Degenerate Principal Series for Symplectic Groups

Chris Jantzen

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ABSTRACT

A brief description of the problem under consideration follows. Let F be a p-adic field (we take char F=0, odd residual characteristic), $G = Sp_{2n}(F)$. Let $P = MU$ be a maximal parabolic subgroup of G, and χ a one-dimensional representation of M. We may extend χ trivially to P. The question we investigate is whether $\pi = \mathrm{Ind}_P^G \chi \otimes 1$ is irreducible or not.

Two different approaches to this problem are used. The first, based on the work of Casselman and subsequent work by Gustafson, reduces the problem to the corresponding question about an associated finite-dimensional representation of a certain Hecke algebra. We use this method to do the case where $M = F^{\times} \times Sp_{2(n-1)}(F)$. The second approach is based on a technique of Tadić, and involves an analysis of Jacquet modules. This is used to prove a more general theorem on induced representations, which may be used to deal with the problem when χ satisfies a regularity condition. We use this method and ad hoc arguments to work out the low rank cases completely.

key words and phrases: p-adic field, symplectic group, induced representation, Jacquet module, Hecke algebra.

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INTRODUCTION

We begin the Introduction with a short description of the problem to be considered in this paper. Let F be a p-adic field with char F=0 and odd residual characteristic; $\mathcal{O}$ its ring of integers. Set $G = Sp_{2n}(F)$. Let $P = MU$ be a maximal parabolic subgroup of G, so $M \cong Gl_k(F) \times Sp_{2(n-k)}(F)$, for some k, $1 \leq k \leq n$. A (quasi)character on M is of the form $\chi \circ det$, $\chi \in \hat{F}^{\times}$, and may be extended trivially to P. We also denote the resulting characters as χ. It will be helpful to decompose χ as $\chi = |\cdot|^s \chi_u$, with χ_u coming from a character of $\mathcal{O}^{\times}$. The question we investigate is whether $\pi = \mathrm{Ind}_P^G \chi$ is irreducible or not, where induction is normalized so that unitary representations induce to unitary representations. In the following paragraphs, we describe some techniques for attacking this problem.

In his unpublished notes on admissible representations [Cas 2], Casselman used Hecke algebras to determine reducibility for $Sl_2(F) = Sp_2(F)$ with χ unramified. In his thesis, Gustafson [Gus] extended these techniques to the maximal parabolic in $Sp_{2n}(F)$ with Levi factor $M \cong Gl_n(F)$, χ unramified. In the second chapter of this thesis (the first consisting of notation and preliminaries), we apply these Hecke algebra methods to determine reducibility for the maximal parabolic in $Sp_{2n}(F)$ with Levi factor $M \cong F^{\times} \times Sp_{2(n-1)}(F)$ and arbitrary χ. In the third and fourth chapters, we use a technique of Tadić [Tad 2] involving Jacquet modules to obtain some general results, as well as some specific ones for small n.

We now describe how Hecke algebras are used to show irreducibility. Let ϖ be a uniformizer of F. Then, $\mathcal{O}/(\varpi\mathcal{O})$ is a finite field which we denote by $\mathbf{F}_q$. To the parabolic subgroup P, we may associate a parahoric subgroup B by taking $P(\mathbf{F}_q)$ and lifting it back to $Sp_{2n}(F)$. To the character χ on P, we can associate an open compact subgroup B_χ contained in B which has the following property:

$*$) every subquotient of π is generated by its B_χ − fixed vectors.

We remark that B_χ depends only on χ_u, and not on s. The proof that B_χ has this property is based on a similar statement in [Gus] for the case where χ is unramified. This, in turn, is based on the results of Borel and Casselman that state that a subquotient of an unramified principal series is generated by its Iwahori-fixed vectors. Let $H = H(G, B_\chi)$ denote the algebra of compactly supported B_χ-biinvariant functions on G. Associated to the induced representation (π, G, V) is the representation (π, H, V^{B_χ}), where H acts on the B_χ-fixed vectors by

$$\pi(f)v = \int_G f(g)\pi(g)v dg.$$

It is known (cf. [Gus]) that $*$) implies the map $W \longrightarrow W^{B_\chi}$ gives a bijective correspondence between subquotients of (π, G, V) and subquotients of (π, H, V^{B_χ}). As V^{B_χ} is a finite-dimensional space, we can explicitly compute certain operators $\pi(h_1), \pi(h_2)$, with $h_1, h_2 \in H$ and show that they can admit no common invariant subspace except at a finite number of potential reducibility points. For order of $\chi_u > 2$, there are no potential reducibility points–π is always irreducible. If order of $\chi_u = 2$, reducibility can only occur if $s = 0$ or $i\pi/\ln q$. For order of $\chi_u = 1$, the points where reducibility is possible are $s = \pm 1, 0, i\pi/\ln q$. The next problem is to show that there is reducibility at the potential reducibility points. We note that in the unramified case, if $s = \pm 1$, the trivial representation is a subquotient, so there is reducibility.

This leaves us with the potential reducibility points where χ is unitary. Now, if χ is unitary, then π is unitary as well. We show that π is reducible by showing that dim $\mathrm{Hom}_G(\pi, \tilde{\pi})$=2, where $\tilde{\pi}$=contragredient of $\pi = \mathrm{Ind}_P^G \chi^{-1}$. To show this, we observe that by Frobenius reciprocity,

$$\mathrm{Hom}_G(\pi, \tilde{\pi}) \cong \mathrm{Hom}_M(\pi_U, \chi^{-1}),$$

where π_U is the Jacquet module of π taken with respect to $P = MU$. If we let $M_\chi = M \cap B_\chi$, we get a representation $(\pi_U, H(M//M_\chi), (V_U)^{M_\chi})$. We can show that

$$\mathrm{Hom}_M(\pi_U, \chi^{-1}) \cong \mathrm{Hom}_{H(M,M_\chi)}(\pi_U, \chi^{-1}),$$

We note that the components of π_U have one-dimensional spaces of M_χ-fixed vectors. By considering the central characters of the components of π_U, one can show that only two of them contain χ^{-1}. The question is whether they arise as a direct sum (to get dim=2) or only as composition factors (so dim=1). For s near the point in question (either 0 or $i\pi/\ln q$), the action of $H(M, M_\chi)$ diagonalizes. It turns out that the eigenvectors may be analytically continued, so the action of $H(M, M_\chi)$ on $(V_U)^{M_\chi}$ will diagonalize at the point in question. Thus, dim=2. The Hecke algebra results are summarized in Theorem 2.32., which we now state. If $n \geq 2$, $\pi = i_{GA}\chi$ is reducible if and only if χ is of the form $|\cdot|^{\pm n}$ or $\chi^2 = 1$. If $n = 2$, in which case the group is just $Sl_2(F)$, reducibility occurs if and only if χ is of the form $\chi = |\cdot|^{\pm 1}$ or $\chi^2 = 1$ with $\chi \neq 1$. When reducible, π has two components.

The computations required to use the Hecke algebra approach are too complicated for an arbitrary maximal parabolic subgroup, so in chapter 3 we introduce a different approach based on a technique of Tadić. As an example, suppose that $\pi = \mathrm{Ind}_P^G \chi$ is a degenerate principal series representation of $Sp_{2n}(F)$ and that the Jacquet module of χ, treating χ as a representation of M, is a regular character on the split torus A of G. The basic idea for showing irreducibility is as follows. Let $r_{AG}(\pi)_{ss}$ denote the semisimplification of the Jacquet module of π with respect to the minimal parabolic subgroup. For P_i an intermediate parabolic subgroup, let $r_{M_iG}(\pi)_{ss}$ denote the semisimplification of the Jacquet module computed with respect to P_i (see chapter 1 for a summary of the Bernstein-Zelevinsky notation for Jacquet modules). Suppose that $\psi_1, \psi_2 \in r_{AG}(\pi)_{ss}$. Now, suppose that we can choose P_1 so that $r_{M_1G}(\pi)_{ss}$ has an irreducible component σ such that both ψ_1 and ψ_2 are in $r_{AM_1}(\sigma)_{ss}$. Then, if π_0 is a composition factor of π with $\psi_1 \in r_{AG}(\pi_0)_{ss}$, we claim that $\psi_2 \in r_{AG}(\pi_0)_{ss}$ as well. To see this, look at $r_{M_1G}(\pi_0)_{ss}$. Since $\psi_1 \in r_{AG}(\pi_0)_{ss}$, $r_{M_1G}(\pi_0)_{ss}$ will have to contain at least part of σ–by regularity, there is only one copy of ψ_1 around. Since σ is irreducible, $r_{M_1G}(\pi_0)_{ss}$ must contain all of σ, so that $\psi_1, \psi_2 \in r_{AG}(\pi_0)_{ss}$. We then argue as follows: suppose $\psi_1 \in r_{AG}(\pi_0)_{ss}$. By working with $r_{M_1G}(\pi_0)_{ss}$, we conclude that $\psi_2 \in r_{AG}(\pi_0)_{ss}$. Then, by using $r_{M_2G}(\pi_0)_{ss}$, we conclude that $\psi_3 \in r_{AG}(\pi_0)_{ss}$, etc.

until we end up with $r_{AG}(\pi_0)_{ss} = r_{AG}(\pi)_{ss}$. Therefore, we conclude that π must have been irreducible.

We now describe how Jacquet modules are used in pursuing reducibility. In this case, one of the $r_{M_iG}\pi$ will have a reducible composition factor which would "normally" (i.e., for generic χ) be irreducible. Suppose the composition factor has components σ_1, σ_2 with $\psi_1 \in r_{AM_\star}(\sigma_1)_{ss}, \psi_2 \in r_{AM_\star}(\sigma_2)_{ss}$, where $P_\star = M_\star U_\star$ is the appropriate parabolic subgroup. Then, let π_i=$\mathrm{Ind}_{P_\star}^G \sigma_i$ We get $\psi_1 \in r_{AG}(\pi_1)_{ss}, \psi_1 \notin r_{AG}(\pi_2)_{ss}$, and similarly for ψ_2. If π were irreducible, then π would have to be a subquotient of either π_1 or π_2. Therefore, $r_{AG}(\pi)_{ss} \subset r_{AG}(\pi_1)_{ss}$ or $r_{AG}(\pi)_{ss} \subset r_{AG}(\pi_2)_{ss}$. But, this is not the case–just look at ψ_1, ψ_2. Thus, we have that π is reducible.

The main theorem in chapter 3 is Theorem 3.1.2, which is based on a generalization of the argument above. The theorem gives necessary and sufficient conditions for reducibility of $\pi = i_{GM}\rho$, where ρ is an irreducible admissible representation of M such that $r_{AM}\rho \neq 0$ and the characters appearing in $r_{AM}\rho$ are regular. First, we associate a graph to π as follows:

vertices: the vertices are the elements of $r_{AG}(\pi)_{ss}$

edges: two vertices ψ_1, ψ_2 are connected by an edge is there is some Levi N and some $\tau \in r_{NG}(\pi)_{ss}$ such that the following hold:

1. τ is an irreducible representation of N.
2. ψ_1, $\psi_2 \in r_{AN}(\tau)_{ss}$.

where $r_{AN}(\tau)_{ss}$ denotes the semisimplification of τ. Under these conditions, Theorem 3.1.2 states that the following are equivalent:

1. π is irreducible.
2. the graph of π is connected
3. the composition factors of $r_{NG}\pi$ as computed using the results of Bernstein-Zelevinsky/Casselman (cf. Theorem 1.2.4) are all irreducible. In particular, it

is sufficient to check those where the Levi N is generated by A and one simple reflection. Note that one must incorporate the reducibility of $r_{AM}\rho$ into these computations.

The third condition reduces the problem to analyzing the reducibility of certain induced representations of $GL_2(F)$ and $Sp_2(F) = Sl_2(F)$, where the results are known.

As indicated, Theorem 3.1.2 is a generalization of the preceding arguments, but it still requires a regularity condition. In the final chapter, we look explicitly at degenerate principal series for $Sp_4(F)$ and $Sp_6(F)$, in general, modifying the kind of arguments used in chapter 3 (including the definition of the graph for π) to allow us to work out the non-regular case.

I would like to take this opportunity to thank some individuals who have contributed to this paper, which is essentially my dissertation. First, I would like to express my gratitude to Paul Sally Jr. for doing a fine job as advisor– it has been a pleasure to work with him. This paper owes much to an idea of Marko Tadić, and I would like to thank him for taking the time and effort to explain it to me. Finally, I would like to thank Timothy Steger for carefully reading the first draft and making many valuable suggestions, and the referee, for similar reasons.

CHAPTER 1

NOTATION AND PRELIMINARIES

1.1. $Sp_{2n}(F)$

The purpose of this chapter is to introduce notation and review some preliminary facts that will be of use in the rest of this thesis.

Let F be an nonarchimedean local field of characteristic zero. Let $\mathcal{O}$ denote the ring of integers, $\mathcal{P}$ the prime ideal in $\mathcal{O}$ and ϖ a uniformizer. Then, $\mathcal{O}/\mathcal{P}$ is a finite field. Let q denote the number of elements in $\mathcal{O}/\mathcal{P}$. We normalize the Haar measure on F so that vol($\mathcal{O}$)=1.

Suppose that χ is a quasicharacter of $F^\times$ (i.e., a multiplicative homomorphism from $F^\times$ to $\mathbf{C}^\times$ - not necessarily unitary). If $x \in F^\times$, x may be decomposed as $x = \varpi^k x_0$, with $x_0 \in \mathcal{O}^\times$. We can then decompose χ as $\chi = |\ |^s \chi_u$ by $\chi(x) = |\varpi^k|^s \chi_u(x_0)$, where χ_u is a character of $\mathcal{O}^\times$, $0 \leq \text{Re}s < 2\pi i/\ln q$. It may be convenient, at times, to view χ_u as a character on $F^\times$ (by $\chi_u(\varpi^k x_0) = \chi_u(x_0)$). We shall use 1 for the trivial character and sgn to denote a nontrivial character satisying sgn^2=1.

As most of this thesis concerns induced representations for $Sp_{2n}(F)$, we next discuss $Sp_{2n}(F)$ and induced representations. In this section, we review some of the structure theory for $Sp_{2n}(F)$.

Recall that we may take

$$Sp_{2n}(F) = \{x \in GL_{2n}(F) | {}^T X J X = J\},$$

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where

$$J = \left(\begin{array}{c|c} & -I_n \\ \hline I_n & \end{array} \right)$$

(entries left vacant are zeros). Let $K = Sp_{2n}(\mathcal{O})$. This is a maximal compact subgroup of $Sp_{2n}(F)$. It has a filtration of open compact normal subgroups

$$\ldots \lhd K_2 \lhd K_1 \lhd K,$$

where

$$K_i = \{X \in K | X \equiv I \mathrm{mod} \mathcal{P}^i\}.$$

Let

$$A = \left\{ \left(\begin{array}{c|c} \begin{matrix} a_1 & & \\ & \ddots & \\ & & a_n \end{matrix} & \\ \hline & \begin{matrix} a_1^{-1} & & \\ & \ddots & \\ & & a_n^{-1} \end{matrix} \end{array} \right) \right\}$$

be a maximal split torus in $Sp_{2n}(F)$. Then, the Weyl group of $Sp_{2n}(F)$ is

$$W = N_G(A)/A,$$

where $N_G(A)$ denotes the normalizer of A in G. The Weyl group of $Sp_{2n}(F)$ has $2^n \cdot n!$ elements and may be viewed as $W = \{\text{permutations and sign changes of} \{e_1, \ldots, e_n\}\}$. W is generated by the simple root reflections $\{s_1, \ldots, s_n\}$, where

$$s_1 = \left(\begin{array}{c|c} \begin{matrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & \ddots \; 1 \end{matrix} & \\ \hline & \begin{matrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & \ddots \; 1 \end{matrix} \end{array} \right), \ldots,$$

$$s_{n-1} = \left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & 1 & & & & & \\ \hline & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ & & & & & & 1 & \end{array}\right), s_n = \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & & & -1 \\ \hline & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ & & 1 & & & \end{array}\right).$$

We now review the parabolic and parahoric subgroups for $Sp_{2n}(F)$. A standard minimal parabolic subgroup for $Sp_{2n}(F)$ is the following:

$$P_{\min} = \left\{ \begin{pmatrix} X & Y \\ & {}^T X^{-1} \end{pmatrix} \in Sp_{2n}(F) | X \in GL_n(F) \text{ is upper triangular} \right\}.$$

Note that ${}^T X^{-1}$ will be lower triangular so that $P_{\min}$ is not the upper triangular matrices in $Sp_{2n}(F)$ (we could arrange for $P_{\min}$ to be upper triangular had we chosen a different J in defining $Sp_{2n}(F)$). Let $\Phi \subset \{s_1, \ldots, s_n\}$. The standard parabolic subgroups of $Sp_{2n}(F)$ are parameterized by such subsets of the simple reflections. Associated to Φ is the parabolic subgroup $P_\Phi =< P_{\min}, \Phi >$.

If $P = MN$ is the Levi factorization of P then

$$M \cong GL_{k_1}(F) \times GL_{k_2}(F) \times \cdots \times GL_{k_{l-1}}(F) \times Sp_{2k_l}(F),$$

with $k_1 + k_2 + \ldots + k_l = n$. This will be embedded in $Sp_{2n}(F)$ as

$$M = \left\{ \left(\begin{array}{cccc|cccc} A_1 & & & & & & & \\ & \ddots & & & & & & \\ & & A_{l-1} & & & & & \\ & & & A & & & & B \\ \hline & & & & {}^T A_1^{-1} & & & \\ & & & & & \ddots & & \\ & & & & & & {}^T A_{l-1}^{-1} & \\ & & & C & & & & D \end{array}\right) \middle| \begin{array}{l} A_i \in GL_{k_i}(F) \\ \text{and } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2k_l}(F) \end{array} \right\}.$$

We note that in terms of the parameterization of parabolic subgroups by subsets of $\{s_1, \ldots, s_n\}$, this parabolic subgroup corresponds to

$$\Phi = \{s_1, \ldots, s_n\} \setminus \{s_{k_1}, s_{k_1+k_2}, \ldots, s_{k_1+\ldots k_{l-1}}, \}.$$

We also remark that the modular function for P is given by

$$\delta(m) = |\det A_1|^{2n+1-k_1} \cdot |\det A_2|^{2(n-k_1)+1-k_2} \cdots |\det A_{l-1}|^{2(n-k_1-\ldots-k_{l-2})+1-k_{l-1}}.$$

The parahoric subgroups may be constructed in a similar fashion. A standard Iwahori subgroup for $Sp_{2n}(F)$ may be constructed as follows: let $\Psi : K \to Sp_{2n}(\mathbf{F}_q)$ be the reduction mod $\mathcal{P}$ homomorphism (where $\mathbf{F}_q$ is the finite field $\mathcal{O}/\mathcal{P}$). Then, take $I = \Psi^{-1}(P_{min}(\mathbf{F}_q))$ as the standard Iwahori subgroup. Let

$$s_0 = \left(\begin{array}{ccc|ccc} & & & -\varpi^{-1} & & \\ 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline \varpi & & & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{array}\right),$$

so that $\{s_0, s_1, \ldots, s_n\}$ generate the affine Weyl group. The standard parahoric subgroups are in bijective correspondence with the subsets Φ of $\{s_0, s_1, \ldots, s_n\}$, the correspondence being given by

$$B_\Phi = < I, \Phi > .$$

Note that if $\Phi \subset \{s_1, \ldots, s_n\}$, then

$$B_\Phi = \Psi^{-1}(P_\Phi(\mathbf{F}_q)).$$

These are the parahoric subgroups we shall be most interested in.

1.2. Induced representations and Jacquet modules

We now review the construction of induced representations and Jacquet modules.

We start by reviewing the construction of induced representations. Let G be a reductive p-adic group and $P = MU$ a parabolic subgroup of G. Let (ρ, X) be an admissible representation of M. Then ρ may be extended trivially to U to get a representation of P, $\rho \otimes 1$. The induced representation $\mathrm{Ind}_P^G \rho \otimes 1$ acts on the space

$$V = \left\{ f : G \to X \,\middle|\, \begin{array}{l} \text{f smooth} \\ f(mug) = \delta^{\frac{1}{2}}(m)\rho(m)f(g) \quad \forall m \in M,\ u \in U,\ g \in G \end{array} \right\},$$

where δ denotes the modular function for P. The action of G on this space is by right translation. We shall give some more convenient notation for induced representation after we discuss Jacquet modules.

We now describe the construction of Jacquet modules. Again, suppose G is a reductive p-adic group and $P = MU$ a parabolic subgroup of G. Let (π, V) be a representation of G. The Jacquet module of π with respect to P, denoted π_U, is a representation of M on a space which is denoted by V_U. The space is

$$V_U = V/V(U),$$

where $V(U) = span\{\pi(u)v - v | v \in V, u \in U\}$. The action of M is given by

$$\pi_U(m)(v + V(U)) = \delta^{-\frac{1}{2}}(m)\pi(m)v + V(U)$$

(one checks that this defines a representation).

We shall frequently use the notation of Bernstein-Zelevinsky [B-Z] for induced representations and Jacquet modules. If $P = MU$ is a parabolic subgroup of $G, (\rho, X)$ an admissible representation of $M, (\pi, V)$ an admissible representation of G, then set

$$i_{GM}(\rho) = \mathrm{Ind}_P^G \rho \otimes 1$$

and

$$r_{MG}(\pi) = \pi_U.$$

The following notation for induced representations in $Sp_{2n}(F)$will also be convenient. It is just an extension (cf. [S-T]) of the shorthand notation of Bernstein-Zelevinsky for induced representation in $GL_n(F)$. Suppose $P = MU$ is a standard parabolic subgroup of $Sp_{2n}(F)$, with

$$M \cong GL_{k_1}(F) \times \cdots \times GL_{k_{l-1}}(F) \times Sp_{2k_l}(F).$$

Let $\rho_1, \ldots, \rho_{l-1}$ be admissible representations of $GL_{k_1}(F), \ldots GL_{k_{l-1}}(F)$, and τ an admissible representation of $Sp_{2k_l}(F)$. Then, let

$$\rho_1 \times \ldots \times \rho_{l-1} \propto \tau = i_{GM}\rho_1 \otimes \ldots \otimes \rho_{l-1} \otimes \tau.$$

The sign $\propto$ is used only to indicate that the last factor comes from a symplectic group. If $k_l = 0$, the representation will be denoted

$$\rho_1 \times \dots \rho_{l-1} \propto 1.$$

The trivial representation of $Sp_{2n}(F)$, $n > 0$, will be denoted tr_n, so there will be no confusion.

We close these preliminaries with four theorems on induced representations and Jacquet modules.

THEOREM **1.2.1 (Frobenius Reciprocity)** *Let G be a connected reductive p-adic group, $P = MU$ a parabolic subgroup, ρ an admissible representation of M, π an admissible representation of G. Then,*

$$\mathrm{Hom}_M(r_{MG}(\pi), \rho) \cong \mathrm{Hom}_G(\pi, i_{GM}\rho).$$

Proof. cf. [B-Z2].

THEOREM **1.2.2** *Let (τ, G, L) be an irreducible subquotient of $\mathrm{Ind}_{P_{\min}}^G \psi$ (ψ a character). Then, there is a $w \in W$ so that τ embeds in $\mathrm{Ind}_{P_{\min}}^G w\psi$. Moreover, all $\mathrm{Ind}_{P_{\min}}^G w\psi$ for $w \in W$ have the same components.*

Proof. See [Cas2] for references for the first part and [B-Z2] for the second.

We next recall the following theorem, which says the constructions of induced representations and Jacquet modules may be done in stages.

THEOREM **1.2.3** *Let $L < M$ be standard Levis for G. Then*

1. $i_{GL} = i_{GM} \circ i_{ML}$
2. $r_{LG} = r_{LM} \circ r_{MG}$.

Proof. cf. [B-Z2].

We close with a theorem of Bernstein-Zelevinsky, Casselman. Let M, N be standard Levis for a connected reductive p-adic group G. Set $W^{MN} = \{w \in W | w(P_{\min} \cap M) \subset P_{\min}, w^{-1}(P_{\min} \cap N) \subset P_{\min}\}$. We remark that these correspond to the elements of shortest length in the double-cosets $W_N \backslash W / W_M$ (W_M = Weyl group of M, etc.).

THEOREM **1.2.4 (Bernstein-Zelevinsky, Casselman)** *Let ρ be an admissible representation of M. Then, $r_{NG} \circ i_{GM}\rho$ has a composition series with factors*

$$i_{NN'} \circ w \circ r_{M'M}\rho, \quad w \in W^{MN}$$

where $M' = M \cap w^{-1}(N), N' = w(M) \cap N$.

Proof. See [Cas2], chapter 6 or [B-Z2] "geometrical lemma" (Lemma 2.12), proved in chapter 6.

CHAPTER 2

THE HECKE ALGEBRA APPROACH

2.1. General theorems

The goal of this chapter is to analyze reducibility/irreducibility in the case where the inducing subgroup has Levi factor $M \cong F^\times \times Sp_{2(n-1)}(F)$. We attack this problem using Hecke algebra methods. In particular, we construct an open compact subgroup B_χ and to the representation (π, G, V), we associate a representation of the Hecke algebra $H(G//B_\chi)$ of compactly supported B_χ-biinvariant functions on the finite-dimensional space V^{B_χ}. This representation, denoted $(\pi, H(G//B_\chi), V^{B_\chi})$, has the property that the subquotients of (π, G, V) and those of $(\pi, H(G//B_\chi), V^{B_\chi})$ are in bijective correspondence, the bijection being implemented by $X \subset V$ G-invariant goes to $X^{B_\chi} \subset V^{B_\chi}$ $H(G//B_\chi)$-invariant. Once this correspondence is established, we can do our analysis on the Hecke algebra side, where the finite-dimensionality of the representation facilitates computations.

We now describe the contents of this chapter in a little more detail. For purposes of this discussion, let us assume that $n > 1$–the case $Sp_2(F) = Sl_2(F)$) behaves a little bit differently. The first section of this chapter consists of constructing B_χ and establishing the aforementioned correspondence. In the second section, we explicitly construct a basis for V^{B_χ}. This construction breaks the problem into three cases: (order of $\chi_u = 1$), (order of $\chi_u = 2$), and (order of $\chi_u > 2$). In section 2.3, we look at the case (order of $\chi_u > 2$). We compute a pair of operators $\pi(h_1)$, $\pi(h_2)$ with $h_1, h_2 \in H(G//B_\chi)$ and observe that they do not admit a common invariant subspace, so that in this case, π must always be irreducible. In section 2.4, we do similar computations for the case (order of χ_u) $= 2$ and conclude that if $\chi^2 \neq 1$, then π is irreducible. The fifth section uses a variation on this theme for the case where χ is unramified to show that if $\chi^2 \neq 1$ or $\chi \neq \nu^{\pm 1}$, then π is irreducible. In the last two sections, we show that if $\chi^2 = 1$ or $\chi = \nu^{\pm 1}$, then π is reducible. If $\chi = \nu^{\pm 1}$, then π has the trivial representation as a subquotient, hence is reducible. If $\chi^2 = 1$, we show that π is reducible by showing that $\mathrm{Hom}_G(\pi, \tilde{\pi})$ is two-dimensional. Again, the analysis is broken up into cases depending on the order of χ_u–the case (order of $\chi_u = 1$) is done in the sixth section, the case (order of $\chi_u = 2$) in the seventh. The results are summarized in Theorem 2.7.2.

The goal of this section is to establish the bijective correspondence between the subquotients of (π, G, V) and those of $(\pi, H(G//B_\chi), V^{B_\chi})$ for a suitably chosen B_χ. We start by choosing $B_\chi$and discussing $(\pi, H(G//B_\chi), V^{B_\chi})$. We then show that any subquotient of (π, G, V) has a nonzero B_χ-fixed vector. The last part is to show that since every subquotient of (π, G, V) has a nonzero B_χ-fixed vector, there is a bijective correspondence between the subquotients of (π, G, V) and those of $(\pi, H(G//B_\chi), V^{B_\chi})$. We note that although we are interested in the degenerate principal series coming from a maximal proper parabolic subgroup in $Sp_{2n}(F)$, the results are more general. Everything should work for an arbitrary parabolic in G a reductive p-adic group. We shall work in the more general situation.

Let $P \subset G$ be a parabolic subgroup, with $P = MU$ its Levi decomposition. Let χ be a character of M and set $\pi = i_{GM}\chi$. The subgroup B_χ which we use may be described by its Iwahori factorization with respect to P. In particular,

$$B_\chi = U_l^- M_\chi U_0$$

where $U_0 = U \cap K, U_l^- = U^- \cap K_l$, with l the smallest positive integer for which $\chi|_{M_l}$ is trivial ($M_l = M \cap K_l$), and $M_\chi = M_0 \cap \ker \chi$. We normalize our measures so that $|B_\chi| = |U_l^-| = |M_\chi| = |U_0| = 1$.

We check that B_χ is a group. In fact, we show that $B_\chi = K_l M_\chi U_0 = M_\chi U_0 K_l$. From this and the fact that K_l is normal in K, it is easy to check that $B_\chi$is closed under multiplication and inverses. To show $B_\chi = K_l M_\chi U_0 = M_\chi U_0 K_l$, it is enough to show $K_l \subset B_\chi$. Suppose $k \in K_l$. Since $K_l \subset K_1 \subset B$, we have a factorization of k as $k = u^- m u$ with $u^- \in U_1^-$, $m \in M_0$, $u \in U_0$. We claim $u^- \in U_l^-$. We use the reduction mod $\mathcal{P}^l$ homomorphism to get $u^- m u \equiv I \bmod \mathcal{P}^l$. Now, suppose $u^- \not\equiv I \bmod \mathcal{P}^l$. Then, $mu \equiv (u^-)^{-1} \mathrm{mod}\ \mathcal{P}^l$. But, this cannot occur since it would imply that $P(\mathcal{O}/\varpi^l\mathcal{O})$ and $U^-(\mathcal{O}/\varpi^l\mathcal{O})$ had nontrivial intersection. Thus,

$u^- \in U_l^-$. A similar argument shows $u \in U_l$. This forces $m \in M_l$. Consequently, $K_l \subset U_l^- M_l U_l \subset B_\chi$, as needed.

We now describe the representation $(\pi, H(G//B_\chi), V^{B_\chi})$. By definition, $H(G//B_\chi)$ is the (convolution) algebra of compactly supported B_χ-biinvariant functions. V^{B_χ} is the space of B_χ-invariant functions in V; finite-dimensional since B_χ is open compact. The action is given by

$$\pi(h)V = \int_G h(g)\pi(g)v dg$$

for

$$h \in H(G//B_\chi), v \in V^{B_\chi}.$$

Our next task it to show that every subquotient of (π, G, V) contains a B_χ-fixed vector. We start with the following result of Casselman.

PROPOSITION **2.1.1 (Casselman)** *Let (π, G, V) be an admissible representation of G (G as above). Let B be a compact subgroup of G with an Iwahori factorization $B = U_B^- M_B U_B$ with respect to $P = MU$ (where $U_B^- = U^- \cap B$, etc.). Then,*

$$r_{MG} : V^B \to (r_{MG}(V))^{M_B}$$

is surjective.

LEMMA **2.1.2 (Jacqet via Casselman)** *With hypotheses as above, suppose that $v \in V^{M_B U_B^-}$. Then, $v_0 = Pr_B(v) = Pr_{U_B}(v)$ satisfies $v - v_0 \in V(U_B)$ (where $Pr_K(v) = \frac{1}{|K|}\int_K \pi(k)vdk$)*

Proof of Lemma. Using the Iwahori factorization,

$$\begin{aligned} v_0 &= \int_B \pi(b)vdb \\ &= \int_{U_B^-}\int_{M_B}\int_{U_B} \pi(b)vdb \\ &= \int_{U_B} \pi(u)vdu \quad (\text{since } v \in V^{M_B U_B^-}) \\ &= Pr_{U_B}(v), \end{aligned}$$

which is the first claim. The second claim follows immediately from this.

LEMMA **2.1.3** *With hypotheses as above,*

$$r_{MG}(V^B) = r_{MG}(V^{M_B U_B^-}).$$

Proof of Lemma. Since $v - v_0 \in V(U_B)$, we have $v - v_0 \in V(U)$. Thus, v and v_0 have the same image in $r_{MG}(V)$.

Proof of Proposition. Let $\bar{X} \subset (r_{MG}(V))^{M_B}$ be finite-dimensional with $X \subset V^{M_B}$ a finite-dimensional subspace mapping onto it. Since X is finite-dimensional, there is a compact $\hat{U}^- \subset U^-$ such that $X \subset V^{M_B \hat{U}_B^-}$. Now, there exists $a \in A_P$, the maximal split torus in the center of M, such that $a^{-1}U_B^- a \subset \hat{U}^-$. This implies that $\pi(a)X \subset V^{M_B U_B^-}$ (Check: $\pi(mu)\pi(a)v = \pi(a)\pi(a^{-1}ma)\pi(a^{-1}ua)v = \pi(a)\pi(m)v = \pi(a)v$ for m, u, a as described.) By the preceding lemma,

$$\pi_U(a)\bar{X} = r_{MG}(\pi(a)X) \subset r_{MG}(V^{M_B U_B^-}) = r_{MG}(V^B)$$

Thus, $\dim \bar{X} \leq \dim V^B < \infty$ by admissibility. Consequently, $\dim(r_{MG}(V))^{M_B} \leq \dim V^B < \infty$.

Next, let $\bar{X} = (r_{MG}(V))^{M_B}$. Then, we must have $\pi_U(a)(V_U)^{M_B} = (V_U)^{M_B}$. Since a was chosen so that $\pi_U(a)(V_U)^{M_B} \subset r_{MG}(V^B)$, we have $(V_U)^{M_B} = (r_{MG}(V))^{M_B} \subset r_{MG}(V^B)$, as desired.

The second ingredient toward establishing that every subquotient of (π, G, V) contains a B_χ-fixed vector consists of extending the results of Borel/Casselman to get the proposition below.

PROPOSITION **2.1.4** *Any subquotient (τ, G, L) of $\mathrm{Ind}_{P_{\min}}^G \psi$ has a nonzero I_ψ-fixed vector. For $P_{\min} = AN$, $I_\psi = N_l^- A_\psi N_0$ with $A_\psi = A_0 \cap \bigcap_{w \in W} \ker w\psi$ and $l =$ smallest positive integer such that $\psi|_{A_l} \equiv 1$.*

Proof. See [Bor] 4.9 for the unramified case (where I_ψ= Iwahori, G semisimple).

By the preceding proposition, it suffices to show that $L_N^{A_\psi} \neq 0$.

First, suppose τ is actually a subrepresentation of $\mathrm{Ind}_{P_{\min}}^G \psi$. Then, Frobenius reciprocity $\Longrightarrow$

$$\mathrm{Hom}_A(\tau_N, \psi) \cong \mathrm{Hom}_G(\tau, \mathrm{Ind}_{P_{\min}}^G \psi) \neq 0$$

Thus, τ_N contains a vector which transforms under A according to ψ, hence is fixed by A_ψ. So, $L_N^{A_\psi} \neq 0$.

Now, suppose that τ is just a subquotient. By Theorem 1.2.2, there is a $w \in W$ so that τ is a subrepresentation of $\mathrm{Ind}_{P_{\min}}^G w\psi$. Then, as above, τ_N contains a vector which transforms under A according to $w\psi$, hence is fixed by A_ψ. So, $L_N^{A_\psi} \neq 0$. The conclusion follows.

This brings us to our proposition.

PROPOSITION **2.1.5** *Any subquotient of* (π, G, V) *has a (nonzero)* B_χ*-fixed vector*

Proof. First, let (ρ, G, Y) be a subquotient of (π, G, Y). We claim that (ρ, G, Y) has an I_χ-fixed vector (using $\chi|_A$ to get I_χ). This follows since χ is a subrepresentation of $\mathrm{Ind}_{M\cap P_{\min}}^M \delta^{-1/2}(\chi|_A)$ with δ the modular function for $M \cap P_{\min}$.

We now proceed as in [Gus].

LEMMA **2.1.6** *Any* K*-invariant subspace* (ω, K, Ω) *of* (π, G, V) *has a nontrivial* $P \cap B_\chi$*-fixed vector.*

Proof of Lemma. Since $G = PK$ (Iwasawa decomposition), $\pi|_K = \mathrm{Ind}_{P\cap K}^K \chi$. By Frobenus reciprocity,

$$[\mathrm{Ind}_{P\cap K}^K \chi : \omega] = [\omega|_{P\cap K} : \chi],$$

so $\omega|_{P\cap K}$ has a vector which translates according to χ, hence is fixed on M_χ (and trivial on U_0). The lemma follows. Note that we are writing χ for $\chi|_{P\cap K}$, etc, above.

LEMMA **2.1.7** *Any nonzero* K*-invariant subspace* X *which contains a nontrivial* K_l*-fixed vector contains a nontrivial* B_χ*-fixed vector.*

Proof of Lemma. $K_l \triangleleft K \Longrightarrow K^{K_l}$ is a K-subrepresentation of χ. By the preceding lemma, it has a $P \cap B_\chi$-fixed vector. Since B_χ is generated by $P \cap B_\chi, U_l^-$ ($U_l^- \subset K_l$), the lemma holds.

Now we finish the proof of the proposition. From the first paragraph, we know that (ρ, G, Y) contains an I_χ-fixed vector. As $K_l \subset I_\chi$, it is K_l-fixed. By the preceding Lemma, (ρ, G, Y) contains a B_χ-fixed vector, as claimed.

We now begin the final task of this section– establishing the bijective correspondence between the subquotients of (π, G, V) and those of $(\pi, H(G//B_\chi), V_\chi^B)$. The lemma and theorem that follow have been culled from various places (cf. [Cas2], [Gus], [Bor]).

LEMMA **2.1.8** *Let V_1, V_2, V_3 be smooth G representations, B an open compact subgroup. If*

$$V_1 \to V_2 \to V_3$$

is an exact sequence of G-modules, then,

$$V_1^B \to V_2^B \to V_3^B$$

is exact as well (as vector spaces).

Proof. See [Cas2], 2.1.7.

LEMMA **2.1.9** *Let V be a smooth finite length representation of G, B an open compact subgroup of G. Suppose that every subquotient of V has a nonzero B-fixed vector. Then, every subquotient of V is generated by its B-fixed vectors.*

Proof. Suppose not. Let V' be a subquotient of V and set $V''= G \cdot V'^B$ = subspace of V' generated by V'^B under the action of G. V'' is a subrepresentation of V'; by assumption, not all of V'. By the previous lemma,

$$\left(\frac{V'}{V''}\right)^B = \frac{(V')^B}{(V'')^B} = 0.$$

But, this cannot happen since $\frac{V'}{V''}$ is a nonzero subquotient of V.

THEOREM **2.1.10** *The map $W \to W^{B_\chi}$ gives a bijective correspondence between subquotient of (π, G, V) and subquotients of $(\pi, H(G//B_\chi), V^{B_\chi})$.*

Proof. By Proposition 2.1.5, every subquotient of V has a nonzero B_χ-fixed vector. Therefore, by the preceding lemma, every subquotient of V is generated by its

B_χ-fixed vectors. We now check that $W \to W^{B_\chi}$ gives a bijective correspondence between subrepresentations of (π, G, V) and those of $(\pi, H(G//B_\chi), V_\chi^B)$.

injective: Suppose that V_1, V_2 are subrepresentations of V satisfying $V_1^{B_\chi} = V_2^{B_\chi}$. Then $G \cdot V_1^{B_\chi} = G \cdot V_2^{B_\chi}$ (where $G \cdot V_i^{B_\chi}$ denotes the subspace of V generated by $V_i^{B_\chi}$ under the action of G). Since every subquotient of V is generated by its B_χ-fixed vectors, $G \cdot V_i^{B_\chi} = V_i$. Thus $V_1 = V_2$, as needed.

surjective: Let $W \subset V^{B_\chi}$ be an $H(G//B_\chi)$-invariant subspace. Set $V_0 = G \cdot W$. We claim $V_0^{B_\chi} = W$. Suppose not - then $V_0^{B_\chi} \supset W$. Suppose $v \in V_0^{B_\chi}$ but $v \notin W$. Since $v \in G \cdot W$, we can write

$$v = \sum_{\text{finite}} \pi(g_i)w_i \quad w_i \in W, g_i \in G$$

We claim $v = \sum \pi(\chi_B * g_i * \chi_B)w_i$, where $\chi_B = \text{char}_{B_\chi}$. Note that $\chi_B * g_i * \chi_B \in H(G//B_\chi)$ so this will express v in such a way that it must lie in W. Check:

$$\sum \pi(\chi_B * g_i * \chi_B)w_i = \sum \pi(\chi_B)\pi(g_i)w_i = \pi(\chi_B)[\sum \pi(g_i)w_i] = \pi(\chi_B)v = v.$$

We also need the following corollary of the proof.

Corollary **2.1.11** *Suppose $(\pi_i, G, V_i), i = 1, 2$ are such that every nonzero subquotient of each V_i contains a nontrivial B-fixed vector. Then,*

$$\text{Hom}_G(V_1, V_2) \cong \text{Hom}_{H(G//B_\chi)}(V_1^B, V_2^B)$$

Proof. (cf. [Cas2], Proposition 2.2.2)

2.2. Basis for V^{B_χ}

The goal of the rest of this chapter is to use the preceding theorem to analyze reducibility for a specific example. Let $G = Sp_{2n}(F)$, F p-adic of characteristic zero, odd residual characteristic. Let $M \cong F^\times \times Sp_{2(n-1)}$ and $P = MU$ the corresponding parabolic subgroup. We identify characters of $F^\times$ with characters on M via the $F^\times$ part. We set $\pi = i_{GM}\chi$, and analyze the reducibility of (π, G, V) by analyzing reducibility for $(\pi, H(G//B_\chi), V^{B_\chi})$. The goal in this section is to construct a basis for V^{B_χ}. The problem breaks itself into three cases. If we write $\chi = |\cdot|^s \chi_u$ (see chapter 1), the cases are

Case 1 (order of χ_u) $= 1$

Case 2 (order of χ_u) $= 2$

Case 3 (order of χ_u) > 2

We shall approach these three separately and summarize the results in Theorem 2.2.4. The reader is advised that the third case is substantially longer than the other two.

We start with the following lemma, which is sort of a Bruhat decomposition. We let B be the parahoric subgroup corresponding to P (obtained by adjoining $s_2, \ldots, s_n$ to the Iwahori). (It is B_χ for $\chi \equiv 1$)

LEMMA **2.2.1** *Let* w_{00} $=$ *identity,*

$$w_{10} = \left(\begin{array}{cccc|cccc} & 1 & & & & & & \\ 1 & & & & & & & \\ & & 1 & & & & & \\ & & & \ddots & & & & \\ & & & & 1 & & & \\ \hline & & & & & 1 & & \\ & & & & 1 & & & \\ & & & & & & 1 & \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{array}\right) \quad w_{11} = \left(\begin{array}{cccc|cccc} 0 & & & & -1 & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & & 0 & & & \\ \hline 1 & & & & 0 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \\ & & & & & & & & 0 \end{array}\right)$$

($w_{10} = s_1$*). Note that for* $n = 1$*,* w_{10} *does not exist. Then,*

$$K = B \cup (Bw_{10}B) \cup (Bw_{11}B) \quad \textit{for } n \geq 2$$

and

$$K = B \cup (Bw_{11}B) \quad \textit{for } n = 1.$$

Proof. See appendix.

At this point, we give a brief description of the process. First, using the preceding Bruhat decomposition as a starting point, we obtain a decomposition $K = \cup_{\alpha \in S} P_0 \alpha B_\chi$, so that $G = PK = \cup_{\alpha \in S} P \alpha B_\chi$ (S finite). Thus, $\Phi \in V^{B_\chi}$ is determined by its values on $\alpha \in S$. The natural first attempt at a basis would then be $\{f_\alpha\}_{\alpha \in S}$, where

$$f_\alpha(g) = \begin{cases} \delta^{1/2}\chi(p) & \text{if } g = p\alpha b \in P\alpha B_\chi \\ 0 & \text{if } g \notin P\alpha B_\chi. \end{cases}$$

However, these are not always well-defined. Thus, once we have determined a set S of representatives of $P_0 \backslash K / B_\chi$, we must determine which of the α's in S actually "work" .

Case 1:

In this case, $B_\chi = B$, so we can just use the Bruhat decomposition $K = \cup_{(i,j)} P_0 w_{ij} B$ from the lemma. We claim that

$$f_{ij}(g) = \begin{cases} \delta^{1/2}\chi(p) & \text{if } g = p w_{ij} b \in P w_{ij} B \\ 0 & \text{if } g \notin P w_{ij} B. \end{cases}$$

are all well-defined, hence constitute a basis for V^B. We remark that $\{(i,j)\} = \{(0,0),(1,0),(1,1)\}$ for $n \geq 2$, $\{(0,0),(1,1)\}$ for $n = 1$.

We check well-definedness. As the double-cosets are disjoint, we have only to show that $p w_{ij} b = p' w_{ij} b' \Longrightarrow \delta^{1/2}\chi(p) = \delta^{1/2}\chi(p')$, or equivalently, $\delta^{1/2}\chi(p'^{-1}p) = 1$. However, $p'^{-1}p = w_{ij} b' b^{-1} w_{ij}^{-1} \in (w_{ij} B w_{ij}^{-1}) \cap P \subset P_0$, so this holds. Thus, we have a basis for V^B.

Case 2:

In this case, $B_\chi = U_1^- M_\chi U_0$, with $M_\chi = M_0 \cap \ker \chi$. We claim that $K = \cup_{i,j} P_0 w_{ij} B_\chi$. In particular, suppose

$$\epsilon = \left(\begin{array}{cccc|cccc} \epsilon & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline & & & & \epsilon^{-1} & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array} \right)$$

satisfies $\chi(\epsilon) = -1$. Then, we have $B = B_\chi \cup (\epsilon B_\chi)$. Thus, we get $K = \cup_{i,j} P_0 w_{ij} (B_\chi \cup (\epsilon B_\chi))$. However, since $w_{ij} \epsilon w_{ij}^{-1} \in P_0$, we have $P_0 w_{ij} B_\chi = P_0 w_{ij}(\epsilon B_\chi) = P_0 w_{ij} B$, and the claim holds.

To address the question of well-definedness, we start with the following lemma.

LEMMA **2.2.2** *Let $\Phi \in V^{B_\chi}$. Then, if $i \neq j$, $\Phi(w_{ij}) = 0$, i.e., $\Phi(w_{10}) = 0$.*

Proof. Let ϵ be as above. Then, since $\chi(\epsilon) = -1$, we have

$$\begin{aligned} -\Phi(w_{10}) &= \Phi(\epsilon w_{10}) \\ &= \Phi(w_{10}\epsilon') \\ &= \Phi(w_{10}), \end{aligned}$$

where

$$\epsilon' = w_{10}^{-1}\epsilon w_{10} = \left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ & \epsilon & & & & & & \\ & & 1 & & & & & \\ & & & \ddots & & & & \\ & & & & 1 & & & \\ \hline & & & & & 1 & & \\ & & & & & & \epsilon^{-1} & \\ & & & & & & & 1 \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{array}\right).$$

Thus, $\Phi(w_{10}) = 0$. Note that for $n = 1$, this lemma is unnecessary.

Now, any $\Phi \in V^{B_\chi}$ must be zero on the double-coset associated to w_{10}. We claim that $\{f_i\}_{i=0,1}$ below constitute a basis for V^{B_χ}:

$$f_i(g) = \begin{cases} \delta^{1/2}\chi(p) & \text{for } g = pw_{ii}b \in Pw_{ii}B_\chi \\ 0 & \text{if } g \notin Pw_{ii}B_\chi. \end{cases}$$

These will be a basis once we know that they are well-defined.

We now check that these are well-defined. Suppose $pw_{ii}b = p'w_{ii}b'$, with $p, p' \in P$ and $b, b' \in B_\chi$. We must show that $\delta^{1/2}\chi(p) = \delta^{1/2}\chi(p')$, or equivalently, $\delta^{1/2}\chi(p'^{-1}p) = 1$. Observe that $p'^{-1}p = w_{ii}b'b^{-1}w_{ii}^{-1} \in M_\chi U_0$ (w_{00}, w_{11} normalize M_χ). The conclusion follows.

Case 3:

In this case, $B_\chi = U_h^- M_\chi U_0$. By the Bruhat decomposition, $K = \cup_{i,j} P_0 w_{ij} B$. Now, $B = \cup_{u^-,m} u^- m B_\chi$, where u^-, m run over representatives of U^-/U_h^- and

M_0/M_χ (M_0 normalizes U_h^-). Next, since any $m \in M_0$ may be written in the form

$$\left(\begin{array}{cccc|cccc} \alpha & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline & & & & \alpha^{-1} & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array}\right) m',$$

with $m' \in M_\chi$, we get $B = \cup_{\alpha,u^-} u^- \alpha B_\chi = \cup_{\alpha,u^-} u^- \alpha B_\chi$ as α runs over $\mathcal{O}^\times / \ker \chi$ and u^- runs over U_1^- / U_h^-. Note that we are not claiming this is a disjoint union (it does not matter- see the upcomming lemma).

If we combine this with the Bruhat decomposition, we get

$$\begin{aligned} K &= \cup_{i,j,\alpha,u^-} P_0 w_{ij} \alpha u^- B_\chi \\ &= \cup_{i,j,u^-} P_0 w_{ij} u^- B_\chi \end{aligned}$$

since $w_{ij} \alpha w_{ij}^{-1} \in P_0$. We now examine $w_{ij} U^- w_{ij}^{-1}$– this allows us to reduce the number of u^-'s needed in the above expression (e.g. some of the u^-'s used above will conjugate into P_0, hence be unnecessary). We do the two nontrivial cases:

$$w_{10} \left(\begin{array}{cc|cc} 1 & & & \\ w_1 & & & \\ \vdots & I_{n-1} & & \\ w_{n-1} & & & \\ \hline \text{x} \quad y_1 \ \cdots \ y_{n-1} & & 1 \ -w_1 \ \cdots \ -w_{n-1} & \\ y_1 & & & \\ \vdots & & & I_{n-1} \\ y_{n-1} & & & \end{array}\right) {w_{10}}^{-1} = \left(\begin{array}{ccccc|cccc} 1 & w_1 & & & & & & & \\ & 1 & & & & & & & \\ & w_2 & 1 & & & & & & \\ & \vdots & & \ddots & & & & & \\ & w_{n-1} & & & 1 & & & & \\ \hline & y_1 & & & & 1 & & & \\ y_1 & x & y_2 & \cdots & y_{n-1} & -w_1 & 1 & \cdots & -w_{n-1} \\ & y_2 & & & & & & & \\ & \vdots & & & & & & \ddots & \\ & y_{n-1} & & & & & & & 1 \end{array}\right)$$

$$w_{11}\left(\begin{array}{cccc|cccc}
1 & & & & & & & \\
w_1 & & & & & & & \\
\vdots & & I_{n-1} & & & & & \\
w_{n-1} & & & & & & & \\
\hline
x & y_1 & \cdots & y_{n-1} & 1 & -w_1 & \cdots & -w_{n-1} \\
y_1 & & & & & & & \\
\vdots & & & & & & I_{n-1} & \\
y_{n-1} & & & & & & &
\end{array}\right) w_{11}^{-1} = \left(\begin{array}{cccc|cccc}
1 & -y_1 & \cdots & -y_{n-1} & x & w_1 & \cdots & w_{n-1} \\
 & & & & w_1 & & & \\
 & & I_{n-1} & & \vdots & & & \\
 & & & & w_{n-1} & & & \\
\hline
 & & & & 1 & & & \\
 & & & & y_1 & & & \\
 & & & & \vdots & & I_{n-1} & \\
 & & & & y_{n-1} & & &
\end{array}\right)$$

Of course, w_{00} acts trivially. We now write down our decomposition more explicitly, removing unnecessary duplications:

$$K = (\cup_{u^-} P_0 w_{00} u^- B_\chi) \cup (\cup_y P_0 w_{10} y B_\chi) \cup (P_0 w_{11} B_\chi)$$

where u^- runs over U_1^-/U_h^- and

$$y = \left(\begin{array}{ccc|ccc}
1 & & & & & \\
 & \ddots & & & & \\
 & & 1 & & & \\
\hline
 & y & & 1 & & \\
y & & & & \ddots & \\
 & & & & & 1
\end{array}\right)$$

with y running over $\mathcal{P}/\mathcal{P}^h$.

LEMMA **2.2.3** *If u^- (resp. y) is a double-coset representative with u^- (resp. y) nonzero* mod $\mathcal{P}^h$ *and $\Phi \in V^{B_\chi}$, then*

$$\Phi(w_{00}u^-) = 0 \quad (resp. \Phi(w_{10}y) = 0).$$

Proof. The general style of the argument is as follows (using u^-'s for u^- or y): we have

$$\Phi(w_{ij}u^-) = \Phi(w_{ij}u^- u) \quad u \in U_0$$

We find a u so that $u^- u \equiv p\hat{u}^- \bmod \mathcal{P}^h$ with $u^- \equiv \hat{u}^- \bmod \mathcal{P}^h$ and $w_{ij} p w_{ij}^{-1} \in P$ but not in $\ker \chi$. Then

$$\begin{aligned}\Phi(w_{ij}u^-) &= \Phi(w_{ij}p\hat{u}^-) \\ &= \Phi((w_{ij}pw_{ij}^{-1})w_{ij}\hat{u}^-) \\ &= \chi(w_{ij}pw_{ij}^{-1})\Phi(w_{ij}u^-)\end{aligned}$$

Since $\chi(w_{ij}pw_{ij}^{-1}) \neq 1$, we must have $\Phi(w_{ij}u^-) = 0$. The difficult part it to find an explicit choice of u (it will depend on u^-). There will be three cases.

To expedite matters, we use the following notation: if X is a matrix, $\max(X)$ is the largest norm that occurs as the norm of an entry of X. Let

$$u^- = u_W^- u_X^- u_Y^- = \left(\begin{array}{cc|cc} 1 & & & \\ W & I_{n-1} & & \\ \hline \hat{X} & {}^TY & 1 & -{}^TW \\ Y & & & I_{n-1} \end{array}\right)$$

with $\hat{X} = X - W^TY$.

Case A: $W, X \equiv 0 (\bmod \mathcal{P}^h)$.

Let

$$u = \left(\begin{array}{cc|cc} 1 & & & {}^TZ \\ & I_{n-1} & Z & \\ \hline & & 1 & \\ & & & I_{n-1} \end{array}\right).$$

We will place conditions on Z as we go along. For now, the entries of Z are in $\mathcal{O}$.

Now, letting $I = I_{n-1}$,

$$\begin{aligned}
\Phi(w_{ij}u^-) &= \Phi(w_{ij}u^-u) \\
&= \Phi\left(w_{ij} \left(\begin{array}{cc|cc} 1 & & & {}^TZ \\ & I & Z & \\ \hline & {}^TY & 1+{}^TYZ & \\ Y & & & I+Y^TZ \end{array} \right) \right) \\
&= \Phi\left(w_{ij} \left(\begin{array}{cc|cc} {}^T(1+{}^TYZ)^{-1} & & & {}^TZ \\ & {}^T(I+Y^TZ)^{-1} & Z & \\ \hline & & 1+{}^TYZ & \\ & & & I+Y^TZ \end{array} \right) \right. \\
& \qquad \left. \left(\begin{array}{cc|cc} 1 & & & \\ & I & & \\ \hline & (1+{}^TYZ)^{-1}\,{}^TY & 1 & \\ Y^T(I+{}^TYZ)^{-1} & & & I \end{array} \right) \right).
\end{aligned}$$

Next, suppose that the largest entry of Y is in $\mathcal{P}^l \backslash \mathcal{P}^{l+1}$ (i.e. $\max Y = q^{-l}$), $1 \leq l \leq h$. Then, we insist that the largest entry of Z lie in $\mathcal{P}^{h-l-1}$ (i.e. $\max Z = q^{-h+l+1}$). We claim that this implies

$$Y^T(1 +^T YZ)^{-1} \equiv Y \bmod \mathcal{P}^h$$

$$(1 +^T YZ)^{-1T} Y \equiv^T Y \bmod \mathcal{P}^h$$

Observe that the largest entry of ${}^TYZ$ must lie in $\mathcal{P}^{h-1}$. Thus, ${}^T(1 +^T YZ)^{-1} \in 1 + \mathcal{P}^{h-1}$. Consequently, $Y^T(1 +^T YZ)^{-1} \equiv Y \bmod \mathcal{P}^h$ (since Y has entries in $\mathcal{P}$).

The other half of the claim is similar. In particular, the second matrix in Φ above is congruent to u^-mod $\mathcal{P}^h$. Thus we have,

$$(*)\qquad \Phi(w_{ij}u^-) = \Phi\left(w_{ij} \left(\begin{array}{cc|cc} {}^T(1+{}^TYZ)^{-1} & & & {}^TZ \\ & {}^T(I+Y^TZ)^{-1} & Z & \\ \hline & & 1+{}^TYZ & \\ & & & I+Y^TZ \end{array} \right) u^- \right).$$

Subcase 1: $w_{ij} = w_{11}$

Then, $(*)$ above gives us

$$\begin{aligned} \Phi(u^-) &= \chi({}^T(1+{}^TYZ)^{-1})\Phi(u^-) \\ &= \chi^{-1}(1+{}^TYZ)\Phi(u^-) \end{aligned}$$

($\delta^{1/2} \equiv 1$ on $P \cap K$). Now, we allow Z to run over its prescribed set ($\max Z \leq q^{-h+l+1}$). Then, $1+{}^TYZ$ will cover all of $1+\mathcal{P}^{h-1}$ Recall that $\chi|_{1+\mathcal{P}^h} \equiv 1$ but $\chi|_{1+\mathcal{P}^h-1} \not\equiv 1$. In particular, there exist choices of Z giving $1+{}^TYZ \notin \ker\chi$. Taking such a Z,

$$\Phi(u^-) = \chi^{-1}(1+{}^TYZ)\Phi(u^-)$$

implies $\Phi(u^-) = 0$, as claimed.

Subcase 2: $w_{ij} = w_{10}$

We note that case A is the only one in which w_{10} occurs for the proof of the lemma. In this case, ${}^TY = (y\ 0 \ldots 0)$. We now further restrict Z to be of a similar form: ${}^TZ = (z\ 0 \ldots 0)$. Then, $(*)$ becomes

$$
\begin{aligned}
\Phi(w_{10}u^-) &= \Phi\left((w_{10})\left(\begin{array}{cccc|cccc}
(1+yz)^{-1} & & & & & z & 0 \ldots & 0 \\
 & (1+yz)^{-1} & & & z & & & \\
 & & 1 & & 0 & & & \\
 & & & \ddots & \vdots & & & \\
 & & & 1 & 0 & & & \\
\hline
 & & & & (1+yz) & & & \\
 & & & & & (1+yz) & & \\
 & & & & & & 1 & \\
 & & & & & & & \ddots \\
 & & & & & & & 1
\end{array}\right) u^-\right) \\
&= \Phi\left(\left(\begin{array}{cccc|cccc}
(1+yz)^{-1} & & & & & z & 0 \ldots & 0 \\
 & (1+yz)^{-1} & & & z & & & \\
 & & 1 & & 0 & & & \\
 & & & \ddots & \vdots & & & \\
 & & & 1 & 0 & & & \\
\hline
 & & & & (1+yz) & & & \\
 & & & & & (1+yz) & & \\
 & & & & & & 1 & \\
 & & & & & & & \ddots \\
 & & & & & & & 1
\end{array}\right) w_{10}u^-\right) \\
&= \chi^{-1}(1+yz)\Phi(w_{10}u^-)
\end{aligned}
$$

Again, as we let z run over $\mathcal{P}^{h-l-1}$, $1+yz$ runs over $1+\mathcal{P}^{h-1}$. In particular, we can find a z so that $\chi^{-1}(1+yz) \neq 1$. Thus, $\Phi(w_{10}u^-) = 0$, as claimed. This finishes case A.

Case B: $\max X \geq \max W$

We note that in this case, we are dealing with $w_{ij} = w_{00}$ only.

Write $U_Z = \left(\begin{array}{cc|cc} 1 & & Z & \\ & I & & \\ \hline & & 1 & \\ & & & I \end{array}\right)$. We restrict $Z \in \mathcal{O}$ for now. More restrictions/specifications about Z will be made as we proceed.

Suppose that $X \in \mathcal{P}^l \backslash \mathcal{P}^{l+1}$. Then, we will want $Z \in \mathcal{P}^{h-l-1}$. We compute:

$$\begin{aligned}
\Phi(u^-) &= \Phi(u^-_X u^-_Y u^-_W) \\
&= \Phi(u^-_X u^-_Y u^-_W u_Z) \\
&= \Phi\left(u^-_X u^-_Y u_Z u^-_W \left(\begin{array}{cc|cc} 1 & & & Z^TW \\ & I & WZ & -WZ^TW \\ \hline & & 1 & \\ & & & I \end{array}\right)\right) \\
&= \Phi(u^-_X u^-_Y u_Z u^-_W) \\
&= \Phi\left(u^-_X u_Z u^-_Y \left(\begin{array}{cc|cc} 1 & -Z^TY & & \\ & I & & \\ \hline & & 1 & \\ & -YZ^TY & -YZ & I \end{array}\right) u^-_W \right)
\end{aligned}$$

Now, if we conjugate this matrix around u^-_W, we get

$$\left(\begin{array}{cc|cc} 1 & -Z^TY & & \\ & I & & \\ \hline & & 1 & \\ & YZ^TY & YZ & I \end{array}\right)\left(\begin{array}{cc|cc} 1 & & & \\ W & I & & \\ \hline & & 1 & -{}^TW \\ & & & I \end{array}\right)\left(\begin{array}{cc|cc} 1 & Z^TY & & \\ & I & & \\ \hline & & 1 & \\ & -YZ^TY & -YZ & I \end{array}\right)$$

$$
\begin{aligned}
&= \left(\begin{array}{cc|cc}
1 - Z^TYW & -Z^TYWZ^TY & & \\
W & I + WZ^TY & & \\
\hline
 & {}^TWYZ^TY & 1 + {}^TWYZ & -{}^TW \\
YZ^TYW & YZ({}^TYW + {}^TWY)Z^TY & YZ^TWYZ & I - YZ^TW
\end{array}\right) \\
&\equiv \left(\begin{array}{cc|cc}
1 & & & \\
W & I & & \\
\hline
 & & 1 & -{}^TW \\
 & & & I
\end{array}\right) \bmod \mathcal{P}^h
\end{aligned}
$$

since $\max X \geq \max W$ and the constraint $Z \in \mathcal{P}^{h-l-1}$ ensures that ZX, ZW, etc., lie in $\mathcal{P}^{h-1}$. Multiplying by Y then puts them in $\mathcal{P}^h$.

Thus, we get

$$
\begin{aligned}
\Phi(u^-) &= \Phi(u_X^- u_Z u_Y^- u_W^-) \\
&= \Phi\left(\left(\begin{array}{cc|cc}
1 & & Z & \\
 & I & & \\
\hline
X & & 1 + XZ & \\
 & & & I
\end{array}\right) u_Y^- u_W^- \right) \\
&= \Phi\left(\left(\begin{array}{cc|cc}
(1 + XZ)^{-1} & & Z & \\
 & I & & \\
\hline
 & & 1 + XZ & \\
 & & & I
\end{array}\right) \left(\begin{array}{cc|cc}
1 & & & \\
 & I & & \\
\hline
(1 + XZ)^{-1} & & 1 & \\
 & & & I
\end{array}\right) u_Y^- u_W^- \right)
\end{aligned}
$$

since $XZ \in \mathcal{P}^{h-1}$, $(1+XZ)^{-1}X \equiv X \bmod \mathcal{P}^h$. Thus,

$$\begin{aligned}\Phi(u^-) &= \Phi\left(\left(\begin{array}{cc|cc} (1+XZ)^{-1} & & Z & \\ & I & & \\ \hline & & 1+XZ & \\ & & & I \end{array}\right) u_X^- u_Y^- u_W^-\right) \\ &= \chi^{-1}(1+XZ)\Phi(u^-)\end{aligned}$$

The rest of the argument is similar to case A. As Z runs over $\mathcal{P}^{h-l-1}$, XZ runs over $\mathcal{P}^{h-1}$. Then, $\Phi(u^-) = \chi^{-1}(1+XZ)\Phi(u^-)$ for all such Z implies $\Phi(u^-) = 0$, since χ is not constant on $1+\mathcal{P}^{h-1}$. This finishes case B.

Case C: $\max W > \max X$

Write $u_Z = \left(\begin{array}{cc|cc} 1 & Z & & \\ & I & & \\ \hline & & 1 & \\ & & -{}^TZ & I \end{array}\right)$. Again, if the largest entry of W is in $\mathcal{P}^l \setminus \mathcal{P}^{l+1}$, we will want the largest entry of Z to be in $\mathcal{P}^{h-l-1}$.

We compute:

$$\begin{aligned}\Phi(u^-) &= \Phi(u_W^- u_X^- u_Y^-) \\ &= \Phi(u_W^- u_X^- u_Y^- u_Z) \\ &= \Phi\left(u_W^- u_X^- u_Z u_Y^- \left(\begin{array}{cc|cc} 1 & & & \\ & I & & \\ \hline & & 1 & \\ & {}^T(YZ)+YZ & & I \end{array}\right)\right) \\ &= \Phi(u_W^- u_X^- u_Z u_Y^-)\end{aligned}$$

since the last matrix is in $M_\chi \subset B_\chi$. Continuing,

$$\Phi(u^-) = \Phi\left(u_W^- u_Z u_X^- \left(\begin{array}{cc|cc} 1 & & & \\ & I & & \\ \hline & XZ & 1 & \\ {}^TZX & {}^TZXZ & & I \end{array} \right) u_Y^- \right)$$

Since $Z \subset \mathcal{P}^{h-l-1}$ and $\max W > \max X$, the matrix above lies in $\mathcal{P}^h$. Thus,

$$\begin{aligned}
\Phi(u^-) &= \Phi(u_W^- u_Z u_X^- u_Y^-) \\
&= \Phi\left(\left(\begin{array}{cc|cc} 1 & Z & & \\ W & I+WZ & & \\ \hline & & 1+ZW & -{}^TW \\ & & -{}^TZ & I \end{array} \right) u_X^- u_Y^- \right) \\
&= \Phi\left(\left(\begin{array}{cc|cc} (1+ZW)^{-1} & Z & & \\ & I+WZ & & \\ \hline & & 1+ZW & \\ & & -{}^TZ & {}^T(I+WZ)^{-1} \end{array} \right) \right. \cdot \\
&\qquad \left. \left(\begin{array}{cc|cc} 1 & & & \\ (I+WZ)^{-1}W & I & & \\ \hline & & 1 & -{}^TW^T(I+WZ)^{-1} \\ & & & I \end{array} \right) u_X^- u_Y^- \right)
\end{aligned}$$

Again, we have $WZ \subset \mathcal{P}^{h-1}$, so the second matrix is congruent to u_W^- mod $\mathcal{P}^h$. Continuing,

$$\begin{aligned}\Phi(u^-) &= \Phi\left(\left(\begin{array}{cc|cc} (1+ZW)^{-1} & Z & & \\ & I+WZ & & \\ \hline & & 1+ZW & \\ & & -{}^{T}Z & {}^{T}(I+WZ)^{-1} \end{array}\right) u_W^- u_X^- u_Y^-\right) \\ &= \chi^{-1}(1+ZW)\Phi(u^-)\end{aligned}$$

As before, under the conditions given, we can arrange that $\chi^{-1}(1+ZW) \neq 1$. Thus, we must have $\Phi(u^-) = 0$. This concludes Case C and the lemma.

Let us now summarize where we are at in Case 3 (i.e., order (χ_0)>2). The above lemma tells us f_α is not well-defined (cf. p.16) for all our double-coset representatives of $P\backslash G/B_\chi$, except possibly $\alpha = w_{00}, w_{10}, w_{11}$. We claim that $f_{w_{10}}$ is not well-defined– the proof is exactly the same as in Case 2. This leaves us with the task of showing that f_α is well-defined for $\alpha = w_{00}, w_{11}$. First we check that $f_0 = f_{w_{00}}$ is well-defined. Suppose $pb = p'b' \in PB_\chi$. We must show that $\delta^{1/2}\chi(p) = \delta^{1/2}\chi(p')$, or equivalently, $\delta^{1/2}\chi(p'^{-1}p) = 1$. However, $p'^{-1}p = b'p^{-1} \in P \cap B_\chi = M_\chi U_0$, so $\delta^{1/2}\chi(p'^{-1}p) = 1$. Thus, f_0 is well-defined.

Finally, we check $f_1 = f_{w_{11}}$. Suppose $pw_{11}b = p'w_{11}b' \in Pw_{11}B_\chi$. We must show that $\delta^{1/2}\chi(p) = \delta^{1/2}\chi(p')$, or equivalently, $\delta^{1/2}\chi(p'^{-1}p) = 1$. However,

$$\begin{aligned} p'^{-1}p &= w_{11}b'b^{-1}w_{11} \\ &\in P \cap (w_{11}B_\chi w_{11}) \\ &= P \cap (U_0^- M_\chi U_h) \\ &= M_\chi U_h \end{aligned}$$

so $\delta^{1/2}\chi(p'^{-1}p) = 1$. Thus, f_1 is well-defined. This concludes Case 3.

We summarize our results in the following theorem:

THEOREM **2.2.4** *The following list gives a basis for* V^{B_χ}.

1. *(order of* χ_u*) = 1*

A basis for V^B consists of

$$f_{ij}(g) = \begin{cases} \delta^{1/2}\chi(p) & \textit{if } g = pw_{ij}b \in Pw_{ij}B \\ 0 & \textit{if } g \notin Pw_{ij}B \end{cases}$$

(note that $B_\chi = B$), for $(i,j) = (0,0),(1,0),(1,1)$. The pair(1,0) does not occur for $n = 1$.

2. *(order of χ_u) = 2*

   A basis for V^{B_χ} consists of

$$f_i(g) = \begin{cases} \delta^{1/2}\chi(p) & \textit{if } g = pw_{ii}b \in Pw_{ii}B_\chi \\ 0 & \textit{if } g \notin Pw_{ii}B_\chi \end{cases}$$

   for $i = 0,1$.

3. *(order of χ_u) > 2*

   A basis for V^{B_χ} consists of

$$f_i(g) = \begin{cases} \delta^{1/2}\chi(p) & \textit{if } g = pw_{ii}b \in Pw_{ii}B_\chi \\ 0 & \textit{if } g \notin Pw_{ii}B_\chi \end{cases}$$

   for $i = 0,1$.

REMARK **2.2.5** *Cases 2 and 3 look the same for the particular parabolic we're looking at. However, that is not the case in general. If the Levi $M \cong GL_k \times Sp_{2(n-k)}$ and χ is a character, identified with a character of M through the determinant, the following describes which f_α's are well-defined: (Cases 1, 2, 3 as above)*

1. $\alpha = w_{ij}$ *for all the* w_{ij}
2. $\alpha = w_{ii}$ $\quad i \in \{0, \ldots, k\}$
3. $\alpha = w_{00}, w_{kk}$.

Thus, the distinction exists in general.

NOTE **2.2.6** *It will be important, for future use, to realize that in Cases 2 and 3, if $\Phi \in V^{B_\chi}$ and $x \notin (P_{w_{00}}B_\chi) \cup (P_{w_{11}}B_\chi)$, then $\Phi(x) = 0$.*

2.3. Irreducibility when (order χ_u) > 2

We now start on the task of analyzing reducibility. The problem splits into the same three cases that occurred in the preceding section, classified by (order of χ_u). The purpose of the present section is to do the analysis for the case (order of χ_u)> 2.

Let us start by describing our analysis for this case. From the first section of this chapter, we need only look at $(\pi, H(G//B_\chi), V^{B_\chi})$. By the preceding section $\{f_0, f_1\}$ constitute a basis for V^{B_χ}. We let

$$\begin{aligned} \mathcal{A}_{w_{11}} &= (|B_\chi w_{11} B_\chi|)^{-1} \mathrm{char}_{B_\chi w_{11} B_\chi} \\ \mathcal{A}_m &= (|B_\chi m B_\chi|)^{-1} \mathrm{char}_{B_\chi m B_\chi} \end{aligned}$$

with

$$m = \left(\begin{array}{cccc|cccc} m & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline & & & & m^{-1} & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array} \right)$$

and $m \in \mathcal{O}^\times$. We observe that $\mathcal{A}_{w_{11}}, \mathcal{A}_m \in H(G//B_\chi)$ (in fact, they are in $H(K//B_\chi)$). We shall explicitly compute $\pi(\mathcal{A}_{w_{11}})$ and $\pi(\mathcal{A}_m)$ with respect to the basis $\{f_0, f_1\}$. It will then be clear that $\pi(\mathcal{A}_{w_{11}})$ and $\pi(\mathcal{A}_m)$ admit no common proper invariant subspace. Therefore, the representation $(\pi, H(G//B_\chi), V^{B_\chi})$ is irreducible. This tells us that (π, G, V) is irreducible.

Lemma **2.3.1** *With respect to the basis* $\{f_0, f_1\}$,

$$\pi(\mathcal{A}_m) = \begin{pmatrix} \chi(m) & 0 \\ 0 & \chi^{-1}(m) \end{pmatrix}.$$

Proof. Let w be w_{00} or w_{11} (by Note 2.2.6, these are the only double-coset representatives of $P \backslash G / B_\chi$ on which $\Phi \in V^{B_\chi}$ can be nonzero). Then,

$$\begin{aligned}\pi(\mathcal{A}_m)\Phi(w) &= \int_{B_\chi} \Phi(wbm)db \\ &= \int_{B_\chi} \Phi(wm)db \quad \text{(since } m \text{ normalizes } U_h^-,\ M_\chi, \text{ and } U_0). \\ &= \Phi(wm)\end{aligned}$$

Thus, $\pi(\mathcal{A}_m)\Phi(w_{00}) = \Phi(w_{00}m) = \chi(m)\Phi(w_{00})$ and $\pi(\mathcal{A}_m)\Phi(w_{11}) = \Phi(w_{11}m) = \Phi(m^{-1}w_{11}) = \chi^{-1}(m)\Phi(w_{11})$. This tells us that

$$\pi(\mathcal{A}_m) = \begin{pmatrix} \chi(m) & 0 \\ 0 & \chi^{-1}(m) \end{pmatrix}.$$

LEMMA **2.3.2** *With respect to the basis* $\{f_0, f_1\}$,

$$\pi(\mathcal{A}_{w_{11}}) = \begin{pmatrix} 0 & 1 \\ \chi(-1)q^{(-2n+1)h} & 0 \end{pmatrix}.$$

Proof. Again, let w be w_{00} or w_{11}. Then,

$$\begin{aligned}\pi(\mathcal{A}_{w_{11}})\Phi(w) &= \int_{B_\chi} \Phi(wbw_{11})db \\ &= \int_{U_0}\int_{M_\chi}\int_{U_h^-} \Phi(wumu^-w_{11})du^-dmdu \\ &= \int_{U_0} \Phi(wuw_{11})du\end{aligned}$$

since $w_{11}^{-1}M_\chi w_{11} \subset B_\chi$ and $w_{11}^{-1}U_h^- w_{11} \subset B_\chi$.

1. $w = w_{00}$

$$\begin{aligned}\pi(\mathcal{A}_{w_{11}})\Phi(w_{00}) &= \int_{U_0} \Phi(uw_{11})du \\ &= \Phi(w_{11}).\end{aligned}$$

2. $w = w_{11}$

$$\begin{aligned}\pi(\mathcal{A}_{w_{11}})\Phi(w_{11}) &= \int_{U_0} \Phi(w_{11}uw_{11})du \\ &= \chi(-1)\int_{U_0^-} \Phi(u^-)du^-.\end{aligned}$$

We write $u^- = \left(\begin{array}{cc|cc} 1 & & & \\ W & I & & \\ \hline \hat{X} & {}^TY & 1 & -{}^TW \\ Y & & & I \end{array}\right)$ as before. Then,

$$\pi(\mathcal{A}w_{11})\Phi(w_{11}) = \chi(-1)[\int_{\hat{X}\in\mathcal{P}}\int \Phi(u^-)du^- + \int_{\hat{X}\in\mathcal{O}^\times}\int \Phi(u^-)du^-]$$

We now observe that $\int_{\hat{X}\in\mathcal{P}}\int f(u^-)du^- = q^{(-2n+1)h}\Phi(w_{00})$. This follows from the fact (see the Appendix) that if all of the entries of W, Y are also in $\mathcal{P}$, then $u^- \in Pw_{00}B_\chi$ (this case contributes $q^{(-2n+1)h}\Phi(w_{00})$). If not, then $u^- \in [(Pw_{00}B) \cup (Pw_{10}B)]\backslash(Pw_{00}B_\chi)$, so $\Phi(u^-) = 0$ by Note 2.2.6. It now remains to show that the second integral is zero.

To see that the second integral is zero, we factor u^- as

$$u^- = \left(\begin{array}{cc|cc} \hat{X}^{-1} & \hat{X}^{-1}{}^TY & 1 & \hat{X}^{-1}{}^TW \\ & I - W\hat{X}^{-1}{}^TY & W & W\hat{X}^{-1}{}^TW \\ \hline & & \hat{X} & \\ & -Y\hat{X}^{-1}{}^TY & Y & I + Y\hat{X}^{-1}{}^TW \end{array}\right).$$

$$w_{11}\left(\begin{array}{cc|cc} 1 & \hat{X}^{-1}{}^TY & -\hat{X}^{-1} & \hat{X}^{-1}{}^TW \\ & I & -W\hat{X}^{-1} & \\ \hline & & 1 & \\ & & -Y\hat{X}^{-1} & I \end{array}\right) \in Pw_{11}U_0$$

Thus,

$$\begin{aligned} \int_{\hat{X}\in\mathcal{O}^\times}\int \Phi(u^-)du^- &= \int_{\hat{X}\in\mathcal{O}^\times} \chi(\hat{X}^{-1})\Phi(w_{11})du^- \\ &= 0 \end{aligned}$$

since χ_u is nontrivial. This finishes the lemma.

It is clear that $\pi(\mathcal{A}_m)$ and $\pi(\mathcal{A}_{w_{11}})$ admit no proper common invariant subspace. Thus $(\pi, H(G//B_\chi), V^{B_\chi})$ is irreducible, so (π, G, V) must be irreducible as well. We close this section with the following remark.

REMARK **2.3.3** *In the general case, where $P = MU$ with $M \cong GL_k \times Sp_{2(n-k)}$, a basis for V^{B_χ} when (order χ_u)> 2 consists of $\{f_\alpha\}$ for $\alpha = w_{00}, w_{kk}$. With respect to this basis,*

$$\begin{aligned}
\pi(\mathcal{A}_m) &= \begin{pmatrix} \chi(m) & 0 \\ 0 & \chi^{-1}(m) \end{pmatrix} \\
\pi(\mathcal{A}_{w_{kk}}) &= \begin{pmatrix} 0 & 1 \\ \chi(-1)^k q^{-kh(2n-\frac{3}{2}k+\frac{1}{2})} & 0 \end{pmatrix},
\end{aligned}$$

with $\mathcal{A}_m = (|B_\chi m B_\chi|)^{-1} char_{B_\chi m B_\chi}$ (m as before), $\mathcal{A}_{w_{kk}} = (|B_\chi w_{kk} B_\chi|)^{-1} char_{B_\chi w_{kk} B_\chi}$. The arguments are similar, except for the following complications. In order to compute $\pi(\mathcal{A}_{kk})\Phi(w_{kk})$, write

$$\begin{aligned}
\pi(\mathcal{A}_{kk})\Phi(w_{kk}) = \chi(-1)[\int_{rank_{\mathcal{P}}\hat{X}=0} \int \Phi(u^-)du^- &+ \int_{0<rank_{\mathcal{P}}\hat{X}<k} \int \Phi(u^-)du^- \\
&+ \int_{rank_{\mathcal{P}}\hat{X}=k} \int \Phi(u^-)du^-].
\end{aligned}$$

where $rank_{\mathcal{P}}\hat{X}$ is the rank of $\hat{X}$ when reduced mod $\mathcal{P}$. *The first integral gives the nonzero contribution, an in the case $k = 1$. The second integral is zero because $1 < rank_{\mathcal{P}}\hat{X} < k$ implies that $u^- \in Pw_{ij}B$ with $i \neq 0, k$– in particular, $u^- \notin Pw_{00}B_\chi$ or $Pw_{kk}B_\chi$. Since $\Phi \equiv 0$ on these double-cosets, the integral is zero. The last integral is also zero, but here a lemma is needed:*

LEMMA **2.3.4** *Let $S = GL_k(\mathcal{O}) \cap \{symmetric\ matrices\}$. Suppose ρ is an irreducible representation of $GL_k(\mathcal{O})$ such that $\rho \not\cong \rho \circ \theta$, where θ is the automorphism given by $\theta(g) = ^T g^{-1}$. Then,*

$$\int_S \pi(g)dg = 0$$

We need to use this and some additional arguments in the general case as a substitute for $\int_{\hat{X}\in\mathcal{O}^\times} \chi(\hat{X}^{-1})\Phi(w_{11}) = 0$ in the case $k = 1$. Thus, we have irreducibility in general for (order χ_u)> 2.

REMARK: We actually have something stronger than $(\pi, H(G//B_\chi), V^{B_\chi})$ irreducible, namely $(\pi, H(K//B_\chi), V^{B_\chi})$ is irreducible. This tells us something about finite groups.

We start by noting that $K/K_h \cong G(\mathcal{O}/\mathcal{P}^h)$. We can interpret χ as a character on $M(\mathcal{O}/\mathcal{P}^h)$ (actually, χ_u is all that will matter). If we look at $\tau = \mathrm{Ind}_{P(\mathcal{O}/\mathcal{P}^h)}^{G(\mathcal{O}/\mathcal{P}^h)}\chi$, we can ask about reducibility. This corresponds to analyzing reducibility for $(\tau, H(G(\mathcal{O}/\mathcal{P}^h)//P(\mathcal{O}/\mathcal{P}^h)), X^{P(\mathcal{O}/\mathcal{P}^h)})$, where X is the induced space. But, this is isomorphic to $(\pi, H(K//B_\chi, V^{B_\chi})$ above, hence irreducible.

2.4. Irreducibility conditions for (order χ_u) $= 2$

In this section we shall analyze irreducibility for the case when (order χ_u) $= 2$. The analysis is similar to the previous case, but more complicated (e.g., there are some reducibility points). The first step is to compute a pair of operators, $\pi(\mathcal{A}_\gamma)$, $\pi(\mathcal{A}_{w_{11}})$ on V^{B_χ}, where

$$\begin{aligned} \mathcal{A}_{w_{11}} &= (|B_\chi w_{11} B_\chi|)^{-1}\mathrm{char}_{B_\chi w_{11} B_\chi} \\ \mathcal{A}_\gamma &= (|B_\chi \gamma B_\chi|)^{-1}\mathrm{char}_{B_\chi \gamma B_\chi} \end{aligned}$$

with

$$\gamma = \left(\begin{array}{ccc|ccc} \varpi & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ \hline & & & \varpi^{-1} & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{array}\right).$$

We then see that these operators admit no common invariant subspace, except for $s = 0, i\pi/\mathrm{lnq}$. This gives us irreducibility except at those points. At $s = 0, i\pi/\mathrm{lnq}$, we can show reducibility by showing $\dim \mathrm{Hom}_G(\pi, \tilde{\pi}) = 2$. This requires some work, so we wait until section 2.7 to show the reducibility at these points. In this section, we just show irreducibility elsewhere.

We start by computing the operators.

LEMMA **2.4.1**

$$\pi(\mathcal{A}_\gamma) = \begin{pmatrix} q^{-s-n} & 0 \\ 0 & q^{-2n}q^{s+n} \end{pmatrix}$$

Proof. Let w denote w_{00} or w_{11}. Then, for $\Phi \in V_\chi^B$

$$\begin{aligned}
\pi(\mathcal{A}_\gamma)\Phi(w) &= \int_{B_\chi} \Phi(wb\gamma)db \\
&= \int_{M_\chi}\int_{U_0} \Phi(wum\gamma)dmdu \quad (\text{since } \gamma^{-1}U_1^-\gamma \subset U_1^-) \\
&= \int_{U_0} \Phi(wu\gamma)du \quad (\text{since } M_\chi \text{ commutes with } \gamma)
\end{aligned}$$

Case 1: $w = w_{00}$

$$\begin{aligned}
\pi(\mathcal{A}_\gamma)\Phi(w_{00}) &= \Phi(\gamma) \\
&= q^{-s-n}\Phi(w_{00})
\end{aligned}$$

Case 2: $w = w_{11}$

For notational convenience, denote $\pi(\mathcal{A}_\gamma)\Phi(w_{11})$ by S.

$$\begin{aligned}
S &= \int_{U_0} \Phi(w_{11}u\gamma)du \\
&= \int_{U_0} \Phi\left(w_{11}\gamma \left(\begin{array}{cccc|cccc}
1 & \varpi^{-1}w_2 & \ldots & \varpi^{-1}w_n & \varpi^{-2}x_1 & \varpi^{-1}x_2 & \ldots & \varpi^{-1}x_n \\
 & 1 & & & \varpi^{-1}x_2 & & & \\
 & & \ddots & & \vdots & & & \\
 & & & 1 & \varpi^{-1}x_n & & & \\
\hline
 & & & & 1 & & & \\
 & & & & -\varpi^{-1}w_2 & 1 & & \\
 & & & & \vdots & & \ddots & \\
 & & & & -\varpi^{-1}w_n & & & 1
\end{array} \right) \right) du
\end{aligned}$$

(conjugating γ around u)

$$
= q^{s+n}\int\limits_{U_0}\Phi\left(w_{11}\left(\begin{array}{cccc|cccc}
1 & \varpi^{-1}w_2 & \dots & \varpi^{-1}w_n & \varpi^{-2}x_1 & \varpi^{-1}x_2 & \dots & \varpi^{-1}x_n \\
 & 1 & & & \varpi^{-1}x_2 & & & \\
 & & \ddots & & \vdots & & & \\
 & & & 1 & \varpi^{-1}x_n & & & \\
\hline
 & & & & 1 & & & \\
 & & & & -\varpi^{-1}w_2 & 1 & & \\
 & & & & \vdots & & \ddots & \\
 & & & & -\varpi^{-1}w_n & & & 1
\end{array}\right)\right) du
$$

$$
= q^{s+n}\int\limits_{U_0}\Phi\left(\left(\begin{array}{cccc|cccc}
1 & & & & & & & \\
\varpi^{-1}x_2 & 1 & & & & & & \\
\vdots & & \ddots & & & & & \\
\varpi^{-1}x_n & & & 1 & & & & \\
\hline
\varpi^{-2}x_1 & \varpi^{-1}w_2 & \dots & \varpi^{-1}w_n & 1 & -\varpi^{-1}x_2 & \dots & -\varpi^{-1}x_n \\
\varpi^{-1}w_2 & & & & & 1 & & \\
\vdots & & & & & & \ddots & \\
\varpi^{-1}w_n & & & & & & & 1
\end{array}\right) w_{11}\right) du
$$

Now, write $M_\chi \cong M'_\chi \times M''_0$, where $M'_\chi = \ker\chi \subset F^\times$, $M''_0 = Sp_{(2n-2)}(\mathcal{O})$. If any of $x_2, \dots, x_n \in \mathcal{O}^\times$, we can conjugate by a matrix in the Weyl group of M''_0 (noting that M''_0 commutes with w_{11}) to arrange that $x_n \in \mathcal{O}^\times$. This will permute the w_i's (as above, if $m \in M''_0$ is the matrix, $\Phi(u^- w_{11}) = \Phi(m u^- w_{11} m^{-1}) = \Phi(m u^- m^{-1} w_{11})$, so we are just conjugating the unipotent matrix) thus, we can split up the integral as

$$
S = q^{-n+1}q^{s+n}\int\limits_{x_1}\int\limits_{w_i\in\mathcal{O}}\Phi\left(\left(\begin{array}{cccc|cccc}
1 & & & & & & & \\
 & \ddots & & & & & & \\
 & & \ddots & & & & & \\
 & & & 1 & & & & \\
\hline
\varpi^{-2}x_1 & \varpi^{-1}w_2 & \dots & \varpi^{-1}w_n & 1 & & & \\
\varpi^{-1}w_2 & & & & & \ddots & & \\
\vdots & & & & & & \ddots & \\
\varpi^{-1}w_n & & & & & & & 1
\end{array}\right) w_{11}\right) dx\,dw
$$

$$
+q^{s+n}\int\int\limits_{R}\Phi\left(\left(\begin{array}{cccc|cccc}
1 & & & & & & & \\
\varpi^{-1}x_2 & 1 & & & & & & \\
\vdots & & \ddots & & & & & \\
\varpi^{-1}x_n & & & 1 & & & & \\
\hline
\varpi^{-2}x_1 & \varpi^{-1}w_2 & \dots & \varpi^{-1}w_n & 1 & -\varpi^{-1}x_2 & \dots & -\varpi^{-1}x_n \\
\varpi^{-1}w_2 & & & & & 1 & & \\
\vdots & & & & & & \ddots & \\
\varpi^{-1}w_n & & & & & & & 1
\end{array}\right) w_{11}\right) dx\,dw
$$

where the first integral simplifies because $x_2, \dots, x_n \in \mathcal{P}$ and the second integral has domain $R' = \{(x_1, \dots, x_n, w_2, \dots, w_n) |$ at least one of $x_2, \dots, x_n$ is in $\mathcal{O}^\times\}$. We can decompose R' as $R' = R'_2 \cup \ldots \cup R'_n$ according to the minimum j for which $x_j \in \mathcal{O}^\times$. Then, after a suitable conjugation from M''_0, each R'_j will have $x_n \in \mathcal{O}^\times$, plus some conditions on the remaining x's. Denote the resulting sets by R_j and let $\int\limits_R = \int\limits_{R_2} + \ldots + \int\limits_{R_n}$ this gives the same result as integrating over R', but it is convenient to have arranged $x_n \in \mathcal{O}^\times$. Note that $|R_2| + \ldots + |R_n| = \frac{1-q^{-n+1}}{1-q^{-1}}$.

As above, we can conjugate the unipotent matrix in the second integral by

$$\left(\begin{array}{cccc|cccc} 1 & & & 0 & & & & \\ & 1 & & -x_n^{-1}x_2 & & & & \\ & & \ddots & \vdots & & & & \\ & & & -x_n^{-1}x_{n-1} & & & & \\ & & & 1 & & & & \\ \hline & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \\ & & & & 0 & x_n^{-1}x_2 & \ldots\, x_n^{-1}x_{n-1} & 1 \end{array}\right) \in M''_0$$

and split up the first integral according to whether any $w_i \in \mathcal{O}^\times$ or not (just as we split it up above according to whether any $x_2, \ldots, x_n \in \mathcal{O}^\times$). Using $\tilde{R}$ to denote the analogue of R for the w's (arranging for $w_2 \in \mathcal{O}^\times$), we get

$$S = q^{-2n+2} q^{s+n} \int\limits_{x \in \mathcal{O}} \Phi\left(\left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline \varpi^{-2}x & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{array}\right) w_{11}\right) dx$$

$$+ q^{-n+1} q^{s+n} \int\int\limits_{\tilde{R}} \Phi\left(\left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline \varpi^{-2}x & \varpi^{-1}w_2 \ldots & \varpi^{-1}w_n & 1 & & \\ \varpi^{-1}w_2 & & & & \ddots & \\ \vdots & & & & & \\ \varpi^{-1}w_n & & & & & 1 \end{array}\right) w_{11}\right) dx\,dw$$

$$+q^{s+n}\int\int_{R}\Phi\left(\left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ \varpi^{-1}x_n & & 1 & & & \\ \hline \varpi^{-2}x_1 & \varpi^{-1}w_2\ldots & \varpi^{-1}w_n & 1 & & -\varpi^{-1}x_n \\ \varpi^{-1}w_2 & & & & \ddots & \\ \vdots & & & & & \\ \varpi^{-1}w_n & & & & & 1 \end{array}\right) w_{11}\right) dx\,dw$$

Now, we conjugate by the following matrices in M_0''

for second integral, for third integral

$$\left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ & 1 & w_2^{-1}w_3\ldots & w_2^{-1}w_n & & & & \\ & & 1 & & & & & \\ & & & \ddots\;1 & & & & \\ \hline & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & -w_2^{-1}w_3 & \ddots & \\ & & & & & \vdots & & \\ & & & & & -w_2^{-1}w_n & & 1 \end{array}\right), \quad \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & 0 & 1 & & \\ & & x_n^{-1}w_2 & & \ddots & \\ & & \vdots & & & \\ 0 & x_n^{-1}w_2\ldots & x_n^{-1}w_n & & & 1 \end{array}\right)$$

We get

$$S = c_1\int_{x\in\mathcal{O}}\Phi\left(\left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline \varpi^{-2}x & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{array}\right) w_{11}\right) dx$$

$$+c_2\int_{x\in\mathcal{O}}\int_{w_2\in\mathcal{O}^\times}\Phi\left(\left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline \varpi^{-2}x & \varpi^{-1}w_2 & & 1 & & \\ \varpi^{-1}w_2 & & & & \ddots & \\ & & & & & 1 \end{array}\right) w_{11}\right) dw\,dx$$

$$+c_3 \int\limits_{x\in\mathcal{O}} \int\limits_{x_n\in\mathcal{O}^\times} \Phi\left(\left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & \ddots & & & & & \\ \varpi^{-1}x_n & & & 1 & & & & \\ \hline \varpi^{-2}x & & & & 1 & & & -\varpi^{-1}x_n \\ & & & & & \ddots & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array}\right) w_{11}\right) dw\,dx$$

If we conjugate the second integral by

$$\left(\begin{array}{ccccc|ccccc} 1 & & & & & & & & & -1 \\ & 0 & & & & & & & & \\ & & 1 & & & & & & & \\ & & & \ddots & & & & & & \\ & & & & 1 & & & & & \\ & & & & 0 & 1 & & & & \\ \hline & & & & 1 & 1 & & & & \\ & & & & & & 0 & & & \\ & & & & & & & 1 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 1 \\ & 1 & & & & & & & & 0 \end{array}\right) \in M_0'',$$

it becomes the same as the third. Thus,

$$S = c_1 \int\limits_{x\in\mathcal{O}} \Phi\left(\left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline \varpi^{-2}x & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array}\right) w_{11}\right) dx$$

$$+(c_2+c_3) \int\limits_{x\in\mathcal{O}} \int\limits_{w\in\mathcal{O}^\times} \Phi\left(\left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & \ddots & & & & & \\ \varpi^{-1}w & & & 1 & & & & \\ \hline \varpi^{-2}x & & & & 1 & & & -\varpi^{-1}w \\ & & & & & \ddots & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array}\right) w_{11}\right) dx\,dw$$

where $c_1 = q^{-2n+2}q^{s+n}$, $c_2 = q^{-n+1}\frac{1-q^{-n+1}}{1-q^{-1}}q^{s+n}$, and $c_3 = \frac{1-q^{-n+1}}{1-q^{-1}}q^{s+n}$ (from $|R|$,etc.). We now break up each of the integrals into three integrals according to whether $x \in \mathcal{P}^2$, $x \in \varpi\mathcal{O}^\times$, or $x \in \mathcal{O}^\times$ – write $x = \varpi\tilde{x}$ if $x \in \varpi\mathcal{O}^\times$, (so $\tilde{x} \in \mathcal{O}^\times$). We then Iwahori factor the matrix. Note that if $x \in \mathcal{P}^2$, x may be removed since

w_{11} will conjugate it into U_0. To save space, we let $\alpha = \varpi^{-1}wx^{-1}$, $\alpha' = \varpi wx^{-1}$, $\tilde{\alpha} = \varpi^{-1}w\tilde{x}^{-1}$, and $\tilde{\alpha}' = \varpi w^{-1}\tilde{x}^{-1}$ for the remainder of this computation. We get

$$
\begin{aligned}
S \; &= d_1\Phi(w_{11}) \\
&+d_2 \int_{\tilde{x}\in\mathcal{O}^\times} \Phi\left(\left(\begin{array}{cccc|cccc}
\varpi & & & & \tilde{x}^{-1} & & & \\
& 1 & & & & & & \\
& & \ddots & & & & & \\
& & & 1 & & & & \\ \hline
& & & & \varpi^{-1} & & & \\
& & & & & 1 & & \\
& & & & & & \ddots & \\
& & & & & & & 1
\end{array}\right)\left(\begin{array}{cccc|cccc}
& & & & -\tilde{x}^{-1} & & & \\
& 1 & & & & & & \\
& & \ddots & & & & & \\
& & & 1 & & & & \\ \hline
\tilde{x} & & & & \varpi & & & \\
& & & & & 1 & & \\
& & & & & & \ddots & \\
& & & & & & & 1
\end{array}\right)(w_{11})\right) d\tilde{x} \\
&+d_3 \int_{x\in\mathcal{O}^\times} \Phi\left(\left(\begin{array}{cccc|cccc}
\varpi^2 & & & & x^{-1} & & & \\
& 1 & & & & & & \\
& & \ddots & & & & & \\
& & & 1 & & & & \\ \hline
& & & & \varpi^{-2} & & & \\
& & & & & 1 & & \\
& & & & & & \ddots & \\
& & & & & & & 1
\end{array}\right)\left(\begin{array}{cccc|cccc}
& & & & -x^{-1} & & & \\
& 1 & & & & & & \\
& & \ddots & & & & & \\
& & & 1 & & & & \\ \hline
x & & & & \varpi^2 & & & \\
& & & & & 1 & & \\
& & & & & & \ddots & \\
& & & & & & & 1
\end{array}\right)(w_{11})\right) dx \\
&+d_4 \int_{w\in\mathcal{O}^\times} \Phi\left(\left(\begin{array}{ccccc|ccccc}
\varpi & & & & w^{-1} & & & & & \\
& 1 & & & & & & & & \\
& & \ddots & & & & & & & \\
& & & 1 & & & & & & \\
& & & & \varpi^{-1} & & & & & \\ \hline
& & & & & \varpi^{-1} & & & & \\
& & & & & & 1 & & & \\
& & & & & & & \ddots & & \\
& & & & & & & & 1 & \\
& & & & & -w^{-1} & & & & \varpi
\end{array}\right)\left(\begin{array}{ccccc|ccccc}
& & & & -w^{-1} & & & & & \\
& 1 & & & & & & & & \\
& & \ddots & & & & & & & \\
& & & 1 & & & & & & \\
w & & & & \varpi & & & & & \\ \hline
& & & & & \varpi & & & & -w \\
& & & & & & 1 & & & \\
& & & & & & & \ddots & & \\
& & & & & & & & 1 & \\
& & & & & w^{-1} & & & &
\end{array}\right)(w_{11})\right) dw \\
&+d_5 \int\int_{w,\tilde{x}\in\mathcal{O}^\times} \Phi\left(\left(\begin{array}{ccccc|ccccc}
\varpi & & & & \tilde{\alpha} & & & & & w \\
& 1 & & & & & & & & \\
& & \ddots & & & & & & & \\
& & & 1 & & & & & & \\
& & & & \tilde{x}^{-1} & & & & & \varpi^{-1}w^2 \\ \hline
& & & & & \varpi^{-1} & & & & \\
& & & & & & 1 & & & \\
& & & & & & & \ddots & & \\
& & & & & & & & 1 & \\
& & & & & -w^{-1} & & & & \tilde{x}
\end{array}\right)\left(\begin{array}{ccccc|ccccc}
& & & & -w^{-1} & & & & & \\
& 1 & & & & & & & & \\
& & \ddots & & & & & & & \\
& & & 1 & & & & & & \\
& & & & \tilde{x} & -w & & & & \\ \hline
\tilde{x} & & & & & \varpi & & & & -w \\
& & & & & & 1 & & & \\
& & & & & & & \ddots & & \\
& & & & & & & & 1 & \\
w^{-1} & & & & & \tilde{\alpha}' & & & &
\end{array}\right)(w_{11})\right) dw\,d\tilde{x} \\
&+d_6 \int\int_{w,x\in\mathcal{O}^\times} \Phi\left(\left(\begin{array}{cccc|cccc}
\varpi^2 & & & & x^{-1} & & & \alpha' \\
& 1 & & & & & & \\
& & \ddots & & & & & \\
& & & 1 & \alpha & & & \\ \hline
& & & & \varpi^{-2} & & & \\
& & & & & 1 & & \\
& & & & & & \ddots & \\
& & & & & & & 1
\end{array}\right)\left(\begin{array}{cccc|cccc}
& & & & -x^{-1} & & & \\
& 1 & & & & & & \\
& & \ddots & & & & & \\
& & & 1 & \alpha' & & & x^{-1}w^2 \\ \hline
x & & & & \varpi^2 & & & -\varpi w \\
& & & & & 1 & & \\
& & & & & & \ddots & \\
& & & & & & & 1
\end{array}\right)(w_{11})\right) dw\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= d_1'\Phi(w_{11}) \\
&+ d_2' \int_{\tilde{x}\in\mathcal{O}^\times} \Phi\left(\left(\begin{array}{cccc|cccc}
-\tilde{x}^{-1} & & & & & & & \\
 & 1 & & & & & & \\
 & & \ddots & & & & & \\
 & & & 1 & & & & \\
\hline
\varpi & & & & -\tilde{x} & & & \\
 & & & & & 1 & & \\
 & & & & & & \ddots & \\
 & & & & & & & 1
\end{array}\right)\right) d\tilde{x} \\
&+ d_3' \int_{x\in\mathcal{O}^\times} \Phi\left(\left(\begin{array}{cccc|cccc}
-x^{-1} & & & & & & & \\
 & 1 & & & & & & \\
 & & \ddots & & & & & \\
 & & & 1 & & & & \\
\hline
\varpi^2 & & & & -x & & & \\
 & & & & & 1 & & \\
 & & & & & & \ddots & \\
 & & & & & & & 1
\end{array}\right)\right) dx \\
&+ d_4' \int_{w\in\mathcal{O}^\times} \Phi\left(\left(\begin{array}{cccc|cccc}
 & & & -w^{-1} & & & & \\
1 & & & & & & & \\
 & \ddots & & & & & & \\
 & & 1 & & & & & \\
 & & & \varpi & & & & \\
\hline
\varpi & & & & & & & -w \\
 & & & & 1 & & & \\
 & & & & & \ddots & & \\
 & & & & & & 1 & \\
w^{-1} & & & & & & &
\end{array}\right)\right) dw \\
&+ d_5' \int \int_{w,\tilde{x}\in\mathcal{O}^\times} \Phi\left(\left(\begin{array}{cccc|ccccc}
 & & & -w^{-1} & & & & & \\
1 & & & & & & & & \\
 & \ddots & & & & & & & \\
 & & 1 & & & & & & \\
-w & & & \tilde{x} & & & & & \\
\hline
\varpi & & & & -\tilde{x} & & & & -w \\
 & & & & & 1 & & & \\
 & & & & & & \ddots & & \\
 & & & & & & & 1 & \\
\varpi w^{-1}\tilde{x}^{-1} & & & & -w^{-1} & & & &
\end{array}\right)\right) dw\,d\tilde{x} \\
&+ d_6' \int \int_{w,\tilde{x}\in\mathcal{O}^\times} \Phi\left(\left(\begin{array}{cccc|cccc}
-x^{-1} & & & & & & & \\
 & 1 & & & & & & \\
 & & \ddots & & & & & \\
-\varpi w x^{-1} & & & 1 & & & & w^2x^{-1} \\
\hline
\varpi^2 & & & & -x & & & -\varpi w \\
 & & & & & 1 & & \\
 & & & & & & \ddots & \\
 & & & & & & & 1
\end{array}\right)\right) dw\,dx
\end{aligned}
$$

where we use $d_1' = q^{-2n}q^{s+n}$, $d_2' = q^{-2n+1}q^{s+n}q^{-s-n}$, $d_3' = q^{-2n+2}q^{s+n}q^{-2s-2n}$, $d_4' = \frac{1-q^{-2n+2}}{1-q^{-1}}q^{-2}q^{s+n}q^{-s-n}$, $d_5' = \frac{1-q^{-2n+2}}{1-q^{-1}}q^{-1}q^{s+n}q^{-s-n}$, $d_6' = \frac{1-q^{-2n+2}}{1-q^{-1}}q^{s+n}q^{-2s-2n}$.

We now claim that the second, third, and sixth terms above are all zero. This is because they all contain an expression like $\int\limits_{x\in\mathcal{O}^\times} \chi^{-1}(x)dx$ when evaluated. For example, we write out the computation for the sixth term. Since

$$\left(\begin{array}{cccc|cccc}
-x^{-1} & & & & & & & \\
& 1 & & & & & & \\
& & \ddots & & & & & \\
-\varpi w x^{-1} & & & 1 & & & & x^{-1}w^2 \\ \hline
\varpi^2 & & & & -x & & & -\varpi w \\
& & & & & 1 & & \\
& & & & & & \ddots & \\
& & & & & & & 1
\end{array}\right) = \left(\begin{array}{cccc|cccc}
-x^{-1} & & & & & & & \\
& 1 & & & & & & \\
& & \ddots & & & & & \\
& & & 1 & & & & \\ \hline
& & & & -x & & & \\
& & & & & 1 & & \\
& & & & & & \ddots & \\
& & & & & & & 1
\end{array}\right)\left(\begin{array}{cccc|cccc}
1 & & & & & & & \\
& \ddots & & & & & & \\
& & \ddots & & & & & \\
-\varpi w x^{-1} & & & 1 & & & & x^{-1}w^2 \\ \hline
-\varpi^2 x^{-1} & & & & 1 & & & \varpi w x^{-1} \\
& & & & & \ddots & & \\
& & & & & & \ddots & \\
& & & & & & & 1
\end{array}\right)$$

and the last matrix is in B_χ

$$\int\int\limits_{w,x\in\mathcal{O}^\times} \Phi\left(\left(\begin{array}{cccc|cccc}
-x^{-1} & & & & & & & \\
& 1 & & & & & & \\
& & \ddots & & & & & \\
-\varpi w x^{-1} & & & 1 & & & & x^{-1}w^2 \\ \hline
-\varpi^2 & & & & -x & & & -\varpi w \\
& & & & & 1 & & \\
& & & & & & \ddots & \\
& & & & & & & 1
\end{array}\right)\right) dwdx = \int\int\limits_{w,x\in\mathcal{O}^\times} \chi(-x^{-1})\Phi(w_{00})dwdx$$

$$= 0$$

since χ is nontrivial on $\mathcal{O}^\times$.

We next claim that the fourth and fifth terms (from page 41) are also zero. This is because the matrices where Φ is being evaluated at all lie in $Bw_{10}B \subset Pw_{10}B_\chi$ (see Appendix), and Φ is zero in $Pw_{10}B_\chi$ (see Note 2.2.6). Thus, we have $\pi(A\gamma)\Phi(w_{11}) = q^{-2n}q^{s+n}\Phi(w_{11})$. This finishes Case 2 and the lemma.

LEMMA **2.4.2** *With respect to the basis* $\{f_0, f_1\}$,

$$\pi(\mathcal{A}_{w_{11}}) = \begin{pmatrix} 0 & 1 \\ q^{-2n+1}\chi(-1) & 0 \end{pmatrix}$$

Proof. Let w denote w_{00} or w_{11}. Then, for $\Phi \in V^{B_\chi}$,

$$\begin{aligned}\pi(\mathcal{A}_{w_{11}})\Phi(w) &= \int_{B_\chi} \Phi(wbw_{11})db \\ &= \int_{U_0}\int_{M_\chi}\int_{U_1^-} \Phi(wumu^-w_{11})du^-dmdu \\ &= \int_{U_0} \Phi(wuw_{11})du\end{aligned}$$

since ${w_{11}}^{-1}M_\chi U_1^- w_{11} \subset B_\chi$.

Case 1: $w = w_{00}$

$$\pi(\mathcal{A}_{w_{11}})\Phi(w_{00}) = \Phi(w_{11})$$

Case 2: $w = w_{11}$

For notational convenience, let $R = \pi(\mathcal{A}_{w_{11}})\Phi(w_{11})$.

$$\begin{aligned} R &= \int_{U_0} \Phi(w_{11}uw_{11})du \\ &= \chi(-1)\int_{U_0} \Phi\left(\left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ x_2 & 1 & & & & & & \\ \vdots & & \ddots & & & & & \\ x_n & & & 1 & & & & \\ \hline x_1 & w_2 & \cdots & w_n & 1 & -x_2 & \cdots & -x_n \\ w_2 & & & & & 1 & & \\ \vdots & & & & & & \ddots & \\ w_n & & & & & & & 1 \end{array}\right)\right) du \end{aligned}$$

since ${w_{11}}^{-1} = \left(\begin{array}{cccc|cccc} -1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline & & & & -1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array}\right) w_{11}.$

We can reduce this using the same reductions that we used in computing $\pi(\mathcal{A}_\gamma)\Phi(w_{11})$ in the last lemma. We get

$$
\begin{aligned}
\pi(\mathcal{A}_{w_{11}})\Phi(w_{11}) &= \chi(-1)q^{-2n+2}\int\limits_{x\in\mathcal{O}}\Phi\left(\left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline x & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{array}\right)\right)dx \\
&= \chi(-1)\frac{1-q^{-2n+2}}{1-q^{-1}}\int\limits_{w\in\mathcal{O}^\times}\int\limits_{x\in\mathcal{O}}\Phi\left(\left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ w & & 1 & & & \\ \hline x & & & 1 & & -w \\ & & & & \ddots & \\ & & & & & 1 \end{array}\right)\right)dx\,dw
\end{aligned}
$$

We split up $\{x\in\mathcal{O}\}$ into $\{x\in\mathcal{P}\}\cup\{x\in\mathcal{O}^\times\}$ and decompose according to P_0wB_χ to get

$$
\begin{aligned}
R &= e_1\Phi(w_{00}) \\
&+e_2\int\limits_{x\in\mathcal{O}^\times}\Phi\left(\left(\begin{array}{cccc|cccc} x^{-1} & & & & 1 & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline & & & & x & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array}\right) w_{11} \left(\begin{array}{ccc|ccc} 1 & & & x^{-1} & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{array}\right)\right)dx \\
&+e_3\int\limits_{w\in\mathcal{O}^\times}\Phi\left(\left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ w & & 1 & & & \\ \hline & & & 1 & & -w \\ & & & & \ddots & \\ & & & & & 1 \end{array}\right)\right)dw
\end{aligned}
$$

$$+e_4\int\int_{w,x\in\mathcal{O}^\times}\Phi\left(\left(\begin{array}{ccc|ccc} x^{-1} & & & 1 & & wx^{-1} \\ & 1 & & & & \\ & & \ddots & & & \\ & & 1 & w & & \\ \hline & & & x & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & 1 \end{array}\right) w_{11} \left(\begin{array}{ccc|cc} 1 & & & x^{-1} & -wx^{-1} \\ & \ddots & & & \\ & & 1 & -wx^{-1} & w^2x^{-1} \\ \hline & & & 1 & \\ & & & & \ddots \\ & & & & 1 \end{array}\right)\right) dx\,dw$$

where $e_1 = \chi(-1)q^{-2n+2}q^{-1}$, $e_2 = \chi(-1)q^{-2n+2}$, $e_3 = \chi(-1)\frac{1-q^{-2n-2}}{1-q^{-1}}q^{-1}$, $e_4 = \chi(-1)\frac{1-q^{-2n-2}}{1-q^{-1}}$. Finally, we claim that that three remaining integrals are zero. The first and third contain $\int_{x\in\mathcal{O}^\times}\chi(x) = 0$. The second integral is zero because Φ is identically zero on the domain of integration (see Note 2.2.6).

PROPOSITION **2.4.3** *For (order χ_u) =2, π will be irreducible unless $s = 0,\ i\pi/lnq$.*

Proof. Clearly, $\pi(\mathcal{A}_\gamma)$ and $\pi(\mathcal{A}_{w_{11}})$ admit no common invariant subspace unless $\pi(\mathcal{A}_\gamma)$ is scalar, which occurs when $s = 0,\ i\pi\text{lnq}$. Thus, $(\pi, H(G//B_\chi), V^{B_\chi})$ is irreducible when $s \neq 0,\ i\pi/\text{lnq}$, so (π, G, V) will be irreducible there too.

2.5. Irreducibility conditions for (order χ_u) $= 1$

This section brings us to the case where χ is unramified, i.e., (order of χ_u) $= 1$. The analysis has a general format similar to that of the last case, but there are differences. As usual, we analyze the Hecke algebra representation, $(\pi, H(G//B_\chi), V^{B_\chi})$ (here, $B_\chi = B$). We start by computing $\pi(\mathcal{A}_\gamma)$, where

$$\mathcal{A}\gamma = (|B\gamma B|)^{-1}\text{char}_{B\gamma B}, \quad \gamma \text{ as before.}$$

Then, we give a lemma of [Gus] which tells us that if $H(G//B)\cdot V^K = V^B$ and $H(G//B)\cdot \tilde{V}^K = \tilde{V}^B$ ($\tilde{V}$ = space of contragredient), then π is irreducible. It turns out that except for $s = 0,\ i\pi/\ln q, \pm n$ (or $s = i\pi/\ln q, \pm 1$ if $n = 1$), we have V^B spanned by V^K under powers of $\pi(\mathcal{A}_\gamma)$. Thus, we have irreducibility at those points. It is easy to check that π is reducible for $s = \pm n$. To show reducibility for Res $= 0$, we use the same kind of argument as for the (order χ_u)=2 case to show $\text{Hom}(\pi, \tilde{\pi})$ has dimension 2, which we do in Section 2.6.

Recall that $\dim V^B = 3$ for $n \geq 2$ and $\dim V^B = 2$ for $n = 1$. We shall do the $n \geq 2$ case and just state the results for the $n = 1$ case. One can find the $n = 1$ case in [Cas2] or [Gus]. The $n \geq 2$ case is very easy to adapt to $n = 1$, anyway.

LEMMA **2.5.1** *For $n \geq 2$, with respect to the basis $\{f_{00}, f_{10}, f_{11}\}$,*

$$\pi(\mathcal{A}_\gamma) = \begin{pmatrix} q^{-s-n} & 0 & 0 \\ (1-q^{-1})q^{-s-n} & q^{-1} & 0 \\ (1-q^{-1})(q^{-s-n}+q^{-2n+1}) & q^{-1}(1-q^{-2n+2}) & q^{-2n}q^{s+n} \end{pmatrix}.$$

For $n = 1$, with respect to the basis $\{f_{00}, f_{11}\}$,

$$\pi(\mathcal{A}_\gamma) = \begin{pmatrix} q^{-s-1} & 0 \\ (1-q^{-1})(q^{-s-1}+q^{-1}) & q^{-2}q^{s+1} \end{pmatrix}.$$

Proof. As indicated above, we shall just do the case $n \geq 2$. Let w denote w_{00}, w_{10}, w_{11}. Then, for $\Phi \in V^B$,

$$\begin{aligned} \pi(\mathcal{A}\gamma)\Phi(w) &= \int_B \Phi(wb\gamma)db \\ &= \int_{U_0}\int_{M_0}\int_{U_1^-} \Phi(wumu^-\gamma)du^-dmdu \\ &= \int_{U_0} \Phi(wu\gamma)du \end{aligned}$$

since $\gamma^{-1}u^-\gamma \in U_1^-$ and $m\gamma = \gamma m$.

Case 1: $w = w_{00}$

$$\begin{aligned} \pi(\mathcal{A}\gamma)\Phi(w_{00}) &= \Phi(\gamma) \\ &= q^{-s-n}\Phi(w_{00}) \end{aligned}$$

Case 2: $w = w_{10}$

For notational convenience, we set $T = \pi(A\gamma)\Phi(w_{10})$.

$$T = \int_{U_0'}\int_{U_0''} \Phi\left((w_{10}) \left(\begin{array}{cccc|cccc} 1 & & & & x_1 & x_2 & \dots & x_n \\ & \ddots & & & x_2 & & & \\ & & \ddots & & \vdots & & & \\ & & & 1 & x_n & & & \\ \hline & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array} \right) \left(\begin{array}{cccc|cccc} 1 & w_2 & \dots & w_n & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline & & & & 1 & & & \\ & & & & -w_2 & 1 & & \\ & & & & \vdots & & \ddots & \\ & & & & -w_n & & & 1 \end{array} \right) \gamma \right) du$$

$$= \int_{U_0''} \Phi\left(w_{10} \left(\begin{array}{cccc|cccc} 1 & w_2 & \dots & w_n & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline & & & & 1 & & & \\ & & & & -w_2 & 1 & & \\ & & & & \vdots & & \ddots & \\ & & & & -w_n & & & 1 \end{array} \right) \gamma \right) du$$

since $w_{10}U_0'w_{10} \subset B$ (where U_0' and U_0'' are as indicated). Since w_{10} conjugates $w_3, \dots, w_n$ into B as well, this reduces to

$$T = \int_{w\in\mathcal{O}} \Phi\left(w_{10} \left(\begin{array}{cccc|cccc} 1 & w & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline & & & & 1 & & & \\ & & & & -w & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array} \right) \gamma \right) dw$$

$$= \int_{w\in\mathcal{O}} \Phi\left(w_{10}\gamma \left(\begin{array}{cccc|cccc} 1 & \varpi^{-1}w & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline & & & & 1 & & & \\ & & & & -\varpi^{-1}w & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array} \right) \right) dw$$

$$= \int_{w \in \mathcal{O}} \Phi \left(w_{10} \left(\begin{array}{cccc|cccc} 1 & \varpi^{-1}w & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline & & & & 1 & & & \\ & & & & -\varpi^{-1}w & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array} \right) \right) dw$$

(since $w_{10}\gamma w_{11}\mathcal{O}^{-1}$ has ϖ in the second diagonal entry). Thus,

$$T = \int_{w \in \mathcal{O}} \Phi \left(\left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ \varpi^{-1}w & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline & & & & 1 & -\varpi^{-1}w & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array} \right) w_{10} \right) dw$$

$$= \int_{w \in \mathcal{P}} \Phi \left(\left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ \varpi^{-1}w & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline & & & & 1 & -\varpi^{-1}w & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array} \right) w_{10} \right) dw$$

$$+ \int_{w \in \mathcal{O}^{\times}} \Phi \left(\left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ \varpi^{-1}w & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline & & & & 1 & -\varpi^{-1}w & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array} \right) w_{10} \right) dw$$

$$= q^{-1}\Phi(w_{10}) + \int_{w\in\mathcal{O}^\times} \Phi\left(\left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ \varpi^{-1}w & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ \hline & & & & 1-\varpi^{-1}w & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{array}\right) w_{10}\right) dw$$

since $w \in \mathcal{P}$ implies $\varpi^{-1}w \in \mathcal{O}$, so that conjugating the w_{10} around the matrix in the $w \in \mathcal{P}$ integral gives a matrix in U_0.

Finally, using an Iwasawa decomposition,

$$\begin{aligned} T &= q^{-1}\Phi(w_{10}) \\ &+ \int_{w\in\mathcal{O}^\times} \Phi\left(\left(\begin{array}{cccc|cccc} \varpi & w^{-1} & & & & & & \\ & \varpi^{-1} & & & & & & \\ & & 1 & & & & & \\ & & & \ddots & & & & \\ & & & & 1 & & & \\ \hline & & & & \varpi^{-1} & & & \\ & & & & -w^{-1} & \varpi & & \\ & & & & & & 1 & \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{array}\right)\left(\begin{array}{cccc|cccc} & -w^{-1} & & & & & & \\ w & \varpi & & & & & & \\ & & 1 & & & & & \\ & & & \ddots & & & & \\ & & & & 1 & & & \\ \hline & & & & \varpi & -w & & \\ & & & & w^{-1} & & & \\ & & & & & & 1 & \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{array}\right) w_{10}\right) dw \\ &= q^{-1}\Phi(w_{10}) + q^{-s-n}\int_{w\in\mathcal{O}^\times} \Phi\left(\left(\begin{array}{cccc|cccc} -w^{-1} & & & & & & & \\ \varpi & w & & & & & & \\ & & 1 & & & & & \\ & & & \ddots & & & & \\ & & & & 1 & & & \\ \hline & & & & -w & \varpi & & \\ & & & & & w^{-1} & & \\ & & & & & & 1 & \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{array}\right) w_{10}\right) dw \\ &= q^{-1}\Phi(w_{10}) + (1-q^{-1})q^{-s-n}\Phi(w_{00}) \end{aligned}$$

as the matrix inside always lies in B (see appendix).

Case 3: $w = w_{11}$

For notational convenience, we set $S = \pi(\mathcal{A}_\gamma)\Phi(w_{11})$.

We can use exactly the same computations as were used in Lemma 2.4.1 to compute $\pi(\mathcal{A}_\gamma)\Phi(w_{11})$ in the case (order of χ_u) $= 2$ to get

$$
\begin{aligned}
S \; &= q^{-2n}q^{s+n}\Phi(w_{11}) \\
&+ q^{-2n+1}q^{s+n}q^{-s-n}\int\limits_{\tilde{x}\in\mathcal{O}^\times}\Phi\left(\left(\begin{array}{cccc|cccc}
-\tilde{x}^{-1} & & & & & & & \\
& 1 & & & & & & \\
& & \ddots & & & & & \\
& & & 1 & & & & \\
\hline
\varpi & & & & -\tilde{x} & & & \\
& & & & & 1 & & \\
& & & & & & \ddots & \\
& & & & & & & 1
\end{array}\right)\right) d\tilde{x} \\
&+ q^{-2n+2}q^{s+n}q^{-2s-2n}\int\limits_{x\in\mathcal{O}^\times}\Phi\left(\left(\begin{array}{cccc|cccc}
-x^{-1} & & & & & & & \\
& 1 & & & & & & \\
& & \ddots & & & & & \\
& & & 1 & & & & \\
\hline
\varpi^2 & & & & -x & & & \\
& & & & & 1 & & \\
& & & & & & \ddots & \\
& & & & & & & 1
\end{array}\right)\right) dx \\
&+ \tfrac{1-q^{-2n+2}}{1-q^{-1}}q^{-2}q^{s+n}q^{-s-n}\int\limits_{w\in\mathcal{O}^\times}\Phi\left(\left(\begin{array}{cccc|cccc}
0 & & & -w^{-1} & & & & \\
& 1 & & & & & & \\
& & \ddots & & & & & \\
& & & 1 & & & & \\
& & & \varpi & -w & & & \\
\hline
\varpi & & & & 0 & & & -w \\
& & & & & 1 & & \\
& & & & & & \ddots & \\
& & & & & & 1 & \\
w^{-1} & & & & & & & 0
\end{array}\right)\right) dw \\
&+ \tfrac{1-q^{-2n+2}}{1-q^{-1}}q^{-1}q^{s+n}q^{-s-n}\int\int\limits_{w,\tilde{x}\in\mathcal{O}^\times}\Phi\left(\left(\begin{array}{cccc|cccc}
0 & & & -w^{-1} & & & & \\
& 1 & & & & & & \\
& & \ddots & & & & & \\
& & & 1 & & & & \\
-w & & & \tilde{x} & & & & \\
\hline
\varpi & & & & -\tilde{x} & & & -w \\
& & & & & 1 & & \\
& & & & & & \ddots & \\
& & & & & & 1 & \\
\varpi w^{-1}\tilde{x}^{-1} & & & & -w^{-1} & & & 0
\end{array}\right)\right) dw\,d\tilde{x} \\
&+ \tfrac{1-q^{-2n+2}}{1-q^{-1}}q^{s+n}q^{-2s-2n}\int\int\limits_{w,x\in\mathcal{O}^\times}\Phi\left(\left(\begin{array}{cccc|cccc}
-x^{-1} & & & & & & & \\
& 1 & & & & & & \\
& & \ddots & & & & & \\
-\varpi w x^{-1} & & & 1 & & & & x^{-1}w^2 \\
\hline
\varpi^2 & & & & -x & & & -\varpi w \\
& & & & & 1 & & \\
& & & & & & \ddots & \\
& & & & & & & 1
\end{array}\right)\right) dw\,dx
\end{aligned}
$$

From the appendix, we can read off which double-cosets $P_0 w_{ij} B = B w_{ij} B$ each of the matrices above lies in. This gives us

$$\begin{aligned}
S &= q^{-2n} q^{s+n} \Phi(w_{11}) \\
&+ q^{-2n+1} q^{s+n} q^{-s-n} (1-q^{-1}) \Phi(w_{00}) \\
&+ q^{-2n+2} q^{s+n} q^{-2s-2n} (1-q^{-1}) \Phi(w_{00}) \\
&+ \tfrac{1-q^{-2n+2}}{1-q^{-1}} q^{-2} q^{s+n} q^{-s-n} (1-q^{-1}) \Phi(w_{10}) \\
&+ \tfrac{1-q^{-2n+2}}{1-q^{-1}} q^{-1} q^{s+n} q^{-s-n} (1-q^{-1})^2 \Phi(w_{10}) \\
&+ \tfrac{1-q^{-2n+2}}{1-q^{-1}} q^{s+n} q^{-2s-2n} (1-q^{-1})^2 \Phi(w_{00}) \\
&= q^{-2n} q^{s+n} \Phi(w_{11}) + q^{-1}(1-q^{-2n+2}) \Phi(w_{10}) + (1-q^{-1})[q^{-2n+1} + q^{-s-n}] \Phi(w_{00}).
\end{aligned}$$

This finishes Case 3 and the lemma.

The next step is to show that π is irreducible except possibly at $s = 0,\ i\pi/\ln q,\ \pm n$ for $n \geq 2$ (or $s = i\pi/\ln q,\ \pm 1$ for $n = 1$). The following lemma from Gustafson will be useful.

LEMMA **2.5.2** *Suppose that V^K, $\tilde{V}^K$ generate V^B, $\tilde{V}^B$, respectively, under the actions of $H(G//B)$. Then π (also $\tilde{\pi}$) is irreducible.*

Proof. Let V_0 be a proper invariant subspace of V. By Theorem 2.1.10, V_0^B is a proper $H(G//B)$-invariant subspace of V^B. Since V^K is one-dimensional, there are two possibilities: $V^K \subset V_0^B$ or $V^K \cap V_0^B = 0$. In the first case, since V^K generates V^B, we get $V_0^B = V^B$. This is a contradiction. In the second case, any $l \in \tilde{V}^K$ annihilates V_o^B. Since $\tilde{V}^K$ generates $\tilde{V}^B$, this tells us that $\tilde{V}^B$ annihilates V_0^B. Thus, $V_0^B = 0$, also a contradiction. Therefore, π (and $\tilde{\pi}$) must be irreducible.

We now state the irreducibility result as a proposition.

PROPOSITION **2.5.3** *For $n \geq 2$, if $s \notin \{0,\ i\pi/\ln q,\ \pm n\}$, then π is irreducible. For $n = 1$, if $s \notin \{i\pi/\ln q,\ \pm 1\}$, then π is irreducible.*

Proof. Let us begin with the $n \geq 2$ case. By the previous lemma, it suffices to check that V^K, $\tilde{V}^K$ generate V^B, $\tilde{V}^B$ under the action of $H(G//B)$.

We start by seeing when $v_0 =$ K-fixed vector and $\pi(\mathcal{A}_\gamma)$ are enough to generate V^B. With respect to our basis for V^B,

$$v_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Then,

$$v_1 = \pi(\mathcal{A}_\gamma)v_0 - q^{-s-n}v_0 = (1-q^{-s-n}) \begin{pmatrix} 0 \\ q^{-1} \\ q^{-1} + q^{-2n}q^{s+n} \end{pmatrix}$$

Also,

$$v_2 = \pi(A\gamma)v_1 - q^{-1}v_1 = q^{-2n}(1-q^{-s-n})(q^{-2n}q^{2s+2n}-1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Clearly, $\{v_0, v_1, v_2,\}$ will span V^B if $(1-q^{-s-n}) \neq 0$ and $(q^{-2n}q^{2s+2n}-1) \neq 0$, that is, for $s \neq n,\ 0,\ i\pi/\ln q$.

Now, recall that if $\pi = \mathrm{Ind}_P^G|\cdot|^s$, then $\tilde{\pi} = \mathrm{Ind}_P^G|\cdot|^{-s}$ (in general, $\mathrm{Ind}_P^G\tilde{\sigma} = \widetilde{\mathrm{Ind}_P^G\sigma}$). Thus, $\tilde{V}^K$ will generate $\tilde{V}^B$ for $s \neq -n, i\pi/\ln q$ ($i\pi/\ln q$ is identified with $-i\pi/\ln q$). Combining these, we see that $V^K, \tilde{V}^K$ generate $V^B, \tilde{V}^B$ as long as $s \notin \{0,\ i\pi/\ln q,\ \pm n\}$

In the case $n = 1$, we have

$$v_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with respect to the usual basis. Then,

$$v_1 = \pi(\mathcal{A}_\gamma)v_0 - q^{-s-1}v_0 = q^{-s-2}(q^{s+1}-1)q^{(s+1)} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, $\{v_1, v_2\}$ span V^B as long as $s \neq i\pi/\ln q,\ -1$. The condition $\tilde{V}^K$ spans $\tilde{V}^B$ gives $s \neq i\pi/\ln q,\ 1$. The proposition follows.

2.6. Reducibility conditions for (order χ_u) = 1

The purpose of this section is to prove the following proposition, which will give us reducibility for $s = 0,\ i\pi/\ln q$ (or $s = i\pi/\ln q$, if $n = 1$) in the case of χ unramified.

PROPOSITION **2.6.1** *χ unramified. Then, for $n \geq 2$.*

$$\dim Hom_G(\pi, \tilde{\pi}) = \begin{cases} 2 & \textit{if } s = 0,\ i\pi/\ln q \\ 1 & \textit{if not.} \end{cases}$$

If $n = 1$, the dimension will be 2 if $s = i\pi/\ln q$, and 1 otherwise.

Proof. First, we claim that if $s \notin \{0,\ i\pi/\ln q,\ \pm n\}$ for $n \geq 2$, or $s \notin \{i\pi/\ln q,\ \pm n\}$, for $n = 1$, then the dimension is one. In this case, π (and $\tilde{\pi}$) is irreducible by Proposition 2.5.3. Now, $\tilde{\pi} = i_{GM}\chi^{-1}$. Let $\psi = r_{AM}(\chi \otimes tr)$ (which is just $\chi \otimes \nu^{-n+1} \otimes \ldots \otimes \nu^{-1}$). Then, $w_{11}\psi = r_{AM}(\chi^{-1} \otimes tr)$. Observe that π is a subrepresentation of $i_{GA}\psi$ and $\tilde{\pi}$ is a subrepresentation of $i_{GA}w_{11}\psi$. Theorem 1.2.2 tells us that these have the same components. Each has exactly one component admitting a K-fixed vector. In one case this is π, in the other, $\tilde{\pi}$. Thus, π and $\tilde{\pi}$ must be equivalent.

We next turn to the problem of showing that dim = 2 at the appropriate places, and return to the case $s = \pm n$ afterwards. Briefly, we reduce the problem to a Jacquet module problem via Frobenius reciprocity, then to a corresponding Hecke algebra question. We will wish to know whether the abelian Hecke algebra $H(M//M_0)$ acts diagonally in a yet-to-be-specified two-dimensional subspace Q of $(V_U)^{M_0}$. (It turns out to be the case that acting diagonally corresponds to having dim Hom = 2). Essentially, we show that it is true for s near 0, $i\pi/\ln q$ (or s near $i\pi/\ln q$, if $n = 1$) by projecting the operator $\pi(\mathcal{A}_\gamma)$ down to $(V_U)^{M_0}$. Then, we shall show that we can analytically continue in s to obtain the result for $s = 0, i\pi/\ln q$

We begin with the following simple fact.

LEMMA **2.6.2 (cf. Gustafson)** *Suppose (ρ, M, X) is an admissible representation of a reductive p-adic group M. If $\lambda \in \hat{Z}$, where Z denotes the center of M, let*

$$X_\lambda = \{x \in X | \textit{there is an } r \in N \textit{ such that } [\pi(z) - \lambda(z)]^r x = 0\ \forall z \in Z\ \}$$

Then, $V = \bigoplus_\lambda V_\lambda$, *a direct sum of* M*- invariant subspaces.*

We now apply Frobenius reciprocity and the preceding lemma to get the following:

$$\begin{aligned}\mathrm{Hom}_G(\pi, \tilde{\pi}) &\cong \mathrm{Hom}_M(\pi_U, \chi^{-1} \otimes tr_{2(n-1)}) \\ &\cong \mathrm{Hom}_M(\pi_U|_Q, \chi^{-1} \otimes tr_{2(n-1)})\end{aligned}$$

where $Q = (V_U)_{\chi^{-1}}$ (recall that $\tilde{\pi} = \mathrm{Ind}_P^G \chi^{-1}$).

We now convert the problem into a Hecke algebra question. By Bernstein-Zelevinsky, Casselman, we know that V_U has composition factors

$$\begin{array}{ccc} \chi \otimes tr_{2(n-1)} & \chi^{-1} \otimes tr_{2(n-1)} & \nu^{-n+1} \otimes (\chi \propto tr_{2(n-2)}) \\ w = w_{00} & w = w_{11} & w = w_{10} \end{array}$$

Note that the third of these does not occur for $n = 1$. When $s \in \{0,\ i\pi/\ln q\}$, we have $\chi = \chi^{-1}$, so the first two (but not the third) occur in $\pi_U|_Q$. In particular, Q is two-dimensional and our question is whether Q is a direct sum of two copies of $\chi^{-1} \otimes tr_{2(n-1)})$, or just a composition series. We observe that every subquotient of Q has an M_0-fixed vector, so by Corollary 2.1.11 we get

$$\mathrm{Hom}_G(\pi, \tilde{\pi}) \cong \mathrm{Hom}_{H(M//M_0)}(\pi_U|_Q, \chi^{-1} \otimes tr_{2(n-1)})$$

(note that $Q^{M_0} = Q$). Thus, our problem is to show that the action of π_U on Q diagonalizes.

We recall the following well-known result of Satake:

LEMMA **2.6.3** *(cf. [Car])* $H(M//M_0)$ ***is abelian.***

We shall also need the following lemma due to Casselman:

LEMMA **2.6.4** *The Jacquet functor* r_{MG} *gives a vector space isomorphism between* V^B *and* $(V_U)^{M_0}$.

Proof. By Proposition 2.1.1, $r_{MG} : V^B \to (V_U)^{M_0}$ is a surjection. We know that V^B is three-dimensional (resp. two-dimensional, for $n = 1$), so if we knew that $(V_U)^{M_0}$ were also three-dimensional (resp. two-dimensional), we would be done. However, this is reasonalbly clear since $\chi \otimes tr_{2(n-1)}$, and $\nu^{-n+1} \otimes (\chi \propto \otimes tr_{2(n-2)})$ (resp. the first two if $n = 1$) each admit exactly one M_0-fixed vector. Note $\nu^{-n+1} \otimes (\chi \propto \otimes tr_{2(n-2)})$ has exactly one since it results from parabolic induction of something 'unramified.'

We are going to define an operator on $(V_U)^{M_0}$ which behaves like $\pi(\mathcal{A}_\gamma)$. We shall want to know the eigenvectors of $\pi(\mathcal{A}_\gamma)$, so we write them down now. For $n \geq 2$, they are

$$v_1 = \begin{pmatrix} 1 \\ \frac{(1-q^{-1})q^{-s-n}}{q^{-s-n}-1} \\ \frac{(1-q^{-1})q^{-s-n}}{q^{-s-n}-q^{-1}} \cdot \frac{q^{-s-n}-q^{-2n}q^{s+n}}{q^{-s-n}-q^{-2n}q^{s+n}} \end{pmatrix} v_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{q^{-1}(1-q^{-2n+2})}{q^{-1}-q^{-2n}q^{s+n}} \end{pmatrix} v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

For $n = 1$, they are

$$\begin{pmatrix} 1 \\ \frac{(1-q^{-1})}{1-q^{-s}} \cdot \frac{q^{-s-1}+q^{-1}}{q^{-s-1}+q^{-1}} \end{pmatrix} \text{and} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We now define the aforementioned operator. We define it via the isomorphism $V^B \cong (V_U)^{M_0}$, but it will turn out to be an operator from $H(M//M_0)$. For expediency, let r denote the Jacquet functor $r_{MG} : V^B \to (V_U)^{M_0}$. We define S on $(V_U)^{M_0}$ by

$$S(r(\Phi)) = r(\pi(\mathcal{A}_\gamma)\Phi) \quad \text{for } \Phi \in V^B.$$

Then,

$$\begin{aligned} S(r(\Phi)) &= r(\pi(\mathcal{A}_\gamma)\Phi) \\ &= r(\int_{U_0} \int_{M_0} \int_{U_1^-} \pi(umu^-\gamma)\Phi du^- dmdu) \\ &= r(\int_{U_0} \int_{M_0} \pi(um\gamma)\Phi dmdu) \quad (\text{since } \gamma^{-1}U_1^-\gamma \subset U_1^-) \\ &= \int_{M_0} \pi_U(m\gamma)r(\Phi)dm \quad (\pi_U|_{U_0} \text{ is trivial}) \\ &= \pi_U(A'_\gamma)r(\Phi) \end{aligned}$$

where $A'_\gamma = \mathrm{char}_{M_0\gamma M_0} \in H(M//M_0)$. Thus, with respect to the basis $r(f_{00})$, $r(f_{10})$, $r(f_{11})$ (or the counterpart for $n = 1$), $\pi_U(A'_\gamma)$ has the same matrix as $\pi(\mathcal{A}_\gamma)$. It must then have the same eigenvectors.

Next, let s_0 be 0 or $i\pi/\ln q$, where we're trying to show $\dim \mathrm{Hom} = 2$. Let T be a suitably small neighborhood of s_0 not containing s_0. Then, for $s \in T$, the eigenvalues of $\pi_U(A'_\gamma) = \pi(\mathcal{A}_\gamma)$ are distinct, and $r(v_1), r(v_2), r(v_3)$ (resp. image of eigenvectors above) give a basis for $(V_U)^{M_0}$. Since $H(M//M_0)$ is abelian, these will be eigenvectors for all of $H(M//M_0)$, i.e, the action diagonalizes. A brief examination of the eigenvectors shows that they have "nice" limits as $s \to s_0$. We wish to use this to show that the action diagonalizes at s_0. This brings us to the following lemma (cf. [Gus]).

LEMMA **2.6.5** *Let $\{\tilde{f}_{00}, \tilde{f}_{10}, \tilde{f}_{11}\}$ correspond to $\{f_{00}, f_{10}, f_{11}\}$ under the isomorphism $V^B \cong (V_U)^{M_0}$. Let $A \in H(M//M_0)$ and consider $\pi_U(A)$ written with respect to the basis $\{\tilde{f}_{00}, \tilde{f}_{10}, \tilde{f}_{11}\}$ of $(V_U)^{M_0}$. Then, the entries of $\pi_U(A)$ are holomorphic as functions of s.*

Proof of Lemma: Before starting the proof proper, we remark that although the space V_U depends on s, the identification of $(V_U)^{M_0}$ with $\mathbf{C}^3$ via the basis $\{\tilde{f}_{00}, \tilde{f}_{10}, \tilde{f}_{11}\}$ does not.

We start by observing that it is sufficient to check the lemma when A is of the form $\mathrm{char}_{M_0\alpha M_0}$, $\alpha \in M$. Fix such an A.

First, we shall find a nice expression for a matrix coefficient of $\pi_U(A)$, then show it is holomorphic in s. Let $\tilde{f} \in \{\tilde{f}_{00}, \tilde{f}_{10}, \tilde{f}_{11}\}$ and $f \in \{f_{00}, f_{10}, f_{11}\}$ corresponding to $\tilde{f}$. Choose m sufficiently large so that $\alpha' = \gamma^m\alpha$ has the property that $\alpha'^{-1}U_1^-\alpha' \subset K_1$. Then

$$\begin{aligned}
(q^{-s-n})^m\pi_U(\mathrm{char}_{M_0\alpha M_0})\tilde{f} &= \pi_U(\mathrm{char}_{M_0\gamma^m\alpha M_0})\tilde{f} \quad \text{(since } \gamma \text{ and } M_0 \text{ commute)}\\
&= r_{MG}(\pi(\mathrm{char}_{U_0M_0\alpha M_0}))\tilde{f}\\
&= r_{MG}(\pi(\mathrm{char}_{P_0\alpha' B})\tilde{f}) \quad \text{(since } f \in V^B\text{)}\\
&= r_{MG}(\pi(\mathrm{char}_{B\alpha' B})\tilde{f})
\end{aligned}$$

since the choice of m ensures $P_0\alpha' B = B\alpha' B$. Thus, the matrix for $\pi_U(\mathrm{char}_{M_0\alpha M_0})$ is just $(q^{s+n})^m$ times the matrix for $\pi(\mathrm{char}_{B\alpha' B})$ (when these matricies are written with respect to the bases above). Therefore, it suffices to check that the matrix coefficients of $\pi(\mathrm{char}_{B\alpha' B})$ (on V^B) are holomorphic in s.

Let $w \in \{w_{00}, w_{10}, w_{11}\}$. A matrix coefficient for $\pi(\mathrm{char}_{B\alpha' B})$ is

$$\sigma(w) \;=\; \int_{B\alpha' B}(\pi(x)f)(w)dx = c \cdot \int_B f(wb\alpha')db,$$

where $c = \mathrm{vol}(B\alpha' B)$. We show that this is holomorphic as a function of s. Suppose $f = f_{ij}$. Then, we set

$$S_l = \left\{ b \in B \;\middle|\; \begin{array}{l} wb\alpha' \in Pw_{ij}B \text{ and for } wb\alpha' = pw_{ij}b, \\ \text{the } F^\times \text{ entry of } m, \text{ where } p = mu \in MU, \\ \text{lies in } \varpi^l \mathcal{O}^\times. \end{array} \right\}.$$

In particular, we will have $f_{ij}(wb\alpha') = q^{l(s+n)}$ for those $b \in S_l$. Thus,

$$\sigma(w) = \pi(\mathrm{char}_{B\alpha' B}) f_{ij}(w) = c \cdot \sum_l \mathrm{vol}\ (S_l) q^{l(s+n)}$$

Once we have argued that this sum is finite, the lemma will follow. However, this is clear: $p \in wB\alpha' B w_{ij}^{-1}$, which is compact (so the number of l's for which S_l is nonempty is finite and determined by α', w, w_{ij}). Thus, the lemma holds.

We now finish the proposition. We observe that the eigenvectors (see p. 55) have nonzero limits as $s \to 0,\ i\pi/\ln q$ ($s \to i\pi/\ln q$ for $n = 1$). Thus, the eigenvectors analytically continue to eigenvectors for $H(M//M_0)$ at $s = 0,\ i\pi/\ln q$ ($s = i\pi/\ln q$ for $n = 1$). In particular, the action of $H(M//M_0)$ diagonalizes there. This finishes the dim $= 2$ case. For $s = \pm n$, Q is one-dimensional so the intertwining algebra can be (at most) one-dimensional, so will be one-dimensional. This finishes the proposition.

2.7. Reducibility conditions for (order χ_u) $= 2$

In this section, we shall give the analogue in the case (order χ_u) $= 2$ to Proposition 2.6.1, and give the main theorem of this chapter, summarizing the reducibility criteria for the degenerate principal series we've been looking at.

PROPOSITION **2.7.1** *(order χ_u) $= 2$. Then*

$$\dim Hom_G(\pi, \tilde{\pi}) = \begin{cases} 2 & \text{if } s = 0,\ i\pi/\ln q, \\ 1 & \text{if not.} \end{cases}$$

Proof. We shall not include the proof because it is essentially the same as that of Proposition 2.6.1 (in fact, a bit easier). The only new fact needed is that $H(M//M_\chi)$

is abelian. If we write $M_\chi = M'_\chi \times M''_0 \subset F^\times \times M''$, it follow from the fact that $F^\times$ commutes with everything and $H(M''//M''_0)$ is abelian.

This brings us to the main theorem of this chapter.

THEOREM **2.7.2** *The representation $\pi = i_{GM}\chi$ ($M \cong F^\times \times Sp_{2(n-1)}(F)$) is reducible if and only if*

$$\chi \in \{\nu^{-n}, 1, sgn, \nu^n\} \quad \textit{for } n \geq 2$$
$$\textit{or} \quad \chi \in \{\nu^{-1}, sgn, \nu\} \qquad \textit{for } n = 1$$

If π is reducible, it has two components.

Proof. By the results of Sections 2.3–2.5, π is irreducible for all other χ. It is reducible with two components for $\chi = 1, sgn$ (or $\chi = sgn$ if $n = 1$) since $\dim \mathrm{Hom}_G(\pi, \tilde{\pi}) = 2$ in this case, from Sections 2.6 and 2.7 (note that $\pi = \tilde{\pi}$ for such χ). π is reducible when $\chi = \nu^{\pm n}$ since the trivial representation is a component. When $n = 1$, $\nu^{\pm 1} \propto 1$ has at most two components (since $\dim V^B = 2$), so there will be exactly two. This leaves us with checking that $\nu^{\pm n} \propto tr_{2(n-1)}$ has two components for $n \geq 2$.

There are at least two ways to check that $\nu^{\pm n} \propto tr_{2(n-1)}$ has two components. One could apply Corollary 3.2.3 (the next chapter is independent of this one). Alternatively, one can compute $\pi(\mathcal{A}_{w_{11}})$ with respect to $\{f_{00}, f_{10}, f_{11}\}$ to get

$$\pi(\mathcal{A}_{w_{11}}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & q^{-1} & 1 - q^{-1} \\ q^{-2n+1} & q^{-1} - q^{-2n+1} & 1 - q^{-1} \end{pmatrix}.$$

One can directly check that the only invariant subspace of $\pi(\mathcal{A}_\gamma)$ and $\pi(\mathcal{A}_{w_{11}})$ for $\chi = \nu^{-n}$ is $\mathbf{C}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (the span of the K-fixed vector). Thus, there can be only two components in $\nu^{-n} \propto tr_{2(n-1)}$. This finishes the theorem.

CHAPTER 3

IRREDUCIBILITY OF CERTAIN REPRESENTATIONS Á LA TADIĆ

3.1. A general theorem on reducibility/irreducibility (in the regular case)

In the first section of this chapter we shall obtain some necessary and sufficient conditions for irreducibility of an induced representation. In the second section, we apply this to the degenerate principal series we are interested in.

For this section, we wish to work in a more general setting. To this end, let G denote a split connected reductive p-adic group, $P = MU$ a parabolic subgroup of G, ρ an irreducible admissible representation of M. Set $\pi = i_{GM}\rho$. We shall use a technique of Tadić (see [Tad2]) involving Jacquet modules as a basis for our theorem.

We introduce the following notation: let $s_1, \ldots, s_n$ be the simple reflections in W. Let $P_{min} = AU_{min}$ denote the minimal parabolic, and set $M_i =< A, s_i >$, which is the Levi factor of a larger parabolic subgroup P_i.

We now recall a theorem of Bernstein-Zelevinsky, Casselman. Let M, N be Levi factors of standard parabolics for G. Set

$$W^{MN} = \{w \in W | w(P_0 \cap M) \subset P_0, \quad w^{-1}(P_0 \cap N) \subset P_0\}.$$

We remark that these correspond to the elements of shortest length in the double cosets $W_N \backslash W / W_M$.

THEOREM **3.1.1** *Let ρ be an admissible representation of M. Then, $r_{NG} \circ i_{GM}\rho$ has a composition series with factors*

$$i_{NN'} \circ w \circ r_{M'M}\rho \quad w \in W^{MN},$$

where $M' = M \cap w^{-1}(N)$, $N' = w(M) \cap N$.

For notational convenience, we shall denote by $BZ_N(\pi)$ the collection of representations $i_{NN'} \circ w \circ \tau$ as τ runs over the components of $r_{M'M}\rho$ and w runs over W^{MN}. (If all the $i_{NN'} \circ w \circ \tau$ were irreducible, this would just be the components of the semisimplification of $r_{NG}(\pi)$, $r_{NG}(\pi)_{ss}$).

We now proceed to set up the main theorem in this chapter. We shall require three things of ρ. First, we want ρ irreducible. Second, we want $r_{AM}(\rho) \neq 0$. Finally, we require a regularity condition on ρ. Let ψ be a character in $r_{AM}(\rho)_{ss}$. We require that ψ be regular with respect to W (not just W_M). Note that if this is true for one character in $r_{AM}(\rho)_{ss}$, it will be true for all of them.

We now associate a graph to π as follows:

vertices: the vertices are the elements of $r_{AG}(\pi)_{ss}$

edges: two vertices ψ_1, ψ_2 are connected by an edge is there is some Levi N and some $\tau \in r_{NG}(\pi)_{ss}$ such that the following hold:

1. τ is an irreducible representation of N.
2. $\psi_1, \psi_2 \in r_{AN}(\tau)_{ss}$.

This brings us to the main theorem.

THEOREM **3.1.2** *Under the conditions above, the following are equivalent:*

1. π is irreducible.

2. the graph of π is connected

3. $\tau \in BZ_{M_i}(\pi) \Longrightarrow \tau$ is irreducible (for any i and τ).

Proof. We show 1 $\Longrightarrow$3 $\Longrightarrow$2 $\Longrightarrow$1.

1 $\Longrightarrow$3: Suppose not– that is, suppose π is irreducible but there is a $\tau \in BZ_{M_i}(\pi)$ which is reducible. Let τ_1, τ_2 denote the components of τ. Let $\psi_j = r_{AM_i}(\tau_j)$, $j = 1, 2$. Note that the Weyl group of M_i has two elements, namely 1 and s_i. Thus, $\psi_2 = s_i\psi_1$. Set $\pi_j = i_{GM_i}\tau_j$.

To obtain a contradiction, we show two things. First, we show that if π is irreducible, we must have $\pi \subset \pi_1$ or $\pi \subset \pi_2$ (as a subquotient). Then, we show that $r_{AG}(\pi)_{ss} \not\subset r_{AG}(\pi_1)_{ss}$ and $r_{AG}(\pi)_{ss} \not\subset r_{AG}(\pi_2)_{ss}$. As these two statements are incompatible, we must have π reducible.

We start by showing that if π is irreducible, it must be a subquotient of π_1 or π_2. First, we show that if $\psi \in r_{AG}(\pi)$ and π is irreducible, then π is a subquotient of $i_{GA}\psi$. By Frobenius reciprocity,

$$\mathrm{Hom}_G(\pi, i_{GA}\phi) \cong \mathrm{Hom}_M(\pi_A, \phi).$$

Take any ϕ so that this is nonzero (such a ϕ exists since π_A is nonzero). Then, π is a subquotient of $i_{GA}\phi$. Also, since $\psi \in r_{AG}(i_{GA}\phi)$, we have $\psi = w\phi$ for some $w \in W$. By Lemma 1.2.2, $i_{GA}\phi$ and $i_{GA}\psi$ have the same components. Therefore, π must be a subquotient of $i_{GA}\psi$.

Now, either $\tau = i_{M_iA}\psi_1$ or $\tau = i_{M_iA}\psi_2$, so suppose $\tau = i_{M_iA}\psi_1$. Thus, inducing in stages, we have $\pi \subset i_{GA}\psi_1 = i_{GM_i}\tau$. Since τ has τ_1 and τ_2 as components, the components of $i_{GM_i}\tau$ are just the components of $i_{GM_i}\tau_1$, $i_{GM_i}\tau_2$. Thus, $\pi \subset \pi_1$ or $\pi \subset \pi_2$, as claimed.

We now show that $r_{AG}(\pi)_{ss} \not\subset r_{AG}(\pi_1)_{ss}$ and $r_{AG}(\pi)_{ss} \not\subset r_{AG}(\pi_2)_{ss}$. First, we show that $r_{AG}(i_{GM_i}\tau)_{ss}$ is the disjoint union of $r_{AG}(\pi_1)_{ss}$ and $r_{AG}(\pi_2)_{ss}$. Since $\psi_2 = s_i\psi_1$, we have $r_{AG}(\pi_1)_{ss} = \{w\psi_1 | w \in W^{M_iA}\}$ and $r_{AG}(\pi_2)_{ss} = \{w\psi_2 | w \in W^{M_iA}\} = \{w\psi_1 | w \in W^{M_iA}s_i\}$. Since W is the disjoint union of W^{M_iA} and $W^{M_iA}s_i$, it follows from regularity that $r_{AG}(i_{GM_i}\tau)_{ss}$ is the disjoint union of $r_{AG}(\pi_1)_{ss}$ and $r_{AG}(\pi_2)_{ss}$. Now, observe that $\psi_1 \in r_{AG}(\pi_1)_{ss}$ but $\psi_1 \notin r_{AG}(\pi_2)_{ss}$, and similarly for ψ_2. Since $\psi_1, \psi_2 \in r_{AG}(\pi)_{ss}$, we cannot have $r_{AG}(\pi)_{ss} \subset r_{AG}(\pi_1)_{ss}$ or $r_{AG}(\pi)_{ss} \subset r_{AG}(\pi_2)_{ss}$. This finishes $1 \Longrightarrow 3$.

3 $\Longrightarrow$2: This is essentially just a Weyl group computation.

First, suppose $\psi \in r_{AG}(\pi)$, but ψ is not in the Jacquet module for $r_{AM}(\rho)$. Then, $\psi = w\psi_0$ for some ψ_0 in $r_{AM}(\rho)$, $w \in W^{MA}$, and by our regularity hypothesis, these are uniquely determined. We shall connect ψ by an edge to some $w'\psi_0 \in r_{AG}(\pi)$ with $l(w') = l(w) - 1$. Iterating this process will then tell us every ψ

is connected to some ψ_0 in $r_{AM}(\rho)$. Since ρ is irreducible, all these ψ_0's will be connected to each other (ρ comes from $1 \in W^{MN}$).

We now determine w' as described. Write $w = s_{i_1}, \ldots, s_{i_k}$ in terms of the simple roots. We claim that $w' = s_{i_2} \ldots s_{i_k}$ works. First, we must check that $w' \in W^{MA}$. This is equivalent to showing that if w is the shortest element in $w \cdot W_M$, w' is the shortest element in $w' \cdot W_M$. (Note that $w \cdot W_M$ and $w' \cdot W_M$ will be distinct since w is of minimal length in $w \cdot W_M$ and w' has $l(w') = l(w) - 1$). This is easy: if $w'' \in w' \cdot W_M$ had $l(w'') < l(w')$, then $s_{i_1} w'' \in w \cdot W_M$ and $l(s_{i_1} w'') = l(w'') + 1 < l(w') + 1 = l(w)$, contradicting the fact that w is the shortest element in that coset. Thus, $w' \in W^{MA}$.

Finally, we claim $\psi = w\psi_0$ shares an edge with $\psi' = w'\psi_0$. In particular, we claim $\tau = i_{M_{i_1}A}\psi'$ is the connecting representation. Certainly, ψ, $\psi' \in r_{AM_{i_1}}(\tau)$. If we show $\tau \in BZ_{M_i}(\pi)$, τ will be irreducible by hypothesis (3), so they will be connected. Observe that $(w \cdot W_M) \cup (w' \cdot W_M) = W_{M_{i_1}} w' W_M$, so w' is the shortest element in that double-coset, hence $w' \in W^{MM_{i_1}}$. We now check that $(w')^{-1} M_{i_1} w' \cap M = A$. If $(w')^{-1} M_{i_1} w' \subset M$, then $(w')^{-1} s_{i_1} w' \in M$, so $w \cdot M = w' \cdot M$. However, w and w' lie in distinct cosets, so we must have $(w')^{-1} M_{i_1} w' \cap M = A$. The B-Z composition factor arising from $w' \in W^{MM_{i_1}}$ is therefore

$$i_{M_{i_1}A}\psi' = i_{M_{i_1}A} \circ w' \circ \psi_0 \subset i_{M_{i_1}A} \circ w' \circ r_{AM}(\rho),$$

as claimed. This finishes $3 \Longrightarrow 2$.

$2 \Longrightarrow 1$: Let π_0 be a subquotient of π. We shall show π is irreducible by showing that π and π_0 have the same semisimplified Jacquet modules, that is $r_{AG}(\pi)_{ss} = r_{AG}(\pi_0)_{ss}$. The key ingredient is the following

LEMMA **3.1.3** *Suppose that ψ_1, $\psi_2 \in r_{AG}(\pi)$ satisfy the following conditions:*

1. *ψ_1, ψ_2 share an edge.*

2. *$\psi_1 \in r_{AG}(\pi_0)$.*

Then, $\psi_2 \in r_{AG}(\pi_0)$.

Proof of Lemma. Suppose that τ is a component of $r_{NG}(\pi)$ which gives rise to the edge, so that τ is an irreducible representation of N with $\psi_1, \psi_2 \in r_{AN}(\tau)$.

Now consider $r_{NG}(\pi_0)$. Since $\psi_1 \in r_{AG}(\pi_0) = r_{AN} \circ r_{NG}(\pi_0)$ and there is only one copy of ψ_1 in $r_{AG}(\pi)$, we must have $r_{NG}(\pi_0) \cap \tau \neq 0$ (as ψ_1 is in there). Since τ is irreducible, we must have $\tau \subset r_{NG}(\pi_0)$. Then, $\psi_2 \in r_{AN}(\tau) \subset r_{AG}(\pi_0)$, as needed. Since the graph of π is connected, repeated application of the lemma shows $r_{AG}(\pi_0)_{ss} = r_{AG}(\pi)_{ss}$, finishing the theorem.

COROLLARY **3.1.4** *The number of components of π is less than or equal to the number of components in graph of π.*

3.2. Applications to degenerate principal series for $Sp_{2n}(F)$ (in the regular case)

In this section, we shall apply the theorem from the last section to the case of degenerate principal series for $G = Sp_{2n}(F)$, with F p-adic of odd residual characteristic.

Let us start by recalling some notation, due to Tadić (see Chapter 1). This will expedite matters somewhat in this section, more so in the next chapter. Suppose $k_1 + \ldots + k_l = n$. Suppose that ρ_i is an admissible representation of $GL_{k_i}(F)$ for $1 \leq i \leq l-1$ and τ is an admissible representation of $Sp_{2k_l}(F)$. Let $P = MN$ be the standard parabolic subgroup of G with

$$M \cong GL_{k_1}(F) \times \ldots \times GL_{k_{l-1}}(F) \times Sp_{2k_l}(F).$$

Then, we set

$$\rho_{k_1} \times \rho_{k_2} \times \cdots \times \rho_{k_{l-1}} \propto \tau = i_{GM}(\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_{l-1} \otimes \tau).$$

We also use the following notation:

$$[\chi]_k = \chi \circ \det \text{ on } GL_k(F)$$

$$\nu = |\cdot|$$

$$tr_k = \text{ trivial representation on } Sp_{2k}(F).$$

In this notation, we can write our degenerate principal series representation as $\pi = [\chi]_k \propto tr_{n-k}$ (recall that we are most concerned with the case where P is a maximal (proper) parabolic).

We start by checking when $r_{AM}([\chi]_k \otimes tr_{n-k})$ is regular (where $M \cong GL_k(F) \times SP_{2(n-k)}(F)$).

LEMMA **3.2.1** *Let*

$$S_1 = \{-n+\tfrac{k+1}{2}, -n+1+\tfrac{k+1}{2}, \ldots, n-\tfrac{k+1}{2}\}, S_2 = \left\{\tfrac{-k+1}{2}, \tfrac{-k+2}{2}, \ldots \tfrac{k-1}{2}\right\}$$

(note that if $n = k$, $S_1 \subset S_2$). We claim that $r_{AM}([\chi]_k \otimes tr_{n-k})$ is nonregular precisely when

$$\chi \in \{\nu^\alpha | \alpha \in S_1\} \cup \{\nu^\alpha \psi | \alpha \in S_2\},$$

where $\psi^2 \equiv 1$.

Proof. We start by claiming

$$r_{AM}([\chi]_k \otimes tr_{n-k}) = \nu^{\frac{-k+1}{2}}\chi \otimes \nu^{\frac{-k+3}{2}}\chi \otimes \ldots \otimes \nu^{\frac{k-1}{2}}\chi\nu^{-n+k} \otimes \ldots \otimes \nu^{-n+k+1} \otimes \ldots \otimes \nu^{-1}$$

This follows from the fact that $[\chi]_k$ is a subrepresentation of $\nu^{\frac{-k+1}{2}}\chi \times \nu^{\frac{-k+3}{2}}\chi \times \ldots \times \nu^{\frac{k-1}{2}}\chi$, tr_{n-k} is a subrepresentation of $\nu^{-n+k} \times \ldots \times \nu^{-1} \propto 1$ and Frobenius reciprocity.

In general, the character $\chi_1 \otimes \chi_2 \otimes \ldots \otimes \chi_n$ will be nonregular for $Sp_{2n}(F)$ if either of the following conditions holds:

$$\begin{array}{lll} & i)\ \chi_i = \chi_j & \text{for some } i,j \text{ with } i \neq j \\ \text{or} & ii)\ \chi_i = \chi_j^{-1} & \text{for some } i,j \end{array}$$

Now, the particular form of our character precludes certain possibilities. Our character will be nonregular if any of the following hold:

1) $\nu^{\frac{-k+1}{2}+l}\chi = \nu^{-n+k+m}$ for some l, m with $0 \le l \le k-1$, $0 \le m \le n-k-1$

2) $\left(\nu^{\frac{-k+1}{2}+l}\chi\right)^{-1} = \nu^{-n+k+m}$ for some l, m with $0 \le l \le k-1$, $0 \le m \le n-k-1$

or 3) $\nu^{\frac{-k+1}{2}+l}\chi = \left(\nu^{\frac{-k+1}{2}+m}\chi\right)^{-1}$ for some l, m with $0 \le l, m \le k-1$.

Note that cases 1) and 2) do not occur when $n = k$. This is accounted for in the lemma by the fact that if $n = k$, $S_1 \subset S_2$.

We now observe that case 1) contributes nonregularity at

$$\chi \in \left\{\nu^{-n+\frac{k+1}{2}}, \nu^{-n+1+\frac{k+1}{2}}, \ldots, \nu^{\frac{k-3}{2}}\right\}$$

and case 2) contributes nonregularity at

$$\chi \in \left\{\nu^{\frac{-k+3}{2}}, \nu^{\frac{-k+3}{2}+1}, \ldots, \nu^{n-\frac{k+1}{2}}\right\}$$

Together, they contribute $\chi \in \{\nu^\alpha | \alpha \in S_1\}$ (when $k = 1$, these both miss ν^0, but it is in S_2 so the lemma still works out). Case 3) contributes $\chi \in \{\nu^\alpha\psi | \alpha \in S_2\}$.

This finishes the regularity lemma. We now give the lemma on reducibility of the B-Z composition factors of the Jacquet modules $r_{M_iG}(\pi)$. This will tell us when condition 3 of the theorem is satisfied.

We pause to note that the representation $\chi_1 \times \chi_2$ is reducible if and only if $\chi_1 = \nu^{\pm 1}\chi_2$. Also, $\chi \rtimes 1$ is reducible if and only if $\chi = \nu^{\pm 1}$ or $\chi^2 = 1$ with $\chi \neq 1$.

LEMMA **3.2.2** *All the B-Z composition factors of the Jacquet modules $r_{M_iG}(\pi)$, $1 \le i \le n$, are irreducible (i.e., condition 3 in the theorem holds) except (possibly) when χ has one of the following forms:*

$$\chi \in \left\{\nu^{-n+\frac{k-1}{2}}, \nu^{-n+\frac{k-1}{2}+1}, \ldots, \nu^{n-\frac{k-1}{2}}\right\}$$

or $\chi^2 \in \left\{\nu^{-k+1}, \nu^{-k+2}, \ldots \nu^{k-1}\right\}$,

with the exception that when $k = 2$, $\chi^2 = 1$ does not satisfy condition 3. Note that the two sets above are not disjoint.

Proof. First, recall that M_i takes the following forms:

$$i < n\text{: } M_i = \underbrace{F^\times \times \ldots \times F^\times}_{i-1} \times GL_2(F) \times \underbrace{F^\times \times \ldots \times F^\times}_{n-i-1}$$

$$i = n\text{: } M_n = F^\times \times \ldots \times F^\times \times Sp_2(F).$$

We remark that $Sp_2(F) = SL_2(F)$.

Now, the B-Z compositions factors of $r_{M_iG}(\pi) = r_{M_iG} \circ i_{GM}([\chi]_k \otimes tr_{n-k})$ are $i_{M_iM_i'} \circ w \circ r_{M'M}([\chi]_k \otimes tr_{n-k})$. This will take one of the following forms:

a) $i_{M_iA} \circ w \circ r_{AM}([\chi]_k \otimes tr_{n-k})$

or b) $w \circ r_{w^{-1}(M_i)M}([\chi]_k \otimes tr_{n-k})$,

depending on whether $M_i' = M_i$ or A. For the moment, let us write $\chi_1 \otimes \ldots \otimes \chi_n$ for $r_{AM}([\chi]_k \otimes tr_{n-k})$ and set $\chi_{w(1)} \otimes \ldots \otimes \chi_{w(n)} = w \cdot (\chi_1 \otimes \ldots \otimes \chi_n)$ for $w \in W$ (note that $w(i) > n$ if the character has been inverted). Then, we claim that the B-Z composition factors take one of the following more explicit forms of a), b) from above:

1. $i < n$

   a) $\chi_{w(1)} \otimes \ldots \otimes \chi_{w(i-1)} \otimes (\chi_{w(i)} \times \chi_{w(i+1)}) \otimes \chi_{w(i+2)} \otimes \ldots \otimes \chi_{w(n)}$

   b) $\chi_{w(1)} \otimes \ldots \otimes \chi_{w(i-1)} \otimes ([\chi']_2) \otimes \chi_{w(i+2)} \otimes \ldots \otimes \chi_{w(n)}$

   for some suitable $\chi' \in \hat{F^\times}$

2. i=n

   a) $\chi_{w(1)} \otimes \ldots \otimes \chi_{w(n-1)} \otimes (\chi_{w(n)} \propto 1)$

   b) $\chi_{w(1)} \otimes \ldots \otimes \chi_{w(n-1)} \otimes (tr_1)$.

For 1a), 2a), this is immediate. For 1b), 2b), it is clear since $r_{M_iM}([\chi]_k \otimes tr_{n-k})$ must be one-dimensional.

Next, we claim that all the B-Z composition factors of the Jacquet modules $r_{M_iG}(\pi)$ are irreducible if the following representations are irreducible:

i)	$(\nu^{\frac{-k+1}{2}+l}\chi) \propto 1$	for	$0 \leq l \leq k-1$
ii)	$(\nu^{\frac{-k+1}{2}+l}\chi) \times \nu^{-m}$	for	$0 \leq l \leq k-1,\ 1 \leq m \leq n-k$
iii)	$(\nu^{\frac{-k+1}{2}+l}\chi) \times \nu^{m}$	for	$0 \leq l \leq k-1,\ 1 \leq m \leq n-k$
iv)	$(\nu^{\frac{-k+1}{2}+l}\chi) \times (\nu^{\frac{-k+1}{2}+m}\chi)^{-1}$	for	$0 \leq l,\ m \leq k-1,\ l \neq m.$

(Note that *ii)* and *iii)* do not occur if $k = n$). In particular, we claim that condition *i)* is sufficient to ensure that a representation of the type in 2a) will be irreducible, while *ii)*-*iv)* guarantee that a representation of the the type in 1a) will be irreducible (1b), 2b) automatically contain only irreducible representations). To see that this is true, we start by remarking that

$$\begin{array}{llcl} & \chi \propto 1 \quad \text{irreducible} & \Longleftrightarrow & \chi^{-1} \propto 1 \text{ irreducible} \\ \text{and} & \chi_1 \times \chi_2 \quad \text{irreducible} & \Longleftrightarrow & \chi_2 \times \chi_1 \text{ irreducible} \\ & & \Longleftrightarrow & \chi_2^{-1} \times \chi_1^{-1} \text{ irreducible} \\ & & \Longleftrightarrow & \chi_1^{-1} \times \chi_2^{-1} \text{ irreducible.} \end{array}$$

With this, we see that conditions *i)*-*iv)* are enough to get irreducibility of the representations from 1a) or 2a) unless the induced part of the representation is either of the form $v^{-m} \propto 1$ for $1 \leq m \leq n-k$ or $\left(\nu^{\frac{-k+1}{2}+l}\chi\right) \times \left(\nu^{\frac{-k+1}{2}+m}\chi\right)$ for $l \leq l < m \leq k-1$ (or something with corresponding irreducibility from above). The first of these will be irreducible for $m \neq 1$, the second for $m \neq l+1$. We now claim that those representations which are automatically reducible, namely, $\nu^{-1} \propto 1$, and $\left(\nu^{\frac{-k+1}{2}+l}\chi\right) \times \left(\nu^{\frac{-k+1}{2}+l+1}\chi\right)$, do not occur as the induced part of one of the B-Z composition factors. Once we have checked this, it will follow that if the representations in i)-iv) above are all irreducible, then all the B-Z composition factors of the $r_{M_iG}(\pi)$'s were irreducible, as claimed.

Next, we argue that a representation of the form $v^{-1} \propto 1$ or $\left(\nu^{\frac{-k+1}{2}+l}\chi\right) \times \left(\nu^{\frac{-k+1}{2}+l+1}\chi\right)$ does not occur generically as the induced part of a B-Z composition factor, that is, these can only arise in the nonregular case where they can arise as the representations in i)-iv). Suppose not– e.g., suppose that $\chi_1 \otimes \ldots \otimes (\nu^{\frac{-k+1}{2}}\chi \times \nu^{\frac{-k+1}{2}+1}\chi) \otimes \ldots \otimes \chi_n$ were a B-Z composition factor of $r_{M_iG}(\pi)$ (leaving the χ_j's unspecified). Then, since $r_{AG} = r_{AM_i} \circ r_{M_iG}$, we must have $\chi_1 \otimes \ldots \otimes (\nu^{\frac{-k+1}{2}}\chi \otimes \nu^{\frac{-k+1}{2}+1}\chi) \otimes \ldots \otimes \chi_n$ and $\chi_1 \otimes \ldots \otimes (\nu^{\frac{-k+1}{2}+1}\chi \otimes \nu^{\frac{-k+1}{2}}\chi) \otimes \ldots \otimes \chi_n$ in $r_{AG}(\pi)$. Suppose that $\chi_1 \otimes \ldots \otimes (\nu^{\frac{-k+1}{2}}\chi \otimes \nu^{\frac{-k+1}{2}+1}\chi) \otimes \ldots \otimes \chi_n$ arises as $w \circ r_{AM}([\chi]_k \otimes tr_{n-k})$, $w \in W^{MA}$. Then, $\chi_1 \otimes \ldots \otimes \nu^{\frac{-k+1}{2}+1}\chi \otimes \nu^{\frac{-k+1}{2}}\chi \otimes \ldots \otimes \chi_n$ must arise as $ws_1 \circ r_{AM}([\chi]_k \otimes tr_{n-k})$ (in the regular case). However, we claim that this cannot happen because $ws_1 \notin W^{MA}$ In particular, ws_1 lies in the same W/W_M coset as w, but only one element from each such coset is in W^{MA}. Thus we have a contradiction, as needed. We note that although this was a specific example that we argued, the reasoning clearly generalizes to cover our claim.

Thus, we now know that all of the B-Z compositions factors of the $r_{M_iG}(\pi)$ will be irreducible if the representations in i)-iv) are all irreducible. It is a straightforward matter to check when this happens. We check iv) as an example– the others are similar. Now, $\nu^{\frac{-k+1}{2}+l}\chi \times \nu^{\frac{k-1}{2}+m}\chi^{-1}$ is reducible if $\nu^{\frac{-k+1}{2}+l}\chi = \nu^{\pm 1}\nu^{\frac{-k+1}{2}+m}\chi^{-1}$, or $\chi^2 = \nu^{k-1+m-l\pm 1}$ This corresponds to

$$\begin{array}{ll} \chi^2 \in \{\nu^{-k+1}, \nu^{-k}, \ldots, \nu^{k-1}\} & \text{if } k \geq 2 \\ \chi^2 \in \{\nu^{-1}, \nu\} & \text{if } k = 2 \end{array}$$

(This case does not occur if $k = 1$). If we do similar computations for i)-iii), they combine to give the lemma.

The "possibly" may be removed from the statement of the lemma. If χ is one of those characters, then there is a reducible B-Z composition factor for some $r_{M_iG}(\pi)$. However, we do not need this for the following corollary.

COROLLARY **3.2.3** *The only values of* χ *for which* $\pi = [\chi]_k \propto tr_{n-k}$ *have* $r_{AM}([\chi]_k \otimes tr_{n-k})$ *regular and* π *reducible are*

$$\chi = \nu^{-n+\frac{k-1}{2}} \text{ or } \nu^{n-\frac{k-1}{2}}$$

In either case, the trivial representation is a component of π.

Proof. Combine the two preceding lemmas to get the first claim. The second is straightforward.

CHAPTER 4

IRREDUCIBILITY CRITERIA FOR DEGENERATE PRINCIPAL SERIES IN $SP_4(F)$, $SP_6(F)$, Á LA TADIĆ

4.1. Extending definitions to the nonregular case

In this chapter, we analyze degenerate principal series in $Sp_4(F)$ and $Sp_6(F)$ using the methods of the last chapter. We start by modifying the definition of the graph associated to the degenerate principal series. We then look at $Sp_4(F)$. In this case, we already run across an irreducible representation with a disconnected graph, so Theorem 3.1.1 does not extend to the nonregular case. Although it does not occur in the examples we consider here, it is also the case that the converse fails; that one can construct a reducible representation with a connected graph, and we briefly describe such a representation. We close by analyzing degenerate principal series for $Sp_6(F)$ (induced from maximal parabolics).

We start by redefining the graph of π in a fashion more suitable for the nonregular case. This definition will have the disadvantage of being useful only when the inducing representation has a one-dimensional Jacquet module (e.g., if the inducing representation is a character or Steinberg). If $\pi = i_{GM}\rho$ for such a ρ, we define the vertices and edges as follows:

vertices: the vertices are the elements of W^{MA}.

edges: two vertices w_1, w_2 share an edge if there is a standard Levi N and a $w \in W^{MN}$ such that

1. $w_1, w_2 \in W_N w W_M$
2. $i_{NN'} \circ w \circ r_{M'M}\rho$ is irreducible

both hold.

4.2. Degenerate Principal Series in $Sp_4(F)$

In this section, we look at degenerate principal series for $Sp_4(F)$ using the techniques of the last chapter. We remark that the analysis of such representations has already been done using Hecke algebra methods, except for $[\chi]_2 \propto 1$ with χ ramified (see chapter 2 of this thesis and [Gus]).

Let $L \cong F^\times \times Sp_2(F)$ and $M \cong GL_2(F)$ be the Levi factors of the standard maximal parabolic subgroups of $G = Sp_4(F)$. Then, the subsets of W which are used in Theorem 1.2.4 are

$$W = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, s_1s_2s_1s_2 = s_2s_1s_2s_1\}$$

$$W^{MA} = \{1, s_2, s_1s_2, s_2s_1s_2\}$$

$$W^{LA} = \{1, s_1, s_2s_1, s_1s_2s_1\}$$

Any of the the other subsets of W which arise may be obtained from these using the following two observations:

1. $W^{XY} = \{w^{-1} | w \in W^{YX}\}$
2. $W^{XY} = W^{XA} \cap W^{AY}$,

where X, Y are standard Levis for G.

We can use these and Theorem 1.2.4 to construct the composition factors of the Jacquet modules for $Sp_4(F)$. The results are summarized in Tables 1 and 2.

Note that the tables are set up to respect taking Jacquet modules in stages. In particular, consider any entry in the right column. It lies in the Jacquet module of the first entry in the left column which is at the same height or higher (e.g., $\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi^{-1}$ is a B-Z composition factor of $r_{MG}([\chi]_2 \propto 1)$ and $r_{AM}(\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi^{-1})$ contains $\nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi^{-1}$ and $\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}}\chi$). One can also read off which Weyl group element a given B-Z compositions factor is constructed from. For r_{AG}, the corresponding Weyl group elements is listed in parentheses. For r_{MG} or r_{LG}, it

Table 1. Jacquet modules for $\chi \propto tr_2$

	$\underline{r_{MG}}$	$\underline{r_{AG}}$	
$\chi \propto tr_2$	$\chi \times \nu^{-1}$	$\chi \otimes \nu^{-1}$	$(w = 1)$
		$\nu^{-1} \otimes \chi$	$(w = s_1)$
	$\nu^{-1} \times \chi$	$\nu^{-1} \otimes \chi^{-1}$	$(w = s_2 s_1)$
		$\chi^{-1} \otimes \nu^{-1}$	$(w = s_1 s_2 s_1)$
	$\underline{r_{LG}}$	$\underline{r_{AG}}$	
$\chi \propto tr_2$	$\chi \otimes tr_2$	$\chi \otimes \nu^{-1}$	$(w = 1)$
	$\nu^{-1} \otimes (\chi \propto 1)$	$\nu^{-1} \otimes \chi$	$(w = s_1)$
		$\nu^{-1} \otimes \chi^{-1}$	$(w = s_2 s_1)$
	$\chi^{-1} \otimes tr_2$	$\chi^{-1} \otimes \nu^{-1}$	$(w = s_1 s_2 s_1)$

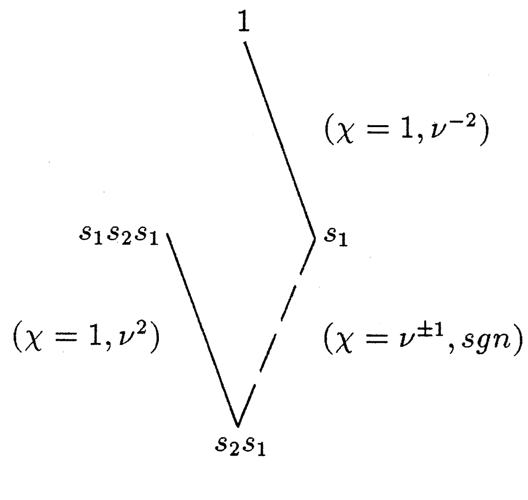

Figure 1. Graph of $\chi \propto tr_2$

Table 2. Jacquet modules for $[\chi]_2 \propto 1$

	$\underline{r_{MG}}$	$\underline{r_{AG}}$	
$[\chi]_2 \propto 1$	$[\chi]_2$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi$	$(w = 1)$
	$\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi^{-1}$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi^{-1}$	$(w = s_2)$
		$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}}\chi$	$(w = s_1 s_2)$
	$[\chi^{-1}]_2$	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1}$	$(w = s_2 s_1 s_2)$
	$\underline{r_{LG}}$	$\underline{r_{AG}}$	
$[\chi]_2 \propto 1$	$\nu^{-\frac{1}{2}}\chi \otimes (\nu^{\frac{1}{2}}\chi \propto 1)$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi$	$(w = 1)$
		$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi^{-1}$	$(w = s_2)$
	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes (\nu^{-\frac{1}{2}}\chi \propto 1)$	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}}\chi$	$(w = s_1 s_2)$
		$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1}$	$(w = s_2 s_1 s_2)$

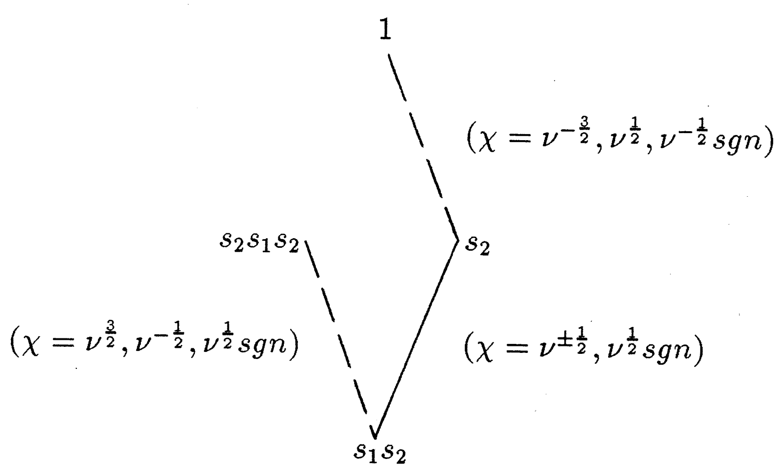

Figure 2. Graph of $[\chi]_2 \propto 1$

will just be the shortest w occurring upon Jacqueting to A (e.g., for $\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi^{-1}$, it will be the shortest of $\{s_2, s_1s_2\}$, i.e., $\{s_2\}$).

We next construct the graphs for $\chi \propto tr_2$ and $[\chi]_2 \propto 1$ for the nonregular cases. Recall from section 3.2 that nonregularity for $\chi \propto tr_2$ occurs at $\chi = \nu^{-1}, 1, sgn, \nu$ and for $[\chi]_2 \propto tr_2$, it occurs at $\chi = \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}sgn, 1, sgn, \nu^{\frac{1}{2}}, \nu^{\frac{1}{2}}sgn$. The graphs are in Figures 1 and 2. They are the graphs for general χ. The edges are "colored" according to whether they come from a B-Z composition factor for $r_{LG}(\pi)$ or $r_{MG}(\pi)$. In parentheses are the values of χ for which that edge disappears, i.e., for which the corresponding B-Z composition factor is reducible.

We start with the following observation. If $\chi \in \{\nu^{-1}, \nu\}$, then the graph of $\chi \propto tr_2$ is disconnected. However, the results of chapter 2 tell us that the representation is irreducible. Thus, it is not true in general that having an irreducible representation implies that the graph is connected.

At this point, we also mention the fact that one can construct a reducible representation with a connected graph. For example, suppose π is obtained by inducing a unitary charater from the minimal parabolic subgroup of $SO_n(F)$. Note that the Levi of a parabolic obtained by adding one simple reflection to the minimal parabolic subgroup is a product of $F^\times$'s and either $GL_2(F)$ or $SO_3(F)$. Since unitary induction in $GL_2(F)$ and $SO_3(F)$ is always irreducible, the graph of π is connected. However, π can be reducible (cf. [Keys] or [Win] for these claims).

The other nonregularity points for $\chi \propto tr_2$ are $\chi = 1, sgn$. These have disconnected graphs and are reducible. We can show reducibility using an argument like that from chapter 3. For $\chi = sgn$, the missing edge corresponds to the reducibility of $\nu^{-1} \otimes (sgn \propto 1)$. Let τ_1 and τ_2 be the components of $\nu^{-1} \otimes (sgn \propto 1)$. Then, $r_{AG}\tau_i = \nu^{-1} \otimes sgn$ (for both i). Set $\pi_i = i_{GL}\tau_i$. By Theorem 1.2.4, $r_{AG}(\pi_1)_{ss}$ consists of $\nu^{-1} \otimes sgn$, $sgn \otimes \nu^{-1}$, $sgn \otimes \nu$, $\nu \otimes sgn$. Now, if $sgn \propto tr_2$ were irreducible, it would have to be a subquotient of π_1 or π_2 since together they contain all the components of $\nu^{-1} \times sgn \propto 1$ (which has the same components as $sgn \times \nu^{-1} \propto 1 \supset sgn \propto tr_2$). However, $r_{AG}(sgn \propto tr_2)$ contains two copies of $\nu^{-1} \otimes sgn$, so this is not the case. Thus, $sgn \propto tr_2$ is irreducible. The argument for reducibility for $1 \propto tr_2$ may be done the same way– the missing lines correspond to the reducibility of $1 \times \nu^{-1}$. Let

τ_1, τ_2 be the components and set $\pi_1 = i_{GM}\tau_i$. A comparison of $r_{AG}(1 \propto tr_2)_{ss}$ with $r_{AG}(\pi_1)_{ss}$ and $r_{AG}(\pi_2)_{ss}$ shows that $1 \propto tr_2$ cannot be a subquotient of either, so it must be reducible.

We now look at $[\chi]_2 \propto 1$. Since $[\chi]_2 \propto 1$ is irreducible iff $[\chi^{-1}]_2 \propto 1$ is irreducible (they are contragredients of each other), we shall consider $\chi = 1, sgn, \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}sgn$. We start by arguing that $[\chi]_2 \propto 1$ is reducible if $\chi = \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}sgn$. The argument is like that above for $1 \propto tr_2$ and $sgn \propto tr_2$. For $\chi = \nu^{-\frac{1}{2}}$, one of the missing lines corresponds to the reducibility of $\nu^{-\frac{1}{2}} \otimes (\nu \propto 1)$. Let τ_1 τ_2 be the components of this and set $\pi_i = i_{GL}\tau_i$. Comparing $r_{AG}([\nu^{-\frac{1}{2}}]_2 \propto 1)_{ss}$ with $r_{AG}(\pi_1)_{ss}$ and $r_{AG}(\pi_2)_{ss}$ tells us $[\nu^{-\frac{1}{2}}]_2 \propto 1$ is not a subquotient of π_1 or π_2, hence must be reducible. Similarly, for $\chi = \nu^{-\frac{1}{2}}sgn$, let τ_1, τ_2 be the components of $\nu^{-1}sgn \otimes (sgn \propto 1)$ (the reducibility of which corresponds to one of the missing lines) and argue the same way.

We now look at $\chi = 1, sgn$, which we claim make $[\chi]_2 \propto 1$ irreducible. The easiest way to show this is essentially the irreducibility argument from chapter 3. Let χ be $1, sgn$ and $\pi = [\chi]_2 \propto 1$. We note that $r_{AG}(\pi)_{ss}$ consists of two copies each of $\nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi$ and $\nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi$. Let π_0 be a subquotient of π such that $r_{AG}(\pi_0)_{ss}$ contains a copy of $\nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi$. Then, $r_{LG}(\pi_0)_{ss}$ must contain a copy of $\nu^{-\frac{1}{2}}\chi \otimes (\nu^{\frac{1}{2}}\chi \times 1)$ (since $\nu^{\frac{1}{2}}\chi \propto 1$ is irreducible). Thus, $\nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi \in r_{AL}(\nu^{-\frac{1}{2}}\chi \otimes (\nu^{\frac{1}{2}}\chi \propto 1))_{ss} \subset r_{AL} \circ r_{LG}(\pi_0)_{ss} = r_{AG}(\pi_0)_{ss}$.

Next, if one copy of $\nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi$ is in $r_{AG}(\pi_0)_{ss}$ then $r_{MG}(\pi_0)_{ss}$ must contain $\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi$ (since $\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi$ is irreducible). Thus, since $r_{AM}(\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi)_{ss} \subset r_{AG}(\pi_0)_{ss}$, we see that $r_{AM}(\pi_0)_{ss}$ will contain both copies of $\nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi$. Finally, if $r_{AG}(\pi_0)_{ss}$ contains both copies of $\nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi$, then $r_{LG}(\pi_0)_{ss}$ must contain both copies of $\nu^{-\frac{1}{2}}\chi \otimes (\nu^{\frac{1}{2}}\chi \propto 1)$. Thus, we conclude that $r_{AG}(\pi_0)_{ss}$ contains both copies of $\nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi$. Since we now have $r_{AG}(\pi_0)_{ss} = r_{AG}(\pi)_{ss}$, π_0 cannot be a proper subquotient. This tells us π must be irreducible. We summarize this in the following theorem:

THEOREM **4.2.1** *1. $\chi \propto tr_2$ is reducible iff $\chi \in \{\nu^{\pm 2}, 1, sgn\}$.*

2. $[\chi]_2 \propto 1$ is reducible iff $\chi \in \{\nu^{\pm\frac{3}{2}}, \nu^{\pm\frac{1}{2}}, \nu^{\pm\frac{1}{2}}sgn\}$.

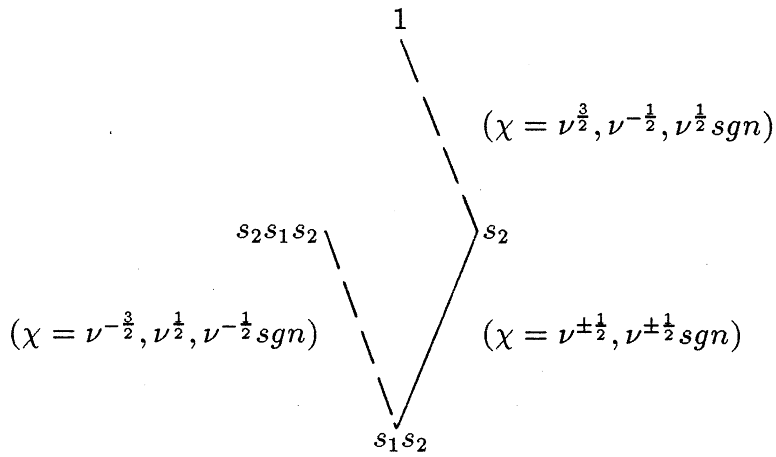

Figure 3. Graph of $St_\chi \propto 1$

Let St_χ be the Steinberg representation of $GL_2(F)$ with central character χ (so that $[\chi]_2$ and St_χ are the components of $\nu^{-\frac{1}{2}}\chi \times \nu^{\frac{1}{2}}\chi$). We can analyze $St_\chi \propto 1$ in exactly the same fashion as $[\chi]_2 \propto 1$. Since $r_{AM}(St_\chi) = \nu^{\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi$, we get the graph in Figure 3.

The same arguments then tell us that $St_\chi \propto 1$ is reducible iff $\chi \in \{\nu^{\pm\frac{3}{2}}, \nu^{\pm\frac{1}{2}}, \nu^{\pm\frac{1}{2}} sgn\}$. A similar analysis for $\chi \propto St_2$ would be inconclusive at $\chi = \nu^{\pm 1}$ (where this approach breaks down for $\chi \propto tr_2$, which is irreducible but has disconnected graph).

4.3. Degenerate Principal Series in $Sp_6(F)$

We now look at $Sp_6(F)$. Let L, M, N be the standard Levis with $L \cong F^\times \times Sp_4(F)$, $M \cong GL_2(F) \times Sp_2(F)$, and $N \cong GL_3(F)$. The following subsets of W are useful in computing Jacquet modules:

$$W^{LA} = \{1, s_1, s_2s_1, s_3s_2s_1, s_2s_3s_2s_1, s_1s_2s_3s_2s_1\}$$

$$W^{MA} = \left\{ \begin{array}{l} 1, s_2, s_1s_2, s_3s_2, s_1s_3s_2, s_2s_3s_2, s_2s_1s_3s_2, \\ s_1s_2s_3s_2, s_3s_2s_1s_3s_2, s_1s_2s_1s_3s_2, \\ s_3s_1s_2s_1s_3s_2, s_2s_3s_1s_2s_1s_3s_2 \end{array} \right\}$$

$$W^{NA} = \{1, s_3, s_2s_3, s_3s_2s_3, s_1s_2s_3, s_1s_3s_2s_3, s_2s_1s_3s_2s_3, s_3s_2s_1s_3s_2s_3\}.$$

As in section 4.2, any other subsets of W which arise may be obtained from these using the following three observations:

1. $W^{XY} = \{w^{-1} | w \in W^{YX}\}$
2. $W^{XY} = W^{XA} \cap W^{AY}$
3. $W^{(X \cap Y)A} = W^{XA} \cap W^{YA}$,

where X, Y are standard Levis for G.

Now, L, M, N are the Levis for maximal parabolics. Let $L_1 = M_1 \cong F^\times \times F^\times \times Sp_2(F)$, $L_2 = N_1 \cong F^\times \times GL_2(F)$, and $M_2 = N_2 \cong GL_2(F) \times F^\times$ be the Levis for the intermediate parabolics. Tables 3–11 give the B-Z composition factors for $\chi \propto tr_4$ (Tables 3, 4, 5), $[\chi]_2 \propto tr_2$ (Tables 6, 7, 8), $[\chi]_3 \propto 1$ (Tables 9, 10, 11).

We next construct some graphs for these (see Figures 4–19. Again, since chapter 3 covers the regular cases, we shall be concerned with the nonregular cases. Note that although the vertices are defined to be elements of W, we label them by the corresponding action on a general character, which is equivalent and and a little more intuitive. By chapter 4, these are

$$\begin{array}{ll} \chi \propto tr_4: & \chi \in \{\nu^{\pm 2}, \nu^{\pm 1}, 1, sgn\} \\ [\chi]_2 \propto tr_2: & \chi \in \{\nu^{\pm\frac{3}{2}}, \nu^{\pm\frac{1}{2}}, \nu^{\pm\frac{1}{2}} sgn, 1, sgn\} \\ [\chi]_3 \propto 1: & \chi \in \{\nu^{\pm 1}, \nu^{\pm 1} sgn, \nu^{\pm\frac{1}{2}} sgn, \nu^{\pm\frac{1}{2}}, 1, sgn\}. \end{array}$$

Since π is irreducible iff $\tilde{\pi}$ is irreducible, it suffices to analyze those χ for which $\mathrm{Re}s \geq 0$. These are in Figures 4–19. Note that to construct the graph, we will need to know when induced representations in $GL_3(F)$ such as $[\chi_1]_2 \times \chi_2$ are irreducible. This is in [Zel]– it is irreducible unless $\chi_2 = \nu^{\pm\frac{3}{2}}\chi_1$.

First, we look at the irreducible cases. All the representations with connected graphs are irreducible. This may be demonstrated using the same arguments as in the last section. We make the following observation: the connectedness of some of these graphs (e.g., $[\nu^{-\frac{1}{2}}]_3 \rtimes 1$) require B-Z composition factors from the maximal parabolics, not just the parabolics obtained by adding one simple reflection to $P_{\min}$. This contrasts with the regular case (see condition # 3 of Theorem 3.1.1).

We now look at reducibility. The arguments will be like those in section 4.2. We shall not go through all the details, but merely indicate a reducible B-Z composition factor which yields suitable choices for τ_1, τ_2. Here is a list:

$1 \rtimes tr_4$	$\nu^{-2} \otimes (\nu^{-1} \times 1)$	in	$r_{L_2G}(\pi)$
$sgn \rtimes tr_4$	$\nu^{-2} \otimes \nu^{-1} \otimes (sgn \rtimes 1)$	in	$r_{L_1G}(\pi)$
$[\nu^{-\frac{1}{2}}]_2 \rtimes tr_2$	$\nu^{-1} \times [\nu^{\frac{1}{2}}]_2$	in	$r_{NG}(\pi)$
$[\nu^{-\frac{1}{2}} sgn]_2 \rtimes tr_2$	$\nu^{-1} sgn \otimes \nu^{-1} \otimes (sgn \rtimes 1)$	in	$r_{L_1G}(\pi)$
$[\nu^{-\frac{3}{2}}]_2 \rtimes tr_2$	$\nu^{-1} \otimes ([\nu^{-\frac{3}{2}}]_2 \rtimes 1)$	in	$r_{LG}(\pi)$
$[1]_3 \rtimes 1$	$[\nu^{-\frac{1}{2}}]_2 \otimes (\nu \rtimes 1)$	in	$r_{MG}(\pi)$
$[sgn]_3 \rtimes 1$	$\nu^{-1} sgn \otimes \nu^{-1} sgn \otimes (sgn \rtimes 1)$	in	$r_{L_1G}(\pi)$
$[\nu^{-1}]_3 \rtimes 1$	$1 \otimes \nu^{-2} \otimes (\nu^{-1} \rtimes 1)$	in	$r_{L_1G}(\pi)$
$[\nu^{-1} sgn]_3 \rtimes 1$	$\nu^{-2} sgn \otimes \nu^{-1} sgn \otimes (sgn \rtimes 1)$	in	$r_{L_1G}(\pi)$

We summarize the observations above and the results on the regular case from chapter 3 in the following theorem:

THEOREM **4.3.1** *1. $\chi \rtimes tr_4$ is reducible if and only if $\chi \in \{\nu^{\pm 3}, 1, sgn\}$.*

2. $[\chi]_2 \rtimes tr_2$ is reducible if and only if $\chi \in \{\nu^{\pm\frac{5}{2}}, \nu^{\pm\frac{3}{2}}, \nu^{\pm\frac{1}{2}}, \nu^{\pm\frac{1}{2}} sgn\}$.

3. $[\chi]_3 \rtimes 1$ is reducible if and only if $\chi \in \{\nu^{\pm 2}, \nu^{\pm 1}, \nu^{\pm 1} sgn, 1, sgn\}$.

Table 3. $F^{\times} \times Sp_4(F)$ parabolic in $Sp_6(F)$, i.e., $\chi \propto tr_4$

$\underline{J_L}$	$\underline{J_{L_1}}$	$\underline{J}$
$\chi \otimes tr_4$	$\chi \otimes \nu^{-2} \otimes tr_2$	$\chi \otimes \nu^{-2} \otimes \nu^{-1}$
$\nu^{-2} \otimes (\chi \propto tr_2)$	$\nu^{-2} \otimes \chi \otimes tr_2$	$\nu^{-2} \otimes \chi \otimes \nu^{-1}$
	$\nu^{-2} \otimes \nu^{-1} \otimes (\chi \propto 1)$	$\nu^{-2} \otimes \nu^{-1} \otimes \chi$
		$\nu^{-2} \otimes \nu^{-1} \otimes \chi^{-1}$
	$\nu^{-2} \otimes \chi^{-1} \otimes tr_2$	$\nu^{-2} \otimes \chi^{-1} \otimes \nu^{-1}$
$\chi^{-1} \otimes tr_4$	$\chi^{-1} \otimes \nu^{-2} \otimes tr_2$	$\chi^{-1} \otimes \nu^{-2} \otimes \nu^{-1}$

$\underline{J_L}$	$\underline{J_{L_2}}$	$\underline{J}$
$\chi \otimes tr_4$	$\chi \otimes [\nu^{-\frac{3}{2}}]_2$	$\chi \otimes \nu^{-2} \otimes \nu^{-1}$
$\nu^{-2} \otimes (\chi \propto tr_2)$	$\nu^{-2} \otimes (\chi \times \nu^{-1})$	$\nu^{-2} \otimes \chi \otimes \nu^{-1}$
		$\nu^{-2} \otimes \nu^{-1} \otimes \chi$
	$\nu^{-2} \otimes (\nu^{-1} \times \chi^{-1})$	$\nu^{-2} \otimes \nu^{-1} \otimes \chi^{-1}$
$\chi^{-1} \otimes tr_4$		$\nu^{-2} \otimes \chi^{-1} \otimes \nu^{-1}$
	$\chi^{-1} \otimes [\nu^{-\frac{3}{2}}]_2$	$\chi^{-1} \otimes \nu^{-2} \otimes \nu^{-1}$

Table 4. $F^\times \times Sp_4(F)$ parabolic in $Sp_6(F)$,i.e., $\chi \propto tr_4$

J_M	J_{M_1}	J
$(\chi \times \nu^{-2}) \otimes tr_2$	$\chi \otimes \nu^{-2} \otimes tr_2$	$\chi \otimes \nu^{-2} \otimes \nu^{-1}$
	$\nu^{-2} \otimes \chi \otimes tr_2$	$\nu^{-2} \otimes \chi \otimes \nu^{-1}$
$[\nu^{-\frac{3}{2}}]_2 \otimes (\chi \propto 1)$	$\nu^{-2} \otimes \nu^{-1} \otimes (\chi \propto 1)$	$\nu^{-2} \otimes \nu^{-1} \otimes \chi$
		$\nu^{-2} \otimes \nu^{-1} \otimes \chi^{-1}$
$(\nu^{-2} \times \chi^{-1}) \otimes tr_2$	$\nu^{-2} \otimes \chi^{-1} \otimes tr_2$	$\nu^{-2} \otimes \chi^{-1} \otimes \nu^{-1}$
	$\chi^{-1} \otimes \nu^{-2} \otimes tr_2$	$\chi^{-1} \otimes \nu^{-2} \otimes \nu^{-1}$

J_M	J_{M_2}	J
$(\chi \times \nu^{-2}) \otimes tr_2$	$(\chi \times \nu^{-2}) \otimes \nu^{-1}$	$\chi \otimes \nu^{-2} \otimes \nu^{-1}$
		$\nu^{-2} \otimes \chi \otimes \nu^{-1}$
$[\nu^{-\frac{3}{2}}]_2 \otimes (\chi \propto 1)$	$[\nu^{-\frac{3}{2}}]_2 \otimes \chi$	$\nu^{-2} \otimes \nu^{-1} \otimes \chi$
	$[\nu^{-\frac{3}{2}}]_2 \otimes \chi^{-1}$	$\nu^{-2} \otimes \nu^{-1} \otimes \chi^{-1}$
$(\nu^{-2} \times \chi^{-1}) \otimes tr_2$	$(\nu^{-2} \times \chi^{-1}) \otimes \nu^{-1}$	$\nu^{-2} \otimes \chi^{-1} \otimes \nu^{-1}$
		$\chi^{-1} \otimes \nu^{-2} \otimes \nu^{-1}$

Table 5. $F^\times \times Sp_4(F)$ parabolic in $Sp_6(F)$,i.e.,$\chi \propto tr_4$

$\underline{J_N}$	$\underline{J_{N_1}}$	$\underline{J}$
$\chi \times [\nu^{-\frac{3}{2}}]_2$	$\chi \otimes [\nu^{-\frac{3}{2}}]_2$	$\chi \otimes \nu^{-2} \otimes \nu^{-1}$
	$\nu^{-2} \otimes (\chi \times \nu^{-1})$	$\nu^{-2} \otimes \chi \otimes \nu^{-1}$
		$\nu^{-2} \otimes \nu^{-1} \otimes \chi$
$[\nu^{-\frac{3}{2}}]_2 \times \chi^{-1}$	$\nu^{-2} \otimes (\nu^{-1} \times \chi^{-1})$	$\nu^{-2} \otimes \nu^{-1} \otimes \chi^{-1}$
		$\nu^{-2} \otimes \chi^{-1} \otimes \nu^{-1}$
	$\chi^{-1} \otimes [\nu^{-\frac{3}{2}}]_2$	$\chi^{-1} \otimes \nu^{-2} \otimes \nu^{-1}$

$\underline{J_N}$	$\underline{J_{N_2}}$	$\underline{J}$
$\chi \times [\nu^{-\frac{3}{2}}]_2$	$(\chi \times \nu^{-2}) \otimes \nu^{-1}$	$\chi \otimes \nu^{-2} \otimes \nu^{-1}$
		$\nu^{-2} \otimes \chi \otimes \nu^{-1}$
	$[\nu^{-\frac{3}{2}}]_2 \otimes \chi$	$\nu^{-2} \otimes \nu^{-1} \otimes \chi$
$[\nu^{-\frac{3}{2}}]_2 \times \chi^{-1}$	$[\nu^{-\frac{3}{2}}]_2 \otimes \chi^{-1}$	$\nu^{-2} \otimes \nu^{-1} \otimes \chi^{-1}$
	$(\nu^{-2} \times \chi^{-1}) \otimes \nu^{-1}$	$\nu^{-2} \otimes \chi^{-1} \otimes \nu^{-1}$
		$\chi^{-1} \otimes \nu^{-2} \otimes \nu^{-1}$

Table 6. $GL_2 \times Sp_{2n}(F)$ parabolic in $Sp_6(F)$,i.e., $[\chi]_2 \rtimes tr_2$

J_L	J_{L_1}	J
$\nu^{-\frac{1}{2}}\chi \otimes (\nu^{\frac{1}{2}}\chi \rtimes tr_2)$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi \otimes tr_2$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi \otimes \nu^{-1}$
	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-1} \otimes (\nu^{\frac{1}{2}}\chi \rtimes 1)$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-1} \otimes \nu^{\frac{1}{2}}\chi$
		$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi^{-1}$
	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes tr_2$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-1}$
$\nu^{-1} \otimes ([\chi]_2 \rtimes 1)$	$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes (\nu^{\frac{1}{2}}\chi \rtimes 1)$	$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi$
		$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi^{-1}$
	$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes (\nu^{-\frac{1}{2}}\chi \rtimes 1)$	$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}}\chi$
		$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1}$
$\nu^{-\frac{1}{2}}\chi^{-1} \otimes (\nu^{-\frac{1}{2}}\chi \rtimes tr_2)$	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes tr_2$	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{-1}$
	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-1} \otimes (\nu^{-\frac{1}{2}}\chi \rtimes 1)$	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi$
		$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1}$
	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1} \otimes tr_2$	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1} \otimes \nu^{-1}$

J_L	J_{L_2}	J
$\nu^{-\frac{1}{2}}\chi \otimes (\nu^{\frac{1}{2}}\chi \rtimes tr_2)$	$\nu^{-\frac{1}{2}}\chi \otimes (\nu^{\frac{1}{2}}\chi \times \nu^{-1})$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi \otimes \nu^{-1}$
		$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-1} \otimes \nu^{\frac{1}{2}}\chi$
	$\nu^{-\frac{1}{2}}\chi \otimes (\nu^{-1} \times \nu^{-\frac{1}{2}}\chi^{-1})$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi^{-1}$
		$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-1}$
$\nu^{-1} \otimes ([\chi]_2 \rtimes 1)$	$\nu^{-1} \otimes [\chi]_2$	$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi$
	$\nu^{-1} \otimes (\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi^{-1})$	$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi^{-1}$
		$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}}\chi$
	$\nu^{-1} \otimes [\chi^{-1}]_2$	$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1}$
$\nu^{-\frac{1}{2}}\chi^{-1} \otimes (\nu^{-\frac{1}{2}}\chi \rtimes tr_2)$	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes (\nu^{-\frac{1}{2}}\chi \times \nu^{-1})$	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{-1}$
		$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi$
	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes (\nu^{-1} \times \nu^{\frac{1}{2}}\chi^{-1})$	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1}$
		$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1} \otimes \nu^{-1}$

Table 7. $GL_2 \times Sp_{2n}(F)$ parabolic in $Sp_6(F)$,i.e., $[\chi]_2 \propto tr_2$

$\underline{J_M}$	$\underline{J_{M_1}}$	$\underline{J}$
$[\chi]_2 \otimes tr_2$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi \otimes tr_2$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi \otimes \nu^{-1}$
$(\nu^{-\frac{1}{2}}\chi \times \nu^{-1}) \otimes (\nu^{\frac{1}{2}}\chi \propto 1)$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-1} \otimes (\nu^{\frac{1}{2}}\chi \propto 1)$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-1} \otimes \nu^{\frac{1}{2}}\chi$
		$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi^{-1}$
	$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes (\nu^{\frac{1}{2}}\chi \propto 1)$	$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi$
		$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi^{-1}$
$(\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi^{-1}) \otimes tr_2$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes tr_2$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-1}$
	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes tr_2$	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{-1}$
$(\nu^{-1} \times \nu^{-\frac{1}{2}}\chi^{-1}) \otimes (\nu^{-\frac{1}{2}}\chi \propto 1)$	$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes (\nu^{-\frac{1}{2}}\chi \propto 1)$	$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}}\chi$
		$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1}$
	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-1} \otimes (\nu^{-\frac{1}{2}}\chi \propto 1)$	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi$
		$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1}$
$[\chi^{-1}]_2 \otimes tr_2$	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1} \otimes tr_2$	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1} \otimes \nu^{-1}$

$\underline{J_M}$	$\underline{J_{M_2}}$	$\underline{J}$
$[\chi]_2 \otimes tr_2$	$[\chi]_2 \otimes \nu^{-1}$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi \otimes \nu^{-1}$
$(\nu^{-\frac{1}{2}}\chi \times \nu^{-1}) \otimes (\nu^{\frac{1}{2}}\chi \propto 1)$	$(\nu^{-\frac{1}{2}}\chi \times \nu^{-1}) \otimes \nu^{\frac{1}{2}}\chi$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-1} \otimes \nu^{\frac{1}{2}}\chi$
		$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi$
	$(\nu^{-\frac{1}{2}}\chi \times \nu^{-1}) \otimes \nu^{-\frac{1}{2}}\chi^{-1}$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi^{-1}$
		$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi^{-1}$
$(\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi^{-1}) \otimes tr_2$	$(\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi^{-1}) \otimes \nu^{-1}$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-1}$
		$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{-1}$
$(\nu^{-1} \times \nu^{-\frac{1}{2}}\chi^{-1}) \otimes (\nu^{-\frac{1}{2}}\chi \propto 1)$	$(\nu^{-1} \times \nu^{-\frac{1}{2}}\chi^{-1}) \otimes \nu^{-\frac{1}{2}}\chi$	$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}}\chi$
		$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi$
	$(\nu^{-1} \times \nu^{-\frac{1}{2}}\chi^{-1}) \otimes \nu^{\frac{1}{2}}\chi^{-1}$	$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1}$
		$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1}$
$[\chi^{-1}]_2 \otimes tr_2$	$[\chi^{-1}]_2 \otimes \nu^{-1}$	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1} \otimes \nu^{-1}$

Table 8. $GL_2 \times Sp_{2n}(F)$ parabolic in $Sp_6(F)$,i.e., $[\chi]_2 \propto tr_2$

J_N	J_{N_1}	J
$[\chi]_2 \times \nu^{-1}$	$\nu^{-\frac{1}{2}}\chi \otimes (\nu^{\frac{1}{2}}\chi \times \nu^{-1})$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi \otimes \nu^{-1}$
		$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-1} \otimes \nu^{\frac{1}{2}}\chi$
	$\nu^{-1} \otimes [\chi]_2$	$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi$
$\nu^{-\frac{1}{2}}\chi \times \nu^{-1} \times \nu^{-\frac{1}{2}}\chi^{-1}$	$\nu^{-\frac{1}{2}}\chi \otimes (\nu^{-1} \times \nu^{-\frac{1}{2}}\chi^{-1})$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi^{-1}$
		$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-1}$
	$\nu^{-1} \otimes (\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi^{-1})$	$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi^{-1}$
		$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}}\chi$
	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes (\nu^{-\frac{1}{2}}\chi \times \nu^{-1})$	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{-1}$
		$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi$
$\nu^{-1} \times [\chi^{-1}]_2$	$\nu^{-1} \otimes [\chi^{-1}]_2$	$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1}$
	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes (\nu^{-1} \times \nu^{\frac{1}{2}}\chi^{-1})$	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1}$
		$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1} \otimes \nu^{-1}$

J_N	J_{N_2}	J
$[\chi]_2 \times \nu^{-1}$	$[\chi]_2 \otimes \nu^{-1}$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi \otimes \nu^{-1}$
	$(\nu^{-\frac{1}{2}}\chi \times \nu^{-1}) \otimes \nu^{\frac{1}{2}}\chi$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-1} \otimes \nu^{\frac{1}{2}}\chi$
		$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi$
$\nu^{-\frac{1}{2}}\chi \times \nu^{-1} \times \nu^{-\frac{1}{2}}\chi^{-1}$	$(\nu^{-\frac{1}{2}}\chi \times \nu^{-1}) \otimes \nu^{-\frac{1}{2}}\chi^{-1}$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi^{-1}$
		$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi^{-1}$
	$(\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi^{-1}) \otimes \nu^{-1}$	$\nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-1}$
		$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{-1}$
	$(\nu^{-1} \times \nu^{-\frac{1}{2}}\chi^{-1}) \otimes \nu^{-\frac{1}{2}}\chi$	$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}}\chi$
		$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi$
$\nu^{-1} \times [\chi^{-1}]_2$	$(\nu^{-1} \times \nu^{-\frac{1}{2}}\chi^{-1}) \otimes \nu^{\frac{1}{2}}\chi^{-1}$	$\nu^{-1} \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1}$
		$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1}$
	$[\chi^{-1}]_2 \otimes \nu^{-1}$	$\nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1} \otimes \nu^{-1}$

Table 9. GL_3 parabolic in $Sp_6(F)$,i.e., $[\chi]_3 \propto 1$

J_L	J_{L_1}	J
$\nu^{-1}\chi \otimes ([\nu^{\frac{1}{2}}\chi]_2 \propto 1)$	$\nu^{-1}\chi \otimes \chi \otimes (\nu\chi \propto 1)$	$\nu^{-1}\chi \otimes \chi \otimes \nu\chi$
		$\nu^{-1}\chi \otimes \chi \otimes \nu^{-1}\chi^{-1}$
	$\nu^{-1}\chi \otimes \nu^{-1}\chi^{-1} \otimes (\chi \propto 1)$	$\nu^{-1}\chi \otimes \nu^{-1}\chi^{-1} \otimes \chi$
		$\nu^{-1}\chi \otimes \nu^{-1}\chi^{-1} \otimes \chi^{-1}$
$\nu^{-1}\chi^{-1} \otimes ([\nu^{-\frac{1}{2}}\chi])_2 \propto 1)$	$\nu^{-1}\chi^{-1} \otimes \nu^{-1}\chi \otimes (\chi \propto 1)$	$\nu^{-1}\chi^{-1} \otimes \nu^{-1}\chi \otimes \chi$
		$\nu^{-1}\chi^{-1} \otimes \nu^{-1}\chi \otimes \chi^{-1}$
	$\nu^{-1}\chi^{-1} \otimes \chi^{-1} \otimes (\nu^{-1}\chi \propto 1)$	$\nu^{-1}\chi^{-1} \otimes \chi^{-1} \otimes \nu^{-1}\chi$
		$\nu^{-1}\chi^{-1} \otimes \chi^{-1} \otimes \nu\chi^{-1}$

J_L	J_{L_2}	J
$\nu^{-1}\chi \otimes ([\nu^{\frac{1}{2}}\chi]_2 \propto 1)$	$\nu^{-1}\chi \otimes [\nu^{\frac{1}{2}}\chi]_2$	$\nu^{-1}\chi \otimes \chi \otimes \nu\chi$
	$\nu^{-1}\chi \otimes (\chi \times \nu^{-1}\chi^{-1})$	$\nu^{-1}\chi \otimes \chi \otimes \nu^{-1}\chi^{-1}$
		$\nu^{-1}\chi \otimes \nu^{-1}\chi^{-1} \otimes \chi$
	$\nu^{-1}\chi \otimes [\nu^{-\frac{1}{2}}\chi^{-1}]_2$	$\nu^{-1}\chi \otimes \nu^{-1}\chi^{-1} \otimes \chi^{-1}$
$\nu^{-1}\chi^{-1} \otimes ([\nu^{-\frac{1}{2}}\chi])_2 \propto 1)$	$\nu^{-1}\chi^{-1} \otimes [\nu^{-\frac{1}{2}}\chi]_2$	$\nu^{-1}\chi^{-1} \otimes \nu^{-1}\chi \otimes \chi$
	$\nu^{-1}\chi^{-1} \otimes (\nu^{-1}\chi \times \chi^{-1})$	$\nu^{-1}\chi^{-1} \otimes \nu^{-1}\chi \otimes \chi^{-1}$
		$\nu^{-1}\chi^{-1} \otimes \chi^{-1} \otimes \nu^{-1}\chi$
	$\nu^{-1}\chi \otimes [\nu^{\frac{1}{2}}\chi^{-1}]_2$	$\nu^{-1}\chi^{-1} \otimes \chi^{-1} \otimes \nu\chi^{-1}$

Table 10. GL_3 parabolic in $Sp_6(F)$,i.e., $[\chi]_3 \propto 1$

$\underline{J_M}$	$\underline{J_{M_1}}$	$\underline{J}$
$[\nu^{-\frac{1}{2}}\chi]_2 \otimes (\nu\chi \propto 1)$	$\nu^{-1}\chi \otimes \chi \otimes (\nu\chi \propto 1)$	$\nu^{-1}\chi \otimes \chi \otimes \nu\chi$
		$\nu^{-1}\chi \otimes \chi \otimes \nu^{-1}\chi^{-1}$
$(\nu^{-1}\chi \times \nu^{-1}\chi^{-1}) \otimes (\chi \propto 1)$	$\nu^{-1}\chi \otimes \nu^{-1}\chi^{-1} \otimes (\chi \propto 1)$	$\nu^{-1}\chi \otimes \nu^{-1}\chi^{-1} \otimes \chi$
		$\nu^{-1}\chi \otimes \nu^{-1}\chi^{-1} \otimes \chi^{-1}$
	$\nu^{-1}\chi^{-1} \otimes \nu^{-1}\chi \otimes (\chi \propto 1)$	$\nu^{-1}\chi^{-1} \otimes \nu^{-1}\chi \otimes \chi$
		$\nu^{-1}\chi^{-1} \otimes \nu^{-1}\chi \otimes \chi^{-1}$
$[\nu^{-\frac{1}{2}}\chi^{-1}]_2 \otimes (\nu^{-1}\chi \propto 1)$	$\nu^{-1}\chi^{-1} \otimes \chi^{-1} \otimes (\nu^{-1}\chi \propto 1)$	$\nu^{-1}\chi^{-1} \otimes \chi^{-1} \otimes \nu^{-1}\chi$
		$\nu^{-1}\chi^{-1} \otimes \chi^{-1} \otimes \nu\chi^{-1}$

$\underline{J_M}$	$\underline{J_{M_2}}$	$\underline{J}$
$[\nu^{-\frac{1}{2}}\chi]_2 \otimes (\nu\chi \propto 1)$	$[\nu^{-\frac{1}{2}}\chi]_2 \otimes \nu\chi$	$\nu^{-1}\chi \otimes \chi \otimes \nu\chi$
	$[\nu^{-\frac{1}{2}}\chi]_2 \otimes \nu^{-1}\chi^{-1}$	$\nu^{-1}\chi \otimes \chi \otimes \nu^{-1}\chi^{-1}$
$(\nu^{-1}\chi \times \nu^{-1}\chi^{-1}) \otimes (\chi \propto 1)$	$(\nu^{-1}\chi \times \nu^{-1}\chi^{-1}) \otimes \chi$	$\nu^{-1}\chi \otimes \nu^{-1}\chi^{-1} \otimes \chi$
		$\nu^{-1}\chi^{-1} \otimes \nu^{-1}\chi \otimes \chi$
	$(\nu^{-1}\chi \times \nu^{-1}\chi^{-1}) \otimes \chi^{-1}$	$\nu^{-1}\chi \otimes \nu^{-1}\chi^{-1} \otimes \chi^{-1}$
		$\nu^{-1}\chi^{-1} \otimes \nu^{-1}\chi \otimes \chi^{-1}$
$[\nu^{-\frac{1}{2}}\chi^{-1}]_2 \otimes (\nu^{-1}\chi \propto 1)$	$[\nu^{-\frac{1}{2}}\chi^{-1}]_2 \otimes \nu^{-1}\chi$	$\nu^{-1}\chi^{-1} \otimes \chi^{-1} \otimes \nu^{-1}\chi$
	$[\nu^{-\frac{1}{2}}\chi^{-1}]_2 \otimes \nu\chi^{-1}$	$\nu^{-1}\chi^{-1} \otimes \chi^{-1} \otimes \nu\chi^{-1}$

Table 11. GL_3 parabolic in $Sp_6(F)$,i.e., $[\chi]_3 \propto 1$

$\underline{J_N}$	$\underline{J_{N_1}}$	$\underline{J}$
$[\chi]_3$	$\nu^{-1}\chi \otimes [\nu^{\frac{1}{2}}\chi]_2$	$\nu^{-1}\chi \otimes \chi \otimes \nu\chi$
$[\nu^{-\frac{1}{2}}\chi]_2 \times \nu^{-1}\chi^{-1}$	$\nu^{-1}\chi \otimes (\chi \times \nu^{-1}\chi^{-1})$	$\nu^{-1}\chi \otimes \chi \otimes \nu^{-1}\chi^{-1}$
		$\nu^{-1}\chi \otimes \nu^{-1}\chi^{-1} \otimes \chi$
	$\nu^{-1}\chi^{-1} \otimes [\nu^{-\frac{1}{2}}\chi]_2$	$\nu^{-1}\chi^{-1} \otimes \nu^{-1}\chi \otimes \chi$
$\nu^{-1}\chi \times [\nu^{-\frac{1}{2}}\chi^{-1}]_2$	$\nu^{-1}\chi^{-1} \otimes (\nu^{-1}\chi \times \chi^{-1})$	$\nu^{-1}\chi^{-1} \otimes \nu^{-1}\chi \otimes \chi^{-1}$
		$\nu^{-1}\chi^{-1} \otimes \chi^{-1} \otimes \nu^{-1}\chi$
	$\nu^{-1}\chi \otimes [\nu^{-\frac{1}{2}}\chi^{-1}]_2$	$\nu^{-1}\chi \otimes \nu^{-1}\chi^{-1} \otimes \chi^{-1}$
$[\chi^{-1}]_3$	$\nu^{-1}\chi^{-1} \otimes [\nu^{\frac{1}{2}}\chi^{-1}]_2$	$\nu^{-1}\chi^{-1} \otimes \chi^{-1} \otimes \nu\chi^{-1}$

$\underline{J_N}$	$\underline{J_{N_2}}$	$\underline{J}$
$[\chi]_3$	$[\nu^{-\frac{1}{2}}\chi]_2 \otimes \nu\chi$	$\nu^{-1}\chi \otimes \chi \otimes \nu\chi$
$[\nu^{-\frac{1}{2}}\chi]_2 \times \nu^{-1}\chi^{-1}$	$(\nu^{-1}\chi \times \nu^{-1}\chi^{-1}) \otimes \chi$	$\nu^{-1}\chi \otimes \nu^{-1}\chi^{-1} \otimes \chi$
		$\nu^{-1}\chi^{-1} \otimes \nu^{-1}\chi \otimes \chi$
	$[\nu^{-\frac{1}{2}}\chi]_2 \otimes \nu^{-1}\chi^{-1}$	$\nu^{-1}\chi \otimes \chi \otimes \nu^{-1}\chi^{-1}$
$\nu^{-1}\chi \times [\nu^{-\frac{1}{2}}\chi^{-1}]_2$	$(\nu^{-1}\chi \times \nu^{-1}\chi^{-1}) \otimes \chi^{-1}$	$\nu^{-1}\chi \otimes \nu^{-1}\chi^{-1} \otimes \chi^{-1}$
		$\nu^{-1}\chi^{-1} \otimes \nu^{-1}\chi \otimes \chi^{-1}$
	$[\nu^{-\frac{1}{2}}\chi^{-1}]_2 \otimes \nu^{-1}\chi$	$\nu^{-1}\chi^{-1} \otimes \chi^{-1} \otimes \nu^{-1}\chi$
$[\chi^{-1}]_3$	$[\nu^{-\frac{1}{2}}\chi^{-1}]_2 \otimes \nu\chi^{-1}$	$\nu^{-1}\chi^{-1} \otimes \chi^{-1} \otimes \nu\chi^{-1}$

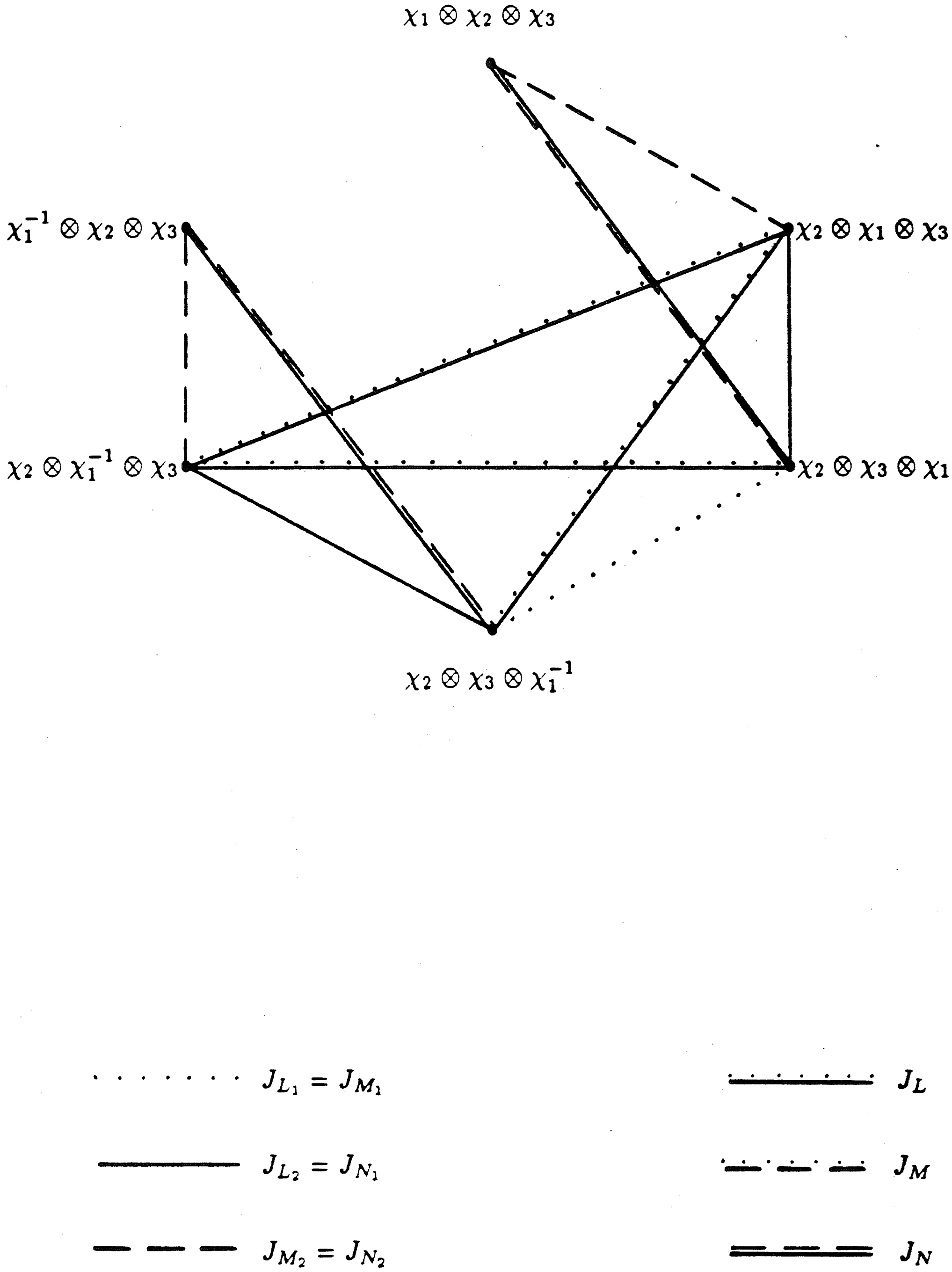

Figure 4. Graph of $\chi \propto tr_4$, χ generic

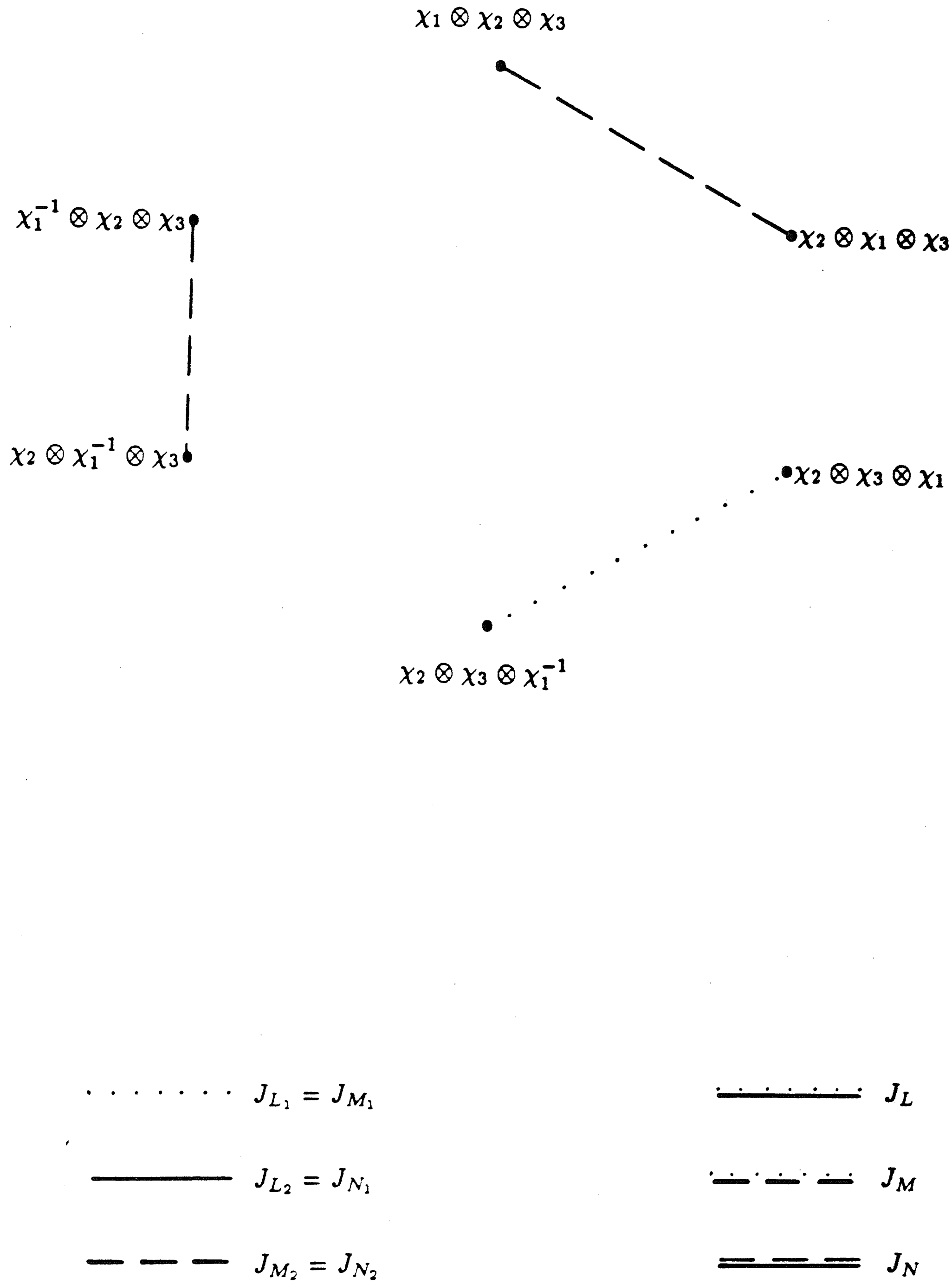

Figure 5. Graph of $\chi \propto tr_4$, $\chi = 1$

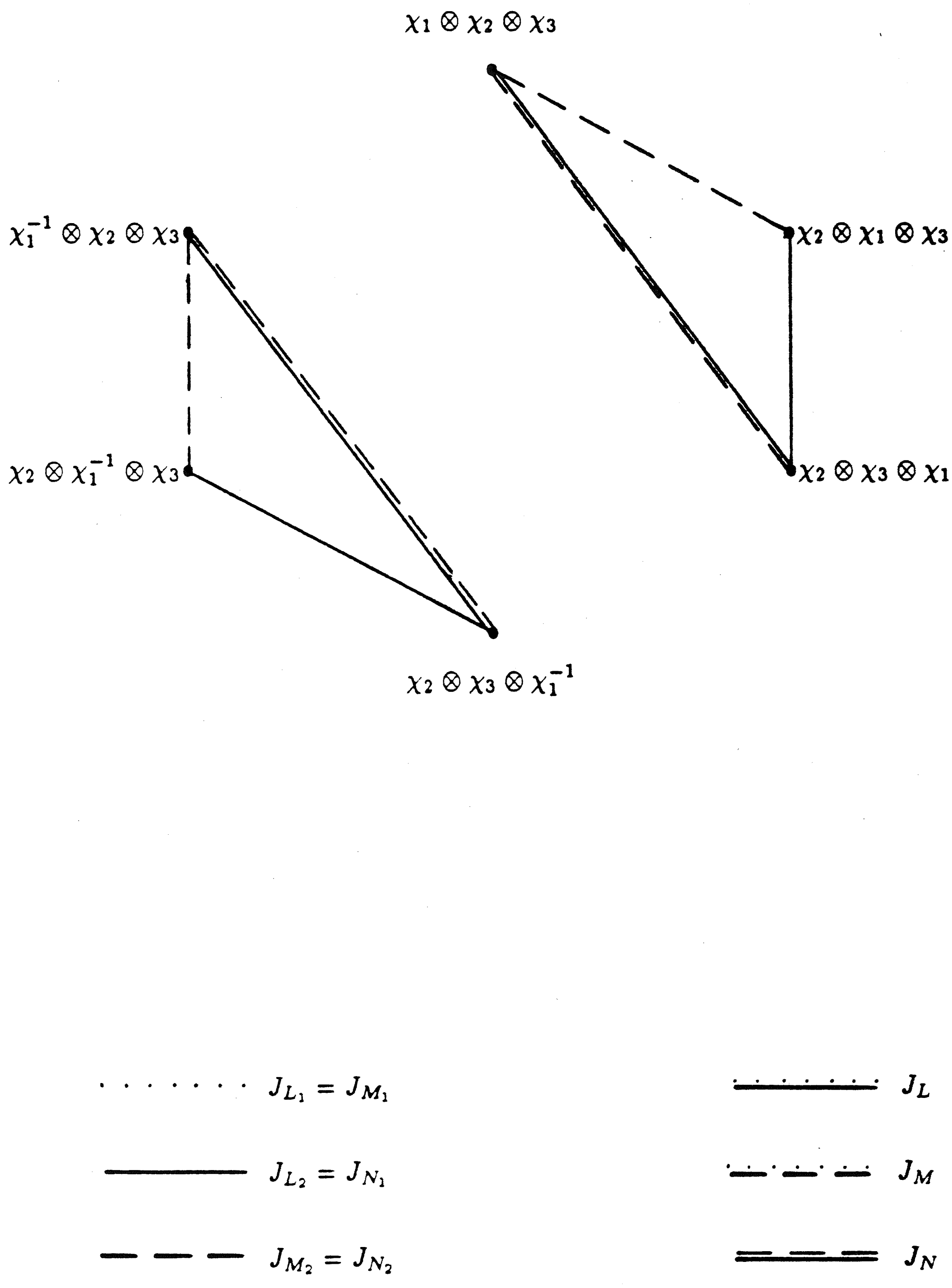

Figure 6. Graph of $\chi \propto tr_4$, $\chi = sgn$

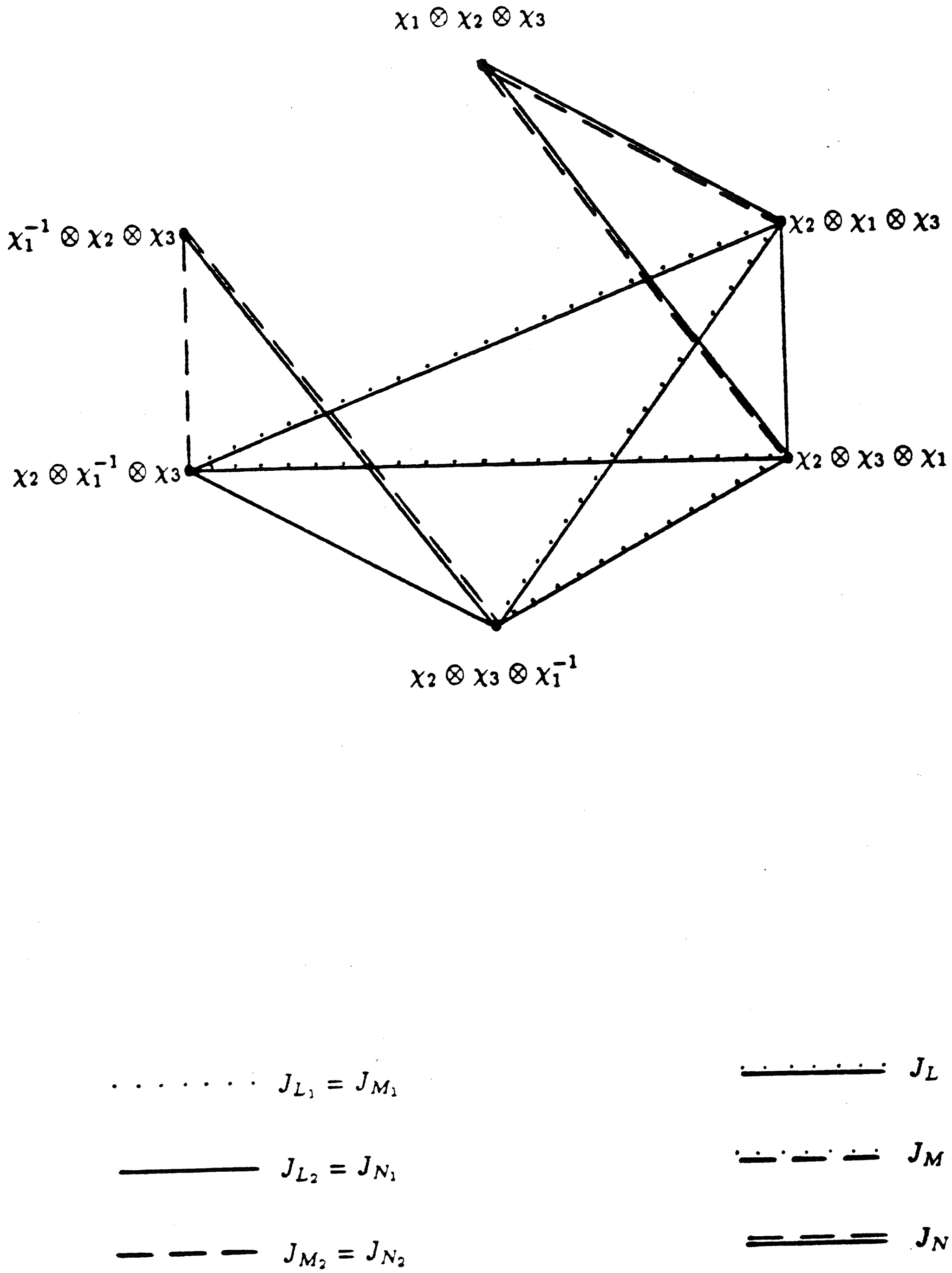

Figure 7. Graph of $\chi \propto tr_4$, $\chi = \nu^{-1}$

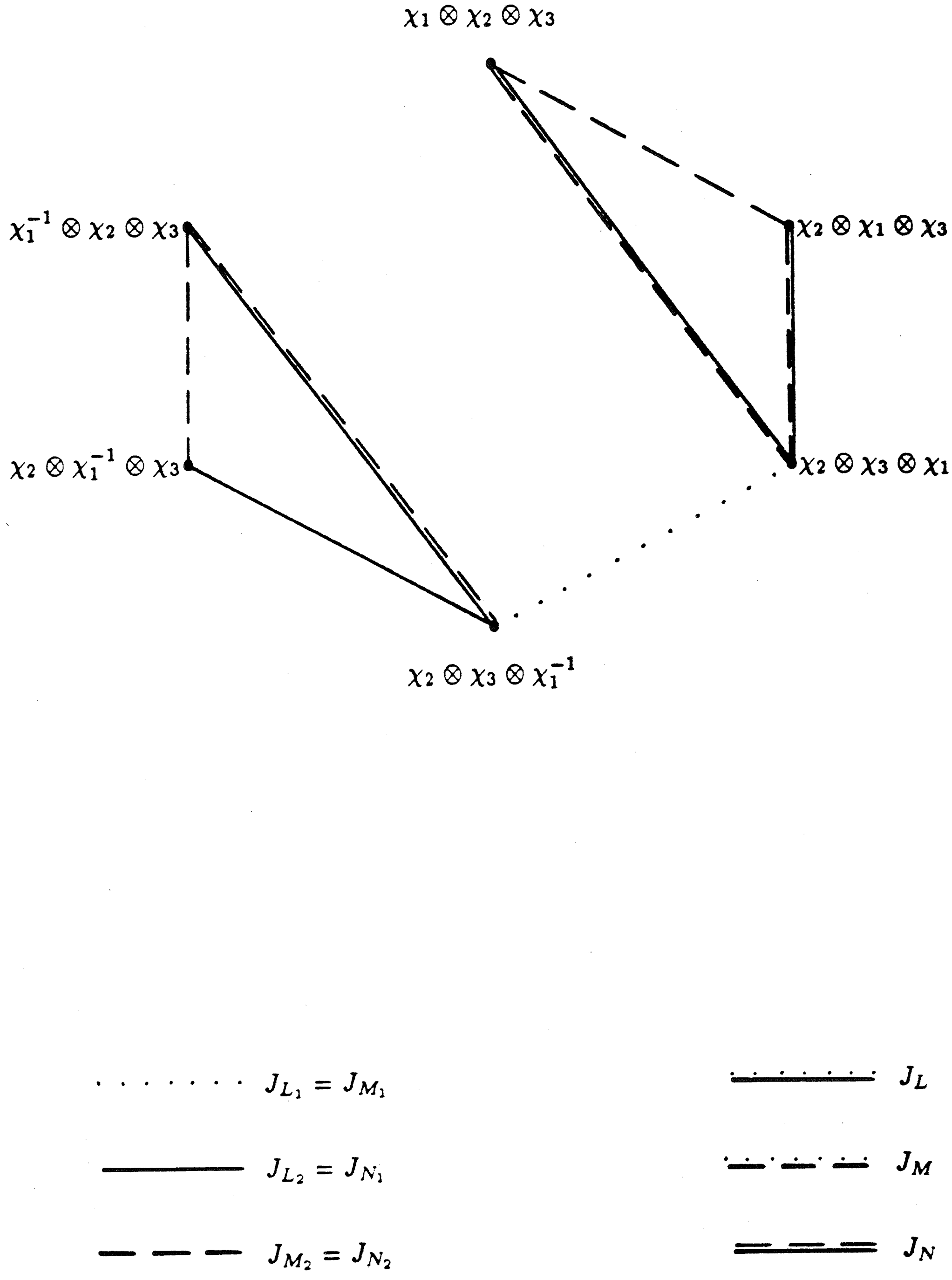

Figure 8. Graph of $\chi \propto tr_4$, $\chi = \nu^{-2}$

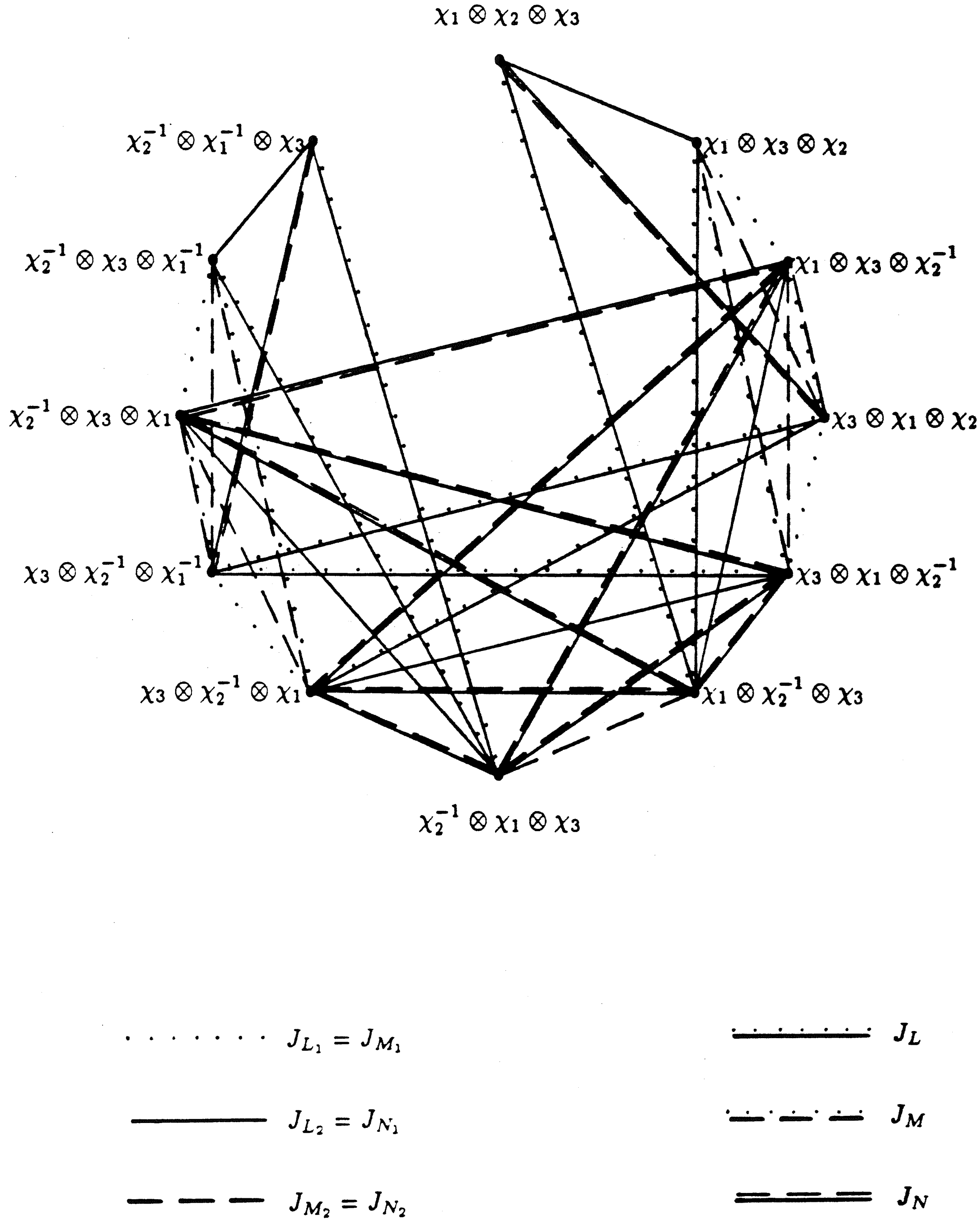

Figure 9. Graph of $[\chi]_2 \propto tr_2$, χ generic (also $\chi = 1, sgn$)

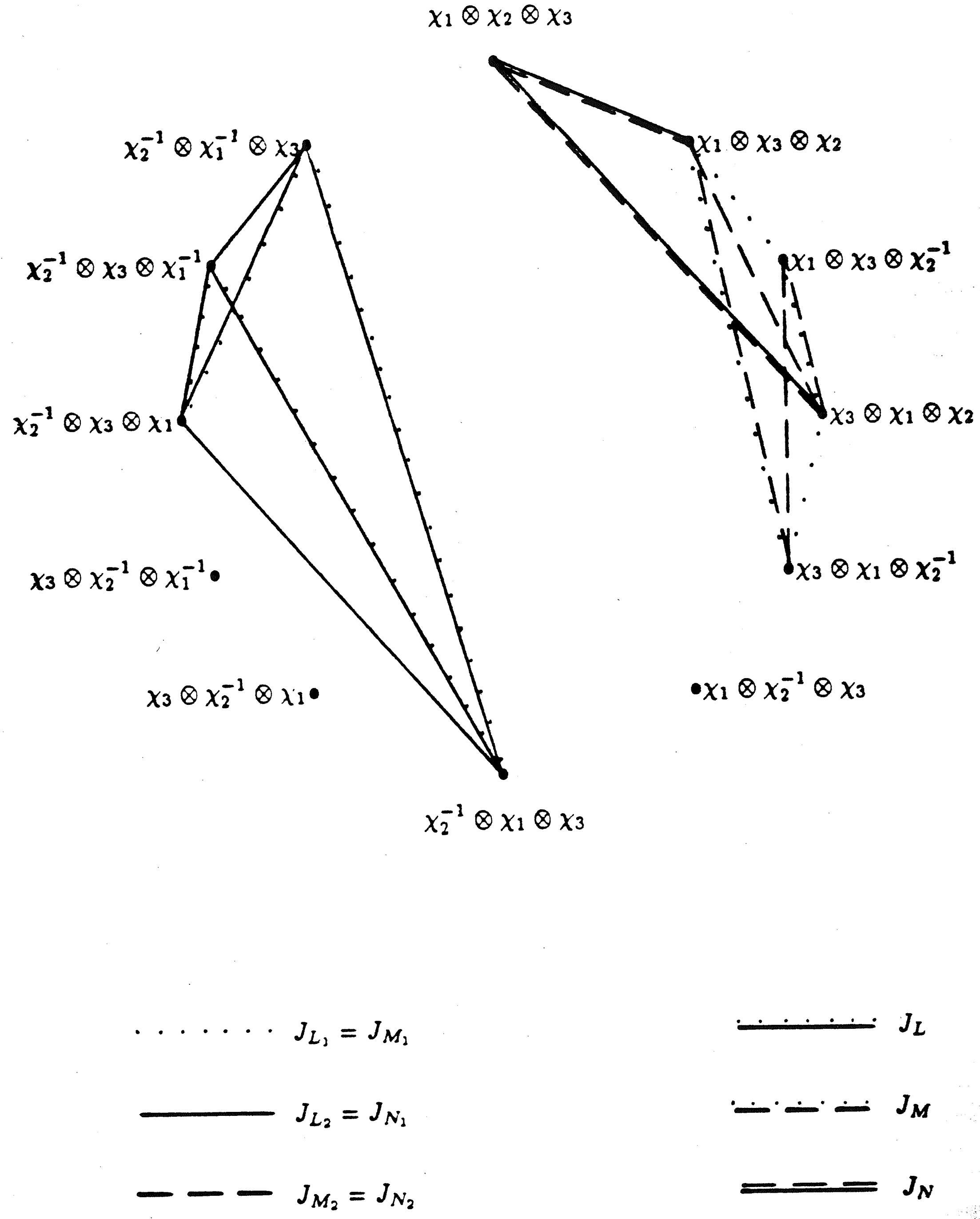

Figure 10. Graph of $[\chi]_2 \propto tr_2$, $\chi = \nu^{-\frac{1}{2}}$

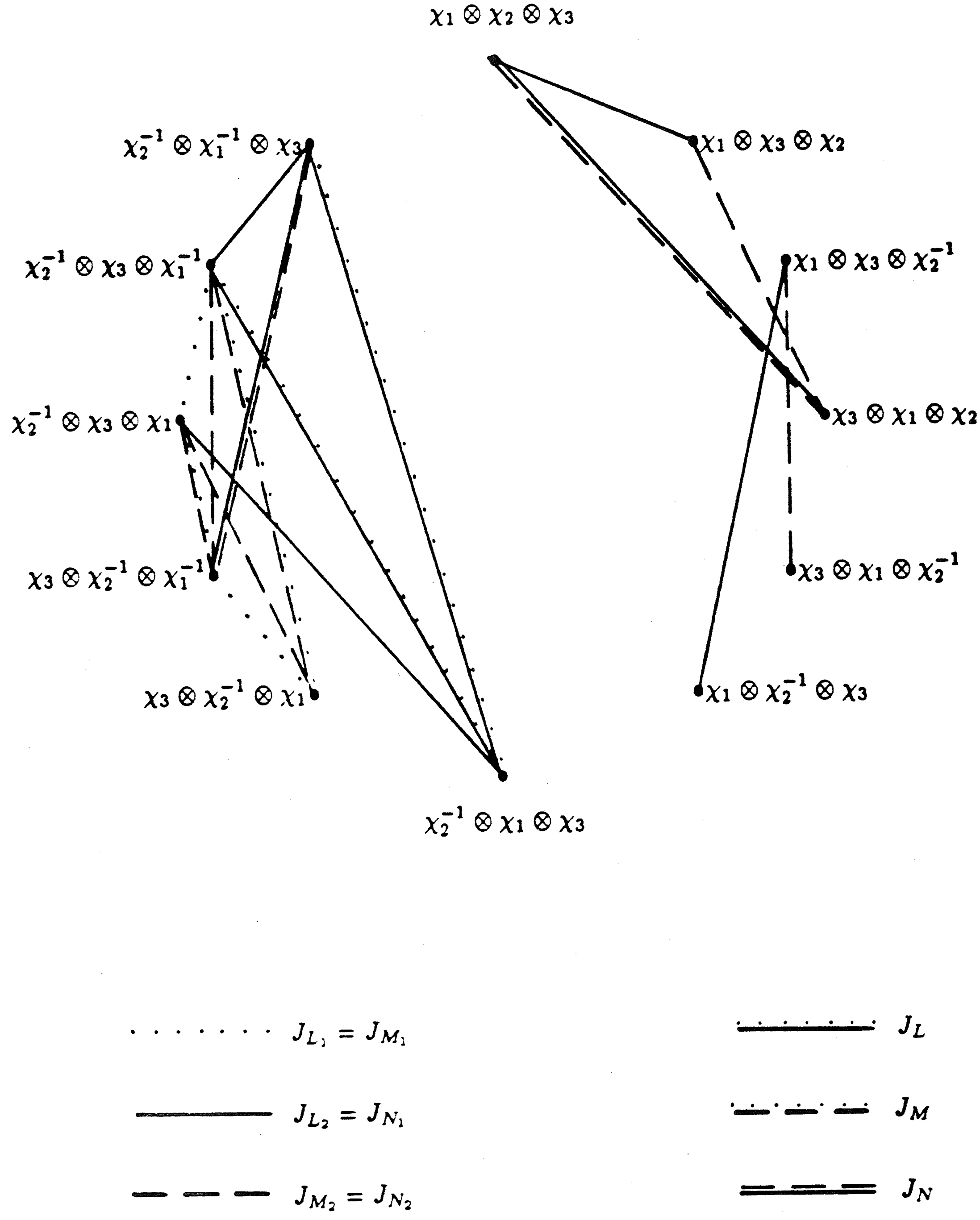

Figure 11. Graph of $[\chi]_2 \propto tr_2$, $\chi = \nu^{-\frac{1}{2}} sgn$

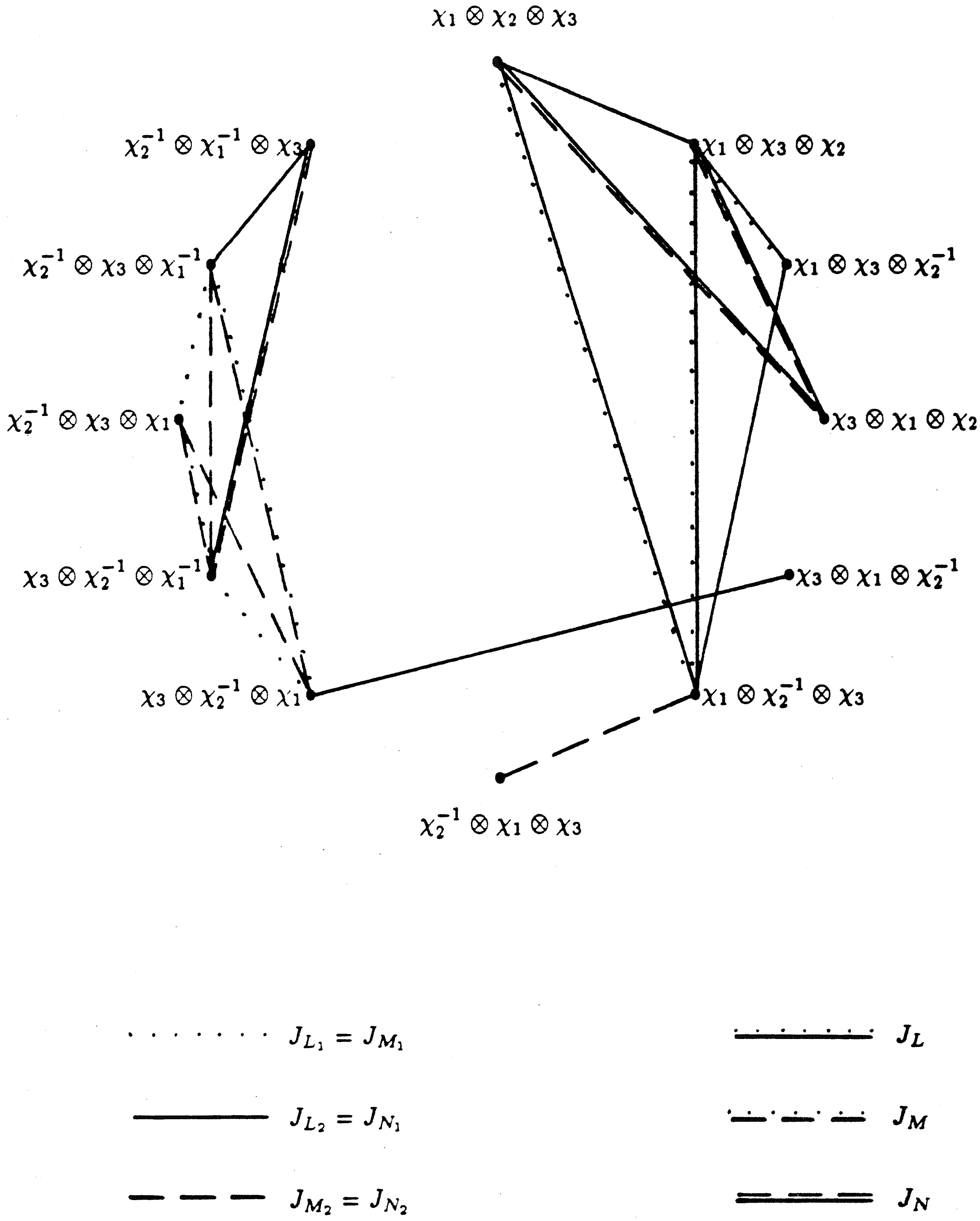

Figure 12. Graph of $[\chi]_2 \propto tr_2$, $\chi = \nu^{-\frac{3}{2}}$

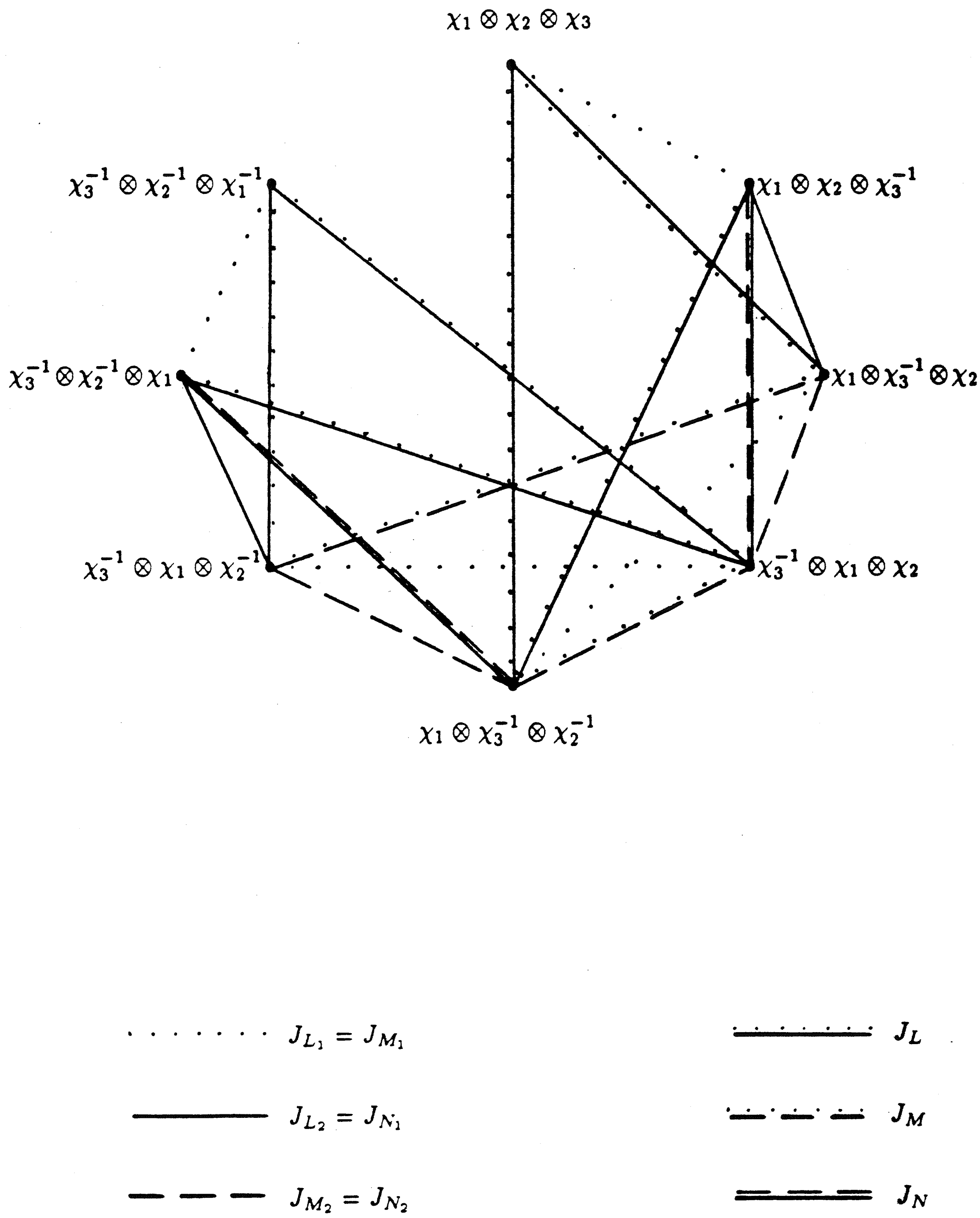

Figure 13. Graph of $[\chi]_3 \propto 1$, χ generic

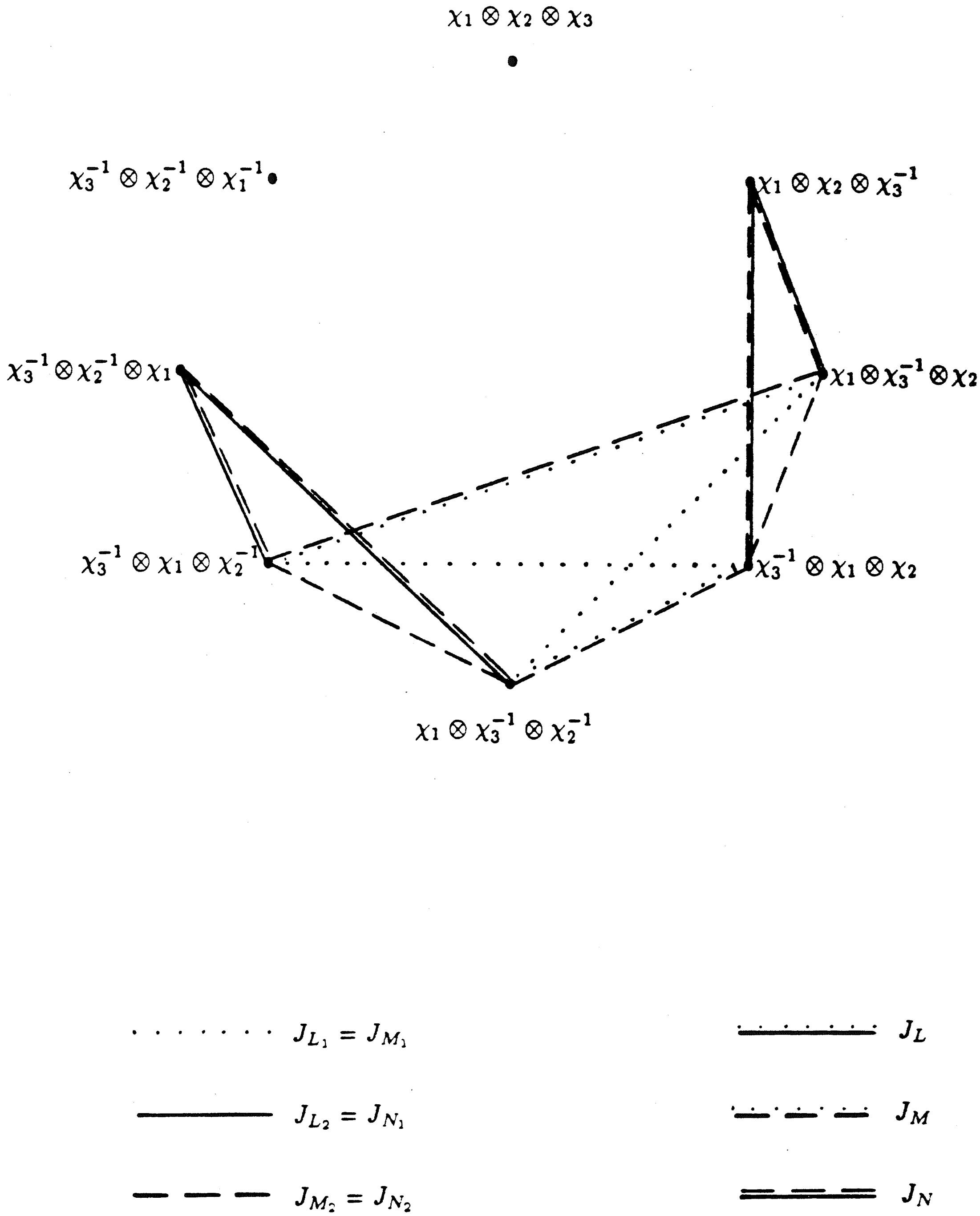

Figure 14. Graph of $[\chi]_3 \rtimes 1$, $\chi = 1$

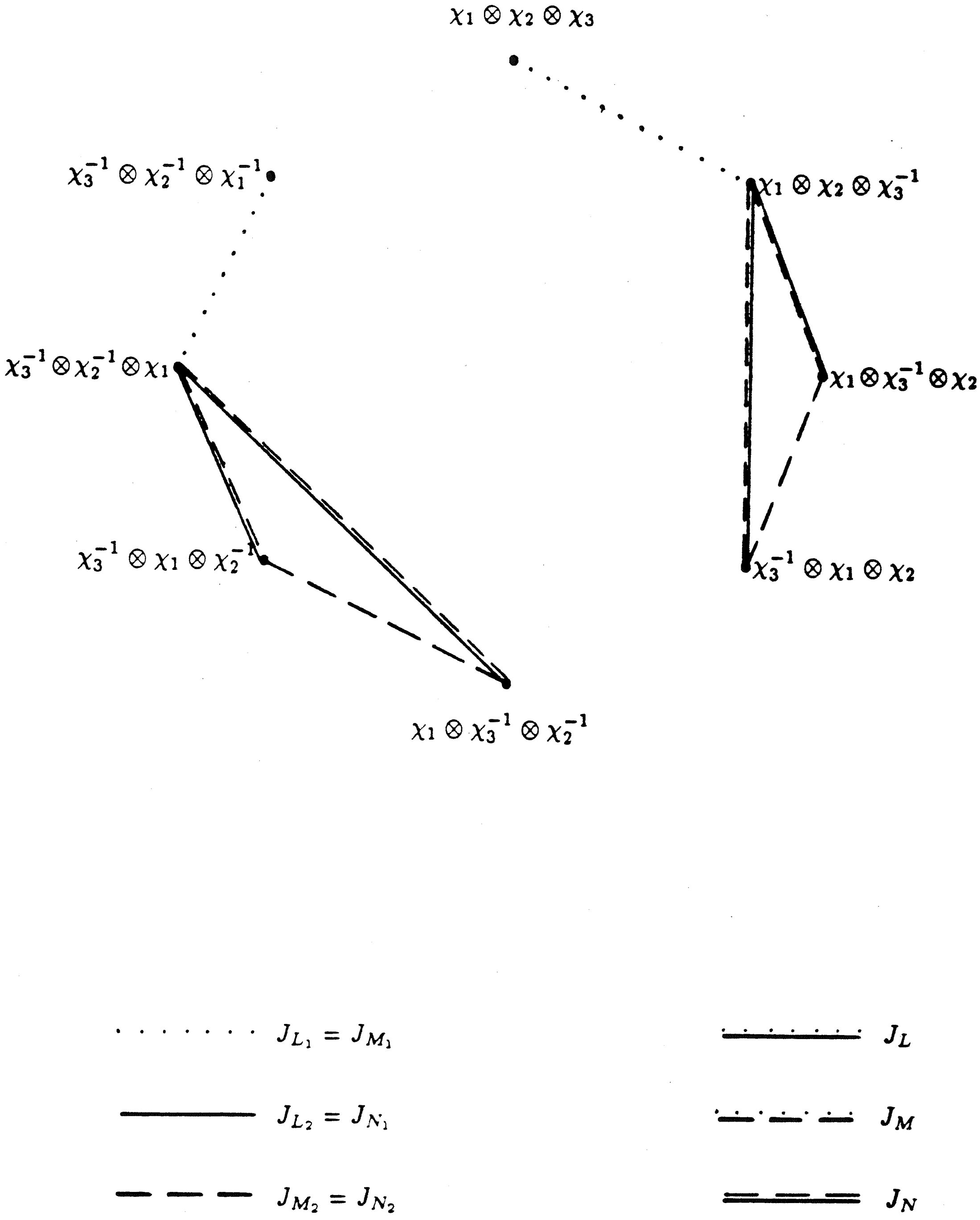

Figure 15. Graph of $[\chi]_3 \propto 1$, $\chi = sgn$

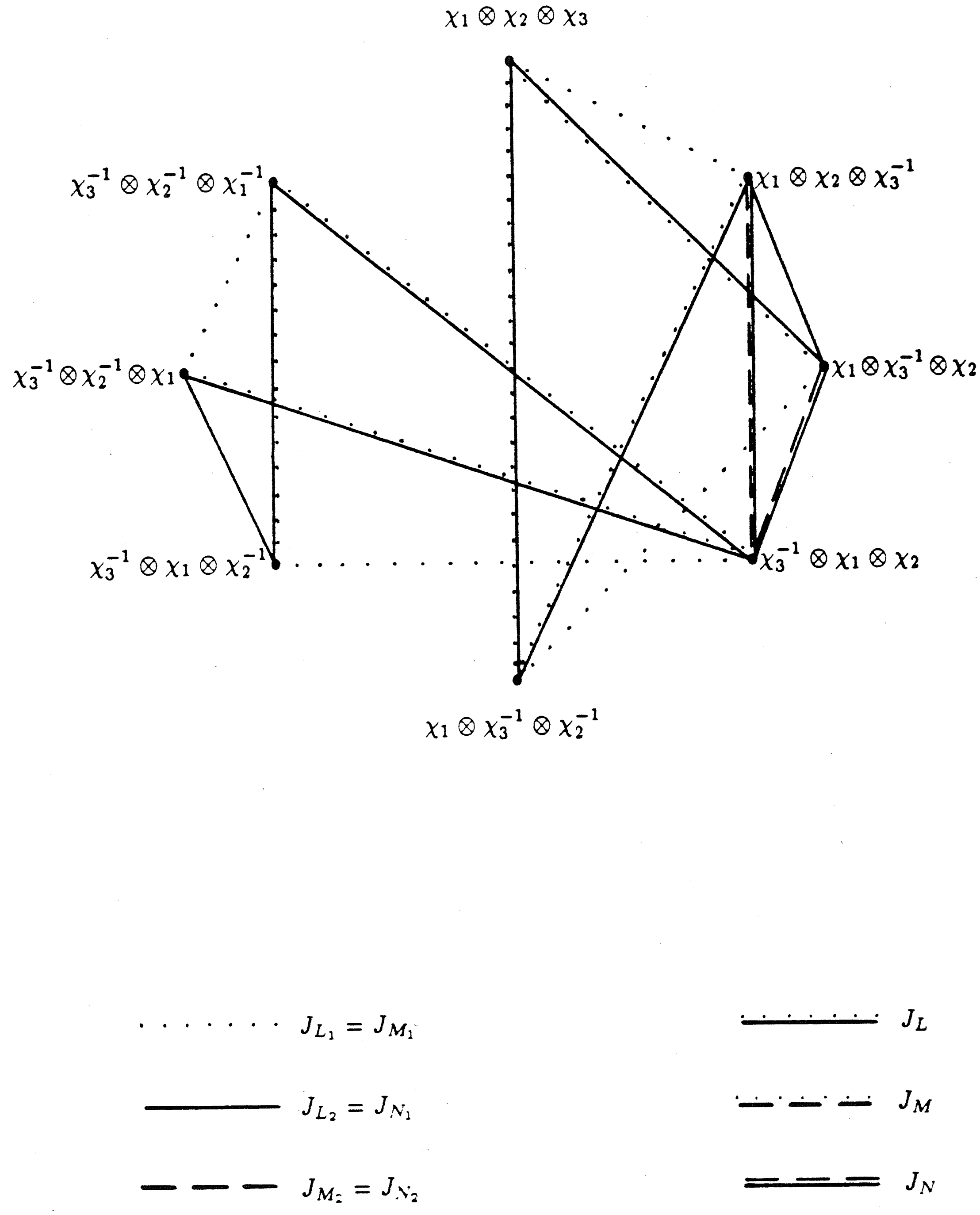

Figure 16. Graph of $[\chi]_3 \rtimes 1$, $\chi = \nu^{-\frac{1}{2}}$

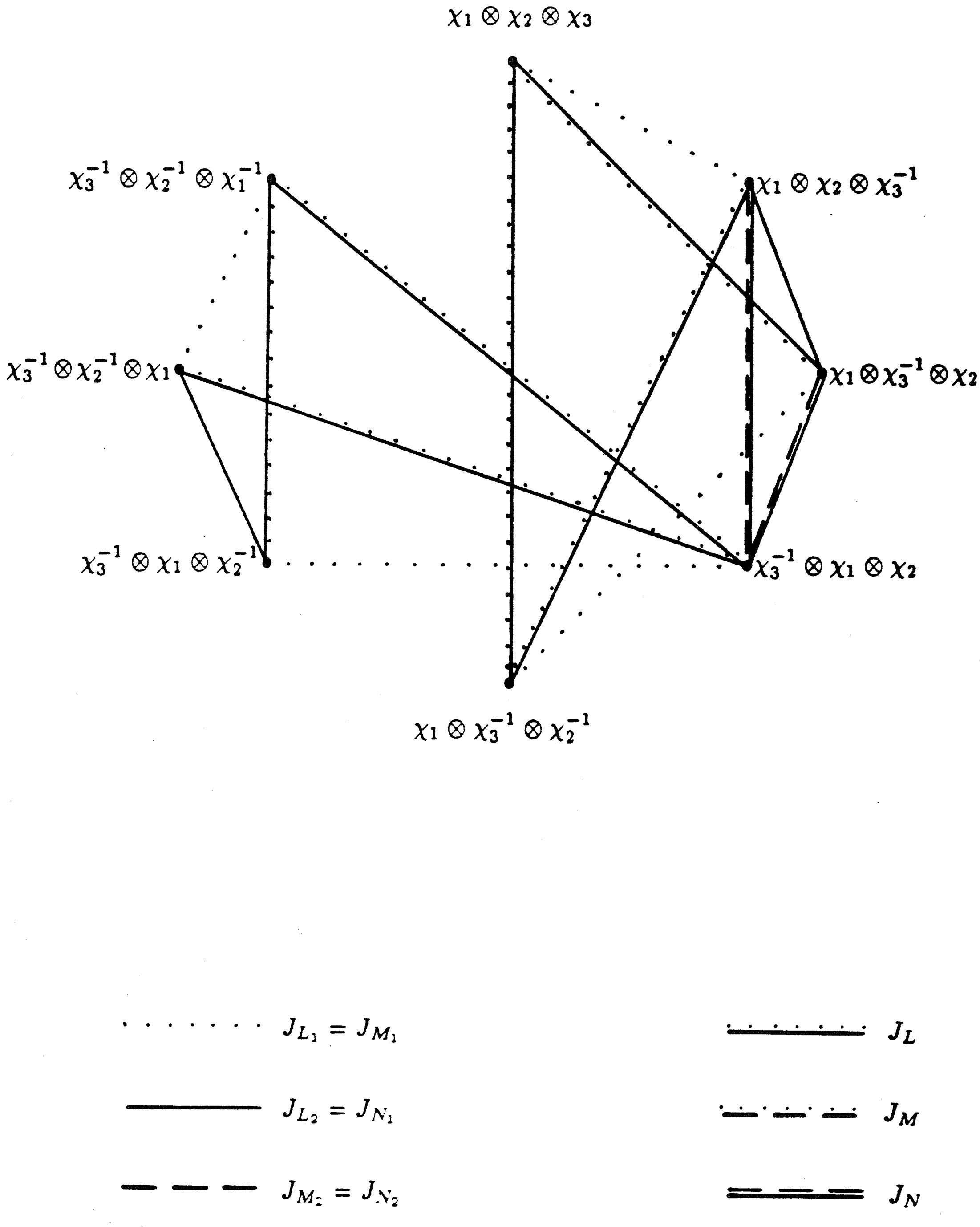

Figure 17. Graph of $[\chi]_3 \propto 1$, $\chi = \nu^{-\frac{1}{2}} sgn$

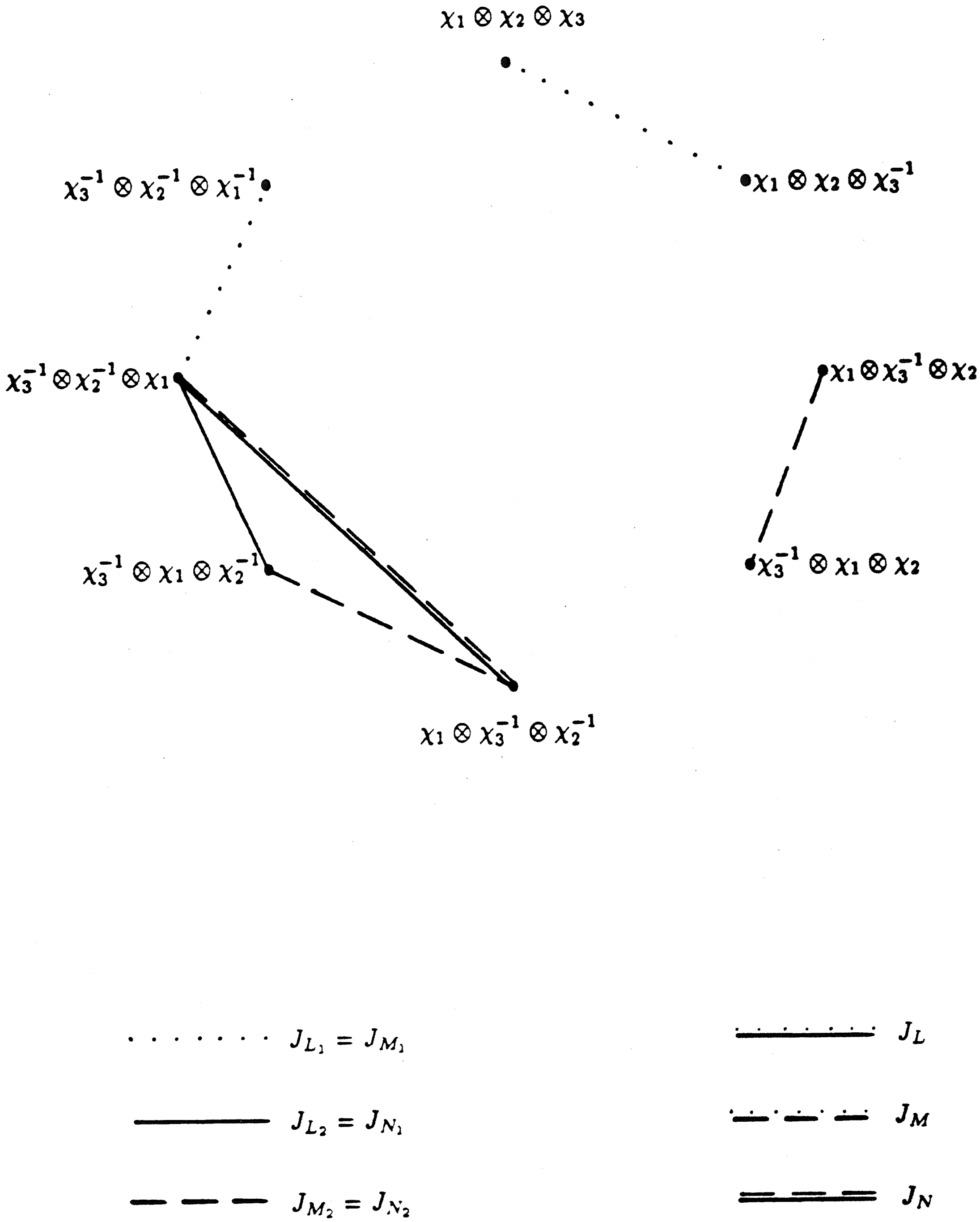

Figure 18. Graph of $[\chi]_3 \rtimes 1$, $\chi = \nu^{-1}$

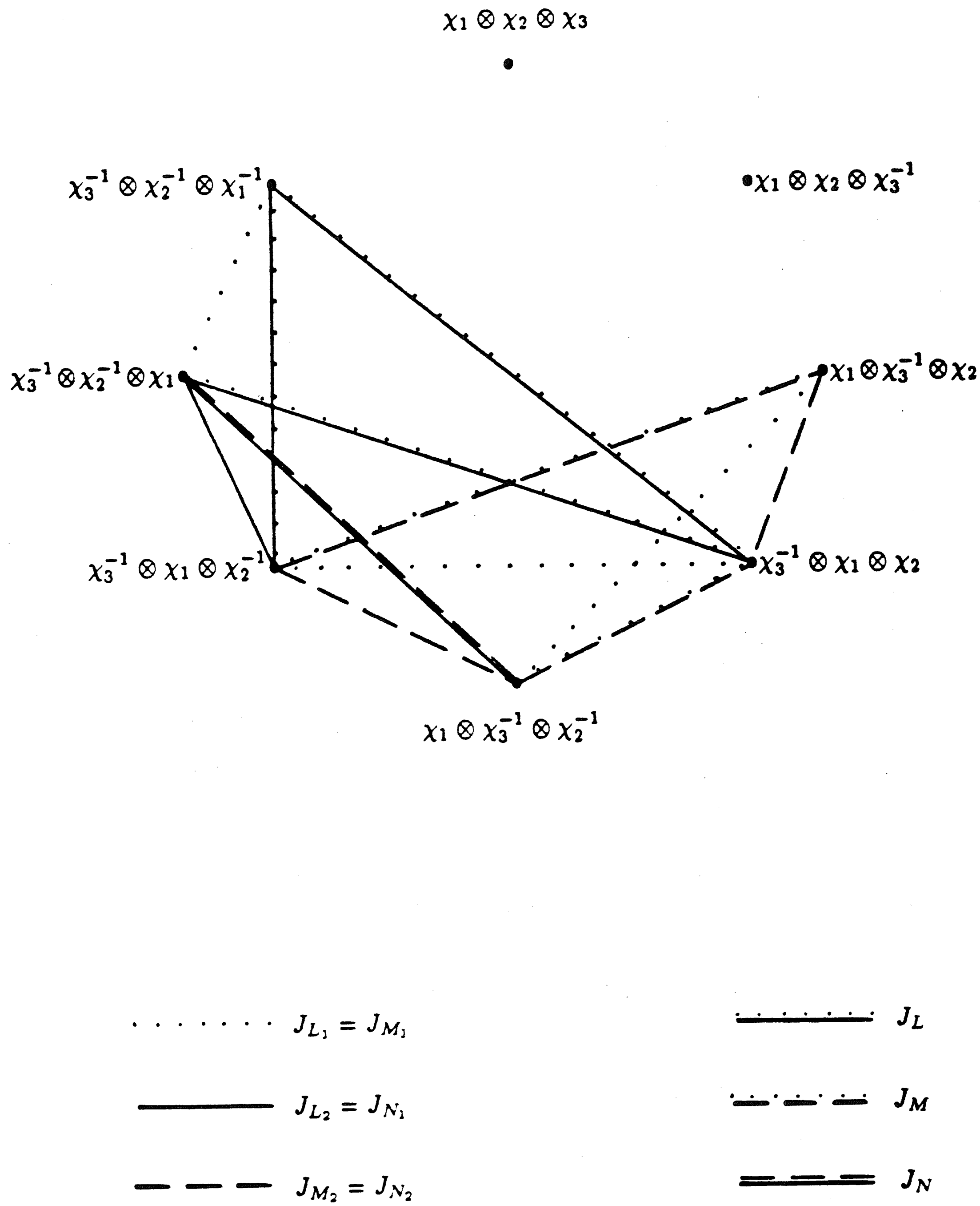

Figure 19. Graph of $[\chi]_3 \propto 1$, $\chi = \nu^{-1} sgn$

APPENDIX

Recall that in $Sp_{2n}(F)$, there is a Bruhat decomposition

$$K = \cup_{w \in W} IwI$$

(this is just the usual Bruhat decomposition of $Sp_{2n}(\mathbf{F}_q)$ lifted back to K). Suppose $B = B_\Phi$ is the parahoric subgroup with $\Phi = \{s_0, s_1, \ldots, s_n\} \backslash \{s_o, s_k\}$, so that B is the inverse image of a maximal parabolic subgroup of $Sp_{2n}(\mathbf{F}_q)$ under the reduction mod$\mathcal{P}$ map. The goal of this appendix is to give a similar decomposition of K into B double cosets, and characterize the elements of K lying in each double-coset. This is contained in the following lemma:

LEMMA **A.3.2** *Let*

$$w_{ij} = \left(\begin{array}{cccc|cccc} & & I_{i-j} & & -I_j & & & \\ & I_{k-i} & & & & & & \\ I_{i-j} & & & I_{n-k-i-j} & & & & \\ \hline I_j & & & & & & & \\ & & & & & & I_{i-j} & \\ & & & & & I_{k-i} & & \\ & & & & I_{i-j} & & & I_{n-k-i-j} \end{array}\right).$$

Then, $K = \cup_{i,j} B w_{ij} B$. *Moreover, suppose* $\tau \in K$ *has the form*

$$\tau = \left(\begin{array}{cc|cc} * & * & * & * \\ W & * & * & * \\ \hline X & * & * & * \\ Y & * & * & * \end{array}\right)$$

(W *is* $(n-k) \times k$*,* X *is* $k \times k$*,* Y *is* $(n-k) \times k$*). Define* $r_1(\tau) = \mathit{rank} \begin{pmatrix} W \\ X \\ Y \end{pmatrix} \bmod \mathcal{P}$*,* $r_2(\tau) = \mathit{rank}\ X \bmod \mathcal{P}$*. Then,* $\tau \in B w_{r_1(\tau) r_2(\tau)} B$*.*

Proof. We start with the following notation:

$$W_\alpha = \text{ Weyl group for } Sp_{2\alpha}(F) \; \hat{W}_\beta = \text{ Weyl group for } GL_\beta(F)$$

Now, since $B =< I, \hat{W}_k \times W_{n-k} >$ ($\hat{W}_k \times W_{n-k} \subset W_n$ in the obvious way), finding a set of representatives for $B\backslash K/B$ is equivalent to finding a set of representatives for $\hat{W}_k \times W_{n-k}\backslash W_n/(\hat{W}_k \times W_{n-k})$. To verify that $\{w_{ij}\}$ is a set of representatives, we must show two things:

1. no two of the w_{ij}'s lie in the same $\hat{W}_k \times w_{n-k}$ double-coset
2. everything in W_n lies in a double-coset associated to one of the w_{ij}'s.

To check #1, we shall show that r_1 and r_2 are constant on double-cosets of B. This suffices since they are different for distinct w_{ij}'s. In particular, we show that r_1 and r_2 are unchanged under right and left multiplication by members of $\hat{W}_k \times W_{n-k}$. This is straightforward:

left:

$$\left(\begin{array}{cc|cc} * & * & * & * \\ W & * & * & * \\ \hline X & * & * & * \\ Y & * & * & * \end{array}\right)\left(\begin{array}{cc|cc} M & & & \\ & A & & B \\ \hline & & {}^TM^{-1} & \\ & C & & D \end{array}\right) = \left(\begin{array}{cc|cc} * & * & * & * \\ WM & * & * & * \\ \hline WX & * & * & * \\ WY & * & * & * \end{array}\right)$$

right:

$$\left(\begin{array}{cc|cc} M & & & \\ & A & & B \\ \hline & & {}^TM^{-1} & \\ & C & & D \end{array}\right)\left(\begin{array}{cc|cc} * & * & * & * \\ W & * & * & * \\ \hline X & * & * & * \\ Y & * & * & * \end{array}\right) = \left(\begin{array}{c|cc} * \quad * & * & * \\ AW + BY & * & * \\ \hline {}^TM^{-1}X* & * & * \\ CW + DY & * & * \end{array}\right)$$

with $M \in \hat{W}_k$, $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in W_{n-k}$. Clearly, r_2 remains unchanged– $X, XM, {}^T M^{-1}$ all have the same rank mod$P$ since M is invertible. Similarly, r_1 remains unchanged since $\begin{pmatrix} WM \\ XM \\ YM \end{pmatrix} = \begin{pmatrix} W \\ X \\ Y \end{pmatrix} \cdot M$ and $\begin{pmatrix} AW+BY \\ {}^T M^{-1} X \\ CW+DY \end{pmatrix} = \begin{pmatrix} A & & B \\ & {}^T M^{-1} & \\ C & & D \end{pmatrix} \begin{pmatrix} W \\ X \\ Y \end{pmatrix}$ and M and $\begin{pmatrix} A & & B \\ & {}^T M^{-1} & \\ C & & D \end{pmatrix}$ are invertible.

Thus, #1 holds. Note that the claim $\tau \in Bw_{r_1(\tau) r_2(\tau)} B$ follows easily once we have established #2. We proceed to that task.

The first step in this process is to show that any two matrices in W_n with the same first k columns lie in the same double-coset of $\hat{W}_k \times W_{n-k}$. We then show that we can take any element in W_n and by suitable left and right multiplications with elements of $\hat{W}_k \times W_{n-k}$, obtain a matrix with the same first k columns as one of the w_{ij}'s.

The first step is fairly straightforward. We claim that for $w \in W_n$, every element of W_n with the same first k columns is of the form $w \cdot \left(\begin{array}{cc|cc} I & & & \\ & A & & B \\ \hline & & I & \\ & C & & D \end{array}\right)$, with $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in W_{n-k}$. If w' has the same first k columns as w, then w^{-1} has the same first rows as $(w')^{-1}$ (since they are permutation matrices). Then, $w^{-1}w'$ is of the desired form, and since $w' = w(w^{-1}w')$, the claim now holds.

Now, we take an element $w \in W_n$ and describe how to use right and left multiplication by $\hat{W}_k \times W_{n-k}$ to get it equal to a w_{ij} (at least up to ± 1's as entries, which suffices since it won't change the Weyl group element being represented). Suppose $r_1(w) = i$, $r_2(w) = j$. We then claim that by multiplying by a suitable element of $\hat{W}_k \times W_{n-k}$ on the left, we get a matrix with the first k columns of the form

$$\left(\begin{array}{c|l} 0 & i \\ * & k-i \\ \hline * & i-j \\ 0 & n+j-k-i \\ \hline * & j \\ 0 & k-j \\ \hline 0 & n-k \end{array}\right).$$

First, we can find $M_1 = \begin{pmatrix} M & & & \\ & I & & \\ & & {}^T M^{-1} & \\ & & & I \end{pmatrix} = \begin{pmatrix} M & & & \\ & I & & \\ & & M & \\ & & & I \end{pmatrix} \in \hat{W}_k$

so that m_1 moves the j nonzero rows in the third block to make them the first j rows there, and moves the $k-i$ nonzero rows in the first block to make them the last $k-i$ rows there. We remark that by symplecticity, the nonzero rows in the first block and those in the third block will not be in corresponding positions, thus we can rearrange

both as claimed. Similarly, we can find $M_2 = \left(\begin{array}{cc|cc} I & & & \\ & A & & B \\ \hline & & I & \\ & C & & D \end{array}\right) \in W_{n-k}$

which moves the $i-j$ nonzero entries in the second/fourth block into the first $i-j$ rows of the second block. Again, note that if a row in the second (resp. fourth) block is nonzero, then the corresponding row in the fourth (resp. second) block is zero. The fact that M_2 is symplectic means that if the αth row of the second block is moved to the βth row of the $\begin{cases} \text{second} \\ \text{fourth} \end{cases}$ block, then the αth row of the fourth block is moved to the βth row of the $\begin{cases} \text{fourth} \\ \text{second} \end{cases}$ block. In any case, as the $i-j$ rows do not occupy corresponding positions, we can find an M_2 as desired.

Next, right multiplication moves columns. A tedious argument similar to the proceeding (which will be omitted) allow us to move the columns to arrange that the first k are the same as those in w_{ij}, as needed. Since any first k columns lie in the same $\hat{W}_k \times W_{n-k}$ double-coset, the $\{w_{ij}\}$ will form a set of representatives for $(\hat{W}_k \times W_{n-k})\backslash W_n/(\hat{W}_k \times W_{n-k})$, hence for $B\backslash K/B$.

Finally, we observe that as no two distinct w_{ij}'s have the same (r_1, r_2) they characterize the B double-coset. In particular, $r \in K$ lies in $Bw_{r_1(\tau)r_2(\tau)}B$, as claimed.

COROLLARY **A.3.3** *Let* $P_0 = P \cap K$. *Then,* $K = \cup_{i,j} P_0 w_{ij} B$.

Proof. Use $Bw_{ij}B = P_0 w_{ij} B$ and the lemma.

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Mathematics Department, Duke University