorem 7.2 (iii) of [Tad3].

Case 2a uses an argument different than those used up to now. So, we do this case in detail. The remaining cases use use arguments similar to those used already; we do case 2c to illustrate this and show the one change necessary.

Case 2a: $j=0=\frac{n-2}{2}$ $(n=2)$

Here, $\pi=\nu^{-\frac{1}{2}}\zeta(\rho,2)\rtimes\sigma$ and

$$\begin{aligned} s_{(p)}\pi &= \nu^{-1}\rho\otimes(\rho\rtimes\sigma)+\rho\otimes(\nu^{-1}\rho\rtimes\sigma) \\ &= \nu^{-1}\rho\otimes T_1+\nu^{-1}\rho\otimes T_2+\rho\otimes L(\nu^{-1}\rho;\sigma) \end{aligned}$$

First, we show that $L(\nu^{-1}\rho;T_i)$, $i=1,2$, are components of π. Observe that

$$\begin{aligned} \nu^{-\frac{1}{2}}\zeta(\rho,2)\rtimes\sigma &\hookrightarrow \nu^{-1}\rho\times\rho\rtimes\sigma \\ &\cong \nu^{-1}\rho\rtimes(T_1\oplus T_2). \end{aligned}$$

Now, by μ^* calculations,

$$s_{(p)}\nu^{-1}\rho\rtimes T_i=\nu^{-1}\rho\otimes T_i+\nu\rho\otimes T_i+\rho\otimes L(\nu^{-1}\rho;\sigma).$$

In particular, this means $\nu^{-1}\rho\otimes T_i$ appears with multiplicity one in $s_{(p)}\pi$, $s_{(p)}\nu^{-1}\rho\rtimes T_i$ and $s_{(p)}\nu^{-1}\rho\times\rho\rtimes\sigma$. Therefore, π and $\nu^{-1}\rho\rtimes T_i$ have a component in common, that component characterized by having $\nu^{-1}\rho\otimes T_i$ in its $s_{(p)}$. Let π_i denote that component. By Lemmma 3.2 and Frobenius reciprocity,

$$\pi_i\hookrightarrow\nu^{-1}\rho\rtimes T_i$$

(n.b. $s_{(p)}\pi_i\leq\nu^{-1}\rho\otimes T_i+\rho\otimes L(\nu^{-1}\rho;\sigma)$). Since $\nu^{-1}\rho\rtimes T_i$ has unique irreducible subrepresentation $L(\nu^{-1}\rho;T_i)$, this forces $\pi_i=L(\nu^{-1}\rho;T_i)$, as needed.

Next, we show that $\pi_1\oplus\pi_2$ is a subrepresentation of π (the reason for doing this will become clear). From above, π_1, π_2 and π are subrepresentations of $\nu^{-1}\rho\times\rho\rtimes\sigma$. Since $\nu^{-1}\rho\otimes T_i$ appears with multiplicity one in $s_{(p)}\pi$ and $s_{(p)}\nu^{-1}\rho\times\rho\rtimes\sigma$, we see that π_i $(i=1,2)$ appears with multiplicity one in π and $\nu^{-1}\rho\times\rho\rtimes\sigma$. By considering the subspace $V_\pi+V_{\pi_1}+V_{\pi_2}\subset V_{\nu^{-1}\rho\times\rho\rtimes\sigma}$, we see that in order for multiplicity one to hold, $V_{\pi_1}+V_{\pi_2}\subset V_\pi$. Thus $\pi_1\oplus\pi_2$ is a subrepresentation of π.

We now use the fact that $\pi_1\oplus\pi_2$ is a subrepresentation of π to show that π has a third component. To see this, observe that $\nu^{\frac{1}{2}}\zeta(\rho,2)\otimes\tilde{\sigma}$ appears in $s_{GL}\tilde{\pi}$ with multiplicity one. Therefore, by Frobenius reciprocity, $\tilde{\pi}$ has a unique irreducible subrepresentation. Consequently, π has a unique irreducible quotient. Therefore, π must have a third component; call it π_3. Note that we necessarily have $s_{(p)}\pi_i=\nu^{-1}\rho\otimes T_i$ for $i=1,2$ and $s_{(p)}\pi_3=\rho\otimes L(\nu^{-1}\rho;\sigma)$.

It remains to identify π_3. Observe that

$$s_{min}\pi_3 = \rho \otimes \nu^{-1}\rho \otimes \sigma + \rho \otimes \nu\rho \otimes \sigma.$$

Therefore,

$$s_{GL}\pi_3 = \nu^{-\frac{1}{2}}\delta(\rho, 2) \otimes \sigma + \nu^{\frac{1}{2}}\zeta(\rho, 2) \otimes \sigma.$$

By Lemma 3.2 and Frobenius reciprocity,

$$\pi_3 \hookrightarrow \nu^{-\frac{1}{2}}\delta(\rho, 2) \rtimes \sigma.$$

Since $\nu^{-\frac{1}{2}}\delta(\rho, 2) \rtimes \sigma$ has $L(\nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma)$ as its unique irreducible subrepresentation, we have $\pi_3 = L(\nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma)$, as needed.

Case 2c: $j = \frac{n-2}{2}$, $n \geq 4$ (n even)

Here, $\pi = \nu^{-\frac{1}{2}}\zeta(\rho, n) \rtimes \sigma)$ and

$$\begin{aligned}
s_{(p)}\pi &= \nu^{-\frac{n}{2}}\rho \otimes \zeta(\rho, n-1) \rtimes \sigma + \nu^{-\frac{n}{2}+1}\rho \otimes \nu^{-1}\zeta(\rho, n-1) \rtimes \sigma \\
&= \nu^{-\frac{n}{2}}\rho \otimes L([\nu^{-\frac{n}{2}+1}\rho, \nu^{-1}\rho], [\nu^{-\frac{n}{2}+1}\rho, \nu^{-1}\rho]; T_1) \\
&+ \nu^{-\frac{n}{2}}\rho \otimes L([\nu^{-\frac{n}{2}+1}\rho, \nu^{-1}\rho], [\nu^{-\frac{n}{2}+1}\rho, \nu^{-1}\rho]; T_2) \\
&+ \nu^{-\frac{n}{2}+1}\rho \otimes L([\nu^{-\frac{n}{2}}\rho, \nu^{-1}\rho], [\nu^{-\frac{n}{2}+2}\rho, \nu^{-1}\rho]; T_1) \\
&+ \nu^{-\frac{n}{2}+1}\rho \otimes L([\nu^{-\frac{n}{2}}\rho, \nu^{-1}\rho], [\nu^{-\frac{n}{2}+2}\rho, \nu^{-1}\rho]; T_2) \\
&+ \nu^{-\frac{n}{2}+1}\rho \otimes L(\nu^{-\frac{n}{2}}\rho, \nu^{\frac{-n+3}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma) \\
&= \tau_1' + \tau_2' + \tau_1'' + \tau_2'' + \tau_3''
\end{aligned}$$

First, the obvious analogue to Lemma 3.5 holds (same proof). From this, it follows that

$$\pi_i = L([\nu^{-\frac{n}{2}}\rho, \nu^{-1}\rho], [\nu^{-\frac{n}{2}+1}\rho, \nu^{-1}\rho]; T_i)$$

for $i = 1, 2$ are components of π.

Next, we claim that $s_{(p)}\pi_i = \tau_i' + \tau_i''$ for $i = 1, 2$. By the analogue to Lemma 3.5, it suffices to show $s_{(p)}\pi_i \geq \tau_i' + \tau_i''$. In general, if $\pi = \nu^{\alpha}\zeta(\rho, n) \rtimes \sigma$,

$$s_{((n-1)p)}\pi = \sum_{i=1}^{n} [\nu^{-\alpha-\frac{i}{2}}\zeta(\rho, n-i) \times \nu^{\alpha+\frac{-n+i-1}{2}}\zeta(\rho, i-1)] \otimes [\nu^{\alpha+\frac{-n+2i-1}{2}}\rho \rtimes \sigma]$$

(a μ^* calculation like that used for Corollary 2.7). In particular, if $\alpha \leq 0$ is a reducibility point,

$$\begin{aligned}
s_{((n-1)p)}\pi &\geq [\nu^{-\frac{\alpha}{2}+\frac{-n-1}{4}}\zeta(\rho, \tfrac{n-1}{2} + \alpha) \times \nu^{\frac{\alpha}{2}+\frac{-n-1}{4}}\zeta(\rho, \tfrac{n-1}{2} - \alpha)] \otimes T_1 \\
&+ [\nu^{-\frac{\alpha}{2}+\frac{-n-1}{4}}\zeta(\rho, \tfrac{n-1}{2} + \alpha) \times \nu^{\frac{\alpha}{2}+\frac{-n-1}{4}}\zeta(\rho, \tfrac{n-1}{2} - \alpha)] \otimes T_2
\end{aligned}$$

and these are the only terms of the form $\ldots \otimes T_1$ or $\ldots \otimes T_2$ in $s_{((n-1)p)}\pi$. Observe that both of these are irreducible. Then, for our particular π,

$$\begin{gathered}
s_{(p)}\pi_1 \geq \tau_1' \\
\Downarrow \\
s_{min}\pi_1 \geq \nu^{-\frac{n}{2}}\rho \otimes (\nu^{-\frac{n}{2}+1}\rho \otimes \nu^{-\frac{n}{2}+1}\rho) \otimes \ldots \otimes (\nu^{-1}\rho \otimes \nu^{-1}\rho) \otimes T_1 \\
\Downarrow \\
s_{((n-1)p)}\pi_1 \geq [\nu^{-\frac{n}{4}}\zeta(\rho, \tfrac{n}{2}-1) \times \nu^{-\frac{n}{4}-\frac{1}{2}}\zeta(\rho, \tfrac{n}{2})] \otimes T_1 \\
\Downarrow \\
s_{(p)}\pi_1 \geq \tau_1' + \tau_1''.
\end{gathered}$$

A similar argument holds for π_2. Note that this is essentially the argument that would be used in the proof of an analogue to Lemma 3.8 (to identify Jacquet modules). We do not separate this out as a lemma as this is the only place it will be used.

Finally, we have $s_{(p)}\pi_3 = \tau_3'''$. The identification of π_3 uses the same argument as in case 2c of Proposition 3.6.

Finally, if one uses the analogue to Lemma 3.5 and the Jacquet module argument involving $s_{((n-1)p)}\pi$ above (which plays the role of Lemma 3.3), the same arguments used in Proposition 3.6 do the remaining cases here. $\square$

We include the following for future use.

Note **3.12.** *Suppose σ is an irreducible supercuspidal representation of S_m and ρ is an irreducible supercuspidal representation of $GL_p(F)$. Further, suppose $\rho \not\cong \tilde{\rho}$. Then $\nu^{\alpha}\zeta(\rho, n) \rtimes \sigma$ is irreducible for all $\alpha \in \mathbb{R}$.*

Proof. See Theorem 7.2 (i) in [Tad3]. $\square$

4. Reducibility points

In this section, we give the reducibility points for $\pi = \nu^{\alpha}\zeta(\rho_0, k) \rtimes \zeta(\rho, \ell; \sigma)$. If ρ satisfies (C1/2), the results are in Theorem 4.1; for (C1), in Theorem 4.3. Note that in this section, we only address the proof for $\rho_0 \cong \rho$. The results for $\rho_0 \not\cong \rho$ are a corollary of the results in section 5. (The arguments used when $\rho_0 \not\cong \rho$ have a somewhat different flavor than those for $\rho_0 \cong \rho$; relying on the irreducibility of $\nu^{\alpha}\rho_0 \times \nu^{\beta}\rho$ for all $\alpha, \beta \in \mathbb{R}$).

In Theorem 4.1 1, most of the work goes into verifying irreducibility. This is also the most important part of the theorem: reducibility could be obtained from Theorem 6.1. (The decomposition into components in Theorem 6.1 is obtained more or less independently from the reducibility results in Theorem 4.1.) Since the proof of reducibility does not add much work, we include it here for the sake of completeness.

Theorem **4.1.** *Let ρ_0, ρ be irreducible unitarizable supercuspidal representations of $GL_{p_0}(F)$, $GL_p(F)$, resp.; σ an irreducible supercuspidal representation of S_m. Further, suppose ρ satisfies (C1/2). Let $\pi = \nu^{\alpha}\zeta(\rho_0, k) \rtimes \zeta(\rho, \ell; \sigma)$, $\alpha \in \mathbb{R}$.*

(1) *Suppose $\rho_0 \cong \rho$. Then, π is reducible if and only if*

$$\alpha \in \left\{-\frac{k}{2}, -\frac{k}{2}+1, \dots, \frac{k}{2}\right\} \cup \left\{\pm(\ell+\frac{k}{2}), \pm(\ell+\frac{k}{2}-1), \dots, \pm(\ell-\frac{k}{2}+1)\right\}$$

(noting that the sets are not necessarily disjoint) with the exception that if $k=2\ell$ and $\alpha=0$, there is irreducibility (i.e., $\zeta(\rho,2\ell) \rtimes \zeta(\rho,\ell;\sigma)$ is irreducible).

(2) *Suppose $\rho_0 \not\cong \rho$. Then, π is reducible if and only if $\nu^\alpha\zeta(\rho_0,k) \rtimes \sigma$ is reducible. If $\rho_0 \not\cong \tilde{\rho}_0$, $\nu^\alpha\zeta(\rho_0,k) \rtimes \sigma$ is irreducible (cf. Note 3.12). For ρ_0 satisfying (C1/2), (C1), or (C0), resp., the reducibility points for $\nu^\alpha\zeta(\rho_0,k) \rtimes \sigma$ are given in Propositions 3.6, 3.10, or 3.11, resp.).*

Proof. Here, we do the case $\rho_0 \cong \rho$. The case $\rho_0 \not\cong \rho$ is covered by Proposition 5.3. The cases $k=1$ and $\ell=0$ are covered by section 3. So, we assume $k \geq 2$, $\ell \geq 1$ here.

To deal with the case $\rho_0 \cong \rho$, we proceed by induction (on $n=\ell+k$), splitting this into three cases. By Lemma 2.9, we may without loss of generality assume $\alpha \leq 0$.

<u>Case 1:</u> $k=2\ell$

Here, we need to show that $\pi = \nu^\alpha\zeta(\rho,2\ell) \rtimes \zeta(\rho,\ell;\sigma)$ is reducible for $\alpha \in \{-2\ell,-2\ell+1,\dots,-1\}$ and irreducible for all other $\alpha \leq 0$.

Let us start by verifying reducibility for $\alpha \in \{-2\ell,-2\ell+1,\dots,-1\}$. Recall that

$$\begin{aligned} s_{(p)}\pi \;&= \nu^{-\alpha-\ell+\frac{1}{2}}\rho \otimes \nu^{\alpha-\frac{1}{2}}\zeta(\rho,2\ell-1) \rtimes \zeta((\rho,\ell;\sigma) \\ &+\nu^{\alpha-\ell+\frac{1}{2}}\rho \otimes \nu^{\alpha+\frac{1}{2}}\zeta(\rho,2\ell-1) \rtimes \zeta((\rho,\ell;\sigma) \\ &+\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{\alpha}\zeta(\rho,2\ell) \rtimes \zeta((\rho,\ell-1;\sigma). \end{aligned}$$

For $\alpha \in \{-2\ell,-2\ell+1,\dots,-1\}$, we may apply Lemma 3.5 to any term in $s_{(p)}\pi$. By induction, we have that $\nu^{\alpha+\frac{1}{2}}\zeta(\rho,2\ell-1) \rtimes \zeta(\rho,\ell;\sigma)$ is reducible for $\alpha \in \{-2\ell,-2\ell+1,\dots,-1\}$, giving π reducible there.

We now turn to the irreducibility points. We save irreducibility for $\alpha=0$ until Lemma 4.2. So, for now we may assume $\alpha<0$. Note that this means $-\alpha-\ell+\frac{1}{2}$, $\alpha-\ell+\frac{1}{2}$, and $-\ell+\frac{1}{2}$ are all distinct. Since $\alpha \notin \{-2\ell,-2\ell+1,\dots,-1\}$, the inductive hypothesis tells us that the three terms in $s_{(p)}\pi$ are all irreducible. Now, by Corollary 2.7,

$$s_{GL}\pi = \sum_{i=0}^{2\ell} \nu^{-\alpha-\frac{i}{2}}\zeta(\rho,2\ell-i) \times \nu^{\alpha+\frac{-2\ell+i}{2}}\zeta(\rho,i) \times \nu^{-\frac{\ell}{2}}\zeta(\rho,\ell) \otimes \sigma.$$

First, if α is not an integer, then the $i=0,2\ell$ terms–i.e., $\nu^{-\alpha}\zeta(\rho,2\ell) \times \nu^{-\frac{\ell}{2}}\zeta(\rho,\ell) \otimes \sigma$ and $\nu^{\alpha}\zeta(\rho,2\ell) \times \nu^{-\frac{\ell}{2}}\zeta(\rho,\ell) \otimes \sigma$–are irreducible (cf. Theorem 2.1). The first of these contains terms of the form $\nu^{-\alpha-\ell+\frac{1}{2}}\rho \otimes \dots$ and $\nu^{-\ell+\frac{1}{2}}\rho \otimes \dots$ in its minimal Jacquet module, s_{min}. This forces $\nu^{-\alpha-\ell+\frac{1}{2}}\rho \otimes \nu^{\alpha-\frac{1}{2}}\zeta(\rho,2\ell-1) \rtimes \zeta(\rho,\ell;\sigma)$ and $\nu^{-\ell+\frac{1}{2}}\rho \otimes \zeta(\rho,2\ell) \rtimes \zeta(\rho,\ell-1;\sigma)$ to both come from the component of π which has $\nu^{-\alpha}\zeta(\rho,2\ell) \times \nu^{-\frac{\ell}{2}}\zeta(\rho,\ell) \otimes \sigma$ in its s_{GL}. Similarly,

$\nu^{\alpha-\ell+\frac{1}{2}}\rho \otimes \nu^{\alpha+\frac{1}{2}}\zeta(\rho,2\ell-1) \rtimes \zeta(\rho,\ell;\sigma)$ and $\nu^{-\ell+\frac{1}{2}}\rho \otimes \zeta(\rho,2\ell) \rtimes \zeta(\rho,\ell-1;\sigma)$ come from the component of π which gives rise to $\nu^{\alpha}\zeta(\rho,2\ell) \times \nu^{-\frac{\ell}{2}}\zeta(\rho,\ell) \otimes \sigma$. Thus, all three terms in $s_{(p)}\pi$ come from the same component of π. Thus, π must be irreducible. If α is an integer, we have $\alpha < -2\ell$. We can then use exactly the same argument as in the noninteger case–the $i = 0, 2\ell$ terms from $s_{GL}\pi$ are still irreducible and the argument goes through. This finishes the $k = 2\ell$ case.

Case 2: $k > 2\ell$

Here, we need to show that π is reducible for $\alpha \in \{-\ell - \frac{k}{2}, -\ell - \frac{k}{2} + 1, \ldots, \ell + \frac{k}{2}\}$ and irreducible for all other α. Again, without loss of generality, we may restrict our attention to $\alpha \leq 0$.

We begin with the reducibility points. First, assume $\alpha \neq 0$, $\ell - \frac{k}{2}$ is a reducibility point. Then, $\alpha + \frac{-k+1}{2}$, $-\alpha + \frac{-k+1}{2}$, and $-\ell + \frac{1}{2}$ are all distinct, so Lemma 3.5 may be applied to any term in $s_{(p)}\pi$. It is easy to check that at least one of them reduces for any of the α we are looking at, making π reducible there. This shows reducibility for all the reducibility points except $\alpha = \ell - \frac{k}{2}$, 0 ($\alpha = 0$ reducible only for k even).

If $\alpha = 0$ we can still apply Lemma 3.5 to the term $\nu^{-\ell+\frac{1}{2}}\rho \otimes \zeta(\rho,k) \rtimes \zeta(\rho,\ell-1;\sigma)$ (but not to the others). So, for k even, Lemma 3.5 and the inductive hypothesis give reducibility for $\alpha = 0$. We could try the same approach for $\alpha = \ell - \frac{k}{2}$–apply Lemma 3.5 to the term $\nu^{\ell-k+\frac{1}{2}}\rho \otimes \nu^{\ell-\frac{k}{2}+\frac{1}{2}}\zeta(\rho,k-1) \rtimes \zeta(\rho,\ell;\sigma)$. However, if $k = 2\ell+1$, this term is irreducible, so the lemma will not be enough to give us reducibility. Thus, we take a different approach for the $\alpha = \ell - \frac{k}{2}$ case.

We show that $\nu^{\ell-\frac{k}{2}}\zeta(\rho,k) \rtimes \zeta(\rho,\ell;\sigma)$ is reducible by comparing it with $\zeta(\rho,2\ell) \rtimes \zeta(\rho,k-\ell;\sigma)$. First, we observe that $L([\nu^{\ell-k+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$ is a component of both (cf. Lemma 3.3). However, since $s_{min}\nu^{\ell-\frac{k}{2}}\zeta(\rho,k) \rtimes \zeta(\rho,\ell;\sigma) \not\leq s_{min}\zeta(\rho,2\ell) \rtimes \zeta(\rho,k-\ell;\sigma)$ (e.g., $\nu^{k-\ell-\frac{1}{2}}\rho$ never appears in any term in $s_{min}\zeta(\rho,2\ell) \rtimes \zeta(\rho,k-\ell;\sigma)$, cf. Theorem 2.3 et seq.), we have that

$$L([\nu^{\ell-k+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) < \nu^{\ell-\frac{k}{2}}\zeta(\rho,k) \rtimes \zeta(\rho,\ell;\sigma)$$

This gives reducibility for $\alpha = \ell - \frac{k}{2}$ and finishes the reducibility points for the case $k > 2\ell$.

We now turn to the irreducibility points. Again, we take $\alpha \leq 0$. For $\alpha \neq 0$, the same basic argument as in the $k = 2\ell$ case works. For $\alpha = 0$ (relevant here only if k is odd), we do not have $\alpha + \frac{-k+1}{2}$, $-\alpha + \frac{-k+1}{2}$, $-\ell + \frac{1}{2}$ all distinct and must be more careful. (Note that $\alpha = 0$ with k even and $\alpha = \ell - \frac{k}{2}$ also have this property but are reducibility points).

Now, suppose we have $\alpha \neq 0$ a point where we need to show irreducibility. Then, $\alpha + \frac{-k+1}{2}$, $-\alpha + \frac{-k+1}{2}$, and $-\ell + \frac{1}{2}$ are all distinct, and further, by the induction hypothesis, the three terms in $s_{(p)}\pi$ are all irreducible. Again, for such α, the $i = 0$ and $i = k$ terms in $s_{GL}\pi$, namely $\nu^{-\alpha}\zeta(\rho,k) \times \nu^{-\frac{\ell}{2}}\zeta(\rho,\ell) \otimes \sigma$ and $\nu^{\alpha}\zeta(\rho,k) \times \nu^{-\frac{\ell}{2}}\zeta(\rho,\ell) \otimes \sigma$, are irreducible. Thus, we (again) have that $\nu^{-\alpha+\frac{-k+1}{2}}\rho \otimes \nu^{\alpha-\frac{1}{2}}\zeta(\rho,k-1) \rtimes \zeta(\rho,\ell;\sigma)$ and $\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{\alpha}\zeta(\rho,k) \rtimes \zeta(\rho,\ell-1;\sigma)$

come from the component of π containing $\nu^{-\alpha}\zeta(\rho,k) \times \nu^{-\frac{\ell}{2}}\zeta(\rho,\ell) \otimes \sigma$ in its s_{GL}. Similarly, $\nu^{\alpha+\frac{-k+1}{2}}\rho \otimes \nu^{\alpha+\frac{1}{2}}\zeta(\rho,k-1) \rtimes \zeta(\rho,\ell;\sigma)$ and $\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{\alpha}\zeta(\rho,k) \rtimes \zeta(\rho,\ell-1;\sigma)$ come from the component of π containing $\nu^{\alpha}\zeta(\rho,k) \times \nu^{-\frac{\ell}{2}}\zeta(\rho,\ell) \otimes \sigma$ in its s_{GL}. Since all the terms in $s_{(p)}\pi$ come from the same component of π, we must have that π is irreducible.

We now turn to the case $\alpha = 0$, k odd. The proof is similar to the general case, but a little more care is required. Note that if $k = 2j+1$,

$$s_{(p)}\pi = 2\nu^{-j}\rho \otimes \nu^{-\frac{1}{2}}\zeta(\rho,2j) \rtimes \zeta(\rho,\ell;\sigma) + \nu^{-\ell+\frac{1}{2}}\rho \otimes \zeta(\rho,2j+1) \rtimes \zeta(\rho,\ell-1;\sigma).$$

By the inductive hypothesis, these are all irreducible. Also,

$$s_{GL}\pi = 2\sum_{i=0}^{j} \nu^{-\frac{i}{2}}\zeta(\rho,2j+1-i) \times \nu^{-j+\frac{i-1}{2}}\zeta(\rho,i) \times \nu^{-\frac{\ell}{2}}\zeta(\rho,\ell) \otimes \sigma.$$

All the terms in $s_{GL}\pi$ are irreducible as well. Now, s_{min} of any term in $s_{GL}\pi$ contains a term of the form $\nu^{-\ell+\frac{1}{2}}\rho \otimes \ldots$. Take any component π_1 of π. Since $s_{GL}\pi_1 \neq 0$, we have that $s_{min}\pi_1$ has a term of the form $\nu^{-\ell+\frac{1}{2}} \otimes \ldots$. Therefore, $s_{(p)}\pi_1 \geq \nu^{-\ell+\frac{1}{2}}\rho \otimes \zeta(\rho,2j+1) \rtimes \zeta(\rho,\ell-1;\sigma)$. Since this holds for any component π_1 of π but $\nu^{-\ell+\frac{1}{2}}\rho \otimes \zeta(\rho,2j+1) \rtimes \zeta(\rho,\ell-1;\sigma)$ occurs in $s_{(p)}\pi$ with multiplicity one, we must have that π is irreducible. This finishes the case $k > 2\ell$.

Case 3: $k < 2\ell$

Here, we need to show that π is reducible for $\alpha \in \{-\frac{k}{2}, \frac{k}{2}+1, \ldots, \frac{k}{2}\} \cup \{\pm(\ell+\frac{k}{2}-1), \pm(\ell+\frac{k}{2}-2), \ldots, \pm(\ell-\frac{k}{2}+1)\}$, irreducible otherwise. Again, without loss of generality, we may restrict our attention to $\alpha \leq 0$.

We begin with the reducibility points. First, suppose α is a reducibility point with $\alpha \neq 0, -\ell+\frac{k}{2}$. Then, $\alpha + \frac{-k+1}{2}$, $-\alpha + \frac{-k+1}{2}$, and $-\ell+\frac{1}{2}$ are all distinct, so Lemma 3.5 may be applied to any term in $s_{(p)}\pi$. It is easy to see that at least one of them is reducible, so that the lemma implies π is reducible.

Now, take $\alpha = 0$ with k even. Unlike the $\alpha = 0$ case for $k > 2\ell$, Lemma 3.5 is not enough to give reducibility–in particular, it fails to work if $k = 2(\ell-1)$. However, for $k = 2j$, we can compare $\zeta(\rho,2j) \rtimes \zeta(\rho,\ell;\sigma)$ and $\nu^{\frac{-\ell+j}{2}}\zeta(\rho,\ell+j) \rtimes \zeta(\rho,j;\sigma)$. They have a component in common, namely $L([\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$ (again, cf. Lemma 3.5). However, $\zeta(\rho,2j) \otimes \zeta(\rho,\ell;\sigma)$ occurs with multiplicity two in $s_{(2jp)}\zeta(\rho,2j) \rtimes \zeta(\rho,\ell;\sigma)$ and multiplicity one in $s_{(2jp)}\nu^{\frac{-\ell+j}{2}}\zeta(\rho,\ell+j) \rtimes \zeta(\rho,j;\sigma)$ (e.g., look at s_{min}). So, $\zeta(\rho,2j) \rtimes \zeta(\rho,\ell;\sigma) > L([\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$, hence is reducible.

We now address the case $\alpha = -\ell+\frac{k}{2}$. If $k \geq \ell$, $-\ell+\frac{k}{2} \geq -\frac{k}{2}$ so we expect reducibility. In this case, we get reducibility by comparing π with $\zeta(\rho,2\ell) \rtimes \zeta(\rho,k-\ell;\sigma)$. First, observe that $L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\ell-k+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$ is a component of both. Further,we have $\nu^{\ell-\frac{k}{2}}\zeta(\rho,k) \otimes \zeta(\rho,\ell;\sigma) < s_{(kp)}\pi$ but $\nu^{\ell-\frac{k}{2}}\zeta(\rho,k) \otimes \zeta(\rho,\ell;\sigma) \not< s_{(kp)}\zeta(\rho,2\ell) \rtimes \zeta(\rho,k-\ell;\sigma)$ (e.g., there is always a $\nu^{-\ell+\frac{1}{2}}\rho$ before $\nu^{\ell-\frac{1}{2}}\rho$ in $s_{min}\zeta(\rho,2\ell) \rtimes \zeta(\rho,k-\ell;\sigma)$). Therefore,

$$\nu^{\ell-\frac{k}{2}}\zeta(\rho,k) \otimes \zeta(\rho,\ell;\sigma) \not< s_{(kp)}L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\ell-k+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$$

hence π must have another component. This finishes the reducibility points.

We now turn to the irreducibility points. Suppose $\alpha \neq 0, -\ell+\frac{k}{2}$ is an (expected) irreducibility point. The same argument as in the case $k > 2\ell$ with $\alpha \neq 0$ shows irreducibility; we do not repeat the argument here. Similarly, for $\alpha = 0$, k odd, the $k < 2\ell$ case is the same as the $k > 2\ell$ case. Again, we do not repeat it. This leaves $\alpha = -\ell + \frac{k}{2}$ (with $k < \ell$).

Suppose $\alpha = -\ell + \frac{k}{2}$ with $k < \ell$, so we expect irreducibility. Then

$$\begin{aligned} s_{(p)}\pi &= \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{k}{2}+\frac{1}{2}}\zeta(\rho, k-1) \rtimes \zeta(\rho, \ell; \sigma) \\ &+\nu^{\ell-k+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{k}{2}-\frac{1}{2}}\zeta(\rho, k-1) \rtimes \zeta(\rho, \ell; \sigma) \\ &+\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{k}{2}}\zeta(\rho, k) \rtimes \zeta(\rho, \ell-1; \sigma). \end{aligned}$$

The first two terms are irreducible; the last reducible. Also,

$$s_{GL}\pi = \sum_{i=0}^{k} \nu^{\ell-\frac{k}{2}-\frac{i}{2}}\zeta(\rho, k-i) \times \nu^{-\ell+\frac{i}{2}}\zeta(\rho, i) \times \nu^{-\frac{\ell}{2}}\zeta(\rho, \ell) \otimes \sigma.$$

All the terms in $s_{GL}\pi$ are irreducible. Every term in $s_{GL}\pi$ for $i < k$ has something of the form $\nu^{\ell-k+\frac{1}{2}}\rho \otimes \dots$ in its (minimal) Jacquet module, therefore they must all come from the component of π containing $\nu^{\ell-k+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{k}{2}-\frac{1}{2}}\zeta(\rho, k-1) \rtimes \zeta(\rho, \ell; \sigma)$ in its $s_{(p)}$. In particular, π can have at most two components, and if $\pi = \pi_1 + \pi_2$,

$$s_{GL}\pi_1 = \sum_{i=0}^{k-1} \nu^{\ell-\frac{k}{2}-\frac{i}{2}}\zeta(\rho, k-i) \times \nu^{-\ell+\frac{i}{2}}\zeta(\rho, i) \times \nu^{-\frac{\ell}{2}}\zeta(\rho, \ell) \otimes \sigma$$

$$s_{GL}\pi_2 = \nu^{-\ell+\frac{k}{2}}\zeta(\rho, k) \times \nu^{-\frac{\ell}{2}}\zeta(\rho, \ell) \otimes \sigma.$$

Therefore, we see that any term in $s_{min}\pi_1$ contains one copy of $\nu^{-\ell+k-\frac{1}{2}}\rho$ and one copy of $\nu^{\ell-k+\frac{1}{2}}\rho$ while any term in $s_{min}\pi_2$ contains two copies of $\nu^{-\ell+k-\frac{1}{2}}\rho$ (and none of $\nu^{\ell-k+\frac{1}{2}}\rho$). However, $\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{k}{2}+\frac{1}{2}}\zeta(\rho, k-1) \rtimes \zeta(\rho, \ell; \sigma) \leq s_{(p)}\pi$ is irreducible and $s_{min}\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{k}{2}+\frac{1}{2}}\zeta(\rho, k-1) \rtimes \zeta(\rho, \ell; \sigma)$ contains both terms with one copy each of $\nu^{-\ell+k-\frac{1}{2}}\rho$ and $\nu^{\ell-k+\frac{1}{2}}\rho$ and terms with two copies of $\nu^{-\ell+k-\frac{1}{2}}\rho$. Therefore, π_1 and π_2 cannot be separate components. This forces π to be irreducible, finishing the case $k < 2\ell$ and the case $\rho_0 = \rho$ in the theorem. □

LEMMA **4.2.** *With notation as in the theorem, $\pi = \zeta(\rho, 2\ell) \rtimes \zeta(\rho, \ell; \sigma)$ is irreducible ($\ell \geq 1$).*

Proof. Let $n = 3\ell$. From Corollary 2.7, we have

$$s_{(p)}\pi = 2\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\frac{1}{2}}\zeta(\rho, 2\ell-1) \rtimes \zeta(\rho, \ell; \sigma) + \nu^{-\ell+\frac{1}{2}}\rho \otimes \zeta(\rho, 2\ell) \rtimes \zeta(\rho, \ell-1; \sigma).$$

These are all reducible.

$$
\begin{aligned}
s_{GL}\pi \quad &= \sum_{i=0}^{2\ell} \nu^{-\frac{i}{2}}\zeta(\rho, 2\ell - i) \times \nu^{-\ell+\frac{i}{2}}\zeta(\rho, i) \times \nu^{-\frac{\ell}{2}}\zeta(\rho, \ell) \otimes \sigma \\
&= 2\sum_{i=0}^{\ell-1} \nu^{-\frac{i}{2}}\zeta(\rho, 2\ell - i) \times \nu^{-\ell+\frac{i}{2}}\zeta(\rho, i) \times \nu^{-\frac{\ell}{2}}\zeta(\rho, \ell) \otimes \sigma \\
&+ \nu^{-\frac{\ell}{2}}\zeta(\rho, \ell) \times \nu^{-\frac{\ell}{2}}\zeta(\rho, \ell) \times \nu^{-\frac{\ell}{2}}\zeta(\rho, \ell) \otimes \sigma
\end{aligned}
$$

All the terms for $s_{GL}\pi$ are irreducible.

$$
\begin{aligned}
s_{(n-1)p}\pi \quad &= \sum_{i=0}^{2\ell} \nu^{-\frac{i}{2}}\zeta(\rho, 2\ell - i) \times \nu^{-\ell+\frac{i}{2}}\zeta(\rho, i) \times \nu^{\frac{-\ell-1}{2}}\zeta(\rho, \ell - 1) \otimes \zeta(\rho, 1; \sigma) \\
&+ \sum_{i=1}^{2\ell} \nu^{-\frac{i}{2}}\zeta(\rho, 2\ell - i) \times \nu^{-\ell+\frac{i-1}{2}}\zeta(\rho, i - 1) \times \nu^{-\frac{\ell}{2}}\zeta(\rho, \ell) \otimes \nu^{-\ell+i-\frac{1}{2}}\rho \rtimes \sigma \\
&= 2\sum_{i=0}^{\ell-1} \nu^{-\frac{i}{2}}\zeta(\rho, 2\ell - i) \times \nu^{-\ell+\frac{i}{2}}\zeta(\rho, i) \times \nu^{\frac{-\ell-1}{2}}\zeta(\rho, \ell - 1) \otimes \zeta(\rho, 1; \sigma) \\
&+ 3\nu^{-\frac{\ell}{2}}\zeta(\rho, \ell) \times \nu^{-\frac{\ell}{2}}\zeta(\rho, \ell) \times \nu^{\frac{-\ell-1}{2}}\zeta(\rho, \ell - 1) \otimes \zeta(\rho, 1; \sigma) \\
&+ 2\sum_{i=1}^{\ell-1} \nu^{-\frac{i}{2}}\zeta(\rho, 2\ell - i) \times \nu^{-\ell+\frac{i-1}{2}}\zeta(\rho, i - 1) \times \nu^{-\frac{\ell}{2}}\zeta(\rho, \ell) \otimes \nu^{-\ell+i-\frac{1}{2}}\rho \rtimes \sigma \\
&+ 2\nu^{-\frac{\ell}{2}}\zeta(\rho, \ell) \times \nu^{\frac{-\ell-1}{2}}\zeta(\rho, \ell - 1) \times \nu^{-\frac{\ell}{2}}\zeta(\rho, \ell) \otimes \delta(\rho, 1; \sigma)
\end{aligned}
$$

This is a decomposition into irreducible components. Two copies of $\nu^{-\frac{\ell}{2}}\zeta(\rho, \ell) \times \nu^{-\frac{\ell}{2}}\zeta(\rho, \ell) \times \nu^{\frac{-\ell-1}{2}}\zeta(\rho, \ell - 1) \otimes \zeta(\rho, 1; \sigma)$ come from the (reducible) $i = \ell, \ell + 1$ terms in the second sum; the other is the $i = \ell$ term in the first sum.

We show irreducibility by showing that all the terms in $s_{GL}\pi$ come from the same component of π. Observe that in the expression for $s_{GL}\pi$ above, we have

$$
i\underline{th} \text{ term} \begin{cases}
r_{min}\nu^{-\frac{i}{2}}\zeta(\rho, 2\ell - i) = \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{3}{2}}\rho \otimes \ldots \otimes \nu^{\ell-i-\frac{1}{2}}\rho \\
r_{min}\nu^{-\ell+\frac{i}{2}}\zeta(\rho, i) = \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{3}{2}}\rho \otimes \ldots \otimes \nu^{-\ell+i-\frac{1}{2}}\rho \\
r_{min}\nu^{-\frac{\ell}{2}}\zeta(\rho, \ell) = \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{3}{2}}\rho \otimes \ldots \otimes \nu^{-\frac{1}{2}}\rho
\end{cases}
$$

$$i-1\underline{st}\text{ term}\begin{cases} r_{min}\nu^{\frac{-i+1}{2}}\zeta(\rho,2\ell+1-i)=\nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{-\ell+\frac{3}{2}}\rho\otimes\ldots\otimes\nu^{\ell-i+\frac{1}{2}}\rho \\ \\ r_{min}\nu^{-\ell+\frac{i-1}{2}}\zeta(\rho,i-1)=\nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{-\ell+\frac{3}{2}}\rho\otimes\ldots\otimes\nu^{-\ell+i-\frac{3}{2}}\rho \\ \\ r_{min}\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)=\nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{-\ell+\frac{3}{2}}\rho\otimes\ldots\otimes\nu^{-\frac{1}{2}}\rho. \end{cases}$$

From this, one can see that the $i\underline{th}$ term may be characterized by the fact that $s_{min}(i\underline{th}\text{ term})$ contains terms of the form $\ldots\otimes\nu^{-\ell+i-\frac{1}{2}}\rho\otimes\sigma$. Similarly, the $i-1\underline{st}$ term may be characterized by the fact that $s_{min}(i-1\underline{st}\text{ term})$ has terms of the form $\ldots\otimes\nu^{\ell-i+\frac{1}{2}}\rho\otimes\sigma$ (for $i>0$). Now, in $s_{((n-1)p)}\pi$, the term $\nu^{-\frac{i}{2}}\zeta(\rho,2\ell-i)\times\nu^{-\ell+\frac{i-1}{2}}\zeta(\rho,i-1)\times\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\otimes\nu^{-\ell+i-\frac{1}{2}}\rho\rtimes\sigma$ is irreducible ($1\leq i\leq\ell-1$). Furthermore, any term in $s_{min}\pi$ of the form $\ldots\otimes\nu^{-\ell+i-\frac{1}{2}}\rho\otimes\sigma$ or $\ldots\otimes\nu^{\ell-i+\frac{1}{2}}\rho\otimes\sigma$ comes from one of the two copies of this. Therefore, if π_0 is a component of π with $s_{GL}\pi_0\geq i\underline{th}$ term, we get

$$\begin{gathered} s_{GL}\pi_0\geq i\underline{th}\text{ term} \\ \Downarrow \\ s_{((n-1)p)}\pi_0\geq\nu^{-\frac{i}{2}}\zeta(\rho,2\ell-i)\times\nu^{-\ell+\frac{i-1}{2}}\zeta(\rho,i-1)\times\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\otimes\nu^{-\ell+i-\frac{1}{2}}\rho\rtimes\sigma \\ \Downarrow \\ s_{GL}\pi_0\geq(i-1)\underline{st}\text{ term}. \end{gathered}$$

That is, each copy of $\nu^{-\frac{i}{2}}\zeta(\rho,2\ell-i)\times\nu^{-\ell+\frac{i-1}{2}}\zeta(\rho,i-1)\times\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\otimes\nu^{-\ell+i-\frac{1}{2}}\rho\rtimes\sigma$ "connects" one copy of the $i\underline{th}$ and $i-1\underline{st}$ terms in $s_{min}\pi$. Using this as i varies for 1 to $\ell-1$, we get that π has at most three components, and further, if $\pi=\pi_1+\pi_2+\pi_3$, then

$$s_{GL}\pi_1=s_{GL}\pi_2=\sum_{i=1}^{\ell-1}\nu^{-\frac{i}{2}}\zeta(\rho,2\ell-i)\times\nu^{-\ell+\frac{i}{2}}\zeta(\rho,i)\times\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\otimes\sigma$$

and

$$s_{GL}\pi_3=\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\times\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\times\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\otimes\sigma.$$

Next, we show that there are at most two components, eliminating π_3 from above. Observe that any term in $s_{min}\pi_3$ has only negative powers of ν appearing (to see this, take Jacquet modules in stages starting with $s_{GL}\pi_3$). This means we cannot have $s_{(2\ell p)}\pi_3\geq\zeta(\rho,2\ell)\otimes\zeta(\rho,\ell;\sigma)$ (which has positive powers of ν appearing). This contradicts Frobenius reciprocity. Thus π has at most two components, and if $\pi=\pi_1+\pi_2$, we have $s_{GL}\pi_1$ as above and $s_{GL}\pi_2=s_{GL}\pi_1+\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\times\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\times\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\otimes\sigma$.

Finally, observe that

$$L=L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)$$

is a component of both $\nu^{-\frac{1}{2}}\zeta(\rho,2\ell-1)\rtimes\zeta(\rho,\ell;\sigma)$ (cf. Lemma 3.3) and $\zeta(\rho,2\ell)\rtimes\zeta(\rho,\ell-1;\sigma)$, and occurs with multiplicity one in each. Therefore, $s_{(p)}\pi$ contains $\nu^{-\ell+\frac{1}{2}}\rho\otimes L$ with multiplicity three. Since $s_{min}\nu^{-\ell+\frac{1}{2}}\rho\otimes L$ has a term with only negative powers of ν appearing, all three copies must come from the component of π containing $\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\times\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\times\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\otimes\sigma$

in its s_{GL}, which contains all of the "all-negative" terms. That is, $s_{(p)}\pi_2 > 3\nu^{-\ell+\frac{1}{2}}\rho\otimes L$. On the other hand, by Frobenius reciprocity, $\nu^{-\ell+\frac{1}{2}}\rho\otimes\zeta(\rho,2\ell)\otimes\zeta(\rho,\ell-1;\sigma) < s_{(p,2\ell p)}\nu^{-\ell+\frac{1}{2}}\rho\otimes L$. We observe (cf. Theorem 2.3 et seq.) that $\nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{-\ell+\frac{1}{2}}\rho\otimes\ldots\otimes\nu^{\ell-\frac{1}{2}}\rho\otimes\nu^{-\ell+\frac{3}{2}}\rho\otimes\ldots\otimes\nu^{-\frac{1}{2}}\rho\otimes\sigma$, which is $s_{min}\nu^{-\ell+\frac{1}{2}}\rho\otimes\zeta(\rho,2\ell)\otimes\zeta(\rho,\ell-1;\sigma)$, occurs with multiplicity four in $s_{min}\pi$. Two copies each come from each $\sum_{i=1}^{\ell-1}\nu^{-\frac{i}{2}}\zeta(\rho,2\ell-i)\times\nu^{-\ell+\frac{i}{2}}\zeta(\rho,i)\times\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\otimes\sigma$ (by symmetry), so that π_1 has two copies. Since there are three copies of $\nu^{-\ell+\frac{1}{2}}\rho\otimes L$, we must have $s_{(p)}\pi_1$ containing at least one copy of $\nu^{-\ell+\frac{1}{2}}\rho\otimes L$. However, we already have that π_2 contains all three copies of $\nu^{-\ell+\frac{1}{2}}\rho\otimes L$, a contradiction. Therefore, π is irreducible. □

We now give the counterpart to Theorem 4.1 for the case where ρ satisfies (C1). The proof is essentially the same.

THEOREM **4.3.** *Let ρ_0, ρ be irreducible unitarizable supercuspidal representations of $GL_{p_0}(F)$, $GL_p(F)$; σ an irreducible supercuspidal representation of S_m. Further, suppose ρ satisfies (C1). Let $\pi = \nu^\alpha\zeta(\rho_0,k)\rtimes\zeta(\rho,\ell;\sigma)$, $\alpha\in\mathbb{R}$.*

(1) *Suppose $\rho_0\cong\rho$. Then, π is reducible if and only if*

$$\alpha\in\left\{\frac{-k+1}{2},\frac{-k+1}{2}+1,\ldots,\frac{k-1}{2}\right\}$$

$$\cup\left\{\pm(\ell+\frac{k+1}{2}),\pm(\ell+\frac{k+1}{2}-1),\ldots,\pm(\ell+\frac{-k+3}{2})\right\}$$

(noting that the sets are not necessarily disjoint) with the exception that if $k=2\ell+1$ and $\alpha=0$, there is irreducibility (i.e., $\zeta(\rho,2\ell+1)\rtimes\zeta(\rho,\ell;\sigma)$ is irreducible).

(2) *Suppose $\rho_0\not\cong\rho$. Then, π is reducible if and only if $\nu^\alpha\zeta(\rho_0,k)\rtimes\sigma$ is reducible. If $\rho_0\not\cong\tilde{\rho}_0$, $\nu^\alpha\zeta(\rho_0,k)\rtimes\sigma$ is irreducible (cf. Note 3.12). For ρ_0 satisfying (C1/2), (C1), or (C0), resp., the reducibility points for $\nu^\alpha\zeta(\rho_0,k)\rtimes\sigma$ are given in Propositions 3.6, 3.10, or 3.11, resp.).*

5. COMPONENTS: THE "RAMIFIED" CASE

In this section, we study $\pi=\nu^\alpha\zeta(\rho_0,k)\rtimes\zeta(\rho,\ell;\sigma)$ for $\rho_0\not\cong\rho$. The main result, Proposition 5.3, relates the components of π to those of $\nu^\alpha\zeta(\rho_0,k)\rtimes\sigma$. (In particular, Proposition 5.3 verifies the reducibility claims from Theorems 4.1 and 4.3.) We are particularly interested in the cases where π is a degenerate principal series representation. Let $\chi=|\cdot|^\alpha\psi_0$, $\alpha\in\mathbb{R}$, be a one-dimensional representation of $F^\times$. Now, $\psi_0^2\neq 1$ corresponds to $\rho_0\not\cong\tilde{\rho}_0$, which is irreducible by Proposition 5.3 and Note 3.12. If $\psi_0=1$, the components of $\chi\circ det_k\rtimes tr_\ell$ (when reducible) are covered by the next section. Suppose $\psi_0=sgn$ (order two). Then we have $\rho_0=\psi_0=sgn$, $\rho=1$, $\sigma=1_{S_0}$. For $SO_{2n+1}(F)$, $\psi_0=sgn$ satisfies (C1/2). Therefore, $\chi\circ det_k\rtimes tr_\ell$ in $SO_{2n+1}(F)$ corresponds to both ρ and ρ_0 satisfying (C1/2). Similarly, in $Sp_{2n}(F)$, $\psi_0=sgn$ satisfies (C0). So, $\chi\circ det_k\rtimes tr_\ell$ in $Sp_{2n}(F)$ corresponds to ρ satisfying (C1) and ρ_0 satisfying (C0). These particular combinations of conditions on ρ and ρ_0 are dealt with explicitly in Corollaries 5.7 and 5.8, respectively. One key fact in proving the results of this section is the following: $\nu^{\alpha}\rho \times \nu^{\beta}\rho_0$ is irreducible for all $\alpha, \beta \in \mathbb{R}$.

We have one small obstacle: if ρ_0 is a representation of $GL_{p_0}(F)$ and ρ is a representation of $GL_p(F)$, we do not necessarily have $p_0 = p$. Therefore, we use the following to play the role that $s_{(p)}$ played in the previous sections.

Definition **5.1.** *For τ a representation of S_n, let $s_{sc}\tau$ denote the sum of all the irreducible terms in $\mu^*\tau$ of the form $\rho' \otimes \tau'$ with ρ' supercuspidal. In addition, allow s_{min} to mean the analogue of s_{min} for this setting–that is, the sum of everything in any Jacquet module which has all of its terms supercuspidal. (Alternatively, if X denotes the set of all permutations of k copies of p_0 and ℓ copies of p, then $s_{min} = \sum_{\alpha \in X} s_{\alpha}$.)*

Lemma **5.2.** *For π as above, if $\pi_0 \leq \pi$ with $\pi_0 \neq 0$, then*

(1) $s_{sc}\pi_0 \leq s_{sc}\pi$
(2) $s_{sc}\pi_0 \neq 0$.

Proof. (1) is trivial. For (2), choose m minimal such that $s_{(m)}\pi_0 \neq 0$. Suppose $\rho_1 \otimes \tau_1 \leq s_{(m)}\pi_0$ with $\rho_1 \otimes \tau_1$ irreducible. Then, ρ_1 is supercuspidal. If not, we have that $s_{(m_1)}\rho_1 \geq \rho_2 \otimes \rho_3$ for some $m_1 < m$. We claim $s_{(m_1)}\pi_0 \neq 0$: by restriction in stages this must be true since $s_{(m_1, m-m_1)}\pi_0 \geq \rho_2 \otimes \rho_3 \otimes \tau \neq 0$. This contradicts the choice of m, finishing (2). □

Proposition **5.3.** *Suppose that σ is an irreducible supercuspidal representation of S_m and ρ is a representation of $GL_p(F)$ satisfying (C1/2). Let $\pi = \nu^{\alpha}\zeta(\rho_0, k) \rtimes \zeta(\rho, \ell; \sigma)$. Write*

$$\nu^{\alpha}\zeta(\rho_0, k) \rtimes \sigma = \sum_{i=1}^{m} L(\Delta_i(k, \alpha)).$$

Further, suppose that this decomposition has multiplicity one, i.e., $\Delta_i(k, \alpha) \neq \Delta_j(k, \alpha)$ for $i \neq j$. Then,

$$\pi = \sum_{i=1}^{m} L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \Delta_i(k, \alpha)).$$

In particular, π is reducible if and only if $\zeta(\rho_0, k) \rtimes \sigma$ is reducible.

Proof. Note that the results holds for $\ell = 0$ (trivially) and for $k = 1$ (Theorem 6.1 (i) of [Tad3]). So, we may make the convenient assumption that $k \geq 2$, $\ell \geq 1$.

We proceed by induction on $k+\ell$. Now, by μ^* (Corollary 2.6) and induction, we have

$$
\begin{aligned}
s_{sc}\pi \;&= \nu^{\alpha+\frac{-k+1}{2}}\rho_0 \otimes \nu^{\alpha+\frac{1}{2}}\zeta(\rho_0,k-1)\rtimes\zeta(\rho,\ell;\sigma)\\
&+\nu^{-\alpha+\frac{-k+1}{2}}\tilde{\rho}_0 \otimes \nu^{\alpha-\frac{1}{2}}\zeta(\rho_0,k-1)\rtimes\zeta(\rho,\ell;\sigma)\\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{\alpha}\zeta(\rho_0,k)\rtimes\zeta(\rho,\ell-1;\sigma)\\
&=\sum_i \nu^{\alpha+\frac{-k+1}{2}}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\Delta_i(k-1,\alpha+\frac{1}{2}))\\
&+\sum_i \nu^{-\alpha+\frac{-k+1}{2}}\tilde{\rho}_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\Delta_i(k-1,\alpha-\frac{1}{2}))\\
&+\sum_i \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\Delta_i(k,\alpha)).
\end{aligned}
$$

We now proceed as in Lemma 3.5. In particular, we show that each component of π contains a term of the form $\nu^{-\ell+\frac{1}{2}}\rho\otimes L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\Delta_i(k,\alpha))$ in its s_{sc}. Suppose π_i is a component of π. By Lemma 5.2, we have $s_{sc}\pi_i\neq 0$. By the same lemma, $s_{sc}\pi_i\leq s_{sc}\pi$. Therefore, suppose $\nu^{\alpha+\frac{-k+1}{2}}\rho_0\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\Delta_i(k-1,\alpha+\frac{1}{2}))\leq s_{sc}\pi_i$. Then, consider $s_{min}\nu^{\alpha+\frac{-k+1}{2}}\rho_0\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\Delta_i(k-1,\alpha+\frac{1}{2}))$ (cf. Definition 5.1). By Lemma 2.8, it has terms of the form $\nu^{\alpha+\frac{-k+1}{2}}\rho_0\otimes\delta_1\ldots\otimes\delta_j\otimes\nu^{-\ell+\frac{1}{2}}\rho\otimes terms$, where $\delta_1,\ldots,\delta_j$ involve only ρ_0's. Therefore, since $\nu^{-\ell+\frac{1}{2}}\rho\times\delta_j=\delta_j\times\nu^{-\ell+\frac{1}{2}}\rho$, etc., we can commute $\nu^{-\ell+\frac{1}{2}}\rho$ and δ_j to argue as follows:

$$
\begin{gathered}
s_{min}\pi_i \geq \nu^{\alpha+\frac{-k+1}{2}}\rho_0\otimes\delta_1\ldots\otimes\delta_j\otimes\nu^{-\ell+\frac{1}{2}}\rho\otimes terms\\
\Downarrow\\
s_{app}\pi_i \geq \nu^{\alpha+\frac{-k+1}{2}}\rho_0\otimes\delta_1\ldots\otimes(\delta_j\times\nu^{-\ell+\frac{1}{2}}\rho)\otimes terms\\
\Downarrow\\
s_{min}\pi_i \geq \nu^{\alpha+\frac{-k+1}{2}}\rho_0\otimes\delta_1\ldots\otimes\delta_{j-1}\otimes\nu^{-\ell+\frac{1}{2}}\rho\otimes\delta_j\otimes terms\\
\Downarrow\\
\vdots\\
\Downarrow\\
s_{min}\pi_i \geq \nu^{\alpha+\frac{-k+1}{2}}\rho_0\otimes\nu^{-\ell+\frac{1}{2}}\rho\otimes\delta_1\otimes\ldots\otimes\delta_j\otimes terms\\
\Downarrow\\
s_{app}\pi_i \geq (\nu^{\alpha+\frac{-k+1}{2}}\rho_0\times\nu^{-\ell+\frac{1}{2}}\rho)\otimes\delta_1\otimes\ldots\otimes\delta_j\otimes terms\\
\Downarrow\\
s_{min}\pi_i \geq \nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{\alpha+\frac{-k+1}{2}}\rho_0\otimes\delta_1\otimes\ldots\otimes\delta_j\otimes terms,
\end{gathered}
$$

where s_{app} denotes the Jacquet module taken with respect to the parabolic subgroup of the appropriate shape. A similar argument holds for $\nu^{-\alpha+\frac{-k+1}{2}}\tilde{\rho}_0\otimes\ldots$.

Next, observe that

$$\pi \hookrightarrow \nu^{\alpha}(\rho_0,k) \rtimes (\nu^{-\ell+\frac{1}{2}}\rho \rtimes \zeta(\rho,\ell-1;\sigma)) \cong \nu^{-\ell+\frac{1}{2}}\rho \times \nu^{\alpha}\zeta(\rho_0,k) \rtimes \zeta(\rho,\ell-1;\sigma).$$

Set $\pi' = \nu^{-\ell+\frac{1}{2}}\rho \times \nu^{\alpha}\zeta(\rho_0,k) \rtimes \zeta(\rho,\ell-1;\sigma)$ and $\pi^{(i)} = \nu^{-\ell+\frac{1}{2}}\rho \rtimes L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\Delta_i(k,\alpha))$. Observe that $\pi^{(i)} < \pi'$. We claim that $\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\Delta_i(k,\alpha))$ occurs with mutiplicity one in $s_{sc}\pi$, $s_{sc}\pi'$ and $s_{sc}\pi^{(i)}$. Multiplicity one in $s_{sc}\pi$ is trivial–$s_{sc}\pi$ is written out above and $\nu^{\alpha}\zeta(\rho_0,k) \rtimes \sigma$ decomposes with multiplicity one. That $s_{sc}\pi'$ contains $\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\Delta_i(k,\alpha))$ with multiplicity one follows from μ^* calculations. This implies multiplicity (at most, and therefore exactly) one in $s_{sc}\pi^{(i)}$, as claimed. From this, it follows that $\pi^{(i)}$ and π have a component in common, characterized by having $\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\Delta_i(k,\alpha))$ in its s_{sc}. Call this representation π_i. By the preceding paragraph, we have $\pi = \sum_i \pi_i$.

We now identify π_i. Since $\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\Delta_i(k,\alpha))$ appears with multiplicity one in $s_{sc}\pi^{(i)}$, by Frobenius reciprocity, $\pi^{(i)}$ has a unique irreducible subrepresentation, necessarily π_i. Thus,

$$\begin{aligned}\pi_i \hookrightarrow\ & \nu^{-\ell+\frac{1}{2}}\rho \rtimes L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\Delta_i(k,\alpha))\\ \hookrightarrow\ & \nu^{-\ell+\frac{1}{2}}\rho \rtimes I([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\Delta_i(k,\alpha))\\ \cong\ & I([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\Delta_i(k,\alpha))\end{aligned}$$

by the usual commuting of ρ and ρ_0 terms (recall that $I(\Delta)$ is the representation obtained by inducing the Langlands data Δ; see section 2). Since $I([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\Delta_i(k,\alpha))$ has unique irreducible subrepresentation $L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\Delta_i(k,\alpha))$, we see that

$$\pi_i = L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\Delta_i(k,\alpha)),$$

as needed. □

Lemma **5.4.** *With hypotheses as in Proposition 5.3, let* $\Delta_i = \Delta_i(k,\alpha)$. *Then, for* $\pi = \nu^{\alpha}\zeta(\rho_0,k) \rtimes \zeta(\rho,\ell;\sigma)$ *write* $\pi = \sum_i \pi_i$ *with* $\pi_i = L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\Delta_i)$. *Then,*

(1)

$$\pi_i \hookrightarrow \nu^{-\frac{\ell}{2}}\zeta(\rho,\ell) \rtimes L(\Delta_i)$$

In particular, this means $s_{(\ell p)}\pi_i \geq \nu^{-\frac{\ell}{2}}\zeta(\rho,\ell) \otimes L(\Delta_i)$.

(2) $\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell) \otimes L(\Delta_i)$ *appears with multiplicity one in* $s_{(\ell p)}\pi$.

Proof. For (1),

$$\begin{aligned}\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell) \rtimes L(\Delta_i) \hookrightarrow\ & \nu^{-\ell+\frac{1}{2}}\rho \times \ldots \times \nu^{-\frac{1}{2}}\rho \rtimes I(\Delta_i)\\ \cong\ & I([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\Delta_i)\end{aligned}$$

by the usual argument commuting ρ-terms around ρ_0-terms ($\nu^{\beta}\rho \times \delta \cong \delta \times \nu^{\beta}\rho$ for any $\beta \in \mathbb{R}$ and any $\delta \in \Delta_i$). Note that π_i is the unique irreducible subrepresentation of the

$I([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \Delta_i)$. Therefore π_i is the unique irreducible subrepresentation of $\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell) \rtimes L(\Delta_i)$. The claim then follows from Frobenius reciprocity.

For (2), observe that in the formula for $\mu^*\pi$ (cf. Corollary 2.6), we see that to have the form $\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell) \otimes \dots$, we must have $j=\ell$, $t=0$, $i=k$. That gives $\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\otimes[\nu^{\alpha}\zeta(\rho_0,k)\rtimes\sigma]$. Since $L(\Delta_i)$ appears with multiplicity one in $\nu^{\alpha}\zeta(\rho_0,k)\rtimes\sigma$, the claim follows. □

Corollary **5.5.** *Suppose that σ is an irreducible supercuspidal representation of S_m and ρ a representation of $GL_p(F)$ satisfying (C1/2). Let $\pi = \nu^{\alpha}\zeta(\rho_0,k)\rtimes\zeta(\rho,\ell;\sigma)$. Suppose that $\nu^{\alpha}\zeta(\rho_0,k)\rtimes\sigma$, $\nu^{\alpha+\frac{1}{2}}\zeta(\rho_0,k-1)\rtimes\sigma$ and $\nu^{\alpha-\frac{1}{2}}\zeta(\rho_0,k)\rtimes\sigma$ satisfy the multiplicity one hypothesis in Proposition 5.3. Then, with notation as in Lemma 5.4, write*

$$s_{(p_0)}L(\Delta_i) = \sum_j \nu^{\alpha_j}\rho_0 \otimes L(\Delta_i^{(j)})$$

(n.b. the α_j need not be distinct, but there are at most two possible values for α_j). Then,

$$\begin{aligned} s_{sc}\pi_i &= \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \Delta_i) \\ &+\sum_j \nu^{\alpha_j}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \Delta_i^{(j)}). \end{aligned}$$

Proof. First, recall that from the proof of Proposition 5.3, we have $s_{sc}\pi_i \geq \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \Delta_i)$.

Next, we claim $s_{sc}\pi_i \geq \sum_j \nu^{\alpha_j}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \Delta_i^{(j)})$. First, from Lemma 5.4 above,

$$\begin{gathered} s_{(\ell p)}\pi_i \geq \nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\otimes L(\Delta_i) \\ \Downarrow \\ s_{(\ell p, p_0)}\pi_i \geq \sum_j \nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\otimes \nu^{\alpha_j}\rho_0 \otimes L(\Delta_i^{(j)}) \\ \Downarrow \\ s_{(p_0,\ell p)}\pi_i \geq \sum_j \nu^{\alpha_j}\rho_0 \otimes \nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\otimes L(\Delta_i^{(j)}). \end{gathered}$$

since $\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\times\nu^{\alpha_j}\rho_0 \cong \nu^{\alpha_j}\rho_0\times\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)$ (as both are irreducible, cf. Theorem 2.1). Now, observe that since

$$\begin{aligned} s_{sc}\pi &= \nu^{\alpha+\frac{-k+1}{2}}\rho_0 \otimes \nu^{\alpha+\frac{1}{2}}\zeta(\rho_0,k-1)\rtimes\zeta(\rho,\ell;\sigma) \\ &+\nu^{-\alpha+\frac{-k+1}{2}}\rho_0 \otimes \nu^{\alpha-\frac{1}{2}}\zeta(\rho_0,k-1)\rtimes\zeta(\rho,\ell;\sigma) \\ &+\nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{\alpha}\zeta(\rho_0,k)\rtimes\zeta(\rho,\ell-1;\sigma), \end{aligned}$$

if $\alpha_j = \alpha + \frac{-k+1}{2}$, then we have that $\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\otimes L(\Delta_i^{(j)})$ must come from a component of $\nu^{\alpha+\frac{1}{2}}\zeta(\rho_0,k-1)\rtimes\zeta(\rho,\ell;\sigma)$ (and similarly for $\alpha_j = -\alpha+\frac{-k+1}{2}$; if $\alpha=0$ their contributions are the same). By Lemma 5.4, this implies

$$s_{(p_0)}\pi_i \geq \sum_j \nu^{\alpha_j}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \Delta_i^{(j)}).$$

As we run through the different values of i, this covers all of

$$\nu^{\alpha+\frac{-k+1}{2}}\rho_0 \otimes \nu^{\alpha+\frac{1}{2}}\zeta(\rho_0, k-1) \rtimes \zeta(\rho, \ell; \sigma) + \nu^{-\alpha+\frac{-k+1}{2}}\rho_0 \otimes \nu^{\alpha-\frac{1}{2}}\zeta(\rho_0, k-1) \rtimes \zeta(\rho, \ell; \sigma),$$

so this covers everything of the form $\nu^{\alpha_j}\rho_0 \otimes \dots$ in $s_{sc}\pi_i$. Therefore,

$$\begin{aligned} s_{sc}\pi_i &= \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \Delta_i) \\ &+ \sum_j \nu^{\alpha_j}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \Delta_i^{(j)}), \end{aligned}$$

as claimed. □

Remark **5.6.** *It is also worth commenting on $s_{min}\pi$, $s_{min}\pi_i$ (cf. Definition 5.1). It is not difficult to see that $s_{min}\pi$ is as described after Theorem 2.3 (without the restriction $p_0 = p$, of course) and*

$$s_{min}\pi_i = \text{ shuffles of } \nu^{-\ell+\frac{1}{2}}\rho \otimes \dots \nu^{-\frac{1}{2}}\rho \text{ and } s_{min}L(\Delta_i).$$

More precisely, if $s_{min}L(\Delta_i) = \sum_j \theta_j$, we mean

$$s_{min}\pi_i = \sum_j \text{ shuffles of } \nu^{-\ell+\frac{1}{2}}\rho \otimes \dots \otimes \nu^{-\frac{1}{2}}\rho \text{ and } \theta_j.$$

Corollary **5.7.** *Suppose σ is an irreducible supercuspidal representation of S_m, ρ, ρ_0 irreducible unitarizable supercuspidal representations of $GL_p(F)$, $GL_{p_0}(F)$. Further, suppose both ρ and ρ_0 satisfy (C1/2). Let $\pi = \nu^{\alpha}\zeta(\rho_0, k) \rtimes \zeta(\rho, \ell; \sigma)$, $\alpha \in \mathbb{R}$. Then, π is reducible if and only if $\alpha \in \{-\frac{k}{2}, -\frac{k}{2}+1, \dots, \frac{k}{2}\}$. Suppose π is reducible. By Lemma 2.9, we may without loss of generality assume that $\alpha \leq 0$. Write $\alpha = -\frac{k}{2} + j$, $0 \leq j \leq \frac{k}{2}$. Since $\ell = 0$ is covered by Proposition 3.6, we may assume $\ell \geq 1$.*

(1) $j = 0$

$\pi = \pi_1 + \pi_2$ *with*

$$\pi_1 = L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-k+\frac{1}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0]; \sigma)$$

$$\pi_2 = L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-k+\frac{1}{2}}\rho_0, \nu^{-\frac{3}{2}}\rho_0]; \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma))$$

(a) $k = 1$

$$s_{sc}\pi_1 = \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\frac{1}{2}}\rho_0; \sigma) + \nu^{-\frac{1}{2}}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$$

$$s_{sc}\pi_2 = \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma)) + \nu^{\frac{1}{2}}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$$

(b) $k > 1$

$$\begin{aligned} s_{sc}\pi_1 &= \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-k+\frac{1}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0]; \sigma) \\ &+\nu^{-k+\frac{1}{2}}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-k+\frac{3}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0]; \sigma) \end{aligned}$$

$$\begin{aligned} s_{sc}\pi_2 &= \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-k+\frac{1}{2}}\rho_0, \nu^{-\frac{3}{2}}\rho_0]; \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma)) \\ &+\nu^{-k+\frac{1}{2}}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-k+\frac{3}{2}}\rho_0, \nu^{-\frac{3}{2}}\rho_0]; \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma)) \\ &+\nu^{\frac{1}{2}}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-k+\frac{1}{2}}\rho_0, \nu^{-\frac{3}{2}}\rho_0]; \sigma) \end{aligned}$$

(2) $1 \leq j < \frac{k}{2}$

$\pi = \pi_1 + \pi_2 + \pi_3$ *with*

$$\pi_1 = L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-k+j+\frac{1}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0], [\nu^{-j+\frac{1}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0]; \sigma)$$

$$\pi_2 = L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-k+j+\frac{1}{2}}\rho_0, \nu^{-\frac{3}{2}}\rho_0], [\nu^{-j+\frac{1}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0]; \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma))$$

$$\pi_3 = L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-k+j+\frac{1}{2}}\rho_0, \nu^{-j-\frac{3}{2}}\rho_0], \nu^{-j}\delta(\rho_0, 2), \ldots, \nu^{-1}\delta(\rho_0, 2); \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma))$$

(a) $j = 1 = \frac{k-1}{2}$ *(so $k = 3$)*

$$\begin{aligned} s_{sc}\pi_1 &= \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\frac{3}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0; \sigma) \\ &+\nu^{-\frac{1}{2}}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\frac{3}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0; \sigma) \\ &+\nu^{-\frac{3}{2}}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\frac{1}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0; \sigma) \end{aligned}$$

$$\begin{aligned} s_{sc}\pi_2 &= \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\frac{3}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0; \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma)) \\ &+\nu^{-\frac{3}{2}}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\frac{1}{2}}\rho_0; \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma)) \end{aligned}$$

$$\begin{aligned} s_{sc}\pi_3 &= \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-1}\delta(\rho_0, 2); \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma)) \\ &+\nu^{-\frac{1}{2}}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\frac{3}{2}}\rho_0; \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma)) \end{aligned}$$

(b) $j = 1$, $k > 3$

$$\begin{aligned} s_{sc}\pi_1 &= \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-k+\frac{3}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0], \nu^{-\frac{1}{2}}\rho_0; \sigma) \\ &+\nu^{-k+\frac{3}{2}}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-k+\frac{5}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0], \nu^{-\frac{1}{2}}\rho_0; \sigma) \\ &+\nu^{-\frac{1}{2}}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-k+\frac{3}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0]; \sigma) \end{aligned}$$

$$\begin{aligned} s_{sc}\pi_2 &= \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-k+\frac{3}{2}}\rho_0, \nu^{-\frac{3}{2}}\rho_0], \nu^{-\frac{1}{2}}\rho_0; \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma)) \\ &+\nu^{-k+\frac{3}{2}}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-k+\frac{5}{2}}\rho_0, \nu^{-\frac{3}{2}}\rho_0], \nu^{-\frac{1}{2}}\rho_0; \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma)) \end{aligned}$$

$$\begin{aligned} s_{sc}\pi_3 &= \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-k+\frac{3}{2}}\rho_0, \nu^{-\frac{5}{2}}\rho_0], \nu^{-1}\delta(\rho_0, 2); \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma)) \\ &+\nu^{-k+\frac{3}{2}}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-k+\frac{5}{2}}\rho_0, \nu^{-\frac{5}{2}}\rho_0], \nu^{-1}\delta(\rho_0, 2); \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma)) \\ &+\nu^{-\frac{1}{2}}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-k+\frac{3}{2}}\rho_0, \nu^{-\frac{3}{2}}\rho_0]; \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma)) \end{aligned}$$

(c) $j=\frac{k-1}{2}$, $k>3$ *(k odd)*

$$\begin{aligned}
s_{sc}\pi_1 &= \nu^{-\ell+\frac{1}{2}}\rho\otimes L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\frac{k}{2}}\rho_0,\nu^{-\frac{1}{2}}\rho_0],[\nu^{-\frac{k}{2}+1}\rho_0,\nu^{-\frac{1}{2}}\rho_0];\sigma)\\
&+\nu^{-\frac{k}{2}}\rho_0\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\frac{k}{2}+1}\rho_0,\nu^{-\frac{1}{2}}\rho_0],[\nu^{-\frac{k}{2}+1}\rho_0,\nu^{-\frac{1}{2}}\rho_0];\sigma)\\
&+\nu^{-\frac{k}{2}+1}\rho_0\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\frac{k}{2}}\rho_0,\nu^{-\frac{1}{2}}\rho_0],[\nu^{-\frac{k}{2}+2}\rho_0,\nu^{-\frac{1}{2}}\rho_0];\sigma)
\end{aligned}$$

$$\begin{aligned}
s_{sc}\pi_2 &= \nu^{-\ell+\frac{1}{2}}\rho\otimes L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\frac{k}{2}}\rho_0,\nu^{-\frac{3}{2}}\rho_0],\\
&\qquad[\nu^{-\frac{k}{2}+1}\rho_0,\nu^{-\frac{1}{2}}\rho_0];\delta(\nu^{-\frac{1}{2}}\rho_0;\sigma))\\
&+\nu^{-\frac{k}{2}}\rho_0\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\frac{k}{2}+1}\rho_0,\nu^{-\frac{3}{2}}\rho_0],\\
&\qquad[\nu^{-\frac{k}{2}+1}\rho_0,\nu^{-\frac{1}{2}}\rho_0];\delta(\nu^{-\frac{1}{2}}\rho_0;\sigma))\\
&+\nu^{-\frac{k}{2}+1}\rho_0\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\frac{k}{2}}\rho_0,\nu^{-\frac{3}{2}}\rho_0],\\
&\qquad[\nu^{-\frac{k}{2}+2}\rho_0,\nu^{-\frac{1}{2}}\rho_0];\delta(\nu^{-\frac{1}{2}}\rho_0;\sigma))
\end{aligned}$$

$$\begin{aligned}
s_{sc}\pi_3 &= \nu^{-\ell+\frac{1}{2}}\rho\otimes L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\nu^{\frac{-k+1}{2}}\delta(\rho_0,2),\dots,\nu^{-1}\delta(\rho_0,2);\delta(\nu^{-\frac{1}{2}}\rho_0;\sigma))\\
&+\nu^{-\frac{k}{2}+1}\rho_0\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\nu^{-\frac{k}{2}}\rho_0,\nu^{\frac{-k+3}{2}}\delta(\rho_0,2),\\
&\qquad\dots,\nu^{-1}\delta(\rho_0,2);\delta(\nu^{-\frac{1}{2}}\rho_0;\sigma))
\end{aligned}$$

(d) $1<j<\frac{k-1}{2}$

$$\begin{aligned}
s_{sc}\pi_1 &= \nu^{-\ell+\frac{1}{2}}\rho\otimes L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-k+j+\frac{1}{2}}\rho_0,\nu^{-\frac{1}{2}}\rho_0],[\nu^{-j+\frac{1}{2}}\rho_0,\nu^{-\frac{1}{2}}\rho_0];\sigma)\\
&+\nu^{-k+j+\frac{1}{2}}\rho_0\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-k+j+\frac{3}{2}}\rho_0,\nu^{-\frac{1}{2}}\rho_0],[\nu^{-j+\frac{1}{2}}\rho_0,\nu^{-\frac{1}{2}}\rho_0];\sigma)\\
&+\nu^{-j+\frac{1}{2}}\rho_0\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-k+j+\frac{1}{2}}\rho_0,\nu^{-\frac{1}{2}}\rho_0],[\nu^{-j+\frac{3}{2}}\rho_0,\nu^{-\frac{1}{2}}\rho_0];\sigma)
\end{aligned}$$

$$\begin{aligned}
s_{sc}\pi_2 &= \nu^{-\ell+\frac{1}{2}}\rho\otimes L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-k+j+\frac{1}{2}}\rho_0,\nu^{-\frac{3}{2}}\rho_0],\\
&\qquad[\nu^{-j+\frac{1}{2}}\rho_0,\nu^{-\frac{1}{2}}\rho_0];\delta(\nu^{-\frac{1}{2}}\rho_0;\sigma))\\
&+\nu^{-k+j+\frac{1}{2}}\rho_0\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-k+j+\frac{3}{2}}\rho_0,\nu^{-\frac{3}{2}}\rho_0],\\
&\qquad[\nu^{-j+\frac{1}{2}}\rho_0,\nu^{-\frac{1}{2}}\rho_0];\delta(\nu^{-\frac{1}{2}}\rho_0;\sigma))\\
&+\nu^{-j+\frac{1}{2}}\rho_0\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-k+j+\frac{1}{2}}\rho_0,\nu^{-\frac{3}{2}}\rho_0],\\
&\qquad[\nu^{-j+\frac{3}{2}}\rho_0,\nu^{-\frac{1}{2}}\rho_0];\delta(\nu^{-\frac{1}{2}}\rho_0;\sigma))
\end{aligned}$$

$$\begin{aligned}
s_{sc}\pi_3 &= \nu^{-\ell+\frac{1}{2}}\rho\otimes L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-k+j+\frac{1}{2}}\rho_0,\nu^{-j-\frac{3}{2}}\rho_0],\nu^{-j}\delta(\rho_0,2),\\
&\qquad\dots,\nu^{-1}\delta(\rho_0,2);\delta(\nu^{-\frac{1}{2}}\rho_0;\sigma))\\
&+\nu^{-k+j+\frac{1}{2}}\rho_0\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-k+j+\frac{3}{2}}\rho_0,\nu^{-j-\frac{3}{2}}\rho_0],\nu^{-j}\delta(\rho_0,2),\\
&\qquad\dots,\nu^{-1}\delta(\rho_0,2);\delta(\nu^{-\frac{1}{2}}\rho_0;\sigma))\\
&+\nu^{-j+\frac{1}{2}}\rho_0\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-k+j+\frac{1}{2}}\rho_0,\nu^{-j-\frac{1}{2}}\rho_0],\nu^{-j+1}\delta(\rho_0,2),\\
&\qquad\dots,\nu^{-1}\delta(\rho_0,2);\delta(\nu^{-\frac{1}{2}}\rho_0;\sigma))
\end{aligned}$$

(3) $j=\frac{k}{2}$ *(k even)*

$\pi = \pi_1 + \pi_2$ *with*

$$\pi_1 = L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\frac{-k+1}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0], [\nu^{\frac{-k+1}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0]; \sigma)$$

$$\pi_2 = L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\frac{-k+1}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0], [\nu^{\frac{-k+1}{2}}\rho_0, \nu^{-\frac{3}{2}}\rho_0]; \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma))$$

(a) $k = 2$

$$\begin{aligned} s_{sc}\pi_1 &= \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\frac{1}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0; \sigma) \\ &+2\nu^{-\frac{1}{2}}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\frac{1}{2}}\rho_0; \sigma) \\ &+\nu^{-\frac{1}{2}}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma)) \end{aligned}$$

$$\begin{aligned} s_{sc}\pi_2 &= \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\frac{1}{2}}\rho_0; \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma)) \\ &+\nu^{-\frac{1}{2}}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma)) \end{aligned}$$

(b) $k > 2$

$$\begin{aligned} s_{sc}\pi_1 &= \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\frac{-k+1}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0], [\nu^{\frac{-k+1}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0]; \sigma) \\ &+2\nu^{\frac{-k+1}{2}}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\frac{-k+1}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0], [\nu^{\frac{-k+3}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0]; \sigma) \\ &+\nu^{\frac{-k+1}{2}}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\frac{k}{2}+1}\delta(\rho_0, 2), \\ &\qquad \ldots, \nu^{-1}\delta(\rho_0, 2); \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma)) \end{aligned}$$

$$\begin{aligned} s_{sc}\pi_2 &= \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\frac{-k+1}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0], \\ &\qquad [\nu^{\frac{-k+1}{2}}\rho_0, \nu^{-\frac{3}{2}}\rho_0]; \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma)) \\ &+2\nu^{\frac{-k+1}{2}}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\frac{-k+3}{2}}\rho_0, \nu^{-\frac{1}{2}}\rho_0], \\ &\qquad [\nu^{\frac{-k+1}{2}}\rho_0, \nu^{-\frac{3}{2}}\rho_0]; \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma)) \\ &+\nu^{\frac{-k+1}{2}}\rho_0 \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\frac{k}{2}+1}\delta(\rho_0, 2), \\ &\qquad \ldots, \nu^{-1}\delta(\rho_0, 2); \delta(\nu^{-\frac{1}{2}}\rho_0; \sigma)) \end{aligned}$$

Proof. This is an immediate consequence of Proposition 3.6 used with Proposition 5.3 and Corollary 5.5. □

We now give the counterpart to this for the case ρ satisfies (C1) and ρ_0 satisfies (C0). Note that if we assume that ρ satisfies (C1) instead of (C1/2), Lemmas 5.3 and 5.5 still hold. The only change required is to replace $\nu^{-\ell+\frac{1}{2}}\rho, \ldots, \nu^{-\frac{1}{2}}\rho$ with $\nu^{-\ell}\rho, \ldots, \nu^{-1}\rho$, resp.

Corollary **5.8.** *Suppose σ is an irreducible supercuspidal representation of S_m, ρ, ρ_0 irreducible unitarizable supercuspidal representations of $GL_p(F)$, $GL_{p_0}(F)$. Further, suppose ρ satisfies (C1) and ρ_0 satisfies (C0). Let $\rho_0 \rtimes \sigma = T_1 + T_2$. Let $\pi = \nu^{\alpha}\zeta(\rho_0, k) \rtimes \zeta(\rho, \ell; \sigma)$, $\alpha \in \mathbb{R}$. Then, π is reducible if and only if $\alpha \in \{\frac{-k+1}{2}, \frac{-k+1}{2} + 1, \ldots, \frac{k-1}{2}\}$. Suppose π is reducible. By Lemma 2.9, we may without loss of generality assume that $\alpha \leq 0$. Write $\alpha = \frac{-k+1}{2} + j$ with $0 \leq j \leq \frac{k-1}{2}$. Since $\ell = 0$ is covered by Proposition 3.11, we may assume $\ell \geq 1$.*

(1) $j = \frac{k-1}{2}$

$\pi=\pi_1+\pi_2$ *with*

$$\pi_i=L([\nu^{-\ell}\rho,\nu^{-1}\rho],[\nu^{\frac{-k+1}{2}}\rho_0,\nu^{-1}\rho_0],[\nu^{\frac{-k+1}{2}}\rho_0,\nu^{-1}\rho_0];T_i)$$

for $i=1,2$.

(a) $k=1$

$$s_{sc}\pi_i=\nu^{-\ell}\rho\otimes L([\nu^{-\ell+1}\rho,\nu^{-1}\rho];T_i)+\rho_0\otimes L([\nu^{-\ell}\rho,\nu^{-1}\rho];\sigma)$$

for $i=1,2$.

(b) $k>1$

$$\begin{aligned}s_{sc}\pi_i &= \nu^{-\ell}\rho\otimes L([\nu^{-\ell+1}\rho,\nu^{-1}\rho],[\nu^{\frac{-k+1}{2}}\rho_0,\nu^{-1}\rho_0],[\nu^{\frac{-k+1}{2}}\rho_0,\nu^{-1}\rho_0];T_i)\\ &+2\nu^{\frac{-k+1}{2}}\rho_0\otimes L([\nu^{-\ell}\rho,\nu^{-1}\rho],[\nu^{\frac{-k+1}{2}}\rho_0,\nu^{-1}\rho_0],[\nu^{\frac{-k+3}{2}}\rho_0,\nu^{-1}\rho_0];T_i)\\ &+\nu^{\frac{-k+1}{2}}\rho_0\otimes L([\nu^{-\ell}\rho,\nu^{-1}\rho],\nu^{-\frac{k}{2}+1}\delta(\rho_0,2),\dots,\nu^{-\frac{1}{2}}\delta(\rho_0,2);\sigma)\end{aligned}$$

for $i=1,2$.

(2) $0\le j<\frac{k-1}{2}$

$\pi=\pi_1+\pi_2+\pi_3$ *with*

$$\pi_i=L([\nu^{-\ell}\rho,\nu^{-1}\rho],[\nu^{-k+j+1}\rho_0,\nu^{-1}\rho_0],[\nu^{-j}\rho_0,\nu^{-1}\rho_0];T_i)$$

for $i=1,2$ *and*

$$\begin{aligned}\pi_3=L(&[\nu^{-\ell}\rho,\nu^{-1}\rho],[\nu^{-k+j+1}\rho_0,\nu^{-j-2}\rho_0],\\ &\nu^{-j-\frac{1}{2}}\delta(\rho_0,2),\nu^{-j+\frac{1}{2}}\delta(\rho_0,2),\dots,\nu^{-\frac{1}{2}}\delta(\rho_0,2);\sigma).\end{aligned}$$

(a) $j=0=\frac{k-2}{2}$ *($k=2$)*

$$\begin{aligned}s_{sc}\pi_i &= \nu^{-\ell}\rho\otimes L([\nu^{-\ell+1}\rho,\nu^{-1}\rho],\nu^{-1}\rho_0;T_i)\\ &+\nu^{-1}\rho_0\otimes L([\nu^{-\ell}\rho,\nu^{-1}\rho];T_i)\end{aligned}$$

for $i=1,2$. *Further,*

$$\begin{aligned}s_{sc}\pi_3 &= \nu^{-\ell}\rho\otimes L([\nu^{-\ell+1}\rho,\nu^{-1}\rho],\nu^{-\frac{1}{2}}\delta(\rho_0,2);\sigma)\\ &+\rho_0\otimes L([\nu^{-\ell}\rho,\nu^{-1}\rho],\nu^{-1}\rho_0;\sigma).\end{aligned}$$

(b) $j=0,\ k>2$

$$\begin{aligned}s_{sc}\pi_i &= \nu^{-\ell}\rho\otimes L([\nu^{-\ell+1}\rho,\nu^{-1}\rho],[\nu^{-k+1}\rho_0,\nu^{-1}\rho_0];T_i)\\ &+\nu^{-k+1}\rho_0\otimes L([\nu^{-\ell}\rho,\nu^{-1}\rho],[\nu^{-k+2}\rho_0,\nu^{-1}\rho_0];T_i)\end{aligned}$$

for $i = 1,2$. Further,

$$\begin{aligned} s_{sc}\pi_3 &= \nu^{-\ell}\rho \otimes L([\nu^{-\ell+1}\rho, \nu^{-1}\rho], [\nu^{-k+1}\rho_0, \nu^{-2}\rho_0], \nu^{-\frac{1}{2}}\delta(\rho_0, 2); \sigma) \\ &+\nu^{-k+1}\rho_0 \otimes L([\nu^{-\ell}\rho, \nu^{-1}\rho], [\nu^{-k+2}\rho_0, \nu^{-2}\rho_0], \nu^{-\frac{1}{2}}\delta(\rho_0, 2); \sigma) \\ &+\rho_0 \otimes L([\nu^{-\ell}\rho, \nu^{-1}\rho], [\nu^{-k+1}\rho_0, \nu^{-1}\rho_0]; \sigma). \end{aligned}$$

(c) $j = \frac{k-2}{2}$, $k \geq 4$ *(k even)*

$$\begin{aligned} s_{sc}\pi_i &= \nu^{-\ell}\rho \otimes L([\nu^{-\ell+1}\rho, \nu^{-1}\rho], [\nu^{-\frac{k}{2}}\rho_0, \nu^{-1}\rho_0], [\nu^{-\frac{k}{2}+1}\rho_0, \nu^{-1}\rho_0]; T_i) \\ &+\nu^{-\frac{k}{2}}\rho_0 \otimes L([\nu^{-\ell}\rho, \nu^{-1}\rho], [\nu^{-\frac{k}{2}+1}\rho_0, \nu^{-1}\rho_0], [\nu^{-\frac{k}{2}+1}\rho_0, \nu^{-1}\rho_0]; T_i) \\ &+\nu^{-\frac{k}{2}+1}\rho_0 \otimes L([\nu^{-\ell}\rho, \nu^{-1}\rho], [\nu^{-\frac{k}{2}}\rho_0, \nu^{-1}\rho_0], [\nu^{-\frac{k}{2}+2}\rho_0, \nu^{-1}\rho_0]; T_i) \end{aligned}$$

for $i = 1,2$. Further,

$$\begin{aligned} s_{sc}\pi_3 &= \nu^{-\ell}\rho \otimes L([\nu^{-\ell+1}\rho, \nu^{-1}\rho], \nu^{\frac{-k+1}{2}}\delta(\rho_0, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho_0, 2); \sigma) \\ &+\nu^{-\frac{k}{2}+1}\rho_0 \otimes L([\nu^{-\ell}\rho, \nu^{-1}\rho], \nu^{-\frac{k}{2}}\rho_0, \nu^{\frac{-k+3}{2}}\delta(\rho_0, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho_0, 2); \sigma). \end{aligned}$$

(d) $0 < j < \frac{k-2}{2}$

$$\begin{aligned} s_{sc}\pi_i &= \nu^{-\ell}\rho \otimes L([\nu^{-\ell+1}\rho, \nu^{-1}\rho], [\nu^{-k+j+1}\rho_0, \nu^{-1}\rho_0], [\nu^{-j}\rho_0, \nu^{-1}\rho_0]; T_i) \\ &+\nu^{-k+j+1}\rho_0 \otimes L([\nu^{-\ell}\rho, \nu^{-1}\rho], [\nu^{-k+j+2}\rho_0, \nu^{-1}\rho_0], [\nu^{-j}\rho_0, \nu^{-1}\rho_0]; T_i) \\ &+\nu^{-j}\rho_0 \otimes L([\nu^{-\ell}\rho, \nu^{-1}\rho], [\nu^{-k+j+1}\rho_0, \nu^{-1}\rho_0], [\nu^{-j+1}\rho_0, \nu^{-1}\rho_0]; T_i) \end{aligned}$$

for $i = 1,2$. Further,

$$\begin{aligned} s_{sc}\pi_3 &= \nu^{-\ell}\rho \otimes L([\nu^{-\ell+1}\rho, \nu^{-1}\rho], [\nu^{-k+j+1}\rho_0, \nu^{-j-2}\rho_0], \\ &\qquad \nu^{-j-\frac{1}{2}}\delta(\rho_0, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho_0, 2); \sigma) \\ &+\nu^{-k+j+1}\rho_0 \otimes L([\nu^{-\ell}\rho, \nu^{-1}\rho], [\nu^{-k+j+2}\rho_0, \nu^{-j-2}\rho_0], \\ &\qquad \nu^{-j-\frac{1}{2}}\delta(\rho_0, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho_0, 2); \sigma) \\ &+\nu^{-j}\rho_0 \otimes L([\nu^{-\ell}\rho, \nu^{-1}\rho], [\nu^{-k+j+1}\rho_0, \nu^{-j-1}\rho_0], \\ &\qquad \nu^{-j+\frac{1}{2}}\delta(\rho_0, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho_0, 2); \sigma). \end{aligned}$$

Proof. This follows from Proposition 3.11 and the analogues of Proposition 5.3 and Proposition 5.5 for the case when ρ satisfies (C1). □

6. Components: the "unramified" case

In this section, we give the components of $\nu^\alpha\zeta(\rho,k)\rtimes\zeta(\rho,\ell;\sigma)$, $\alpha\in\mathbb{R}$. For ρ satisfying (C1/2), the results are in Theorem 6.1; for ρ satisfying (C1), the results are in Theorem 6.2. The techniques are basically the same as those used in section 3 to get the components of $\nu^\alpha\zeta(\rho,n)\rtimes\sigma$–the argument is inductive; $s_{(p)}$ is the main tool. Of course, things are more complicated here. Note that Jacquet modules for $k\geq 2$, $\ell\geq 1$ are given in abbreviated form in the tables in this section (to save space). The cases $k=1$ and $\ell=0$ are both covered in section 3.

Theorem **6.1.** *Suppose σ is an irreducible supercuspidal representation of S_m and ρ is a representation of $GL_p(F)$ satisfying (C1/2). Let $\pi=\nu^\alpha\zeta(\rho,k)\rtimes\zeta(\rho,\ell;\sigma)$ with $\alpha\in\mathbb{R}$. Recall that π is reducible if and only if $\alpha\in\{\pm(\ell+\frac{k}{2}),\pm(\ell+\frac{k}{2}-1),\dots,\pm(\ell-\frac{k}{2}+1)\}\cup\{\{-\frac{k}{2},-\frac{k}{2}+1,\dots,\frac{k}{2}\}\setminus\{0 \text{ if } k=2\ell\}\}$. Let S_1 denote the first set; S_2 the second. Suppose π is reducible. By Lemma 2.9, without loss of generality, we may restrict our attention to $\alpha\leq 0$. Note: $\mathcal{T}$ is described in Proposition 3.1.*

(1) $\alpha\in S_1$, $\alpha\notin S_2$
  $\pi=\pi_1+\pi_2$ *with*

$$\pi_1=L([\nu^{\alpha+\frac{-k+1}{2}}\rho,\nu^{\alpha+\frac{k-1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)$$

$$\pi_2=L([\nu^{\alpha+\frac{-k+1}{2}}\rho,\nu^{-\ell-\frac{3}{2}}\rho],\nu^{-\ell}\delta(\rho,2),\nu^{-\ell+1}\delta(\rho,2),\dots,\nu^{\alpha+\frac{k}{2}}\delta(\rho,2),[\nu^{\alpha+\frac{k+3}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)$$

(2) $\alpha\in S_2$, $\alpha\notin S_1$
  Write $\alpha=-\frac{k}{2}+j$, $0\leq j\leq\frac{k}{2}$.
  (a) $j=k-\ell$ ($j<k-j=\ell$)
   $\pi=\pi_1+\pi_2$ *with*

$$\pi_1=L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-k+\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)$$

$$\pi_2=L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-k+\ell-\frac{3}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{3}{2}}\rho],\nu^{-k+\ell}\delta(\rho,2),\nu^{-k+\ell+1}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\mathcal{T})$$

  (b) $k-\ell<j<\frac{k}{2}$ ($j<k-j<\ell$)
   (i) $j=0$
    $\pi=\pi_1+\pi_2$ *with*

$$\pi_1=L([\nu^{-k+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)$$

$$\pi_2=L([\nu^{-k+\frac{1}{2}}\rho,\nu^{-\frac{3}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{3}{2}}\rho];\mathcal{T})$$

   (ii) $j>0$

$\pi = \pi_1 + \pi_2 + \pi_3$ *with*

$$\pi_1 = L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$$

$$\pi_2 = L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-j-\frac{3}{2}}\rho], [\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-j}\delta(\rho,2), \nu^{-j+1}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \mathcal{T})$$

$$\pi_3 = L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-k+j-\frac{3}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-k+j}\delta(\rho,2), \nu^{-k+j+1}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \mathcal{T})$$

(c) $j = \frac{k}{2}$, k *even* $(j = k - j < \ell)$
$\pi = \pi_1 + \pi_2$ *with*

$$\pi_1 = L([\nu^{\frac{-k+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$$

$$\pi_2 = L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{k}{2}-\frac{3}{2}}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\frac{k}{2}}\delta(\rho,2), \nu^{-\frac{k}{2}+1}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \mathcal{T})$$

(3) $\alpha \in S_1 \cap S_2$
Write $\alpha = -\frac{k}{2} + j$, $0 \le j \le \frac{k}{2}$.
(a) $j \le \ell - 1$ $(j < \ell < k - j)$
$\pi = \pi_1 + \pi_2 + \pi_3 + \pi_4$ *with*

$$\pi_1 = L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$$

$$\pi_2 = L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-j-\frac{3}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-j}\delta(\rho,2), \nu^{-j+1}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \mathcal{T})$$

$$\pi_3 = L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\ell-\frac{3}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell}\delta(\rho,2), \nu^{-\ell+1}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho; \sigma))$$

$$\pi_4 = L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\ell-\frac{3}{2}}\rho], \nu^{-\ell}\delta(\rho,2), \nu^{-\ell+1}\delta(\rho,2), \dots, \nu^{-j-2}\delta(\rho,2), \nu^{-j-\frac{1}{2}}\delta(\rho,3), \nu^{-j+\frac{1}{2}}\delta(\rho,3), \dots, \nu^{-\frac{1}{2}}\delta(\rho,3); \sigma)$$

(b) $j = \ell$ $(j = \ell < k - j)$
$\pi = \pi_1 + \pi_2$ *with*

$$\pi_1 = L([\nu^{-k+\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$$

$$\pi_2 = L([\nu^{-k+\ell+\frac{1}{2}}\rho, \nu^{-\ell-\frac{3}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell}\delta(\rho,2), \nu^{-\ell+1}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho; \sigma))$$

(c) $\ell < j < \frac{k}{2}$ $(\ell < j < k - j)$

$\pi=\pi_1+\pi_2+\pi_3$ *with*

$$\pi_1=L([\nu^{-k+j+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-j+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)$$

$$\begin{aligned}\pi_2=L([\nu^{-k+j+\frac{1}{2}}\rho,\nu^{-\ell-\frac{3}{2}}\rho],[\nu^{-j+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\\ \nu^{-\ell}\delta(\rho,2),\nu^{-\ell+1}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma))\end{aligned}$$

$$\begin{aligned}\pi_3=L([\nu^{-k+j+\frac{1}{2}}\rho,\nu^{-j-\frac{3}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\\ \nu^{-j}\delta(\rho,2),\nu^{-j+1}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma))\end{aligned}$$

(d) $j=\frac{k}{2}$, k *even* $(\ell<j=k-j)$

$\pi=\pi_1+\pi_2$ *with*

$$\pi_1=L([\nu^{\frac{-k+1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{\frac{-k+1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)$$

$$\begin{aligned}\pi_2=L([\nu^{\frac{-k+1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{\frac{-k+1}{2}}\rho,\nu^{-\ell-\frac{3}{2}}\rho],\\ \nu^{-\ell}\delta(\rho,2),\nu^{-\ell+1}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma))\end{aligned}$$

The proof of the theorem also gives us $s_{(p)}\pi_i$ for each component π_i of π (summarized in the tables); for reasons of space, we do not write them out here. (Note that the tables only give the Jacquet modules for $k\geq 2$, $\ell\geq 1$; for $k=1$ or $\ell=0$, see section 3.)

Proof. The proof is by induction on $k+\ell$ (not exactly the rank, but the same basic idea). Note that the cases $k=1$ and $\ell=0$ are done in section 3, so we may assume $k\geq 2$, $\ell\geq 1$. Also, we do case 1 ($\alpha\in S_1$, $\alpha\notin S_2$) last, so we may restrict our attention to cases 2 and 3 for the time being.

As in Proposition 3.6, the induction focuses on $s_{(p)}\pi$. For $\pi=\nu^{-\frac{k}{2}+j}\zeta(\rho,k)\rtimes\zeta(\rho,\ell;\sigma)$, we have

$$\begin{aligned}s_{(p)}\pi\;&=\nu^{-k+j+\frac{1}{2}}\rho\otimes\nu^{-\frac{k}{2}+j+\frac{1}{2}}\zeta(\rho,k-1)\rtimes\zeta(\rho,\ell;\sigma)\\ &+\nu^{-j+\frac{1}{2}}\rho\otimes\nu^{-\frac{k}{2}+j-\frac{1}{2}}\zeta(\rho,k-1)\rtimes\zeta(\rho,\ell;\sigma)\\ &+\nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{-\frac{k}{2}+j}\zeta(\rho,k)\rtimes\zeta(\rho,\ell-1;\sigma)\end{aligned}$$

Let

$$\tau'=\nu^{-k+j+\frac{1}{2}}\rho\otimes\nu^{-\frac{k}{2}+j+\frac{1}{2}}\zeta(\rho,k-1)\rtimes\zeta(\rho,\ell;\sigma)$$

$$\tau''=\nu^{-j+\frac{1}{2}}\rho\otimes\nu^{-\frac{k}{2}+j-\frac{1}{2}}\zeta(\rho,k-1)\rtimes\zeta(\rho,\ell;\sigma)$$

$$\tau'''=\nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{-\frac{k}{2}+j}\zeta(\rho,k)\rtimes\zeta(\rho,\ell-1;\sigma)$$

For τ', we have $k'=k-1$, $\ell'=\ell$, $j'=j$ (in the obvious notation) so that $k'-j'=k-j-1$. Similarly, for τ'' we have $k''-j''=k-j$, $j''=j-1$, $\ell''=\ell$ and for τ''', $k'''-j'''=k-j$, $j'''=j$, $\ell'''=\ell-1$. Further, by inductive hypothesis, we know that τ', τ'', τ''' decompose according to the theorem.

The proof of the theorem is broken into subcases based on how τ', τ'', τ''' decompose (with respect to the theorem). The particular case of the theorem governing the decomposition of τ' is given in the second column in the table below, and is easily determined from $k'-j'$, j', ℓ'. One note: if $j=\frac{k}{2}$, in order to avoid having $\alpha'>0$, we replace $\tau'=\nu^{\frac{-k+1}{2}}\rho\otimes\nu^{\frac{1}{2}}\zeta(\rho,k-1)\rtimes\zeta(\rho,\ell;\sigma)$ with $\nu^{\frac{-k+1}{2}}\rho\otimes\nu^{-\frac{1}{2}}\zeta(\rho,k-1)\rtimes\zeta(\rho,\ell;\sigma)=\tau''$ (so then $k'-j'=k''-j''$, $j'=j''$, $\ell'=\ell''$). The third and fourth columns have the corresponding information for τ'' and τ''', respectively. The final column indicates which components of τ', τ'', τ''' are contained in $s_{(p)}\pi_i$ for each component π_i of π. Note that this is part of the induction–we assume the table gives the Jacquet modules for lower values of $k+\ell$ and verify it for $k+\ell$ under consideration.

We note that the notation in the tables is the obvious notation–e.g., if τ' decomposes according to case 3a, then τ'_2 is the second component in part 3a of the statement of the theorem.

Case (for π)	for τ'	for τ''	for τ'''	components
1.				
(α) $\alpha=-\ell-\frac{k}{2}$	1	irr	irr	$s_{(p)}\pi_1=\tau_1'$ $s_{(p)}\pi_2=\tau_2'+\tau''+\tau'''$
(β) $\alpha=-\ell+\frac{k}{2}-1$	irr	1	1	$s_{(p)}\pi_1=\tau'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2''+\tau_2'''$
(γ) $-\ell-\frac{k}{2}<\alpha$, $\alpha<-\ell+\frac{k}{2}-1$	1	1	1	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2'+\tau_2''+\tau_2'''$
2a.				
(α) $k=\ell$	2b(i)	irr	3a	$s_{(p)}\pi_1=\tau_1'+\tau_1'''+\tau_3'''$ $s_{(p)}\pi_2=\tau_2'+\tau''+\tau_2'''+\tau_4'''$
(β) $\ell<k<2\ell-1$	2b(ii)	2a	3a	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_1'''+\tau_3'''$ $s_{(p)}\pi_2=\tau_2'+\tau_3'+\tau_2''+\tau_2'''+\tau_4'''$
(γ) $k=2\ell-1$	2c	2a	3b	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_1'''+\tau_2'''$ $s_{(p)}\pi_2=\tau_2'+\tau_2''$
2b(i).				
(α) $k=\ell-1$	2b(i)	irr	2a	$s_{(p)}\pi_1=\tau_1'+\tau_1'''$ $s_{(p)}\pi_2=\tau_2'+\tau''+\tau_2'''$
(β) $k<\ell-1$	2b(i)	irr	2b(i)	$s_{(p)}\pi_1=\tau_1'+\tau_1'''$ $s_{(p)}\pi_2=\tau_2'+\tau''+\tau_2'''$
2b(ii).				
(α) $j=1,\ k=3,\ \ell=3$	2c	2b(i)	2a	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2''+\tau_2'''$ $s_{(p)}\pi_3=\tau_2'$
(β) $j=1,\ k=3,\ \ell>3$	2c	2b(i)	2b(ii)	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2''+\tau_2'''$ $s_{(p)}\pi_3=\tau_2'+\tau_3'''$
(γ) $j=1,\ k=\ell,\ \ell>3$	2b(ii)	2b(i)	2a	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2'+\tau_2''+\tau_2'''$ $s_{(p)}\pi_3=\tau_3'$
(δ) $j=1,\ 3<k<\ell$	2b(ii)	2b(i)	2b(ii)	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2'+\tau_2''+\tau_2'''$ $s_{(p)}\pi_3=\tau_3'+\tau_3'''$
(ϵ) $j=\frac{k-1}{2}$, $k=2\ell-3,\ \ell>3$	2c	2b(ii)	2a	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2''+\tau_2'''$ $s_{(p)}\pi_3=\tau_2'+\tau_3''$
(ζ) $j=\frac{k-1}{2}$, $3<k<2\ell-3$	2c	2b(ii)	2b(ii)	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2''+\tau_2'''$ $s_{(p)}\pi_3=\tau_2'+\tau_3''+\tau_3'''$

Case (for π)	for τ'	for τ''	for τ'''	components
(η) $j = k - \ell + 1$, $\ell < k < 2\ell - 3$	2b(ii)	2b(ii)	2a	$s_{(p)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(p)}\pi_2 = \tau_2' + \tau_2'' + \tau_2'''$ $s_{(p)}\pi_3 = \tau_3' + \tau_3''$
(θ) $1 < j < \frac{k-1}{2}$, $j \neq k - \ell + 1$	2b(ii)	2b(ii)	2b(ii)	$s_{(p)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(p)}\pi_2 = \tau_2' + \tau_2'' + \tau_2'''$ $s_{(p)}\pi_1 = \tau_3' + \tau_3'' + \tau_3'''$
2c.				(n.b. $\tau' = \tau''$)
(α) $k = 2$, $\ell = 2$	2b(i)	2b(i)	irr	$s_{(p)}\pi_1 = \tau_1' + \tau_2' + \tau_1'' + \tau'''$ $s_{(p)}\pi_2 = \tau_2''$
(β) $k = 2$, $\ell > 2$	2b(i)	2b(i)	2c	$s_{(p)}\pi_1 = \tau_1' + \tau_2' + \tau_1'' + \tau_1'''$ $s_{(p)}\pi_2 = \tau_2'' + \tau_2'''$
(γ) $k = 2\ell - 2$, $\ell > 2$	2b(ii)	2b(ii)	irr	$s_{(p)}\pi_1 = \tau_1' + \tau_2' + \tau_1'' + \tau'''$ $s_{(p)}\pi_2 = \tau_3' + \tau_2'' + \tau_3''$
(δ) $2 < k < 2\ell - 2$	2b(ii)	2b(ii)	2c	$s_{(p)}\pi_1 = \tau_1' + \tau_2' + \tau_1'' + \tau_1'''$ $s_{(p)}\pi_2 = \tau_3' + \tau_2'' + \tau_3'' + \tau_2'''$
3a.				
(α) $j = 0$, $\ell = 1$, $k = 2$	2a	1	3b	$s_{(p)}\pi_1 = \tau_1' + \tau_1'''$ $s_{(p)}\pi_2 = \tau_2' + \tau_1''$ $s_{(p)}\pi_3 = \tau_2'''$ $s_{(p)}\pi_4 = \tau_2''$
(β) $j = 0$, $\ell = 1$, $k > 2$	3a	1	3b	$s_{(p)}\pi_1 = \tau_1' + \tau_1'''$ $s_{(p)}\pi_2 = \tau_2' + \tau_1''$ $s_{(p)}\pi_3 = \tau_3' + \tau_2'''$ $s_{(p)}\pi_4 = \tau_4' + \tau_2''$
(γ) $j = 0$, $\ell > 1$, $k = \ell + 1$	2a	1	3a	$s_{(p)}\pi_1 = \tau_1' + \tau_1'''$ $s_{(p)}\pi_2 = \tau_2' + \tau_1'' + \tau_2'''$ $s_{(p)}\pi_3 = \tau_3'''$ $s_{(p)}\pi_4 = \tau_2'' + \tau_4'''$
(δ) $j = 0$, $\ell > 1$, $k > \ell + 1$	3a	1	3a	$s_{(p)}\pi_1 = \tau_1' + \tau_1'''$ $s_{(p)}\pi_2 = \tau_2' + \tau_1'' + \tau_2'''$ $s_{(p)}\pi_3 = \tau_3' + \tau_3'''$ $s_{(p)}\pi_4 = \tau_4' + \tau_2'' + \tau_4'''$
(ϵ) $j = k - \ell - 1 > 0$, $k = 2\ell$	2a	3a	3b	$s_{(p)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(p)}\pi_2 = \tau_2' + \tau_2''$ $s_{(p)}\pi_3 = \tau_3'' + \tau_2'''$ $s_{(p)}\pi_4 = \tau_4''$
(ζ) $j = k - \ell - 1 > 0$, $k < 2\ell$	2a	3a	3a	$s_{(p)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(p)}\pi_2 = \tau_2' + \tau_2'' + \tau_2'''$ $s_{(p)}\pi_3 = \tau_3'' + \tau_3'''$ $s_{(p)}\pi_4 = \tau_4'' + \tau_4'''$

Case (for π)	for τ'	for τ''	for τ'''	components
(η) $j=\ell-1>0$, $j\neq k-\ell-1$	3a	3a	3b	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2'+\tau_2''$ $s_{(p)}\pi_3=\tau_3'+\tau_3''+\tau_2'''$ $s_{(p)}\pi_4=\tau_4'+\tau_4''$
(θ) $0<j<\ell-1$, $j\neq k-\ell-1$	3a	3a	3a	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2'+\tau_2''+\tau_2'''$ $s_{(p)}\pi_3=\tau_3'+\tau_3''+\tau_3'''$ $s_{(p)}\pi_4=\tau_4'+\tau_4''+\tau_4'''$
3b.				
(α) $k=2\ell+1$	irr	3a	3c	$s_{(p)}\pi_1=\tau'+\tau_1''+\tau_2''+\tau_1'''$ $s_{(p)}\pi_2=\tau_3''+\tau_4''+\tau_2'''+\tau_3'''$
(β) $k>2\ell+1$	3b	3a	3c	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_2''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2'+\tau_3''+\tau_4''+\tau_2'''+\tau_3'''$
3c.				
(α) $j=\ell+1$, $k=2\ell+3$	3d	3b	3c	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2'+\tau_2'''$ $s_{(p)}\pi_3=\tau_2''+\tau_3'''$
(β) $j=\ell+1$, $k>2\ell+3$	3c	3b	3c	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2'+\tau_2'''$ $s_{(p)}\pi_3=\tau_3'+\tau_2''+\tau_3'''$
(γ) $j>\ell+1$, $j=\frac{k-1}{2}$	3d	3c	3c	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2'+\tau_2''+\tau_2'''$ $s_{(p)}\pi_3=\tau_3''+\tau_3'''$
(δ) $\ell+1<j<\frac{k-1}{2}$	3c	3c	3c	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2'+\tau_2''+\tau_2'''$ $s_{(p)}\pi_3=\tau_3'+\tau_3''+\tau_3'''$
3d.				(n.b. $\tau'=\tau''$)
(α) $k=2\ell+2$	3b	3b	3d	$s_{(p)}\pi_1=\tau_1'+\tau_2'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2''+\tau_2'''$
(β) $k>2\ell+2$	3c	3c	3d	$s_{(p)}\pi_1=\tau_1'+\tau_3'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2'+\tau_2''+\tau_3''+\tau_2'''$

For convenience, we begin with the unitary cases, cases 2c and 3d.

<u>3d (β):</u> $k>2\ell+2$

We have $\pi = \zeta(\rho, k) \rtimes \zeta(\rho, \ell; \sigma)$. Again, since $\nu^{\frac{1}{2}}\zeta(\rho, k-1) \rtimes \zeta(\rho, \ell; \sigma)$ and $\nu^{-\frac{1}{2}}\zeta(\rho, k-1) \rtimes \zeta(\rho, \ell; \sigma)$ have the same components, $\tau' = \tau''$. So, by inductive hypothesis, we have (see table)

$$\begin{aligned}
s_{(p)}\pi &= 2\nu^{\frac{-k+1}{2}}\rho \otimes \nu^{-\frac{1}{2}}\zeta(\rho, k-1) \rtimes \zeta(\rho, \ell; \sigma) + \nu^{-\ell+\frac{1}{2}}\rho \otimes \zeta(\rho, k) \rtimes \zeta(\rho, \ell-1; \sigma) \\
&= 2\nu^{\frac{-k+1}{2}}\rho \otimes L([\nu^{\frac{-k+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\frac{k}{2}+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\
&+2\nu^{\frac{-k+1}{2}}\rho \otimes L([\nu^{\frac{-k+1}{2}}\rho, \nu^{-\ell-\frac{3}{2}}\rho], [\nu^{-\frac{k}{2}+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell}\delta(\rho, 2), \\
&\qquad \ldots, \nu^{-1}\delta(\rho, 2); \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \\
&+2\nu^{\frac{-k+1}{2}}\rho \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\frac{k}{2}+1}\delta(\rho, 2), \ldots, \nu^{-1}\delta(\rho, 2); \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{\frac{-k+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{\frac{-k+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-\ell-\frac{1}{2}}\rho], \nu^{-\ell+1}\delta(\rho, 2), \\
&\qquad \ldots, \nu^{-1}\delta(\rho, 2); \delta(\nu^{-\frac{1}{2}}\rho; \sigma))
\end{aligned}$$

First, we check that π has two components. Set $\theta = s_{min}\zeta(\rho, k) \otimes \zeta(\rho, \ell; \sigma)$, i.e.,

$$\theta = \nu^{\frac{-k+1}{2}}\rho \otimes \nu^{\frac{-k+3}{2}}\rho \otimes \ldots \otimes \nu^{\frac{k-1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{3}{2}}\rho \otimes \ldots \nu^{-\frac{1}{2}}\rho \otimes \sigma.$$

From the description of $s_{min}\pi$ (cf. Theorem 2.3 et seq.), we see that θ occurs in $s_{min}\pi$ with multiplicity two. Therefore, $\zeta(\rho, k) \otimes \zeta(\rho, \ell; \sigma)$ occurs in $s_{(kp)}\pi$ with multiplicity two. Since π is unitary and reducible, Frobenius reciprocity tells us that π must have (exactly) two components. By Lemma 3.3, one component is

$$\pi_1 = L([\nu^{\frac{-k+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma).$$

Let π_2 denote the other component.

We now determine $s_{(p)}\pi_1$ and $s_{(p)}\pi_2$. By Lemma 3.3, $s_{(p)}\pi_1 \geq \tau_1' + \tau_1'' + \tau_1'''$ (n.b. $\tau_1' = \tau_1''$). Frobenius reciprocity tells us $s_{min}\pi_1 \geq \theta$ and $s_{min}\pi_2 \geq \theta$. Since both copies of τ_1' are in $s_{(p)}\pi_1$, we have $\theta \not\leq s_{min}\tau_1'$. Therefore, $\theta \leq s_{min}\tau_2'$ or $\theta \leq s_{min}\tau_3'$ (again, note $\tau_2' = \tau_2''$ and $\tau_3' = \tau_3''$). We determine which by a comparison: by Lemma 3.3, π_1 is also a component of

$\hat{\pi}=\nu^{-\frac{k}{4}+\frac{\ell}{2}}\zeta(\rho,\ell+\frac{k}{2})\rtimes\zeta(\rho,\frac{k}{2};\sigma)$. Further, by inductive hypothesis, we can decompose $s_{(p)}\hat{\pi}$:

$$\begin{aligned}
s_{(p)}\hat{\pi} &= \nu^{\frac{-k+1}{2}}\rho\otimes\nu^{-\frac{k}{4}+\frac{\ell}{2}+\frac{1}{2}}\zeta(\rho,\ell+\frac{k}{2}-1)\rtimes\zeta(\rho,\frac{k}{2};\sigma)\\
&+\nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{-\frac{k}{4}+\frac{\ell}{2}-\frac{1}{2}}\zeta(\rho,\ell+\frac{k}{2}-1)\rtimes\zeta(\rho,\frac{k}{2};\sigma)\\
&+\nu^{\frac{-k+1}{2}}\rho\otimes\nu^{-\frac{k}{4}+\frac{\ell}{2}}\zeta(\rho,\ell+\frac{k}{2})\rtimes\zeta(\rho,\frac{k}{2}-1;\sigma)\\
&= \nu^{\frac{-k+1}{2}}\rho\otimes L([\nu^{\frac{-k+3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{\frac{-k+1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\
&+\nu^{\frac{-k+1}{2}}\rho\otimes L([\nu^{\frac{-k+1}{2}}\rho,\nu^{-\ell-\frac{3}{2}}\rho],[\nu^{\frac{-k+3}{2}}\rho,\nu^{-\frac{3}{2}}\rho],\nu^{-\ell}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\mathcal{T})\\
&+\nu^{\frac{-k+1}{2}}\rho\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{3}{2}}\rho],\nu^{-\frac{k}{2}+1}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\mathcal{T})\\
&+\nu^{-\ell+\frac{1}{2}}\rho\otimes L([\nu^{\frac{-k+1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{\frac{-k+1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\
&+\nu^{-\ell+\frac{1}{2}}\rho\otimes L([\nu^{\frac{-k+1}{2}}\rho,\nu^{-\ell-\frac{1}{2}}\rho],[\nu^{\frac{-k+1}{2}}\rho,\nu^{-\frac{3}{2}}\rho],\nu^{-\ell+1}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\mathcal{T})\\
&+\nu^{\frac{-k+1}{2}}\rho\otimes L([\nu^{\frac{-k+1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{\frac{-k+3}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\
&+\nu^{\frac{-k+1}{2}}\rho\otimes L([\nu^{\frac{-k+1}{2}}\rho,\nu^{-\ell-\frac{3}{2}}\rho],[\nu^{\frac{-k+3}{2}}\rho,\nu^{-\frac{3}{2}}\rho],\nu^{-\ell}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\mathcal{T})\\
&+\nu^{\frac{-k+1}{2}}\rho\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\nu^{-\frac{k}{2}+1}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma))\\
&+\nu^{\frac{-k+1}{2}}\rho\otimes L(\nu^{-\frac{k}{2}+1}\delta(\rho,2),\dots,\nu^{-\ell-2}\delta(\rho,2),\nu^{-\ell-\frac{1}{2}}\delta(\rho,3),\dots,\nu^{-\frac{1}{2}}\delta(\rho,3);\sigma)
\end{aligned}$$

A comparison of $s_{(p)}\pi$ and $s_{(p)}\hat{\pi}$ tells us that $s_{(p)}\pi_1\leq\tau_1'+\tau_3'+\tau_1''+\tau_1'''$. Therefore, we must have $\theta\leq s_{min}\tau_3'$ and $s_{(p)}\pi_1=\tau_1'+\tau_3'+\tau_1''+\tau_1'''$. This gives $s_{(p)}\pi_2=\tau_2'+\tau_2''+\tau_3''+\tau_2'''$. Thus the Jacquet modules work out as needed.

It remains to identify π_2. First, observe that by Lemma 2.8 (or the $s_{(p)}$ results applied to τ_2')

$$\begin{aligned}
s_{(p,p)}\pi_2\geq\nu^{\frac{-k+1}{2}}\rho\otimes\nu^{\frac{-k+1}{2}}\rho\otimes L([&\nu^{\frac{-k+3}{2}}\rho,\nu^{-\ell-\frac{3}{2}}\rho],[\nu^{\frac{-k+3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\\
&\nu^{-\ell}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma)).
\end{aligned}$$

Furthermore, we claim that (up to multiplicity) this is the only term of the form $\nu^{\frac{-k+1}{2}}\rho\otimes\nu^{\frac{-k+1}{2}}\rho\otimes\dots$ in $s_{(p,p)}\pi_2$. To see this, consider $\tau'=\tau''=\nu^{\frac{-k+1}{2}}\rho\otimes\nu^{-\frac{1}{2}}\zeta(\rho,k-1)\rtimes\zeta(\rho,\ell;\sigma)$ (the induced part of which is case 3c(α) or (γ), i.e., $j'=\frac{k'-1}{2}$). Then, by the inductive hypothesis

$$\begin{aligned}
s_{(p,p)}\tau' &= \nu^{\frac{-k+1}{2}}\rho\otimes\{\nu^{\frac{-k+1}{2}}\rho\otimes\zeta(\rho,k-2)\rtimes\zeta(\rho,\ell;\sigma)+\nu^{\frac{-k+3}{2}}\rho\otimes\nu^{-1}\zeta(\rho,k-2)\rtimes\zeta(\rho,\ell;\sigma)\\
&\quad+\nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{-\frac{1}{2}}\zeta(\rho,k-1)\rtimes\zeta(\rho,\ell-1;\sigma)\}\\
&= \nu^{\frac{-k+1}{2}}\rho\otimes\nu^{\frac{-k+1}{2}}\rho\otimes L([\nu^{\frac{-k+3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{\frac{-k+3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\
&\quad+\nu^{\frac{-k+1}{2}}\rho\otimes\nu^{\frac{-k+1}{2}}\rho\otimes L([\nu^{\frac{-k+3}{2}}\rho,\nu^{-\ell-\frac{3}{2}}\rho],[\nu^{\frac{-k+3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\\
&\qquad\qquad\nu^{-\ell}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma))\\
&\quad+\nu^{\frac{-k+1}{2}}\rho\otimes\nu^{\frac{-k+3}{2}}\rho\text{ terms }+\nu^{\frac{-k+1}{2}}\rho\otimes\nu^{-\ell+\frac{1}{2}}\rho\text{ terms.}
\end{aligned}$$

Now, we know (from case 3c (α) and (γ), by induction)

$$\begin{aligned}
s_{(p,p)}\tau_1' &\geq \nu^{\frac{-k+1}{2}}\rho\otimes\nu^{\frac{-k+1}{2}}\rho\otimes L([\nu^{\frac{-k+3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{\frac{-k+3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\
s_{(p,p)}\tau_2' &\geq \nu^{\frac{-k+1}{2}}\rho\otimes\nu^{\frac{-k+1}{2}}\rho\otimes L([\nu^{\frac{-k+3}{2}}\rho,\nu^{-\ell-\frac{3}{2}}\rho],[\nu^{\frac{-k+3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\\
&\qquad\nu^{-\ell}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma)).
\end{aligned}$$

From this, it follows that

$$\nu^{\frac{-k+1}{2}}\rho\otimes\nu^{\frac{-k+1}{2}}\rho\otimes L([\nu^{\frac{-k+3}{2}}\rho,\nu^{-\ell-\frac{3}{2}}\rho],[\nu^{\frac{-k+3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\nu^{-\ell}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma))$$

is the only term of the form $\nu^{\frac{-k+1}{2}}\rho\otimes\nu^{\frac{-k+1}{2}}\rho\otimes\dots$ in $s_{(p,p)}\pi_2$ (up to multiplicity). By Lemma 3.2 and Frobenius reciprocity, we have

$$\begin{gathered}
\pi_2\hookrightarrow\nu^{\frac{-k+1}{2}}\rho\times\nu^{\frac{-k+1}{2}}\rho\rtimes L([\nu^{\frac{-k+3}{2}}\rho,\nu^{-\ell-\frac{3}{2}}\rho],[\nu^{\frac{-k+3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\\
\nu^{-\ell}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma))\\
\Downarrow\\
\pi_2\hookrightarrow\nu^{\frac{-k+1}{2}}\rho\times\nu^{\frac{-k+1}{2}}\rho\rtimes[(\nu^{\frac{-k+3}{2}}\rho\times\nu^{\frac{-k+3}{2}}\rho)\times\dots\times(\nu^{-\ell-\frac{3}{2}}\rho\times\nu^{-\ell-\frac{3}{2}}\rho)\times\nu^{-\ell-\frac{1}{2}}\rho\\
\times\nu^{-\ell}\delta(\rho,2)\times\nu^{-\ell+\frac{1}{2}}\rho\times\nu^{-\ell+1}\delta(\rho,2)\times\dots\times\nu^{-1}\delta(\rho,2)\times\nu^{-\frac{1}{2}}\rho\rtimes\delta(\nu^{-\frac{1}{2}}\rho;\sigma)]\\
\Downarrow\\
\pi_2=L([\nu^{\frac{-k+1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{\frac{-k+1}{2}}\rho,\nu^{-\ell-\frac{3}{2}}\rho],\nu^{-\ell}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma))
\end{gathered}$$

by the uniqueness of the Langlands subrepresentation. With π_2 properly identified, 3d(β) is done.

<u>3d(α):</u> $k=2\ell+2$

The same arguments work as those used for 3d(β), save for the identification of π_2, which is easier. In particular, we get

$$\begin{aligned}
s_{(p)}\pi_2 &= \nu^{-\ell-\frac{1}{2}}\rho\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\nu^{-\ell}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma))\\
&\quad+\nu^{-\ell+\frac{1}{2}}\rho\otimes L([\nu^{-\ell-\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\nu^{-\ell-\frac{1}{2}}\rho,\nu^{-\ell+1}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma))
\end{aligned}$$

Therefore, Lemma 3.2 and Frobenius reciprocity imply

$$\pi_2 \hookrightarrow \nu^{-\ell-\frac{1}{2}}\rho \rtimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma))$$

which then forces

$$\pi_2 = L([\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma))$$

as above.

2c(δ): $2 < k < 2\ell - 2$

Again, we have $\tau' = \tau''$, so that (by inductive hypothesis)

$$\begin{aligned} s_{(p)}\pi &= 2\nu^{\frac{-k+1}{2}}\rho \otimes \nu^{-\frac{1}{2}}\zeta(\rho,k-1) \rtimes \zeta(\rho,\ell;\sigma) + \nu^{-\ell+\frac{1}{2}}\rho \otimes \zeta(\rho,k) \rtimes \zeta(\rho,\ell-1;\sigma) \\ &= 2\nu^{\frac{-k+1}{2}}\rho \otimes L([\nu^{\frac{-k+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\frac{-k+3}{2}}, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\ &+2\nu^{\frac{-k+1}{2}}\rho \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{\frac{-k-1}{2}}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\frac{k}{2}+1}\delta(\rho,2) \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ &+2\nu^{\frac{-k+1}{2}}\rho \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{\frac{-k-3}{2}}\rho], [\nu^{\frac{-k+3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\frac{k}{2}}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ &+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{\frac{-k+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\ &+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{\frac{-k-3}{2}}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\frac{k}{2}}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \end{aligned}$$

At this point, Lemma 3.5 (applied to the $\nu^{-\ell+\frac{1}{2}}\rho$–terms) tells us that

$$\pi_1 = L([\nu^{\frac{-k+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$$
$$\pi_2 = L([\nu^{-\ell+\frac{1}{2}}, \nu^{\frac{-k-3}{2}}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\frac{k}{2}}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T})$$

are both components of π. Next, we can use the same arguments as in 3d(β) to see that π has only two components and to show that $s_{(p)}\pi_1 = \tau_1' + \tau_2' + \tau_1'' + \tau_1'''$, $s_{(p)}\pi_2 = \tau_3' + \tau_2'' + \tau_3'' + \tau_2'''$, finishing this subcase.

2c(β): $k = 2$, $\ell > 2$

The proof for this subcase parallels that for 2c(δ).

2c(γ): $k = 2\ell - 2$, $\ell > 2$

Here, we have

$$\begin{aligned} s_{(p)}\pi \;&= 2\nu^{-\ell+\frac{3}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{5}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\ &+2\nu^{-\ell+\frac{3}{2}}\rho \otimes L(\nu^{-\ell+\frac{1}{2}}\rho, [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+2}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ &+2\nu^{-\ell+\frac{3}{2}}\rho \otimes L([\nu^{-\ell+\frac{5}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+1}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ &+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \end{aligned}$$

(n.b. τ''' is irreducible). The same arguments as in 3d(β) tell us that π has two components and $s_{(p)}\pi_1 = \tau_1' + \tau_2' + \tau_1'' + \tau'''$ and $s_{(p)}\pi_2 = \tau_3' + \tau_2'' + \tau_3''$.

The identification of π_2 is similar to that in 3d(β) (π_1 is, of course, given by Lemma 3.3). First, we claim

$$s_{(p,p)}\pi_2 \geq \nu^{-\ell+\frac{3}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+2}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T})$$

and that this is the only term of the form $\nu^{-\ell+\frac{3}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes \ldots$ in $s_{(p,p)}\pi_2$. This follows from the same basic argument used in 3d(β). Therefore,

$$s_{(2p)}\pi_2 \geq \nu^{-\ell+1}\delta(\rho,2) \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+2}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}).$$

Since this is the only term in $s_{(2p)}\pi_2$ with this central character, Lemma 3.2 and Frobenius reciprocity imply

$$\begin{gathered} \pi_2 \hookrightarrow \nu^{-\ell+1}\delta(\rho,2) \rtimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+2}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ \Downarrow \\ \pi_2 \hookrightarrow \nu^{-\ell+1}\delta(\rho,2) \rtimes [\nu^{-\ell+\frac{3}{2}}\rho \times \nu^{-\ell+2}\delta(\rho,2) \times \nu^{-\ell+\frac{5}{2}}\rho \times \nu^{-\ell+3}\delta(\rho,2) \times \\ \ldots \times \nu^{-\frac{3}{2}}\rho \times \nu^{-1}\delta(\rho,2) \rtimes \mathcal{T}] \\ \Downarrow \\ \pi_2 = L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+1}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \end{gathered}$$

by the uniqueness of the Langlands subrepresentation. This finishes 2c(γ).

<u>2c(α):</u> $k = 2$, $\ell = 2$

The proof for this subcase parallels that for 2c (γ).

This finishes up the unitary cases. We next turn to cases 2a and 3b; the other cases in which $k - j$, j, ℓ are not all distinct.

<u>3b(α):</u> $k = 2\ell + 1$

In this subcase, $\pi=\nu^{-\frac{1}{2}}\zeta(\rho,2\ell+1)\rtimes\zeta(\rho,\ell;\sigma)$. By induction,

$$\begin{aligned}
s_{(p)}\pi &= \nu^{-\ell-\frac{1}{2}}\rho\otimes\zeta(\rho,2\ell)\rtimes\zeta(\rho,\ell;\sigma)+\nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{-1}\zeta(\rho,2\ell)\rtimes\zeta(\rho,\ell;\sigma)\\
&\quad+\nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{-\frac{1}{2}}\zeta(\rho,2\ell+1)\rtimes\zeta(\rho,\ell-1;\sigma)\\
&= \nu^{-\ell-\frac{1}{2}}\rho\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\
&\quad+\nu^{-\ell+\frac{1}{2}}\rho\otimes L([\nu^{-\ell-\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\
&\quad+\nu^{-\ell+\frac{1}{2}}\rho\otimes L(\nu^{-\ell-\frac{1}{2}}\rho,[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{3}{2}}\rho],\nu^{-\ell+1}\delta(\rho,2),\ldots,\nu^{-1}\delta(\rho,2);\mathcal{T})\\
&\quad+\nu^{-\ell+\frac{1}{2}}\rho\otimes L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\nu^{-\ell}\delta(\rho,2),\ldots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma))\\
&\quad+\nu^{-\ell+\frac{1}{2}}\rho\otimes L(\nu^{-\ell+\frac{1}{2}}\delta(\rho,3),\ldots,\nu^{-\frac{1}{2}}\delta(\rho,3);\sigma)\\
&\quad+\nu^{-\ell+\frac{1}{2}}\rho\otimes L([\nu^{-\ell-\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\
&\quad+\nu^{-\ell+\frac{1}{2}}\rho\otimes L(\nu^{-\ell-\frac{1}{2}}\rho,[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\nu^{-\ell+1}\delta(\rho,2),\ldots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma))\\
&\quad+\nu^{-\ell+\frac{1}{2}}\rho\otimes L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\nu^{-\ell}\delta(\rho,2),\ldots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma)).
\end{aligned}$$

First, by Lemma 3.3,

$$\pi_1=L([\nu^{-\ell-\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)$$

is a component of π. Observe that π_1 is also a component of $\zeta(\rho,2\ell)\rtimes\zeta(\rho,\ell+1;\sigma)$ (covered by case 2c). Since case 2c has already been done, we know that $s_{(p)}\pi_1=\tau'+\tau_1''+\tau_2''+\tau_1'''$.

The next step is to show there is only one more component. To this end, observe that τ'' is 3a(α) or (ϵ) and τ''' is 3c(α) or falls under Proposition 3.6. In particular, this tells us that

$$\begin{aligned}
s_{(p,p)}\tau_3'' &= \nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{-\ell+\frac{1}{2}}\rho\otimes L(\nu^{-\ell-\frac{1}{2}}\rho,[\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\nu^{-\ell+1}\delta(\rho,2),\\
&\qquad\qquad\ldots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma))\\
&\quad+\nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{-\ell+\frac{3}{2}}\rho\otimes\ldots\ \text{(this term is missing if } \ell=1)\\
s_{(p,p)}\tau_4'' &= \nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{-\ell+\frac{3}{2}}\rho\otimes L(\nu^{-\ell}\delta(\rho,2),\nu^{-\ell+\frac{3}{2}}\delta(\rho,3),\ldots,\nu^{-\frac{1}{2}}\delta(\rho,3);\sigma)\\
s_{(p,p)}\tau_2''' &= \nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{-\ell-\frac{1}{2}}\rho\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\nu^{-\ell+1}\delta(\rho,2),\ldots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma))\\
&\quad+\nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{-\ell+\frac{3}{2}}\rho\otimes\ldots\ \text{(this term is missing if } \ell=1)\\
s_{(p,p)}\tau_3''' &= s_{(p,p)}\tau_3''\ \text{(since } \tau_3'''=\tau_3'').
\end{aligned}$$

First, let π_2 be a component of π such that $s_{(p)}\pi_2 \geq \tau_3''$. Then,

$$\begin{gathered}
s_{(p,p)}\pi_2 \geq \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes L(\nu^{-\ell-\frac{1}{2}}\rho, [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell+1}\delta(\rho,2), \\
\qquad \ldots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\
\Downarrow \text{ (since } \nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-\ell+\frac{1}{2}}\rho \text{ is irreducible)} \\
s_{(2p)}\pi_2 \geq (\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-\ell+\frac{1}{2}}\rho) \otimes L(\nu^{-\ell-\frac{1}{2}}\rho, [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell+1}\delta(\rho,2), \\
\qquad \ldots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\
\Downarrow \text{ (since } r_{(p,p)}\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-\ell+\frac{1}{2}}\rho = 2\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho) \\
s_{(p,p)}\pi_2 \geq 2\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes L(\nu^{-\ell-\frac{1}{2}}\rho, [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell+1}\delta(\rho,2), \\
\qquad \ldots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma)).
\end{gathered}$$

Thus, $s_{(p)}\pi_2 \geq \tau_3'' + \tau_3'''$. Note that these are the only terms of the form $\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes \ldots$ in $s_{(p,p)}(\pi - \pi_1)$.

Next, observe that

$$\begin{gathered}
s_{(p,p,p)}\tau_2''' \geq \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell-\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell+1}\delta(\rho,2), \\
\ldots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma)).
\end{gathered}$$

Now,

$$\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-\ell-\frac{1}{2}}\rho \times \nu^{-\ell+\frac{1}{2}}\rho = \nu^{-\ell}\zeta(\rho,2) \times \nu^{-\ell+\frac{1}{2}}\rho + \nu^{-\ell}\delta(\rho,2) \times \nu^{-\ell+\frac{1}{2}}\rho,$$

where both terms on the right-hand side are irreducible. Therefore, if π_0 is the component of π such that $s_{(p)}\pi_0 \geq \tau_2'''$, we must have either

$$s_{(3p)}\pi_0 \geq (\nu^{-\ell}\zeta(\rho,2) \times \nu^{-\ell+\frac{1}{2}}\rho) \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell+1}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma))$$

or

$$s_{(3p)}\pi_0 \geq (\nu^{-\ell}\delta(\rho,2) \times \nu^{-\ell+\frac{1}{2}}\rho) \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell+1}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma)).$$

Since $r_{min}\nu^{-\ell}\zeta(\rho,2) \times \nu^{-\ell+\frac{1}{2}}\rho \geq \nu^{-\ell-\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho$ and there are no terms of the form $\nu^{-\ell-\frac{1}{2}}\rho \otimes \ldots$ in $s_{(p)}(\pi - \pi_1)$, it must be the latter. Then, since $r_{(2p)}\nu^{-\ell}\delta(\rho,2) \times \nu^{-\ell+\frac{1}{2}}\rho \geq (\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-\ell+\frac{1}{2}}\rho) \otimes \nu^{-\ell-\frac{1}{2}}\rho$, we have

$$\begin{gathered}
s_{(3p)}\pi_0 \geq (\nu^{-\ell}\delta(\rho,2) \times \nu^{-\ell+\frac{1}{2}}\rho) \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell+1}\delta(\rho,2), \\
\qquad \ldots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\
\Downarrow \\
s_{(2p,p)}\pi_0 \geq (\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-\ell+\frac{1}{2}}\rho) \otimes \nu^{-\ell-\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell+1}\delta(\rho,2), \\
\qquad \ldots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\
\Downarrow \\
s_{(2p)}\pi_0 \geq (\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-\ell+\frac{1}{2}}\rho) \otimes L(\nu^{-\ell-\frac{1}{2}}\rho, [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell+1}\delta(\rho,2), \\
\qquad \ldots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma))
\end{gathered}$$

since this is the only term of the form $(\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-\ell+\frac{1}{2}}\rho) \otimes \ldots$ in $s_{(2p)}(\pi - \pi_1)$. Therefore $s_{(p)}\pi_0 \geq \tau_3'' + \tau_3'''$–i.e., $\pi_0 = \pi_2$. This gives us $s_{(p)}\pi_2 \geq \tau_3'' + \tau_2''' + \tau_3'''$.

We use the same basic argument to deal with τ_4'':

$$s_{(p,p,p)}\tau_4'' = \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{3}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes L(\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\ell+\frac{3}{2}}\delta(\rho,3), \dots, \nu^{-\frac{1}{2}}\delta(\rho,3);\sigma).$$

Now,

$$\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-\ell+\frac{3}{2}}\rho \times \nu^{-\ell+\frac{1}{2}}\rho = \nu^{-\ell+1}\zeta(\rho,2) \times \nu^{-\ell+\frac{1}{2}}\rho + \nu^{-\ell+1}\delta(\rho,2) \times \nu^{-\ell+\frac{1}{2}}\rho.$$

Both terms on the right-hand side are irreducible. Again, let π_0 be the component of π such that $s_{(p)}\pi_0 \geq \tau_4''$. Then either

$$s_{(3p)}\pi_0 \geq (\nu^{-\ell+1}\zeta(\rho,2) \times \nu^{-\ell+\frac{1}{2}}\rho) \otimes L(\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\ell+\frac{3}{2}}\delta(\rho,3), \dots, \nu^{-\frac{1}{2}}\delta(\rho,3);\sigma)$$

or

$$s_{(3p)}\pi_0 \geq (\nu^{-\ell+1}\delta(\rho,2) \times \nu^{-\ell+\frac{1}{2}}\rho) \otimes L(\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\ell+\frac{3}{2}}\delta(\rho,3), \dots, \nu^{-\frac{1}{2}}\delta(\rho,3);\sigma).$$

Since $r_{min}\nu^{-\ell+1}\delta(\rho,2) \times \nu^{-\ell+\frac{1}{2}}\rho \geq \nu^{-\ell+\frac{3}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho$ and $s_{(p)}\pi - \pi_1$ (in fact, $s_{(p)}\pi$) has no terms of the form $\nu^{-\ell+\frac{3}{2}} \otimes \dots$, it must be the former. We have

$$\begin{gathered}
s_{(3p)}\pi_0 \geq (\nu^{-\ell+1}\zeta(\rho,2) \times \nu^{-\ell+\frac{1}{2}}\rho) \otimes L(\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\ell+\frac{3}{2}}\delta(\rho,3), \dots, \nu^{-\frac{1}{2}}\delta(\rho,3);\sigma) \\
\Downarrow \\
s_{(2p,p)}\pi_0 \geq (\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-\ell+\frac{1}{2}}\rho) \otimes \nu^{-\ell+\frac{3}{2}}\rho \otimes L(\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\ell+\frac{3}{2}}\delta(\rho,3), \dots, \nu^{-\frac{1}{2}}\delta(\rho,3);\sigma) \\
\Downarrow \\
s_{(p)}\pi_0 \geq \tau_3'' + \tau_3'''
\end{gathered}$$

again since $s_{(2p)}(\pi - \pi_1)$ has only one term of the form $\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-\ell+\frac{1}{2}}\rho \otimes \dots$. Thus $\pi_0 = \pi_2$ and we have $s_{(p)}\pi_2 = \tau_3'' + \tau_4'' + \tau_2''' + \tau_3'''$. Note that since we have now accounted for all of $s_{(p)}\pi$, we know there are no additional components; $\pi = \pi_1 + \pi_2$.

It remains to identify π_2. Observe that

$$\begin{gathered}
s_{(p,p)}\pi_2 \geq \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell-\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell+1}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\
\Downarrow \\
s_{(2p)}\pi_2 \geq \nu^{-\ell}\delta(\rho,2) \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell+1}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma)).
\end{gathered}$$

A glance at $s_{(p,p)}\tau_3''$, etc., calculated earlier tells us that this is the only term in $s_{(2p)}\pi_2$ with this central character. Therefore, by Lemma 3.2 and Frobenius reciprocity

$$\begin{gathered}
\pi_2 \hookrightarrow \nu^{-\ell}\delta(\rho,2) \rtimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell+1}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\
\Downarrow \\
\pi_2 \hookrightarrow \nu^{-\ell}\delta(\rho,2) \rtimes (\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-\ell+1}\delta(\rho,2) \times \nu^{-\ell+\frac{3}{2}}\rho \times \dots \times \nu^{-1}\delta(\rho,2) \times \nu^{-\frac{1}{2}}\rho \rtimes \delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\
\Downarrow \\
\pi_2 = L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma))
\end{gathered}$$

by the uniqueness of the Langlands subrepresentation. This finishes 3b(α).

<u>3b(β):</u> $k > 2\ell + 1$

We have $\pi = \nu^{-\frac{k}{2}+\ell}\zeta(\rho,k) \rtimes \zeta(\rho,\ell;\sigma)$ and by inductive hypothesis,

$$\begin{aligned}
s_{(p)}\pi \; &= \nu^{-k+\ell+\frac{1}{2}}\rho \otimes \nu^{-\frac{k}{2}+\ell+\frac{1}{2}}\zeta(\rho,k-1) \rtimes \zeta(\rho,\ell;\sigma) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\frac{k}{2}+\ell-\frac{1}{2}}\zeta(\rho,k-1) \rtimes \zeta(\rho,\ell;\sigma) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\frac{k}{2}+\ell}\zeta(\rho,k) \rtimes \zeta(\rho,\ell-1;\sigma) \\
&= \nu^{-k+\ell+\frac{1}{2}}\rho \otimes L([\nu^{-k+\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma) \\
&+\nu^{-k+\ell+\frac{1}{2}}\rho \otimes L([\nu^{-k+\ell+\frac{3}{2}}\rho,\nu^{-\ell-\frac{3}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\nu^{-\ell}\delta(\rho,2), \\
&\qquad\qquad \ldots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-k+\ell+\frac{1}{2}},\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-k+\ell+\frac{1}{2}}\rho,\nu^{-\ell-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{3}{2}}\rho],\nu^{-\ell+1}\delta(\rho,2),\ldots,\nu^{-1}\delta(\rho,2);\mathcal{T}) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-k+\ell+\frac{1}{2}}\rho,\nu^{-\ell-\frac{3}{2}}\rho],[\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\nu^{-\ell}\delta(\rho,2), \\
&\qquad\qquad \ldots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-k+\ell+\frac{1}{2}}\rho,\nu^{-\ell-\frac{3}{2}}\rho],\nu^{-\ell+\frac{1}{2}}\delta(\rho,3),\ldots,\nu^{-\frac{1}{2}}\delta(\rho,3);\sigma) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-k+\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-k+\ell+\frac{1}{2}}\rho,\nu^{-\ell-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\nu^{-\ell+1}\delta(\rho,2), \\
&\qquad\qquad \ldots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-k+\ell+\frac{1}{2}}\rho,\nu^{-\ell-\frac{3}{2}}\rho],[\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\nu^{-\ell}\delta(\rho,2), \\
&\qquad\qquad \ldots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma))
\end{aligned}$$

By Lemma 3.5 3 applied to the $\nu^{-k+\ell+\frac{1}{2}}\rho$-terms, we have $\pi = \pi_1 + \pi_2$ with

$$\pi_1 = L([\nu^{-k+\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)$$
$$\pi_2 = L([\nu^{-k+\ell+\frac{1}{2}}\rho,\nu^{-\ell-\frac{3}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\nu^{-\ell}\delta(\rho,2),\ldots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma)).$$

Finally, observe that π_1 is also a component of $\zeta(\rho,2\ell) \rtimes \zeta(\rho,k-\ell;\sigma)$ (case 2c (β) or (δ)). This tells us (since case 2c is already done) $s_{(p)}\pi_1 = \tau_1' + \tau_1'' + \tau_2'' + \tau_1'''$ and therefore $s_{(p)}\pi_2 =$ the rest $= \tau_2' + \tau_3'' + \tau_4'' + \tau_2''' + \tau_3'''$. This finishes 3b($\beta$).

<u>2a(β):</u> $\ell < k < 2\ell - 1$

Here, $\pi = \nu^{-\ell+\frac{k}{2}}\zeta(\rho,k) \rtimes \zeta(\rho,\ell;\sigma)$ and, by inductive hypothesis,

$$\begin{aligned}
s_{(p)}\pi \; &= \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{k}{2}+\frac{1}{2}}\zeta(\rho,k-1) \rtimes \zeta(\rho,\ell;\sigma) \\
&+\nu^{-k+\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{k}{2}-\frac{1}{2}}\zeta(\rho,k-1) \rtimes \zeta(\rho,\ell;\sigma) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{k}{2}}\zeta(\rho,k) \rtimes \zeta(\rho,\ell-1;\sigma) \\
&= \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-k+\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-k+\ell-\frac{3}{2}}\rho],[\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{3}{2}}\rho],\nu^{-k+\ell}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\mathcal{T}) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-k+\ell+\frac{1}{2}}\rho,\nu^{-\frac{3}{2}}\rho],\nu^{-\ell+1}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\mathcal{T}) \\
&+\nu^{-k+\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-k+\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma) \\
&+\nu^{-k+\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-k+\ell-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{3}{2}}\rho],\nu^{-k+\ell+1}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\mathcal{T}) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-k+\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-k+\ell-\frac{3}{2}}\rho],[\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{3}{2}}\rho],\nu^{-k+\ell}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\mathcal{T}) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-k+\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\nu^{-\ell+1}\delta(\rho,2),\dots,\nu^{-1}\delta(\rho,2);\delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L(\nu^{-\ell+1}\delta(\rho,2),\dots,\nu^{-k+\ell-2}\delta(\rho,2),\nu^{-k+\ell-\frac{1}{2}}\delta(\rho,3),\dots,\nu^{-\frac{1}{2}}\delta(\rho,3);\sigma)
\end{aligned}$$

By Lemma 3.3,

$$\pi_1 = L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-k+\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)$$

is a component. Furthermore, π_1 is also a component of $\zeta(\rho,2\ell) \rtimes \zeta(\rho,k-\ell;\sigma)$ (which is case 3d(β); already done), so we know $s_{(p)}\pi_1 = \tau_1' + \tau_1'' + \tau_1''' + \tau_3'''$.

Let π_2 be the component of π with $s_{(p)}\pi_2 \geq \tau_2''$. We claim $s_{(p)}\pi_2 = \tau_2' + \tau_3' + \tau_2'' + \tau_2''' + \tau_4'''$. For example, consider τ_2'. Let π_0 be the component of π with $s_{(p)}\pi_0 \geq \tau_2'$. Then, noting that

$\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-k+\ell+\frac{1}{2}}\rho$ is irreducible,

$$\begin{gathered} s_{(p,p)}\tau_2' \geq \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-k+\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-k+\ell-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \\ \nu^{-k+\ell+1}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ \Downarrow \\ s_{(2p)}\pi_0 \geq (\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-k+\ell+\frac{1}{2}}\rho) \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-k+\ell-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \\ \nu^{-k+\ell+1}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ \Downarrow \\ s_{(p,p)}\pi_0 \geq \nu^{-k+\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-k+\ell-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \\ \nu^{-k+\ell+1}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ \Downarrow \\ s_{(p)}\pi_0 \geq \tau_2'' \end{gathered}$$

Thus $\pi_0 = \pi_2$, so $s_{(p)}\pi_2 \geq \tau_2' + \tau_2''$. The same argument applied to τ_2''' (n.b. $\tau_2''' = \tau_2'$) and τ_4''' gives $s_{(p)}\pi_2 \geq \tau_2' + \tau_2'' + \tau_2''' + \tau_4'''$.

Next, if $j > 1$, the same argument gives us $\tau_3' \leq s_{(p)}\pi_2$. If $j = 1$, we show that $\tau_3' \leq s_{(p)}\pi_2$ in a similar, though slightly more involved way (using $s_{(p,p,p)}$, as in the proof for 3b(α)). In particular, if $j = 1$, we have $k = \ell + 1$, so that

$$\tau_3' = \nu^{-\ell+\frac{1}{2}}\rho \otimes L(\nu^{-\ell+1}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T})$$

and

$$s_{(p,2p)}\tau_3' \geq \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+1}\delta(\rho,2) \otimes L(\nu^{-\ell+2}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}).$$

Therefore, if π_0 is the component of π with $\tau_3' \leq s_{(p)}\pi_0$,

$$s_{(p,p,p)}\pi_0 \geq \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{3}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes L(\nu^{-\ell+2}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}).$$

Now,

$$\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-\ell+\frac{3}{2}}\rho \times \nu^{-\ell+\frac{1}{2}}\rho = \nu^{-\ell+1}\zeta(\rho,2) \times \nu^{-\ell+\frac{1}{2}}\rho + \nu^{-\ell+1}\delta(\rho,2) \times \nu^{-\ell+\frac{1}{2}}\rho$$

with both terms on the right-hand side irreducible. Therefore, we must have

$$s_{(3p)}\pi_0 \geq (\nu^{-\ell+1}\zeta(\rho,2) \times \nu^{-\ell+\frac{1}{2}}\rho) \otimes L(\nu^{-\ell+2}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T})$$

or

$$s_{(3p)}\pi_0 \geq (\nu^{-\ell+1}\delta(\rho,2) \times \nu^{-\ell+\frac{1}{2}}\rho) \otimes L(\nu^{-\ell+2}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}).$$

Since $r_{min}\nu^{-\ell+1}\delta(\rho,2) \times \nu^{-\ell+\frac{1}{2}}\rho \geq \nu^{-\ell+\frac{3}{2}} \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho$ and $s_{(p)}\pi$ has no terms of the form $\nu^{-\ell+\frac{3}{2}}\rho \otimes \ldots$, it must be the former. Now,

$$\begin{gathered} s_{(3p)}\pi_0 \geq (\nu^{-\ell+1}\zeta(\rho,2) \times \nu^{-\ell+\frac{1}{2}}\rho) \otimes L(\nu^{-\ell+2}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ \Downarrow \\ s_{(p,p,p)}\pi_0 \geq \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{3}{2}}\rho \otimes L(\nu^{-\ell+2}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}). \end{gathered}$$

Observe that (noting that τ' is 2b(ii) (α) or (γ))

$$s_{(p,p)}\tau_3' = \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{3}{2}}\rho \otimes L(\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\ell+2}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}).$$

In particular, $s_{(p,p)}\tau_3'$ has no terms of the form $\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes \dots$. Since $s_{(p,p)}\pi_0$ does, we must have $s_{(p)}\pi_0 > \tau_3'$. This forces $\pi_0 = \pi_2$, so that $s_{(p)}\pi_2 = \tau_2' + \tau_3' + \tau_2'' + \tau_2''' + \tau_4'''$.

The last step is to identify π_2. This follows a familiar pattern; it is the same argument used to identify π_2 in 3b (α). Observe that

$$s_{(p,p)}\pi_2 \geq \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-k+\ell-\frac{3}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-k+\ell}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \mathcal{T}).$$

Further, this is the only term of the form $\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes \dots$ (up to multiplicity) in $s_{(p,p)}\pi_2$. So, by Lemma 3.2 and Frobenius reciprocity,

$$\begin{gathered}\pi_2 \hookrightarrow \nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-\ell+\frac{1}{2}}\rho \rtimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-k+\ell-\frac{3}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-k+\ell}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ \Downarrow \\ \pi_2 = L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-k+\ell-\frac{3}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-k+\ell}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \mathcal{T})\end{gathered}$$

by the uniqueness of the Langlands subrepresentation. This finishes 2a (β).

<u>2a(α):</u> $k = \ell$

This is essentially the same as 2a (β).

<u>2a(γ):</u> $k = 2\ell - 1$

Here, we have $\pi = \nu^{-\frac{1}{2}}\zeta(\rho, 2\ell-1) \rtimes \zeta(\rho, \ell; \sigma)$ and

$$\begin{aligned} s_{(p)}\pi &= \nu^{-\ell+\frac{1}{2}}\rho \otimes \zeta(\rho, 2\ell-2) \rtimes \zeta(\rho,\ell;\sigma) + \nu^{-\ell+\frac{3}{2}}\rho \otimes \nu^{-1}\zeta(\rho,2\ell-2) \rtimes \zeta(\rho,\ell;\sigma) \\ &\quad + \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\frac{1}{2}}\zeta(\rho,2\ell-1) \rtimes \zeta(\rho,\ell-1;\sigma) \\ &= \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\ &\quad + \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+1}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ &\quad + \nu^{-\ell+\frac{3}{2}}\rho \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{5}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\ &\quad + \nu^{-\ell+\frac{3}{2}}\rho \otimes L(\nu^{-\ell+\frac{1}{2}}\rho, [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+2}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ &\quad + \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\ &\quad + \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell+1}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma)). \end{aligned}$$

Now, by Lemma 3.3,

$$\pi_1 = L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma).$$

Comparison with $\zeta(\rho, 2\ell) \rtimes \zeta(\rho, \ell-1; \sigma)$ gives us $s_{(p)}\pi_1 = \tau_1' + \tau_1'' + \tau_1''' + \tau_2'''$.

Let π_2 be the component of π with $s_{(p)}\pi_2 \geq \tau_2''$. Then, the same sort of argument used in 3b (α) (using $s_{(p,p,p)}/s_{(3p)}$) gives us $s_{(p)}\pi_2 = \tau_2' + \tau_2''$:

$$\begin{gathered} s_{(p,p,p)}\tau_2'' \geq \nu^{-\ell+\frac{3}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+2}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ \Downarrow \\ s_{(3p)}\pi_2 \geq (\nu^{-\ell+1}\delta(\rho,2) \times \nu^{-\ell+\frac{1}{2}}\rho) \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+2}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ \Downarrow \\ s_{(p,p,p)}\pi_2 \geq \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{3}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+2}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}). \end{gathered}$$

Therefore, $s_{(p)}\pi_2$ has a term of the form $\nu^{-\ell+\frac{1}{2}}\rho \otimes \ldots$, necessarily τ_2'. Finally, by Lemma 3.2 and Frobenius reciprocity,

$$\begin{gathered} \pi_2 \hookrightarrow \nu^{-\ell+\frac{1}{2}}\rho \rtimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+1}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ \Downarrow \\ \pi_2 = L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+1}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \end{gathered}$$

(uniqueness of Langlands subrepresentation). This finishes 2a (γ).

2b(ii) (α): $j=1$, $k=3$, $\ell=3$

Here, $\pi = \nu^{-\frac{1}{2}}\zeta(\rho,3) \rtimes \zeta(\rho,3;\sigma)$ and

$$\begin{aligned} s_{(p)}\pi \;&= \nu^{-\frac{3}{2}}\rho \otimes \zeta(\rho,2) \rtimes \zeta(\rho,3;\sigma) + \nu^{-\frac{1}{2}}\rho \otimes \nu^{-1}\zeta(\rho,2) \rtimes \zeta(\rho,3;\sigma) \\ &+\nu^{-\frac{5}{2}}\rho \otimes \nu^{-\frac{1}{2}}\zeta(\rho,3) \rtimes \zeta(\rho,2;\sigma) \\ &= \nu^{-\frac{3}{2}}\rho \otimes L(\nu^{-\frac{5}{2}}\rho, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho; \sigma) \\ &+\nu^{-\frac{3}{2}}\rho \otimes L(\nu^{-\frac{5}{2}}\rho, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ &+\nu^{-\frac{1}{2}}\rho \otimes L(\nu^{-\frac{5}{2}}\rho, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho; \sigma) \\ &+\nu^{-\frac{1}{2}}\rho \otimes L(\nu^{-\frac{5}{2}}\rho, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho; \mathcal{T}) \\ &+\nu^{-\frac{5}{2}}\rho \otimes L(\nu^{-\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho; \sigma) \\ &+\nu^{-\frac{5}{2}}\rho \otimes L(\nu^{-\frac{3}{2}}\rho, \nu^{-1}\delta(\rho,2); \mathcal{T}). \end{aligned}$$

First, by Lemma 3.5 2,

$$\begin{gathered} \pi_1 = L(\nu^{-\frac{5}{2}}\rho, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho; \sigma) \\ \pi_2 = L(\nu^{-\frac{5}{2}}\rho, \nu^{-\frac{3}{2}}\rho, \nu^{-1}\delta(\rho,2); \mathcal{T}) \end{gathered}$$

are components of π. Lemma 3.3 implies $s_{(p)}\pi_1 \geq \tau_1' + \tau_1'' + \tau_1'''$. Then, Lemma 3.5 1 tells us $s_{(p)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$.

Next, consider $\hat{\pi} = \nu^{-1}\zeta(\rho,4) \rtimes \zeta(\rho,2;\sigma)$. By induction,

$$\begin{aligned} s_{(p)}\hat{\pi} &= \nu^{-\frac{5}{2}}\rho \otimes \nu^{-\frac{1}{2}}\zeta(\rho,3) \rtimes \zeta(\rho,2;\sigma) + \nu^{-\frac{1}{2}}\rho \otimes \nu^{-\frac{3}{2}}\zeta(\rho,3) \rtimes \zeta(\rho,2;\sigma) \\ &+\nu^{-\frac{3}{2}}\rho \otimes \nu^{-1}\zeta(\rho,4) \rtimes \zeta(\rho,1;\sigma) \\ &= \nu^{-\frac{5}{2}}\rho \otimes L(\nu^{-\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho; \sigma) \\ &+\nu^{-\frac{5}{2}}\rho \otimes L(\nu^{-\frac{3}{2}}\rho, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ &+\nu^{-\frac{1}{2}}\rho \otimes L(\nu^{-\frac{5}{2}}\rho, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho; \sigma) \\ &+\nu^{-\frac{1}{2}}\rho \otimes L(\nu^{-\frac{5}{2}}\rho, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho; \mathcal{T}) \\ &+\nu^{-\frac{1}{2}}\rho \otimes L(\nu^{-2}\delta(\rho,2), \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\ &+\nu^{-\frac{1}{2}}\rho \otimes L(\nu^{-2}\delta(\rho,2), \nu^{-\frac{1}{2}}\delta(\rho,3); \sigma) \\ &+\nu^{-\frac{3}{2}}\rho \otimes L(\nu^{-\frac{5}{2}}\rho, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho; \sigma) \\ &+\nu^{-\frac{3}{2}}\rho \otimes L(\nu^{-\frac{5}{2}}\rho, \nu^{-1}\delta(\rho,2), \nu^{-\frac{1}{2}}\rho; \delta(\nu^{-\frac{1}{2}}\rho;\sigma)). \end{aligned}$$

By Lemma 3.5, π_2 is also a component of $\hat{\pi}$ (as is π_1). A comparison of $s_{(p)}\pi$ and $s_{(p)}\hat{\pi}$ tells us that $s_{(p)}\pi_2 \leq \tau_2'' + \tau_2'''$. We can use the same sort of $s_{(p,p)}/s_{(2p)}$ argument as in 2a (β) to get $s_{(p)}\pi_2 = \tau_2'' + \tau_2'''$. In particular, let π_0 denote the component of π with $s_{(p)}\pi_0 \geq \tau_2''$. Then

$$\begin{gathered} s_{(p,p)}\tau_2'' \geq \nu^{-\frac{1}{2}}\rho \otimes \nu^{-\frac{5}{2}}\rho \otimes L(\nu^{-\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho; \mathcal{T}) \\ \Downarrow \\ s_{(2p)}\pi_0 \geq (\nu^{-\frac{1}{2}}\rho \times \nu^{-\frac{5}{2}}\rho) \otimes L(\nu^{-\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho; \mathcal{T}) \\ \Downarrow \\ s_{(p,p)}\pi_0 \geq \nu^{-\frac{5}{2}}\rho \otimes \nu^{-\frac{1}{2}}\rho \otimes L(\nu^{-\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho; \mathcal{T}) \end{gathered}$$

so that $s_{(p)}\pi_0 \geq \tau_2'''$. Thus $\pi_0 = \pi_2$ and $s_{(p)}\pi_2 = \tau_2'' + \tau_2'''$.

Let π_3 denote the final component, so that $s_{(p)}\pi_3 = \tau_2'$. We identify π_3 using now-familiar methods (cf. 2c (γ), 3b (α)):

$$\begin{gathered} s_{(p,p)}\pi_3 = \nu^{-\frac{3}{2}}\rho \otimes \nu^{-\frac{1}{2}}\rho \otimes L(\nu^{-\frac{5}{2}}\rho, \nu^{-\frac{3}{2}}\rho; \mathcal{T}) + \nu^{-\frac{3}{2}}\rho \otimes \nu^{-\frac{5}{2}}\rho \otimes L(\nu^{-1}\delta(\rho,2); \mathcal{T}) \\ \Downarrow \\ s_{(2p)}\pi_3 = \nu^{-1}\zeta(\rho,2) \otimes L(\nu^{-\frac{5}{2}}\rho, \nu^{-\frac{3}{2}}\rho; \mathcal{T}) + \nu^{-2}\delta(\rho,2) \otimes L(\nu^{-1}\delta(\rho,2); \mathcal{T}) \\ \Downarrow \\ \pi_2 \hookrightarrow \nu^{-2}\delta(\rho,2) \rtimes L(\nu^{-1}\delta(\rho,2); \mathcal{T}) \\ \Downarrow \\ \pi_2 = L(\nu^{-2}\delta(\rho,2), \nu^{-1}\delta(\rho,2); \mathcal{T}), \end{gathered}$$

finishing 2b(ii) (α).

2b(ii) (θ): $1 < j < \frac{k-1}{2}$, $j \neq k-\ell+1$

By induction, we have

$$\begin{aligned} s_{(p)}\pi \;&= \nu^{-k+j+\frac{1}{2}} \otimes L([\nu^{-k+j+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\ &+\nu^{-k+j+\frac{1}{2}} \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-j-\frac{3}{2}}\rho], [\nu^{-k+j+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-j}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ &+\nu^{-k+j+\frac{1}{2}} \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-k+j-\frac{1}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-k+j+1}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ &+\nu^{-j+\frac{1}{2}}\rho \otimes L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-j+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\ &+\nu^{-j+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-j-\frac{1}{2}}\rho], [\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-j+1}\delta(\rho,2) \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ &+\nu^{-j+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-k+j-\frac{3}{2}}\rho], [\nu^{-j+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-k+j}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ &+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\ &+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-j-\frac{3}{2}}\rho], [\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-j}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ &+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-k+j-\frac{3}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-k+j}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}). \end{aligned}$$

By Lemma 3.5 3, $\pi = \pi_1 + \pi_2 + \pi_3$, where

$$\begin{aligned} \pi_1 &= L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\ \pi_2 &= L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-j-\frac{3}{2}}\rho], [\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-j}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ \pi_3 &= L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-k+j-\frac{3}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-k+j}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}). \end{aligned}$$

By Lemma 3.3, $s_{(p)}\pi_1 \geq \tau_1' + \tau_1'' + \tau_1'''$. By Lemma 3.5 1, $s_{(p)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$. To identify $s_{(p)}\pi_2$ and $s_{(p)}\pi_3$, we use the same sort of $s_{(p,p)}/s_{(2p)}$ argument already used a couple of times. By induction (noting that τ' is 2b(ii) (ζ) or (θ), τ'' is 2b(ii) (δ) or (θ), τ''' is 2b(ii) (η) or (θ)), we have

$$\begin{aligned}
s_{(p,p)}\tau_2' &\geq \nu^{-k+j+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-j-\frac{3}{2}}\rho], [\nu^{-k+j+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \\
&\qquad \nu^{-j}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T})
\end{aligned}$$

$$\begin{aligned}
s_{(p,p)}\tau_3' &\geq \nu^{-k+j+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-k+j-\frac{1}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \\
&\qquad \nu^{-k+j+1}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T})
\end{aligned}$$

$$\begin{aligned}
s_{(p,p)}\tau_2'' &\geq \nu^{-j+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-j-\frac{1}{2}}\rho], [\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \\
&\qquad \nu^{-j+1}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T})
\end{aligned}$$

$$\begin{aligned}
s_{(p,p)}\tau_3'' &\geq \nu^{-j+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-k+j-\frac{3}{2}}\rho], [\nu^{-j+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \\
&\qquad \nu^{-k+j}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T})
\end{aligned}$$

$$\begin{aligned}
s_{(p,p)}\tau_2''' &\geq \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-k+j+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-j-\frac{3}{2}}\rho], [\nu^{-k+j+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \\
&\qquad \nu^{-j}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\
&\quad + \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-j+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-j-\frac{1}{2}}\rho], [\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \\
&\qquad \nu^{-j+1}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T})
\end{aligned}$$

$$\begin{aligned}
s_{(p,p)}\tau_3''' &\geq \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-k+j+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-k+j-\frac{1}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \\
&\qquad \nu^{-k+j+1}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\
&\quad + \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-j+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-k+j-\frac{3}{2}}\rho], [\nu^{-j+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \\
&\qquad \nu^{-k+j}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}).
\end{aligned}$$

By Lemma 3.5 1 applied to τ', τ'', τ''', we see that each term above appears only once in $s_{(p,p)}\pi$. Now, we argue as usual. For example, we know $s_{(p)}\pi_2 \geq \tau_2'''$. Therefore, since $\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-k+j+\frac{1}{2}}\rho$

and $\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-j+\frac{1}{2}}\rho$ are irreducible, we have

$$\begin{aligned} s_{(p,p)}\pi_2 \;&\geq \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-k+j+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-j-\frac{3}{2}}\rho], [\nu^{-k+j+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \\ &\qquad\qquad \nu^{-j}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ &+\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-j+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-j-\frac{1}{2}}\rho], [\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \\ &\qquad\qquad \nu^{-j+1}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} s_{(2p)}\pi_2 \;&\geq (\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-k+j+\frac{1}{2}}\rho) \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-j-\frac{3}{2}}\rho], [\nu^{-k+j+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \\ &\qquad\qquad \nu^{-j}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ &+(\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-j+\frac{1}{2}}\rho) \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-j-\frac{1}{2}}\rho], [\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \\ &\qquad\qquad \nu^{-j+1}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} s_{(p,p)}\pi_2 \;&\geq \nu^{-k+j+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-j-\frac{3}{2}}\rho], [\nu^{-k+j+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \\ &\qquad\qquad \nu^{-j}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\ &+\nu^{-j+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-j-\frac{1}{2}}\rho], [\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \\ &\qquad\qquad \nu^{-j+1}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \end{aligned}$$

$$\Downarrow$$

$$s_{(p)}\pi_2 \geq \tau_2' + \tau_2''.$$

Thus $s_{(p)}\pi_2 \geq \tau_2' + \tau_2'' + \tau_2'''$. By Lemma 3.5 1, $s_{(p)}\pi_2 = \tau_2' + \tau_2'' + \tau_2'''$. A similar argument gives $s_{(p)}\pi_3 = \tau_3' + \tau_3'' + \tau_3'''$. (One could also use this argument to get $s_{(p)}\pi_1$.) This finishes 2b(ii) (θ).

2b(ii) (β): $j = 1$, $k = 3$, $\ell > 3$

This subcase parallels 2b(ii) (θ).

2b(ii) (ϵ): $j = \ell - 2$, $k = 2\ell - 3$, $\ell > 3$

Here, $\pi = \nu^{-\frac{1}{2}}\zeta(\rho, 2\ell-3) \rtimes \zeta(\rho, \ell; \sigma)$. By inductive hypothesis,

$$\begin{aligned}
s_{(p)}\pi &= \nu^{-\ell+\frac{3}{2}}\rho \otimes \zeta(\rho, 2\ell-4) \rtimes \zeta(\rho, \ell; \sigma) + \nu^{-\ell+\frac{5}{2}}\rho \otimes \nu^{-1}\zeta(\rho, 2\ell-4) \rtimes \zeta(\rho, \ell; \sigma) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\frac{1}{2}}\zeta(\rho, 2\ell-3) \rtimes \zeta(\rho, \ell-1; \sigma) \\
&= \nu^{-\ell+\frac{3}{2}}\rho \otimes L([\nu^{-\ell+\frac{5}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{5}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\
&+\nu^{-\ell+\frac{3}{2}}\rho \otimes L(\nu^{-\ell+\frac{1}{2}}\rho, [\nu^{-\ell+\frac{5}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+2}\delta(\rho, 2), \dots, \nu^{-1}\delta(\rho, 2); \mathcal{T}) \\
&+\nu^{-\ell+\frac{5}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{7}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\
&+\nu^{-\ell+\frac{5}{2}}\rho \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\ell+\frac{3}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+3}\delta(\rho, 2), \dots, \nu^{-1}\delta(\rho, 2); \mathcal{T}) \\
&+\nu^{-\ell+\frac{5}{2}}\rho \otimes L([\nu^{-\ell+\frac{7}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+1}\delta(\rho, 2), \dots, \nu^{-1}\delta(\rho, 2); \mathcal{T}) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{5}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+2}\delta(\rho, 2), \dots, \nu^{-1}\delta(\rho, 2); \mathcal{T}).
\end{aligned}$$

By Lemma 3.5 2,

$$\pi_1 = L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{5}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$$
$$\pi_2 = L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+2}\delta(\rho, 2), \dots, \nu^{-1}\delta(\rho, 2); \mathcal{T})$$

are components of π. By Lemma 3.3, $s_{(p)}\pi_1 \geq \tau_1'+\tau_1''+\tau_1'''$; by Lemma 3.5 1, $s_{(p)}\pi_1 = \tau_1'+\tau_1''+\tau_1'''$. The same $s_{(p,p)}/s_{(2p)}$ argument as above (cf. 2b(ii) (θ)) tells us $s_{(p)}\pi_2 \geq \tau_2''+\tau_2'''$. Observe that π_2 is also a component of $\nu^{-1}\zeta(\rho, 2\ell-2) \rtimes \zeta(\rho, \ell-1; \sigma)$ by Lemma 3.5 2. A comparison of $s_{(p)}\pi$ and $s_{(p)}\nu^{-1}\zeta(\rho, 2\ell-2) \rtimes \zeta(\rho, \ell-1; \sigma)$ shows that $s_{(p)}\pi_2 \leq \tau_2''+\tau_2'''$; therefore $s_{(p)}\pi_2 = \tau_2''+\tau_2'''$.

Next, take π_3 with $s_{(p)}\pi_3 \geq \tau_3''$. We claim $s_{(p)}\pi_3 = \tau_2'+\tau_3''$. To see this, use the same $s_{(p,p,p)}/s_{(3p)}$ argument as earlier (cf. 3b (α), 2a (β)). If π_0 is the component of π with $s_{(p)}\pi_0 \geq \tau_2'$, then (by inductive hypothesis)

$$\begin{aligned}
s_{(p,p,p)}\pi_0 \geq 2\nu^{-\ell+\frac{3}{2}}\rho \otimes \nu^{-\ell+\frac{5}{2}}\rho \otimes \nu^{-\ell+\frac{5}{2}}\rho \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\ell+\frac{3}{2}}\rho], [\nu^{-\ell+\frac{7}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \\
\nu^{-\ell+3}\delta(\rho, 2), \dots, \nu^{-1}\delta(\rho, 2); \mathcal{T}).
\end{aligned}$$

This implies $s_{(p)}\pi_0$ contains a term of the form $\nu^{-\ell+\frac{5}{2}}\rho \otimes \dots$, i.e., τ_1'', τ_2'', or τ_3''. As τ_1'' and τ_2'' are accounted for, we have $s_{(p)}\pi_3 = \tau_2'+\tau_3''$.

Finally, we show that

$$\pi_3 = L([\nu^{-\ell+\frac{5}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+1}\delta(\rho, 2), \dots, \nu^{-1}\delta(\rho, 2); \mathcal{T})$$

in the usual way for something having the lowest exponent in the Langlands data attached to a $\delta(\rho, 2)$ term (cf. 3b (α), et. al.). From $s_{(p,p)}\pi_3$, we can see that

$$s_{(2p)}\pi_3 \geq \nu^{-\ell+1}\delta(\rho, 2) \otimes L([\nu^{-\ell+\frac{5}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+2}\delta(\rho, 2), \dots, \nu^{-1}\delta(\rho, 2); \mathcal{T}).$$

Since this is the only component of $s_{(2p)}\pi_3$ with this central character, Lemma 3.2 and Frobenius reciprocity imply

$$\pi_3 \hookrightarrow \nu^{-\ell+1}\delta(\rho,2) \rtimes L([\nu^{-\ell+\frac{5}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+2}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \mathcal{T}).$$

By the uniqueness of Langlands subrepresentations, π_3 is as claimed. This finishes 2b(ii) (ϵ).

2b(ii) (γ): $j=1$, $k=\ell$, $\ell>3$

This subcase is similar to 2b(ii) (ϵ) above, but a bit easier. Here, we can determine $s_{(p)}\pi_2$, $s_{(p)}\pi_3$ just using $s_{(p,p)}/s_{(2p)}$ arguments; there is no need to go to $s_{(p,p,p)}/s_{(3p)}$.

2b(ii) (δ): $j=1$, $3<k<\ell$

This subcase parallels 2b(ii) (θ).

2b(ii) (ζ): $j=\frac{k-1}{2}$, $3<k<2\ell-3$

This subcase also parallels 2b(ii) (θ).

2b(ii) (η): $j=k-\ell+1$, $\ell<k<2\ell-3$

This subcase parallels 2b(ii) (γ).

2b(i) (α): $k=\ell-1$

Here, $\pi=\nu^{\frac{-\ell+1}{2}}\zeta(\rho,\ell-1)\rtimes\zeta(\rho,\ell;\sigma)$ and

$$\begin{aligned}
s_{(p)}\pi &= \nu^{-\ell+\frac{3}{2}}\rho\otimes\nu^{\frac{-\ell+2}{2}}\zeta(\rho,\ell-2)\rtimes\zeta(\rho,\ell;\sigma)+\nu^{\frac{1}{2}}\rho\otimes\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell-2)\rtimes\zeta(\rho,\ell;\sigma)\\
&\quad+\nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{\frac{-\ell+1}{2}}\zeta(\rho,\ell-1)\rtimes\zeta(\rho,\ell-1;\sigma)\\
&= \nu^{-\ell+\frac{3}{2}}\rho\otimes L([\nu^{-\ell+\frac{5}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\
&\quad+\nu^{-\ell+\frac{3}{2}}\rho\otimes L([\nu^{-\ell+\frac{5}{2}}\rho,\nu^{-\frac{3}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{3}{2}}\rho];\mathcal{T})\\
&\quad+\nu^{\frac{1}{2}}\rho\otimes L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{3}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\
&\quad+\nu^{-\ell+\frac{1}{2}}\rho\otimes L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\
&\quad+\nu^{-\ell+\frac{1}{2}}\rho\otimes L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{3}{2}}\rho],[\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{3}{2}}\rho];\mathcal{T}).
\end{aligned}$$

Lemma 3.5 2 tells us

$$\begin{aligned}
\pi_1 &= L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\
\pi_2 &= L([\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{3}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{3}{2}}\rho];\mathcal{T})
\end{aligned}$$

are components of π. Lemma 3.3 gives $s_{(p)}\pi_1\geq\tau_1'+\tau_1'''$. We claim $s_{(p)}\pi_2=\tau_2'+\tau''+\tau_2'''$, (implying $s_{(p)}\pi_1=\tau_1'+\tau_1'''$). The usual $s_{(p,p)}/s_{(2p)}$ argument gives $s_{(p)}\pi_2\geq\tau_2'+\tau''+\tau_2'''$–in particular, it shows that the component of π containing τ_2' (resp. τ_2''') necessarily contains τ''. This finishes 2b(i) (α).

2b(i) (β): $k < \ell - 1$

This can be done the same way as 2b(i) (α).

3c (α): $j = \ell + 1$, $k = 2\ell + 3$

We have $\pi = \nu^{-\frac{1}{2}}\zeta(\rho, 2\ell+3) \rtimes \zeta(\rho, \ell; \sigma)$ and

$$\begin{aligned}
s_{(p)}\pi \ &= \nu^{-\ell-\frac{3}{2}}\rho \otimes \zeta(\rho, 2\ell+2) \rtimes \zeta(\rho, \ell; \sigma) + \nu^{-\ell-\frac{1}{2}}\rho \otimes \nu^{-1}\zeta(\rho, 2\ell+2) \rtimes \zeta(\rho, \ell; \sigma) \\
&\quad + \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\frac{1}{2}}\zeta(\rho, 2\ell+3) \rtimes \zeta(\rho, \ell-1; \sigma) \\
&= \nu^{-\ell-\frac{3}{2}}\rho \otimes L([\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\
&\quad + \nu^{-\ell-\frac{3}{2}}\rho \otimes L([\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell}\delta(\rho, 2), \dots, \nu^{-1}\delta(\rho, 2); \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \\
&\quad + \nu^{-\ell-\frac{1}{2}}\rho \otimes L([\nu^{-\ell-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\
&\quad + \nu^{-\ell-\frac{1}{2}}\rho \otimes L(\nu^{-\ell-\frac{3}{2}}\rho, [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell}\delta(\rho, 2), \dots, \nu^{-1}\delta(\rho, 2); \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \\
&\quad + \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\
&\quad + \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell-\frac{3}{2}}\rho, \nu^{-\ell-\frac{1}{2}}\rho], [\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell+1}\delta(\rho, 2), \\
&\qquad\qquad \dots, \nu^{-1}\delta(\rho, 2); \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \\
&\quad + \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell-1}\delta(\rho, 2), \dots, \nu^{-1}\delta(\rho, 2); \delta(\nu^{-\frac{1}{2}}\rho; \sigma)).
\end{aligned}$$

The argument here is like that in 2b(ii) (ϵ). By Lemma 3.5 2,

$$\pi_1 = L([\nu^{-\ell-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$$
$$\pi_2 = L([\nu^{-\ell-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell}\delta(\rho, 2), \dots, \nu^{-1}\delta(\rho, 2); \delta(\nu^{-\frac{1}{2}}\rho; \sigma))$$

are components of π. By Lemmas 3.3 and 3.5 1, $s_{(p)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$. An $s_{(p,p)}/s_{(2p)}$ argument shows $s_{(p)}\pi_2 \geq \tau_2' + \tau_2'''$. Suppose π_3 is the component of π such that $s_{(p)}\pi_3 \geq \tau_3'''$. Then, we may use an $s_{(p,p,p)}/s_{(3p)}$ argument to get $s_{(p)}\pi_3 \geq \tau_2'' + \tau_3'''$:

$$\begin{gathered}
s_{(p,p,p)}\tau_3''' \geq \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell-\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell-\frac{3}{2}}\rho, \nu^{-\ell-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \\
\nu^{-\ell+1}\delta(\rho, 2), \dots, \nu^{-1}\delta(\rho, 2); \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \\
\Downarrow \\
s_{(3p)}\pi_3 \geq (\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-\ell}\zeta(\rho, 2)) \otimes L([\nu^{-\ell-\frac{3}{2}}\rho, \nu^{-\ell-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \\
\nu^{-\ell+1}\delta(\rho, 2), \dots, \nu^{-1}\delta(\rho, 2); \delta(\nu^{-\frac{1}{2}}\rho; \sigma)),
\end{gathered}$$

choosing $\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-\ell}\zeta(\rho,2)$ over $\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{-\ell}\delta(\rho,2)$ because $s_{(p,p)}\pi$ contains no terms of the form $\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes \ldots$. This implies $s_{(p)}\pi_3$ contains a term of the form $\nu^{-\ell-\frac{1}{2}}\rho \otimes \ldots$; necessarily τ_2''. Thus, we have $s_{(p)}\pi_2 = \tau_2' + \tau_2'''$ and $s_{(p)}\pi_3 = \tau_2'' + \tau_3'''$. The usual argument for identifying a component whose lowest exponent in the Langlands data is attached to a $\delta(\rho,2)$ gives

$$\pi_3 = L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell-1}\delta(\rho,2), \ldots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma)).$$

This finishes 3c (α).

<u>3c (β):</u> $j = \ell+1$, $k > 2\ell+3$

This subcase may be done the same way as 2b(ii) (θ).

<u>3c (γ):</u> $j = \frac{k-1}{2}$ (k odd), $j > \ell+1$

This follows 3c (α) except that the $s_{(p,p)}/s_{(2p)}$ arguments are enough to determine $s_{(p)}\pi_2$, $s_{(p)}\pi_3$ (i.e., it is not necessary to use any $s_{(p,p,p)}/s_{(3p)}$ arguments).

<u>3c (δ):</u> $\ell+1 < j < \frac{k-1}{2}$

This subcase is done like 3c (β).

<u>3a (θ):</u> $0 < j < \ell-1$, $j \neq k-\ell-1$

Here, $\pi = \nu^{-\frac{k}{2}+j}\zeta(\rho,k) \rtimes \zeta(\rho,\ell;\sigma)$ and

$$\begin{aligned}
s_{(p)}\pi &= \nu^{-k+j+\frac{1}{2}}\rho \otimes \nu^{-\frac{k}{2}+j+\frac{1}{2}}\zeta(\rho,k-1) \rtimes \zeta(\rho,\ell;\sigma) \\
&+\nu^{-j+\frac{1}{2}}\rho \otimes \nu^{-\frac{k}{2}+j-\frac{1}{2}}\zeta(\rho,k-1) \rtimes \zeta(\rho,\ell;\sigma) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\frac{k}{2}+j}\zeta(\rho,k) \rtimes \zeta(\rho,\ell-1;\sigma) \\
&= \nu^{-k+j+\frac{1}{2}}\rho \otimes L([\nu^{-k+j+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\
&+\nu^{-k+j+\frac{1}{2}}\rho \otimes L([\nu^{-k+j+\frac{3}{2}}\rho, \nu^{-j-\frac{3}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-j}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\
&+\nu^{-k+j+\frac{1}{2}}\rho \otimes L([\nu^{-k+j+\frac{3}{2}}\rho, \nu^{-\ell-\frac{3}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \\
&\qquad \nu^{-\ell}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\
&+\nu^{-k+j+\frac{1}{2}}\rho \otimes L([\nu^{-k+j+\frac{3}{2}}\rho, \nu^{-\ell-\frac{3}{2}}\rho], \nu^{-\ell}\delta(\rho,2), \dots, \nu^{-j-2}\delta(\rho,2), \\
&\qquad \nu^{-j-\frac{1}{2}}\delta(\rho,3), \dots, \nu^{-\frac{1}{2}}\delta(\rho,3); \sigma) \\
&+\nu^{-j+\frac{1}{2}}\rho \otimes L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}], [\nu^{-j+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\
&+\nu^{-j+\frac{1}{2}}\rho \otimes L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-j-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-j+1}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\
&+\nu^{-j+\frac{1}{2}}\rho \otimes L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\ell-\frac{3}{2}}\rho], [\nu^{-j+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \\
&\qquad \nu^{-\ell}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\
&+\nu^{-j+\frac{1}{2}}\rho \otimes L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\ell-\frac{3}{2}}\rho], \nu^{-\ell}\delta(\rho,2), \dots, \nu^{-j-1}\delta(\rho,2), \\
&\qquad \nu^{-j+\frac{1}{2}}\delta(\rho,3), \dots, \nu^{-\frac{1}{2}}\delta(\rho,3); \sigma) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-j-\frac{3}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-j}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \mathcal{T}) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\ell-\frac{1}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \\
&\qquad \nu^{-\ell+1}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\ell-\frac{1}{2}}\rho], \nu^{-\ell+1}\delta(\rho,2), \dots, \nu^{-j-2}\delta(\rho,2), \\
&\qquad \nu^{-j-\frac{1}{2}}\delta(\rho,3), \dots, \nu^{-\frac{1}{2}}\delta(\rho,3); \sigma).
\end{aligned}$$

Lemma 3.5 3 (applied to the $\nu^{-k+j+\frac{1}{2}}\rho$–terms) gives $\pi = \pi_1 + \pi_2 + \pi_3 + \pi_4$, where

$$\pi_1 = L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$$

$$\pi_2 = L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-j-\frac{3}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-j}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \mathcal{T})$$

$$\pi_3 = L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\ell-\frac{3}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma))$$
$$\pi_4 = L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\ell-\frac{3}{2}}\rho], \nu^{-\ell}\delta(\rho,2), \dots, \nu^{-j-2}\delta(\rho,2), \nu^{-j-\frac{1}{2}}\delta(\rho,3), \dots, \nu^{-\frac{1}{2}}\delta(\rho,3); \sigma).$$

We now identify $s_{(p)}\pi_i$. Rather than using an $s_{(p,p)}/s_{(2p)}$ argument, we can argue as follows: observe that π_1 and π_2 are both components of $\nu^{\frac{-\ell+j}{2}}\zeta(\rho,\ell+j) \rtimes \zeta(\rho,k-j;\sigma)$ (case 2b) and π_1 and π_3 are both components of $\nu^{\frac{-k+j+\ell}{2}}\zeta(\rho,k-j+\ell) \rtimes \zeta(\rho,j;\sigma)$ (case 3c). Since cases 2b and 3c are already done, we know $s_{(p)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$, $s_{(p)}\pi_2 = \tau_2' + \tau_2'' + \tau_2'''$, $s_{(p)}\pi_3 = \tau_3' + \tau_3'' + \tau_3'''$. Necessarily, $s_{(p)}\pi_4 = \tau_4' + \tau_4'' + \tau_4'''$. (Note that the comparison of these generalized degenerate principal series plays an important role in section 7.)

<u>3a (α):</u> $j=0$, $\ell=1$, $k=2$

This subcase for ordinary degenerate principal series (i.e., $\rho = 1$ on $F^\times$ and σ the trivial representation of $SO_1(F)$) is done in Theorem 4.5 of [Jan2]. The same argument works here. (Note: the argument is very similar to that used in 3a (γ) below; the comparison used in 3a (θ) above plays an essential role.)

<u>3a (β):</u> $j=0$, $\ell=1$, $k>2$

The same argument as in 3a (θ) works here.

<u>3a (γ):</u> $j=0$, $\ell>1$, $k=\ell+1$

Here $\pi = \nu^{\frac{-\ell-1}{2}}\zeta(\rho,\ell+1) \rtimes \zeta(\rho,\ell;\sigma)$ and

$$\begin{aligned} s_{(p)}\pi &= \nu^{-\ell-\frac{1}{2}}\rho \otimes \nu^{-\frac{\ell}{2}}\zeta(\rho,\ell) \rtimes \zeta(\rho,\ell;\sigma) + \nu^{\frac{1}{2}}\rho \otimes \nu^{-\frac{\ell}{2}-1}\zeta(\rho,\ell) \rtimes \zeta(\rho,\ell;\sigma) \\ &\quad + \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{\frac{-\ell-1}{2}}\zeta(\rho,\ell+1) \rtimes \zeta(\rho,\ell-1;\sigma) \\ &= \nu^{-\ell-\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\ &\quad + \nu^{-\ell-\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho]; \mathcal{T}) \\ &\quad + \nu^{\frac{1}{2}}\rho \otimes L([\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\ &\quad + \nu^{\frac{1}{2}}\rho \otimes L(\nu^{-\ell}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \sigma) \\ &\quad + \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\ &\quad + \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho]; \mathcal{T}) \\ &\quad + \nu^{-\ell+\frac{1}{2}}\rho \otimes L(\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\ell+1}\delta(\rho,2), \dots, \nu^{-1}\delta(\rho,2); \delta(\nu^{-\frac{1}{2}}\rho;\sigma)) \\ &\quad + \nu^{-\ell+\frac{1}{2}}\rho \otimes L(\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\ell+1}\delta(\rho,2), \dots, \nu^{-2}\delta(\rho,2), \nu^{-\frac{1}{2}}\delta(\rho,3); \sigma). \end{aligned}$$

First, Lemma 3.5 2 tells us

$$\pi_1 = L([\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$$
$$\pi_2 = L([\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho]; \mathcal{T})$$

are components of π. The first comparison used in 3a (θ) gives $s_{(p)}\pi_1 = \tau_1' + \tau_1'''$ and $s_{(p)}\pi_2 = \tau_2' + \tau_1'' + \tau_2'''$. Further, Lemma 3.5 1 tells us that π has at least four components. Choose π_3 and π_4 such that $s_{(p)}\pi_3 \geq \tau_3'''$ and $s_{(p)}\pi_4 \geq \tau_4'''$. The usual argument for identifying components when that lowest exponent in the Langlands data is attached to a $\delta(\rho, 2)$ (cf. 3b (α), et. al.) tells us

$$\pi_3 = L(\nu^{-\ell}\delta(\rho, 2), \ldots, \nu^{-1}\delta(\rho, 2); \delta(\nu^{-\frac{1}{2}}\rho; \sigma))$$
$$\pi_4 = L(\nu^{-\ell}\delta(\rho, 2), \ldots, \nu^{-2}\delta(\rho, 2), \nu^{-\frac{1}{2}}\delta(\rho, 3); \sigma).$$

Then, the second comparison from 3a (θ) gives $s_{(p)}\pi_3 = \tau_3'''$. Finally, the usual $s_{(p,p)}/s_{(2p)}$ argument shows that τ_2'' comes from the same component as τ_4''', i.e., $s_{(p)}\pi_4 = \tau_2'' + \tau_4'''$.

<u>3a (δ):</u> $j = 0$, $\ell > 1$, $k = \ell + 1$

This subcase may be done the same way as 3a (θ).

<u>3a (ϵ):</u> $j = k - \ell - 1$, $k = 2\ell$

Here, $\pi = \nu^{-1}\zeta(\rho, 2\ell) \rtimes \zeta(\rho, \ell; \sigma)$ and

$$\begin{aligned}
s_{(p)}\pi &= \nu^{-\ell-\frac{1}{2}}\rho \otimes \nu^{-\frac{1}{2}}\zeta(\rho, 2\ell-1) \rtimes \zeta(\rho, \ell; \sigma) + \nu^{-\ell+\frac{3}{2}}\rho \otimes \nu^{-\frac{3}{2}}\zeta(\rho, 2\ell-1) \rtimes \zeta(\rho, \ell; \sigma) \\
&\quad + \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-1}\zeta(\rho, 2\ell) \rtimes \zeta(\rho, \ell-1; \sigma) \\
&= \nu^{-\ell-\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\
&\quad + \nu^{-\ell-\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+1}\delta(\rho, 2), \ldots, \nu^{-1}\delta(\rho, 2); \mathcal{T}) \\
&\quad + \nu^{-\ell+\frac{3}{2}}\rho \otimes L([\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{5}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\
&\quad + \nu^{-\ell+\frac{3}{2}}\rho \otimes L([\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\ell+\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+2}\delta(\rho, 2), \ldots, \nu^{-1}\delta(\rho, 2); \mathcal{T}) \\
&\quad + \nu^{-\ell+\frac{3}{2}}\rho \otimes L([\nu^{-\ell+\frac{5}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell}\delta(\rho, 2), \ldots, \nu^{-1}\delta(\rho, 2); \delta(\nu^{-\frac{1}{2}}\rho; \sigma)) \\
&\quad + \nu^{-\ell+\frac{3}{2}}\rho \otimes L(\nu^{-\ell}\delta(\rho, 2), \nu^{-\ell+\frac{3}{2}}\delta(\rho, 3), \ldots, \nu^{-\frac{1}{2}}\delta(\rho, 3); \sigma) \\
&\quad + \nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma) \\
&\quad + \nu^{-\ell+\frac{1}{2}}\rho \otimes L(\nu^{-\ell-\frac{1}{2}}\rho, [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell+1}\delta(\rho, 2), \ldots, \nu^{-1}\delta(\rho, 2); \delta(\nu^{-\frac{1}{2}}\rho; \sigma)).
\end{aligned}$$

By Lemma 3.5 2,

$$\pi_1 = L([\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-\ell+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$$
$$\pi_2 = L([\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+1}\delta(\rho, 2), \dots, \nu^{-1}\delta(\rho, 2); \mathcal{T})$$

are components of π. Since π_1 and π_2 are also both components of $\nu^{-\frac{1}{2}}\zeta(\rho, 2\ell-1) \rtimes \zeta(\rho, \ell+1; \sigma)$ (one of the comparisons used in 3a (θ)), we get that $s_{(p)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ and $s_{(p)}\pi_2 = \tau_2' + \tau_2''$. If we choose π_3 such that $s_{(p)}\pi_3 \geq \tau_2'''$, the usual argument for identifying components when the lowest exponent in the Langlands data is attached to a $\delta(\rho, 2)$ tells us (cf. 3b (α), et. al.)

$$\pi_3 = L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \nu^{-\ell}\delta(\rho, 2), \dots, \nu^{-1}\delta(\rho, 2); \delta(\nu^{-\frac{1}{2}}\rho; \sigma)).$$

Again, as in 3a (θ), π_3 is also a component of $\nu^{-\frac{1}{2}}\zeta(\rho, 2\ell + 1) \rtimes \zeta(\rho, \ell - 1; \sigma)$, so we get $s_{(p)}\pi_3 = \tau_3'' + \tau_3'''$.

This leaves a fourth component, with $s_{(p)}\pi_4 = \tau_4''$. We use the same basic idea to identify π_4–whose lowest exponent in its Langlands data is attached to a $\delta(\rho, 3)$–as we would if the lowest exponent were attached to a $\delta(\rho, 2)$. The only difference is that we have to use $s_{(3p)}$ instead of $s_{(2p)}$. In particular,

$$\begin{aligned} s_{(p,p)}\pi_4 &= \nu^{-\ell+\frac{3}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes L(\nu^{-\ell-\frac{1}{2}}\rho, \nu^{-\ell+\frac{3}{2}}\delta(\rho, 3), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 3); \sigma) \\ &\quad + \nu^{-\ell+\frac{3}{2}}\rho \otimes \nu^{-\ell+\frac{5}{2}}\rho \otimes L(\dots) \\ &\Downarrow \\ s_{(p,p,p)}\pi_4 &\geq \nu^{-\ell+\frac{3}{2}}\rho \otimes \nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{-\ell-\frac{1}{2}}\rho \otimes L(\nu^{-\ell+\frac{3}{2}}\delta(\rho, 3), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 3); \sigma) \\ &\Downarrow \\ s_{(3p)}\pi_4 &\geq \nu^{-\ell+\frac{1}{2}}\delta(\rho, 3) \otimes L(\nu^{-\ell+\frac{3}{2}}\delta(\rho, 3), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 3); \sigma) \end{aligned}$$

and this is the only term in $s_{(3p)}\pi_4$ with this central character. Therefore, by Lemma 3.2 and Frobenius reciprocity,

$$\pi_4 \hookrightarrow \nu^{-\ell+\frac{1}{2}}\delta(\rho, 3) \rtimes L(\nu^{-\ell+\frac{3}{2}}\delta(\rho, 3), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 3); \sigma).$$

The uniqueness of Langlands subrepresentations gives

$$\pi_4 = L(\nu^{-\ell+\frac{1}{2}}\delta(\rho, 3), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 3); \sigma).$$

This finishes 3a (ϵ).

<u>3a (ζ):</u> $j = k - \ell - 1 > 0$, $k < 2\ell$

The proof for this subcase parallels that for 3a (γ).

<u>3a (η):</u> $j = \ell - 1$, $j \neq k - \ell - 1$

The proof for this subcase parallels that for 3a (θ).

We now return to case 1.

<u>1 (γ):</u> $-\ell - \frac{k}{2} < \alpha < -\ell + \frac{k}{2} - 1$

Here,

$$\begin{aligned}
s_{(p)}\pi &= \nu^{\alpha+\frac{-k+1}{2}}\rho \otimes \nu^{\alpha+\frac{1}{2}}\zeta(\rho,k-1) \rtimes \zeta(\rho,\ell;\sigma) \\
&+\nu^{-\alpha+\frac{-k+1}{2}}\rho \otimes \nu^{\alpha-\frac{1}{2}}\zeta(\rho,k-1) \rtimes \zeta(\rho,\ell;\sigma) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes \nu^{\alpha}\zeta(\rho,k) \rtimes \zeta(\rho,\ell-1;\sigma) \\
&= \nu^{\alpha+\frac{-k+1}{2}}\rho \otimes L([\nu^{\alpha+\frac{-k+3}{2}}\rho, \nu^{\alpha+\frac{k-1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma) \\
&+\nu^{\alpha+\frac{-k+1}{2}}\rho \otimes L([\nu^{\alpha+\frac{-k+3}{2}}\rho, \nu^{-\ell-\frac{3}{2}}\rho], \nu^{-\ell}\delta(\rho,2),\dots,\nu^{\alpha+\frac{k}{2}}\delta(\rho,2),[\nu^{\alpha+\frac{k+3}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma) \\
&+\nu^{-\alpha+\frac{-k+1}{2}}\rho \otimes L([\nu^{\alpha+\frac{-k+1}{2}}\rho, \nu^{\alpha+\frac{k-3}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma) \\
&+\nu^{-\alpha+\frac{-k+1}{2}}\rho \otimes L([\nu^{\alpha+\frac{-k+1}{2}}\rho, \nu^{-\ell-\frac{3}{2}}\rho], \nu^{-\ell}\delta(\rho,2), \\
&\qquad \dots,\nu^{\alpha+\frac{k}{2}-1}\delta(\rho,2),[\nu^{\alpha+\frac{k+1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{\alpha+\frac{-k+1}{2}}\rho, \nu^{\alpha+\frac{k-1}{2}}\rho],[\nu^{-\ell-\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma) \\
&+\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{\alpha+\frac{-k+1}{2}}\rho, \nu^{-\ell-\frac{1}{2}}\rho], \nu^{-\ell+1}\delta(\rho,2),\dots,\nu^{\alpha+\frac{k}{2}}\delta(\rho,2),[\nu^{\alpha+\frac{k+3}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)
\end{aligned}$$

First, by Lemma 3.5 3 (applied to the $\nu^{\alpha+\frac{-k+1}{2}}\rho$–terms), we get $\pi=\pi_1+\pi_2$ with

$$\pi_1 = L([\nu^{\alpha+\frac{-k+1}{2}}\rho, \nu^{\alpha+\frac{k-1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)$$

$$\pi_2 = L([\nu^{\alpha+\frac{-k+1}{2}}\rho, \nu^{-\ell-\frac{3}{2}}\rho], \nu^{-\ell}\delta(\rho,2),\dots,\nu^{\alpha+\frac{k}{2}}\delta(\rho,2),[\nu^{\alpha+\frac{k+3}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma).$$

Further, since

$$\pi_1 = \nu^{\frac{\alpha}{2}-\frac{\ell}{2}+\frac{k}{4}}\zeta(\rho,\ell+\alpha+\frac{k}{2}) \rtimes \zeta(\rho,-\alpha+\frac{k}{2};\sigma)$$

(irreducible), we have $s_{(p)}\pi_1 = \tau_1'+\tau_1''+\tau_1'''$. Necessarily, $s_{(p)}\pi_2 = \tau_2'+\tau_2''+\tau_2'''$.

<u>1 (α):</u> $\alpha=-\ell-\frac{k}{2}$

In this case, the same argument as in 1 (γ) works. Note that here we have $\pi_1=\zeta(\rho,k+\ell;\sigma)$, which gives $s_{(p)}\pi_1=\tau_1'$.

<u>1 (β):</u> $\alpha=-\ell+\frac{k}{2}-1$

Here

$$\begin{aligned}
s_{(p)}\pi &= \nu^{-\ell-\frac{1}{2}}\rho\otimes\nu^{-\ell+\frac{k}{2}-\frac{1}{2}}\zeta(\rho,k-1)\rtimes\zeta(\rho,\ell;\sigma)\\
&+\nu^{\ell-k+\frac{3}{2}}\rho\otimes\nu^{-\ell+\frac{k}{2}-\frac{3}{2}}\zeta(\rho,k-1)\rtimes\zeta(\rho,\ell;\sigma)\\
&+\nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{-\ell+\frac{k}{2}-1}\zeta(\rho,k)\rtimes\zeta(\rho,\ell-1;\sigma)\\
&= \nu^{-\ell-\frac{1}{2}}\rho\otimes L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\ell+k-\frac{3}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\
&+\nu^{\ell-k+\frac{3}{2}}\rho\otimes L([\nu^{-\ell-\frac{1}{2}}\rho,\nu^{-\ell+k-\frac{5}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\
&+\nu^{\ell-k+\frac{3}{2}}\rho\otimes L(\nu^{-\ell}\delta(\rho,2),\dots,\nu^{-\ell+k-2}\delta(\rho,2),[\nu^{-\ell+k-\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\
&+\nu^{-\ell+\frac{1}{2}}\rho\otimes L([\nu^{-\ell-\frac{1}{2}}\rho,\nu^{-\ell+k-\frac{3}{2}}\rho],[\nu^{-\ell+\frac{3}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)\\
&+\nu^{-\ell+\frac{1}{2}}\rho\otimes L(\nu^{-\ell-\frac{1}{2}}\rho,\nu^{-\ell+1}\delta(\rho,2),\dots,\nu^{-\ell+k-1}\delta(\rho,2),[\nu^{-\ell+k+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)
\end{aligned}$$

By Lemma 3.3, $\pi_1 = L([\nu^{-\ell-\frac{1}{2}}\rho,\nu^{-\ell+k-\frac{3}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)$ and $s_{(p)}\pi_1\geq\tau'+\tau_1'''$. The usual $s_{(p,p)}/s_{(2p)}$ argument tells us $s_{(p)}\pi_1\geq\tau'+\tau_1''+\tau_1'''$ so $s_{(p)}\pi_1=\tau'+\tau_1''+\tau_1'''$.

The usual $s_{(p,p)}/s_{(2p)}$ argument tells us that $s_{(p)}\pi_2=\tau_2''+\tau_2'''$. The identification of π_2 follows the usual argument for a component which has the lowest exponent in its Langlands data attached to a $\delta(\rho,2)$.

This finishes the proof of the theorem. □

We now give the counterpart for the case where ρ satisfies (C1).

THEOREM **6.2.** *Let σ be an irreducible supercuspidal representation of S_m and suppose ρ is a representation of $GL_p(F)$ satisfying (C1). Let $\pi=\nu^{\alpha}\zeta(\rho,k)\rtimes\zeta(\rho,\ell;\sigma)$ with $\alpha\in\mathbb{R}$. Recall that π is reducible if and only if $\alpha\in\{\pm(\ell+\frac{k+1}{2}),\pm(\ell+\frac{k+1}{2}-1),\dots,\pm(\ell+\frac{-k+3}{2})\}\cup\{\{\frac{-k+1}{2},\frac{-k+1}{2}+1,\dots,\frac{k-1}{2}\}\setminus\{0 \text{ if } k=2\ell+1\}\}$. Let S_1 denote the first set; S_2 the second. Suppose π is reducible. By Lemma 2.9, without loss of generality, we may restrict our attention to $\alpha\leq 0$. Note: $\mathcal{T}$ is described in Proposition 3.10.*

(1) $\alpha\in S_1$, $\alpha\notin S_2$
  (a) $\alpha<\frac{-k-1}{2}$
  $\pi=\pi_1+\pi_2$ with

$$\pi_1=L([\nu^{\alpha+\frac{-k+1}{2}}\rho,\nu^{\alpha+\frac{k-1}{2}}\rho],[\nu^{-\ell}\rho,\nu^{-1}\rho];\sigma)$$

$$\begin{aligned}\pi_2=L([&\nu^{\alpha+\frac{-k+1}{2}}\rho,\nu^{-\ell-2}\rho],\nu^{-\ell-\frac{1}{2}}\delta(\rho,2),\\ &\nu^{-\ell+\frac{1}{2}}\delta(\rho,2),\dots,\nu^{\alpha+\frac{k}{2}}\delta(\rho,2),[\nu^{\alpha+\frac{k+3}{2}}\rho,\nu^{-1}\rho];\sigma)\end{aligned}$$

  (b) $\alpha=\frac{-k-1}{2}$

$\pi = \pi_1 + \pi_2$ *with*

$$\pi_1 = L([\nu^{-k}\rho, \nu^{-1}\rho], [\nu^{-\ell}\rho, \nu^{-1}\rho]; \sigma)$$

$$\pi_2 = L([\nu^{-k}\rho, \nu^{-\ell-2}\rho], \nu^{-\ell-\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{3}{2}}\delta(\rho, 2); \delta(\nu^{-1}\rho; \sigma))$$

(2) $\alpha \in S_2$, $\alpha \notin S_1$

Write $\alpha = \frac{-k+1}{2} + j$, $0 \leq j \leq \frac{k-1}{2}$.

(a) $j = k - \ell - 1$ $(j < k - j - 1 = \ell)$

$\pi = \pi_1 + \pi_2$ *with*

$$\pi_1 = L([\nu^{-\ell}\rho, \nu^{-1}\rho], [\nu^{-k+\ell+1}\rho, \nu^{-1}\rho], [\nu^{-\ell}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma)$$

$$\pi_2 = L([\nu^{-\ell}\rho, \nu^{-1}\rho], [\nu^{-\ell}\rho, \nu^{-k+\ell-1}\rho], \nu^{-k+\ell+\frac{1}{2}}\delta(\rho, 2), \nu^{-k+\ell+\frac{3}{2}}\delta(\rho, 2), \dots \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma)$$

(b) $k - \ell - 1 < j < \frac{k-1}{2}$ $(j < k - j - 1 < \ell)$

$\pi = \pi_1 + \pi_2 + \pi_3$ *with*

$$\pi_1 = L([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho], [\nu^{-\ell}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma)$$

$$\pi_2 = L([\nu^{-\ell}\rho, \nu^{-j-2}\rho], [\nu^{-k+j+1}\rho, \nu^{-1}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho, 2), \nu^{-j+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma)$$

$$\pi_3 = L([\nu^{-\ell}\rho, \nu^{-k+j-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho], \nu^{-k+j+\frac{1}{2}}\delta(\rho, 2), \nu^{-k+j+\frac{3}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma)$$

(c) $j = \frac{k-1}{2}$, k *odd* $(j = k - j - 1 < \ell)$

$\pi = \pi_1 + \pi_2$ *with*

$$\pi_1 = L([\nu^{\frac{-k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{-\ell}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma)$$

$$\pi_2 = L([\nu^{-\ell}\rho, \nu^{\frac{-k-3}{2}}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-1}\rho], \nu^{-\frac{k}{2}}\delta(\rho, 2), \nu^{-\frac{k}{2}+1}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma)$$

(3) $\alpha \in S_1 \cap S_2$

Write $\alpha = \frac{-k+1}{2} + j$, $0 \leq j \leq \frac{k-1}{2}$.

(a) $j \leq \ell - 1$ $(j < \ell < k - j - 1)$

$\pi = \pi_1 + \pi_2 + \pi_3 + \pi_4$ *with*

$$\pi_1 = L([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho], [\nu^{-\ell}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma)$$

$$\pi_2 = L([\nu^{-k+j+1}\rho, \nu^{-j-2}\rho], [\nu^{-\ell}\rho, \nu^{-1}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho, 2), \nu^{-j+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma)$$

$$\pi_3 = L([\nu^{-k+j+1}\rho, \nu^{-\ell-2}\rho], [\nu^{-j}\rho, \nu^{-1}\rho], \nu^{-\ell-\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{3}{2}}\delta(\rho, 2); \mathcal{T})$$

$$\pi_4 = L([\nu^{-k+j+1}\rho, \nu^{-\ell-2}\rho], \nu^{-\ell-\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-j-\frac{5}{2}}\delta(\rho, 2), \nu^{-j-1}\delta(\rho, 3), \nu^{-j}\delta(\rho, 3), \dots, \nu^{-1}\delta(\rho, 3); \delta(\nu^{-1}\rho; \sigma))$$

(b) $j = \ell$ $(j = \ell < k - j - 1)$

$\pi = \pi_1 + \pi_2$ *with*

$$\pi_1 = L([\nu^{-k+\ell+1}\rho, \nu^{-1}\rho], [\nu^{-\ell}\rho, \nu^{-1}\rho], [\nu^{-\ell}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma)$$

$$\pi_2 = L([\nu^{-k+\ell+1}\rho, \nu^{-\ell-2}\rho], [\nu^{-\ell}\rho, \nu^{-1}\rho], \nu^{-\ell-\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{3}{2}}\delta(\rho, 2); \mathcal{T})$$

(c) $\ell < j < \frac{k-1}{2}$ $(\ell < j < k - j - 1)$

$\pi = \pi_1 + \pi_2 + \pi_3$ *with*

$$\pi_1 = L([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho], [\nu^{-\ell}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma)$$

$$\pi_2 = L([\nu^{-k+j+1}\rho, \nu^{-\ell-2}\rho], [\nu^{-j}\rho, \nu^{-1}\rho], \nu^{-\ell-\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{3}{2}}\delta(\rho, 2); \mathcal{T})$$

$$\pi_3 = L([\nu^{-k+j+1}\rho, \nu^{-j-2}\rho], [\nu^{-\ell}\rho, \nu^{-1}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho, 2), \nu^{-j+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{3}{2}}\delta(\rho, 2); \mathcal{T})$$

(d) $j = \frac{k-1}{2}$, k *odd* $(\ell < j = k - j - 1)$

$\pi = \pi_1 + \pi_2$ *with*

$$\pi_1 = L([\nu^{\frac{-k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{-\ell}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma)$$

$$\pi_2 = L([\nu^{\frac{-k+1}{2}}\rho, \nu^{-\ell-2}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-1}\rho], \nu^{-\ell-\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{3}{2}}\delta(\rho, 2); \mathcal{T})$$

The proof of the theorem also gives us $s_{(p)}\pi_i$ for each component π_i of π (summarized in the tables); for reasons of space, we do not write them out here. (Note that the tables only give the Jacquet modules for $k \geq 2$, $\ell \geq 1$; for $k = 1$ or $\ell = 0$, see section 3.)

Proof. The proof is basically the same as that for Theorem 6.1, so we will not go into any detail. However, we do include the counterpart to the tables of Jacquet modules used in the proof. Note that for $\pi=\nu^{\frac{-k+1}{2}+j}\zeta(\rho,k)\rtimes\zeta(\rho,\ell;\sigma)$, we have

$$\begin{aligned} s_{(p)}\pi \;&= \nu^{-k+j+1}\rho\otimes\nu^{-\frac{k}{2}+j+1}\zeta(\rho,k-1)\rtimes\zeta(\rho,\ell;\sigma) \\ &+\nu^{-j}\rho\otimes\nu^{-\frac{k}{2}+j}\zeta(\rho,k-1)\rtimes\zeta(\rho,\ell;\sigma) \\ &+\nu^{-\ell}\rho\otimes\nu^{\frac{-k+1}{2}+j}\zeta(\rho,k)\rtimes\zeta(\rho,\ell-1;\sigma). \end{aligned}$$

Case (for π)	for τ'	for τ''	for τ'''	components
1a.				
(α) $\alpha = -\ell + \frac{-k-1}{2}$	1a	irr	irr	$s_{(p)}\pi_1 = \tau_1'$ $s_{(p)}\pi_2 = \tau_2' + \tau'' + \tau'''$
(β) $\alpha = -\ell + \frac{k-3}{2}$	irr	1a	1a	$s_{(p)}\pi_1 = \tau' + \tau_1'' + \tau_1'''$ $s_{(p)}\pi_2 = \tau_2'' + \tau_2'''$
(γ) $-\ell + \frac{-k-1}{2} < \alpha$, $\alpha < -\ell + \frac{k-3}{2}$	1a	1a	1a	$s_{(p)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(p)}\pi_2 = \tau_2' + \tau_2'' + \tau_2'''$
1b.				
(α) $\ell = k-1$	irr	1a	1b	$s_{(p)}\pi_1 = \tau' + \tau_1'' + \tau_1'''$ $s_{(p)}\pi_2 = \tau_2'' + \tau_2'''$
(β) $\ell < k-1$	1b	1a	1b	$s_{(p)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(p)}\pi_2 = \tau_2' + \tau_2'' + \tau_2'''$
2a.				
(α) $k = 2$, $\ell = 1$	2c	irr	3b	$s_{(p)}\pi_1 = \tau_1' + \tau_1''' + \tau_2'''$ $s_{(p)}\pi_2 = \tau_2' + \tau''$
(β) $k = \ell + 1$, $\ell > 1$	2b	irr	3a	$s_{(p)}\pi_1 = \tau_1' + \tau_1''' + \tau_3'''$ $s_{(p)}\pi_2 = \tau_2' + \tau_3' + \tau'' + \tau_2''' + \tau_4'''$
(γ) $k = 2\ell$, $\ell > 1$	2c	2a	3b	$s_{(p)}\pi_1 = \tau_1' + \tau_1'' + \tau_1''' + \tau_2'''$ $s_{(p)}\pi_2 = \tau_2' + \tau_2''$
(δ) $\ell + 1 < k < 2\ell$	2b	2a	3a	$s_{(p)}\pi_1 = \tau_1' + \tau_1'' + \tau_1''' + \tau_3'''$ $s_{(p)}\pi_2 = \tau_2' + \tau_3' + \tau_2'' + \tau_2''' + \tau_4'''$
2b.				
(α) $j = 0$, $k = 2$, $\ell = 2$	2c	irr	2a	$s_{(p)}\pi_1 = \tau_1' + \tau_1'''$ $s_{(p)}\pi_2 = \tau'' + \tau_2'''$ $s_{(p)}\pi_3 = \tau_2'$
(β) $j = 0$, $k = 2$, $\ell > 2$	2c	irr	2b	$s_{(p)}\pi_1 = \tau_1' + \tau_1'''$ $s_{(p)}\pi_2 = \tau'' + \tau_2'''$ $s_{(p)}\pi_3 = \tau_2' + \tau_3'''$
(γ) $j = 0$, $k = \ell$, $\ell > 2$	2b	irr	2a	$s_{(p)}\pi_1 = \tau_1' + \tau_1'''$ $s_{(p)}\pi_2 = \tau_2' + \tau'' + \tau_2'''$ $s_{(p)}\pi_3 = \tau_3'$
(δ) $j = 0$, $2 < k < \ell$	2b	irr	2b	$s_{(p)}\pi_1 = \tau_1' + \tau_1'''$ $s_{(p)}\pi_2 = \tau_2' + \tau'' + \tau_2'''$ $s_{(p)}\pi_3 = \tau_3' + \tau_3'''$
(ϵ) $j = \ell - 2$, $k = 2\ell - 2$, $\ell > 2$	2c	2b	2a	$s_{(p)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''$ $s_{(p)}\pi_2 = \tau_2'' + \tau_2'''$ $s_{(p)}\pi_3 = \tau_2' + \tau_3''$

Case (for π)	for τ'	for τ''	for τ'''	components
(ζ) $j=\frac{k}{2}-1$, $2<k<2\ell-2$	2c	2b	2b	$s_{(p)}\pi_1=\tau'_1+\tau''_1+\tau'''_1$ $s_{(p)}\pi_2=\tau''_2+\tau'''_2$ $s_{(p)}\pi_3=\tau'_2+\tau''_3+\tau'''_3$
(η) $j=k-\ell$, $\ell<k<2\ell-2$	2b	2b	2a	$s_{(p)}\pi_1=\tau'_1+\tau''_1+\tau'''_1$ $s_{(p)}\pi_2=\tau'_2+\tau''_2+\tau'''_2$ $s_{(p)}\pi_3=\tau'_3+\tau''_3$
(θ) $0<j<\frac{k}{2}-1$, $j>k-\ell$	2b	2b	2b	$s_{(p)}\pi_1=\tau'_1+\tau''_1+\tau'''_1$ $s_{(p)}\pi_2=\tau'_2+\tau''_2+\tau'''_2$ $s_{(p)}\pi_3=\tau'_3+\tau''_3+\tau'''_3$
2c.				(n.b. $\tau'=\tau''$)
(α) $k=2\ell-1$	2b	2b	irr	$s_{(p)}\pi_1=\tau'_1+\tau'_2+\tau''_1+\tau'''$ $s_{(p)}\pi_2=\tau'_3+\tau''_2+\tau''_3$
(β) $k<2\ell-1$	2b	2b	2c	$s_{(p)}\pi_1=\tau'_1+\tau'_2+\tau''_1+\tau'''_1$ $s_{(p)}\pi_2=\tau'_3+\tau''_2+\tau''_3+\tau'''_2$
3a.				
(α) $j=0$, $\ell=1$, $k=3$	2a	1b	3b	$s_{(p)}\pi_1=\tau'_1+\tau'''_1$ $s_{(p)}\pi_2=\tau'_2+\tau''_1$ $s_{(p)}\pi_3=\tau'''_2$ $s_{(p)}\pi_4=\tau''_2$
(β) $j=0$, $\ell=1$, $k>3$	3a	1b	3b	$s_{(p)}\pi_1=\tau'_1+\tau'''_1$ $s_{(p)}\pi_2=\tau'_2+\tau''_1$ $s_{(p)}\pi_3=\tau'_3+\tau'''_2$ $s_{(p)}\pi_4=\tau'_4+\tau''_2$
(γ) $j=0$, $\ell=k-2$, $k>3$	2a	1b	3a	$s_{(p)}\pi_1=\tau'_1+\tau'''_1$ $s_{(p)}\pi_2=\tau'_2+\tau''_1+\tau'''_2$ $s_{(p)}\pi_3=\tau'''_3$ $s_{(p)}\pi_4=\tau''_2+\tau'''_4$
(δ) $j=0$, $1<\ell<k-2$	3a	1b	3a	$s_{(p)}\pi_1=\tau'_1+\tau'''_1$ $s_{(p)}\pi_2=\tau'_2+\tau''_1+\tau'''_2$ $s_{(p)}\pi_3=\tau'_3+\tau'''_3$ $s_{(p)}\pi_4=\tau'_4+\tau''_2+\tau'''_4$
(ϵ) $j=\ell-1$, $k=2\ell+1$, $\ell>1$	2a	3a	3b	$s_{(p)}\pi_1=\tau'_1+\tau''_1+\tau'''_1$ $s_{(p)}\pi_2=\tau'_2+\tau''_2$ $s_{(p)}\pi_3=\tau''_3+\tau'''_2$ $s_{(p)}\pi_4=\tau''_4$
(ζ) $j=k-\ell-2$, $\ell+2<k<2\ell+1$	2a	3a	3a	$s_{(p)}\pi_1=\tau'_1+\tau''_1+\tau'''_1$ $s_{(p)}\pi_2=\tau'_2+\tau''_2+\tau'''_2$ $s_{(p)}\pi_3=\tau''_3+\tau'''_3$ $s_{(p)}\pi_4=\tau''_4+\tau'''_4$

Case (for π)	for τ'	for τ''	for τ'''	components
(η) $j=\ell-1$, $1<\ell<\frac{k-1}{2}$	3a	3a	3b	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2'+\tau_2''$ $s_{(p)}\pi_3=\tau_3'+\tau_3''+\tau_2'''$ $s_{(p)}\pi_4=\tau_4'+\tau_4''$
(θ) $0<j<\ell-1$, $j<k-\ell-2$	3a	3a	3a	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2'+\tau_2''+\tau_2'''$ $s_{(p)}\pi_3=\tau_3'+\tau_3''+\tau_3'''$ $s_{(p)}\pi_4=\tau_4'+\tau_4''+\tau_4'''$
3b.				
(α) $k=2\ell+2$	irr	3a	3c	$s_{(p)}\pi_1=\tau'+\tau_1''+\tau_2''+\tau_1'''$ $s_{(p)}\pi_2=\tau_3''+\tau_4''+\tau_2'''+\tau_3'''$
(β) $k>2\ell+2$	3b	3a	3c	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_2''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2'+\tau_3''+\tau_4''+\tau_2'''+\tau_3'''$
3c.				
(α) $j=\ell+1$, $k=2\ell+4$	3d	3b	3c	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2'+\tau_2'''$ $s_{(p)}\pi_3=\tau_2''+\tau_3'''$
(β) $j=\ell+1$, $k>2\ell+4$	3c	3b	3c	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2'+\tau_2'''$ $s_{(p)}\pi_3=\tau_3'+\tau_2''+\tau_3'''$
(γ) $j=\frac{k}{2}-1$, $k>2\ell+4$	3d	3c	3c	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2'+\tau_2''+\tau_2'''$ $s_{(p)}\pi_3=\tau_3''+\tau_3'''$
(δ) $\ell+1<j<\frac{k}{2}-1$	3c	3c	3c	$s_{(p)}\pi_1=\tau_1'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2'+\tau_2''+\tau_2'''$ $s_{(p)}\pi_3=\tau_3'+\tau_3''+\tau_3'''$
3d.				(n.b. $\tau'=\tau''$)
(α) $k=2\ell+3$	3b	3b	3d	$s_{(p)}\pi_1=\tau_1'+\tau_2'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2''+\tau_2'''$
(β) $k>2\ell+3$	3c	3c	3d	$s_{(p)}\pi_1=\tau_1'+\tau_3'+\tau_1''+\tau_1'''$ $s_{(p)}\pi_2=\tau_2'+\tau_2''+\tau_3''+\tau_2'''$

7. Composition series

In this section, we give the composition series for $\pi=\nu^{\alpha}\zeta(\rho_0,k)\rtimes\zeta(\rho,\ell;\sigma)$. If ρ satisfies (C1/2) and $\rho_0\cong\rho$, the components of π are given in Theorem 6.1; the composition series are described in Theorem 7.1 below. Theorem 7.1 also gives composition series when $\rho_0 \not\cong \rho$ but ρ_0 satisfies (C1/2) (i.e., the representations whose components are described in Corollary 5.7). Similarly, if ρ satisfies (C1) and $\rho_0 \cong \rho$, the components are given in Theorem 6.2; the composition series in Theorem 7.2 below. Theorem 7.2 also gives the composition series when $\rho_0 \not\cong \rho$ has ρ_0 satisfying (C0) (i.e., the representations whose components are described in Corollary 5.8). The main tool is Frobenius reciprocity, using a comparison between different generalized degenerate principal series which have common components (cf. proof of 3a in Theorem 6.1) to isolate key Jacquet module components.

Theorem **7.1.** *Let σ be an irreducible supercuspidal representation of S_m and suppose that ρ is a representation of $GL_p(F)$ satisfying (C1/2). Let $\pi = \nu^{\alpha}\zeta(\rho, k) \rtimes \zeta(\rho, \ell; \sigma)$, $\alpha \leq 0$, be reducible. The components of π are described in Theorem 6.1. In the notation of that theorem, we have the following:*

case 1: π_1 *is the unique irreducible subrepresentation;* π_2 *the unique irreducible quotient.*
case 2a: π_1 *is the unique irreducible subrepresentation;* π_2 *the unique irreducible quotient.*
case 2b(i): π_1 *is the unique irreducible subrepresentation;* π_2 *the unique irreducible quotient.*
case 2b(ii): $\pi_1 \oplus \pi_3$ *is a subrepresentation;* π_2 *the unique irreducible quotient.*
case 2c: $\pi \cong \pi_1 \oplus \pi_2$.
case 3a: π_1 *is the unique irreducible subrepresentation;* π_4 *the unique irreducible quotient.* $\pi_2 \oplus \pi_3$ *is a subquotient.*
case 3b: π_1 *is the unique irreducible subrepresentation;* π_2 *the unique irreducible quotient.*
case 3c: $\pi_1 \oplus \pi_2$ *is a subrepresentation;* π_3 *the unique irreducible quotient.*
case 3d: $\pi \cong \pi_1 \oplus \pi_2$

Suppose $\rho_0 \not\cong \rho$ is a representation of $GL_{p_0}(F)$ which also satisfies (C1/2). Let $\pi = \nu^{\alpha}\zeta(\rho_0, k) \rtimes \zeta(\rho, \ell; \sigma)$, $\alpha \leq 0$, be reducible. Then, the components of π are described in Corollary 5.7. In the notation of that corollary, we have the following:

case 1: π_1 *is the unique irreducible subrepresentation;* π_2 *the unique irreducible quotient.*
case 2: $\pi_1 \oplus \pi_2$ *is a subrepresentation;* π_3 *the unique irreducible quotient.*
case 3: $\pi \cong \pi_1 \oplus \pi_2$.

For $\alpha > 0$, the order of composition series is reversed from that of $-\alpha$.

Proof. We start with the last claim first–in particular, we begin by relating composition series for $\nu^{-\alpha}\zeta(\rho_0, k) \rtimes \zeta(\rho, \ell; \sigma)$ to composition series for $\nu^{\alpha}\zeta(\rho_0, k) \rtimes \zeta(\rho, \ell; \sigma)$. For example, suppose $\pi = \nu^{-\frac{k}{2}+j}\zeta(\rho, k) \rtimes \zeta(\rho, \ell; \sigma)$ $0 \leq j < \frac{k}{2}$, decomposes according to case 3a. Then π_1 is the unique irreducible subrepresentation, π_4 the unique irreducible quotient, and $\pi_2 \oplus \pi_3$ a subquotient. We claim that $\nu^{\frac{k}{2}-j}\zeta(\rho, k) \rtimes \zeta(\rho, \ell; \sigma)$ has π_1 as unique irreducible quotient, π_4 as unique irreducible subrepresentation, and $\pi_2 \oplus \pi_3$ as a subquotient. To see this, consider $\tau = \nu^{-\frac{k}{2}+j}\zeta(\rho, k) \rtimes \zeta(\rho, \ell; \tilde{\sigma})$. Then τ also decomposes according to case 3a. So, τ_1 is the unique irreducible subrepresentation, τ_4 the unique irreducible quotient, and $\tau_2 \oplus \tau_3$ a subquotient. Now, by Lemma 2.9, $\tau_i = \tilde{\pi}_i$ for $i = 1,2,3,4$. Since $\tilde{\tau} = \nu^{\frac{k}{2}-j}\zeta(\rho,k) \rtimes \zeta(\rho,\ell;\sigma)$, taking contragredients gives the composition series claimed.

We address the case $\rho_0 \cong \rho$ first. Note that cases 2c and 3d are clear.

We start with the most difficult part of the proof: cases 2b, 3a, and 3c. We do these together to make use of the following observation (which was also used in the proof for 3a in Theorem 6.1): if $\pi = \nu^{-\frac{k}{2}+j}\zeta(\rho,k) \rtimes \zeta(\rho,\ell;\sigma)$ is case 3a, then $\pi' = \nu^{\frac{-\ell+j}{2}}\zeta(\rho,\ell+j) \rtimes \zeta(\rho,k-j;\sigma)$ and $\pi'' = \nu^{\frac{-k+j+\ell}{2}}\zeta(\rho,k+\ell-j) \rtimes \zeta(\rho,j;\sigma)$ (cases 2b and 3c, respectively) have $\pi_1 = \pi_1' = \pi_1''$, $\pi_2 = \pi_2'$, and $\pi_3 = \pi_3''$. Further, every representation from case 2b (resp. case 3c) arises as such a π' (resp. π'').

We start by addressing π'. For the time being, let us assume π' is case 2b(ii). We begin by showing that π_2 is the unique irreducible quotient. Recall that $\tilde{\pi}' = \nu^{\frac{\ell-j}{2}}\zeta(\rho,\ell+j) \rtimes \zeta(\rho,k-j;\tilde{\sigma})$, $\tilde{\pi} = \nu^{\frac{k}{2}-j}\zeta(\rho,k) \rtimes \zeta(\rho,\ell;\tilde{\sigma})$, and $\tilde{\pi}'' = \nu^{\frac{k-j-\ell}{2}}\zeta(\rho,k+\ell-j) \rtimes \zeta(\rho,j;\tilde{\sigma})$. Now, observe that

(1) $s_{((\ell+j)p)}\tilde{\pi}'$ contains $\nu^{\frac{\ell-j}{2}}\zeta(\rho,\ell+j) \otimes \zeta(\rho,k-j;\tilde{\sigma})$ with multiplicity one.

(2) $s_{((\ell+j)p)}\tilde{\pi}$ contains $\nu^{\frac{\ell-j}{2}}\zeta(\rho,\ell+j) \otimes \zeta(\rho,k-j;\tilde{\sigma})$ with multiplicity one.

(3) $s_{((\ell+j)p)}\tilde{\pi}''$ does not contain $\nu^{\frac{\ell-j}{2}}\zeta(\rho,\ell+j) \otimes \zeta(\rho,k-j;\tilde{\sigma})$.

(To see this, consider $s_{min}\tilde{\pi}'$, $s_{min}\tilde{\pi}$, $s_{min}\tilde{\pi}''$; cf. Theorem 2.3 et seq.) Now, (3) implies that $s_{((\ell+j)p)}\tilde{\pi}_1'$ does not contain $\nu^{\frac{\ell-j}{2}}\zeta(\rho,\ell+j) \otimes \zeta(\rho,k-j;\tilde{\sigma})$. Next, observe that since

$$\pi' \hookrightarrow \nu^{\frac{-\ell+j}{2}}\zeta(\rho,\ell+j) \rtimes (\nu^{\frac{-k+j-\ell}{2}}\zeta(\rho,k-j-\ell) \rtimes \zeta(\rho,\ell;\sigma))$$

$$\pi \hookrightarrow (\nu^{\frac{-k+j-\ell}{2}}\zeta(\rho,k-j-\ell) \times \nu^{\frac{-\ell+j}{2}}\zeta(\rho,\ell+j)) \rtimes \zeta(\rho,\ell;\sigma),$$

we have $\tilde{\pi}, \tilde{\pi}' < \nu^{\frac{\ell-j}{2}}\zeta(\rho,\ell+j) \times \nu^{\frac{k-j+\ell}{2}}\zeta(\rho,k-j-\ell) \rtimes \zeta(\rho,\ell;\tilde{\sigma})$. Since $s_{((\ell+j)p)}\nu^{\frac{\ell-j}{2}}\zeta(\rho,\ell+j) \times \nu^{\frac{k-j+\ell}{2}}\zeta(\rho,k-j-\ell) \rtimes \zeta(\rho,\ell;\tilde{\sigma})$ contains $\nu^{\frac{\ell-j}{2}}\zeta(\rho,\ell+j) \rtimes \zeta(\rho,k-j;\tilde{\sigma})$ with multiplicity one (again, cf. Theorem 2.3 et seq.), we see that the copies of $\nu^{\frac{\ell-j}{2}}\zeta(\rho,\ell+j) \otimes \zeta(\rho,k-j;\tilde{\sigma})$ in $s_{((\ell+j)p)}\tilde{\pi}'$ and $s_{((\ell+j)p)}\tilde{\pi}$ must come from a common component of $\tilde{\pi}$ and $\tilde{\pi}'$, necessarily $\tilde{\pi}_2 = \tilde{\pi}_2'$. By Frobenius reciprocity, an irreducible subrepresentation of $\tilde{\pi}'$ must contain $\nu^{\frac{\ell-j}{2}}\zeta(\rho,\ell+j) \otimes \zeta(\rho,k-j;\tilde{\sigma})$ in its $s_{((\ell+j)p)}$. Therefore, $\tilde{\pi}_2'$ is the only possible irreducible subrepresentation of $\tilde{\pi}'$, making π_2' the unique irreducible quotient of π'.

Next, we turn to the task of showing that $\pi_1' \oplus \pi_3'$ is a subrepresentation of π'. We work inductively, so we begin by assuming $\ell = j+1$. Then, $\pi' = \nu^{-\frac{1}{2}}\zeta(\rho,2j+1) \rtimes \zeta(\rho,k-j;\sigma)$. Observe that

$$\begin{aligned} \pi' \hookrightarrow \;& (\nu^{-j-\frac{1}{2}}\rho \times \zeta(\rho,2j)) \rtimes \zeta(\rho,k-j;\sigma) \\ \cong \;& \nu^{-j-\frac{1}{2}}\rho \rtimes (\zeta(\rho,2j) \rtimes \zeta(\rho,k-j;\sigma)) \end{aligned}$$

which admits both

$$\nu^{-j-\frac{1}{2}}\rho \rtimes L([\nu^{-j+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-j+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],[\nu^{-k+j+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho];\sigma)$$

and

$$\nu^{-j-\frac{1}{2}}\rho \rtimes L([\nu^{-k+j+\frac{1}{2}}\rho,\nu^{-j-\frac{3}{2}}\rho],[\nu^{-j+\frac{1}{2}}\rho,\nu^{-\frac{3}{2}}\rho],\nu^{-j}\delta(\rho,2),\ldots,\nu^{-1}\delta(\rho,2);\mathcal{T})$$

as subrepresentations (noting that $\zeta(\rho, 2j) \rtimes \zeta(\rho, k-j; \sigma)$ is case 2c). The usual argument using Lemma 3.2 and Frobenius reciprocity tells us that π_1' is a subrepresentation of the first and π_3' is a subrepresentation of the second (n.b. Theorem 6.1 gives us $s_{(p)}\pi_1'$, $s_{(p)}\pi_3'$; see 2b(ii) (α), (β), (ϵ), (ζ) in the table). Further, a μ^* calculation tells us that

$$\nu^{-j-\frac{1}{2}}\rho \otimes L([\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$$

and

$$\nu^{-j-\frac{1}{2}}\rho \otimes L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-j-\frac{3}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-j}\delta(\rho, 2), \dots, \nu^{-1}\delta(\rho, 2); \mathcal{T})$$

each appear in $s_{(p)}\nu^{-j-\frac{1}{2}}\rho \times \zeta(\rho, 2j) \rtimes \zeta(\rho, k-j; \sigma)$ with multiplicity one. Therefore π_1' and π_3' each appear in $\nu^{-j-\frac{1}{2}}\rho \times \zeta(\rho, 2j) \rtimes \zeta(\rho, k-j; \sigma)$ exactly once. Since π', $\pi_1' \oplus \pi_3'$ are subrepresentations of $\nu^{-j-\frac{1}{2}}\rho \times \zeta(\rho, 2j) \rtimes \zeta(\rho, k-j; \sigma)$ and π_1', π_3' appear only once in $\nu^{-j-\frac{1}{2}}\rho \times \zeta(\rho, 2j) \rtimes \zeta(\rho, k-j; \sigma)$, we see that π' must contain $\pi_1' \oplus \pi_3'$ as a subrepresentation, as needed. (To see this, just consider the subspace of the larger representation formed by image$(\pi') + (V_{\pi_1'} \oplus V_{\pi_3'})$.)

Next, suppose $\ell > j+1$. Observe that

$$\pi' \hookrightarrow \nu^{-\ell+\frac{1}{2}}\rho \rtimes (\nu^{\frac{-\ell+j+1}{2}}\zeta(\rho, \ell+j-1) \rtimes \zeta(\rho, k-j; \sigma)),$$

with $\nu^{\frac{-\ell+j+1}{2}}\zeta(\rho, \ell+j-1) \rtimes \zeta(\rho, k-j; \sigma)$ a case 2b(ii) representation. By the inductive hypothesis and exactness, we see that $\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{\frac{-\ell+j+1}{2}}\zeta(\rho, \ell+j-1) \rtimes \zeta(\rho, k-j; \sigma)$ has

$$\nu^{-\ell+\frac{1}{2}}\rho \rtimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$$

and

$$\nu^{-\ell+\frac{1}{2}}\rho \rtimes L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\ell-\frac{1}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+1}\delta(\rho, 2), \dots, \nu^{-1}\delta(\rho, 2); \mathcal{T})$$

as subrepresentations. Again, the usual argument using Frobenius reciprocity and Lemma 3.2 tells us that π_1' is a subrepresentation of the first and π_3' a subrepresentation of the second. Further, a μ^* calculation tells us that

$$\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-\ell+\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho], [\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho]; \sigma)$$

and

$$\nu^{-\ell+\frac{1}{2}}\rho \otimes L([\nu^{-k+j+\frac{1}{2}}\rho, \nu^{-\ell-\frac{1}{2}}\rho], [\nu^{-j+\frac{1}{2}}\rho, \nu^{-\frac{3}{2}}\rho], \nu^{-\ell+1}\delta(\rho, 2), \dots, \nu^{-1}\delta(\rho, 2); \mathcal{T})$$

each appear in $s_{(p)}\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{\frac{-\ell+j+1}{2}}\zeta(\rho, \ell+j-1) \rtimes \zeta(\rho, k-j; \sigma)$ with multiplicity one. Therefore, π_1' and π_3' each appear only once in $\nu^{-\ell+\frac{1}{2}}\rho \times \nu^{\frac{-\ell+j+1}{2}}\zeta(\rho, \ell+j-1) \rtimes \zeta(\rho, k-j; \sigma)$; as subrepresentations. Since $\pi' \hookrightarrow \nu^{-\ell+\frac{1}{2}}\rho \times \nu^{\frac{-\ell+j+1}{2}}\zeta(\rho, \ell+j-1) \rtimes \zeta(\rho, k-j; \sigma)$, we again get that $\pi_1' \oplus \pi_3'$ is a subrepresentation of π', as required. This finishes up the analysis for π' when π' is 2b(ii).

When π' is 2b(i), things are much easier. The same proof shows that π_2' is the unique irreducible quotient. This forces π_1' to be the unique irreducible subrepresentation.

The argument for π'' (case 3c) is the same as that for π' above (when π' is 2b(ii)).

We now consider π. The fact that π_4 is the unique irreducible quotient is similar to the proof that π_2' is the unique irreducible quotient of π', only easier. Here, we have

(1) $s_{(kp)}\tilde{\pi'}$ does not contain $\nu^{\frac{k}{2}-j}\zeta(\rho,k)\otimes\zeta(\rho,\ell;\tilde{\sigma})$.

(2) $s_{(kp)}\tilde{\pi}$ contains $\nu^{\frac{k}{2}-j}\zeta(\rho,k)\otimes\zeta(\rho,\ell;\tilde{\sigma})$ with multiplicity one.

(3) $s_{(kp)}\tilde{\pi''}$ does not contain $\nu^{\frac{k}{2}-j}\zeta(\rho,k)\otimes\zeta(\rho,\ell;\tilde{\sigma})$.

Therefore, $\nu^{\frac{k}{2}-j}\zeta(\rho,k)\otimes\zeta(\rho,\ell;\tilde{\sigma})$ comes from $\tilde{\pi}_4$. By Frobenius reciprocity, this forces $\tilde{\pi}_4$ to be the unique irreducible subrepresentation of $\tilde{\pi}$, hence π_4 is the unique irreducible quotient of π.

Next, we show that π_1 is the unique irreducible subrepresentation. We do this by showing that none of the other components can appear as subrepresentations. First, observe that

(1) $s_{(kp)}\pi'$ contains $\nu^{-\frac{k}{2}+j}\zeta(\rho,k)\otimes\zeta(\rho,\ell;\sigma)$ with $\begin{cases}\text{multiplicity 2 if } j=0\\ \text{multiplicity 3 if } j>0\end{cases}$

(2) $s_{(kp)}\pi$ contains $\nu^{-\frac{k}{2}+j}\zeta(\rho,k)\otimes\zeta(\rho,\ell;\sigma)$ with $\begin{cases}\text{multiplicity 2 if } j=0\\ \text{multiplicity 3 if } j>0\end{cases}$

(3) $s_{(kp)}\pi''$ contains $\nu^{-\frac{k}{2}+j}\zeta(\rho,k)\otimes\zeta(\rho,\ell;\sigma)$ with multiplicity 2.

Further, as before, we have $\pi',\pi<\pi^*$, with

$$\pi^*=\nu^{\frac{-k+j-\ell}{2}}\zeta(\rho,k-j-\ell)\times\nu^{\frac{-\ell+j}{2}}\zeta(\rho,\ell+j)\rtimes\zeta(\rho,\ell;\sigma)$$

and $\pi,\pi''<\pi^{**}$, with

$$\pi^{**}=\nu^{-\frac{k}{2}+j}\zeta(\rho,k)\times\nu^{\frac{-\ell-j}{2}}\zeta(\rho,\ell-j)\rtimes\zeta(\rho,j;\sigma).$$

We also observe that

(4) $s_{(kp)}\pi^*$ contains $\nu^{-\frac{k}{2}+j}\zeta(\rho,k)\otimes\zeta(\rho,\ell;\sigma)$ with $\begin{cases}\text{multiplicity 2 if } j=0\\ \text{multiplicity 3 if } j>0\end{cases}$

(5) $s_{(kp)}\pi^{**}$ contains $\nu^{-\frac{k}{2}+j}\zeta(\rho,k)\otimes\zeta(\rho,\ell;\sigma)$ with $\begin{cases}\text{multiplicity 2 if } j=0\\ \text{multiplicity 3 if } j>0\end{cases}$

Now, (1), (2), and (4) imply that $\pi_1+\pi_2$ contain all the copies of $\nu^{-\frac{k}{2}+j}\zeta(\rho,k)\otimes\zeta(\rho,\ell;\sigma)$ in $s_{(kp)}\pi$. Then, (2), (3), and (5) imply that $s_{(kp)}\pi_1$ contains two copies of $\nu^{-\frac{k}{2}+j}\zeta(\rho,k)\otimes\zeta(\rho,\ell;\sigma)$. In short, we have

$s_{(kp)}\pi_1$ contains $\nu^{-\frac{k}{2}+j}\zeta(\rho,k)\otimes\zeta(\rho,\ell;\sigma)$ with multiplicity 2

$s_{(kp)}\pi_2$ contains $\nu^{-\frac{k}{2}+j}\zeta(\rho,k)\otimes\zeta(\rho,\ell;\sigma)$ with $\begin{cases}\text{multiplicity 0 if } j=0\\ \text{multiplicity 1 if } j>0\end{cases}$

$s_{(kp)}\pi_3$ and $s_{(kp)}\pi_4$ do not contain $\nu^{-\frac{k}{2}+j}\zeta(\rho,k)\otimes\zeta(\rho,\ell;\sigma)$.

Therefore, by Frobenius reciprocity, the only other possible irreducible subrepresentation of π is π_2.

Next, we show that π_2 is also not a subrepresentation of π. First, observe that

$$\pi\hookrightarrow\nu^{-\frac{k}{2}+j}\zeta(\rho,k)\times\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\rtimes\sigma.$$

Therefore, by Frobenius reciprocity, if π_0 is an irreducible subrepresentation of π, we have

$$s_{GL}\pi_0\geq\nu^{-\frac{k}{2}+j}\zeta(\rho,k)\times\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\otimes\sigma$$

(note that this is irreducible). By taking Jacquet modules in stages, this means $s_{min}\pi_0$ contains terms of the form

$$\nu^{-k+j+\frac{1}{2}}\rho\otimes\nu^{-k+j+\frac{3}{2}}\rho\otimes\ldots\otimes\nu^{-\ell-\frac{1}{2}}\rho\otimes(\nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{-\ell+\frac{1}{2}}\rho)\otimes(\nu^{-\ell+\frac{3}{2}}\rho\otimes\nu^{-\ell+\frac{3}{2}}\rho)\otimes\ldots$$
$$\ldots\otimes(\nu^{-j-\frac{1}{2}}\rho\otimes\nu^{-j-\frac{1}{2}}\rho)\otimes\ldots.$$

On the other hand, we claim that $s_{min}\pi_2$ does not have any terms of this form. Suppose this were not the case. Then, we would have

$$s_{(p,p,\ldots,p)}\pi_2\geq\nu^{-k+j+\frac{1}{2}}\rho\otimes\ldots\otimes\nu^{-\ell-\frac{1}{2}}\rho\otimes(\nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{-\ell+\frac{1}{2}}\rho)\otimes\ldots\otimes(\nu^{-j-\frac{1}{2}}\rho\otimes\nu^{-j-\frac{1}{2}}\rho)\otimes L(\Delta)$$

for some Langlands data Δ. By Lemma 3.2 and Frobenius reciprocity,

$$\pi_2\hookrightarrow\nu^{-k+j+\frac{1}{2}}\rho\times\ldots\times\nu^{-\ell-\frac{1}{2}}\rho\times(\nu^{-\ell+\frac{1}{2}}\rho\times\nu^{-\ell+\frac{1}{2}}\rho)\times\ldots\times(\nu^{-j-\frac{1}{2}}\rho\times\nu^{-j-\frac{1}{2}}\rho)\rtimes L(\Delta)$$

for some such Δ. Note that $s_{min}L(\Delta)$ contains no terms of the form $\nu^{\alpha}\rho$ with $\alpha<-j+\frac{1}{2}$. Therefore, $\nu^{-k+j+\frac{1}{2}}\rho\times\ldots\times\nu^{-\ell-\frac{1}{2}}\rho\times(\nu^{-\ell+\frac{1}{2}}\rho\times\nu^{-\ell+\frac{1}{2}}\rho)\times\ldots\times(\nu^{-j-\frac{1}{2}}\rho\times\nu^{-j-\frac{1}{2}}\rho)\rtimes L(\Delta)$ has $L([\nu^{-k+j+\frac{1}{2}}\rho,\nu^{-j-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-j-\frac{1}{2}}\rho],\Delta)$ as its unique irreducible subrepresentation, so that

$$\pi_2=L([\nu^{-k+j+\frac{1}{2}}\rho,\nu^{-j-\frac{1}{2}}\rho],[\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-j-\frac{1}{2}}\rho],\Delta),$$

a contradiction. Therefore, $s_{min}\pi_2$ has no terms of the form $\nu^{-k+j+\frac{1}{2}}\rho\otimes\ldots\otimes\nu^{-\ell-\frac{1}{2}}\rho\otimes(\nu^{-\ell+\frac{1}{2}}\rho\otimes\nu^{-\ell+\frac{1}{2}}\rho)\otimes\ldots\otimes(\nu^{-j-\frac{1}{2}}\rho\otimes\nu^{-j-\frac{1}{2}}\rho)\otimes\ldots$, hence cannot be a subrepresentation of π. This leaves π_1 as the unique irreducible subrepresentation of π.

Let π_{res} denote the restriction of π to the maximal proper invariant subspace, so that $\pi_{res}=\pi_1+\pi_2+\pi_3$. We show that $\pi_2\oplus\pi_3$ is a quotient of π_{res}. First, we claim that $\nu^{\frac{\ell-j}{2}}\zeta(\rho,\ell+j)\otimes\zeta(\rho,k-j;\sigma)$ is the only term in $s_{((\ell+j)p)}\pi'$ with this central character; from above it is part of $s_{((\ell+j)p)}\pi_2$. Further, we claim that no term in $s_{((\ell+j)p)}\pi''$ has this central character. To see these claims, consider the description of $s_{min}\pi'$ and $s_{min}\pi''$ as shuffles. From this, one can see that $\nu^{\frac{\ell-j}{2}}\zeta(\rho,\ell+j)\otimes\zeta(\rho,k-j;\sigma)$ and $\nu^{\frac{-\ell+j}{2}}\zeta(\rho,\ell+j)\otimes\zeta(\rho,k-j;\sigma)$ have the "highest" central characters in $s_{((\ell+j)p)}\pi'$, $s_{((\ell+j)p)}\pi''$, resp. Therefore, $\nu^{\frac{\ell-j}{2}}\zeta(\rho,\ell+j)\otimes\zeta(\rho,k-j;\sigma)$ is the only term in $s_{((\ell+j)p)}\pi_{res}$ with this central character. Therefore, by Lemma 3.2 and Frobenius reciprocity,

$$\mathrm{Hom}_G(\pi_{res},\nu^{\frac{\ell-j}{2}}\zeta(\rho,\ell+j)\rtimes\zeta(\rho,k-j;\sigma))\neq 0,$$

i.e., $\nu^{\frac{\ell-j}{2}}\zeta(\rho,\ell+j)\rtimes\zeta(\rho,k-j;\sigma)$ has a subrepresentation which is a quotient of π_{res}. Now, $\nu^{\frac{\ell-j}{2}}\zeta(\rho,\ell+j)\rtimes\zeta(\rho,k-j;\sigma)$ has π_2 as its unique irreducible subrepresentation (from above). So, the quotient of π_{res} must be either π_2 or $\begin{array}{c}\pi_1\\ |\\ \pi_2\end{array}$ (i.e., a representation having π_2 as its unique subrepresentation and π_1 as its unique quotient). Since we can rule out the latter, we have π_2 as a quotient of π_{res}. We apply a similar argument for π_3. We claim $\nu^{\frac{k-j-\ell}{2}}\zeta(\rho,k+\ell-j)\otimes\zeta(\rho,j;\sigma)$ is the only term in $s_{((k+\ell-j)p)}\pi''$ with this central character, and is part of $s_{((k+\ell-j)p)}\pi_3$. Further, nothing in $s_{((k+\ell-j)p)}\pi'$ has this central character. Thus, as above,

$$\mathrm{Hom}_G(\pi_{res},\nu^{\frac{k-j-\ell}{2}}\zeta(\rho,k+\ell-j)\rtimes\zeta(\rho,j;\sigma))\neq 0.$$

So, $\nu^{\frac{k-j-\ell}{2}}\zeta(\rho,k+\ell-j)\rtimes\zeta(\rho,j;\sigma)$ has a subrepresentation which is a quotient of π_{res}, necessarily π_3 or $\begin{array}{c}\pi_1\\ \mid\\ \pi_3\end{array}$. Again, we can rule out the latter to conclude that π_3 is also a quotient of π_{res}. Therefore, $\pi_2 \oplus \pi_3$ is a quotient of π_{res}, as claimed.

We now consider case 2a. Let $\pi = \nu^{\frac{k}{2}-\ell}\zeta(\rho,k) \rtimes \zeta(\rho,\ell;\sigma)$ (case 2a) and $\pi' = \zeta(\rho,2\ell) \rtimes \zeta(\rho,k-\ell;\sigma)$ (case 3d). We note that $\pi_1' = \pi_1$. We show that π_2 is the unique irreducible quotient of π using the same kind of argument as above. Observe that

(1) $s_{(kp)}\tilde{\pi}$ contains $\nu^{\ell-\frac{k}{2}}\zeta(\rho,k)\otimes\zeta(\rho,\ell;\tilde{\sigma})$ with multiplicity one

(2) $s_{(kp)}\tilde{\pi}'$ does not contain $\nu^{\ell-\frac{k}{2}}\zeta(\rho,k)\otimes\zeta(\rho,\ell;\tilde{\sigma})$.

Therefore, $s_{(kp)}\tilde{\pi}_2$ contains the copy of $\nu^{\ell-\frac{k}{2}}\zeta(\rho,k)\otimes\zeta(\rho,\ell;\tilde{\sigma})$. By Frobenius reciprocity, $\tilde{\pi}_2$ is the unique irreducible subrepresentation of $\tilde{\pi}$, hence π_2 is the unique irreducible quotient of π. Therefore, π_1 is the unique irreducible subrepresentation of π, finishing case 2a. A similar argument works for case 3b (use $\pi = \nu^{\ell-\frac{k}{2}}\zeta(\rho,k)\rtimes\zeta(\rho,\ell;\sigma)$ (case 3b) and $\pi' = \zeta(\rho,2\ell)\rtimes\zeta(\rho,k-\ell;\sigma)$ (case 2c)) and case 1 (use $\pi = \nu^{\alpha}\zeta(\rho,k)\rtimes\zeta(\rho,\ell;\sigma)$ (case 1) and $\pi' = \nu^{-\frac{\ell}{2}+\frac{\alpha}{2}+\frac{k}{4}}\zeta(\rho,\ell+\alpha+\frac{k}{2})\rtimes\zeta(\rho,-\alpha+\frac{k}{2};\sigma)$ (irreducible)). This finishes the last of the cases for $\rho_0\cong\rho$.

Now, suppose $\rho_0 \not\cong \rho$. We focus on the case where $\pi = \nu^{\alpha}\zeta(\rho_0,k)\rtimes\zeta(\rho,\ell;\sigma)$ has three components (i.e., $\alpha = -\frac{k}{2}+j$ with $1\le j<\frac{k}{2}$). Then, write

$$\nu^{-\frac{k}{2}+j}\zeta(\rho_0,k)\rtimes\sigma = L(\Delta_1)+L(\Delta_2)+L(\Delta_3),$$

with

$$L(\Delta_1) = L([\nu^{-k+j+\frac{1}{2}}\rho_0,\nu^{-\frac{1}{2}}\rho_0],[\nu^{-j+\frac{1}{2}}\rho_0,\nu^{-\frac{1}{2}}\rho_0];\sigma)$$

$$L(\Delta_2) = L([\nu^{-k+j+\frac{1}{2}}\rho_0,\nu^{-\frac{3}{2}}\rho_0],[\nu^{-j+\frac{1}{2}}\rho_0,\nu^{-\frac{1}{2}}\rho_0];\delta(\nu^{-\frac{1}{2}}\rho_0;\sigma))$$

$$L(\Delta_3) = L([\nu^{-k+j+\frac{1}{2}}\rho_0,\nu^{-j-\frac{3}{2}}\rho_0],\nu^{-j}\delta(\rho_0,2),\ldots,\nu^{-1}\delta(\rho_0,2);\delta(\nu^{-\frac{1}{2}}\rho_0;\sigma)).$$

Note that by the results already verified above, we have that $L(\Delta_1)\oplus L(\Delta_2)$ is a subrepresentation of $\nu^{-\frac{k}{2}+j}\zeta(\rho_0,k)\rtimes\sigma$; $L(\Delta_3)$ is the unique irreducible quotient.

We now turn our attention back to π. First, we claim that $\pi_3 = L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\Delta_3)$ is the unique irreducible quotient of π. We do this by showing $\tilde{\pi}_3 = L([\nu^{-\ell+\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho],\tilde{\Delta}_3)$ (where $\tilde{\Delta}_3$ is defined by $L(\tilde{\Delta}_3) = \widetilde{L(\Delta_3)}$) is the unique irreducible subrepresentation of $\tilde{\pi} = \nu^{-\alpha}\zeta(\rho_0,k)\rtimes\zeta(\rho,\ell;\tilde{\sigma})$ (n.b. π reducible implies $\tilde{\rho}_0\cong\rho_0$). Now, since $L(\Delta_3)$ is the unique irreducible quotient of $\nu^{\alpha}\zeta(\rho_0,k)\rtimes\sigma$, we have $\nu^{-\alpha}\zeta(\rho_0,k)\otimes\tilde{\sigma}\le s_{(kp_0)}L(\tilde{\Delta}_3)$. Thus, by Lemma

5.4,

$$\begin{gathered} s_{(\ell p)}\tilde{\pi}_3 \geq \nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\otimes L(\tilde{\Delta}_3) \\ \Downarrow \\ s_{(\ell p,kp_0)}\tilde{\pi}_3 \geq \nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\otimes \nu^{-\alpha}\zeta(\rho_0,k)\otimes\tilde{\sigma} \\ \Downarrow \\ s_{(\ell p+kp_0)}\tilde{\pi}_3 \geq \nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\times \nu^{-\alpha}\zeta(\rho_0,k)\otimes\tilde{\sigma} \\ \Downarrow \\ s_{(kp_0,\ell p)}\tilde{\pi}_3 \geq \nu^{-\alpha}\zeta(\rho_0,k)\otimes \nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\otimes\tilde{\sigma}. \end{gathered}$$

Now, by μ^* computations (or by considering s_{min}), we have that $\nu^{-\alpha}\zeta(\rho_0,k)\otimes\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\otimes\tilde{\sigma}$ appears with multiplicity one in $s_{(kp_0,\ell p)}\tilde{\pi}$. Therefore, the above implies

$$s_{(kp_0)}\tilde{\pi}_3 \geq \nu^{-\alpha}\zeta(\rho_0,k)\otimes\zeta(\rho,\ell;\tilde{\sigma}).$$

Since $s_{(kp_0)}\tilde{\pi}_3$ contains the only copy of $\nu^{-\alpha}\zeta(\rho_0,k)\otimes\zeta(\rho,\ell;\tilde{\sigma})$, it must be the unique irreducible subrepresentation of $\tilde{\pi}$, as needed.

Finally, we show that $\pi_1\oplus\pi_2$ is a subrepresentation of π. Now, from above, $L(\Delta_1)\oplus L(\Delta_2)$ is a subrepresentation of $\nu^{\alpha}\zeta(\rho_0,k)\rtimes\sigma$. Next,

$$\pi\hookrightarrow\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\rtimes(\nu^{\alpha}\zeta(\rho_0,k)\rtimes\sigma).$$

Then, by exactness and induction in stages, $\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\rtimes L(\Delta_1)$ and $\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\rtimes L(\Delta_2)$ are subrepresentations of $\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\rtimes(\nu^{\alpha}\zeta(\rho_0,k)\rtimes\sigma)$. Further, by Lemma 5.4, $\pi_1\hookrightarrow\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\rtimes L(\Delta_1)$ and $\pi_2\hookrightarrow\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\rtimes L(\Delta_2)$. Now, we note that $\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\otimes\nu^{\alpha}\zeta(\rho_0,k)\otimes\sigma$ occurs with multiplicity two in $s_{(\ell p,kp_0)}\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\times\nu^{\alpha}\zeta(\rho_0,k)\rtimes\sigma$. Therefore, π_1 and π_2 appear only once in $\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\times\nu^{\alpha}\zeta(\rho_0,k)\rtimes\sigma$. Since the only copies of π_1 and π_2 in $\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\times\nu^{\alpha}\zeta(\rho_0,k)\rtimes\sigma$ appear as subrepresentations, and $\pi\hookrightarrow\nu^{-\frac{\ell}{2}}\zeta(\rho,\ell)\rtimes\nu^{\alpha}\zeta(\rho_0,k)\rtimes\sigma$, we see that $\pi_1\oplus\pi_2$ is a subrepresentation of π, as claimed.

The two-component cases are easy; their proofs are omitted. □

Theorem **7.2.** *Suppose that σ is an irreducible supercuspidal representation of S_m and ρ is a representation of $GL_p(F)$ satisfying (C1). Let $\pi=\nu^{\alpha}\zeta(\rho,k)\rtimes\zeta(\rho,\ell;\sigma)$, $\alpha\leq 0$, be reducible. The components of π are described in Theorem 6.2. In the notation of that theorem, we have the following:*

case 1a: π_1 *is the unique irreducible subrepresentation;* π_2 *the unique irreducible quotient.*
case 1b: π_1 *is the unique irreducible subrepresentation;* π_2 *the unique irreducible quotient.*
case 2a: π_1 *is the unique irreducible subrepresentation;* π_2 *the unique irreducible quotient.*
case 2b: $\pi_1 \oplus \pi_3$ *is a subrepresentation;* π_2 *the unique irreducible quotient.*
case 2c: $\pi \cong \pi_1 \oplus \pi_2$.
case 3a: π_1 *is the unique irreducible subrepresentation;* π_4 *the unique irreducible quotient.* $\pi_2 \oplus \pi_3$ *is a subquotient.*
case 3b: π_1 *is the unique irreducible subrepresentation;* π_2 *the unique irreducible quotient.*
case 3c: $\pi_1 \oplus \pi_2$ *is a subrepresentation;* π_3 *the unique irreducible quotient.*
case 3d: $\pi \cong \pi_1 \oplus \pi_2$

Suppose ρ_0 *is a representation of* $GL_{p_0}(F)$ *satisfying (C0). Let* $\pi = \nu^{\alpha}\zeta(\rho_0, k) \rtimes \zeta(\rho, \ell; \sigma)$, $\alpha \leq 0$, *be reducible. Then, the components of* π *are described in Corollary 5.8. In the notation of that corollary, we have the following:*
case 1: $\pi \cong \pi_1 \oplus \pi_2$.
case 2: $\pi_1 \oplus \pi_2$ *is a subrepresentation;* π_3 *the unique irreducible quotient.*
For $\alpha > 0$, *the order of composition series is reversed from that for* $-\alpha$.

Proof. The proof for $\rho_0 \cong \rho$ is essentially the same as its counterpart in Theorem 7.1. We note that if $\pi = \nu^{\frac{-k+1}{2}+j}\zeta(\rho, k) \rtimes \zeta(\rho, \ell; \sigma)$ is case 3a, then $\pi' = \nu^{\frac{-\ell+j}{2}}\zeta(\rho, j+\ell+1) \rtimes \zeta(\rho, k-j-1; \sigma)$ (case 2b) and $\pi'' = \nu^{\frac{-k+j+\ell+1}{2}}\zeta(\rho, k+\ell-j) \rtimes \zeta(\rho, j; \sigma)$ (case 3c) have $\pi_1 = \pi_1' = \pi_1''$, $\pi_2 = \pi_2'$ and $\pi_3 = \pi_3''$. Thus, the same sort of comparisons used in Theorem 7.1 may be used here.

Suppose $\rho_0 \not\cong \rho$. We start by considering the case $\ell = 0$. In this case, $\pi = \nu^{\frac{-k+1}{2}+j}\zeta(\rho_0, k) \rtimes \sigma$. We claim that $\pi_1 \oplus \pi_2$ is a subrepresentation of π and π_3 is the unique irreducible quotient. If $j = 0$, this follows from the same argument used in the proof of Proposition 3.11, case 2a. For $j \geq 1$, observe that

$$\begin{aligned} \pi &\hookrightarrow \nu^{-\frac{k}{2}+j}\zeta(\rho_0, k-1) \times \nu^{j}\rho_0 \rtimes \sigma \\ &\cong \nu^{-\frac{k}{2}+j}\zeta(\rho_0, k-1) \times \nu^{-j}\rho_0 \rtimes \sigma \\ &\cong \nu^{-j}\rho_0 \times \nu^{-\frac{k}{2}+j}\zeta(\rho_0, k-1) \rtimes \sigma. \end{aligned}$$

Proposition 3.11 tells us

$$\begin{aligned} s_{(p)}\pi_i &= \nu^{-k+j+1}\rho_0 \otimes L([\nu^{-k+j+2}\rho_0, \nu^{-1}\rho_0], [\nu^{-j}\rho_0, \nu^{-1}\rho_0]; T_i) \\ &+\nu^{-j}\rho_0 \otimes L([\nu^{-k+j+1}\rho_0, \nu^{-1}\rho_0], [\nu^{-j+1}\rho_0, \nu^{-1}\rho_0]; T_i) \end{aligned}$$

for $i = 1, 2$. By induction (on j), we have that $L([\nu^{-k+j+1}\rho_0, \nu^{-1}\rho_0], [\nu^{-j+1}\rho_0, \nu^{-1}\rho_0]; T_i)$ is a subrepresentation of $\nu^{-\frac{k}{2}+j}\zeta(\rho_0, k-1) \rtimes \sigma$. By Lemma 3.2 and Frobenius reciprocity, we know that

$$\begin{aligned}\pi_i \hookrightarrow\ & \nu^{-j}\rho_0 \rtimes L([\nu^{-k+j+1}\rho_0, \nu^{-1}\rho_0], [\nu^{-j+1}\rho_0, \nu^{-1}\rho_0]; T_i) \\ \hookrightarrow\ & \nu^{-j}\rho_0 \rtimes (\nu^{-\frac{k}{2}+j}\zeta(\rho_0, k-1) \rtimes \sigma).\end{aligned}$$

Now, π_i, $i = 1, 2$, has multiplicity one in $\nu^{-j}\rho_0 \times \nu^{-\frac{k}{2}+j}\zeta(\rho_0, k-1) \rtimes \sigma$ (in fact, π_i has multiplicity one in $\nu^{-k+j+1}\rho_0 \times \ldots \times \nu^{-j-1}\rho_0 \times (\nu^{-j}\rho_0 \times \nu^{-j}\rho_0) \times \ldots \times (\nu^{-1}\rho_0 \times \nu^{-1}\rho_0) \times \rho_0 \rtimes \sigma$). Then, by considering the subspace $V_\pi + V_{\pi_1} + V_{\pi_2}$ in the space of $\nu^{-j}\rho_0 \times \nu^{-\frac{k}{2}+j}\zeta(\rho_0, k-1) \rtimes \sigma$, we see that $\pi_1 \oplus \pi_2$ is a subrepresentation of π. That π_3 is the unique irreducible quotient is then easy–the usual Jacquet module argument shows that there is a unique irreducible quotient; necessarily it is π_3.

From this point, the argument now follows that of the $\rho_0 \ncong \rho$ case in Theorem 7.1. (In Theorem 7.1, it was not necessary to do the case $\ell = 0$ separately–since ρ_0 satisfied (C1/2), the $\ell = 0$ results followed from the work already done.) □

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Department of Mathematics, University of Chicago, Chicago, IL 60637