Degenerate Principal Series for Symplectic Groups

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# TABLE OF CONTENTS

ABSTRACT ......................................................................................... iv

LIST OF FIGURES ................................................................................. v

LIST OF TABLES ..................................................................................... vi

INTRODUCTION ....................................................................................... vii

Chapter

1. NOTATION AND PRELIMINARIES .................................................. 1
   1.1. $Sp_{2n}(F)$ ........................................................................... 1
   1.2. Induced representations and Jacquet modules ......................... 4

2. THE HECKE ALGEBRA APPROACH ............................................. 8
   2.1. General theorems ................................................................... 8
   2.2. Basis for $V_{Bx}$ .................................................................... 14
   2.3. Irreducibility when (order $\chi_u > 2$) ..................................... 30
   2.4. Irreducibility conditions for (order $\chi_u = 2$) ....................... 34
   2.5. Irreducibility conditions for (order $\chi_u = 1$) ....................... 45
   2.6. Reducibility conditions for (order $\chi_u = 1$) ....................... 53
   2.7. Reducibility conditions for (order $\chi_u = 2$) ....................... 57

3. IRREDUCIBILITY OF CERTAIN REPRESENTATIONS À LA TADIĆ 59
   3.1. A general theorem on reducibility/irreducibility (in the regular case) .......................................................................... 59
   3.2. Applications to degenerate principal series for $Sp_{2n}(F)$ (in the regular case) ................................................................. 63

4. IRREDUCIBILITY CRITERIA FOR DEGENERATE PRINCIPAL SERIES IN $SP_4(F)$, $SP_6(F)$, À LA TADIĆ .................................................. 70
   4.1. Extending definitions to the nonregular case ......................... 70
   4.2. Degenerate Principal Series in $Sp_4(F)$ .................................. 71
   4.3. Degenerate Principal Series in $Sp_6(F)$ .................................. 76

APPENDIX .............................................................................................. 104

REFERENCES ....................................................................................... 109
ABSTRACT

A brief description of the problem under consideration follows. Let $F$ be a $p$-adic field (we take char $F=0$, odd residual characteristic), $G = Sp_{2n}(F)$. Let $P = MU$ be a maximal parabolic subgroup of $G$, and $\chi$ a one-dimensional representation of $M$. We may extend $\chi$ trivially to $P$. The question we investigate is whether $\pi = \text{Ind}_P^G \chi \otimes 1$ is irreducible or not.

Two different approaches to this problem are used. The first, based on the work of Casselman and subsequent work by Gustafson, reduces the problem to the corresponding question about an associated finite-dimensional representation of a certain Hecke algebra. We use this method to do the case where $M = F^\times \times Sp_{2(n-1)}(F)$. The second approach is based on a technique of Tadić, and involves an analysis of Jacquet modules. This is used to prove a more general theorem on induced representations, which may be used to deal with the problem when $\chi$ satisfies a regularity condition. We use this method and ad hoc arguments to work out the low rank cases completely.

key words and phrases: $p$-adic field, symplectic group, induced representation, Jacquet module, Hecke algebra.
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Graph of $\chi \propto tr_2$</td>
<td>72</td>
</tr>
<tr>
<td>2. Graph of $[\chi]_2 \propto 1$</td>
<td>73</td>
</tr>
<tr>
<td>3. Graph of $St_\chi \propto 1$</td>
<td>76</td>
</tr>
<tr>
<td>4. Graph of $\chi \propto tr_4$, $\chi$ generic</td>
<td>88</td>
</tr>
<tr>
<td>5. Graph of $\chi \propto tr_4$, $\chi = 1$</td>
<td>89</td>
</tr>
<tr>
<td>6. Graph of $\chi \propto tr_4$, $\chi = sgn$</td>
<td>90</td>
</tr>
<tr>
<td>7. Graph of $\chi \propto tr_4$, $\chi = \nu^{-1}$</td>
<td>91</td>
</tr>
<tr>
<td>8. Graph of $\chi \propto tr_4$, $\chi = \nu^{-2}$</td>
<td>92</td>
</tr>
<tr>
<td>9. Graph of $[\chi]_2 \propto tr_2$, $\chi$ generic (also $\chi = 1, sgn$)</td>
<td>93</td>
</tr>
<tr>
<td>10. Graph of $[\chi]_2 \propto tr_2$, $\chi = \nu^{-\frac{1}{2}}$</td>
<td>94</td>
</tr>
<tr>
<td>11. Graph of $[\chi]_2 \propto tr_2$, $\chi = \nu^{-\frac{1}{2}} sgn$</td>
<td>95</td>
</tr>
<tr>
<td>12. Graph of $[\chi]_2 \propto tr_2$, $\chi = \nu^{-\frac{1}{2}}$</td>
<td>96</td>
</tr>
<tr>
<td>13. Graph of $[\chi]_3 \propto 1$, $\chi$ generic</td>
<td>97</td>
</tr>
<tr>
<td>14. Graph of $[\chi]_3 \propto 1$, $\chi = 1$</td>
<td>98</td>
</tr>
<tr>
<td>15. Graph of $[\chi]_3 \propto 1$, $\chi = sgn$</td>
<td>99</td>
</tr>
<tr>
<td>16. Graph of $[\chi]_3 \propto 1$, $\chi = \nu^{-\frac{1}{2}}$</td>
<td>100</td>
</tr>
<tr>
<td>17. Graph of $[\chi]_3 \propto 1$, $\chi = \nu^{-\frac{1}{2}} sgn$</td>
<td>101</td>
</tr>
<tr>
<td>18. Graph of $[\chi]_3 \propto 1$, $\chi = \nu^{-1}$</td>
<td>102</td>
</tr>
<tr>
<td>19. Graph of $[\chi]_3 \propto 1$, $\chi = \nu^{-1} sgn$</td>
<td>103</td>
</tr>
</tbody>
</table>
# List of Tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Jacquet modules for $\chi \propto tr_2$</td>
<td>72</td>
</tr>
<tr>
<td>2. Jacquet modules for $[\chi]_2 \propto 1$</td>
<td>73</td>
</tr>
<tr>
<td>3. $F^x \times Sp_4(F)$ parabolic in $Sp_6(F)$, i.e., $\chi \propto tr_4$</td>
<td>79</td>
</tr>
<tr>
<td>4. $F^x \times Sp_4(F)$ parabolic in $Sp_6(F)$, i.e., $\chi \propto tr_4$</td>
<td>80</td>
</tr>
<tr>
<td>5. $F^x \times Sp_4(F)$ parabolic in $Sp_6(F)$, i.e., $\chi \propto tr_4$</td>
<td>81</td>
</tr>
<tr>
<td>6. $GL_2 \times Sp_{2n}(F)$ parabolic in $Sp_6(F)$, i.e., $[\chi]_2 \propto tr_2$</td>
<td>82</td>
</tr>
<tr>
<td>7. $GL_2 \times Sp_{2n}(F)$ parabolic in $Sp_6(F)$, i.e., $[\chi]_2 \propto tr_2$</td>
<td>83</td>
</tr>
<tr>
<td>8. $GL_2 \times Sp_{2n}(F)$ parabolic in $Sp_6(F)$, i.e., $[\chi]_2 \propto tr_2$</td>
<td>84</td>
</tr>
<tr>
<td>9. $GL_3$ parabolic in $Sp_6(F)$, i.e., $[\chi]_3 \propto 1$</td>
<td>85</td>
</tr>
<tr>
<td>10. $GL_3$ parabolic in $Sp_6(F)$, i.e., $[\chi]_3 \propto 1$</td>
<td>86</td>
</tr>
<tr>
<td>11. $GL_3$ parabolic in $Sp_6(F)$, i.e., $[\chi]_3 \propto 1$</td>
<td>87</td>
</tr>
</tbody>
</table>
INTRODUCTION

We begin the Introduction with a short description of the problem to be considered in this paper. Let $F$ be a $p$-adic field with char $F=0$ and odd residual characteristic; $\mathcal{O}$ its ring of integers. Set $G = Sp_{2n}(F)$. Let $P = MU$ be a maximal parabolic subgroup of $G$, so $M \cong G\ell_k(F) \times Sp_{2(n-k)}(F)$, for some $k$, $1 \leq k \leq n$. A (quasi)character on $M$ is of the form $\chi \circ det$, $\chi \in \hat{F}^\times$, and may be extended trivially to $P$. We also denote the resulting characters as $\chi$. It will be helpful to decompose $\chi$ as $\chi = | \cdot |^s \chi_u$, with $\chi_u$ coming from a character of $\mathcal{O}^\times$. The question we investigate is whether $\pi = Ind_P^G \chi$ is irreducible or not, where induction is normalized so that unitary representations induce to unitary representations. In the following paragraphs, we describe some techniques for attacking this problem.

In his unpublished notes on admissible representations [Cas 2], Casselman used Hecke algebras to determine reducibility for $Sl_2(F) = Sp_2(F)$ with $\chi$ unramified. In his thesis, Gustafson [Gus] extended these techniques to the maximal parabolic in $Sp_{2n}(F)$ with Levi factor $M \cong G\ell_n(F)$, $\chi$ unramified. In the second chapter of this thesis (the first consisting of notation and preliminaries), we apply these Hecke algebra methods to determine reducibility for the maximal parabolic in $Sp_{2n}(F)$ with Levi factor $M \cong F^\times \times Sp_{2(n-1)}(F)$ and arbitrary $\chi$. In the third and fourth chapters, we use a technique of Tadić [Tad 2] involving Jacquet modules to obtain some general results, as well as some specific ones for small $n$.

We now describe how Hecke algebras are used to show irreducibility. Let $\varpi$ be a uniformizer of $F$. Then, $\mathcal{O}/(\varpi \mathcal{O})$ is a finite field which we denote by $F_q$. To the parabolic subgroup $P$, we may associate a parahoric subgroup $B$ by taking $P(F_q)$ and lifting it back to $Sp_{2n}(F)$. To the character $\chi$ on $P$, we can associate an open compact subgroup $B_\chi$ contained in $B$ which has the following property:

*) every subquotient of $\pi$ is generated by its $B_\chi$ – fixed vectors.
We remark that $B_x$ depends only on $\chi_u$, and not on $s$. The proof that $B_x$ has
this property is based on a similar statement in [Gus] for the case where $\chi$ is
unramified. This, in turn, is based on the results of Borel and Casselman that state
that a subquotient of an unramified principal series is generated by its Iwahori-
fixed vectors. Let $H = H(G, B_x)$ denote the algebra of compactly supported $B_x$-
iinvariant functions on $G$. Associated to the induced representation $(\pi, G, V)$ is
the representation $(\pi, H, V^{B_x})$, where $H$ acts on the $B_x$-fixed vectors by

$$\pi(f)v = \int_G f(g)\pi(g)v \, dg.$$ 

It is known (cf. [Gus]) that *) implies the map $W \rightarrow W^{B_x}$ gives a bijective
correspondence between subquotients of $(\pi, G, V)$ and subquotients of $(\pi, H, V^{B_x})$.
As $V^{B_x}$ is a finite-dimensional space, we can explicitly compute certain operators
$\pi(h_1), \pi(h_2)$, with $h_1, h_2 \in H$ and show that they can admit no common invariant
subspace except at a finite number of potential reducibility points. For order of
$\chi_u > 2$, there are no potential reducibility points—$\pi$ is always irreducible. If order
of $\chi_u = 2$, reducibility can only occur if $s = 0$ or $i\pi / \ln q$. For order of $\chi_u = 1$, the
points where reducibility is possible are $s = \pm 1, 0, i\pi / \ln q$. The next problem is to
show that there is reducibility at the potential reducibility points. We note that in
the unramified case, if $s = \pm 1$, the trivial representation is a subquotient, so there
is reducibility.

This leaves us with the potential reducibility points where $\chi$ is unitary.
Now, if $\chi$ is unitary, then $\pi$ is unitary as well. We show that $\pi$ is reducible by
showing that $\text{dim} \, \text{Hom}_G(\pi, \tilde{\pi}) = 2$, where $\tilde{\pi}$=contragredient of $\pi = \text{Ind}_G^H \chi^{-1}$. To show
this, we observe that by Frobenius reciprocity,

$$\text{Hom}_G(\pi, \tilde{\pi}) \cong \text{Hom}_M(\pi_U, \chi^{-1}),$$

where $\pi_U$ is the Jacquet module of $\pi$ taken with respect to $P = MU$. If we let
$M_x = M \cap B_x$, we get a representation $(\pi_U, H(M//M_x), (V_U)^{M_x})$. We can show that

$$\text{Hom}_M(\pi_U, \chi^{-1}) \cong \text{Hom}_{H(M,M_x)}(\pi_U, \chi^{-1}),$$
Degenerate Principal Series

We note that the components of \( \pi_U \) have one-dimensional spaces of \( M_\chi \)-fixed vectors. By considering the central characters of the components of \( \pi_U \), one can show that only two of them contain \( \chi^{-1} \). The question is whether they arise as a direct sum (to get \( \text{dim}=2 \)) or only as composition factors (so \( \text{dim}=1 \)). For \( s \) near the point in question (either \( 0 \) or \( i\pi/\ln q \)), the action of \( H(M, M_\chi) \) diagonalizes. It turns out that the eigenvectors may be analytically continued, so the action of \( H(M, M_\chi) \) on \((V_U)^{M_\chi}\) will diagonalize at the point in question. Thus, \( \text{dim}=2 \). The Hecke algebra results are summarized in Theorem 2.32., which we now state. If \( n \geq 2 \), \( \pi = iG\chi \) is reducible if and only if \( \chi \) is of the form \( |\cdot|^\pm n \) or \( \chi^2 = 1 \). If \( n = 2 \), in which case the group is just \( SL_2(F) \), reducibility occurs if and only if \( \chi \) is of the form \( \chi = |\cdot|^\pm 1 \) or \( \chi^2 = 1 \) with \( \chi \neq 1 \). When reducible, \( \pi \) has two components.

The computations required to use the Hecke algebra approach are too complicated for an arbitrary maximal parabolic subgroup, so in chapter 3 we introduce a different approach based on a technique of Tadić. As an example, suppose that \( \pi = \text{Ind}_P^G \chi \) is a degenerate principal series representation of \( SP_{2n}(F) \) and that the Jacquet module of \( \chi \), treating \( \chi \) as a representation of \( M \), is a regular character on the split torus \( A \) of \( G \). The basic idea for showing irreducibility is as follows. Let \( r_{AG}(\pi)_{ss} \) denote the semisimplification of the Jacquet module of \( \pi \) with respect to the minimal parabolic subgroup. For \( P_1 \) an intermediate parabolic subgroup, let \( r_{M,G}(\pi)_{ss} \) denote the semisimplification of the Jacquet module computed with respect to \( P_1 \) (see chapter 1 for a summary of the Bernstein-Zelevinsky notation for Jacquet modules). Suppose that \( \psi_1, \psi_2 \in r_{AG}(\pi)_{ss} \). Now, suppose that we can choose \( P_1 \) so that \( r_{M,G}(\pi)_{ss} \) has an irreducible component \( \sigma \) such that both \( \psi_1 \) and \( \psi_2 \) are in \( r_{AM_1}(\sigma)_{ss} \). Then, if \( \pi_0 \) is a composition factor of \( \pi \) with \( \psi_1 \in r_{AG}(\pi_0)_{ss} \), we claim that \( \psi_2 \in r_{AG}(\pi_0)_{ss} \) as well. To see this, look at \( r_{M,G}(\pi_0)_{ss} \). Since \( \psi_1 \in r_{AG}(\pi_0)_{ss} \), \( r_{M,G}(\pi_0)_{ss} \) will have to contain at least part of \( \sigma \)-by regularity, there is only one copy of \( \psi_1 \) around. Since \( \sigma \) is irreducible, \( r_{M,G}(\pi_0)_{ss} \) must contain all of \( \sigma \), so that \( \psi_1, \psi_2 \in r_{AG}(\pi_0)_{ss} \). We then argue as follows: suppose \( \psi_1 \in r_{AG}(\pi_0)_{ss} \). By working with \( r_{M,G}(\pi_0)_{ss} \), we conclude that \( \psi_2 \in r_{AG}(\pi_0)_{ss} \). Then, by using \( r_{M,G}(\pi_0)_{ss} \), we conclude that \( \psi_3 \in r_{AG}(\pi_0)_{ss} \), etc.
until we end up with $r_{AG}(\pi_0)_{ss} = r_{AG}(\pi)_{ss}$. Therefore, we conclude that $\pi$ must have been irreducible.

We now describe how Jacquet modules are used in pursuing reducibility. In this case, one of the $r_{M,G} \pi$ will have a reducible composition factor which would “normally” (i.e., for generic $\chi$) be irreducible. Suppose the composition factor has components $\sigma_1, \sigma_2$ with $\psi_1 \in r_{AM_*}(\sigma_1)_{ss}, \psi_2 \in r_{AM_*}(\sigma_2)_{ss}$, where $P_* = M_* U_*$ is the appropriate parabolic subgroup. Then, let $\pi_i = Ind_{P_*}^{G} \sigma_i$. We get $\psi_1 \in r_{AG}(\pi_1)_{ss}, \psi_1 \notin r_{AG}(\pi_2)_{ss}$, and similarly for $\psi_2$. If $\pi$ were irreducible, then $\pi$ would have to be a subquotient of either $\pi_1$ or $\pi_2$. Therefore, $r_{AG}(\pi)_{ss} \subset r_{AG}(\pi_1)_{ss}$ or $r_{AG}(\pi)_{ss} \subset r_{AG}(\pi_2)_{ss}$. But, this is not the case—just look at $\psi_1, \psi_2$. Thus, we have that $\pi$ is reducible.

The main theorem in chapter 3 is Theorem 3.1.2, which is based on a generalization of the argument above. The theorem gives necessary and sufficient conditions for reducibility of $\pi = i_{GM} \rho$, where $\rho$ is an irreducible admissible representation of $M$ such that $r_{AM} \rho \neq 0$ and the characters appearing in $r_{AM} \rho$ are regular. First, we associate a graph to $\pi$ as follows:

**vertices:** the vertices are the elements of $r_{AG}(\pi)_{ss}$

**edges:** two vertices $\psi_1, \psi_2$ are connected by an edge if there is some Levi $N$ and some $\tau \in r_{NG}(\pi)_{ss}$ such that the following hold:

1. $\tau$ is an irreducible representation of $N$.
2. $\psi_1, \psi_2 \in r_{AN}(\tau)_{ss}$.

where $r_{AN}(\tau)_{ss}$ denotes the semisimplification of $\tau$. Under these conditions, Theorem 3.1.2 states that the following are equivalent:

1. $\pi$ is irreducible.
2. the graph of $\pi$ is connected
3. the composition factors of $r_{NG} \pi$ as computed using the results of Bernstein-Zelevinsky/Casselman (cf. Theorem 1.2.4) are all irreducible. In particular, it
is sufficient to check those where the Levi $N$ is generated by $A$ and one simple reflection. Note that one must incorporate the reducibility of $r_{AM\rho}$ into these computations.

The third condition reduces the problem to analyzing the reducibility of certain induced representations of $GL_2(F)$ and $Sp_2(F) = Sl_2(F)$, where the results are known.

As indicated, Theorem 3.1.2 is a generalization of the preceding arguments, but it still requires a regularity condition. In the final chapter, we look explicitly at degenerate principal series for $Sp_4(F)$ and $Sp_6(F)$, in general, modifying the kind of arguments used in chapter 3 (including the definition of the graph for $\pi$) to allow us to work out the non-regular case.

I would like to take this opportunity to thank some individuals who have contributed to this paper, which is essentially my dissertation. First, I would like to express my gratitude to Paul Sally Jr. for doing a fine job as advisor—it has been a pleasure to work with him. This paper owes much to an idea of Marko Tadić, and I would like to thank him for taking the time and effort to explain it to me. Finally, I would like to thank Timothy Steger for carefully reading the first draft and making many valuable suggestions, and the referee, for similar reasons.
CHAPTER 1

NOTATION AND PRELIMINARIES

1.1. \( Sp_{2n}(F) \)

The purpose of this chapter is to introduce notation and review some preliminary facts that will be of use in the rest of this thesis.

Let \( F \) be an nonarchimedean local field of characteristic zero. Let \( \mathcal{O} \) denote the ring of integers, \( \mathcal{P} \) the prime ideal in \( \mathcal{O} \) and \( \varpi \) a uniformizer. Then, \( \mathcal{O}/\mathcal{P} \) is a finite field. Let \( q \) denote the number of elements in \( \mathcal{O}/\mathcal{P} \). We normalize the Haar measure on \( F \) so that \( \text{vol}(\mathcal{O}) = 1 \).

Suppose that \( \chi \) is a quasicharacter of \( F^\times \) (i.e., a multiplicative homomorphism from \( F^\times \) to \( \mathbb{C}^\times \) - not necessarily unitary). If \( x \in F^\times \), \( x \) may be decomposed as \( x = \varpi^k x_0 \), with \( x_0 \in \mathcal{O}^\times \). We can then decompose \( \chi \) as \( \chi = | \cdot |^s \chi_u \) by \( \chi(x) = |\varpi^k|^s \chi_u(x_0) \), where \( \chi_u \) is a character of \( \mathcal{O}^\times \), \( 0 \leq \text{Res} < 2\pi i / \ln q \). It may be convenient, at times, to view \( \chi_u \) as a character on \( F^\times \) (by \( \chi_u(\varpi^k x_0) = \chi_u(x_0) \)). We shall use \( 1 \) for the trivial character and \( sgn \) to denote a nontrivial character satisfying \( sgn^2 = 1 \).

As most of this thesis concerns induced representations for \( Sp_{2n}(F) \), we next discuss \( Sp_{2n}(F) \) and induced representations. In this section, we review some of the structure theory for \( Sp_{2n}(F) \).

Recall that we may take

\[
Sp_{2n}(F) = \{ x \in GL_{2n}(F) | ^TXJX = J \},
\]

\(^1\)Received by the editor Jan. 18, 1991
where

\[
J = \begin{pmatrix}
1 & \cdots & 1 \\
& \ddots & \ddots \\
& & 1
\end{pmatrix}
\begin{pmatrix}
-I_n \\
\end{pmatrix}
\begin{pmatrix}
1 & \cdots & 1 \\
& \ddots & \ddots \\
& & 1
\end{pmatrix}
\]

(entries left vacant are zeros). Let \( K = Sp_{2n}(O) \). This is a maximal compact subgroup of \( Sp_{2n}(F) \). It has a filtration of open compact normal subgroups

\[\ldots \triangleleft K_2 \triangleleft K_1 \triangleleft K,\]

where

\[K_i = \{ X \in K | X \equiv 1 \mod P^i \}.\]

Let

\[
A = \left\{ \begin{pmatrix}
1 & \cdots & 1 \\
& \ddots & \ddots \\
& & 1
\end{pmatrix} \begin{pmatrix}
a_1 & \cdots & a_n \\
& \ddots & \ddots \\
& & a_1^{-1} \cdots a_n^{-1}
\end{pmatrix} \right\}
\]

be a maximal split torus in \( Sp_{2n}(F) \). Then, the Weyl group of \( Sp_{2n}(F) \) is

\[
W = N_G(A)/A,
\]

where \( N_G(A) \) denotes the normalizer of \( A \) in \( G \). The Weyl group of \( Sp_{2n}(F) \) has \( 2^n \cdot n! \) elements and may be viewed as \( W = \{ \text{permutations and sign changes of} \{e_1, \ldots, e_n\} \} \). \( W \) is generated by the simple root reflections \( \{s_1, \ldots, s_n\} \), where

\[
s_1 = \begin{pmatrix}
1 & 1 \\
1 & \ddots & 1 \\
\end{pmatrix}, \ldots,
\]

\[
\begin{pmatrix}
1 & 1 \\
\end{pmatrix}, \ldots,
\]

\[
\begin{pmatrix}
1 & 1 \\
\end{pmatrix}
\]
We now review the parabolic and parahoric subgroups for $S_{2n}(F)$. A standard minimal parabolic subgroup for $S_{2n}(F)$ is the following:

$$P_{\min} = \left\{ \begin{pmatrix} X & Y \\ T X^{-1} \end{pmatrix} \in S_{2n}(F) | X \in GL_n(F) \text{ is upper triangular} \right\}.$$ 

Note that $TX^{-1}$ will be lower triangular so that $P_{\min}$ is not the upper triangular matrices in $S_{2n}(F)$ (we could arrange for $P_{\min}$ to be upper triangular had we chosen a different $J$ in defining $S_{2n}(F)$). Let $\Phi \subset \{s_1, \ldots, s_n\}$. The standard parabolic subgroups of $S_{2n}(F)$ are parameterized by such subsets of the simple reflections. Associated to $\Phi$ is the parabolic subgroup $P_{\Phi} = < P_{\min}, \Phi >$.

If $P = MN$ is the Levi factorization of $P$ then

$$M \cong GL_{k_1}(F) \times GL_{k_2}(F) \times \cdots \times GL_{k_{i-1}}(F) \times S_{2k_i}(F),$$

with $k_1 + k_2 + \cdots + k_i = n$. This will be embedded in $S_{2n}(F)$ as

$$M = \begin{cases} \begin{pmatrix} A_1 & \cdots & A_{i-1} \\ & \ddots & \vdots \\ & & A \\ \vdots & \ddots & B \\ & & TA_i^{-1} \\ & & \vdots \\ & & C \\ & & D \end{pmatrix} & A_i \in GL_{k_i}(F) \\ \text{and} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in S_{2k_i}(F) \end{cases}.$$ 

We note that in terms of the parameterization of parabolic subgroups by subsets of $\{s_1, \ldots, s_n\}$, this parabolic subgroup corresponds to

$$\Phi = \{s_1, \ldots, s_n\} \setminus \{s_{k_1}, s_{k_1+k_2}, \ldots, s_{k_1+\cdots+k_{i-1}}\}.$$
We also remark that the modular function for $P$ is given by
\[
\delta(m) = |\det A_1|^{2n+1-k_1} \cdot |\det A_2|^{2(n-k_1)+1-k_2} \cdots |\det A_{i-1}|^{2(n-k_1-\cdots-k_{i-2})+1-k_{i-1}}.
\]

The parahoric subgroups may be constructed in a similar fashion. A standard Iwahori subgroup for $Sp_{2n}(F)$ may be constructed as follows: let $\Psi : K \to Sp_{2n}(\mathbb{F}_q)$ be the reduction mod $\mathcal{P}$ homomorphism (where $\mathbb{F}_q$ is the finite field $\mathcal{O}/\mathcal{P}$). Then, take $I = \Psi^{-1}(P_{\text{min}}(\mathbb{F}_q))$ as the standard Iwahori subgroup. Let
\[
s_0 = \begin{pmatrix}
1 & \cdots & \cdots & -\omega^{-1} \\
\vdots & & & \\
\omega & & & 1 \\
& & & \\
\omega^{-1} & & & 1
\end{pmatrix},
\]
so that $\{s_0, s_1, \ldots, s_n\}$ generate the affine Weyl group. The standard parahoric subgroups are in bijective correspondence with the subsets $\Phi$ of $\{s_0, s_1, \ldots, s_n\}$, the correspondence being given by
\[
B_{\Phi} = \langle I, \Phi \rangle.
\]
Note that if $\Phi \subset \{s_1, \ldots, s_n\}$, then
\[
B_{\Phi} = \Psi^{-1}(P_{\Phi}(\mathbb{F}_q)).
\]
These are the parahoric subgroups we shall be most interested in.

### 1.2. Induced representations and Jacquet modules

We now review the construction of induced representations and Jacquet modules.

We start by reviewing the construction of induced representations. Let $G$ be a reductive $p$-adic group and $P = MU$ a parabolic subgroup of $G$. Let $\mathcal{O}$ be an admissible representation of $M$ with $\rho$ and $\chi$ be $\mathbb{C}$-valued characters. Then $\rho$ may be extended trivially to $U$ to get a representation of $P$, $\rho \otimes 1$. The induced representation $\text{Ind}_P^G \rho \otimes 1$ acts on the space
\[
V = \left\{ f : G \to X \mid f \text{ smooth} \quad f(mug) = \delta^\chi(m)\rho(m)f(g) \quad \forall m \in M, \ u \in U, \ g \in G \right\}.
\]
where $\delta$ denotes the modular function for $P$. The action of $G$ on this space is by right translation. We shall give some more convenient notation for induced representation after we discuss Jacquet modules.

We now describe the construction of Jacquet modules. Again, suppose $G$ is a reductive $p$-adic group and $P = MU$ a parabolic subgroup of $G$. Let $(\pi, V)$ be a representation of $G$. The Jacquet module of $\pi$ with respect to $P$, denoted $\pi_U$, is a representation of $M$ on a space which is denoted by $V_U$. The space is

$$V_U = V/V(U),$$

where $V(U) = \text{span} \{ \pi(u)v - v | v \in V, u \in U \}$. The action of $M$ is given by

$$\pi_U(m)(v + V(U)) = \delta^{-\frac{1}{2}}(m)\pi(m)v + V(U)$$

(one checks that this defines a representation).

We shall frequently use the notation of Bernstein-Zelevinsky [B-Z] for induced representations and Jacquet modules. If $P = MU$ is a parabolic subgroup of $G$, $(\rho, X)$ an admissible representation of $M$, $(\pi, V)$ an admissible representation of $G$, then set

$$i_{GM}(\rho) = \text{Ind}_P^G \rho \otimes 1$$

and

$$r_{MG}(\pi) = \pi_U.$$

The following notation for induced representations in $S_{p_{2n}}(F)$ will also be convenient. It is just an extension (cf. [S-T]) of the shorthand notation of Bernstein-Zelevinsky for induced representation in $GL_n(F)$. Suppose $P = MU$ is a standard parabolic subgroup of $S_{p_{2n}}(F)$, with

$$M \cong GL_{k_1}(F) \times \cdots \times GL_{k_{l-1}}(F) \times S_{p_{2k_l}}(F).$$

Let $\rho_1, \ldots, \rho_{l-1}$ be admissible representations of $GL_{k_1}(F) \ldots GL_{k_{l-1}}(F)$, and $\tau$ an admissible representation of $S_{p_{2k_l}}(F)$. Then, let

$$\rho_1 \times \cdots \times \rho_{l-1} \times \tau = i_{GM} \rho_1 \otimes \cdots \otimes \rho_{l-1} \otimes \tau.$$
The sign \( \propto \) is used only to indicate that the last factor comes from a symplectic group. If \( k_l = 0 \), the representation will be denoted
\[
\rho_1 \times \ldots \rho_{l-1} \propto 1.
\]
The trivial representation of \( Sp_{2n}(F) \), \( n > 0 \), will be denoted \( tr_n \), so there will be no confusion.

We close these preliminaries with four theorems on induced representations and Jacquet modules.

**Theorem 1.2.1 (Frobenius Reciprocity)** Let \( G \) be a connected reductive \( p \)-adic group, \( P = MU \) a parabolic subgroup, \( \rho \) an admissible representation of \( M \), \( \pi \) an admissible representation of \( G \). Then,
\[
\text{Hom}_M(r_{MG}(\pi), \rho) \cong \text{Hom}_G(\pi, i_{GMN}\rho).
\]

**Proof.** cf. [B-Z2].

**Theorem 1.2.2** Let \( (\tau, G, L) \) be an irreducible subquotient of \( \text{Ind}^{G}_{P_{\min}} \psi \) (\( \psi \) a character). Then, there is a \( w \in W \) so that \( \tau \) embeds in \( \text{Ind}^{G}_{P_{\min}} w\psi \). Moreover, all \( \text{Ind}^{G}_{P_{\min}} w\psi \) for \( w \in W \) have the same components.

**Proof.** See [Cas2] for references for the first part and [B-Z2] for the second.

We next recall the following theorem, which says the constructions of induced representations and Jacquet modules may be done in stages.

**Theorem 1.2.3** Let \( L < M \) be standard Levis for \( G \). Then

1. \( i_{GL} = i_{GM} \circ i_{ML} \)

2. \( r_{LG} = r_{LM} \circ r_{MG} \).

**Proof.** cf. [B-Z2].

We close with a theorem of Bernstein-Zelevinsky, Casselman. Let \( M, N \) be standard Levis for a connected reductive \( p \)-adic group \( G \). Set \( W^{MN} = \{ w \in W | w(P_{\min} \cap M) \subset P_{\min}, w^{-1}(P_{\min} \cap N) \subset P_{\min} \} \). We remark that these correspond to the elements of shortest length in the double-cosets \( W_N \backslash W / W_M \) (\( W_M = \) Weyl group of \( M \), etc.).
Theorem 1.2.4 (Bernstein-Zelevinsky, Casselman) Let $\rho$ be an admissible representation of $M$. Then, $r_{NG} \circ i_{GM\rho}$ has a composition series with factors

$$i_{NN'} \circ w \circ r_{M'M\rho}, \quad w \in W^{MN}$$

where $M' = M \cap w^{-1}(N), N' = w(M) \cap N$.

Proof. See [Cas2], chapter 6 or [B-Z2] "geometrical lemma" (Lemma 2.12), proved in chapter 6.
CHAPTER 2
THE HECKE ALGEBRA APPROACH

2.1. General theorems

The goal of this chapter is to analyze reducibility/irreducibility in the case where the inducing subgroup has Levi factor \( M \cong F^x \times Sp_{2(n-1)}(F) \). We attack this problem using Hecke algebra methods. In particular, we construct an open compact subgroup \( B_x \) and to the representation \((\pi,G,V)\), we associate a representation of the Hecke algebra \( H(G//B_x) \) of compactly supported \( B_x \)-biinvariant functions on the finite-dimensional space \( V^{B_x} \). This representation, denoted \((\pi,H(G//B_x),V^{B_x})\), has the property that the subquotients of \((\pi,G,V)\) and those of \((\pi,H(G//B_x),V^{B_x})\) are in bijective correspondence, the bijection being implemented by \( X \subset V \) \( G \)-invariant goes to \( X^{B_x} \subset V^{B_x} \) \( H(G//B_x) \)-invariant. Once this correspondence is established, we can do our analysis on the Hecke algebra side, where the finite-dimensionality of the representation facilitates computations.

We now describe the contents of this chapter in a little more detail. For purposes of this discussion, let us assume that \( n > 1 \)–the case \( Sp_2(F) = Sl_2(F) \) behaves a little bit differently. The first section of this chapter consists of constructing \( B_x \) and establishing the aforementioned correspondence. In the second section, we explicitly construct a basis for \( V^{B_x} \). This construction breaks the problem into three cases: (order of \( \chi_u = 1 \)), (order of \( \chi_u = 2 \)), and (order of \( \chi_u > 2 \)). In section 2.3, we look at the case (order of \( \chi_u > 2 \)). We compute a pair of operators \( \pi(h_1), \pi(h_2) \) with \( h_1, h_2 \in H(G//B_x) \) and observe that they do not admit a common invariant subspace, so that in this case, \( \pi \) must always be irreducible. In section 2.4, we do similar computations for the case (order of \( \chi_u = 2 \)) and conclude that if \( \chi^2 \neq 1 \), then \( \pi \) is irreducible. The fifth section uses a variation on this theme for the case where \( \chi \) is unramified to show that if \( \chi^2 \neq 1 \) or \( \chi \neq \nu^{\pm1} \), then \( \pi \) is irreducible. In the last two sections, we show that if \( \chi^2 = 1 \) or \( \chi = \nu^{\pm1} \), then \( \pi \) is
Degenerate Principal Series

reducible. If $\chi = \nu^{\pm 1}$, then $\pi$ has the trivial representation as a subquotient, hence is reducible. If $\chi^2 = 1$, we show that $\pi$ is reducible by showing that $\text{Hom}_G(\pi, \tilde{\pi})$ is two-dimensional. Again, the analysis is broken up into cases depending on the order of $\chi_u$—the case (order of $\chi_u = 1$) is done in the sixth section, the case (order of $\chi_u = 2$) in the seventh. The results are summarized in Theorem 2.7.2.

The goal of this section is to establish the bijective correspondence between the subquotients of $(\pi, G, V)$ and those of $(\pi, H(G//B_\chi), V^{B_\chi})$ for a suitably chosen $B_\chi$. We start by choosing $B_\chi$ and discussing $(\pi, H(G//B_\chi), V^{B_\chi})$. We then show that any subquotient of $(\pi, G, V)$ has a nonzero $B_\chi$-fixed vector. The last part is to show that since every subquotient of $(\pi, G, V)$ has a nonzero $B_\chi$-fixed vector, there is a bijective correspondence between the subquotients of $(\pi, G, V)$ and those of $(\pi, H(G//B_\chi), V^{B_\chi})$. We note that although we are interested in the degenerate principal series coming from a maximal proper parabolic subgroup in $Sp_{2n}(F)$, the results are more general. Everything should work for an arbitrary parabolic in $G$ a reductive $p$-adic group. We shall work in the more general situation.

Let $P \subset G$ be a parabolic subgroup, with $P = MU$ its Levi decomposition. Let $\chi$ be a character of $M$ and set $\pi = i_{GM \chi}$. The subgroup $B_\chi$ which we use may be described by its Iwahori factorization with respect to $P$. In particular,

$$B_\chi = U^- M_\chi U_0$$

where $U_0 = U \cap K, U^- = U^- \cap K_I$, with $l$ the smallest positive integer for which $\chi|_{M_I}$ is trivial ($M_I = M \cap K_I$), and $M_\chi = M_0 \cap \ker \chi$. We normalize our measures so that $|B_\chi| = |U^-| = |M_\chi| = |U_0| = 1$.

We check that $B_\chi$ is a group. In fact, we show that $B_\chi = K_I M_\chi U_0 = M_\chi U_0 K_I$. From this and the fact that $K_I$ is normal in $K$, it is easy to check that $B_\chi$ is closed under multiplication and inverses. To show $B_\chi = K_I M_\chi U_0 = M_\chi U_0 K_I$, it is enough to show $K_I \subset B_\chi$. Suppose $k \in K_I$. Since $K_I \subset K_I \subset B$, we have a factorization of $k$ as $k = u^{-m}ru$ with $r \in U_I^-$, $m \in M_0$, $u \in U_0$. We claim $u^- \in U_I^-$. We use the reduction mod $P^l$ homomorphism to get $u^{-m}ru \equiv I \mod P^l$. Now, suppose $u^- \not\equiv I \mod P^l$. Then, $mu \equiv (u^-)^{-1} \mod P^l$. But, this cannot occur since it would imply that $P(O/\omega^lO)$ and $U^-(O/\omega^lO)$ had nontrivial intersection. Thus,
$u^\sim \in U_i^\sim$. A similar argument shows $u \in U_i$. This forces $m \in M_i$. Consequently, $K_i \subset U_i^\sim M_i U_i \subset B_\chi$, as needed.

We now describe the representation $(\pi, H(G//B_\chi), V^{B_\chi})$. By definition, $H(G//B_\chi)$ is the (convolution) algebra of compactly supported $B_\chi$-biinvariant functions. $V^{B_\chi}$ is the space of $B_\chi$-invariant functions in $V$; finite-dimensional since $B_\chi$ is open compact. The action is given by

$$\pi(h)V = \int_G h(g)\pi(g)v \, dg$$

for

$$h \in H(G//B_\chi), \, v \in V^{B_\chi}.$$

Our next task it to show that every subquotient of $(\pi, G, V)$ contains a $B_\chi$-fixed vector. We start with the following result of Casselman.

**Proposition 2.1.1 (Casselman)** Let $(\pi, G, V)$ be an admissible representation of $G$ ($G$ as above). Let $B$ be a compact subgroup of $G$ with an Iwahori factorization $B = U_B^{-} M_B U_B$ with respect to $P = MU$ (where $U_B^{-} = U^{-} \cap B$, etc.). Then,

$$r_{MG} : V^B \rightarrow (r_{MG}(V))^{M_B}$$

is surjective.

**Lemma 2.1.2 (Jacquet via Casselman)** With hypotheses as above, suppose that $v \in V^{M_BU_B^{-}}$. Then, $v_0 = Pr_B(v) = Pr_{U_B}(v)$ satisfies $v - v_0 \in V(U_B)$ (where $Pr_K(v) = \frac{1}{|K|} \int_K \pi(k)v \, dk$)

**Proof of Lemma.** Using the Iwahori factorization,

$$v_0 = \int_B \pi(b)v \, db = \int_{U_B^{-}} \int_{M_B} \int_{U_B} \pi(b)v \, db = \int_{U_B^{-}} \pi(u)v \, du \quad \text{(since $v \in V^{M_BU_B^{-}}$)}$$

$$= Pr_{U_B}(v),$$

which is the first claim. The second claim follows immediately from this.
Lemma 2.1.3 With hypotheses as above,

\[ r_{MG}(V^B) = r_{MG}(V^{MBU_B}). \]

Proof of Lemma. Since \( v - v_0 \in V(U_B) \), we have \( v - v_0 \in V(U) \). Thus, \( v \) and \( v_0 \) have the same image in \( r_{MG}(V) \).

Proof of Proposition. Let \( \tilde{X} \subset (r_{MG}(V))^{MB} \) be finite-dimensional with \( X \subset V^{MB} \) a finite-dimensional subspace mapping onto it. Since \( X \) is finite-dimensional, there is a compact \( \tilde{U}^- \subset U^- \) such that \( X \subset V^{MB\tilde{U}^-} \). Now, there exists \( a \in A_F \), the maximal split torus in the center of \( M \), such that \( a^{-1}U_B a \subset \tilde{U}^- \). This implies that \( \pi(a)X \subset V^{MBU_B} \) (Check: \( \pi(mu)\pi(a)v = \pi(a)\pi(a^{-1}ma)\pi(a^{-1}ua)v = \pi(a)\pi(m)v = \pi(a)v \) for \( m, u, a \) as described.) By the preceding lemma,

\[ \pi_U(a)\tilde{X} = r_{MG}(\pi(a)X) \subset r_{MG}(V^{MBU_B}) = r_{MG}(V^B) \]

Thus, \( \dim \tilde{X} \leq \dim V^B < \infty \) by admissibility. Consequently, \( \dim(r_{MG}(V))^{MB} \leq \dim V^B < \infty \).

Next, let \( \tilde{X} = (r_{MG}(V))^{MB} \). Then, we must have \( \pi_U(a)(V_U)^{MB} = (V_U)^{MB} \).
Since \( a \) was chosen so that \( \pi_U(a)(V_U)^{MB} \subset r_{MG}(V^B) \), we have \( (V_U)^{MB} = (r_{MG}(V))^{MB} \subset r_{MG}(V^B) \), as desired.

The second ingredient toward establishing that every subquotient of \( (\pi, G, V) \) contains a \( B^- \)-fixed vector consists of extending the results of Borel/Casselman to get the proposition below.

Proposition 2.1.4 Any subquotient \((\tau, G, L)\) of \( \text{Ind}^G_{P_{\text{min}}} \psi \) has a nonzero \( I_{\psi}^- \)-fixed vector. For \( P_{\text{min}} = AN \), \( I_{\psi} = N_I^-A_{\psi}N_0 \) with \( A_{\psi} = A_0 \cap \bigcap_{w \in W} \ker w\psi \) and \( l = \) smallest positive integer such that \( \psi|_{A_1} \equiv 1 \).

Proof. See [Bor] 4.9 for the unramified case (where \( I_{\psi}= Iwahori, G \) semisimple).

By the preceding proposition, it suffices to show that \( L_{N}^{A_\psi} \neq 0 \).

First, suppose \( \tau \) is actually a subrepresentation of \( \text{Ind}^G_{P_{\text{min}}} \psi \). Then, Frobenius reciprocity \( \Rightarrow \)

\[ \text{Hom}_A(\tau_N, \psi) \cong \text{Hom}_G(\tau, \text{Ind}^G_{P_{\text{min}}} \psi) \neq 0 \]
Thus, $\tau_N$ contains a vector which transforms under $A$ according to $\psi$, hence is fixed by $A_\psi$. So, $L_N^{A_\psi} \neq 0$.

Now, suppose that $\tau$ is just a subquotient. By Theorem 1.2.2, there is a $w \in W$ so that $\tau$ is a subrepresentation of $\text{Ind}^G_{P_{\text{min}}} w\psi$. Then, as above, $\tau_N$ contains a vector which transforms under $A$ according to $w\psi$, hence is fixed by $A_\psi$. So, $L_N^{A_\psi} \neq 0$. The conclusion follows.

This brings us to our proposition.

**Proposition 2.1.5** Any subquotient of $(\pi, G, V)$ has a (nonzero) $B_\chi$-fixed vector.

**Proof.** First, let $(\rho, G, Y)$ be a subquotient of $(\pi, G, Y)$. We claim that $(\rho, G, Y)$ has an $I_\chi$-fixed vector (using $\chi|_A$ to get $I_\chi$). This follows since $\chi$ is a subrepresentation of $\text{Ind}^M_{M \cap P_{\text{min}}} \delta^{-1/2}(\chi|_A)$ with $\delta$ the modular function for $M \cap P_{\text{min}}$.

We now proceed as in [Gus].

**Lemma 2.1.6** Any $K$-invariant subspace $(\omega, K, \Omega)$ of $(\pi, G, V)$ has a nontrivial $P \cap B_\chi$-fixed vector.

**Proof of Lemma.** Since $G = PK$ (Iwasawa decomposition), $\pi|_K = \text{Ind}^K_{P \cap K} \chi$. By Frobenius reciprocity,

$$[\text{Ind}^K_{P \cap K} \chi : \omega] = [\omega|_{P \cap K} : \chi],$$

so $\omega|_{P \cap K}$ has a vector which translates according to $\chi$, hence is fixed on $M_\chi$ (and trivial on $U_0$). The lemma follows. Note that we are writing $\chi$ for $\chi|_{P \cap K}$, etc, above.

**Lemma 2.1.7** Any nonzero $K$-invariant subspace $X$ which contains a nontrivial $K_l$-fixed vector contains a nontrivial $B_\chi$-fixed vector.

**Proof of Lemma.** $K_l \triangleleft K \implies K^{K_l}$ is a $K$-subrepresentation of $\chi$. By the preceding lemma, it has a $P \cap B_\chi$-fixed vector. Since $B_\chi$ is generated by $P \cap B_\chi, U_l^-$ ($U_l^- \subset K_l$), the lemma holds.

Now we finish the proof of the proposition. From the first paragraph, we know that $(\rho, G, Y)$ contains an $I_\chi$-fixed vector. As $K_l \subset I_\chi$, it is $K_l$-fixed. By the preceding Lemma, $(\rho, G, Y)$ contains a $B_\chi$-fixed vector, as claimed.
We now begin the final task of this section—establishing the bijective correspondence between the subquotients of $(\pi, G, V)$ and those of $(\pi, H(G//B_x), V^B_x)$. The lemma and theorem that follow have been culled from various places (cf. [Cas2], [Gus], [Bor]).

**Lemma 2.1.8** Let $V_1, V_2, V_3$ be smooth $G$ representations, $B$ an open compact subgroup. If

$$V_1 \rightarrow V_2 \rightarrow V_3$$

is an exact sequence of $G$-modules, then,

$$V^B_1 \rightarrow V^B_2 \rightarrow V^B_3$$

is exact as well (as vector spaces).

**Proof.** See [Cas2], 2.1.7.

**Lemma 2.1.9** Let $V$ be a smooth finite length representation of $G$, $B$ an open compact subgroup of $G$. Suppose that every subquotient of $V$ has a nonzero $B$-fixed vector. Then, every subquotient of $V$ is generated by its $B$-fixed vectors.

**Proof.** Suppose not. Let $V'$ be a subquotient of $V$ and set $V'' = G \cdot V'^B = $ subspace of $V'$ generated by $V'^B$ under the action of $G$. $V''$ is a subrepresentation of $V'$; by assumption, not all of $V'$. By the previous lemma,

$$\left(\frac{V'}{V''}\right)^B = \frac{(V')^B}{(V'')^B} = 0.$$ 

But, this cannot happen since $\frac{V'}{V''}$ is a nonzero subquotient of $V$.

**Theorem 2.1.10** The map $W \rightarrow W^B_x$ gives a bijective correspondence between subquotient of $(\pi, G, V)$ and subquotients of $(\pi, H(G//B_x), V^{B_x})$.

**Proof.** By Proposition 2.1.5, every subquotient of $V$ has a nonzero $B_x$-fixed vector. Therefore, by the preceding lemma, every subquotient of $V$ is generated by its
$B_x$-fixed vectors. We now check that $W \to W^{B_x}$ gives a bijective correspondence between subrepresentations of $(\pi, G, V)$ and those of $(\pi, H(G//B_x), V_x^B)$.

**injective:** Suppose that $V_1, V_2$ are subrepresentations of $V$ satisfying $V_1^{B_x} = V_2^{B_x}$. Then $G \cdot V_1^{B_x} = G \cdot V_2^{B_x}$ (where $G \cdot V_i^{B_x}$ denotes the subspace of $V$ generated by $V_i^{B_x}$ under the action of $G$). Since every subquotient of $V$ is generated by its $B_x$-fixed vectors, $G \cdot V_i^{B_x} = V_i$. Thus $V_1 = V_2$, as needed.

**surjective:** Let $W \subset V^{B_x}$ be an $H(G//B_x)$-invariant subspace. Set $V_0 = G \cdot W$. We claim $V_0^{B_x} = W$. Suppose not - then $V_0^{B_x} \supset W$. Suppose $v \in V_0^{B_x}$ but $v \notin W$. Since $v \in G \cdot W$, we can write

$$v = \sum_{\text{finite}} \pi(g_i)w_i \quad w_i \in W, g_i \in G$$

We claim $v = \sum \pi(\chi_B * g_i \ast \chi_B)w_i$, where $\chi_B = \text{char}_{B_x}$. Note that $\chi_B * g_i \ast \chi_B \in H(G//B_x)$ so this will express $v$ in such a way that it must lie in $W$. Check:

$$\sum \pi(\chi_B * g_i \ast \chi_B)w_i = \sum \pi(\chi_B)\pi(g_i)w_i = \pi(\chi_B)(\sum \pi(g_i)w_i) = \pi(\chi_B)v = v.$$

We also need the following corollary of the proof.

**Corollary 2.1.11** Suppose $(\pi_i, G, V_i), i = 1, 2$ are such that every nonzero subquotient of each $V_i$ contains a nontrivial $B_x$-fixed vector. Then,

$$\text{Hom}_G(V_1, V_2) \cong \text{Hom}_{H(G//B_x)}(V_1^{B_x}, V_2^{B_x})$$

**Proof.** (cf. [Cas2], Proposition 2.2.2)

### 2.2. Basis for $V^{B_x}$

The goal of the rest of this chapter is to use the preceding theorem to analyze reducibility for a specific example. Let $G = Sp_{2n}(F)$, $F$ $p$-adic of characteristic zero, odd residual characteristic. Let $M \cong F^x \times Sp_{2(n-1)}$ and $P = MU$ the corresponding parabolic subgroup. We identify characters of $F^x$ with characters on $M$ via the $F^x$ part. We set $\pi = i_{GM} \chi$, and analyze the reducibility of $(\pi, G, V)$ by analyzing reducibility for $(\pi, H(G//B_x), V^{B_x})$. The goal in this section is to
construct a basis for $V^B_x$. The problem breaks itself into three cases. If we write $\chi = | \cdot |^s \chi_u$ (see chapter 1), the cases are

**Case 1** (order of $\chi_u$) = 1

**Case 2** (order of $\chi_u$) = 2

**Case 3** (order of $\chi_u$) > 2

We shall approach these three separately and summarize the results in Theorem 2.2.4. The reader is advised that the third case is substantially longer than the other two.

We start with the following lemma, which is sort of a Bruhat decomposition. We let $B$ be the parahoric subgroup corresponding to $P$ (obtained by adjoining $s_2, \ldots, s_n$ to the Iwahori). (It is $B_x$ for $\chi \equiv 1$)

**Lemma 2.2.1** Let $w_{00} =$ identity,

$$
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
\end{pmatrix}
$$

$$(w_{10} = s_1). \text{ Note that for } n = 1, w_{10} \text{ does not exist. Then,}$$

$$K = B \cup (Bw_{10}B) \cup (Bw_{11}B) \text{ for } n \geq 2$$

and

$$K = B \cup (Bw_{11}B) \text{ for } n = 1.$$

**Proof.** See appendix.

At this point, we give a brief description of the process. First, using the preceding Bruhat decomposition as a starting point, we obtain a decomposition $K = \cup_{\alpha \in S} P_\alpha B_x$, so that $G = PK = \cup_{\alpha \in S} P_\alpha B_x \text{ (S finite). Thus, } \Phi \in V^B_x$ is
determined by its values on $\alpha \in S$. The natural first attempt at a basis would then be $\{f_\alpha\}_{\alpha \in S}$, where

$$f_\alpha(g) = \begin{cases} \delta^{1/2}\chi(p) & \text{if } g = p\alpha b \in P\alpha B_x \\ 0 & \text{if } g \notin P\alpha B_x. \end{cases}$$

However, these are not always well-defined. Thus, once we have determined a set $S$ of representatives of $P_0 \backslash K/B_x$, we must determine which of the $\alpha$'s in $S$ actually "work".

**Case 1:**

In this case, $B_x = B$, so we can just use the Bruhat decomposition $K = \cup_{(i,j)} P_0w_{ij}B$ from the lemma. We claim that

$$f_{ij}(g) = \begin{cases} \delta^{1/2}\chi(p) & \text{if } g = pw_{ij}b \in Pw_{ij}B \\ 0 & \text{if } g \notin Pw_{ij}B. \end{cases}$$

are all well-defined, hence constitute a basis for $V^B$. We remark that $\{(i,j)\} = \{(0,0),(1,0),(1,1)\}$ for $n \geq 2$, $\{(0,0),(1,1)\}$ for $n = 1$.

We check well-definedness. As the double-cosets are disjoint, we have only to show that $pw_{ij}b = p'w_{ij}b' \implies \delta^{1/2}\chi(p) = \delta^{1/2}\chi(p')$, or equivalently, $\delta^{1/2}\chi(p'^{-1}p) = 1$. However, $p'^{-1}p = w_{ij}b^{-1}w_{ij}^{-1} \in (w_{ij}Bw_{ij}^{-1}) \cap P \subset P_0$, so this holds. Thus, we have a basis for $V^B$.

**Case 2:**

In this case, $B_x = U_1^- M_x U_0$, with $M_x = M_0 \cap \ker \chi$. We claim that $K = \cup_{ij} P_0w_{ij}B_x$. In particular, suppose

$$\epsilon = \begin{pmatrix} \epsilon & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & \epsilon^{-1} & 1 \\ 1 & \cdots & 1 & \end{pmatrix}$$

satisfies $\chi(\epsilon) = -1$. Then, we have $B = B_x \cup (\epsilon B_x)$. Thus, we get $K = \cup_{ij} P_0w_{ij}(B_x \cup (\epsilon B_x))$. However, since $w_{ij}\epsilon w_{ij}^{-1} \in P_0$, we have $P_0w_{ij}B_x = P_0w_{ij}(\epsilon B_x) = P_0w_{ij}B$, and the claim holds.
To address the question of well-definedness, we start with the following lemma.

**Lemma 2.2.2** Let $\Phi \in V^B_x$. Then, if $i \neq j$, $\Phi(w_{ij}) = 0$, i.e., $\Phi(w_{10}) = 0$.

**Proof.** Let $\epsilon$ be as above. Then, since $\chi(\epsilon) = -1$, we have

\[-\Phi(w_{10}) = \Phi(\epsilon w_{10}) = \Phi(w_{10} \epsilon) = \Phi(w_{10}),\]

where

\[
\epsilon' = w_{10}^{-1} \epsilon w_{10} = \begin{pmatrix}
1 & \epsilon & 1 \\
\epsilon & 1 & 1 \\
1 & 1 & \epsilon^{-1}
\end{pmatrix}.
\]

Thus, $\Phi(w_{10}) = 0$. Note that for $n = 1$, this lemma is unnecessary.

Now, any $\Phi \in V^B_x$ must be zero on the double-coset associated to $w_{10}$. We claim that $\{f_i\}_{i=0,1}$ below constitute a basis for $V^B_x$:

\[
f_i(g) = \begin{cases}
\delta^{1/2} \chi(p) & \text{for } g = pw_{ii}b \in Pw_{ii}B_x \\
0 & \text{if } g \notin Pw_{ii}B_x.
\end{cases}
\]

These will be a basis once we know that they are well-defined.

We now check that these are well-defined. Suppose $pw_{ii}b = p'w_{ii}b'$, with $p, p' \in P$ and $b, b' \in B_x$. We must show that $\delta^{1/2} \chi(p) = \delta^{1/2} \chi(p')$, or equivalently, $\delta^{1/2} \chi(p'^{-1}p) = 1$. Observe that $p'^{-1}p = w_{ii}b'b^{-1}w_{ii}^{-1} \in M_x U_0$ ($w_{00}, w_{11}$ normalize $M_x$). The conclusion follows.

**Case 3:**

In this case, $B_x = U_0^{-} M_x U_0$. By the Bruhat decomposition, $K = \bigcup_{i,j} P_0 w_{ij} B$. Now, $B = \bigcup_{u^{-}, m} u^{-} m B_x$, where $u^{-}, m$ run over representatives of $U^- / U^-_h$ and
$M_0/M_\chi$ ($M_0$ normalizes $U^-_h$). Next, since any $m \in M_0$ may be written in the form

$$\begin{pmatrix}
\alpha & 1 \\
\vdots & \ddots & \ddots & 1 \\
\alpha^{-1} & 1 & \ddots & \\
1 & \ddots & \ddots & \ddots & 1
\end{pmatrix} m',
$$

with $m' \in M_\chi$, we get $B = \cup_{\alpha,u^-} u^- \alpha B_\chi = \cup_{\alpha,u^-} u^- \alpha B_\chi$ as $\alpha$ runs over $\mathcal{O}_x/\ker \chi$ and $u^-$ runs over $U^-_1/U^-_h$. Note that we are not claiming this is a disjoint union (it does not matter—see the upcoming lemma).

If we combine this with the Bruhat decomposition, we get

$$K = \cup_{i,j;u^-} P_0 w_i; u^- B_\chi = \cup_{i,j;u^-} P_0 w_i; u^- B_\chi$$

since $w_{ij} \alpha w_{ij}^{-1} \in P_0$. We now examine $w_{ij} U^- w_{ij}^{-1}$—this allows us to reduce the number of $u^-$'s needed in the above expression (e.g. some of the $u^-$'s used above will conjugate into $P_0$, hence be unnecessary). We do the two nontrivial cases:

$$w_{10} = \begin{pmatrix} 1 \\ w_1 \\ \vdots \\ w_{n-1} \end{pmatrix} I_{n-1} \quad w_{10}^{-1} = \begin{pmatrix} 1 & w_1 \\ 1 \\ \vdots \\ 1 & w_{n-1} \end{pmatrix} I_{n-1}$$
Of course, \( w_{00} \) acts trivially. We now write down our decomposition more explicitly, removing unnecessary duplications:

\[
K = (U_u P_0 w_{00} u^- B_x) \cup (U_y P_0 w_{10} y B_x) \cup (P_0 w_{11} B_x)
\]

where \( u^- \) runs over \( U_1^- / U_h^- \) and

\[
y = \begin{vmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
y & \cdots & 1
\end{vmatrix}
\]

with \( y \) running over \( \mathcal{P} / \mathcal{P}^h \).

**Lemma 2.2.3** If \( u^- \) (resp. \( y \)) is a double-coset representative with \( u^- \) (resp. \( y \)) nonzero mod \( \mathcal{P}^h \) and \( \Phi \in V^{B_x} \), then

\[
\Phi(w_{00} u^-) = 0 \quad (\text{resp.} \Phi(w_{10} y) = 0).
\]

**Proof.** The general style of the argument is as follows (using \( u^- \)'s for \( u^- \) or \( y \)): we have

\[
\Phi(w_{ij} u^-) = \Phi(w_{ij} u^- u) \quad u \in U_0
\]
We find a $u$ so that $u^- u \equiv \hat{p}^- \mod \mathcal{P}^h$ with $u^- \equiv \hat{u}^- \mod \mathcal{P}^h$ and $w_{ij}p w_{ij}^{-1} \in P$ but not in $\ker \chi$. Then

\[
\Phi(w_{ij} u^-) = \Phi(w_{ij} \hat{p}^-) = \Phi((w_{ij} p w_{ij}^{-1}) w_{ij} \hat{u}^-) = \chi(w_{ij} p w_{ij}^{-1}) \Phi(w_{ij} u^-)
\]

Since $\chi(w_{ij} p w_{ij}^{-1}) \neq 1$, we must have $\Phi(w_{ij} u^-) = 0$. The difficult part is to find an explicit choice of $u$ (it will depend on $u^-$). There will be three cases.

To expedite matters, we use the following notation: if $X$ is a matrix, $\max(X)$ is the largest norm that occurs as the norm of an entry of $X$. Let

\[
u^- = u^- W u^- X u^- Y = \begin{pmatrix} 1 & & \\ W & I_{n-1} & \\ \hline \hat{X} & T_Y & 1 \\ \hline Y & I_{n-1} & \\ \end{pmatrix}
\]

with $\hat{X} = X - W^T Y$.

**Case A:** $W, X \equiv 0 \mod \mathcal{P}^h$.

Let

\[
u = \begin{pmatrix} 1 & & \\ I_{n-1} & Z & \\ \hline 1 & & \\ \hline \end{pmatrix}
\]

We will place conditions on $Z$ as we go along. For now, the entries of $Z$ are in $\mathcal{O}$. 

Now, letting $I = I_{n-1}$,

\[
\Phi(w_{ij} u^-) = \Phi(w_{ij} u^- u)
\]

\[
= \Phi \left( w_{ij} \begin{pmatrix}
1 & T_Z \\
I & Z \\
T_Y & 1 + TYZ \\
Y & I + Y^TZ
\end{pmatrix} \right)
\]

\[
= \Phi \left( w_{ij} \begin{pmatrix}
T(1 + TYZ)^{-1} & T_Z \\
(I + Y^TZ)^{-1} & Z \\
(1 + TYZ)^{-1} & I + TYZ \\
Y^T(I + TYZ)^{-1} & I
\end{pmatrix} \right)
\]

Next, suppose that the largest entry of $Y$ is in $\mathcal{P}^l \setminus \mathcal{P}^{l+1}$ (i.e. $\max Y = q^{-l}$), $1 \leq l \leq h$. Then, we insist that the largest entry of $Z$ lie in $\mathcal{P}^{h-l-1}$ (i.e. $\max Z = q^{-h+l+1}$). We claim that this implies

\[
Y^T(1 + TYZ)^{-1} \equiv Y \mod \mathcal{P}^h
\]

\[
(1 + TYZ)^{-1} T Y \equiv T Y \mod \mathcal{P}^h
\]

Observe that the largest entry of $TYZ$ must lie in $\mathcal{P}^{h-1}$. Thus, $T(1 + TYZ)^{-1} \in 1 + \mathcal{P}^{h-1}$. Consequently, $Y^T(1 + TYZ)^{-1} \equiv Y \mod \mathcal{P}^h$ (since $Y$ has entries in $\mathcal{P}$).
The other half of the claim is similar. In particular, the second matrix in $\Phi$ above is congruent to $u^{-}\mod \mathcal{P}^h$. Thus we have,

\[
(*) \quad \Phi(w_{ij}u^{-}) = \Phi \begin{pmatrix} w_{ij} & (T(1+^TYZ)^{-1} \quad T_Z \\ Z & 1+^TYZ & I+^TY^TZ \end{pmatrix} u^{-}.
\]

**Subcase 1:** $w_{ij} = w_{11}$

Then, (*) above gives us

\[
\Phi(u^{-}) = \chi(T(1+^TYZ)^{-1})\Phi(u^{-}) = \chi^{-1}(1+^TYZ)\Phi(u^{-})
\]

($\delta^{1/2} \equiv 1$ on $P \cap K$). Now, we allow $Z$ to run over its prescribed set ($\max Z \leq q^{-h+i+1}$). Then, $1+^TYZ$ will cover all of $1+\mathcal{P}^{h-1}$ Recall that $\chi|_{1+\mathcal{P}^h} \equiv 1$ but $\chi|_{1+\mathcal{P}^{h-1}} \neq 1$. In particular, there exist choices of $Z$ giving $1+^TYZ \notin \ker \chi$. Taking such a $Z$,

\[
\Phi(u^{-}) = \chi^{-1}(1+^TYZ)\Phi(u^{-})
\]

implies $\Phi(u^{-}) = 0$, as claimed.

**Subcase 2:** $w_{ij} = w_{10}$
We note that case A is the only one in which \( w_{10} \) occurs for the proof of the lemma. In this case, \( T_Y = (y \ 0 \ldots \ 0) \). We now further restrict \( Z \) to be of a similar form: \( T_Z = (z \ 0 \ldots \ 0) \). Then, (*) becomes

\[
\Phi(w_{10}u^-) = \Phi\begin{pmatrix}
\begin{array}{ccc}
(1 + yz)^{-1} & (1 + yz)^{-1} & z \\
(1 + yz)^{-1} & 1 & 0 \\
& 0 & \vdots \\
& & 1 \\
& & \vdots \\
& & 1
\end{array}
\end{pmatrix}
\begin{pmatrix}
(1 + yz) \\
(1 + yz) \\
1 \\
1
\end{pmatrix} u^-
\]

\[
= \Phi\begin{pmatrix}
\begin{array}{ccc}
(1 + yz)^{-1} & (1 + yz)^{-1} & z \\
(1 + yz)^{-1} & 1 & 0 \\
& 0 & \vdots \\
& & 1 \\
& & \vdots \\
& & 1
\end{array}
\end{pmatrix}
\begin{pmatrix}
(1 + yz) \\
(1 + yz) \\
1 \\
1
\end{pmatrix} w_{10}u^-
\]

\[
= \chi^{-1}(1 + yz)\Phi(w_{10}u^-)
\]

Again, as we let \( z \) run over \( \mathcal{P}^{h-l-1} \), \( 1 + yz \) runs over \( 1 + \mathcal{P}^{h-1} \). In particular, we can find a \( z \) so that \( \chi^{-1}(1 + yz) \neq 1 \). Thus, \( \Phi(w_{10}u^-) = 0 \), as claimed. This finishes case A.

**Case B**: \( \max X \geq \max W \)

We note that in this case, we are dealing with \( w_{ij} = w_{00} \) only.
Write $U_Z = \begin{pmatrix} 1 & Z \\ I & 1 \\ 1 & I \end{pmatrix}$. We restrict $Z \in \mathcal{O}$ for now. More restrictions/specifications about $Z$ will be made as we proceed.

Suppose that $X \in \mathcal{P}^l \setminus \mathcal{P}^{l+1}$. Then, we will want $Z \in \mathcal{P}^{h-l-1}$. We compute:

$$
\Phi(u^-) = \Phi(u_X^- u_Y^- u_w^-)
= \Phi(u_X^- u_Y^- u_w^- u_Z^-)

= \Phi \begin{pmatrix} u_X^- u_Y^- u_Z^- u_w^- \end{pmatrix}
\begin{pmatrix}
1 & z^T w \\
I & w^T z - w z^T w \\
1 & I \\
\end{pmatrix}

= \Phi(u_X^- u_Y^- u_Z^- u_w^-)

= \Phi \begin{pmatrix} u_X^- u_Z^- \end{pmatrix}
\begin{pmatrix}
1 & -z y^- \\
I & 1 \\
-y z y^- & -y z & I \\
\end{pmatrix}

u_w^T

Now, if we conjugate this matrix around $u_w^T$, we get

$$
\begin{pmatrix}
1 & -z y^- \\
I & 1 \\
y z y^- & y z & I \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & z y^- \\
I & 1 \\
y z y^- & y z & I \\
\end{pmatrix}
\begin{pmatrix}
1 & -z y^- \\
I & 1 \\
y z y^- & y z & I \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & z y^- \\
I & 1 \\
y z y^- & y z & I \\
\end{pmatrix}
$$
\[ \Phi(u^-) = \Phi(u^-_X u^-_Y u^-_W) \]

\[ = \Phi \left( \begin{array}{cc|c}
1 & Z & u^-_Y u^-_W \\
I & 1 + XZ & I \\
X & 1 + XZ & I \\
\hline 
1 + XZ & 1 + XZ & I \\
I & I & 1 + XZ \\
\end{array} \right) \]

since \( \max X \geq \max W \) and the constraint \( Z \in \mathcal{P}^{h-l-1} \) ensures that \( ZX, ZW, \) etc., lie in \( \mathcal{P}^{h-1} \). Multiplying by \( Y \) then puts them in \( \mathcal{P}^h \).

Thus, we get

\[ \Phi(u^-) = \Phi(u^-_X u^-_Y u^-_W) \]

\[ = \Phi \left( \begin{array}{cc|c}
1 & Z & u^-_Y u^-_W \\
I & 1 + XZ & I \\
X & 1 + XZ & I \\
\hline 
1 + XZ & 1 + XZ & I \\
I & I & 1 + XZ \\
\end{array} \right) \]
since $XZ \in \mathcal{P}^{h-1}, (1+XZ)^{-1}X \equiv X \text{mod } \mathcal{P}^h$. Thus,

$$
\Phi(u^-) = \Phi\left(\begin{array}{c|c|c}
1+XZ & Z & \hline
I & I & \hline
1+XZ & I & \hline
\end{array}\right) u_X u_Y u_W
$$

$$
= \chi^{-1}(1+XZ)\Phi(u^-)
$$

The rest of the argument is similar to case A. As $Z$ runs over $\mathcal{P}^{h-1}$, $XZ$ runs over $\mathcal{P}^{h-1}$. Then, $\Phi(u^-) = \chi^{-1}(1+XZ)\Phi(u^-)$ for all such $Z$ implies $\Phi(u^-) = 0$, since $\chi$ is not constant on $1+\mathcal{P}^{h-1}$. This finishes case B.

**Case C: max $W > max X$**

Write $u_Z = \begin{pmatrix} 1 & Z \\ I & 1 \end{pmatrix}$. Again, if the largest entry of $W$ is in $\mathcal{P}^l \setminus \mathcal{P}^{l+1}$, we will want the largest entry of $Z$ to be in $\mathcal{P}^{h-l-1}$.

We compute:

$$
\Phi(u^-) = \Phi(u_W u_X u_Y)
$$

$$
= \Phi(u_W u_X u_Y u_Z)
$$

$$
= \Phi\left(\begin{array}{c|c|c}
1 & Z & \hline
I & I & \hline
\tau(YZ)+YZ & 1 & \hline
\end{array}\right)
$$

$$
= \Phi(u_W u_X u_Y u_Z u_Y)
$$

$$
= \Phi(u_W u_X u_Z u_Y)
$$
since the last matrix is in $M_x \subseteq B_x$. Continuing,

$$
\Phi(u^-) = \Phi\left( u_W^{-1}u_Zu_X^{-1} \begin{pmatrix}
1 & I \\
XZ & 1 \\
\tau_{ZX} & \tau_{ZXX} & I
\end{pmatrix} u_Y^- \right)
$$

Since $Z \subseteq \mathcal{P}^{h-I-1}$ and $\max W > \max X$, the matrix above lies in $\mathcal{P}^h$. Thus,

$$
\Phi(u^-) = \Phi(u_W^{-1}u_Zu_X^{-1}u_Y^-) = \Phi\left( \begin{pmatrix}
1 & Z \\
W & I + WZ \\
1 + ZW & -\tau W \\
-\tau Z & I \\
(1 + ZW)^{-1} & Z \\
I + WZ & I + ZW \\
1 + ZW & -\tau Z & \tau(I + WZ)^{-1} \\
(I + WZ)^{-1}W & I & 1 & -\tau W(T(I + WZ)^{-1}) \\
\end{pmatrix} u_X^-u_Y^- \right)
$$
Again, we have $WZ \subset \mathcal{P}^{h-1}$, so the second matrix is congruent to $u_W^{-1} \mod \mathcal{P}^h$. Continuing,

$$
\Phi(u^-) = \Phi \left( \begin{pmatrix} (1+ZW)^{-1} & Z \\ I+WZ & u_W^{-1} \end{pmatrix} \begin{pmatrix} 1+ZW \\ -T \end{pmatrix} \begin{pmatrix} u_W^{-1} \\ u_X^{-1} u_Y^{-1} \end{pmatrix} \right) = \chi^{-1}(1+ZW)\Phi(u^-)
$$

As before, under the conditions given, we can arrange that $\chi^{-1}(1+ZW) \neq 1$. Thus, we must have $\Phi(u^-) = 0$. This concludes Case C and the lemma.

Let us now summarize where we are at in Case 3 (i.e., order $(\chi_0) > 2$). The above lemma tells us $f_\alpha$ is not well-defined (cf. p.16) for all our double-coset representatives of $P\backslash G/B_X$, except possibly $\alpha = w_{00}, w_{10}, w_{11}$. We claim that $f_{w_{10}}$ is not well-defined--the proof is exactly the same as in Case 2. This leaves us with the task of showing that $f_\alpha$ is well-defined for $\alpha = w_{00}, w_{11}$. First we check that $f_0 = f_{w_{00}}$ is well-defined. Suppose $pb = p'b' \in PB_X$. We must show that $\delta^{1/2} \chi(p) = \delta^{1/2} \chi(p')$, or equivalently, $\delta^{1/2} \chi(p^{-1}p) = 1$. However, $p^{-1}p = b'b^{-1} \in P \cap B_X = M_X U_0$, so $\delta^{1/2} \chi(p^{-1}p) = 1$. Thus, $f_0$ is well-defined.

Finally, we check $f_1 = f_{w_{11}}$. Suppose $pw_{11}b = p'w_{11}b' \in Pw_{11}B_X$. We must show that $\delta^{1/2} \chi(p) = \delta^{1/2} \chi(p')$, or equivalently, $\delta^{1/2} \chi(p^{-1}p) = 1$. However,

$$
p^{-1}p = w_{11}b'b^{-1}w_{11} \\
\in P \cap (w_{11}B_X w_{11}) \\
= P \cap (U_0^{-1} M_X U_h) \\
= M_X U_h
$$

so $\delta^{1/2} \chi(p^{-1}p) = 1$. Thus, $f_1$ is well-defined. This concludes Case 3.

We summarize our results in the following theorem:

**Theorem 2.2.4** The following list gives a basis for $V^{B_X}$.

1. (order of $\chi_u$) = 1
A basis for $V^B$ consists of

$$f_{ij}(g) = \begin{cases} 
\delta^{1/2} \chi(p) & \text{if } g = pw_{ij}b \in Pw_{ij}B \\
0 & \text{if } g \notin Pw_{ij}B 
\end{cases}$$

(note that $B_x = B$), for $(i, j) = (0, 0), (1, 0), (1, 1)$. The pair $(1, 0)$ does not occur for $n = 1$.

2. (order of $\chi_u) = 2$

A basis for $V^{B_x}$ consists of

$$f_i(g) = \begin{cases} 
\delta^{1/2} \chi(p) & \text{if } g = pw_{ii}b \in Pw_{ii}B_x \\
0 & \text{if } g \notin Pw_{ii}B_x 
\end{cases}$$

for $i = 0, 1$.

3. (order of $\chi_u) > 2$

A basis for $V^{B_x}$ consists of

$$f_i(g) = \begin{cases} 
\delta^{1/2} \chi(p) & \text{if } g = pw_{ii}b \in Pw_{ii}B_x \\
0 & \text{if } g \notin Pw_{ii}B_x 
\end{cases}$$

for $i = 0, 1$.

**Remark 2.2.5** Cases 2 and 3 look the same for the particular parabolic we're looking at. However, that is not the case in general. If the Levi $M \cong GL_k \times Sp_{2(n-k)}$ and $\chi$ is a character, identified with a character of $M$ through the determinant, the following describes which $f_{\alpha}$'s are well-defined: (Cases 1, 2, 3 as above)

1. $\alpha = w_{ij}$ for all the $w_{ij}$

2. $\alpha = w_{ii} \quad i \in \{0, \ldots, k\}$

3. $\alpha = w_{00}, w_{kk}$

Thus, the distinction exists in general.

**Note 2.2.6** It will be important, for future use, to realize that in Cases 2 and 3, if $\Phi \in V^{B_x}$ and $x \notin (Pw_{00}B_x) \cup (Pw_{11}B_x)$, then $\Phi(x) = 0$. 
2.3. Irreducibility when \((\text{order } \chi_u) > 2\)

We now start on the task of analyzing reducibility. The problem splits into the same three cases that occurred in the preceding section, classified by (order of \(\chi_u\)). The purpose of the present section is to do the analysis for the case (order of \(\chi_u\)) > 2.

Let us start by describing our analysis for this case. From the first section of this chapter, we need only look at \((\pi, H(G//B_\chi), V^{B_\chi})\). By the preceding section \(\{f_0, f_1\}\) constitute a basis for \(V^{B_\chi}\). We let

\[
A_{w_{11}} = (|B_\chi w_{11} B_\chi|^{-1} \text{char}_{B_\chi w_{11} B_\chi} \\
A_m = (|B_\chi m B_\chi|^{-1} \text{char}_{B_\chi m B_\chi})
\]

with

\[
m = \begin{pmatrix}
m & 1 \\
\vdots & \ddots & \vdots \\
1 & \ddots & \ddots & 1 \\
m^{-1} & 1 & \ddots & \ddots
\end{pmatrix}
\]

and \(m \in \mathcal{O}^\chi\). We observe that \(A_{w_{11}}, A_m \in H(G//B_\chi)\) (in fact, they are in \(H(K//B_\chi)\)). We shall explicitly compute \(\pi(A_{w_{11}})\) and \(\pi(A_m)\) with respect to the basis \(\{f_0, f_1\}\). It will then be clear that \(\pi(A_{w_{11}})\) and \(\pi(A_m)\) admit no common proper invariant subspace. Therefore, the representation \((\pi, H(G//B_\chi), V^{B_\chi})\) is irreducible. This tells us that \((\pi, G, V)\) is irreducible.

**Lemma 2.3.1** With respect to the basis \(\{f_0, f_1\}\),

\[
\pi(A_m) = \begin{pmatrix}
\chi(m) & 0 \\
0 & \chi^{-1}(m)
\end{pmatrix}
\]

**Proof.** Let \(w\) be \(w_{00}\) or \(w_{11}\) (by Note 2.2.6, these are the only double-coset representatives of \(P\backslash G/B_\chi\) on which \(\Phi \in V^{B_\chi}\) can be nonzero). Then,
\[ \pi(A_m)\Phi(w) = \int_{B_x} \Phi(wbn) dB \]
\[ = \int_{B_x} \Phi(wm) dB \quad (\text{since } m \text{ normalizes } U^-_h, M_x, \text{ and } U_0). \]
\[ = \Phi(wm) \]

Thus, \( \pi(A_m)\Phi(w_{00}) = \Phi(w_{00}m) = \chi(m)\Phi(w_{00}) \) and \( \pi(A_m)\Phi(w_{11}) = \Phi(w_{11}m) = \Phi(m^{-1}w_{11}) = \chi^{-1}(m)\Phi(w_{11}). \) This tells us that

\[ \pi(A_m) = \begin{pmatrix} \chi(m) & 0 \\ 0 & \chi^{-1}(m) \end{pmatrix}. \]

**Lemma 2.3.2** With respect to the basis \( \{f_0, f_1\}, \)

\[ \pi(A_{w_{11}}) = \begin{pmatrix} 0 & 1 \\ \chi(-1)q^{-2(n+1)h} & 0 \end{pmatrix}. \]

**Proof.** Again, let \( w \) be \( w_{00} \) or \( w_{11} \). Then,

\[ \pi(A_{w_{11}})\Phi(w) = \int_{B_x} \Phi(wbw_{11}) dB \]
\[ = \int_{U_0} \int_{M_x} \int_{U^-_h} \Phi(wumu^{-1}w_{11}) du^- dm du \]
\[ = \int_{U_0} \Phi(wuw_{11}) du \]

since \( w_{11}^{-1}M_xw_{11} \subset B_x \) and \( w_{11}^{-1}U^-_hw_{11} \subset B_x. \)

1. \( w = w_{00} \)

\[ \pi(A_{w_{11}})\Phi(w_{00}) = \int_{U_0} \Phi(uw_{11}) du \]
\[ = \Phi(w_{11}). \]

2. \( w = w_{11} \)

\[ \pi(A_{w_{11}})\Phi(w_{11}) = \int_{U_0} \Phi(w_{11}uw_{11}) du \]
\[ = \chi(-1)\int_{U^-_0} \Phi(u^-) du^- .\]
We write \( u^- = \begin{pmatrix} 1 & \tau W \\ W & I \\ \tau Y & 1 - \tau W \\ Y & I \end{pmatrix} \) as before. Then,

\[
\pi(Aw_{11})\Phi(w_{11}) = \chi(-1)[ \int_{x \in \mathcal{P}} \int_{x \in \mathcal{O}^x} \Phi(u^-) du^- + \int_{x \in \mathcal{O}^x} \int_{x \in \mathcal{O}^x} \Phi(u^-) du^- ]
\]

We now observe that \( \int_{x \in \mathcal{P}} \int_{x \in \mathcal{O}^x} \Phi(u^-) du^- = q^{(-2n+1)h} \Phi(w_{00}) \). This follows from the fact (see the Appendix) that if all of the entries of \( W, Y \) are also in \( \mathcal{P} \), then \( u^- \in Pw_{00}B_x \) (this case contributes \( q^{(-2n+1)h} \Phi(w_{00}) \)). If not, then \( u^- \in [(Pw_{00}B) \cup (Pw_{10}B)](Pw_{00}B_x) \), so \( \Phi(u^-) = 0 \) by Note 2.2.6. It now remains to show that the second integral is zero.

To see that the second integral is zero, we factor \( u^- \) as

\[
u^- = \begin{pmatrix} \check{x}^{-1} & \check{x}^{-1}\tau Y & 1 & \check{x}^{-1}\tau W \\ & I - \check{x}^{-1}\tau Y & W & \check{x}^{-1}\tau W \\ & \check{x} & \check{x}^{-1}\tau Y & Y \\ & & & I + Y \check{x}^{-1}\tau W \end{pmatrix}
\]

Thus,

\[
\int_{x \in \mathcal{O}^x} \int_{x \in \mathcal{O}^x} \Phi(u^-) du^- = \int_{x \in \mathcal{O}^x} \chi(\check{x}^{-1}) \Phi(w_{11}) du^- = 0
\]

since \( \chi_u \) is nontrivial. This finishes the lemma.
It is clear that \( \pi(A_m) \) and \( \pi(A_{w_{11}}) \) admit no proper common invariant subspace. Thus \( (\pi, H(G/B_X), V^{B_X}) \) is irreducible, so \( (\pi, G, V) \) must be irreducible as well. We close this section with the following remark.

**Remark 2.3.3** In the general case, where \( P = MU \) with \( M \cong GL_k \times Sp_{2(n-k)} \), a basis for \( V^{B_X} \) when \( \text{order } \chi_u > 2 \) consists of \( \{f_\alpha\} \) for \( \alpha = w_{00}, w_{kk} \). With respect to this basis,

\[
\pi(A_m) = \begin{pmatrix} \chi(m) & 0 \\ 0 & \chi^{-1}(m) \end{pmatrix} \\
\pi(A_{w_{kk}}) = \begin{pmatrix} 0 & 1 \\ \chi(-1)^kq^{-kh(2n-\frac{3}{2}k+\frac{1}{2})} & 0 \end{pmatrix},
\]

with \( A_m = |B_xmB_X|^{-1} \text{char}_{B_xmB_X} \) (as before), \( A_{w_{kk}} = |B_xw_{kk}B_X|^{-1} \text{char}_{B_xw_{kk}B_X} \). The arguments are similar, except for the following complications. In order to compute \( \pi(A_{kk})\Phi(w_{kk}) \), write

\[
\pi(A_{kk})\Phi(w_{kk}) = \chi(-1)[\int_{\text{rank}_P \hat{X} = 0} \int \Phi(u^-)du^- + \int_{0<\text{rank}_P \hat{X} < k} \int \Phi(u^-)du^- + \int_{\text{rank}_P \hat{X} = k} \int \Phi(u^-)du^-],
\]

where \( \text{rank}_P \hat{X} \) is the rank of \( \hat{X} \) when reduced mod \( P \). The first integral gives the nonzero contribution, an in the case \( k = 1 \). The second integral is zero because \( 1 < \text{rank}_P \hat{X} < k \) implies that \( u^- \in Pw_{ij}B \) with \( i \neq 0, k \) in particular, \( u^- \notin Pw_{00}B_X \) or \( Pw_{kk}B_X \). Since \( \Phi \equiv 0 \) on these double-cosets, the integral is zero. The last integral is also zero, but here a lemma is needed:

**Lemma 2.3.4** Let \( S = GL_k(O) \cap \{\text{symmetric matrices}\} \). Suppose \( \rho \) is an irreducible representation of \( GL_k(O) \) such that \( \rho \not\cong \rho \circ \theta \), where \( \theta \) is the automorphism given by \( \theta(g) = T^Tg^{-1} \). Then,

\[
\int_S \pi(g)dg = 0
\]

We need to use this and some additional arguments in the general case as a substitute for \( \int_{\hat{X} \in O^X} \chi(\hat{X}^{-1})\Phi(w_{11}) = 0 \) in the case \( k = 1 \). Thus, we have irreducibility in general for \( \text{order } \chi_u > 2 \).
REMARK: We actually have something stronger than \( (\pi, H(G//B_\chi), V^{B_\chi}) \) irreducible, namely \( (\pi, H(K//B_\chi), V^{B_\chi}) \) is irreducible. This tells us something about finite groups.

We start by noting that \( K/K_h \cong G(\mathcal{O}/\mathcal{P}^h) \). We can interpret \( \chi \) as a character on \( M(\mathcal{O}/\mathcal{P}^h) \) (actually, \( \chi_u \) is all that will matter). If we look at \( \tau = \text{Ind}_{P(\mathcal{O}/\mathcal{P}^h)}^{G(\mathcal{O}/\mathcal{P}^h)} \chi \), we can ask about reducibility. This corresponds to analyzing reducibility for \( (\tau, H(G(\mathcal{O}/\mathcal{P}^h)//P(\mathcal{O}/\mathcal{P}^h)), X^{P(\mathcal{O}/\mathcal{P}^h)}) \), where \( X \) is the induced space. But, this is isomorphic to \( (\pi, H(K//B_\chi), V^{B_\chi}) \) above, hence irreducible.

### 2.4. Irreducibility conditions for (order \( \chi_u \)) = 2

In this section we shall analyze irreducibility for the case when (order \( \chi_u \)) = 2. The analysis is similar to the previous case, but more complicated (e.g., there are some reducibility points). The first step is to compute a pair of operators, \( \pi(A_\gamma), \pi(A_{\omega_{11}}) \) on \( V^{B_\chi} \), where

\[
A_{\omega_{11}} = (|B_\chi w_{11} B_\chi|)^{-1} \text{char}_{B_\chi w_{11} B_\chi}
\]

\[
A_\gamma = (|B_\chi \gamma B_\chi|)^{-1} \text{char}_{B_\chi \gamma B_\chi}
\]

with

\[
\gamma = \begin{pmatrix}
\varpi \\
\cdots \\
1 \\
\frac{\varpi^{-1}}{1} \\
\cdots \\
1
\end{pmatrix}
\]

We then see that these operators admit no common invariant subspace, except for \( s = 0, i\pi/\ln q \). This gives us irreducibility except at those points. At \( s = 0, i\pi/\ln q \), we can show reducibility by showing \( \dim \text{Hom}_G(\pi, \bar{\pi}) = 2 \). This requires some work, so we wait until section 2.7 to show the reducibility at these points. In this section, we just show irreducibility elsewhere.

We start by computing the operators.
Lemma 2.4.1

\[ \pi(A_\gamma) = \begin{pmatrix} q^{-s-n} & 0 \\ 0 & q^{-2n}q^{s+n} \end{pmatrix} \]

Proof. Let \( w \) denote \( w_{00} \) or \( w_{11} \). Then, for \( \Phi \in V^B_x \)

\[ \pi(A_\gamma)\Phi(w) = \int_{B_x} \Phi(wb\gamma)db \]

\[ = \int_{M_xU_0} \Phi(wum\gamma)dmdu \quad (\text{since } \gamma^{-1}U_1^{-}\gamma \subset U_1^{-}) \]

\[ = \int_{U_0} \Phi(wu\gamma)du \quad (\text{since } M_x \text{ commutes with } \gamma) \]

Case 1: \( w = w_{00} \)

\[ \pi(A_\gamma)\Phi(w_{00}) = \Phi(\gamma) = q^{-s-n}\Phi(w_{00}) \]

Case 2: \( w = w_{11} \)

For notational convenience, denote \( \pi(A_\gamma)\Phi(w_{11}) \) by \( S \).

\[ S = \int_{U_0} \Phi(w_{11}u\gamma)du \]

\[ = \int_{U_0} \Phi \begin{pmatrix} \begin{pmatrix} 1 & \omega^{-1}w_2 & \cdots & \omega^{-1}w_n \\ \omega^{-1}x_1 & 1 & \cdots & \omega^{-1}x_n \\ \vdots & \vdots & \ddots & \vdots \\ \omega^{-1}x_n & \omega^{-1}x_2 & \cdots & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} 1 & -\omega^{-1}w_2 & 1 & \cdots \\ -\omega^{-1}x_2 & 1 & \cdots & \omega^{-1}x_n \\ \vdots & \vdots & \ddots & \vdots \\ -\omega^{-1}x_n & -\omega^{-1}x_2 & \cdots & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} 1 \\ -\omega^{-1}w_2 \\ \vdots \\ -\omega^{-1}w_n \\ 1 \end{pmatrix} \]

(conjugating \( \gamma \) around \( u \))
$= q^{s+n} \int_{\mathcal{U}_0} \Phi \left( \begin{array}{c|c} \begin{array}{ccccc} 1 & \omega^{-1}w_2 & \ldots & \omega^{-1}w_n & \omega^{-2}x_1 & \omega^{-1}x_2 & \ldots & \omega^{-1}x_n \\ \omega^{-1}x_2 & 1 & \ldots & 1 & 1 & \omega^{-1}w_2 & \ldots & \omega^{-1}w_n \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \omega^{-1}x_n & \ldots & 1 & 1 & \omega^{-1}w_n & \ldots & \omega^{-1}w_n & 1 \end{array} \end{array} \right) du$

$= q^{s+n} \int_{\mathcal{U}_0} \Phi \left( \begin{array}{c|c} \begin{array}{ccccc} 1 & \omega^{-1}x_2 & \ldots & \omega^{-1}x_n & \omega^{-2}x_1 & \omega^{-1}w_2 & \ldots & \omega^{-1}w_n \\ \omega^{-1}x_2 & 1 & \ldots & 1 & \omega^{-1}w_2 & \ldots & \omega^{-1}w_n \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \omega^{-1}x_n & \ldots & 1 & 1 & \omega^{-1}w_n & \ldots & \omega^{-1}w_n & 1 \end{array} \end{array} \right) w_{11} du$

Now, write $M_\chi \cong M'_\chi \times M''_\chi$, where $M'_\chi = \ker \chi \subset F^\times$, $M''_\chi = SP_{2n-2}(\mathcal{O})$. If any of $x_2, \ldots, x_n \in \mathcal{O}^\times$, we can conjugate by a matrix in the Weyl group of $M''_0$ (noting that $M''_0$ commutes with $w_{11}$) to arrange that $x_n \in \mathcal{O}^\times$. This will permute the $w_i$'s (as above, if $m \in M'_0$ is the matrix, $\Phi(u^-w_{11}) = \Phi(mu^-w_{11}m^{-1}) = \Phi(mu^-m^{-1}w_{11})$, so we are just conjugating the unipotent matrix). Thus, we can split up the integral as

$S = q^{-n+1}q^{s+n} \int_{x_1, w_i \in \mathcal{O}} \int \Phi \left( \begin{array}{c|c} \begin{array}{ccccc} 1 & \omega^{-1}x_2 & \ldots & \omega^{-1}x_n & \omega^{-2}x_1 & \omega^{-1}w_2 & \ldots & \omega^{-1}w_n \\ \omega^{-1}x_2 & 1 & \ldots & 1 & \omega^{-1}w_2 & \ldots & \omega^{-1}w_n \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \omega^{-1}x_n & \ldots & 1 & 1 & \omega^{-1}w_n & \ldots & \omega^{-1}w_n & 1 \end{array} \end{array} \right) w_{11} d\!x \!d\!w$

$+ q^{s+n} \int_{R} \int \Phi \left( \begin{array}{c|c} \begin{array}{ccccc} 1 & \omega^{-1}x_2 & \ldots & \omega^{-1}x_n & \omega^{-2}x_1 & \omega^{-1}w_2 & \ldots & \omega^{-1}w_n \\ \omega^{-1}x_2 & 1 & \ldots & 1 & \omega^{-1}w_2 & \ldots & \omega^{-1}w_n \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \omega^{-1}x_n & \ldots & 1 & 1 & \omega^{-1}w_n & \ldots & \omega^{-1}w_n & 1 \end{array} \end{array} \right) w_{11} d\!x \!d\!w$

where the first integral simplifies because $x_2, \ldots, x_n \in \mathcal{P}$ and the second integral has domain $R' = \{(x_1, \ldots, x_n, w_2, \ldots, w_n) \mid \text{at least one of } x_2, \ldots, x_n \text{ is in } \mathcal{O}^\times\}$. We
can decompose $R'$ as $R' = R'_2 \cup \ldots \cup R'_n$ according to the minimum $j$ for which $x_j \in O^x$. Then, after a suitable conjugation from $M''_0$, each $R'_j$ will have $x_n \in O^x$, plus some conditions on the remaining $x$'s. Denote the resulting sets by $R_j$ and let $\int_{R_1} + \ldots + \int_{R_n}$ this gives the same result as integrating over $R'$, but it is convenient to have arranged $x_n \in O^x$. Note that $|R_2| + \ldots + |R_n| = \frac{1-q^{-n+1}}{1-q^{-1}}$.

As above, we can conjugate the unipotent matrix in the second integral by

$$\begin{pmatrix}
1 & 0 & \frac{-x^{-1}_1 x_2}{1} & \frac{-x^{-1}_1 x_{n-1}}{1} \\
1 & \ldots & \frac{-x^{-1}_n x_2}{1} & \frac{-x^{-1}_n x_{n-1}}{1} \\
& & 1 & \ldots \\
& & 1 & \frac{0 x^{-1}_n x_2 \ldots x^{-1}_n x_{n-1}}{1}
\end{pmatrix} \in M''_0$$

and split up the first integral according to whether any $w_i \in O^x$ or not (just as we split it up above according to whether any $x_2, \ldots, x_n \in O^x$). Using $\tilde{R}$ to denote the analogue of $R$ for the $w$'s (arranging for $w_2 \in O^x$), we get

$$S = q^{-n+2} q^{n+n} \int_{x \in O} \Phi \begin{pmatrix}
1 & \ldots & \frac{-x^{-2}_x}{1} & 1 \\
1 & \ldots & 1 & \frac{-w^{-1} x_2 \ldots x^{-1}_n x_{n-1}}{1} \\
& & 1 & \ldots \\
& & 1 & \frac{0 x^{-2}_x \ldots x^{-1}_n x_{n-1}}{1}
\end{pmatrix} w_{11} \, dx$$

$$+ q^{-n+1} q^{n+n} \int_{\tilde{R}} \Phi \begin{pmatrix}
1 & \ldots & \frac{-x^{-2}_x}{1} & 1 \\
1 & \ldots & 1 & \frac{-w^{-1} x_2 \ldots x^{-1}_n x_{n-1}}{1} \\
& & 1 & \ldots \\
& & 1 & \frac{0 x^{-2}_x \ldots x^{-1}_n x_{n-1}}{1}
\end{pmatrix} w_{11} \, dx \, dw$$
\[ +q^{s+n} \int \int_R \Phi \left( \begin{array}{cccc}
1, & \cdots & \cdots & 1 \\
\omega^{-1}z_n & & & \omega^{-1}w_n \\
\omega^{-2}z_1 & \omega^{-1}w_2 & & \omega^{-1}w_n \\
\omega^{-1}w_2 & & & \\
\omega^{-1}w_n & & & 
\end{array} \right) \\
w_{11} \right) \right) dx dw \]

Now, we conjugate by the following matrices in \( M''_0 \)

for second integral

\[
\left( \begin{array}{cccc}
1 & w_2^{-1}w_3 & \cdots & w_n^{-1}w_n \\
1 & \cdots & & \\
1 & & & \\
\frac{1}{w_2^{-1}w_3} & \frac{1}{w_2^{-1}w_n} & & \\
\frac{1}{w_2^{-1}w_3} & \frac{1}{w_2^{-1}w_n} & & \\
1 & & & 
\end{array} \right)
\]

for third integral

\[
\left( \begin{array}{cccc}
1 & & & \\
& & & \frac{1}{z_n^{-1}w_2} \cdots \frac{1}{z_n^{-1}w_n} \\
& & & \\
\frac{1}{z_n^{-1}w_2} & \cdots & & \\
\frac{1}{z_n^{-1}w_2} & \cdots & & \\
1 & & & 
\end{array} \right)
\]

We get

\[ S = c_1 \int_{x \in \mathcal{O}} \Phi \left( \begin{array}{cccc}
1, & \cdots & \cdots & 1 \\
\omega^{-2}x & & & \\
\omega^{-2}x & \omega^{-1}w_2 & & \\
\omega^{-1}w_2 & & & \\
\omega^{-1}w_n & & & 
\end{array} \right) w_{11} \right) dx \]

\[ +c_2 \int_{x \in \mathcal{O} \ w_2 \in \mathcal{O}^x} \Phi \left( \begin{array}{cccc}
1, & \cdots & \cdots & 1 \\
\omega^{-2}x & \omega^{-1}w_2 & & \\
\omega^{-1}w_2 & & & \\
\omega^{-1}w_n & & & 
\end{array} \right) w_{11} \right) dw dx \]
\[ +c_3 \int_{x \in \mathcal{O}} \int_{x_n \in \mathcal{O}^x} \Phi \left( \begin{array}{c|c} 1 & \cdots & 1 \\ \hline \omega^{-1} x_n & \cdots & 1 \\ \hline \omega^{-2} x & & 1 \end{array} \right) w_{11} \, dw \, dx \]

If we conjugate the second integral by
\[
\left( \begin{array}{ccc} 1 & 0 & 1 \\ \cdot & 1 & 0 \\ 1 & 1 & 0 \end{array} \right) \cdot (-1) \in M''_0,
\]

it becomes the same as the third. Thus,
\[
S = c_1 \int_{x \in \mathcal{O}} \Phi \left( \begin{array}{c|c} 1 & \cdots & 1 \\ \hline \omega^{-2} x & & 1 \end{array} \right) w_{11} \, dx
\]
\[
+(c_2 + c_3) \int_{x \in \mathcal{O}} \int_{w \in \mathcal{O}^x} \Phi \left( \begin{array}{c|c} 1 & \cdots & 1 \\ \hline \omega^{-1} w & \cdots & 1 \\ \hline \omega^{-2} x & & 1 \end{array} \right) w_{11} \, dx \, dw
\]

where \( c_1 = q^{-2n+2} q^{s+n} \), \( c_2 = q^{-n+1} \frac{q^{n+1} - q^{-1}}{1 - q^{-1}} q^{s+n} \), and \( c_3 = \frac{q^{-n+1}}{1 - q^{-1}} q^{s+n} \) (from \(|R|\), etc.).

We now break up each of the integrals into three integrals according to whether \( x \in \mathcal{P}^2 \), \( x \in \varpi \mathcal{O}^x \), or \( x \in \mathcal{O}^x \) – write \( x = \varpi \tilde{x} \) if \( x \in \varpi \mathcal{O}^x \), (so \( \tilde{x} \in \mathcal{O}^x \)). We then Iwahori factor the matrix. Note that if \( x \in \mathcal{P}^2 \), \( x \) may be removed since
$w_{11}$ will conjugate it into $U_0$. To save space, we let $\alpha = \varpi^{-1}wx^{-1}$, $\alpha' = \varpi wx^{-1}$, $\tilde{\alpha} = \varpi^{-1}w\tilde{x}^{-1}$, and $\tilde{\alpha}' = \varpi w^{-1}\tilde{x}^{-1}$ for the remainder of this computation. We get

\[ S = d_1 \Phi (w_{11}) \]

\[
\begin{array}{c}
+ d_2 \int_{\tilde{x} \in \mathcal{O}^x} \Phi \\
\begin{array}{ccc}
\varpi & 1 & \tilde{x}^{-1} \\
1 & \varpi^{-1} & 1 \\
\tilde{x} & 1 & 1 \\
\end{array} \\
\end{array} 
\begin{array}{c}
+ d_3 \int_{x \in \mathcal{O}^x} \Phi \\
\begin{array}{ccc}
\varpi^2 & 1 & x^{-1} \\
1 & \varpi^{-2} & 1 \\
x & 1 & 1 \\
\end{array} \\
\end{array} 
\begin{array}{c}
\begin{array}{c}
(w_{11}) \\
d\tilde{x} \\
\end{array} \\
\end{array}
\]

\[
\begin{array}{c}
+ d_4 \int_{w \in \mathcal{O}^x} \Phi \\
\begin{array}{ccc}
\varpi & w^{-1} & 1 \\
1 & \varpi^{-1} & 1 \\
w^{-1} & 1 & 1 \\
\end{array} \\
\end{array} 
\begin{array}{c}
\begin{array}{c}
(w_{11}) \\
dw \\
\end{array} \\
\end{array}
\]

\[
\begin{array}{c}
+ d_5 \int_{w, \tilde{x} \in \mathcal{O}^x} \Phi \\
\begin{array}{ccc}
\varpi & \tilde{x}^{-1} & 1 \\
\varpi^{-1}w^2 & \varpi^{-1} & 1 \\
\tilde{x} & 1 & 1 \\
\end{array} \\
\end{array} 
\begin{array}{c}
\begin{array}{c}
(w_{11}) \\
dwd\tilde{x} \\
\end{array} \\
\end{array}
\]

\[
\begin{array}{c}
+ d_6 \int_{w, x \in \mathcal{O}^x} \Phi \\
\begin{array}{ccc}
\varpi^2 & x^{-1} & \alpha' \\
1 & \varpi^{-2} & 1 \\
x & 1 & 1 \\
\end{array} \\
\end{array} 
\begin{array}{c}
\begin{array}{c}
(w_{11}) \\
dwdx \\
\end{array} \\
\end{array}
\]
\[ = d_1' \Phi(w_{11}) \]

\[ + d_2' \int_{\tilde{z} \in \mathcal{O}^+} \Phi \begin{pmatrix} \begin{array}{cc}
-\tilde{z}^{-1} & 1 \\
\varpi & -\tilde{z} \\
\end{array}
\end{pmatrix} d\tilde{z} \]

\[ + d_3' \int_{z \in \mathcal{O}^+} \Phi \begin{pmatrix} \begin{array}{cc}
-x^{-1} & 1 \\
\varpi^2 & -x \\
\end{array}
\end{pmatrix} dx \]

\[ + d_4' \int_{w \in \mathcal{O}^+} \Phi \begin{pmatrix} \begin{array}{cc}
-w^{-1} & 1 \\
\varpi & -w \\
\end{array}
\end{pmatrix} dw \]

\[ + d_5' \int_{w, \tilde{z} \in \mathcal{O}^+} \Phi \begin{pmatrix} \begin{array}{cc}
-w^{-1} & 1 \\
-w & \tilde{z} \\
\varpi & -\tilde{z} \\
\varpi w^{-1} \tilde{z}^{-1} & -w^{-1} \end{array}
\end{pmatrix} dwd\tilde{z} \]

\[ + d_6' \int_{w, \tilde{z} \in \mathcal{O}^+} \Phi \begin{pmatrix} \begin{array}{cc}
-w_{z} \tilde{z}^{-1} & 1 \\
\varpi^2 & -w_{z} \\
\end{array}
\end{pmatrix} dwdx \]
where we use $d'_1 = q^{-2n} q^{s+n}$, $d'_2 = q^{-2n+1} q^{s+n} q^{-s-n}$, $d'_3 = q^{-2n+2} q^{s+n} q^{-2s-2n}$, $d'_4 = \frac{1 - q^{-2n+2}}{1 - q^{-1}} q^{-2} q^{s+n} q^{-s-n}$, $d'_5 = \frac{1 - q^{-2n+2}}{1 - q^{-1}} q^{-1} q^{s+n} q^{-s-n}$, $d'_6 = \frac{1 - q^{-2n+2}}{1 - q^{-1}} q^{s+n} q^{-2s-2n}$.

We now claim that the second, third, and sixth terms above are all zero. This is because they all contain an expression like $\int_{x \in O^x} \chi^{-1}(x) dx$ when evaluated. For example, we write out the computation for the sixth term. Since

\[
\begin{pmatrix}
-x^{-1} & 1 \\
-w x^{-1} & -x^{-1} w^2 \\
-w^2 & -x & -w \\
\end{pmatrix}
= \begin{pmatrix}
-x^{-1} & 1 \\
-w x^{-1} & -x^{-1} w^2 \\
-w^2 & -x & -w \\
\end{pmatrix}
= \begin{pmatrix}
1 \\
-w x^{-1} & -x^{-1} w^2 \\
-w^2 x^{-1} & 1 & -w x^{-1} \\
\end{pmatrix}
\]

and the last matrix is in $B_x$

\[
\int \int_{w,x \in O^x} \Phi \begin{pmatrix}
-x^{-1} & 1 \\
-w x^{-1} & -x^{-1} w^2 \\
-w^2 & -x & -w \\
\end{pmatrix} dwdx = \int \int_{w,x \in O^x} \chi(-x^{-1}) \Phi(w_{00}) dwdx
\]

\[
= 0
\]

since $\chi$ is nontrivial on $O^x$.

We next claim that the fourth and fifth terms (from page 41) are also zero. This is because the matrices where $\Phi$ is being evaluated at all lie in $Bw_{10}B \subset Pw_{10}B_x$ (see Appendix), and $\Phi$ is zero in $Pw_{10}B_x$ (see Note 2.2.6). Thus, we have $\pi(A_\gamma) \Phi(w_{11}) = q^{-2n} q^{s+n} \Phi(w_{11})$. This finishes Case 2 and the lemma.

**Lemma 2.4.2** With respect to the basis $\{f_0, f_1\}$,

\[
\pi(A_{w_{11}}) = \begin{pmatrix}
0 & 1 \\
q^{-2n+1} \chi(-1) & 0 \\
\end{pmatrix}
\]
Proof. Let $w$ denote $w_{00}$ or $w_{11}$. Then, for $\Phi \in V^B$,

$$
\pi(A_{w_{11}})\Phi(w) = \int_{B_x} \Phi(wbw_{11})db \\
= \int \int \int_{U_0^x M_x U_{11}^-} \Phi(\mu_{w_{11}}^{-1} du - du dmu} \\
= \int \Phi(wuw_{11})du
$$

since $w_{11}^{-1}M_x U_{11}^- w_{11} \subset B_x$.

**Case 1:** $w = w_{00}$

$$
\pi(A_{w_{11}})\Phi(w_{00}) = \Phi(w_{11})
$$

**Case 2:** $w = w_{11}$

For notational convenience, let $R = \pi(A_{w_{11}})\Phi(w_{11})$.

$$
R = \int_{U_0} \Phi(w_{11}uw_{11})du \\
= \chi(-1) \int_{U_0} \Phi \left( \begin{array}{cccc}
1 & 1 & \cdots & 1 \\
-1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots \\
1 & \cdots & \cdots & 1
\end{array} \right) du
$$

since $w_{11}^{-1} = \left( \begin{array}{cc}
-1 & 1 \\
1 & -1 \\
\vdots & \vdots \\
1 & 1
\end{array} \right) w_{11}$. 


We can reduce this using the same reductions that we used in computing $\pi(A_x)\Phi(w_{11})$ in the last lemma. We get

$$\pi(A_{w_{11}})\Phi(w_{11}) = \chi(-1)q^{-2n+2} \int_{x \in \mathcal{O}} \Phi \left( \begin{array}{cc}
1 & \\
\cdots & 1 \\
x & \end{array} \right) dx$$

$$= \chi(-1)\frac{1-q^{-2n+2}}{1-q^{-1}} \int_{w \in \mathcal{O}^x} \int_{x \in \mathcal{O}} \Phi \left( \begin{array}{cc}
1 & \\
\cdots & 1 \\
x & \\
\end{array} \right) \left( \begin{array}{cc}
w & \\
\cdots & 1 \\
x & \\
1 & \end{array} \right) dxdw$$

We split up $\{x \in \mathcal{O}\}$ into $\{x \in \mathcal{P}\} \cup \{x \in \mathcal{O}^x\}$ and decompose according to $P_0 w B_x$ to get

$$R = e_1 \Phi(w_{00})$$

$$+ e_2 \int_{x \in \mathcal{O}^x} \Phi \left( \begin{array}{cc}
x^{-1} & 1 \\
1 & \cdots \\
1 & \end{array} \right) \left( \begin{array}{cc}
1 & \\
\cdots & 1 \\
x & \end{array} \right) w_{11} \left( \begin{array}{cc}
1 & \\
\cdots & 1 \\
1 & \end{array} \right) dx$$

$$+ e_3 \int_{w \in \mathcal{O}^x} \Phi \left( \begin{array}{cc}
w & \\
\cdots & 1 \\
w & \end{array} \right) dw$$
\[ + \epsilon_4 \int_{v \in \mathcal{O}_K} \int_{w \in \mathcal{O}_K} \Phi \left( \begin{array}{c|cc} \frac{x^{-1}}{1} & 1 & \frac{wz^{-1}}{1} \\ \hline 1 & w \\ \frac{z}{1} & 1 \end{array} \right) w_{11} \left( \begin{array}{c|cc} \frac{1}{z^{-1}} & \frac{z^{-1}}{1} & -\frac{wz^{-1}}{1} \\ \hline 1 & \frac{-wz^{-1}}{1} & \frac{w^2z^{-1}}{1} \end{array} \right) \right) dx dw \]

where \( \epsilon_1 = \chi(-1)q^{-2n+2}q^{-1} \), \( \epsilon_2 = \chi(-1)q^{-2n+2} \), \( \epsilon_3 = \chi(-1)\frac{1-q^{-2n-2}}{1-q^{-1}}q^{-1} \), \( \epsilon_4 = \chi(-1)\frac{1-q^{-2n-2}}{1-q^{-1}} \). Finally, we claim that the three remaining integrals are zero. The first and third contain \( \int_{x \in \mathcal{O}_K} \chi(x) = 0 \). The second integral is zero because \( \Phi \) is identically zero on the domain of integration (see Note 2.2.6).

**Proposition 2.4.3** For (order \( \chi_u \)) = 2, \( \pi \) will be irreducible unless \( s = 0, \ i\pi/\ln q \).

**Proof.** Clearly, \( \pi(A_{\gamma}) \) and \( \pi(A_{w_{11}}) \) admit no common invariant subspace unless \( \pi(A_{\gamma}) \) is scalar, which occurs when \( s = 0, \ i\pi/\ln q \). Thus, \( (\pi, H(G//B_\chi), V^{B_\chi}) \) is irreducible when \( s \neq 0, \ i\pi/\ln q \), so \( (\pi, G, V) \) will be irreducible there too.

### 2.5. Irreducibility conditions for (order \( \chi_u \)) = 1

This section brings us to the case where \( \chi \) is unramified, i.e., (order of \( \chi_u \)) = 1. The analysis has a general format similar to that of the last case, but there are differences. As usual, we analyze the Hecke algebra representation, \( (\pi, H(G//B_\chi), V^{B_\chi}) \) (here, \( B_\chi = B \)). We start by computing \( \pi(A_{\gamma}) \), where

\[ A_{\gamma} = (|B\gamma B|)^{-1} \text{char}_{B\gamma B}, \quad \gamma \text{ as before.} \]

Then, we give a lemma of [Gus] which tells us that if \( H(G//B) \cdot V^K = V^B \) and \( H(G//B) \cdot V^K = V^B \) (\( \hat{V} \) = space of contragredient), then \( \pi \) is irreducible. It turns out that except for \( s = 0, \ i\pi/\ln q, \pm n \) (or \( s = i\pi/\ln q, \pm 1 \) if \( n = 1 \)), we have \( V^B \) spanned by \( V^K \) under powers of \( \pi(A_{\gamma}) \). Thus, we have irreducibility at those points. It is easy to check that \( \pi \) is reducible for \( s = \pm n \). To show reducibility for \( \text{Res} = 0 \), we use the same kind of argument as for the (order \( \chi_u \))=2 case to show \( \text{Hom}(\pi, \bar{\pi}) \) has dimension 2, which we do in Section 2.6.
Recall that \( \dim V^B = 3 \) for \( n \geq 2 \) and \( \dim V^B = 2 \) for \( n = 1 \). We shall do the \( n \geq 2 \) case and just state the results for the \( n = 1 \) case. One can find the \( n = 1 \) case in [Cas2] or [Gus]. The \( n \geq 2 \) case is very easy to adapt to \( n = 1 \), anyway.

**Lemma 2.5.1** For \( n \geq 2 \), with respect to the basis \( \{f_{00}, f_{10}, f_{11}\} \),

\[
\pi(A_\gamma) = \begin{pmatrix}
q^{-s-n} & 0 & 0 \\
(1 - q^{-1})q^{-s-n} & q^{-1} & 0 \\
(1 - q^{-1})(q^{-s-n} + q^{-2n+1}) & q^{-1}(1 - q^{-2n+2}) & q^{-2n}s^{n+1}
\end{pmatrix}.
\]

For \( n = 1 \), with respect to the basis \( \{f_{00}, f_{11}\} \),

\[
\pi(A_\gamma) = \begin{pmatrix}
q^{-s-1} & 0 \\
(1 - q^{-1})(q^{-s-1} + q^{-1}) & q^{-2}s^{+1}
\end{pmatrix}.
\]

**Proof.** As indicated above, we shall just do the case \( n \geq 2 \). Let \( w \) denote \( w_{00}, w_{10}, w_{11} \). Then, for \( \Phi \in V^B \),

\[
\pi(A_\gamma)\Phi(w) = \int_B \Phi(wb\gamma)db \\
= \int_{U_0} \int_{M_0} \int_{U_1^-} \Phi(wmu\gamma)du - dmdu \\
= \int_{U_0} \Phi(wu\gamma)du
\]

since \( \gamma^{-1}u^{-1} \gamma \in U_1^- \) and \( m\gamma = \gamma m \).

**Case 1:** \( w = w_{00} \)

\[
\pi(A_\gamma)\Phi(w_{00}) = \Phi(\gamma) = q^{-s-n}\Phi(w_{00})
\]
Degenerate Principal Series

Case 2: $w = w_{10}$

For notational convenience, we set $T = \pi(A\gamma)\Phi(w_{10})$.

\[
T = \int_{U_0^\prime U_0^{''}} \Phi(w_{10}) \begin{pmatrix}
1 & 1 & \ldots & 1 \\
\vdots & x_1 & \ldots & x_n \\
1 & \ddots & \ddots & \ddots \\
\end{pmatrix} \begin{pmatrix}
1 & w_2 & \ldots & w_n \\
\vdots & \vdots & \ddots & \vdots \\
1 & -w_2 & \ddots & \ddots \\
\end{pmatrix} \begin{pmatrix}
1 \\
\vdots \\
1 \\
\end{pmatrix} \gamma \, du
\]

\[
= \int_{U_0^{''}} \Phi(w_{10}) \begin{pmatrix}
1 & w_2 & \ldots & w_n \\
\vdots & \ddots & \ddots & \ddots \\
1 & -w_2 & \ddots & \ddots \\
\end{pmatrix} \gamma \, du
\]

since $w_{10}U_0^\prime w_{10} \subset B$ (where $U_0^\prime$ and $U_0^{''}$ are as indicated). Since $w_{10}$ conjugates $w_3, \ldots, w_n$ into $B$ as well, this reduces to

\[
T = \int_{w \in \mathcal{O}} \Phi(w_{10}) \begin{pmatrix}
1 & w \\
\vdots & \vdots \\
1 & -w \\
\end{pmatrix} \gamma \, dw
\]

\[
= \int_{w \in \mathcal{O}} \Phi(w_{10} \gamma) \begin{pmatrix}
1 & \omega^{-1}w \\
\vdots & \vdots \\
1 & -\omega^{-1}w \\
\end{pmatrix} \, dw
\]
\[
= \int_{w \in \mathcal{O}} \Phi \begin{pmatrix}
1 & \omega^{-1} w & \ldots & 1 \\
1 & \omega^{-1} w & \ldots & 1 \\
\vdots & \ddots & \ddots & \vdots \\
1 & \omega^{-1} w & \ldots & 1
\end{pmatrix} w_{10} dw
\]

(since \(w_{10} \gamma w_{11} \mathcal{O}^{-1}\) has \(\omega\) in the second diagonal entry). Thus,

\[
T = \int_{w \in \mathcal{O}} \Phi \begin{pmatrix}
1 & \omega^{-1} w & \ldots & 1 \\
\omega^{-1} w & 1 & \ldots & 1 \\
\vdots & \ddots & \ddots & \vdots \\
\omega^{-1} w & \omega^{-1} w & \ldots & 1
\end{pmatrix} w_{10} dw
\]

\[
= \int_{w \in \mathcal{P}} \Phi \begin{pmatrix}
1 & \omega^{-1} w & \ldots & 1 \\
\omega^{-1} w & 1 & \ldots & 1 \\
\vdots & \ddots & \ddots & \vdots \\
\omega^{-1} w & \omega^{-1} w & \ldots & 1
\end{pmatrix} w_{10} dw
\]

\[
+ \int_{w \in \mathcal{O}^X} \Phi \begin{pmatrix}
1 & \omega^{-1} w & \ldots & 1 \\
\omega^{-1} w & 1 & \ldots & 1 \\
\vdots & \ddots & \ddots & \vdots \\
\omega^{-1} w & \omega^{-1} w & \ldots & 1
\end{pmatrix} w_{10} dw
\]
Since $w \in \mathcal{P}$ implies $\varpi^{-1} w \in \mathcal{O}$, so that conjugating the $w_{10}$ around the matrix in the $w \in \mathcal{P}$ integral gives a matrix in $\mathcal{U}_0$.

Finally, using an Iwasawa decomposition,

\[
T = q^{-1} \Phi(w_{10}) + \int_{w \in \mathcal{O} \times} \Phi \begin{pmatrix}
\begin{array}{c|c}
\varpi^{-1} w & 1 \\
\hline 
\varpi^{-1} w & 1 \\
\end{array}
\end{pmatrix} w_{10} dw
\]

\[
= q^{-1} \Phi(w_{10}) + q^{-s-n} \int_{w \in \mathcal{O} \times} \Phi \begin{pmatrix}
\begin{array}{c|c}
\varpi^{-1} w & 1 \\
\hline 
\varpi^{-1} w & 1 \\
\end{array}
\end{pmatrix} w_{10} dw
\]

\[
= q^{-1} \Phi(w_{10}) + (1 - q^{-1}) q^{-s-n} \Phi(w_{00})
\]

as the matrix inside always lies in $B$ (see appendix).

**Case 3: $w = w_{11}$**

For notational convenience, we set $S = \pi(\mathcal{A}_r) \Phi(w_{11})$. 
We can use exactly the same computations as were used in Lemma 2.4.1 to compute $\pi(A_\gamma) \Phi(w_{11})$ in the case (order of $\chi_u) = 2$ to get

$$S = q^{-2n} s^n \Phi(w_{11})$$

$$+ q^{-2n+1} q s^n q^{-n} \int_{\overline{z} \in \mathcal{O}^x} \Phi \begin{pmatrix} -\overline{z}^{-1} & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} d\overline{z}$$

$$+ q^{-2n+2} q s^n q^{-2s-2n} \int_{x \in \mathcal{O}^x} \Phi \begin{pmatrix} -x^{-1} & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} dx$$

$$+ \frac{1 - x^{-2n+2}}{1 - x^{-2n+2}} q^{-2} q s^n q^{-n} \int_{w \in \mathcal{O}^x} \Phi \begin{pmatrix} 0 & -w^{-1} & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} dw$$

$$+ \frac{1 - x^{-2n+2}}{1 - x^{-2n+2}} q^{-1} q s^n q^{-n} \int_{w, \overline{z} \in \mathcal{O}^x} \Phi \begin{pmatrix} 0 & -w^{-1} & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} dwd\overline{z}$$

$$+ \frac{1 - x^{-2n+2}}{1 - x^{-2n+2}} q s^n q^{-2s-2n} \int_{w, x \in \mathcal{O}^x} \Phi \begin{pmatrix} -x^{-1} & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} dw dx$$
From the appendix, we can read off which double-cosets $P_0w_{ij}B = Bw_{ij}B$ each of the matrices above lies in. This gives us

$$S = q^{-2n}q^{s+n}\Phi(w_{11})$$

$$+q^{-2n+1}q^{s+n}q^{-s-n}(1 - q^{-1})\Phi(w_{00})$$

$$+q^{-2n+2}q^{s+n}q^{-2s-2n}(1 - q^{-1})\Phi(w_{00})$$

$$+\frac{1-q^{-2n+2}}{1-q^{-1}}q^{-2}q^{s+n}q^{-s-n}(1 - q^{-1})\Phi(w_{10})$$

$$+\frac{1-q^{-2n+2}}{1-q^{-1}}q^{-1}q^{s+n}q^{-s-n}(1 - q^{-1})^2\Phi(w_{10})$$

$$+\frac{1-q^{-2n+2}}{1-q^{-1}}q^{s+n}q^{-2s-2n}(1 - q^{-1})^2\Phi(w_{00})$$

$$= q^{-2n}q^{s+n}\Phi(w_{11}) + q^{-1}(1 - q^{-2n+2})\Phi(w_{10}) + (1 - q^{-1})[q^{-2n+1} + q^{-s-n}]\Phi(w_{00}).$$

This finishes Case 3 and the lemma.

The next step is to show that $\pi$ is irreducible except possibly at $s = 0$, $i\pi/\ln q$, $\pm n$ for $n \geq 2$ (or $s = i\pi/\ln q$, $\pm 1$ for $n = 1$). The following lemma from Gustafson will be useful.

**Lemma 2.5.2** Suppose that $V^K$, $\hat{V}^K$ generate $V^B$, $\hat{V}^B$, respectively, under the actions of $H(G//B)$. Then $\pi$ (also $\hat{\pi}$) is irreducible.

**Proof.** Let $V_0$ be a proper invariant subspace of $V$. By Theorem 2.1.10, $V^B_0$ is a proper $H(G//B)$-invariant subspace of $V^B$. Since $V^K$ is one-dimensional, there are two possibilities: $V^K \subset V^B_0$ or $V^K \cap V^B_0 = 0$. In the first case, since $V^K$ generates $V^B$, we get $V^B_0 = V^B$. This is a contradiction. In the second case, any $l \in \hat{V}^K$ annihilates $V^B_0$. Since $\hat{V}^K$ generates $\hat{V}^B$, this tells us that $\hat{V}^B$ annihilates $V^B_0$. Thus, $V^B_0 = 0$, also a contradiction. Therefore, $\pi$ (and $\hat{\pi}$) must be irreducible.

We now state the irreducibility result as a proposition.

**Proposition 2.5.3** For $n \geq 2$, if $s \not\in \{0, i\pi/\ln q, \pm n\}$, then $\pi$ is irreducible. For $n = 1$, if $s \not\in \{i\pi/\ln q, \pm 1\}$, then $\pi$ is irreducible.

**Proof.** Let us begin with the $n \geq 2$ case. By the previous lemma, it suffices to check that $V^K$, $\hat{V}^K$ generate $V^B$, $\hat{V}^B$ under the action of $H(G//B)$.
We start by seeing when $v_0 = K$-fixed vector and $\pi(A_\gamma)$ are enough to generate $V^B$. With respect to our basis for $V^B$,

$$v_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Then,

$$v_1 = \pi(A_\gamma)v_0 - q^{-s-n}v_0 = (1 - q^{-s-n}) \begin{pmatrix} 0 \\ q^{-1} \\ q^{-1} + q^{-2n}q^{s+n} \end{pmatrix}.$$

Also,

$$v_2 = \pi(A_\gamma)v_1 - q^{-1}v_1 = q^{-2n}(1 - q^{-s-n})(q^{-2n}q^{2s+2n} - 1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Clearly, $\{v_0, v_1, v_2\}$ will span $V^B$ if $(1 - q^{-s-n}) \neq 0$ and $(q^{-2n}q^{2s+2n} - 1) \neq 0$, that is, for $s \neq n$, $0$, $i\pi/\ln q$.

Now, recall that if $\pi = \text{Ind}^G_{\tilde{P}}|\cdot|^s$, then $\tilde{\pi} = \text{Ind}^G_{\tilde{P}}|\cdot|^{-s}$ (in general, $\text{Ind}^G_{\tilde{P}}\tilde{\sigma} = \text{Ind}^G_{\tilde{P}}\sigma$). Thus, $\hat{V}_K$ will generate $\hat{V}^B$ for $s \neq -n, i\pi/\ln q$ ($i\pi/\ln q$ is identified with $-i\pi/\ln q$). Combining these, we see that $V^K, \hat{V}_K$ generate $V^B, \hat{V}^B$ as long as $s \notin \{0, i\pi/\ln q, \pm n\}$.

In the case $n = 1$, we have

$$v_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with respect to the usual basis. Then,

$$v_1 = \pi(A_\gamma)v_0 - q^{-s-1}v_0 = q^{-s-2}(q^{s+1} - 1)q^{(s+1)} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, $\{v_1, v_2\}$ span $V^B$ as long as $s \neq i\pi/\ln q, -1$. The condition $\hat{V}_K$ spans $\hat{V}^B$ gives $s \neq i\pi/\ln q, 1$. The proposition follows.
2.6. Reducibility conditions for (order $\chi_u = 1$

The purpose of this section is to prove the following proposition, which will give us reducibility for $s = 0$, $i\pi/\ln q$ (or $s = i\pi/\ln q$, if $n = 1$) in the case of $\chi$ unramified.

**Proposition 2.6.1** $\chi$ unramified. Then, for $n \geq 2$.

$$\dim \text{Hom}_G(\pi, \tilde{\pi}) = \begin{cases} 
2 & \text{if } s = 0, \ i\pi/\ln q \\
1 & \text{if not.} 
\end{cases}$$

If $n = 1$, the dimension will be 2 if $s = i\pi/\ln q$, and 1 otherwise.

*Proof.* First, we claim that if $s \notin \{0, \ i\pi/\ln q, \ \pm n\}$ for $n \geq 2$, or $s \notin \{i\pi/\ln q, \ \pm n\}$ for $n = 1$, then the dimension is one. In this case, $\pi$ (and $\tilde{\pi}$) is irreducible by Proposition 2.5.3. Now, $\tilde{\pi} = i_{GM} \chi^{-1}$. Let $\psi = r_{AM}(\chi \otimes tr)$ (which is just $\chi \otimes \nu^{-n+1} \otimes \ldots \otimes \nu^{-1}$). Then, $w_{11} \psi = r_{AM}(\chi^{-1} \otimes tr)$. Observe that $\pi$ is a subrepresentation of $i_{GA} \psi$ and $\tilde{\pi}$ is a subrepresentation of $i_{GA} w_{11} \psi$. Theorem 1.2.2 tells us that these have the same components. Each has exactly one component admitting a $K$-fixed vector. In one case this is $\pi$, in the other, $\tilde{\pi}$. Thus, $\pi$ and $\tilde{\pi}$ must be equivalent.

We next turn to the problem of showing that $\dim = 2$ at the appropriate places, and return to the case $s = \pm n$ afterwards. Briefly, we reduce the problem to a Jacquet module problem via Frobenius reciprocity, then to a corresponding Hecke algebra question. We will wish to know whether the abelian Hecke algebra $H(M//M_0)$ acts diagonally in a yet-to-be-specified two-dimensional subspace $Q$ of $(V_U)^{M_0}$. (It turns out to be the case that acting diagonally corresponds to having $\dim \text{Hom} = 2$). Essentially, we show that it is true for $s$ near $0$, $i\pi/\ln q$ (or $s$ near $i\pi/\ln q$, if $n = 1$) by projecting the operator $\pi(\mathcal{A}_s)$ down to $(V_U)^{M_0}$. Then, we shall show that we can analytically continue in $s$ to obtain the result for $s = 0, i\pi/\ln q$.

We begin with the following simple fact.

**Lemma 2.6.2** (cf. Gustafson) Suppose $(\rho, M, X)$ is an admissible representation of a reductive $p$-adic group $M$. If $\lambda \in \hat{Z}$, where $Z$ denotes the center of $M$, let

$$X_\lambda = \{ x \in X \mid \text{there is an } r \in N \text{ such that } [\pi(z) - \lambda(z)]^r x = 0 \ \forall z \in Z \}$$
Then, \( V = \bigoplus_{\lambda} V_{\lambda} \), a direct sum of \(M\)-invariant subspaces.

We now apply Frobenius reciprocity and the preceding lemma to get the following:

\[
\text{Hom}_G(\pi, \bar{\pi}) \cong \text{Hom}_M(\pi_U, \chi^{-1} \otimes tr_{2(n-1)}) \\
\cong \text{Hom}_M(\pi_U|_Q, \chi^{-1} \otimes tr_{2(n-1)})
\]

where \( Q = (V_U)_{\chi^{-1}} \) (recall that \( \bar{\pi} = \text{Ind}_F^G \chi^{-1} \)).

We now convert the problem into a Hecke algebra question. By Bernstein-Zelevinsky, Casselman, we know that \( V_U \) has composition factors

\[
\chi \otimes tr_{2(n-1)} \quad \chi^{-1} \otimes tr_{2(n-1)} \quad \nu^{-n+1} \otimes (\chi \otimes tr_{2(n-2)}) \\
w = w_{00} \quad w = w_{11} \quad w = w_{10}
\]

Note that the third of these does not occur for \( n = 1 \). When \( s \in \{0, i\pi/\ln q\} \), we have \( \chi = \chi^{-1} \), so the first two (but not the third) occur in \( \pi_U|_Q \). In particular, \( Q \) is two-dimensional and our question is whether \( Q \) is a direct sum of two copies of \( \chi^{-1} \otimes tr_{2(n-1)} \), or just a composition series. We observe that every subquotient of \( Q \) has an \( M_0 \)-fixed vector, so by Corollary 2.1.11 we get

\[
\text{Hom}_G(\pi, \bar{\pi}) \cong \text{Hom}_{H(M//M_0)}(\pi_U|_Q, \chi^{-1} \otimes tr_{2(n-1)})
\]

(note that \( Q^{M_0} = Q \)). Thus, our problem is to show that the action of \( \pi_U \) on \( Q \) diagonalizes.

We recall the following well-known result of Satake:

**Lemma 2.6.3** (cf. [Car] ) \( H(M//M_0) \) is abelian.

We shall also need the following lemma due to Casselman:

**Lemma 2.6.4** The Jacquet functor \( r_{MG} \) gives a vector space isomorphism between \( V^B \) and \((V_U)^{M_0}\).

**Proof.** By Proposition 2.1.1, \( r_{MG} : V^B \to (V_U)^{M_0} \) is a surjection. We know that \( V^B \) is three-dimensional (resp. two-dimensional, for \( n = 1 \)), so if we knew that \((V_U)^{M_0}\) were also three-dimensional (resp. two-dimensional), we would be done. However,
this is reasonably clear since \( \chi \otimes tr_{2(n-1)} \) and \( \nu^{-n+1} \otimes (\chi \propto \otimes tr_{2(n-2)}) \) (resp. the first two if \( n = 1 \)) each admit exactly one \( M_0 \)-fixed vector. Note \( \nu^{-n+1} \otimes (\chi \propto \otimes tr_{2(n-2)}) \) has exactly one since it results from parabolic induction of something 'unramified.'

We are going to define an operator on \((V_U)^{M_0}\) which behaves like \(\pi(A_\gamma)\).

We shall want to know the eigenvectors of \(\pi(A_\gamma)\), so we write them down now. For \( n \geq 2 \), they are

\[
v_1 = \begin{pmatrix} 1 \\ \frac{(1-q^{-1})q^{-s-n}}{q^{-s-n} - 1} \\ \frac{(1-q^{-1})q^{-s-n}}{q^{-s-n} - q^{-1}} \\ \frac{q^{-s-n} - q^{-2n+1}}{q^{-s-n} - q^{-1}} \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{q^{1-n}q^{-2n+2}}{q^{-1}q^{-2n+2}} \\ \frac{q^{-1}q^{-2n+2}}{q^{-1}q^{-2n+2}} \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

For \( n = 1 \), they are

\[
\begin{pmatrix} 1 \\ \frac{(1-q^{-1})}{1-q^{-s}} \\ \frac{q^{-s-1}+q^{-1}}{q^{-s-1}+q^{-1}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

We now define the aforementioned operator. We define it via the isomorphism \(V^B \cong (V_U)^{M_0}\), but it will turn out to be an operator from \(H(M/M_0)\). For expediency, let \( r \) denote the Jacquet functor \( r_{MG} : V^B \rightarrow (V_U)^{M_0} \). We define \( S \) on \((V_U)^{M_0}\) by

\[
S(r(\Phi)) = r(\pi(A_\gamma)\Phi) \quad \text{for} \ \Phi \in V^B.
\]

Then,

\[
S(r(\Phi)) = r(\pi(A_\gamma)\Phi) \\
= r\left( \int_{U_0 M_0^{-1}} \int_{U^{-1}} \pi(umu^{-1}\gamma)\Phi du^{-1} dm du \right) \\
= r\left( \int_{U_0 M_0^{-1}} \pi(um\gamma)\Phi dmu \right) \quad (\text{since } \gamma^{-1}U_1^{-1}\gamma \subset U_1^{-1}) \\
= \int_{M_0^{-1}} \pi_U(m\gamma)r(\Phi)dm \quad (\pi_U|_{U_0} \text{ is trivial}) \\
= \pi_U(A'_\gamma)r(\Phi)
\]

where \( A'_\gamma = \text{char}_{M_0\gamma M_0} \in H(M/M_0) \). Thus, with respect to the basis \( r(f_{00}), r(f_{10}), r(f_{11}) \) (or the counterpart for \( n = 1 \)), \( \pi_U(A'_\gamma) \) has the same matrix as \( \pi(A_\gamma) \). It must then have the same eigenvectors.

Next, let \( s_0 \) be \( 0 \) or \( i\pi/\ln q \), where we're trying to show \( \dim \text{Hom} = 2 \).

Let \( T \) be a suitably small neighborhood of \( s_0 \) not containing \( s_0 \). Then, for \( s \in T \),
the eigenvalues of \( \pi_U(A_\gamma') = \pi(A_\gamma) \) are distinct, and \( r(v_1), r(v_2), r(v_3) \) (resp. image of eigenvectors above) give a basis for \((V_U)^{M_0}\). Since \( H(M//M_0) \) is abelian, these will be eigenvectors for all of \( H(M//M_0) \), i.e., the action diagonalizes. A brief examination of the eigenvectors shows that they have “nice” limits as \( s \to s_0 \). We wish to use this to show that the action diagonalizes at \( s_0 \). This brings us to the following lemma (cf. [Gus]).

**Lemma 2.6.5** Let \( \{\tilde{f}_{00}, \tilde{f}_{10}, \tilde{f}_{11}\} \) correspond to \( \{f_{00}, f_{10}, f_{11}\} \) under the isomorphism \( V^B \cong (V_U)^{M_0} \). Let \( A \in H(M//M_0) \) and consider \( \pi_U(A) \) written with respect to the basis \( \{\tilde{f}_{00}, \tilde{f}_{10}, \tilde{f}_{11}\} \) of \((V_U)^{M_0}\). Then, the entries of \( \pi_U(A) \) are holomorphic as functions of \( s \).

**Proof of Lemma:** Before starting the proof proper, we remark that although the space \( V_U \) depends on \( s \), the identification of \((V_U)^{M_0}\) with \( \mathbb{C}^3 \) via the basis \( \{\tilde{f}_{00}, \tilde{f}_{10}, \tilde{f}_{11}\} \) does not.

We start by observing that it is sufficient to check the lemma when \( A \) is of the form \( \text{char}_{M_0 \alpha M_0}, \alpha \in M \). Fix such an \( A \).

First, we shall find a nice expression for a matrix coefficient of \( \pi_U(A) \), then show it is holomorphic in \( s \). Let \( \tilde{f} \in \{\tilde{f}_{00}, \tilde{f}_{10}, \tilde{f}_{11}\} \) and \( f \in \{f_{00}, f_{10}, f_{11}\} \) corresponding to \( \tilde{f} \). Choose \( m \) sufficiently large so that \( \alpha' = \gamma^m \alpha \) has the property that \( \alpha'^{-1}U^- \alpha' \subset K_1 \). Then

\[
(q^{-s-n})^m \pi_U(\text{char}_{M_0 \alpha M_0})\tilde{f} = \pi_U(\text{char}_{M_0 \gamma^m \alpha M_0})\tilde{f} \quad \text{(since } \gamma \text{ and } M_0 \text{ commute)}
= r_{MG}(\pi(\text{char}_{U_0 M_0 \alpha M_0}))\tilde{f}
= r_{MG}(\pi(\text{char}_{B_0 \alpha' B}))\tilde{f} \quad \text{(since } f \in V^B)\]

since the choice of \( m \) ensures \( P_0 \alpha' B = B \alpha' B \). Thus, the matrix for \( \pi_U(\text{char}_{M_0 \alpha M_0}) \) is just \((q^{s+n})^m \) times the matrix for \( \pi(\text{char}_{B_0 \alpha' B}) \) (when these matrices are written with respect to the bases above). Therefore, it suffices to check that the matrix coefficients of \( \pi(\text{char}_{B_0 \alpha' B}) \) (on \( V^B \)) are holomorphic in \( s \).

Let \( w \in \{w_{00}, w_{10}, w_{11}\} \). A matrix coefficient for \( \pi(\text{char}_{B_0 \alpha' B}) \) is

\[
\sigma(w) = \int_{B_0 \alpha' B} \pi(x)f(w)dx = c \cdot \int_B f(wb\alpha')db,
\]
where \( c = \text{vol}(B\alpha'B) \). We show that this is holomorphic as a function of \( s \). Suppose \( f = f_{ij} \). Then, we set

\[
S_i = \begin{cases} 
  b \in B & \text{whenever } wb\alpha' \in Pw_{ij}B \text{ and for } wb\alpha' = pw_{ij}b, \\
  \text{the } F^x \text{ entry of } m, \text{ where } p = mu \in MU, \\
  \text{lies in } \mathfrak{w}^i \mathcal{O}^x.
\end{cases}
\]

In particular, we will have \( f_{ij}(wb\alpha') = q^{(s+n)} \) for those \( b \in S_i \). Thus,

\[
\sigma(w) = \pi(\text{char}_{B\alpha'B})f_{ij}(w) = c \cdot \sum_i \text{vol } (S_i)q^{(s+n)}
\]

Once we have argued that this sum is finite, the lemma will follow. However, this is clear: \( p \in wB\alpha'Bw_{ij}^{-1} \), which is compact (so the number of \( l \)'s for which \( S_i \) is nonempty is finite and determined by \( \alpha', w, w_{ij} \)). Thus, the lemma holds.

We now finish the proposition. We observe that the eigenvectors (see p. 55) have nonzero limits as \( s \to 0 \), \( i\pi/\ln q \) (\( s \to i\pi/\ln q \) for \( n = 1 \)). Thus, the eigenvectors analytically continue to eigenvectors for \( H(M//M_0) \) at \( s = 0, i\pi/\ln q \) (\( s = i\pi/\ln q \) for \( n = 1 \)). In particular, the action of \( H(M//M_0) \) diagonalizes there. This finishes the \( \dim = 2 \) case. For \( s = \pm n \), \( Q \) is one-dimensional so the intertwining algebra can be (at most) one-dimensional, so will be one-dimensional. This finishes the proposition.

### 2.7. Reducibility conditions for (order \( \chi_u \) = 2)

In this section, we shall give the analogue in the case (order \( \chi_u \) = 2) to Proposition 2.6.1, and give the main theorem of this chapter, summarizing the reducibility criteria for the degenerate principal series we've been looking at.

**Proposition 2.7.1** (order \( \chi_u \) = 2). Then

\[
\dim \text{Hom}_G(\pi, \bar{\pi}) = \begin{cases} 
  2 & \text{if } s = 0, i\pi/\ln q, \\
  1 & \text{if not.}
\end{cases}
\]

**Proof.** We shall not include the proof because it is essentially the same as that of Proposition 2.6.1 (in fact, a bit easier). The only new fact needed is that \( H(M//M_x) \)
is abelian. If we write \( M_\chi = M'_\chi \times M''_0 \subset F^x \times M'' \), it follow from the fact that \( F^x \) commutes with everything and \( H(M''/M''_0) \) is abelian.

This brings us to the main theorem of this chapter.

**Theorem 2.7.2** The representation \( \pi = i_{GM} \chi (M \cong F^x \times Sp_{2(n-1)}(F)) \) is reducible if and only if

\[
\chi \in \{\nu^{-n}, 1, sgn, \nu^n\} \quad \text{for } n \geq 2
\]

or \( \chi \in \{\nu^{-1}, sgn, \nu\} \) for \( n = 1 \)

If \( \pi \) is reducible, it has two components.

**Proof.** By the results of Sections 2.3–2.5, \( \pi \) is irreducible for all other \( \chi \). It is reducible with two components for \( \chi = 1, sgn \) (or \( \chi = sgn \) if \( n = 1 \)) since \( \dim \text{Hom}_G(\pi, \tilde{\pi}) = 2 \) in this case, from Sections 2.6 and 2.7 (note that \( \pi = \tilde{\pi} \) for such \( \chi \)). \( \pi \) is reducible when \( \chi = \nu^{\pm n} \) since the trivial representation is a component. When \( n = 1 \), \( \nu^{\pm 1} \propto 1 \) has at most two components (since \( \dim V^B = 2 \)), so there will be exactly two. This leaves us with checking that \( \nu^{\pm n} \propto tr_{2(n-1)} \) has two components for \( n \geq 2 \).

There are at least two ways to check that \( \nu^{\pm n} \propto tr_{2(n-1)} \) has two components. One could apply Corollary 3.2.3 (the next chapter is independent of this one). Alternatively, one can compute \( \pi(\mathcal{A}_{w_{11}}) \) with respect to \( \{f_{00}, f_{10}, f_{11}\} \) to get

\[
\pi(\mathcal{A}_{w_{11}}) = \begin{pmatrix}
0 & 0 & 1 \\
0 & q^{-1} & 1 - q^{-1} \\
q^{-2n+1} & q^{-1} - q^{-2n+1} & 1 - q^{-1}
\end{pmatrix}.
\]

One can directly check that the only invariant subspace of \( \pi(\mathcal{A}_\gamma) \) and \( \pi(\mathcal{A}_{w_{11}}) \) for \( \chi = \nu^{-n} \) is \( \mathbb{C} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \) (the span of the \( K \)-fixed vector). Thus, there can be only two components in \( \nu^{-n} \propto tr_{2(n-1)} \). This finishes the theorem.
CHAPTER 3

IRREDUCIBILITY OF CERTAIN REPRESENTATIONS Á LA TADIČ

3.1. A general theorem on reducibility/irreducibility (in the regular case)

In the first section of this chapter we shall obtain some necessary and sufficient conditions for irreducibility of an induced representation. In the second section, we apply this to the degenerate principal series we are interested in.

For this section, we wish to work in a more general setting. To this end, let $G$ denote a split connected reductive $p$-adic group, $P = MU$ a parabolic subgroup of $G$, $\rho$ an irreducible admissible representation of $M$. Set $\pi = i_{GM}\rho$. We shall use a technique of Tadić (see [Tad2]) involving Jacquet modules as a basis for our theorem.

We introduce the following notation: let $s_1, \ldots, s_n$ be the simple reflections in $W$. Let $P_{min} = AU_{min}$ denote the minimal parabolic, and set $M_i = < A, s_i >$, which is the Levi factor of a larger parabolic subgroup $P_i$.

We now recall a theorem of Bernstein-Zelevinsky, Casselman. Let $M, N$ be Levi factors of standard parabolics for $G$. Set

$$W^{MN} = \{w \in W | w(P_0 \cap M) \subset P_0, \quad w^{-1}(P_0 \cap N) \subset P_0\}.$$ 

We remark that these correspond to the elements of shortest length in the double cosets $W_N \setminus W/W_M$.

**Theorem 3.1.1** Let $\rho$ be an admissible representation of $M$. Then, $r_{NG} \circ i_{GM}\rho$ has a composition series with factors

$$i_{NN'} \circ w \circ r_{M'M} \rho \quad w \in W^{MN},$$

where $M' = M \cap w^{-1}(N)$, $N' = w(M) \cap N$. 

59
For notational convenience, we shall denote by $BZ_N(\pi)$ the collection of representations $i_{NN} \circ w \circ \tau$ as $\tau$ runs over the components of $r_{M'M'}\rho$ and $w$ runs over $W^{MN}$.

(If all the $i_{NN} \circ w \circ \tau$ were irreducible, this would just be the components of the semisimplification of $\pi_{NG}$, $\pi_{NG(\pi)_{ss}}$).

We now proceed to set up the main theorem in this chapter. We shall require three things of $\rho$. First, we want $\rho$ irreducible. Second, we want $r_{AM}(\rho) \neq 0$. Finally, we require a regularity condition on $\rho$. Let $\psi$ be a character in $r_{AM(\rho)_{ss}}$. We require that $\psi$ be regular with respect to $W$ (not just $W_M$). Note that if this is true for one character in $r_{AM(\rho)_{ss}}$, it will be true for all of them.

We now associate a graph to $\pi$ as follows:

**vertices:** the vertices are the elements of $r_{AG}(\pi)_{ss}$

**edges:** two vertices $\psi_1, \psi_2$ are connected by an edge is there is some Levi $N$ and some $\tau \in r_{NG}(\pi)_{ss}$ such that the following hold:

1. $\tau$ is an irreducible representation of $N$.
2. $\psi_1, \psi_2 \in r_{AN}(\tau)_{ss}$.

This brings us to the main theorem.

**Theorem 3.1.2** Under the conditions above, the following are equivalent:

1. $\pi$ is irreducible.
2. the graph of $\pi$ is connected
3. $\tau \in BZ_{M_i}(\pi) \implies \tau$ is irreducible (for any $i$ and $\tau$).

**Proof.** We show $1 \implies 3 \implies 2 \implies 1$.

1 $\implies$ 3: Suppose not— that is, suppose $\pi$ is irreducible but there is a $\tau \in BZ_{M_i}(\pi)$ which is reducible. Let $\tau_1, \tau_2$ denote the components of $\tau$. Let $\psi_j = r_{AM_i}(\tau_j)$, $j = 1, 2$. Note that the Weyl group of $M_i$ has two elements, namely 1 and $s_i$. Thus, $\psi_2 = s_i \psi_1$. Set $\pi_j = i_{GM_i} \tau_j$. 
Degenerate Principal Series

To obtain a contradiction, we show two things. First, we show that if $\pi$ is irreducible, we must have $\pi \subset \pi_1$ or $\pi \subset \pi_2$ (as a subquotient). Then, we show that $r_{AG}(\pi)_{ss} \not\subset r_{AG}(\pi_1)_{ss}$ and $r_{AG}(\pi)_{ss} \not\subset r_{AG}(\pi_2)_{ss}$. As these two statements are incompatible, we must have $\pi$ reducible.

We start by showing that if $\pi$ is irreducible, it must be a subquotient of $\pi_1$ or $\pi_2$. First, we show that if $\psi \in r_{AG}(\pi)$ and $\pi$ is irreducible, then $\pi$ is a subquotient of $i_{GA}\psi$. By Frobenius reciprocity,

$$\text{Hom}_G(\pi, i_{GA}\phi) \cong \text{Hom}_M(\pi_A, \phi).$$

Take any $\phi$ so that this is nonzero (such a $\phi$ exists since $\pi_A$ is nonzero). Then, $\pi$ is a subquotient of $i_{GA}\phi$. Also, since $\psi \in r_{AG}(i_{GA}\phi)$, we have $\psi = w\phi$ for some $w \in W$. By Lemma 1.2.2, $i_{GA}\phi$ and $i_{GA}\psi$ have the same components. Therefore, $\pi$ must be a subquotient of $i_{GA}\psi$.

Now, either $\tau = i_{M,A}\psi_1$ or $\tau = i_{M,A}\psi_2$, so suppose $\tau = i_{M,A}\psi_1$. Thus, inducing in stages, we have $\pi \subset i_{GA}\psi_1 = i_{GM}\tau$. Since $\tau$ has $\tau_1$ and $\tau_2$ as components, the components of $i_{GM}\tau$ are just the components of $i_{GM}\tau_1$, $i_{GM}\tau_2$. Thus, $\pi \subset \pi_1$ or $\pi \subset \pi_2$, as claimed.

We now show that $r_{AG}(\pi)_{ss} \not\subset r_{AG}(\pi_1)_{ss}$ and $r_{AG}(\pi)_{ss} \not\subset r_{AG}(\pi_2)_{ss}$. First, we show that $r_{AG}(i_{GM}\tau)_{ss}$ is the disjoint union of $r_{AG}(\pi_1)_{ss}$ and $r_{AG}(\pi_2)_{ss}$. Since $\psi_2 = si_1\psi_1$, we have $r_{AG}(\pi_1)_{ss} = \{w\psi_1|w \in W^{M,A}\}$ and $r_{AG}(\pi_2)_{ss} = \{w\psi_2|w \in W^{M,A}s_i\}$. Since $W$ is the disjoint union of $W^{M,A}$ and $W^{M,A}s_i$, it follows from regularity that $r_{AG}(i_{GM}\tau)_{ss}$ is the disjoint union of $r_{AG}(\pi_1)_{ss}$ and $r_{AG}(\pi_2)_{ss}$. Now, observe that $\psi_1 \in r_{AG}(\pi_1)_{ss}$ but $\psi_1 \not\in r_{AG}(\pi_2)_{ss}$, and similarly for $\psi_2$. Since $\psi_1, \psi_2 \in r_{AG}(\pi)_{ss}$, we cannot have $r_{AG}(\pi)_{ss} \subset r_{AG}(\pi_1)_{ss}$ or $r_{AG}(\pi)_{ss} \subset r_{AG}(\pi_2)_{ss}$. This finishes $1 \Rightarrow 3$.

$3 \Rightarrow 2$: This is essentially just a Weyl group computation.

First, suppose $\psi \in r_{AG}(\pi)$, but $\psi$ is not in the Jacquet module for $r_{AM}(\rho)$. Then, $\psi = w\psi_0$ for some $\psi_0$ in $r_{AM}(\rho)$, $w \in W^{MA}$, and by our regularity hypothesis, these are uniquely determined. We shall connect $\psi$ by an edge to some $w'\psi_0 \in r_{AG}(\pi)$ with $l(w') = l(w) - 1$. Iterating this process will then tell us every $\psi$
is connected to some $\psi_0$ in $r_{AM}(\rho)$. Since $\rho$ is irreducible, all these $\psi_0$'s will be connected to each other ($\rho$ comes from $1 \in W^{MN}$).

We now determine $w'$ as described. Write $w = s_{i_1} \ldots s_{i_k}$ in terms of the simple roots. We claim that $w' = s_{i_2} \ldots s_{i_k}$ works. First, we must check that $w' \in W^{MA}$. This is equivalent to showing that if $w$ is the shortest element in $w \cdot W_{M}$, $w'$ is the shortest element in $w' \cdot W_{M}$. (Note that $w \cdot W_{M}$ and $w' \cdot W_{M}$ will be distinct since $w$ is of minimal length in $w \cdot W_{M}$ and $w'$ has $l(w') = l(w) - 1$.) This is easy: if $w'' \in w' \cdot W_{M}$ had $l(w'') < l(w')$, then $s_i w'' \in w \cdot W_{M}$ and $l(s_i w'') = l(w'') + 1 < l(w') + 1 = l(w)$, contradicting the fact that $w$ is the shortest element in that coset. Thus, $w' \in W^{MA}$.

Finally, we claim $\psi = w \psi_0$ shares an edge with $\psi' = w' \psi_0$. In particular, we claim $\tau = i_{M_i, A} \psi'$ is the connecting representation. Certainly, $\psi, \psi' \in r_{AM_i}(\tau)$. If we show $\tau \in BZ_{M_i}(\pi)$, $\tau$ will be irreducible by hypothesis (3), so they will be connected. Observe that $(w \cdot W_{M}) \cup (w' \cdot W_{M}) = W_{M_i} w' W_{M}$, so $w'$ is the shortest element in that double-coset, hence $w' \in W^{MM_i}$. We now check that $(w')^{-1} M_i w' \cap M = A$. If $(w')^{-1} M_i w' \subset M$, then $(w')^{-1} s_i w' \in M$, so $w \cdot M = w' \cdot M$. However, $w$ and $w'$ lie in distinct cosets, so we must have $(w')^{-1} M_i w' \cap M = A$. The B-Z composition factor arising from $w' \in W^{MM_i}$ is therefore

$$i_{M_i, A} \psi' = i_{M_i, A} \circ w' \circ \psi_0 \subset i_{M_i, A} \circ w' \circ r_{AM}(\rho),$$

as claimed. This finishes $3 \implies 2$.

$2 \implies 1$: Let $\pi_0$ be a subquotient of $\pi$. We shall show $\pi$ is irreducible by showing that $\pi$ and $\pi_0$ have the same semisimplified Jacquet modules, that is $r_{AG}(\pi)_{ss} = r_{AG}(\pi_0)_{ss}$. The key ingredient is the following

**Lemma 3.1.3** Suppose that $\psi_1, \psi_2 \in r_{AG}(\pi)$ satisfy the following conditions:

1. $\psi_1, \psi_2$ share an edge.

2. $\psi_1 \in r_{AG}(\pi_0)$.

Then, $\psi_2 \in r_{AG}(\pi_0)$.
Proof of Lemma. Suppose that $\tau$ is a component of $r_{NG}(\pi)$ which gives rise to the edge, so that $\tau$ is an irreducible representation of $N$ with $\psi_1, \psi_2 \in r_{AN}(\tau)$.

Now consider $r_{NG}(\pi_0)$. Since $\psi_1 \in r_{AG}(\pi_0) = r_{AN} \circ r_{NG}(\pi_0)$ and there is only one copy of $\psi_1$ in $r_{AG}(\pi)$, we must have $r_{NG}(\pi_0) \cap \tau \neq 0$ (as $\psi_1$ is in there). Since $\tau$ is irreducible, we must have $\tau \subset r_{AG}(\pi_0)$. Then, $\psi_2 \in r_{AN}(\tau) \subset r_{AG}(\pi_0)$, as needed. Since the graph of $\pi$ is connected, repeated application of the lemma shows $r_{AG}(\pi_0)_{ss} = r_{AG}(\pi)_{ss}$, finishing the theorem.

**Corollary 3.1.4** The number of components of $\pi$ is less than or equal to the number of components in graph of $\pi$.

### 3.2. Applications to degenerate principal series for $Sp_{2n}(F)$ (in the regular case)

In this section, we shall apply the theorem from the last section to the case of degenerate principal series for $G = Sp_{2n}(F)$, with $F$ $p$-adic of odd residual characteristic.

Let us start by recalling some notation, due to Tadić (see Chapter 1). This will expedite matters somewhat in this section, more so in the next chapter. Suppose $k_1 + \ldots + k_l = n$. Suppose that $\rho_i$ is an admissible representation of $GL_{k_i}(F)$ for $1 \leq i \leq l - 1$ and $\tau$ is an admissible representation of $Sp_{2k_l}(F)$. Let $P = MN$ be the standard parabolic subgroup of $G$ with

$$M \cong GL_{k_1}(F) \times \ldots \times GL_{k_{l-1}}(F) \times Sp_{2k_l}(F).$$

Then, we set

$$\rho_{k_1} \times \rho_{k_2} \times \ldots \times \rho_{k_{l-1}} \times \tau \cong i_G M(\rho_1 \otimes \rho_2 \otimes \ldots \otimes \rho_{l-1} \otimes \tau).$$

We also use the following notation:

$$[\chi]_k = \chi \circ \det \text{ on } GL_k(F)$$

$$\nu = | \cdot |$$

$$tr_k = \text{ trivial representation on } Sp_{2k}(F).$$
In this notation, we can write our degenerate principal series representation as \( \pi = [\chi]_k \otimes \tau_{n-k} \) (recall that we are most concerned with the case where \( P \) is a maximal (proper) parabolic).

We start by checking when \( r_{AM}([\chi]_k \otimes \tau_{n-k}) \) is regular (where \( M \cong GL_k(F) \times SP_{2(n-k)}(F) \)).

**Lemma 3.2.1** Let

\[
S_1 = \{-n + \frac{k+1}{2}, -n + 1 + \frac{k+1}{2}, \ldots, n - \frac{k+1}{2}\}, \quad S_2 = \{ -\frac{k+1}{2}, -\frac{k+2}{2}, \ldots, -\frac{k-1}{2} \}
\]

(note that if \( n = k \), \( S_1 \subset S_2 \)). We claim that \( r_{AM}([\chi]_k \otimes \tau_{n-k}) \) is nonregular precisely when

\[
\chi \in \{ \nu^a | \alpha \in S_1 \} \cup \{ \nu^a \psi | \alpha \in S_2 \},
\]

where \( \psi^2 \equiv 1 \).

**Proof.** We start by claiming

\[
r_{AM}([\chi]_k \otimes \tau_{n-k}) = \nu^{-\frac{k+1}{2}} \chi \otimes \nu^{-\frac{k+3}{2}} \chi \otimes \ldots \otimes \nu^{\frac{k+1}{2}} \chi \nu^{-n+k} \otimes \ldots \otimes \nu^{-n+k+1} \otimes \ldots \otimes \nu^{-1}
\]

This follows from the fact that \( [\chi]_k \) is a subrepresentation of \( \nu^{-\frac{k+1}{2}} \chi \times \nu^{-\frac{k+3}{2}} \chi \times \ldots \times \nu^{\frac{k+1}{2}} \chi \), \( \tau_{n-k} \) is a subrepresentation of \( \nu^{-n+k} \times \ldots \times \nu^{-1} \propto 1 \) and Frobenius reciprocity.

In general, the character \( \chi_1 \otimes \chi_2 \otimes \ldots \otimes \chi_n \) will be nonregular for \( SP_{2n}(F) \) if either of the following conditions holds:

i) \( \chi_i = \chi_j \) for some \( i, j \) with \( i \neq j \)

or

ii) \( \chi_i = \chi_j^{-1} \) for some \( i, j \)

Now, the particular form of our character precludes certain possibilities. Our character will be nonregular if any of the following hold:
Degenerate Principal Series

1) $\nu^{\frac{-k+1}{2}+l} \chi = \nu^{-n+k+m}$ for some $l, m$ with $0 \leq l \leq k - 1$, $0 \leq m \leq n - k - 1$

2) $(\nu^{\frac{-k+1}{2}+l} \chi)^{-1} = \nu^{-n+k+m}$ for some $l, m$ with $0 \leq l \leq k - 1$, $0 \leq m \leq n - k - 1$

or 3) $\nu^{\frac{-k+1}{2}+l} \chi = (\nu^{\frac{-k+1}{2}+m} \chi)^{-1}$ for some $l, m$ with $0 \leq l, m \leq k - 1$.

Note that cases 1) and 2) do not occur when $n = k$. This is accounted for in the lemma by the fact that if $n = k$, $S_1 \subset S_2$.

We now observe that case 1) contributes nonregularity at

$$\chi \in \left\{ \nu^{-n+\frac{k+1}{2}}, \nu^{-n+1+\frac{k+1}{2}}, \ldots, \nu^{\frac{k-1}{2}} \right\}$$

and case 2) contributes nonregularity at

$$\chi \in \left\{ \nu^{\frac{-k+1}{2}}, \nu^{\frac{-k+3}{2}+1}, \ldots, \nu^{n-\frac{k+1}{2}} \right\}$$

Together, they contribute $\chi \in \{ \nu^\alpha | \alpha \in S_1 \}$ (when $k = 1$, these both miss $\nu^0$, but it is in $S_2$ so the lemma still works out). Case 3) contributes $\chi \in \{ \nu^\alpha \psi | \alpha \in S_2 \}$.

This finishes the regularity lemma. We now give the lemma on reducibility of the B-Z composition factors of the Jacquet modules $r_{M,G}(\pi)$. This will tell us when condition 3 of the theorem is satisfied.

We pause to note that the representation $\chi_1 \times \chi_2$ is reducible if and only if $\chi_1 = \nu^{\pm 1} \chi_2$. Also, $\chi \propto 1$ is reducible if and only if $\chi = \nu^{\pm 1}$ or $\chi^2 = 1$ with $\chi \neq 1$.

**Lemma 3.2.2** All the B-Z composition factors of the Jacquet modules $r_{M,G}(\pi)$, $1 \leq i \leq n$, are irreducible (i.e., condition 3 in the theorem holds) except (possibly) when $\chi$ has one of the following forms:

$$\chi \in \left\{ \nu^{-n+\frac{k-1}{2}}, \nu^{-n+\frac{k-1}{2}+1}, \ldots, \nu^{n-\frac{k-1}{2}} \right\}$$

or $\chi^2 \in \{ \nu^{-k+1}, \nu^{-k+2}, \ldots, \nu^{k-1} \}$,

with the exception that when $k = 2$, $\chi^2 = 1$ does not satisfy condition 3. Note that the two sets above are not disjoint.
Proof. First, recall that $M_i$ takes the following forms:

\[ i < n: \quad M_i = \underbrace{F^x \times \ldots \times F^x}_{i-1} \times GL_2(F) \times \underbrace{F^x \times \ldots \times F^x}_{n-i-1} \]

\[ i = n: \quad M_n = F^x \times \ldots \times F^x \times Sp_2(F) \]

We remark that $Sp_2(F) = SL_2(F)$.

Now, the B-Z compositions factors of $r_{M_iG}(\pi) = r_{M_iG} \circ i_{GM}([\chi]_k \otimes tr_{n-k})$ are $i_{M_i\bar{M}_i} \circ w \circ r_{\bar{M}_iM}([\chi]_k \otimes tr_{n-k})$. This will take one of the following forms:

a) $i_{M_iA} \circ w \circ r_{AM}([\chi]_k \otimes tr_{n-k})$

or b) $w \circ r_{w^{-1}(M_i)M}([\chi]_k \otimes tr_{n-k})$,

depending on whether $\bar{M}_i = M_i$ or $A$. For the moment, let us write $\chi_1 \otimes \ldots \otimes \chi_n$ for $r_{AM}([\chi]_k \otimes tr_{n-k})$ and set $\chi_{w(1)} \otimes \ldots \otimes \chi_{w(n)} = w \cdot (\chi_1 \otimes \ldots \otimes \chi_n)$ for $w \in W$ (note that $w(i) > n$ if the character has been inverted). Then, we claim that the B-Z composition factors take one of the following more explicit forms of a), b) from above:

1. $i < n$
   a) $\chi_{w(1)} \otimes \ldots \otimes \chi_{w(i-1)} \otimes (\chi_{w(i)} \times \chi_{w(i+1)}) \otimes \chi_{w(i+2)} \otimes \ldots \otimes \chi_{w(n)}$

   b) $\chi_{w(1)} \otimes \ldots \otimes \chi_{w(i-1)} \otimes ([\chi']_2) \otimes \chi_{w(i+2)} \otimes \ldots \otimes \chi_{w(n)}$

   for some suitable $\chi' \in F^x$

2. $i=n$
   a) $\chi_{w(1)} \otimes \ldots \otimes \chi_{w(n-1)} \otimes (\chi_{w(n)} \propto 1)$

   b) $\chi_{w(1)} \otimes \ldots \otimes \chi_{w(n-1)} \otimes (tr_1)$. 
Degenerate Principal Series

For 1a), 2a), this is immediate. For 1b), 2b), it is clear since \( r_{M,M}([\chi]_k \otimes tr_{n-k}) \) must be one-dimensional.

Next, we claim that all the B-Z composition factors of the Jacquet modules \( r_{M,G}(\pi) \) are irreducible if the following representations are irreducible:

\[
\begin{align*}
\text{i) } & \quad \left( \nu^{-\frac{k+1}{2}+l} \chi \right) \propto 1 \quad \text{for } 0 \leq l \leq k - 1 \\
\text{ii) } & \quad \left( \nu^{-\frac{k+1}{2}+l} \chi \right) \times \nu^{-m} \quad \text{for } 0 \leq l \leq k - 1, 1 \leq m \leq n - k \\
\text{iii) } & \quad \left( \nu^{-\frac{k+1}{2}+l} \chi \right) \times \nu^{m} \quad \text{for } 0 \leq l \leq k - 1, 1 \leq m \leq n - k \\
\text{iv) } & \quad \left( \nu^{-\frac{k+1}{2}+l} \chi \right) \times \left( \nu^{-\frac{k+1}{2}+m} \chi \right)^{-1} \quad \text{for } 0 \leq l, m \leq k - 1, l \neq m.
\end{align*}
\]

(Note that ii) and iii) do not occur if \( k = n \). In particular, we claim that condition i) is sufficient to ensure that a representation of the type in 2a) will be irreducible, while ii)-iv) guarantee that a representation of the type in 1a) will be irreducible (1b), 2b) automatically contain only irreducible representations). To see that this is true, we start by remarking that

\[
\chi \propto 1 \quad \text{irreducible} \iff \chi^{-1} \propto 1 \text{ irreducible}
\]

and \( \chi_1 \times \chi_2 \text{ irreducible} \iff \chi_2 \times \chi_1 \text{ irreducible} \]

\[
\iff \chi_2^{-1} \times \chi_1^{-1} \text{ irreducible}
\]

\[
\iff \chi_1^{-1} \times \chi_2^{-1} \text{ irreducible}.
\]

With this, we see that conditions i)-iv) are enough to get irreducibility of the representations from 1a) or 2a) unless the induced part of the representation is either of the form \( \nu^{-m} \propto 1 \) for \( 1 \leq m \leq n - k \) or \( \left( \nu^{-\frac{k+1}{2}+l} \chi \right) \times \left( \nu^{-\frac{k+1}{2}+m} \chi \right) \) for \( l \leq l < m \leq k - 1 \) (or something with corresponding irreducibility from above).

The first of these will be irreducible for \( m \neq 1 \), the second for \( m \neq l + 1 \). We now claim that those representations which are automatically reducible, namely, \( \nu^{-1} \propto 1 \), and \( \left( \nu^{-\frac{k+1}{2}+l} \chi \right) \times \left( \nu^{-\frac{k+1}{2}+l+1} \chi \right) \), do not occur as the induced part of one of the B-Z composition factors. Once we have checked this, it will follow that if
the representations in i)-iv) above are all irreducible, then all the B-Z composition factors of the $r_{M_G}(\pi)$'s were irreducible, as claimed.

Next, we argue that a representation of the form $\nu^{-1} \otimes 1$ or $(\nu^{\frac{k+1}{2}+i} \chi \times \nu^{\frac{k+1}{2}+i} \chi) \otimes \ldots \otimes \chi_n$ does not occur generically as the induced part of a B-Z composition factor, that is, these can only arise in the nonregular case where they can arise as the representations in i)-iv). Suppose not—e.g., suppose that $\chi_1 \otimes \ldots \otimes (\nu^{\frac{k+1}{2}} \chi \times \nu^{\frac{k+1}{2}+1} \chi) \otimes \ldots \otimes \chi_n$ were a B-Z composition factor of $r_{M_G}(\pi)$ (leaving the $\chi_i$'s unspecified). Then, since $r_{AG} = r_{AM, i} \circ r_{M_G}$, we must have $\chi_1 \otimes \ldots \otimes (\nu^{\frac{k+1}{2}} \chi \times \nu^{\frac{k+1}{2}+1} \chi) \otimes \ldots \otimes \chi_n$ and $\chi_1 \otimes \ldots \otimes (\nu^{\frac{k+1}{2}} \chi \otimes \nu^{\frac{k+1}{2}+1} \chi) \otimes \ldots \otimes \chi_n$ in $r_{AG}(\pi)$. Suppose that $\chi_1 \otimes \ldots \otimes (\nu^{\frac{k+1}{2}} \chi \otimes \nu^{\frac{k+1}{2}+1} \chi) \otimes \ldots \otimes \chi_n$ arises as $w \circ r_{AM}(\chi_k \otimes \tau_{n-k})$, $w \in W^{MA}$. Then, $\chi_k \otimes \ldots \otimes (\nu^{\frac{k+1}{2}} \chi \otimes \nu^{\frac{k+1}{2}+1} \chi) \otimes \ldots \otimes \chi_n$ must arise as $w s_1 \circ r_{AM}(\chi_k \otimes \tau_{n-k})$ (in the regular case). However, we claim that this cannot happen because $w s_1 \not\in W^{MA}$.

In particular, $w s_1$ lies in the same $W/W_M$ coset as $w$, but only one element from each such coset is in $W^{MA}$. Thus we have a contradiction, as needed. We note that although this was a specific example that we argued, the reasoning clearly generalizes to cover our claim.

Thus, we now know that all of the B-Z composition factors of the $r_{M_G}(\pi)$ will be irreducible if the representations in i)-iv) are all irreducible. It is a straightforward matter to check when this happens. We check iv) as an example—the others are similar. Now, $(\nu^{\frac{k+1}{2}+1} \chi \times \nu^{\frac{k+1}{2}+m} \chi)^{-1}$ is reducible if $\nu^{\frac{k+1}{2}+1} \chi = \nu^{\pm 1} \nu^{\frac{k+1}{2}+m} \chi^{-1}$, or $\chi^2 = \nu^{k-1+m-l+1}$. This corresponds to

$$\chi^2 \in \{\nu^{-k+1}, \nu^{-k}, \ldots, \nu^{k-1}\} \quad \text{if } k \geq 2$$

$$\chi^2 \in \{\nu^{-1}, \nu\} \quad \text{if } k = 2$$

(This case does not occur if $k = 1$). If we do similar computations for i)-iii), they combine to give the lemma.

The "possibly" may be removed from the statement of the lemma. If $\chi$ is one of those characters, then there is a reducible B-Z composition factor for some $r_{M_G}(\pi)$. However, we do not need this for the following corollary.
Corollary 3.2.3 The only values of $\chi$ for which $\pi = [\chi]_k \propto tr_{n-k}$ have $r_{AM}([\chi]_k \otimes tr_{n-k})$ regular and $\pi$ reducible are

$$\chi = \nu^{-n+\frac{k-1}{2}} \text{ or } \nu^{n-\frac{k-1}{2}}$$

In either case, the trivial representation is a component of $\pi$.

Proof. Combine the two preceding lemmas to get the first claim. The second is straightforward.
CHAPTER 4

IRREDUCIBILITY CRITERIA FOR DEGENERATE PRINCIPAL SERIES IN $SP_4(F)$, $SP_6(F)$, Á LA TADIĆ

4.1. Extending definitions to the nonregular case

In this chapter, we analyze degenerate principal series in $Sp_4(F)$ and $Sp_6(F)$ using the methods of the last chapter. We start by modifying the definition of the graph associated to the degenerate principal series. We then look at $Sp_4(F)$. In this case, we already run across an irreducible representation with a disconnected graph, so Theorem 3.1.1 does not extend to the nonregular case. Although it does not occur in the examples we consider here, it is also the case that the converse fails; that one can construct a reducible representation with a connected graph, and we briefly describe such a representation. We close by analyzing degenerate principal series for $Sp_6(F)$ (induced from maximal parabolics).

We start by redefining the graph of $\pi$ in a fashion more suitable for the nonregular case. This definition will have the disadvantage of being useful only when the inducing representation has a one-dimensional Jacquet module (e.g., if the inducing representation is a character or Steinberg). If $\pi = i_{GM}\rho$ for such a $\rho$, we define the vertices and edges as follows:

**vertices:** the vertices are the elements of $W^{MA}$.

**edges:** two vertices $w_1, w_2$ share an edge if there is a standard Levi $N$ and a $w \in W^{MN}$ such that

1. $w_1, w_2 \in W_N w W_M$

2. $i_{NN'} \circ w \circ r_{M'M}\rho$ is irreducible

both hold.
4.2. Degenerate Principal Series in $Sp_4(F)$

In this section, we look at degenerate principal series for $Sp_4(F)$ using the techniques of the last chapter. We remark that the analysis of such representations has already been done using Hecke algebra methods, except for $[\chi] \propto 1$ with $\chi$ ramified (see chapter 2 of this thesis and [Gus]).

Let $L \cong F^x \times Sp_2(F)$ and $M \cong GL_2(F)$ be the Levi factors of the standard maximal parabolic subgroups of $G = Sp_4(F)$. Then, the subsets of $W$ which are used in Theorem 1.2.4 are

$$W = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, s_1s_2s_1s_2 = s_2s_1s_2s_1\}$$

$$W^{MA} = \{1, s_2, s_1s_2, s_2s_1s_2\}$$

$$W^{LA} = \{1, s_1, s_2s_1, s_1s_2s_1\}$$

Any of the other subsets of $W$ which arise may be obtained from these using the following two observations:

1. $W^{XY} = \{w^{-1} | w \in W^{YX}\}$
2. $W^{XY} = W^{XA} \cap W^{AY}$,

where $X, Y$ are standard Levis for $G$.

We can use these and Theorem 1.2.4 to construct the composition factors of the Jacquet modules for $Sp_4(F)$. The results are summarized in Tables 1 and 2.

Note that the tables are set up to respect taking Jacquet modules in stages. In particular, consider any entry in the right column. It lies in the Jacquet module of the first entry in the left column which is at the same height or higher (e.g., $\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi^{-1}$ is a B-Z composition factor of $r_{MG}([\chi] \propto 1)$ and $r_{AM}(\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi^{-1})$ contains $\nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi^{-1}$ and $\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}}\chi$). One can also read off which Weyl group element a given B-Z compositions factor is constructed from. For $r_{AG}$, the corresponding Weyl group elements is listed in parentheses. For $r_{MG}$ or $r_{LG}$, it
### Table 1. Jacquet modules for $\chi \propto \text{tr}_2$

<table>
<thead>
<tr>
<th>$r^{MG}$</th>
<th>$r^{AG}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi \propto \text{tr}_2$</td>
<td>$\chi \times \nu^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$\nu^{-1} \otimes \chi$ $(w = s_1)$</td>
</tr>
<tr>
<td>$\nu^{-1} \times \chi$</td>
<td>$\nu^{-1} \otimes \chi^{-1}$ $(w = s_2s_1)$</td>
</tr>
<tr>
<td></td>
<td>$\chi^{-1} \otimes \nu^{-1}$ $(w = s_1s_2s_1)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$r^{LG}$</th>
<th>$r^{AG}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi \propto \text{tr}_2$</td>
<td>$\chi \otimes \text{tr}_2$</td>
</tr>
<tr>
<td></td>
<td>$\nu^{-1} \otimes (\chi \propto 1)$</td>
</tr>
<tr>
<td></td>
<td>$\nu^{-1} \otimes \chi^{-1}$ $(w = s_2s_1)$</td>
</tr>
<tr>
<td>$\chi^{-1} \otimes \text{tr}_2$</td>
<td>$\chi^{-1} \otimes \nu^{-1}$ $(w = s_1s_2s_1)$</td>
</tr>
</tbody>
</table>

---

**Figure 1.** Graph of $\chi \propto \text{tr}_2$
Table 2. Jacquet modules for $[\chi]_2 \propto 1$

<table>
<thead>
<tr>
<th>$r_{MG}$</th>
<th>$r_{AG}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[\chi]_2 \propto 1$</td>
<td>$\nu^{-\frac{1}{2}} \chi \otimes \nu^\frac{1}{2} \chi$ ($w = 1$)</td>
</tr>
<tr>
<td>$\nu^{-\frac{1}{2}} \chi \times \nu^{-\frac{1}{2}} \chi^{-1}$</td>
<td>$\nu^{-\frac{1}{2}} \chi \otimes \nu^{-\frac{1}{2}} \chi^{-1}$ ($w = s_2$)</td>
</tr>
<tr>
<td>$[\chi^{-1}]_2$</td>
<td>$\nu^{-\frac{1}{2}} \chi^{-1} \otimes \nu^\frac{1}{2} \chi^{-1}$ ($w = s_2 s_1 s_2$)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$r_{LG}$</th>
<th>$r_{AG}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[\chi]_2 \propto 1$</td>
<td>$\nu^{-\frac{1}{2}} \chi \otimes (\nu^\frac{1}{2} \chi \propto 1)$</td>
</tr>
<tr>
<td>$\nu^{-\frac{1}{2}} \chi^{-1} \otimes (\nu^{-\frac{1}{2}} \chi \propto 1)$</td>
<td>$\nu^{-\frac{1}{2}} \chi \otimes \nu^\frac{1}{2} \chi$ ($w = 1$)</td>
</tr>
<tr>
<td>$\nu^{-\frac{1}{2}} \chi^{-1} \otimes \nu^{-\frac{1}{2}} \chi^{-1}$ ($w = s_2$)</td>
<td></td>
</tr>
<tr>
<td>$\nu^{-\frac{1}{2}} \chi^{-1} \otimes \nu^\frac{1}{2} \chi^{-1}$ ($w = s_2 s_1 s_2$)</td>
<td></td>
</tr>
</tbody>
</table>

Figure 2. Graph of $[\chi]_2 \propto 1$
will just be the shortest $w$ occurring upon Jacquetizing to $A$ (e.g., for $\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi^{-1}$, it will be the shortest of $\{s_2, s_1 s_2\}$, i.e., $\{s_2\}$).

We next construct the graphs for $\chi \propto tr_2$ and $[\chi]_2 \propto 1$ for the non-regular cases. Recall from section 3.2 that nonregularity for $\chi \propto tr_2$ occurs at $\chi = \nu^{-1}, 1, sgn, \nu$ and for $[\chi]_2 \propto tr_2$, it occurs at $\chi = \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}sgn, 1, sgn, \nu^{\frac{1}{2}}, \nu^{\frac{1}{2}}sgn$. The graphs are in Figures 1 and 2. They are the graphs for general $\chi$. The edges are "colored" according to whether they come from a $B-Z$ composition factor for $r_{LG}(\pi)$ or $r_{MG}(\pi)$. In parentheses are the values of $\chi$ for which that edge disappears, i.e., for which the corresponding $B-Z$ composition factor is reducible.

We start with the following observation. If $\chi \in \{\nu^{-1}, \nu\}$, then the graph of $\chi \propto tr_2$ is disconnected. However, the results of chapter 2 tell us that the representation is irreducible. Thus, it is not true in general that having an irreducible representation implies that the graph is connected.

At this point, we also mention the fact that one can construct a reducible representation with a connected graph. For example, suppose $\pi$ is obtained by inducing a unitary character from the minimal parabolic subgroup of $SO_n(F)$. Note that the Levi of a parabolic obtained by adding one simple reflection to the minimal parabolic subgroup is a product of $F^\times$'s and either $GL_2(F)$ or $SO_3(F)$. Since unitary induction in $GL_2(F)$ and $SO_3(F)$ is always irreducible, the graph of $\pi$ is connected. However, $\pi$ can be reducible (cf. [Keys] or [Win] for these claims).

The other nonregularity points for $\chi \propto tr_2$ are $\chi = 1, sgn$. These have disconnected graphs and are reducible. We can show reducibility using an argument like that from chapter 3. For $\chi = sgn$, the missing edge corresponds to the reducibility of $\nu^{-1} \otimes (sgn \propto 1)$. Let $\tau_1$ and $\tau_2$ be the components of $\nu^{-1} \otimes (sgn \propto 1)$. Then, $r_{AG}\tau_i = \nu^{-1} \otimes sgn$ (for both $i$). Set $\pi_i = i_{GL}\tau_i$. By Theorem 1.2.4, $r_{AG}(\pi_1)_{ss}$ consists of $\nu^{-1} \otimes sgn, sgn \otimes \nu^{-1}, sgn \otimes \nu, \nu \otimes sgn$. Now, if $sgn \propto tr_2$ were irreducible, it would have to be a subquotient of $\pi_1$ or $\pi_2$ since together they contain all the components of $\nu^{-1} \times sgn \propto 1$ (which has the same components as $sgn \times \nu^{-1} \propto 1 \supset sgn \propto tr_2$). However, $r_{AG}(sgn \propto tr_2)$ contains two copies of $\nu^{-1} \otimes sgn$, so this is not the case. Thus, $sgn \propto tr_2$ is irreducible. The argument for reducibility for $1 \propto tr_2$ may be done the same way-- the missing lines correspond to the reducibility of $1 \times \nu^{-1}$. Let
\( \tau_1, \tau_2 \) be the components and set \( \pi_i = i_{GM} \tau_i \). A comparison of \( r_{AG}(1 \times tr_2)_{ss} \) with \( r_{AG}(\pi_1)_{ss} \) and \( r_{AG}(\pi_2)_{ss} \) shows that \( 1 \times tr_2 \) cannot be a subquotient of either, so it must be reducible.

We now look at \( [\chi]_2 \propto 1 \). Since \( [\chi]_2 \propto 1 \) is irreducible iff \( [\chi^{-1}]_2 \propto 1 \) is irreducible (they are contragredients of each other), we shall consider \( \chi = 1, sgn, \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}} sgn \). We start by arguing that \( [\chi]_2 \propto 1 \) is reducible if \( \chi = \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}} sgn \). The argument is like that above for \( 1 \propto tr_2 \) and \( sgn \propto tr_2 \). For \( \chi = \nu^{-\frac{1}{2}} \), one of the missing lines corresponds to the reducibility of \( \nu^{-\frac{1}{2}} \otimes (\nu \propto 1) \). Let \( \tau_1, \tau_2 \) be the components of this and set \( \pi_i = i_{GL} \tau_i \). Comparing \( r_{AG}(\nu^{-\frac{1}{2}})_{2} \propto 1)_{ss} \) with \( r_{AG}(\pi_1)_{ss} \) and \( r_{AG}(\pi_2)_{ss} \) tells us \( \nu^{-\frac{1}{2}} \propto 1 \) is not a subquotient of \( \pi_1 \) or \( \pi_2 \), hence must be reducible. Similarly, for \( \chi = \nu^{-\frac{1}{2}} sgn \), let \( \tau_1, \tau_2 \) be the components of \( \nu^{-1} sgn \otimes (sgn \propto 1) \) (the reducibility of which corresponds to one of the missing lines) and argue the same way.

We now look at \( \chi = 1, sgn \), which we claim make \( [\chi]_2 \propto 1 \) irreducible. The easiest way to show this is essentially the irreducibility argument from chapter 3. Let \( \chi \) be \( 1, sgn \) and \( \pi = [\chi]_2 \propto 1 \). We note that \( r_{AG}(\pi)_{ss} \) consists of two copies each of \( \nu^{-\frac{1}{2}} \chi \otimes \nu^{-\frac{1}{2}} \chi \) and \( \nu^{-\frac{1}{2}} \chi \otimes \nu^{-\frac{1}{2}} \chi \). Let \( \pi_0 \) be a subquotient of \( \pi \) such that \( r_{AG}(\pi_0)_{ss} \) contains a copy of \( \nu^{-\frac{1}{2}} \chi \otimes \nu^{-\frac{1}{2}} \chi \). Then, \( r_{LG}(\pi_0)_{ss} \) must contain a copy of \( \nu^{-\frac{1}{2}} \chi \otimes (\nu^{-\frac{1}{2}} \chi \times 1) \) (since \( \nu^{-\frac{1}{2}} \chi \propto 1 \) is irreducible). Thus, \( \nu^{-\frac{1}{2}} \chi \otimes \nu^{-\frac{1}{2}} \chi \in r_{AL}(\nu^{-\frac{1}{2}} \chi \otimes (\nu^{-\frac{1}{2}} \chi \propto 1))_{ss} \subset r_{AL} \circ r_{LG}(\pi_0)_{ss} = r_{AG}(\pi_0)_{ss} \).

Next, if one copy of \( \nu^{-\frac{1}{2}} \chi \otimes \nu^{-\frac{1}{2}} \chi \) is in \( r_{AG}(\pi_0)_{ss} \), then \( r_{MG}(\pi_0)_{ss} \) must contain \( \nu^{-\frac{1}{2}} \chi \times \nu^{-\frac{1}{2}} \chi \) (since \( \nu^{-\frac{1}{2}} \chi \times \nu^{-\frac{1}{2}} \chi \) is irreducible). Thus, since \( r_{AM}(\nu^{-\frac{1}{2}} \chi \times \nu^{-\frac{1}{2}} \chi)_{ss} \subset r_{AG}(\pi_0)_{ss} \), we see that \( r_{AM}(\pi_0)_{ss} \) will contain both copies of \( \nu^{-\frac{1}{2}} \chi \otimes \nu^{-\frac{1}{2}} \chi \).

Finally, if \( r_{AG}(\pi_0)_{ss} \) contains both copies of \( \nu^{-\frac{1}{2}} \chi \otimes \nu^{-\frac{1}{2}} \chi \), then \( r_{LG}(\pi_0)_{ss} \) must contain both copies of \( \nu^{-\frac{1}{2}} \chi \otimes (\nu^{-\frac{1}{2}} \chi \propto 1) \). Thus, we conclude that \( r_{AG}(\pi_0)_{ss} \) contains both copies of \( \nu^{-\frac{1}{2}} \chi \otimes \nu^{-\frac{1}{2}} \chi \). Since we now have \( r_{AG}(\pi_0)_{ss} = r_{AG}(\pi)_{ss} \), \( \pi_0 \) cannot be a proper subquotient. This tells us \( \pi \) must be irreducible. We summarize this in the following theorem:

**Theorem 4.2.1**

1. \( \chi \propto tr_2 \) is reducible iff \( \chi \in \{\nu^{\pm 2}, 1, sgn\} \).

2. \( [\chi]_2 \propto 1 \) is reducible iff \( \chi \in \{\nu^{\pm \frac{1}{2}}, \nu^{\pm \frac{1}{2}}, \nu^{\pm \frac{1}{2}} sgn\} \).
Let $St_\chi$ be the Steinberg representation of $GL_2(F)$ with central character $\chi$ (so that $[\chi]_2$ and $St_\chi$ are the components of $\nu^{-\frac{1}{2}} \chi \times \nu^\frac{1}{2} \chi$). We can analyze $St_\chi \propto 1$ in exactly the same fashion as $[\chi]_2 \propto 1$. Since $r_{AM}(St_\chi) = \nu^\frac{1}{2} \chi \otimes \nu^{-\frac{1}{2}} \chi$, we get the graph in Figure 3.

The same arguments then tell us that $St_\chi \propto 1$ is reducible iff $\chi \in \{\nu^{\frac{3}{2}}, \nu^{\frac{1}{2}}, \nu^{\pm \frac{3}{2}} \text{sgn}\}$. A similar analysis for $\chi \propto St_2$ would be inconclusive at $\chi = \nu^{\pm 1}$ (where this approach breaks down for $\chi \propto tr_2$, which is irreducible but has disconnected graph).

### 4.3. Degenerate Principal Series in $Sp_6(F)$

We now look at $Sp_6(F)$. Let $L, M, N$ be the standard Levis with $L \cong F^* \times Sp_4(F)$, $M \cong GL_2(F) \times Sp_4(F)$, and $N \cong GL_3(F)$. The following subsets of $W$ are useful in computing Jacquet modules:

$$W^{LA} = \{1, s_1, s_2s_1, s_3s_2s_1, s_2s_3s_2s_1, s_1s_2s_3s_2s_1\}$$

$$W^{MA} = \{ s_1s_2s_3s_2, s_3s_2s_3s_2s_1s_3s_2, s_1s_2s_3s_2s_3s_2, s_2s_3s_2s_1s_3s_2, s_3s_1s_2s_3s_2, s_2s_3s_1s_2s_3s_2\}$$
\[ W^{NA} = \{1, s_3, s_2s_3, s_3s_2s_3, s_1s_2s_3, s_1s_3s_2s_3, s_2s_1s_3s_2s_3, s_3s_2s_1s_3s_2s_3 \}. \]

As in section 4.2, any other subsets of \( W \) which arise may be obtained from these using the following three observations:

1. \( W^{XY} = \{w^{-1} | w \in W^{YX} \} \)
2. \( W^{XY} = W^{XA} \cap W^{AY} \)
3. \( W^{(X \cap Y)A} = W^{XA} \cap W^{YA} \),

where \( X, Y \) are standard Levi's for \( G \).

Now, \( L, M, N \) are the Levi's for maximal parabolics. Let \( L_1 = M_1 \cong F^x \times F^x \times Sp_2(F) \), \( L_2 = N_1 \cong F^x \times GL_2(F) \), and \( M_2 = N_2 \cong GL_2(F) \times F^x \) be the Levi for the intermediate parabolics. Tables 3–11 give the B-Z composition factors for \( \chi \propto tr_4 \) (Tables 3, 4, 5), \([\chi]_2 \propto tr_2 \) (Tables 6, 7, 8), \([\chi]_3 \propto 1 \) (Tables 9, 10, 11).

We next construct some graphs for these (see Figures 4–19. Again, since chapter 3 covers the regular cases, we shall be concerned with the nonregular cases. Note that although the vertices are defined to be elements of \( W \), we label them by the corresponding action on a general character, which is equivalent and and a little more intuitive. By chapter 4, these are

\[
\begin{align*}
\chi \propto tr_4: & \quad \chi \in \{\nu^{\pm 2}, \nu^{\pm 1}, 1, \text{sgn} \} \\
[\chi]_2 \propto tr_2: & \quad \chi \in \{\nu^{\pm \frac{3}{2}}, \nu^{\pm \frac{1}{2}}, \nu^{\pm \frac{1}{2}} \text{sgn}, 1, \text{sgn} \} \\
[\chi]_3 \propto 1: & \quad \chi \in \{\nu^{\pm 1}, \nu^{\pm 1} \text{sgn}, \nu^{\pm \frac{3}{2}} \text{sgn}, \nu^{\pm \frac{1}{2}}, 1, \text{sgn} \}.
\end{align*}
\]

Since \( \pi \) is irreducible iff \( \tilde{\pi} \) is irreducible, it suffices to analyze those \( \chi \) for which \( \text{Res} \geq 0 \). These are in Figures 4–19. Note that to construct the graph, we will need to know when induced representations in \( GL_3(F) \) such as \([\chi_1]_2 \times \chi_2 \) are irreducible. This is in [Zel]– it is irreducible unless \( \chi_2 = \nu^{\pm \frac{3}{2}} \chi_1 \).

First, we look at the irreducible cases. All the representations with connected graphs are irreducible. This may be demonstrated using the same arguments as in the last section. We make the following observation: the connectedness of some
of these graphs (e.g., $[\nu^{-\frac{1}{2}}]_3 \propto 1$) require B-Z composition factors from the maximal parabolics, not just the parabolics obtained by adding one simple reflection to $P_{\text{min}}$. This contrasts with the regular case (see condition \# 3 of Theorem 3.1.1).

We now look at reducibility. The arguments will be like those in section 4.2. We shall not go through all the details, but merely indicate a reducible B-Z composition factor which yields suitable choices for $\tau_1, \tau_2$. Here is a list:

\[
\begin{align*}
1 \propto tr_4 & \quad \nu^{-2} \otimes (\nu^{-1} \times 1) & \text{in } r_{L_2G}(\pi) \\
sgn \propto tr_4 & \quad \nu^{-2} \otimes \nu^{-1} \otimes (sgn \propto 1) & \text{in } r_{L_1G}(\pi) \\
[\nu^{-\frac{1}{2}}]_2 \propto tr_2 & \quad \nu^{-1} \times [\nu^{\frac{1}{2}}]_2 & \text{in } r_{NG}(\pi) \\
[\nu^{-\frac{1}{2}} sgn]_2 \propto tr_2 & \quad \nu^{-1} sgn \otimes \nu^{-1} \otimes (sgn \propto 1) & \text{in } r_{L_1G}(\pi) \\
[\nu^{-\frac{3}{2}}]_2 \propto tr_2 & \quad \nu^{-1} \otimes ([\nu^{-\frac{3}{2}}]_2 \propto 1) & \text{in } r_{LG}(\pi) \\
[1]_3 \propto 1 & \quad [\nu^{-\frac{1}{2}}]_2 \otimes (\nu \propto 1) & \text{in } r_{MG}(\pi) \\
[sgn]_3 \propto 1 & \quad \nu^{-1} sgn \otimes \nu^{-1} sgn \otimes (sgn \propto 1) & \text{in } r_{L_2G}(\pi) \\
[\nu^{-1}]_3 \propto 1 & \quad 1 \otimes \nu^{-2} \otimes (\nu^{-1} \propto 1) & \text{in } r_{L_1G}(\pi) \\
[\nu^{-1} sgn]_3 \propto 1 & \quad \nu^{-2} sgn \otimes \nu^{-1} sgn \otimes (sgn \propto 1) & \text{in } r_{L_1G}(\pi)
\end{align*}
\]

We summarize the observations above and the results on the regular case from chapter 3 in the following theorem:

**Theorem 4.3.1**

1. $\chi \propto tr_4$ is reducible if and only if $\chi \in \{\nu^{\pm 3}, 1, sgn\}$.

2. $[\chi]_2 \propto tr_2$ is reducible if and only if $\chi \in \{\nu^{\pm \frac{3}{2}}, \nu^{\pm 2}, \nu^{\pm \frac{1}{2}}, \nu^{\pm \frac{1}{2}} sgn\}$.

3. $[\chi]_3 \propto 1$ is reducible if and only if $\chi \in \{\nu^{\pm 2}, \nu^{\pm 1}, \nu^{\pm 1} sgn, 1, sgn\}$.
Table 3. $F^x \times Sp_4(F)$ parabolic in $Sp_6(F)$, i.e., $\chi \propto tr_4$

<table>
<thead>
<tr>
<th>$J_L$</th>
<th>$J_{L_1}$</th>
<th>$J$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$\chi \otimes \nu^{-2} \otimes \nu^{-1}$</td>
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Table 4. \( F^* \times \text{Sp}_4(F) \) parabolic in \( \text{Sp}_6(F) \), i.e., \( \chi \propto tr_4 \)

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Table 5. $F^x \times Sp_4(F)$ parabolic in $Sp_6(F)$, i.e., $\chi \propto tr_4$

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Table 6. $GL_2 \times Sp_{2n}(F)$ parabolic in $Sp_6(F)$, i.e., $[\chi]_2 \propto tr_2$

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### Table 7. $GL_2 \times Sp_{2n}(F)$ parabolic in $Sp_6(F)$, i.e., $[\chi]_2 \propto tr_2$

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Table 8. $GL_2 \times Sp_{2n}(F)$ parabolic in $Sp_6(F)$, i.e., $[\chi]_2 \propto tr_2$

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Table 9. \( GL_3 \) parabolic in \( Sp_6(F) \), i.e., \([\chi]_3 \propto 1\)

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<tr>
<th>( J_L )</th>
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<th>( J )</th>
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<tbody>
<tr>
<td>( \nu^{-1} \chi \otimes ([\nu^{\frac{1}{2}} \chi]_2 \propto 1) )</td>
<td>( \nu^{-1} \chi \otimes \chi \otimes (\nu \chi \propto 1) )</td>
<td>( \nu^{-1} \chi \otimes \chi \otimes \nu \chi )</td>
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<td>( \nu^{-1} \chi \otimes \nu^{-1} \chi^{-1} \otimes (\nu \chi \propto 1) )</td>
<td>( \nu^{-1} \chi \otimes \nu^{-1} \chi^{-1} \otimes \nu^{-1} \chi )</td>
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<td>( \nu^{-1} \chi^{-1} \otimes \nu^{-1} \chi \otimes (\chi \propto 1) )</td>
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<td>( \nu^{-1} \chi^{-1} \otimes \chi^{-1} \otimes (\nu^{-1} \chi \propto 1) )</td>
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<td>( \nu^{-1} \chi \otimes \nu^{-1} \chi^{-1} \otimes \chi )</td>
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<tr>
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Table 10. $GL_3$ parabolic in $Sp_6(F)$, i.e., $[\chi]_3 \propto 1$

<table>
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<th>$J_M$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$[\nu^{-\frac{1}{2}}\chi]_2 \otimes (\nu \propto 1)$</td>
<td>$\nu^{-1}\chi \otimes \chi \otimes (\nu \propto 1)$</td>
<td>$\nu^{-1}\chi \otimes \chi \otimes \nu\chi$</td>
</tr>
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<td>$\nu^{-1}\chi \otimes \nu^{-1}\chi^{-1} \otimes \chi$</td>
</tr>
<tr>
<td>$\nu^{-1}\chi^{-1} \otimes \nu^{-1}\chi \otimes (\chi \propto 1)$</td>
<td>$\nu^{-1}\chi^{-1} \otimes \nu^{-1}\chi \otimes \chi^{-1}$</td>
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</tbody>
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Table 11. \( GL_3 \) parabolic in \( Sp_6(F) \), i.e., \([\chi]_3 \propto 1\)

\[
\begin{array}{ccc}
J_N & J_N & J \\
[\chi]_3 & \nu^{-1} \chi \otimes [\nu^{\frac{1}{2}} \chi]_2 & \nu^{-1} \chi \otimes \chi \otimes \nu \chi \\
[\nu^{-\frac{1}{2}} \chi]_2 \times \nu^{-1} \chi^{-1} & \nu^{-1} \chi \otimes (\chi \times \nu^{-1} \chi^{-1}) & \nu^{-1} \chi \otimes \chi \otimes \nu^{-1} \chi^{-1} \\
& \nu^{-1} \chi^{-1} \otimes [\nu^{-\frac{1}{2}} \chi]_2 & \nu^{-1} \chi^{-1} \otimes \nu^{-1} \chi \otimes \chi \\
\nu^{-1} \chi \times [\nu^{-\frac{1}{2}} \chi^{-1}]_2 & \nu^{-1} \chi^{-1} \otimes (\nu^{-1} \chi \times \chi^{-1}) & \nu^{-1} \chi^{-1} \otimes \nu^{-1} \chi \otimes \chi^{-1} \\
& \nu^{-1} \chi \otimes [\nu^{-\frac{1}{2}} \chi^{-1}]_2 & \nu^{-1} \chi \otimes \nu^{-1} \chi^{-1} \otimes \chi^{-1} \\
[\chi^{-1}]_3 & \nu^{-1} \chi^{-1} \otimes [\nu^{\frac{1}{2}} \chi^{-1}]_2 & \nu^{-1} \chi^{-1} \otimes \chi^{-1} \otimes \nu \chi^{-1} \\

\end{array}
\]
\[ \chi_1 \otimes \chi_2 \otimes \chi_3 \]

\[ \chi^{-1}_1 \otimes \chi_2 \otimes \chi_3 \]

\[ \chi_2 \otimes \chi^{-1}_1 \otimes \chi_3 \]

\[ \chi_2 \otimes \chi_3 \otimes \chi^{-1}_1 \]

\[ \chi_2 \otimes \chi_1 \otimes \chi_3 \]

\[ \chi_2 \otimes \chi_3 \otimes \chi_1 \]

\[ \cdots \cdots \ J_{L_2} = J_{M_1} \]

\[ \cdots \cdots \ J_L \]

\[ \cdots \cdots \ J_{L_1} = J_{N_1} \]

\[ \cdots \cdots \ J_M \]

\[ \cdots \cdots \ J_{M_2} = J_{N_2} \]

\[ \cdots \cdots \ J_N \]

Figure 4. Graph of \( \chi \propto tr_4, \chi \) generic
Figure 5. Graph of $\chi \propto tr_4, \chi = 1$
Figure 6. Graph of $\chi \propto tr_4, \chi = sgn$
Degenerate Principal Series

$\chi_1 \otimes \chi_2 \otimes \chi_3$

$\chi_1^{-1} \otimes \chi_2 \otimes \chi_3$

$\chi_2 \otimes \chi_1^{-1} \otimes \chi_3$

$\chi_2 \otimes \chi_3 \otimes \chi_1$

$\chi_2 \otimes \chi_3 \otimes \chi_1^{-1}$

$\cdots \cdots J_{L_1} = J_{M_1}$

$\cdots J_{L_2} = J_{N_1}$

$\cdots \cdots J_{M_2} = J_{N_2}$

$\cdots \cdots J_L$

$\cdots \cdots J_M$

$\cdots \cdots J_N$

Figure 7. Graph of $\chi \propto tr_4$, $\chi = \nu^{-1}$
Figure 8. Graph of $\chi \propto tr_4, \chi = \nu^{-2}$
Figure 9. Graph of $[\chi]_2 \circ tr_2$, $\chi$ generic (also $\chi = 1, sgn$)
Figure 10. Graph of \([\chi]_2 \propto tr_2, \chi = \nu^{-\frac{1}{2}}\)
Degenerate Principal Series

\( X_1 \otimes X_2 \otimes X_3 \)

\( \chi_2^{-1} \otimes \chi_1^{-1} \otimes X_3 \)

\( \chi_2^{-1} \otimes X_3 \otimes \chi_1^{-1} \)

\( \chi_3 \otimes \chi_2^{-1} \otimes \chi_1^{-1} \)

\( \chi_3 \otimes \chi_2^{-1} \otimes \chi_1 \)

\( X_2^{-1} \otimes \chi_1 \otimes X_3 \)

\( X_1 \otimes X_3 \otimes X_2 \)

\( X_1 \otimes X_3 \otimes X_2^{-1} \)

\( X_3 \otimes X_1 \otimes X_2 \)

\( X_3 \otimes X_1 \otimes X_2^{-1} \)

\( X_1 \otimes X_2^{-1} \otimes X_3 \)

\( \cdots \cdots \cdot \)

\( J_{L_3} = J_{M_1} \)

\( J_{L_2} = J_{N_1} \)

\( J_{M_2} = J_{N_2} \)

\( \cdots \cdots \cdot \)

\( J_L \)

\( J_M \)

\( J_N \)

Figure 11. Graph of \([\chi]_2 \propto tr_2, \chi = \nu^{-\frac{1}{2}}sgn\)
\[ J_{L_1} = J_{M_1}, \quad J_L = \ldots \]
\[ J_{L_2} = J_{N_1}, \quad J_M = \ldots \]
\[ J_{M_2} = J_{N_2}, \quad J_N = \ldots \]

Figure 12. Graph of \([\chi]_2 \propto tr_2, \chi = \nu^{-\frac{3}{2}}\)
Figure 13. Graph of $[\chi_3] \alpha 1, \chi$ generic
Figure 14. Graph of $[\chi_3] \propto 1, \chi = 1$
Figure 15. Graph of $[\chi]_3 \propto 1$, $\chi = sgn$
\[ J_{L_1} = J_{M_1} \]
\[ J_{L_2} = J_{N_1} \]
\[ J_{M_2} = J_{N_2} \]

\[ J_L \]
\[ J_M \]
\[ J_N \]

Figure 16. Graph of \([\chi_3 \alpha 1, \chi = \nu^{-\frac{1}{2}}]\)
Figure 17. Graph of $[\chi_3 \propto 1, \chi = \nu^{-\frac{1}{2}} \operatorname{sgn}$
Figure 18. Graph of $[\chi]_3 \propto 1$, $\chi = \nu^{-1}$
Figure 19. Graph of $[\chi]_3 \propto 1$, $\chi = \nu^{-1} \text{sgn}$
APPENDIX

Recall that in $Sp_{2n}(F)$, there is a Bruhat decomposition

$$K = \cup_{w \in W} IwI$$

(this is just the usual Bruhat decomposition of $Sp_{2n}(F_q)$ lifted back to $K$). Suppose $B = B_\Phi$ is the parahoric subgroup with $\Phi = \{s_0, s_1, \ldots, s_n\} \setminus \{s_0, s_k\}$, so that $B$ is the inverse image of a maximal parabolic subgroup of $Sp_{2n}(F_q)$ under the reduction mod $\mathcal{P}$ map. The goal of this appendix is to give a similar decomposition of $K$ into $B$ double cosets, and characterize the elements of $K$ lying in each double-coset. This is contained in the following lemma:

**Lemma A.3.2** Let

$$w_{ij} = \begin{pmatrix}
I_{i-j} & -I_j \\
I_{k-i} & I_{i-j} \\
I_{i-j} & I_{n-k-i-j} \\
I_j & \end{pmatrix}.$$

Then, $K = \cup_{i,j} Bw_{ij}B$. Moreover, suppose $\tau \in K$ has the form

$$\tau = \begin{pmatrix}
* & * & * & * \\
W & * & * & * \\
X & * & * & * \\
Y & * & * & * \\
\end{pmatrix}$$

($W$ is $(n-k) \times k$, $X$ is $k \times k$, $Y$ is $(n-k) \times k$). Define $r_1(\tau) = \text{rank} \begin{pmatrix} W \\ X \end{pmatrix} \mod \mathcal{P}$,

$$r_2(\tau) = \text{rank} X \mod \mathcal{P}.$$ Then, $\tau \in Bw_{r_1(\tau)r_2(\tau)}B$. 

104
Proof. We start with the following notation:

\[ W_\alpha = \text{Weyl group for } Sp_{2\alpha}(F), \quad \hat{W}_\beta = \text{Weyl group for } GL_\beta(F) \]

Now, since \( B = \langle I, \hat{W}_k \times W_{n-k} \rangle \) (\( \hat{W}_k \times W_{n-k} \subset W_n \) in the obvious way), finding a set of representatives for \( B \setminus K/B \) is equivalent to finding a set of representatives for \( \hat{W}_k \times W_{n-k} \setminus W_n / (\hat{W}_k \times W_{n-k}) \). To verify that \( \{w_{ij}\} \) is a set of representatives, we must show two things:

1. no two of the \( w_{ij} \)'s lie in the same \( \hat{W}_k \times w_{n-k} \) double-coset
2. everything in \( W_n \) lies in a double-coset associated to one of the \( w_{ij} \)'s.

To check \#1, we shall show that \( r_1 \) and \( r_2 \) are constant on double-cosets of \( B \). This suffices since they are different for distinct \( w_{ij} \)'s. In particular, we show that \( r_1 \) and \( r_2 \) are unchanged under right and left multiplication by members of \( \hat{W}_k \times W_{n-k} \). This is straightforward:

left:

\[
\begin{pmatrix}
* & * & * & * \\
W & * & * & * \\
X & * & * & * \\
Y & * & * & * \\
\end{pmatrix}
\begin{pmatrix}
M & A & B \\
& A & B \\
& C & D \\
\end{pmatrix}
= 
\begin{pmatrix}
* & * & * & * \\
WM & * & * & * \\
WX & * & * & * \\
WY & * & * & * \\
\end{pmatrix}
\]

right:

\[
\begin{pmatrix}
M & A & B \\
& A & B \\
& C & D \\
\end{pmatrix}
\begin{pmatrix}
* & * & * & * \\
W & * & * & * \\
X & * & * & * \\
Y & * & * & * \\
\end{pmatrix}
= 
\begin{pmatrix}
* & * & * & * \\
AW + BW & * & * \\
TM^{-1}X & * & * \\
CW + DV & * & * \\
\end{pmatrix}
\]
with $M \in \hat{W}_k$, \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in W_{n-k} \). Clearly, \( r_2 \) remains unchanged—\( X, XM, T M^{-1} \) all have the same rank mod $P$ since $M$ is invertible. Similarly, \( r_1 \) remains unchanged since $\begin{pmatrix} WM \\ XM \\ YM \end{pmatrix} = \begin{pmatrix} W \\ X \\ Y \end{pmatrix} \cdot M$ and $\begin{pmatrix} AW + BY \\ T M^{-1} X \\ CW + DY \end{pmatrix} = \begin{pmatrix} A & B \\ T M^{-1} & X \\ C & D \end{pmatrix}$.

and $M$ and $\begin{pmatrix} A & B \\ T M^{-1} & X \\ C & D \end{pmatrix}$ are invertible.

Thus, #1 holds. Note that the claim $\tau \in BW_{r_1(\tau)r_2(\tau)}B$ follows easily once we have established #2. We proceed to that task.

The first step in this process is to show that any two matrices in $W_n$ with the same first $k$ columns lie in the same double-coset of $\hat{W}_k \times W_{n-k}$. We then show that we can take any element in $W_n$ and by suitable left and right multiplications with elements of $\hat{W}_k \times W_{n-k}$, obtain a matrix with the same first $k$ columns as one of the $w_{ij}$'s.

The first step is fairly straightforward. We claim that for $w \in W_n$, every element of $W_n$ with the same first $k$ columns is of the form $w \cdot \begin{pmatrix} I \\ A & B \\ C & D \end{pmatrix}$, with $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in W_{n-k}$. If $w'$ has the same first $k$ columns as $w$, then $w^{-1}$ has the same first rows as $(w')^{-1}$ (since they are permutation matrices). Then, $w^{-1}w'$ is of the desired form, and since $w' = w(w^{-1}w')$, the claim now holds.

Now, we take an element $w \in W_n$ and describe how to use right and left multiplication by $\hat{W}_k \times W_{n-k}$ to get it equal to a $w_{ij}$ (at least up to $\pm 1$'s as entries, which suffices since it won't change the Weyl group element being represented). Suppose $r_1(w) = i$, $r_2(w) = j$. We then claim that by multiplying by a suitable
element of $\hat{W}_k \times W_{n-k}$ on the left, we get a matrix with the first $k$ columns of the form

\[
\begin{pmatrix}
0 & i \\
* & k - i \\
* & i - j \\
0 & n + j - k - i \\
* & j \\
0 & k - j \\
0 & n - k
\end{pmatrix}
\]

First, we can find $M_1 = \begin{pmatrix} M & I \\ T_{M^{-1}} & I \end{pmatrix} = \begin{pmatrix} M & I \\ M & I \end{pmatrix} \in \hat{W}_k$

so that $m_1$ moves the $j$ nonzero rows in the third block to make them the first $j$ rows there, and moves the $k - i$ nonzero rows in the first block to make them the last $k - i$ rows there. We remark that by symplecticity, the nonzero rows in the first block and those in the third block will not be in corresponding positions, thus we can rearrange both as claimed. Similarly, we can find $M_2 = \begin{pmatrix} I & A \\ B & I \\ C & D \end{pmatrix} \in W_{n-k}$

which moves the $i - j$ nonzero entries in the second/fourth block into the first $i - j$ rows of the second block. Again, note that if a row in the second (resp. fourth) block is nonzero, then the corresponding row in the fourth (resp. second) block is zero. The fact that $M_2$ is symplectic means that if the $\alpha$th row of the second block is moved to the $\beta$th row of the second/fourth block, then the $\alpha$th row of the fourth block is moved to the $\beta$th row of the fourth block. In any case, as the $i - j$ rows do not occupy corresponding positions, we can find an $M_2$ as desired.
Next, right multiplication moves columns. A tedious argument similar to the proceeding (which will be omitted) allow us to move the columns to arrange that the first $k$ are the same as those in $w_{ij}$, as needed. Since any first $k$ columns lie in the same $\tilde{W}_k \times W_{n-k}$ double-coset, the $\{w_{ij}\}$ will form a set of representatives for $(\tilde{W}_k \times W_{n-k})/W_n/(\tilde{W}_k \times W_{n-k})$, hence for $B\backslash K/B$.

Finally, we observe that as no two distinct $w_{ij}$'s have the same $(r_1, r_2)$ they characterize the $B$ double-coset. In particular, $r \in K$ lies in $Bw_{r_1(r), r_2(r)}B$, as claimed.

**Corollary A.3.3** Let $P_0 = P \cap K$. Then, $K = \cup_{i,j} P_0 w_{ij} B$.

**Proof.** Use $Bw_{ij}B = P_0 w_{ij} B$ and the lemma.
REFERENCES


[Cop] M. Copper, The Fourier transform and intertwining operators for certain degenerate principal series representations of $Sp(n; F)$, $F = \mathbb{R}$ or $\mathbb{C}$ (preprint).


Mathematics Department, Duke University