manuscripta math. 156, 23–55 (2018)

Chris Jantzen

Jacquet modules and irreducibility of induced representations for classical p-adic groups

Received: 2 March 2017 / Accepted: 27 June 2017
Published online: 10 July 2017

Abstract. Let G be a classical p-adic group. If T is an irreducible tempered representation of such a group and ρ an irreducible unitary supercuspidal representation of a general linear group, we can form the parabolically induced representation $\mathrm{Ind}_P^G(|det|^y\rho \otimes T)$. The main result in this paper is the determination for which $y \in \mathbb{R}$ the induced representation is reducible. The key technical result in establishing this is the determination of a certain Jacquet module subquotient.

1. Introduction

Let G be a classical p-adic group of the type consided in [20]. If T is an irreducible tempered representation of such a group and ρ an irreducible unitary supercuspidal representation of a general linear group, we can form the parabolically induced representation $\mathrm{Ind}_P^G(\nu^y\rho \otimes T)$, where $\nu = |det|$ as in [6]. The main result in this paper is the determination for which $y \in \mathbb{R}$ the induced representation is reducible. The key technical result in establishing this is the determination of a certain Jacquet module subquotient $\mu^*_{\{\nu^x\rho\}}$ (discussed later in this introduction and dealt with at length in the paper).

The results are somewhat reminiscent of those on representations induced from discrete series in [22], though the approach is a bit different. Part of the purpose of this paper is, in fact, to demonstrate a different approach. The results could also serve as a starting point to a more general analysis of induced representations, with a long-term goal of understanding when a standard module reduces. However, our immediate purpose is in analyzing the duality operator of [2,23] for the groups under consideration. In fact, both the $\mu^*_{\{\nu^x\rho\}}(T)$ results and the reducibility results obtained from them are central to [13].

To start, we recall a key definition. Let $G_n(F)$ be from one of the families of classical groups under consideration (symplectic, odd special orthogonal, even orthogonal, unitary—see Sect. 2). For $1 \leq m \leq n$, we have a (maximal proper)

Research supported in part by NSA Grant H98230-13-1-0237.
C. Jantzen (✉): Department of Mathematics, East Carolina University, Greenville, NC 27858, USA
e-mail: jantzenc@ecu.edu

Mathematics Subject Classification: 22E50

https://doi.org/10.1007/s00229-017-0955-2

standard parabolic subgroup with Levi factor $M_{(m)} \cong GL(m, F) \times G_{n-m}(F)$. For π a representation of G, set

$$\mu^*(\pi) = \sum_{i=0}^{n} r_{M_{(i)},G}(\pi),$$

with the sum in $R \otimes R[S]$—see Sect. 2.1. This was originally defined in [28], and has many useful properties, discussed in more detail in the next section. However, the definition is sufficient for the purposes at hand. In particular, it enables us to define the following (Definition 3.1.1 [12]):

Definition 1.1. For X a set of (not necessarily unitary) supercuspidal representations of general linear groups, let $f = f_\pi(X)$ be the largest value such that a (minimal nonzero) Jacquet module of π has a term of the form $\nu^{x_1}\rho_1 \otimes \cdots \otimes \nu^{x_f}\rho_f \otimes \cdots$ with $\nu^{x_1}\rho_1, \ldots, \nu^{x_f}\rho_f \in X$ (where $\nu = |det|$ as in [33]). We let

$$\mu_X^*(\pi) = \sum_i \lambda_i \otimes \theta_i,$$

where the sum is over all irreducible $\lambda_i \otimes \theta_i \leq \mu^*(\pi)$ for which the (minimal nonzero) Jacquet module of λ_i contains a term of the form $\nu^{x_1}\rho_1 \otimes \cdots \otimes \nu^{x_f}\rho_f$, with $\nu^{x_1}\rho_1, \ldots, \nu^{x_f}\rho_f \in X$

Suppose X has the property that $\nu^x\rho \in X \Rightarrow \nu^{-x}\check{\rho} \notin X$. If π is irreducible, we have the following key properties (Sect. 3.1 [12]):

- $\mu_X^*(\pi)$ consists of a single representation, denoted $\lambda_\pi(X) \otimes \theta_\pi(X)$.
- $\pi \hookrightarrow i_{G,M}(\lambda_\pi(X) \otimes \theta_\pi(X))$ as unique irreducible subrepresentation. (1.1)
- If $\mu_X^*(\pi_1) = \mu_X^*(\pi_2)$, then $\pi_1 \cong \pi_2$ (follows from above).

Note that we have $|X| = 1$ for most applications.

To illustrate how these results can be used to analyze reducibility, we look at the example of $i_{G,M}(\nu^{\frac{1}{2}}\rho \otimes T)$, T tempered (in the more interesting case, when the cuspidal reducibility is in $-\frac{1}{2} + \mathbb{N}$; see Sect. 2.1). First, observe that we have $\pi = L_{quot}(\nu^{\frac{1}{2}}\rho; T)$—the Langlands quotient of $i_{G,M}(\nu^{\frac{1}{2}}\rho \otimes T)$—as the unique irreducible quotient. In the reducible case, we show there is also a tempered subrepresentation T'. We do so by providing the data for T', then using Theorem 3.1—which gives $\mu^*_{\{\nu^{\frac{1}{2}}\rho\}}(T')$—and (1.1) to show that $T' \hookrightarrow i_{G,M}(\nu^{\frac{1}{2}}\rho \otimes T)$. In the irreducible case, Proposition 4.1 tells us that if $i_{G,M}(\nu^{\frac{1}{2}}\rho \otimes T)$ has another irreducible subquotient, it must be tempered. Were there such a T', it is not difficult to show that for $X = \{\nu^{\frac{1}{2}}\rho\}$, one would have $\theta_{T'}(X) = \theta_T(X)$. However, one can also use Theorem 3.1 and the observation that $f_{T'}(X) = f_T(X) + 1$ (with $f(X)$ the f in Definition 1.1) to show that exactly one of $\theta_{T'}(X)$ and $\theta_T(X)$ is tempered, a contradiction.

The arguments in this paper are ultimately built from the Mœglin–Tadić classification of discrete series ([20]) and the machinery needed for that classification. We therefore make the assumptions needed for [20]; the results then apply in the generality of [20], that is, to the symplectic, odd special orthogonal, even orthogonal, and unitary groups considered there.

We close by briefly describing the contents of this paper. In the next section, we introduce notation and give some background results. In Sect. 3, we give the result for $\mu^*_{\{\nu^x\rho\}}(T)$, where T is tempered (Theorem 3.1). Section 4 contains the main reducibility results (Theorem 4.7; also Note 4.8). We close with an appendix, which offers a characterization of admissibility (in the sense of [20]) designed to make claims of admissibility more transparent. We also take the opportunity to thank the referee for suggestions and corrections which helped improve the paper.

2. Notation and preliminaries

2.1. Notation and preliminaries

We first discuss some structure theory from [33] and [3,28]. First, let $S(n, F)$ denote the rank n member of one of the families of classical groups under consideration and set

$$R = \bigoplus_{n\geq 0} \mathcal{R}(GL(n, F)) \text{ and } R[S] = \bigoplus_{n\geq 0} \mathcal{R}(S(n, F)),$$

where $\mathcal{R}(G)$ denotes the Grothendieck group of the category of smooth finite-length representations of G. We define multiplication on R as follows: suppose ρ_1, ρ_2 are representations of $GL(n_1, F), GL(n_2, F)$, resp. We have $M = GL(n_1, F) \times GL(n_2, F)$ the Levi factor of a standard parabolic subgroup of $G = GL(n, F)$, where $n = n_1 + n_2$, and set $\tau_1 \times \tau_2 = i_{G,M}(\tau_1 \otimes \tau_2)$ (normalized parabolic induction—see [6]). This extends (after semisimplification) to give the multiplication $\times : R \times R \longrightarrow R$. To describe the comultiplication on R, let $M_{(i)}$ denote the standard Levi factor for $G = GL(n, F)$ having $M_{(i)} = GL(i, F) \times GL(n-i, F)$. For a representation τ of $GL(n, F)$, we define

$$m^*(\tau) = \sum_{i=0}^{n} r_{M_{(i)},G}(\tau),$$

the sum of semisimplified Jacquet modules (lying in $R \otimes R$). This extends to a map $m^* : R \longrightarrow R \otimes R$. We note that with this multiplication and comultiplication (and antipode map given by the Zelevinsky involution, a special case of the general duality operator of [2,23]), R is a Hopf algebra. There are two analogues for general linear groups of the μ^*_X discussed in the introduction: m^*_X and ${}_Xm^*$. For an irreducible representation π, we let $f = f_\pi(X)$ (resp., $g = g_\pi(X)$) be the largest value such that a minimal nonzero Jacquet module of π has a term of the form $\nu^{x_1}\rho_1 \otimes \cdots \otimes \nu^{x_f}\rho_f \otimes \cdots$ (resp., of the form $\cdots \otimes \nu^{x_g}\rho_g \otimes \cdots \otimes \nu^{x_1}\rho_1$) with all $\nu^{x_i}\rho_i \in X$, $1 \leq i \leq f$ (resp., $1 \leq i \leq g$). The analogue of (1.1) holds without restriction on X (Lemma 2.1.2 [10]); we define m^*_X and ${}_Xm^*$ accordingly.

Recall that for a, b with $a \leq b$ and $b - a \in \mathbb{Z}$, $\delta([\nu^a\rho, \nu^b\rho])$ denotes the generalized Steinberg representation associated to the segment $[\nu^a\rho, \nu^b\rho]$, i.e., the unique irreducible subrepresentation of $\nu^b\rho \times \nu^{b-1}\rho \times \cdots \times \nu^a\rho$ ([33]). The unique irreducible subrepresentation of $\nu^a\rho \times \nu^{a+1}\rho \times \cdots \times \nu^b\rho$ is denoted $\zeta([\nu^a\rho, \nu^b\rho])$. Note that $\zeta([\nu^a\rho, \nu^b\rho])$ is dual to $\delta([\nu^a\rho, \nu^b\rho])$.

Next, suppose τ is a representation of $GL(n_1, F)$ and θ a representation of $S(n_2, F)$. We have $M = GL(n_1, F) \times S(n_2, F)$ the Levi factor of a standard parabolic subgroup of $G = S(n, F)$, with $n = n_1+n_2$, and set $\tau \rtimes \theta = i_{G,M}(\tau \otimes \theta)$. If one extends $\rtimes$ to a map $\rtimes : R \otimes R[S] \longrightarrow R[S]$, we have $R[S]$ as a module over R. To describe its comodule structure, let $M_{(i)} = GL(i, F) \times S(n-i, F)$, a standard Levi factor for $G = S(n, F)$. For a representation π of $S(n, F)$, we define

$$\mu^*(\pi) = \sum_{i=0}^{n} r_{M_{(i)},G}(\pi),$$

the sum of (normalized) semisimplified Jacquet modules (lying in $R \otimes R[S]$). This extends to a map $\mu^* : R[S] \longrightarrow R \otimes R[S]$. In addition to μ^*_X introduced earlier, there is another variant of this which is needed occasionally in what follows. For an irreducible representation λ of a general linear group and a representation π of one of the classical groups under consideration, we let $\mu^*_\lambda(\pi)$ be the sum of everything in $\mu^*(\pi)$ having first factor isomorphic to λ. More precisely, if $\mu^*(\pi) = \sum_i \lambda_i \otimes \xi_i$, we let $\mu^*_\lambda(\pi) = \sum_{i \in I_\lambda} \lambda_i \otimes \xi_i$, where $I_\lambda = \{i \mid \lambda_i \cong \lambda\}$.

For unitary groups, let ξ denote the nontrivial element of the Galois group of the underlying quadratic extension. For a representation π of $S(n, F)$, we then define

$$\check{\pi} = \begin{cases} \tilde{\pi} \circ \xi & \text{for unitary groups,} \\ \tilde{\pi} & \text{otherwise,} \end{cases}$$

where $\tilde{}$ denotes contragredient. Using this, we may give $R[S]$ the structure of an M^*-module over R ([3,20,28]):

Theorem 2.1. *Define $M^* : R \longrightarrow R \otimes R$ by*

$$M^* = (m \otimes 1) \circ (\check{} \otimes m^*) \circ s \circ m^*,$$

where m denotes the multiplication $\times : R \otimes R \longrightarrow R$ and $s : R \otimes R \longrightarrow R \otimes R$ the extension of the map defined on representations by $s : \tau_1 \otimes \tau_2 \longmapsto \tau_2 \otimes \tau_1$. Then

$$\mu^*(\tau \rtimes \pi) = M^*(\tau) \rtimes \mu^*(\pi),$$

where $\rtimes$ on the right hand side is determined by $(\tau_1 \otimes \tau_2) \rtimes (\tau \otimes \theta) = (\tau_1 \times \tau) \otimes (\tau_2 \rtimes \theta)$.

We now take a moment to review cuspidal reducibility values. Suppose ρ is an irreducible unitary supercuspidal representation of a general linear group and σ an irreducible supercuspidal representation of a classical group. If $\rho \not\cong \check{\rho}$, then $\nu^x\rho \rtimes \sigma$ is irreducible for all $x \in \mathbb{R}$; if $\rho \cong \check{\rho}$, then there is a unique nonnegative $x \in \mathbb{R}$ such that $\nu^x\rho \rtimes \sigma$ reduces ([27] and Corollary 4.4 [4]), which we denote by $red(\rho; \sigma)$. The values for $red(\rho; \sigma)$ for $Sp(2n, F)$ and $SO(2n+1, F)$ have been determined (assuming certain conjectures) in [18] and [34]; in the generic case, it is known that they must lie in $\{0, \frac{1}{2}, 1\}$ ([24,25]). Further, in the quasi-split, characteristic zero case, the reducibility values are now known to be half-integral ([1,19]).

We next review the Casselman criterion for $S(n, F)$ (see [8,32], which extends easily to the non-connected group $O(2n, F)$). Suppose π is an irreducible representation of $S(n, F)$. Suppose $\nu^{x_1}\rho_1 \otimes \cdots \otimes \nu^{x_k}\rho_k \otimes \sigma \leq r_{M,G}(\pi)$ has ρ_i an irreducible unitary supercuspidal representation of $GL(m_i, F)$ for $i = 1, \ldots, k$, σ an irreducible supercuspidal representation of $S(m, F)$, and $x_1, \ldots, x_k \in \mathbb{R}$. The Casselman criterion tells us that if π is tempered, the following hold:

$$\begin{aligned} m_1x_1 &\geq 0 \\ m_1x_1 + m_2x_2 &\geq 0 \\ &\vdots \\ m_1x_1 + m_2x_2 + \cdots + m_kx_k &\geq 0. \end{aligned}$$

Conversely, if these inequalities hold for any such $\nu^{x_1}\rho_1\otimes\cdots\otimes\nu^{x_k}\rho_k\otimes\sigma$ (i.e., ρ_i an irreducible unitary supercuspidal representation of $GL(m_i, F)$ and σ an irreducible supercuspidal representation of $S(m, F)$) appearing in a Jacquet module of π, then π is tempered. The criterion for square-integrability is the same except that the inequalities are strict.

We also take a moment to review the Langlands classification ([7,14,26]; also the appendix of [4] for the non-connected group $O(2n, F)$). We work in the subrepresentation setting of the Langlands classification as it is the most convenient for applying Jacquet module methods. Suppose $\tau_1, \ldots, \tau_k$ are irreducible tempered representations of general linear groups and $x_1 < \cdots < x_k$. Then the induced representation $\nu^{x_1}\tau_1 \times \cdots \times \nu^{x_k}\tau_k$ has a unique irreducible subrepresentation which we denote $\mathcal{L}(\nu^{x_1}\tau_1, \ldots, \nu^{x_k}\tau_k)$. Every irreducible admissible representation of a general linear group may be written in this way, and the data $\nu^{x_1}\tau_1 \otimes \cdots \otimes \nu^{x_k}\tau_k$ are unique. Turning to classical groups, if $\tau_1, \ldots \tau_k$ are irreducible tempered representations of general linear groups, τ an irreducible tempered representation of $S(n, F)$, and $x_1 < \cdots < x_k < 0$, the representation $\nu^{x_1}\tau_1 \times \cdots \times \nu^{x_k}\tau_k \rtimes \tau$ has a unique irreducible subrepresentation which we denote $L(\nu^{x_1}\tau_1, \ldots, \nu^{x_k}\tau_k; \tau)$. Further, any irreducible admissible representation of a classical group may be written in this way, and the data $\nu^{x_1}\tau_1 \otimes \cdots \otimes \nu^{x_k}\tau_k \otimes \tau$ are again unique.

The next lemma is Lemma 5.5 of [9].

Lemma 2.2. *Suppose π is an irreducible representation of G, λ an irreducible representation of M and $\pi \hookrightarrow i_{G,M}(\lambda)$. If $L > M$, then there is an irreducible representation ρ of L such that*

(1) $\pi \hookrightarrow i_{G,L}(\rho)$
(2) *ρ is a subquotient of $i_{L,M}(\lambda)$.*

We pause to note that for $X = \{\rho\}$ with $\rho \cong \check{\rho}$, the hypotheses for (1.1) do not hold. In this case, the first property in (1.1) holds, but only up to multiplicity. The second property also holds, but the third does not. For the first two, the proofs are essentially the same; for the third, an easy counterexample occurs when $\rho \rtimes \sigma$ is reducible—the two components are inequvalent but have $\mu^*_{\{\rho\}}$ the same.

Lemma 2.3. *Suppose* $f_\pi(\nu^x\rho) = f$. *If*

$$\pi \hookrightarrow (\nu^x\rho)^f \times \lambda_1 \times \cdots \times \lambda_k \rtimes T$$

with $\lambda_1 \otimes \cdots \otimes \lambda_k \otimes T$ *satisfying the conditions for Langlands data (subrepresentation setting), then*

$$\mu^*_{\{\nu^x\rho\}}(\pi) = (\nu^x\rho)^f \otimes L(\lambda_1, \ldots, \lambda_k; T)$$

(up to multiplicity if $x = 0$*).*

Proof. By Lemma 2.2, $\pi \hookrightarrow (\nu^x\rho)^f \rtimes \theta$ for some irreducible $\theta \leq \lambda_1 \times \cdots \times \lambda_k \rtimes T$; necessarily $\theta = \theta_\pi(\nu^x\rho)$. Also, as $\mu^*_{\{\nu^x\rho\}}(\pi) = (\nu^x\rho)^f \otimes \theta$ (up to multiplicity if $x = 0$), and $r_{M,G}(\pi) \geq (\nu^x\rho)^f \otimes \lambda_1 \otimes \cdots \otimes \lambda_k \otimes T$ (by Frobenius reciprocity), it follows from Lemma 3.1.3 and Remark 3.1.4 [12] that $r_{M',G'}(\theta) \geq \lambda_1 \otimes \cdots \otimes \lambda_k \otimes T$ (where θ is a representation of G' and $M' < G'$ is the appropriate standard Levi). However, the only irreducible subquotient of $\lambda_1 \times \cdots \times \lambda_k \rtimes T$ containing $\lambda_1 \otimes \cdots \otimes \lambda_k$ in its Jacquet module is $L(\lambda_1, \ldots, \lambda_k; T)$ ([5] or [9]), finishing the lemma. □

2.2. The extended Mœglin–Tadić classification

In this section, we review the extension of the construction of [20] to tempered representations. The extension used here is from [12]; we also note the somewhat different extension available in [31].

Recall that the Mœglin–Tadić classification is a bijective correspondence between (equivalence classes of) discrete series for a family of classical groups and (equivalence classes of) admissible triples. An admissible triple is a triple of the form $(Jord, \sigma, \varepsilon)$. Here $Jord$ consists of pairs (ρ, a), with ρ an irreducible unitary supercuspidal representation of a general linear group and $a \in \mathbb{N}$ subject to a parity condition from ρ, σ the "partial cuspidal support" (the supercuspidal representation of a classical group which appears in any minimal nozero Jacquet module term), and ε a function defined on a subset of $Jord \cup (Jord \times Jord)$ taking values in $\{\pm 1\}$ which essentially distinguishes between discrete series having the same supercuspidal support (or equivalently, by Lemma 2.1.1 [12], the same $Jord$). Information about induced representations into which the discrete series embeds is also encoded in the data. This classification and its properties have been summarized in [20], many of the references for this paper (e.g., [11,12,21,22,29–31]), as well as many other places. We forgo doing so again and simply refer the reader to these sources, as well as to the characterization of admissibility in the appendix of this paper.

To extend the Mœglin–Tadić classification to tempered representations, we first consider the elliptic case. Suppose

$$T_{ell} \hookrightarrow \delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-c_\ell+1}{2}}\rho_\ell, \nu^{\frac{c_\ell-1}{2}}\rho_\ell\right]\right) \rtimes \delta, \quad (2.1)$$

with δ a discrete series for a classical group. Let $(Jord(\delta), \sigma, \varepsilon_\delta)$ be the Mœglin–Tadić data for δ, with $S_\delta \subset Jord(\delta) \cup (Jord(\delta) \times Jord(\delta))$ the domain for ε_δ. Intuitively, we construct $Jord(T)$ from $Jord(\delta)$ by adding two copies each of

$(\rho_1, c_1), \dots, (\rho_\ell, c_\ell)$ (one for each end of $\delta([\nu^{\frac{-c_i+1}{2}}\rho_i, \nu^{\frac{c_i-1}{2}}\rho_i])$, even if $c_i = 1$). More precisely, we introduce a fourth datum, m_T—the multiplicity—so have T associated to $(Jord(T), \sigma, \varepsilon_T, m_T)$. Then,

$$Jord(T) = Jord(\delta) \cup \{(\rho_1, c_1), \dots, (\rho_\ell, c_\ell)\}$$

and

$$m_T(\rho, a) = \begin{cases} 1 \text{ if } (\rho, a) \in Jord(\delta), \\ 2 \text{ if } (\rho, a) \cong (\rho_i, c_i) \text{ for some } i. \end{cases}$$

Again, we have $\varepsilon_T : S_T \longrightarrow \{\pm 1\}$, with the domain $S_T \subset Jord(T) \cup (Jord(T) \times Jord(T))$. We have $S_T \supset S_\delta$, and $\varepsilon_T|_{S_\delta} = \varepsilon_\delta$. The additional values of ε_T effectively distinguish the 2^ℓ components of $\delta([\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1]) \times \cdots \times \delta([\nu^{\frac{-c_\ell+1}{2}}\rho_\ell, \nu^{\frac{c_\ell-1}{2}}\rho_\ell]) \rtimes \delta$. In particular, we have the following extension of the basic embedding property of [20]: for $(\rho, a) \in Jord(T)$, let a_- be the largest value of $b < a$ satisfying $(\rho, b) \in Jord(\delta)$ if it exists. Then (Proposition 2.3.2 [12])

$$\begin{aligned}
&(1) \text{ if } \quad m(\rho, a) = 1, \\
&\quad \varepsilon\left((\rho, a), (\rho, a_-)\right) \quad = 1 \Leftrightarrow T \hookrightarrow \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \\
&\quad\quad \rtimes\theta \text{ for some irreducible } \theta. \\
&(2) \text{ if } \quad m(\rho, a) = 2, \\
&\quad \varepsilon\left((\rho, a), (\rho, a_-)\right) = 1 \Leftrightarrow T \hookrightarrow \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \\
&\quad \times\delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \theta \text{ for some irreducible } \theta.
\end{aligned} \tag{2.2}$$

Other basic properties of the Mœglin–Tadić classification also have counterparts in the extension to the tempered case; we forgo including them here but include citations when used.

For more general tempered representations—i.e., no longer assuming elliptic—we construct $Jord(T), m_T$ in a similar manner. Write

$$T \cong \delta\left(\left[\nu^{\frac{-d_1+1}{2}}\rho'_1, \nu^{\frac{d_1-1}{2}}\rho'_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-d_m+1}{2}}\rho'_m, \nu^{\frac{d_m-1}{2}}\rho'_m\right]\right) \rtimes T_{ell} \tag{2.3}$$

(irreducibly induced) with T_{ell} elliptic tempered. We construct $Jord(T), m_T$ from $Jord(T_{ell}), m_{T_{ell}}$ by adding one copy each of $(\rho'_1, d_1), (\check{\rho}'_1, d_1), \dots, (\rho'_m, d_m)$, $(\check{\rho}'_m, d_m)$ (so two copies of (ρ'_i, d_i) are added if $\check{\rho}'_i \cong \rho'_i$); S_T and ε_T match $S_{T_{ell}}$ and $\varepsilon_{T_{ell}}$ (noting that the corresponding induced representation is irreducible so we do not have components to distinguish). We then have an extension of the above embedding: for $(\rho, a) \in Jord(T_{ell})$, let $a_- < a$ be the largest value such that $(\rho, a_-) \in Jord(T_{ell})$ if it exists. Then

$$\begin{aligned}
&\varepsilon\left((\rho, a), (\rho, a_-)\right) \\
&\quad = 1 \Leftrightarrow T \hookrightarrow \underbrace{\delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)}_{m(\rho,a)} \rtimes\theta
\end{aligned}$$

This is not explicitly stated in [12], but is a straightforward consequence of (2.2) and the irreducibility of (2.3).

We add one notational convention: as defined in [20], $Jord_\rho = \{a \mid (\rho, a) \in Jord\}$. By abuse of notation, we also allow this to be interpreted as $\{(\rho, a) \mid (\rho, a) \in Jord\}$ as this interpretation is useful when dealing with the domain of ε.

Before closing, we note one immediate consequence of the definitions in Sect. 2.3 [12] which is needed later:

Lemma 2.4. *Suppose T is an elliptic tempered representation with $(\rho, a) \in Jord(T)$ and $m(\rho, a) = 2$. Let T^* be the elliptic tempered representation with $T \hookrightarrow \delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes T^*$. Then, T^* satisfies $Jord(T^*) = Jord(T) \setminus \{(\rho, a)\}$ with m_{T^*} and ε_{T^*} given by restriction (and partial cuspidal supports the same).*

3. Jacquet modules of tempered representations

In this section, we prove the main Jacquet module result of this paper, Theorem 3.1. The elliptic case is covered by Proposition 3.4 (for $a > 2$) and Lemma 3.7 (for $a = 1, 2$). The general tempered case follows from (2.3) and the elliptic tempered case. Note that the discrete series case is already known (Theorem 3.2.2 [12] or Theorem 8.5 [17]). We begin by stating the result, then proceed to prove it in a sequence of lemmas.

Theorem 3.1. *Suppose T is irreducible tempered with $(\rho, a) \in Jord(T)$. If $a > 2$, we have the following:*

(1) *If $\rho \not\cong \check{\rho}$, then*

$$\mu^*_{\left\{\nu^{\frac{a-1}{2}}\rho\right\}}(T) = (\nu^{\frac{a-1}{2}}\rho)^{m_T(\rho,a)} \otimes L(\delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right)^{m_T(\rho,a)}; T_1),$$

where $m_{T_1}(\rho, a) = m_{T_1}(\check{\rho}, a) = 0$, $m_{T_1}(\rho', b) = m_T(\rho', b)$ for all other (ρ', b), and $\varepsilon_{T_1} = \varepsilon_T$ (noting that $(\rho, a) \notin S_T$).

(2) *If $\rho \cong \check{\rho}$ but $\frac{a-1}{2} \not\equiv red(\rho; \sigma) \bmod 1$,*

$$\mu^*_{\left\{\nu^{\frac{a-1}{2}}\rho\right\}}(T) = \left(\nu^{\frac{a-1}{2}}\rho\right)^{m_T(\rho,a)} \otimes T_2$$

where $m_{T_2}(\rho, a) = 0$, $m_{T_2}(\rho, a-2) = m_T(\rho, a)$, $m_{T_2}(\rho', b) = m_T(\rho', b)$ for all other (ρ', b), and $\varepsilon_{T_2} = \varepsilon_T$ (noting that $(\rho, a) \notin S_T$).

(3) *If $\rho \cong \check{\rho}$ with $\frac{a-1}{2} \equiv red(\rho; \sigma) \bmod 1$ and either (i) $(\rho, a-2) \notin Jord$, or (ii) $\varepsilon_T(\rho, a)\varepsilon_T(\rho, a_-)^{-1} = 1$, then*

$$\mu^*_{\left\{\nu^{\frac{a-1}{2}}\rho\right\}}(T) = \left(\nu^{\frac{a-1}{2}}\rho\right)^{m_T(\rho,a)} \otimes T_3,$$

where $m_{T_3}(\rho, a) = 0$, $m_{T_3}(\rho, a-2) = m_T(\rho, a) + m_T(\rho, a-2)$, and $m_{T_2}(\rho', b) = m_T(\rho', b)$ for all other (ρ', b). If $m_T(\rho, a-2) > 0$, ε_{T_3} is *just given by restriction; if* $m_T(\rho, a-2) = 0$*, it is given by substituting* $(\rho, a-2)$ *for* (ρ, a)*. More precisely, if* $m(\rho, a-2) = 0$*,* ε_{T_3} *is determined by the following changes:* $\varepsilon_{T_3}(\rho, a-2) = \varepsilon_T(\rho, a)$ *if defined, and* $\varepsilon_{T_3}(\rho, a-2)\varepsilon_{T_3}(\rho, b)^{-1} = \varepsilon_T(\rho, a)\varepsilon_T(\rho, b)^{-1}$ *for all other* b*.*

(4) *If* $\rho \cong \check{\rho}$ *with* $\frac{a-1}{2} \equiv red(\rho; \sigma) \bmod 1$*,* $a_- = a-2$*, and* $\varepsilon_T(\rho, a)\varepsilon_T(\rho, a_-)^{-1} = -1$*, then*

$$\mu^*_{\left\{\nu^{\frac{a-1}{2}}\rho\right\}}(T) = \begin{cases} \left(\nu^{\frac{a-1}{2}}\rho\right)^{m_T(\rho,a)-1} \otimes T_4 \text{ if } m_T(\rho, a) \text{ odd} \\ \left(\nu^{\frac{a-1}{2}}\rho\right)^{m_T(\rho,a)-1} \\ \quad \otimes L(\delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right); T_5) \text{ if } m_T(\rho, a) \text{ even.} \end{cases}$$

Here, $m_{T_4}(\rho, a) = 1$*,* $m_{T_4}(\rho, a-2) = m_T(\rho, a-2) + m_T(\rho, a) - 1$*,* $m_{T_4}(\rho', b) = m_T(\rho', b)$ *for all other* (ρ', b)*, and* $\varepsilon_{T_4} = \varepsilon_T$*; for* T_5*,* $m_{T_5}(\rho, a) = 0$*,* $m_{T_5}(\rho, a-2) = m_T(\rho, a-2) + m_T(\rho, a) - 2$*,* $m_{T_5}(\rho', b) = m_T(\rho', b)$ *for all other* (ρ', b)*, and* ε_{T_5} *the restriction of* ε_T*.*

If $a = 2$*, we have the following:*

(5) *If* $\rho \ncong \check{\rho}$*, then*

$$\mu^*_{\left\{\nu^{\frac{1}{2}}\rho\right\}}(T) = \left(\nu^{\frac{1}{2}}\rho\right)^{m_T(\rho,2)} \otimes L((\nu^{-\frac{1}{2}}\rho)^{m_T(\rho,2)}; T_1),$$

where $m_{T_1}(\rho, a) = m_{T_1}(\check{\rho}, a) = 0$*,* $m_{T_1}(\rho', b) = m_T(\rho', b)$ *for all other* (ρ', b)*, and* $\varepsilon_{T_1} = \varepsilon_T$ *(noting that* $(\rho, 2) \notin S_T$*).*

(6) *If* $\rho \cong \check{\rho}$ *but* $red(\rho; \sigma) \not\equiv \frac{1}{2} \bmod 1$*,*

$$\mu^*_{\left\{\nu^{\frac{1}{2}}\rho\right\}}(T) = \left(\nu^{\frac{1}{2}}\rho\right)^{m_T(\rho,2)} \otimes T_2$$

where $m_{T_2}(\rho, 2) = 0$*,* $m_{T_2}(\rho', b) = m_T(\rho', b)$ *for all other* (ρ', b)*, and* $\varepsilon_{T_2} = \varepsilon_T$ *(noting that* $(\rho, a) \notin S_T$*).*

(7) *If* $\rho \cong \check{\rho}$ *with* $red(\rho; \sigma) \equiv \frac{1}{2} \bmod 1$ *and* $\varepsilon_T(\rho, 2) = 1$*, then*

$$\mu^*_{\left\{\nu^{\frac{1}{2}}\rho\right\}}(T) = \left(\nu^{\frac{1}{2}}\rho\right)^{m_T(\rho,2)} \otimes T_3,$$

where $m_{T_3}(\rho, 2) = 0$*,* $m_{T_2}(\rho', b) = m_T(\rho', b)$ *for all other* (ρ', b)*, and* ε_{T3} *just the restriction of* ε_T*.*

(8) *If* $\rho \cong \check{\rho}$ *with* $red(\rho; \sigma) \equiv \frac{1}{2} \bmod 1$ *and* $\varepsilon_T(\rho, 2) = -1$*, then*

$$\mu^*_{\{\nu^{\frac{1}{2}}\rho\}}(T) = \begin{cases} \left(\nu^{\frac{1}{2}}\rho\right)^{m_T(\rho,2)-1} \otimes T_4 \text{ if } m_T(\rho, 2) \text{ odd,} \\ \left(\nu^{\frac{1}{2}}\rho\right)^{m_T(\rho,2)-1} \otimes L(\nu^{-\frac{1}{2}}\rho; T_5) \text{ if } m_T(\rho, 2) \text{ even.} \end{cases}$$

Here, $m_{T_4}(\rho, 2) = 1$*,* $m_{T_4}(\rho', b) = m_T(\rho', b)$ *for all other* (ρ', b)*, and* $\varepsilon_{T_4} = \varepsilon_T$*; for* T_5*,* $m_{T_5}(\rho, 2) = 0$*,* $m_{T_5}(\rho', b) = m_T(\rho', b)$ *for all other* (ρ', b)*, and* ε_{T_5} *just the restriction of* ε_T*.*

If $a = 1$*, we have the following:*

(9) *If* $\rho \not\cong \check{\rho}$, *then*

$$\mu^*_{\{\rho\}}(T) = (\rho)^{m_T(\rho,1)} \otimes T_1,$$

where $m_{T_1}(\rho,1) = m_{T_1}(\check{\rho},1) = 0$, $m_{T_1}(\rho',b) = m_T(\rho',b)$ *for all other* (ρ',b), *and* $\varepsilon_{T_1} = \varepsilon_T$ *(noting that* $(\rho,1) \notin S_T$*).*

(10) *If* $\rho \cong \check{\rho}$ *but* $red(\rho;\sigma) \not\equiv 0 \, mod\, 1$,

$$\mu^*_{\{\rho\}}(T) = 2^{\frac{m_T(\rho,1)}{2}} \cdot (\rho)^{\frac{m_T(\rho,1)}{2}} \otimes T_2$$

where $m_{T_2}(\rho,1) = 0$, $m_{T_2}(\rho',b) = m_T(\rho',b)$ *for all other* (ρ',b), *and* $\varepsilon_{T_2} = \varepsilon_T$ *(noting that* $(\rho,1) \notin S_T$*).*

(11) *If* $\rho \cong \check{\rho}$ *with* $red(\rho;\sigma) \equiv 0 \, mod\, 1$, *then*

$$\mu^*_{\{\rho\}}(T) = \begin{cases} 2^{\frac{m_T(\rho,1)}{2}} \cdot (\rho)^{\frac{m_T(\rho,1)}{2}} \otimes T_3 \text{ if } m_T(\rho,1) \text{ even}, \\ 2^{\frac{m_T(\rho,1)-1}{2}} \cdot (\rho)^{\frac{m_T(\rho,1)-1}{2}} \otimes T_3 \text{ if } m_T(\rho,1) \text{ odd}, \end{cases}$$

where $m_{T_3}(\rho,1) = \begin{cases} 0 & \text{if } m_T(\rho,1) \text{ even} \\ 1 & \text{if } m_T(\rho,1) \text{ odd} \end{cases}$, $m_{T_3}(\rho',b) = m_T(\rho',b)$ *for all other* (ρ',b), *and* $\varepsilon_{T_3} = \varepsilon_T$ *if* $m_T(\rho,1)$ *odd,* ε_{T_3} *the restriction of* ε_T *if* $m_T(\rho,1)$ *even.*

The following lemma is covered by the results of [22], but corrects an error. We remark that the results could also be obtained using the approach from Sect. 4 for the tempered case, noting that the Jacquet module results obtained in this section for tempered representations (and used in the arguments in Sect. 4) are known in the discrete series case (Theorem 3.2.2 [12]).

Lemma 3.2. *Let* δ *be a discrete series representation and* $a \geq 1$. *Then* $\nu^{\frac{a-1}{2}}\rho \rtimes \delta$ *is reducible if and only if one of the following occurs:*

(1) $a > 2$, $(\rho, a-2) \in Jord(\delta)$, *and either (i)* $(\rho,a) \notin Jord(\delta)$ *or (ii)* $\varepsilon_\delta(\rho,a)\varepsilon_\delta(\rho,a-2) = 1$,

(2) $a = 2$ *and either (i)* $(\rho,2) \notin Jord(\delta)$ *with* $red(\rho;\sigma) \equiv \frac{1}{2} \, mod\, 1$, *or (ii)* $\varepsilon_\delta(\rho,2) = 1$,

(3) $a = 1$ *and* $(\rho,a) \notin Jord(\delta)$ *with* $red(\rho;\sigma) \equiv 0 \, mod\, 1$.

Lemma 3.3. *Let* T *be an elliptic tempered representation. If* $(\rho,a) \notin Jord(T)$, *we have* $f_T(\nu^{\frac{a-1}{2}}\rho) = 0$. *If* $(\rho,a) \in Jord(T)$, *we have the following:*

(1) *If* a_- *exists,*

$$f_T\left(\nu^{\frac{a-1}{2}}\rho\right) = \begin{cases} m_T(\rho,a) \text{ if either } a > a_- + 2 \text{ or } \varepsilon_T(\rho,a)\varepsilon_T(\rho,a_-)^{-1} = 1, \\ m_T(\rho,a) - 1 \text{ if } a_- = a-2 \text{ and } \varepsilon_T(\rho,a)\varepsilon_T(\rho,a_-)^{-1} = -1. \end{cases}$$

(2) *If* a_- *does not exist,*

$$f_T\left(\nu^{\frac{a-1}{2}}\rho\right) = \begin{cases} m_T(\rho,a) \text{ if either } a > 2, \text{ or } a = 2 \text{ and } \varepsilon_T(\rho,a) = 1, \\ m_T(\rho,a) - 1 \text{ if either } a = 1, \text{ or } a = 2 \text{ and } \varepsilon_T(\rho,a) = -1. \end{cases}$$

Proof. That $f_T(\nu^{\frac{a-1}{2}}\rho) = 0$ when $(\rho, a) \notin Jord(T)$ follows immediately from the embeddings of T into induced representations [(Eq. (2.1)] and the corresponding property for discrete series (e.g., Remark 1.3.2 [12]). More generally, we have $f_T(\nu^{\frac{a-1}{2}}\rho) \leq m_T(\rho, a)$ for $a > 1$.

For (1), if $a > a_- + 2$, we use

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-c_h+1}{2}}\rho_h, \nu^{\frac{c_h-1}{2}}\rho_h\right]\right) \rtimes \delta, \quad (3.1)$$

with δ discrete series. If $m_T(\rho, a) = 1$, then $(\rho, a) \in Jord(\delta)$; the result then follows from Proposition 3.2.2 [12] and a commuting argument (resp., μ^* argument) for $f_T(\nu^{\frac{a-1}{2}}\rho) = 1$ (resp., $f_T(\nu^{\frac{a-1}{2}}\rho) = 0$). If $m_T(\rho, a) = 2$, then without loss of generality, suppose $(\rho, a) = (\rho_h, c_h)$. Then,

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-c_{h-1}+1}{2}}\rho_{h-1}, \nu^{\frac{c_{h-1}-1}{2}}\rho_{h-1}\right]\right) \rtimes T_h, \quad (3.2)$$

for some irreducible $T_h \leq \delta([\nu^{\frac{-c_h+1}{2}}\rho_h, \nu^{\frac{c_h-1}{2}}\rho_h]) \rtimes \delta$. The result when $f_T(\nu^{\frac{a-1}{2}}\rho) = m_T(\rho, a)$ then follows from Lemma 3.3.3 [12] and a commuting argument. Now, suppose $a = a_- + 2$ and $\varepsilon_T(\rho, a)\varepsilon_T(\rho, a_-)^{-1} = -1$. It follows from Proposition 2.3.2 [12] that $f_T(\rho, a) < m_T(\rho, a)$; an easy commuting argument with (3.1) then tells us $f_T(\nu^{\frac{a-1}{2}}\rho) \geq 1 \Rightarrow f_T(\nu^{\frac{a-1}{2}}\rho) = 1$.

We now look at (2). For $a = 1$, the result follows from (3.1): if $m_T(\rho, 1) = 2$, then $(\rho_i, c_i) = (\rho, 1)$ for some i, and the result is clear. If $m_T(\rho, 1) = 1$, then $(\rho, 1) \in Jord(\delta)$. Therefore, $\mu^*_{\{\rho\}}(\delta) = 0$ by the Casselman criterion, from which $f_T(\rho) = 0$ follows. For $a = 2$ and $\varepsilon_T(\rho, 2) = 1$, the result follows from Lemma 2.3.5 [12]. If $\varepsilon_T(\rho, 2) = -1$, it follows from Lemma 2.3.5 [12] that $f_T(\nu^{\frac{1}{2}}\rho) < m_T(\rho, 2)$. If $\varepsilon_T(\rho, 2) = -1$ and $m_T(\rho, 2) = 1$, we have $(\rho, 2) \in Jord(\delta)$. That $\varepsilon_\delta(\rho, 2) = \varepsilon_T(\rho, 2) = -1 \Rightarrow f_\delta(\nu^{\frac{1}{2}}\rho) = 0$ follows directly from the Mœglin–Tadić classification; that $f_T(\nu^{\frac{1}{2}}\rho) = 0$ then follows from (3.1). If $m_T(\rho, 2) = 2$ and $\varepsilon_T(\rho, 2) = -1$, we must have $(\rho_i, c_i) = (\rho, 2)$ in (3.2). It then follows (easy commuting argument) that $f_T(\nu^{\frac{1}{2}}\rho) \geq 1 \Rightarrow f_T(\nu^{\frac{1}{2}}\rho) = 1$. If $a > 2$ and $m_T(\rho, a) = 1$, it follows from (3.1), Proposition 2.1.2 [12] and and easy commuting argument that $f_T(\nu^{\frac{a-1}{2}}\rho) = 1$. If $m_T(\rho, a) = 2$, the result follows from (3.2), Lemma 3.3.3 [12], and an easy commuting argument. □

Proposition 3.4. *Let T be an elliptic tempered representation, $(\rho, a) \in Jord(T)$. Suppose $f_T(\nu^{\frac{a-1}{2}}\rho) \neq 0$ (see Lemma 3.3) and $a > 2$.*

(1) *If $(\rho, a-2) \notin Jord(T)$,*

$$\mu^*_{\left\{\nu^{\frac{a-1}{2}}\rho\right\}}(T) = \left(\nu^{\frac{a-1}{2}}\rho\right)^{m_T(\rho,a)} \otimes T_1$$

where T_1 has $Jord(T_1) = (Jord(T) \cup \{(\rho, a-2)\}) \setminus \{(\rho, a)\}$, $m_{T_1}(\rho, a-2) = m_T(\rho, a)$, and remaining multiplicities matching those for T. We obtain ε_{T_1} by substituting $(\rho, a-2)$ for (ρ, a): ε_{T_1} is determined by $\varepsilon_{T_1}(\rho, a-2) =$

$\varepsilon_T(\rho,a)$ if defined, $\varepsilon_{T_1}(\rho,b)\varepsilon_{T_1}(\rho,a-2)^{-1} = \varepsilon_T(\rho,b)\varepsilon_T(\rho,a)^{-1}$, and $\varepsilon_{T_1}|_{S_{T_1}\cap S_T} = \varepsilon_T|_{S_{T_1}\cap S_T}$.

(2) *If $(\rho,a-2)\in Jord(T)$ with $\varepsilon_T(\rho,a)\varepsilon_T(\rho,a-2)^{-1}=1$,*

$$\mu^*_{\left\{\nu^{\frac{a-1}{2}}\rho\right\}}(T) = \left(\nu^{\frac{a-1}{2}}\rho\right)^{m_T(\rho,a)} \otimes T_2,$$

where $Jord(T_2) = Jord(T)\setminus\{(\rho,a)\}$, $m_{T_2}(\rho,a-2) = m_T(\rho,a-2) + m_T(\rho,a)$, remaining multiplicities the same as for T, and $\varepsilon_{T_2} = \varepsilon_T|_{S_{T_2}}$.

(3) *If $(\rho,a-2)\in Jord(T)$ with $\varepsilon_T(\rho,a)\varepsilon_T(\rho,a-2)^{-1}=-1$, in order to have $f_T(\nu^{\frac{a-1}{2}}\rho)\neq 0$, we must have $m_T(\rho,a)=2$ (and then $f_T(\nu^{\frac{a-1}{2}}\rho)=1$). Further,*

$$\mu^*_{\left\{\nu^{\frac{a-1}{2}}\rho\right\}}(T) = \nu^{\frac{a-1}{2}}\rho \otimes L(\delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right); T_3),$$

where T_3 has $Jord(T_3) = Jord(T)\setminus\{(\rho,a)\}$ and m_{T_3}, ε_{T_3} given by restriction.

Proof. The values of $f_T(\nu^{\frac{a-1}{2}}\rho)$ are given in Lemma 3.3.

We start with (3), which is the easiest of the remaining cases. Here, we have

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes T_3 \hookrightarrow \nu^{\frac{a-1}{2}}\rho \times \delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right) \rtimes T_3$$

with T_3 as in the statement of the proposition. Lemma 2.3 then gives the result for (3).

We now turn to (1), in which case $f_T(\nu^{\frac{a-1}{2}}\rho) = m_T(\rho,a)$. Write

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-a_k+1}{2}}\rho_k, \nu^{\frac{a_k-1}{2}}\rho_k\right]\right) \rtimes \delta.$$

In what follows, we essentially reduce to the case $k=1$, which is then addressed in Lemma 3.5 below. Thus, suppose $k>1$.

First, for $1\le j\le k$, let

$$\begin{aligned}\Lambda_j = \delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-a_1+j-1}{2}}\rho_{j-1}, \nu^{\frac{a_{j-1}-1}{2}}\rho_{j-1}\right]\right) \\ \times\delta\left(\left[\nu^{\frac{-a_{j+1}+1}{2}}\rho_{j+1}, \nu^{\frac{a_{j+1}-1}{2}}\rho_{j+1}\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-a_k+1}{2}}\rho_k, \nu^{\frac{a_k-1}{2}}\rho_k\right]\right).\end{aligned}$$

Then,

$$\begin{gathered} T \hookrightarrow \Lambda_j \times \delta\left(\left[\nu^{\frac{-a_j+1}{2}}\rho_j, \nu^{\frac{a_j-1}{2}}\rho_j\right]\right) \rtimes \delta \\ \Downarrow \\ T \hookrightarrow \Lambda_j \rtimes T_j \end{gathered}$$

for some irreducible $T_j \le \delta([\nu^{\frac{-a_j+1}{2}}\rho_j, \nu^{\frac{a_j-1}{2}}\rho_j]) \rtimes \delta$. By definition, $\varepsilon_T|_{S_j} = \varepsilon_{T_j}$, where S_j denotes the domain for T_j.

Write $\mu^*_{\{\nu^{\frac{a-1}{2}}\rho\}}(T) = (\nu^{\frac{a-1}{2}}\rho)^m \otimes T'$, where $m = m_T(\rho, a)$. Set

$$\delta'_j = \begin{cases} \delta & \text{if } m_T(\rho, a) = 2, \\ \delta' & \text{if } m_T(\rho, a) = 1, \end{cases}$$

where for $m_T(\rho, a) = 1$, the δ' is that of Theorem 3.2.2 [12] (noting that this δ' is elliptic tempered, but need not be square-integrable). We define $\Lambda'_j = \Lambda_j$ if $m_T(\rho, a) = 1$ or $(\rho_j, a_j) = (\rho, a)$. If not, we let i be such that $(\rho_i, a_i) = (\rho, a)$ and replace $\delta([\nu^{\frac{-a_i+1}{2}}\rho_i, \nu^{\frac{a_i-1}{2}}\rho_i])$ with $\delta([\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho])$ in the definition of Λ_j (with the rest of the terms remaining the same). Finally, set $a'_j = \begin{cases} a_j - 2 \text{ if } (\rho_j, a_j) = (\rho, a) \\ a_j \text{ if not.} \end{cases}$ Then

$$T' \hookrightarrow \Lambda'_j \times \delta\left(\left[\nu^{\frac{-a'_j+1}{2}}\rho_j, \nu^{\frac{a'_j-1}{2}}\rho_j\right]\right) \rtimes \delta'_j$$
$$\Downarrow$$
$$T' \hookrightarrow \Lambda'_j \rtimes T'_j$$

for some irreducible $T'_j \le \delta([\nu^{\frac{-a'_j+1}{2}}\rho_j, \nu^{\frac{a'_j-1}{2}}\rho_j]) \rtimes \delta'_j$. Again, we have $\varepsilon_{T'}|_{S'_j} = \varepsilon_{T'_j}$.

Fix j. First, suppose $m_T(\rho, a) = 1$ or $(\rho_j, a_j) = (\rho, a)$. Note that in this case, $Jord(T'_j) = (Jord(T_j)\backslash\{(\rho, a)\}) \cup \{(\rho, a-2)\}$. If we knew that $\mu^*_{\{\nu^{\frac{a-1}{2}}\rho\}}(T_j) = (\nu^{\frac{a-1}{2}}\rho)^{m_T(\rho,a)} \otimes T'_j$, the $k = 1$ case (Lemma 3.5 below) would imply that $\varepsilon_{T'}|_{S'_j}$ is as claimed. As the values of $\varepsilon_{T'}$ on $\cup_j S'_j$ determine $\varepsilon_{T'}$ (Sect. 2.3 [12]), the result follows for this case. Thus to finish this case, it remains to show $\mu^*_{\{\nu^{\frac{a-1}{2}}\rho\}}(T_j) = (\nu^{\frac{a-1}{2}}\rho)^{m_T(\rho,a)} \otimes T'_j$.

Observe that we have $\Lambda'_j = \Lambda_j$ and

$$\begin{aligned} T &\hookrightarrow \Lambda_j \rtimes T_j \\ &\hookrightarrow \Lambda_j \times \left(\nu^{\frac{a-1}{2}}\rho\right)^{m_T(\rho,a)} \rtimes \theta_{T_j} \\ &\cong \left(\nu^{\frac{a-1}{2}}\rho\right)^{m_T(\rho,a)} \times \Lambda_j \rtimes \theta_{T_j}, \end{aligned}$$

noting that θ_{T_j} is tempered. Since $\mu^*_{\{\nu^{\frac{a-1}{2}}\rho\}}(T) = (\nu^{\frac{a-1}{2}}\rho)^{m_T(\rho,a)} \otimes T'$, we have

$$T' \hookrightarrow \Lambda_j \rtimes \theta_{T_j}$$
$$\Downarrow$$
$$\theta_{T_j} = T'_j,$$

as needed.

Now, suppose $(\rho_j, a_j) \neq (\rho, a)$ and $m_T(\rho, a) = 2$. Note that in this case, we have $Jord(T_j') = Jord(T_j)$ and must show $T_j' = T_j$. We have

$$\begin{gathered}
T \hookrightarrow \Lambda_j \rtimes T_j \\
\Downarrow \\
\mu^*_{\left\{\nu^{\frac{a-1}{2}}\rho\right\}}(T) \leq \left(\nu^{\frac{a-1}{2}}\rho\right)^2 \otimes \Lambda_j' \rtimes T_j \\
\Downarrow \\
T' \leq \Lambda_j' \rtimes T_j \\
\Downarrow \\
T_j' = T_j,
\end{gathered}$$

as needed. This finishes (1).

For (2), the argument is fairly simple if either (ρ, a) or $(\rho, a-2)$ has multiplicity two. If $m_T(\rho, a) = 2$, we have

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes T',$$

where the data for T' are obtained by removing both copies of (ρ, a) and restricting ε_T (Lemma 2.4). As

$$\begin{aligned}
&\mu^*_{\left\{\nu^{\frac{a-1}{2}}\rho\right\}}\left(\delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes T'\right) \\
&\quad = \left(\nu^{\frac{a-1}{2}}\rho\right)^2 \otimes \left(\delta\left(\left[\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right) \rtimes T'\right),
\end{aligned}$$

and $f_T(\nu^{\frac{a-1}{2}}\rho) = 2$, it follows that (noting the irreducibility of $\delta([\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]) \rtimes T'$)

$$\mu^*_{\left\{\nu^{\frac{a-1}{2}}\rho\right\}}(T) = \left(\nu^{\frac{a-1}{2}}\rho\right)^2 \otimes \left(\delta\left(\left[\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right) \rtimes T'\right).$$

As $\delta([\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]) \rtimes T' \cong T_2$, the result follows. If $m_T(\rho, a) = 1$ but $m_T(\rho, a-2) = 2$, the argument is similar: by Proposition 3.4.3 [12], we have

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes T'',$$

where the data for T'' are obtained by removing one copy each of (ρ, a) and $(\rho, a-2)$ and restricting ε_T. Here, we have

$$\begin{gathered}
\mu^*_{\left\{\nu^{\frac{a-1}{2}}\rho\right\}}\left(\delta\left(\left[\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes T''\right) = \nu^{\frac{a-1}{2}}\rho \otimes \left(\delta\left(\left[\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right) \rtimes T''\right) \\
\Downarrow \\
\mu^*_{\left\{\nu^{\frac{a-1}{2}}\rho\right\}}(T) = \nu^{\frac{a-1}{2}}\rho \otimes \left(\delta\left(\left[\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right) \rtimes T''\right).
\end{gathered}$$

Again, the conclusion follows from the observation that $T_2 \cong \delta([\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]) \rtimes T''$.

If $m_T(\rho, a) = m_T(\rho, a-2) = 1$, the argument is based on combining a pair of embeddings. First, by Proposition 3.4.3 [12], we have

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes T',$$

with T' having $Jord(T') = Jord(T)\backslash\{(\rho, a), (\rho, a-2)\}$ and $m_{T'}, \varepsilon_{T'}$ given by restriction. Then,

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes T' \hookrightarrow \nu^{\frac{a-1}{2}} \times \delta\left(\left[\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right) \rtimes T'.$$

By Lemma 2.3, letting $\theta_T = \theta_T(\nu^{\frac{a-1}{2}}\rho)$, we then have

$$\theta_T \leq \delta\left(\left[\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right) \rtimes T'. \tag{3.3}$$

Note that it follows from Eq. (3.3) that $\theta_T(\nu^{\frac{a-1}{2}}\rho)$ is tempered.

In the embedding

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-a_k+1}{2}}\rho_k, \nu^{\frac{a_k-1}{2}}\rho_k\right]\right) \rtimes \delta_T,$$

we have $(\rho, a), (\rho, a-2) \in Jord(\delta_T)$. By Theorem 3.2.2 [12], we have

$$\delta_T \hookrightarrow \nu^{\frac{a-1}{2}}\rho \rtimes T'',$$

with T'' having $Jord(T'') = Jord(\delta_T)\backslash\{(\rho, a)\}, m_{T''}(\rho, a-2) = 2$ with remaining multiplicities and $\varepsilon_{T''}$ given by restriction. Then, by a commuting argument (noting the obvious irreducibilities involved)

$$\begin{aligned} T &\hookrightarrow \delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-a_k+1}{2}}\rho_k, \nu^{\frac{a_k-1}{2}}\rho_k\right]\right) \times \nu^{\frac{a-1}{2}}\rho \rtimes T'' \\ &\cong \nu^{\frac{a-1}{2}}\rho \times \delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-a_k+1}{2}}\rho_k, \nu^{\frac{a_k-1}{2}}\rho_k\right]\right) \rtimes T''. \end{aligned}$$

Again, by Lemma 2.3,

$$\theta_T \leq \delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-a_k+1}{2}}\rho_k, \nu^{\frac{a_k-1}{2}}\rho_k\right]\right) \rtimes T''. \tag{3.4}$$

Recall ([12]) that ε_{θ_T} is defined as follows: we have

$$\begin{aligned} \theta_T \leq\ &\delta\left(\left[\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right) \\ &\times \delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-a_k+1}{2}}\rho_k, \nu^{\frac{a_k-1}{2}}\rho_k\right]\right) \rtimes \delta_{\theta_T}. \end{aligned}$$

For notational convenience, let $(\rho_0, a_0) = (\rho, a-2)$. Then, for $i = 0, \ldots, k$, there is a unique $\mathcal{T}_i \leq \delta([\nu^{\frac{-a_i+1}{2}}\rho_i, \nu^{\frac{a_i-1}{2}}\rho_i]) \rtimes \delta_{\theta_T}$ such that

$$\begin{aligned} \theta_T \leq\ &\delta\left(\left[\nu^{\frac{-a_0+1}{2}}\rho_0, \nu^{\frac{a_0-1}{2}}\rho_0\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-a_{i-1}+1}{2}}\rho_{i-1}, \nu^{\frac{a_{i-1}-1}{2}}\rho_{i-1}\right]\right) \\ &\times \delta\left(\left[\nu^{\frac{-a_{i+1}+1}{2}}\rho_{i+1}, \nu^{\frac{a_{i+1}-1}{2}}\rho_{i+1}\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-a_k+1}{2}}\rho_k, \nu^{\frac{a_k-1}{2}}\rho_k\right]\right) \rtimes \mathcal{T}_i. \end{aligned}$$

If S_i denotes the domain for $\delta([\nu^{\frac{-a_i+1}{2}}\rho_i, \nu^{\frac{a_i-1}{2}}\rho_i]) \rtimes \delta_{\theta_T}$, by definition (Sect. 2.3 [12]) we have

$$\varepsilon_{\theta_T}\big|_{S_i} = \varepsilon_{\mathcal{T}_i}.$$

Observe that it then follows directly from (3.4) and the description of the data of T'' that

$$\varepsilon_{\theta_T}\big|_{S_0} = \varepsilon_{T''} = \varepsilon_T\big|_{S_0}.$$

Also, it follows directly from (3.3) and the description of the data of T' that

$$\varepsilon_{\theta_T}\big|_{S_i} = \varepsilon_{T'}|_{S_i} = \varepsilon_T\big|_{S_i}$$

for $i = 1, \dots, k$. Combining these, we have

$$\varepsilon_{\theta_T}\big|_{S_i} = \varepsilon_T\big|_{S_i}$$

for $i = 0, \dots, k$.

As ε_{θ_T} is determined by its values on $S_{\theta_T} \cap [Jord_{\rho_i}(\theta_T) \cup Jord_{\rho_i}(\theta_T) \times Jord_{\rho_i}(\theta_T)]$, $i = 0, 1, \dots, k$, it suffices to show that $\varepsilon_{\theta_T}|_{S_i}$ determines ε_{θ_T} on $S_{\theta_T} \cap [Jord_{\rho_i}(\theta_T) \cup Jord_{\rho_i}(\theta_T) \times Jord_{\rho_i}(\theta_T)]$ for $i = 0, 1, \dots, k$. Fix $i \in \{0, 1, \dots, k\}$. If $red(\rho_i; \sigma) = 0$ or $red(\rho_i; \sigma) \equiv \frac{1}{2} \mod 1$, then ε_{θ_T} is defined on $Jord_{\rho_i}(\theta_T)$; by Lemma 2.3.1 [12], its values there determine ε_{θ_T} on $S_{\theta_T} \cap [(Jord_{\rho_i}(\theta_T) \cup (Jord_{\rho_i}(\theta_T) \times Jord_{\rho_i}(\theta_T))]$. If $red(\rho_i; \sigma) \in \mathbb{N}$, then ε_{θ_T} is not defined on individual elements of $Jord_{\rho_i}(\theta_T)$. However, it still follows from Lemma 2.3.1 [12] that ε_{θ_T} is determined on $S_{\theta_T} \cap [Jord_{\rho_i}(\theta_T) \cup (Jord_{\rho_i}(\theta_T) \times Jord_{\rho_i}(\theta_T))]$ by its values on S_i, $i = 0, \dots, k$ (more precisely, those having $\rho_j \cong \rho_i$) as long as $Jord_{\rho_i}(\delta_\theta) \neq \emptyset$. (To see this, fix $(\rho_i, b) \in Jord(\delta_{\theta_T})$ and observe that for $\rho_j \cong \rho_i$, $\varepsilon_{\theta_T}(\rho_i, a_i)\varepsilon_{\theta_T}(\rho_i, a_j)^{-1} = \varepsilon_{\theta_T}(\rho_i, a_i)\varepsilon_{\theta_T}(\rho_i, b)^{-1} \cdot \varepsilon_{\theta_T}(\rho_i, b)\varepsilon_{\theta_T}(\rho_i, a_j)^{-1}$.) However, that $Jord_{\rho_i}(\delta_{\theta_T}) \neq \emptyset$ follows for $red(\rho_i; \sigma) \geq 1$ as in Example 14.4.0 [20], finishing this case and the proposition. □

Lemma 3.5. *Proposition 3.4 holds in the case $k = 1$ and (ρ, a) satisfies either* (i) *$a_- < a - 2$ (or does not exist), or* (ii) $\varepsilon_T(\rho, a)\varepsilon_T(\rho, a_-)^{-1} = 1$.

Proof. Write $\mu^*_{\{\nu^{\frac{a-1}{2}}\rho\}}(T) = (\nu^{\frac{a-1}{2}}\rho)^m \otimes T'$, where $m = m_T(\rho, a)$. Note that as in the proof of Proposition 3.4, it suffices to show that $\varepsilon_{T'}$ is as claimed. Further, if we let $S_T(\rho)$ be that part of S_T supported on $Jord_\rho(T) \cup (Jord_\rho(T) \times Jord_\rho(T))$, and similarly for $S_\delta(\rho)$, $S_{T'}(\rho)$, etc., it suffices to show that $\varepsilon_{T'}|_{S_{T'}(\rho)}$ is as claimed.

We note that the case where (ρ, a_-) exists is covered by Lemma 3.3.3 [12]. Thus we assume (ρ, a_-) does not exist. The proof is broken into three main cases: (1) $Jord_\rho(T) = \{(\rho, a)\}$ with $red(\rho; \sigma)$ half-integral, (2) $Jord_\rho(T) = \{(\rho, a)\}$ with $red(\rho; \sigma)$ integral, (3) (ρ, a_+) exists, where a_+ is the smallest value of $b > a$ satisfying $(\rho, b) \in Jord$ if it exists.

Case 1 $Jord_\rho(T) = \{(\rho, a)\}$ with $red(\rho; \sigma)$ half-integral

Observe that by supercuspidal support considerations (Lemma 2.4.1 [12]), $m_{T'}(\rho, a-2) = m$. Now, in one direction,

$$\begin{gathered} \varepsilon_{T'}(\rho, a-2) = 1 \\ \Downarrow \text{ (Lemma 2.3.5 [12])} \\ T' \hookrightarrow \delta\left(\left[\nu^{\frac{1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right)^m \rtimes \theta' \\ \Downarrow \\ T \hookrightarrow \left(\nu^{\frac{a-1}{2}}\rho\right)^m \times \delta\left(\left[\nu^{\frac{1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right)^m \rtimes \theta'. \end{gathered}$$

By Frobenius reciprocity, $r_{M,G}(T) \geq (\nu^{\frac{a-1}{2}}\rho)^m \otimes \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho])^m \otimes \theta'$ for the appropriate standard Levi factor M. As the only irreducible representation of a general linear group having $(\nu^{\frac{a-1}{2}}\rho)^m \otimes \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho])^m$ in its Jacquet module is $\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])^m$, we get $\mu^*(T) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])^m \otimes \theta'$. It now follows from Note 2.4.2 and Lemma 2.3.5 [12] that $\varepsilon_T(\rho, a) = 1$.

In the converse direction,

$$\begin{gathered} \varepsilon_T(\rho, a) = 1 \\ \Downarrow \text{ (Lemma 2.3.5 [12])} \\ T \hookrightarrow \delta\left(\left[\nu^{\frac{1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)^m \rtimes \theta \hookrightarrow \left(\nu^{\frac{a-1}{2}}\rho\right)^m \times \delta\left(\left[\nu^{\frac{1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right)^m \rtimes \theta \\ \Downarrow \\ r_{M,G}(T) \geq \left(\nu^{\frac{a-1}{2}}\rho\right)^m \otimes \delta\left(\left[\nu^{\frac{1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right)^m \otimes \theta \end{gathered}$$

for the appropriate standard Levi factor M. As $\mu^*_{\{\nu^{\frac{a-1}{2}}\rho\}}(T) = (\nu^{\frac{a-1}{2}}\rho)^m \otimes T'$. It follows that $\mu^*(T') \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho])^m \otimes \theta$. As above, this then implies $\varepsilon_{T'}(\rho, a-2) = 1$, as needed.

Case 2 $Jord_\rho(T) = \{(\rho, a)\}$ with $red(\rho; \sigma)$ integral

First, observe that $|Jord_\rho(T)| \leq 1$, so $|Jord_\rho(\delta)| \leq 1$. This implies $red(\rho; \sigma) \in \{0, 1\}$ (e.g., see Sect. 14.4 [20]). If $red(\rho; \sigma) = 1$, we have $S_T(\rho) \subset Jord_\rho(T) \times Jord_\rho(T)$. Thus, $\varepsilon = \emptyset$ (see Example 14.4.1 [20], e.g.) and there is nothing to prove. We therefore assume $red(\rho; \sigma) = 0$ in what follows. Recall that in this case, one makes an arbitrary enumeration of components $\rho \rtimes \sigma = \tau_1(\rho; \sigma) \oplus \tau_{-1}(\rho; \sigma)$ (see [29,30]) as part of the classification of discrete series.

In the notation of Definition 2.3.6 [12],

$$\begin{gathered} \varepsilon_{T'}(\rho, a-2) = \eta \\ \Downarrow \\ T' \hookrightarrow \lambda \rtimes T(\delta\left(\left[\nu\rho, \nu^{\frac{a-3}{2}}\rho\right]\right)^m; \tau_\eta(\rho; \sigma)) \\ \Downarrow \\ T \hookrightarrow \left(\nu^{\frac{a-1}{2}}\rho\right)^m \times \lambda \rtimes T(\delta\left(\left[\nu\rho, \nu^{\frac{a-3}{2}}\rho\right]\right)^m; \tau_\eta(\rho; \sigma)) \\ \cong \lambda \times \left(\nu^{\frac{a-1}{2}}\rho\right)^m \rtimes T\left(\delta\left(\left[\nu\rho, \nu^{\frac{a-3}{2}}\rho\right]\right)^m; \tau_\eta(\rho; \sigma)\right) \\ \Downarrow \\ r_{M,G}(T) \geq \lambda \otimes \left(\nu^{\frac{a-1}{2}}\rho\right)^m \otimes \left(\nu^{\frac{a-3}{2}}\rho\right)^m \otimes \cdots \otimes (\nu\rho)^m \otimes \tau_\eta(\rho; \sigma), \end{gathered}$$

with the irreducibility of $(\nu^{\frac{a-1}{2}}\rho)^m \times \lambda$ immediate from supercuspidal support considerations. Now, observe that by the Langlands classification, $L((\nu^{\frac{-a+1}{2}}\rho)^m, (\nu^{\frac{-a+3}{2}}\rho)^m, \dots, (\nu^{-1}\rho)^m; \tau_{-\eta}(\rho;\sigma))$ is the unique irreducible representation containing $(\nu^{\frac{-a+1}{2}}\rho)^m \otimes (\nu^{\frac{-a+3}{2}}\rho)^m \otimes \cdots \otimes (\nu^{-1}\rho)^m \otimes \tau_{-\eta}(\rho;\sigma)$ in its Jacquet module. It then follows from duality that $T(\delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho])^m; \tau_\eta(\rho;\sigma))$ is the unique irreducible representation containing $(\nu^{\frac{a-1}{2}}\rho)^m \otimes (\nu^{\frac{a-3}{2}}\rho)^m \otimes \cdots \otimes (\nu\rho)^m \otimes \tau_\eta(\rho;\sigma)$ in its Jacquet module. Thus,

$$\mu^*(T) \geq \lambda \otimes T(\delta\left(\left[\nu\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)^m; \tau_\eta(\rho;\sigma)).$$

By Lemma 2.2.8 [12] for $m = 2$ and a similar argument when $m = 1$, this then implies

$$\varepsilon_T(\rho, a) = \eta,$$

as needed.

Case 3 (ρ, a_+) exists

We first consider the case $m_T(\rho, a_+) = 1$.

In one direction,

$$\begin{gathered}
\varepsilon_{T'}(\rho, a_+)\varepsilon_{T'}(\rho, a-2) = 1 \\
\Downarrow \\
T' \hookrightarrow \delta\left(\left[\nu^{\frac{a-1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right) \rtimes \lambda \\
\Downarrow \\
\begin{aligned}
T &\hookrightarrow \left(\nu^{\frac{a-1}{2}}\rho\right)^m \times \delta\left(\left[\nu^{\frac{a-1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right) \rtimes \lambda \\
&\cong \delta\left(\left[\nu^{\frac{a-1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right) \times \left(\nu^{\frac{a-1}{2}}\rho\right)^m \rtimes \lambda \\
&\hookrightarrow \delta\left(\left[\nu^{\frac{a-3}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right) \times (\nu^{\frac{a-1}{2}}\rho)^{m+1} \rtimes \lambda
\end{aligned} \\
\Downarrow \\
\varepsilon_T(\rho, a_+)\varepsilon_T(\rho, a)^{-1} = 1.
\end{gathered}$$

In the other direction, by Proposition 3.4.3 [12]

$$\begin{gathered}
\varepsilon_T(\rho, a_+)\varepsilon_T(\rho, a)^{-1} = 1 \\
\Downarrow \\
T \hookrightarrow \delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right) \rtimes T^* \hookrightarrow \delta([\nu^{\frac{a-1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho]) \times \delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right) \rtimes T^* \\
\Downarrow \text{ (Lemma 2.2)} \\
T \hookrightarrow \delta\left(\left[\nu^{\frac{a-1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right) \rtimes \xi
\end{gathered}$$

for some irreducible $\xi \leq \delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]) \rtimes T^*$. From this embedding and the irreducibility of $\delta([\nu^{\frac{a-1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho]) \times \nu^{\frac{a-1}{2}}\rho$, we see that $f_T(\nu^{\frac{a-1}{2}}\rho) = f_\xi(\nu^{\frac{a-1}{2}}\rho)$, so $f_\xi(\nu^{\frac{a-1}{2}}\rho) = m$. It then follows that $\xi \hookrightarrow (\nu^{\frac{a-1}{2}}\rho)^m \rtimes \xi'$ for some irreducible ξ'. Then,

$$\begin{gathered} T \hookrightarrow \delta\left(\left[\nu^{\frac{a-1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right) \times \left(\nu^{\frac{a-1}{2}}\rho\right)^m \rtimes \xi' \\ \cong \left(\nu^{\frac{a-1}{2}}\rho\right)^m \times \delta\left(\left[\nu^{\frac{a-1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right) \rtimes \xi' \\ \Downarrow \\ r_{M,G}(T) \geq (\nu^{\frac{a-1}{2}}\rho)^m \otimes \delta([\nu^{\frac{a-1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho]) \otimes \xi' \end{gathered}$$

for the appropriate standard Levi factor M. It then follows from (1.1) that

$$\begin{gathered} \mu^*(T') \geq \delta\left(\left[\nu^{\frac{a-1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right) \otimes \xi' \\ \Downarrow \text{ (Note 2.4.2 [12])} \\ \varepsilon_{T'}(\rho, a_+)\varepsilon_{T'}(\rho, a-2)^{-1} = 1, \end{gathered}$$

as needed.

It remains to address the case $m_T(\rho, a_+) = 2$. Note that as $k = 1$, this has $m_T(\rho, a) = 1$, so $(\rho, a) \in Jord(\delta)$. The implication $\varepsilon_{T'}(\rho, a_+)\varepsilon_{T'}(\rho, a-2)^{-1} = 1 \Rightarrow \varepsilon_T(\rho, a_+)\varepsilon_T(\rho, a)^{-1} = 1$ is similar to the argument when $m_T(\rho, a_+) = 1$. In the other direction,

$$\begin{gathered} \varepsilon_T(\rho, a_+)\varepsilon_T(\rho, a)^{-1} = 1 \\ \Downarrow \\ \mu^*_{\delta\left(\left[\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right)^2}(T) = \delta\left(\left[\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right)^2 \otimes \left(\delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \delta\right), \end{gathered}$$

noting that

$$\begin{aligned} &\mu^*_{\delta\left(\left[\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right)^2}\left(\left(\delta\left(\left[\nu^{\frac{-a_++1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right) \rtimes \delta\right)\right. \\ &= \delta\left(\left[\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right)^2 \otimes \left(\delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \delta\right). \end{aligned}$$

has the right-hand side irreducible (since $(\rho, a) \in Jord(\delta)$). As $f_{\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \delta}$ $(\nu^{\frac{a-1}{2}}\rho) = 3$, we get

$$\begin{gathered} T \hookrightarrow \delta\left(\left[\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right)^2 \times \left(\nu^{\frac{a-1}{2}}\rho\right)^3 \rtimes \theta \\ \Downarrow \text{ (Lemma 2.2)} \\ T \hookrightarrow \delta\left(\left[\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right) \times \lambda_1 \times \left(\nu^{\frac{a-1}{2}}\rho\right)^2 \rtimes \theta \end{gathered}$$

for some irreducible θ and some irreducible $\lambda_1 \leq \delta([\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho]) \times \nu^{\frac{a-1}{2}}\rho$.

The two possibilities for λ_1 are $\delta([\nu^{\frac{a-1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho])$ and $\mathcal{L}(\nu^{\frac{a-1}{2}}\rho, \delta([\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho]))$. In either case, we have $\nu^{\frac{a-1}{2}}\rho \times \lambda_1$ is irreducible ([33] and Lemma 3.6 below). We now have

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right) \times \left(\nu^{\frac{a-1}{2}}\rho\right)^2 \times \lambda_1 \rtimes \theta$$
$$\Downarrow \text{ (Lemma 2.2)}$$
$$T \hookrightarrow \lambda_2 \times \nu^{\frac{a-1}{2}}\rho \times \lambda_1 \rtimes \theta \cong \nu^{\frac{a-1}{2}}\rho \times \lambda_2 \times \lambda_1 \rtimes \theta$$

for some irreducible $\lambda_2 \leq \delta([\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho]) \times \nu^{\frac{a-1}{2}}\rho$. Now, if $\lambda_2 = \mathcal{L}(\nu^{\frac{a-1}{2}}\rho, \delta([\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho]))$, the embedding above would immediately imply $f_T(\nu^{\frac{a-1}{2}}\rho) \geq 2$, a contradiction. Thus, $\lambda_2 = \delta([\nu^{\frac{a-1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho])$. Since $\delta([\nu^{\frac{a-1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho]) \times \lambda_1$ is irreducible for either possibility for λ_1 ([33], Lemma 1.3.3 [10]), we have

$$T \hookrightarrow \nu^{\frac{a-1}{2}}\rho \times \delta\left(\left[\nu^{\frac{a-1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right) \times \lambda_1 \rtimes \theta$$
$$\cong \nu^{\frac{a-1}{2}}\rho \times \lambda_1 \times \delta\left(\left[\nu^{\frac{a-1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right) \rtimes \theta.$$

Applying the same argument again tells us $\lambda_1 = \delta([\nu^{\frac{a-1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho])$. Arguing as in the case $m_T(\rho, a_+) = 1$, we now get

$$T \hookrightarrow \nu^{\frac{a-1}{2}}\rho \times \delta\left(\left[\nu^{\frac{a-1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right)^2 \rtimes \theta$$
$$\Downarrow$$
$$r_{M,G}(T) \geq \nu^{\frac{a-1}{2}}\rho \otimes \delta\left(\left[\nu^{\frac{a-1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right)^2 \otimes \theta$$
$$\Downarrow$$
$$\mu^*(T') \geq \delta\left(\left[\nu^{\frac{a-1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right)^2 \otimes \theta$$
$$\Downarrow \text{ (Note 2.4.2 [12])}$$
$$\varepsilon_{T'}(\rho, a_+)\varepsilon_{T'}(\rho, a-2)^{-1} = 1,$$

as needed. $\square$

Lemma 3.6. *The following representations are irreducible:*

(1) $\nu^{\frac{a-1}{2}}\rho \times \mathcal{L}(\nu^{\frac{a-1}{2}}\rho, \delta([\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho]))$ *for* $b \geq a$,
(2) $\mathcal{L}(\nu^{\frac{-a+1}{2}}\rho, \delta([\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]), \nu^{\frac{a-1}{2}}\rho) \times \nu^{\frac{a-1}{2}}\rho$ *for* $a \geq 2$,
(3) $\mathcal{L}(\nu^{\frac{-a+1}{2}}\rho, \delta([\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])) \times \nu^{\frac{a-1}{2}}\rho$ *for* $a \geq 2$.

Proof. (1) is a special case of Lemma 4.1 (or more generally, Proposition 4.3) of [15]. For $a = 2$, (2) and (3) follow from [33]. For $a > 2$, (3) is immediate from Lemma 1.3.3 [10].

This leaves (2) when $a > 2$. In this case, observe that by Lemma 1.3.1 [10], the only possible irreducible subquotients of $\mathcal{L}(\nu^{\frac{-a+1}{2}}\rho, \delta([\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]), \nu^{\frac{a-1}{2}}\rho) \times \nu^{\frac{a-1}{2}}\rho$ are the following:

(1) $\pi_1 = \mathcal{L}(\nu^{\frac{-a+1}{2}}\rho, \delta([\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]), \nu^{\frac{a-1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho)$,
(2) $\pi_2 = \mathcal{L}(\nu^{\frac{-a+1}{2}}\rho, \delta([\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]), \nu^{\frac{a-1}{2}}\rho)$
(3) $\pi_3 = \mathcal{L}(\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]), \nu^{\frac{a-1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho)$

(4) $\pi_4 = \mathcal{L}(\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]), \nu^{\frac{a-1}{2}}\rho)$ (i.e., $\pi_4 = \delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \times \nu^{\frac{a-1}{2}}\rho$).

By the Langlands classification, π_1 appears with multiplicity one. We now show that the remaining possibilities cannot occur. Observe that by Proposition 2.1.4 [10], $\mathcal{L}(\nu^{\frac{-a+1}{2}}\rho, \delta([\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]), \nu^{\frac{a-1}{2}}\rho)$ has $f(\nu^{\frac{a-1}{2}}\rho) = 0$, so any irreducible subquotient of $\mathcal{L}(\nu^{\frac{-a+1}{2}}\rho, \delta([\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]), \nu^{\frac{a-1}{2}}\rho) \times \nu^{\frac{a-1}{2}}\rho$ has $f(\nu^{\frac{a-1}{2}}\rho) \leq 1$. However, Proposition 2.1.4 [10] tells us $f_{\pi_2}(\nu^{\frac{a-1}{2}}\rho) = f_{\pi_4}(\nu^{\frac{a-1}{2}}\rho) = 2$. Thus, they do not occur. Further, $m^*(\pi_3)$ contains a term of the form $\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]) \otimes \cdots$, whereas $m^*\left(\mathcal{L}(\nu^{\frac{-a+1}{2}}\rho, \delta([\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]), \nu^{\frac{a-1}{2}}\rho) \times \nu^{\frac{a-1}{2}}\rho\right)$ does not (as the $\nu^{\frac{-a+3}{3}}\rho$ appears after the $\nu^{\frac{-a+1}{2}}\rho$ in any Jacquet module term). Irreducibility follows. □

Lemma 3.7. *Suppose T is elliptic tempered.*

(1) *If $(\rho, 2) \in Jord(T)$, then*

$$\mu^*_{\left\{\nu^{\frac{1}{2}}\rho\right\}}(T) = \begin{cases} \left(\nu^{\frac{1}{2}}\rho\right)^{m_T(\rho,2)} \otimes T_1 \text{ if } \varepsilon_T(\rho, 2) = 1 \\ \left(\nu^{\frac{1}{2}}\rho\right)^{m_T(\rho,2)-1} \otimes L(\nu^{-\frac{1}{2}}\rho; T_1) \text{ if } \varepsilon_T(\rho, 2) = -1 \end{cases}$$

where $Jord(T_1) = Jord(T)\backslash\{(\rho, 2)\}$ with remaining multiplicities matching those of T and ε_{T_1} given by restriction. Note that if $\varepsilon_T(\rho, 2) = -1$, we must have $m_T(\rho, 2) = 2$ to have $f_T(\nu^{\frac{1}{2}}\rho) \neq 0$.

(2) *If $(\rho, 1) \in Jord(T)$, then we must have $m_T(\rho, 1) = 2$ to have $f_T(\rho) = m_T(\rho, 1) - 1$ nonzero. In this case*

$$\mu^*_{\{\rho\}}(T) = \rho \otimes T_2,$$

where $Jord(T_2) = Jord(T)\backslash\{(\rho, 1)\}$ with remaining multiplicities matching those of T and ε_{T_2} given by restriction.

Proof. The claim for (2) follows directly from the definition of $Jord(T)$.

For (1), the proof in the case where $\varepsilon_T(\rho, 2) = -1$ is similar to that for Proposition 3.4 (3); the proof in the case where $\varepsilon_T(\rho, 2) = 1$ and $m_T(\rho, 2) = 2$ is similar to that of Proposition 3.4 (2) in the case where $m(\rho, a) = 2$. The proof of (1) in the case where $\varepsilon_T(\rho, 2) = 1$ and $m_T(\rho, 2) = 1$ reduces to $k = 1$ in the same manner as in the proof of Proposition 3.4 (2) when $m(\rho, a) = 1$. Thus we are reduced to showing (1) in the case where $\varepsilon_T(\rho, 2) = 1$, $m_T(\rho, 2) = 1$, and $k = 1$.

In this case, we have

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \delta$$

for some discrete series δ. Then, writing $\mu^*_{\{\nu^{\frac{1}{2}}\rho\}}(\delta) = \nu^{\frac{1}{2}}\rho \otimes \delta'$ as in Theorem 3.2.2 [12], we have

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \times \nu^{\frac{1}{2}}\rho \rtimes \delta' \cong \nu^{\frac{1}{2}}\rho \times \delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \delta'$$
$$\Downarrow$$
$$\mu^*_{\left\{\nu^{\frac{1}{2}}\rho\right\}}(T) = \nu^{\frac{1}{2}}\rho \otimes T'$$

for some irreducible $T' \leq \delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta'$. T' clearly has the same partial cuspidal support as T; that $Jord(T')$ and $m_{T'}$ are as described follows from supercuspidal support considerations (Lemma 2.4.1 [12]). It remains to show that $\varepsilon_{T'}$ is as described. For this, it suffices to show that $\varepsilon_{T'}(\rho, b) = \varepsilon_T(\rho, b)$ for $(\rho, b) \in Jord(T')$ (so $b > 2$). We do this in two cases.
Case 1 (ρ, b_-) does not exist in $Jord(T')$.

Noting that $m_{T'}(\rho, b) = m_T(\rho, b)$, Lemma 2.3.5 [12] tells us

$$\begin{gathered}
\varepsilon_{T'}(\rho, b) = 1 \\
\Downarrow \\
T' \hookrightarrow \delta\left(\left[\nu^{\frac{1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right)^{m_T(\rho,b)} \rtimes \theta' \\
\Downarrow \\
\begin{aligned}
T &\hookrightarrow \nu^{\frac{1}{2}}\rho \times \delta\left(\left[\nu^{\frac{1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right)^{m_T(\rho,b)} \rtimes \theta' \\
&\cong \delta\left(\left[\nu^{\frac{1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right)^{m_T(\rho,b)} \times \nu^{\frac{1}{2}}\rho \rtimes \theta' \\
&\hookrightarrow \delta\left(\left[\nu^{\frac{3}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right)^{m_T(\rho,b)} \times \left(\nu^{\frac{1}{2}}\rho\right)^{m_T(\rho,b)+1} \rtimes \theta'
\end{aligned} \\
\Downarrow \text{ (Lemma 2.2)} \\
T \hookrightarrow \delta\left(\left[\nu^{\frac{3}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right)^{m_T(\rho,b)} \rtimes \theta'' \\
\Downarrow \\
\varepsilon_T(\rho, b)\varepsilon_T(\rho, 2)^{-1} = 1 \\
\Downarrow \text{ (Lemma 2.3.1 [12])} \\
\varepsilon_T(\rho, b) = 1
\end{gathered}$$

as we have $\varepsilon_T(\rho, 2) = 1$. Conversely,

$$\begin{gathered}
\varepsilon_T(\rho, b) = 1 \\
\Downarrow \\
\varepsilon_T(\rho, b)\varepsilon_T(\rho, 2)^{-1} = 1 \\
\Downarrow \\
T \hookrightarrow \delta\left(\left[\nu^{\frac{3}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right)^{m_T(\rho,b)} \rtimes \theta''.
\end{gathered}$$

Further, by iterating Proposition 3.4, we see that θ'' is tempered with $m_{\theta''}(\rho, 2) = m_T(\rho, b) + 1$ and $\varepsilon_{T''}(\rho, 2) = 1$. In particular, $f_{\theta''}(\nu^{\frac{1}{2}}\rho) = 1 + m_T(\rho, b)$. Thus, noting $T \hookrightarrow \nu^{\frac{1}{2}}\rho \rtimes T'$ by (1.1),

$$\begin{gathered}
T \hookrightarrow \delta\left(\left[\nu^{\frac{3}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right)^{m_T(\rho)} \times \left(\nu^{\frac{1}{2}}\rho\right)^{m_T(\rho,b)+1} \rtimes \theta''' \\
\Downarrow \\
\mu^*(T) \geq \delta\left(\left[\nu^{\frac{1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right)^{m_T(\rho)} \times \nu^{\frac{1}{2}}\rho \otimes \lambda \text{ for some irreducible } \lambda \\
\Downarrow \\
\mu^*\left(\nu^{\frac{1}{2}}\rho \rtimes T'\right) \geq \delta\left(\left[\nu^{\frac{1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right)^{m_T(\rho,b)} \times \nu^{\frac{1}{2}}\rho \otimes \lambda \\
\Downarrow \\
\mu^*(T') \geq \delta\left(\left[\nu^{\frac{1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right)^{m_T(\rho,b)} \otimes \lambda
\end{gathered}$$

by a standard μ^* analysis. As $m_{T'}(\rho, b) = m_T(\rho, b)$, this implies $\varepsilon_{T'}(\rho, b) = 1$ by the same argument as in Corollary 2.2.7 [12], as needed.

Case 2 (ρ, b_-) exists in $Jord(T')$.

By Lemma 2.3.1 [12] and the preceding case, to show $\varepsilon_{T'}(\rho, b) = \varepsilon_T(\rho, b)$, it suffices to show $\varepsilon_{T'}(\rho, b)\varepsilon_{T'}(\rho, b_-)^{-1} = \varepsilon_T(\rho, b)\varepsilon_T(\rho, b_-)^{-1}$. The proof that $\varepsilon_{T'}(\rho, b)\varepsilon_{T'}(\rho, b_-)^{-1} = 1 \Rightarrow \varepsilon_T(\rho, b)\varepsilon_T(\rho, b_-)^{-1} = 1$ is very similar to the proof that $\varepsilon_{T'}(\rho, b) = \varepsilon_T(\rho, b)$ in the previous case. The converse direction is easier than in the previous case:

$$\varepsilon_T(\rho, b)\varepsilon_T(\rho, b_-)^{-1} = 1$$
$$\Downarrow$$
$$\mu^*(T) \geq \delta\left(\left[\nu^{\frac{b_-+1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right)^{m(\rho,b)} \otimes \lambda \text{ for some irreducible } \lambda$$
$$\Downarrow$$
$$\mu^*\left(\nu^{\frac{1}{2}}\rho \rtimes T'\right) \geq \delta\left(\left[\nu^{\frac{b_-+1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right)^{m(\rho,b)} \otimes \lambda$$
$$\Downarrow$$
$$\mu^*(T') \geq \delta\left(\left[\nu^{\frac{b_-+1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right)^{m(\rho,b)} \otimes \lambda' \text{ for some irreducible } \lambda',$$

(as $\frac{1}{2} \notin \{\frac{b_-+1}{2}, \frac{b_-+3}{2}, \ldots, \frac{b-1}{2}\}$). As $m_{T'}(\rho, b) = m_T(\rho, b)$, it then follows that $\varepsilon_{T'}(\rho, b)\varepsilon_{T'}(\rho, b_-)^{-1} = 1$, as needed. □

4. Reducibility results

In this section, we determine when $\nu^{\frac{a-1}{2}}\rho \rtimes T$ is reducible. We focus on the case where $(\rho, a) \in Jord$ with $a \equiv 2red(\rho; \sigma) + 1 \bmod 2$, with the final result given in Theorem 4.7; Note 4.8 discusses the (much simpler) case when this fails. We start with a general lemma, then turn to the task of analyzing $\nu^{\frac{a-1}{2}}\rho \rtimes T$.

Proposition 4.1. *Suppose* $\pi \leq (\nu^{\frac{-a+1}{2}}\rho)^k \times \delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho])^\ell \rtimes T$ *is irreducible. Then,* π *has the form*

$$\pi = L(\left(\nu^{\frac{-a+1}{2}}\rho\right)^{k'}, \delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right)^{\ell'}; T')$$

with $k' + \ell' \leq k + \ell$.

Proof. Write $\pi = L(\Delta; T')$. If $\Delta = \emptyset$, we are done. Thus, suppose $\delta([\nu^{\frac{-d+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])$, $c < d$, is a generalized Steinberg representation appearing in Δ. Then, if we write $\mu^*(T) = \sum_h \lambda_h \otimes \theta_h$, it follows from Theorem 2.1 that

$$\delta\left(\left[\nu^{\frac{-d+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \leq \left(\nu^{\frac{-a+1}{2}}\rho\right)^{x_1} \times \left(\nu^{\frac{a-1}{2}}\rho\right)^{x_2}$$
$$\times \left(\prod_{s=1}^{k} \delta\left(\left[\nu^{-i_s+1}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{j_s}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)\right) \times \lambda_h,$$

where $\frac{-a+1}{2} \leq i_s \leq \frac{a+1}{2}$ and $i_s \leq j_s \leq \frac{a+1}{2}$. We first claim that $d = a$. If $\nu^{\frac{-d+1}{2}}\rho$ came from $(\nu^{\frac{-a+1}{2}}\rho)^{x_1}$, we would have $d = a$, as claimed. We clearly cannot have $\nu^{\frac{-d+1}{2}}\rho$ coming from $(\nu^{\frac{a-1}{2}}\rho)^{x_2}$. If $\frac{-d+1}{2} = -i_s + 1$, then we must have $\frac{c-1}{2} \geq \frac{a-3}{2} \Rightarrow c \geq a-2$. Therefore, $d \geq a$, so $i_s \geq \frac{a+1}{2} \Rightarrow i_s = \frac{a+1}{2}$. This again corresponds to $d = a$. If $j_s = \frac{-d+1}{2}$, we must have $\frac{c-1}{2} \geq \frac{a-1}{2} \Rightarrow c \geq a$. Therefore, $d \geq a+2$. This forces $j_s \leq \frac{-a-1}{2}$, a contradiction. Thus we cannot have $j_s = \frac{-d+1}{2}$. Similarly, $\nu^{\frac{-d+1}{2}}\rho$ cannot come from λ_h as λ_h must have nonnegative central exponent (by the Casselman criterion).

Next, we claim that $\frac{c-1}{2} = \frac{-a+1}{2}$ or $\frac{a-3}{2}$. Again, consider the possible sources of $\nu^{\frac{c-1}{2}}\rho$. If it came from $(\nu^{\frac{-a+1}{2}}\rho)^{x_1}$ or $\delta([\nu^{-i_s+1}\rho, \nu^{\frac{a-3}{2}}\rho])$, we would have $\frac{c-1}{2} = \frac{-a+1}{2}$ or $\frac{a-3}{2}$, as claimed. The condition $c < d = a$ immediately eliminates $(\nu^{\frac{a-1}{2}}\rho)^{x_2}$ and $\delta([\nu^{j_s}\rho, \nu^{\frac{a-1}{2}}\rho])$ as possible sources. The remaining possibility is that $\nu^{\frac{c-1}{2}}\rho$ comes from λ_h. As λ_h must contribute to $\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])$ and must satisfy the Casselman criterion, we have $\lambda_h = \delta([\nu^{\frac{-e+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])$ with $e \leq c < a$. This leaves $\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{-e-1}{2}}\rho])$ to be accounted for. We have $\frac{-e+1}{2} \leq \frac{c-1}{2} \leq \frac{a-3}{2}$, so $\frac{-e-1}{2} < \frac{a-3}{2}$. Therefore, none of $(\nu^{\frac{a-1}{2}}\rho)^{x_2}$, $\delta([\nu^{-i_s+1}\rho, \nu^{\frac{a-3}{2}}\rho])$, $\delta([\nu^{j_s}\rho, \nu^{\frac{a-1}{2}}\rho])$ can contribute to $\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{-e-1}{2}}\rho])$. The only remaining possibility is that $\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{-e-1}{2}}\rho]) = \nu^{\frac{-a+1}{2}}\rho$. Then $e = a-2$. Since $a-2 \leq c < a$, we must also have $c = a-2$, finishing the claim.

As for the bounds on k', ℓ', from the discussion above, each lower segment end of $\nu^{\frac{-a+1}{2}}\rho$ appearing in (the GL part of) $L((\nu^{\frac{-a+1}{2}}\rho)^{k'}, \delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho])^{\ell'}; T')$ comes from a $\nu^{\frac{-a+1}{2}}\rho$ in $(\nu^{\frac{-a+1}{2}}\rho)^k$ or $\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho])^{\ell}$. It then follows that $k' + \ell' \leq k + \ell$. □

We attack the reducibility conditions for $\nu^{\frac{a-1}{2}}\rho \rtimes T$, $a > 2$, through a sequence of lemmas.

Lemma 4.2. *If $m_T(\rho, a-2) > 2$, then $\nu^{\frac{a-1}{2}}\rho \rtimes T$ is reducible.*

Proof. Write $T \cong \delta([\nu^{\frac{-a+3}{2}}, \nu^{\frac{a-3}{2}}\rho]) \rtimes T'$ (irreducible as $m_T(\rho, a-2) > 2$). Then

$$\nu^{\frac{a-1}{2}}\rho \rtimes T \cong \nu^{\frac{a-1}{2}}\rho \times \delta\left(\left[\nu^{\frac{-a+3}{2}}, \nu^{\frac{a-3}{2}}\rho\right]\right) \rtimes T',$$

whose reducibility follows from that of $\nu^{\frac{a-1}{2}}\rho \times \delta([\nu^{\frac{-a+3}{2}}, \nu^{\frac{a-3}{2}}\rho])$. □

Lemma 4.3. *Suppose $(\rho, a-2) \notin Jord$. Then $\nu^{\frac{a-1}{2}}\rho \rtimes T$ is irreducible.*

Proof. From Proposition 4.1, the only possible irreducible subquotients of $\nu^{\frac{a-1}{2}}\rho \rtimes T$ are $L(\nu^{\frac{a-1}{2}}\rho; T)$, $L(\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]); T')$, and T''. The first occurs with multiplicity one by the Langlands classification. The second has a term of the form $\nu^{\frac{a-3}{2}}\rho \otimes \cdots$ in its $s_{(1)}$, so cannot occur. Thus if the third does not occur, irreducibility follows.

That the third cannot occur essentially reduces to supercuspidal support considerations. In the notation of Lemma 2.4.1 [12], if $\frac{a-3}{2} \geq red(\rho;\sigma)$,

$$0 = m_T(\rho, a-2) = n_T\left(\rho, \frac{a-3}{2}\right) - n_T\left(\rho, \frac{a-1}{2}\right)$$
$$\Rightarrow n_T\left(\rho, \frac{a-3}{2}\right) = n_T\left(\rho, \frac{a-1}{2}\right),$$

where $n_T(\rho, x)$ denotes the number of times $\nu^{\pm x}\rho$ appears in a term in the supercuspidal support (i.e., if $\nu^{x_1}\rho_1 \otimes \nu^{x_2}\rho_2 \otimes \cdots \otimes \nu^{x_k-1}\rho_k \otimes \sigma$ is in the minimal Jacquet module for T, then $n_T(\rho, x) = |\{i \mid \rho_i \cong \rho \text{ and } x_i = \pm x\}|$). A tempered subquotient of T'' would have an extra $\nu^{\pm\frac{a-1}{2}}\rho$ in its supercuspidal support, so

$$\begin{aligned} m_{T''}(\rho, a-2) &= n_{T''}\left(\rho, \frac{a-3}{2}\right) - n_{T''}\left(\rho, \frac{a-1}{2}\right) \\ &= n_T\left(\rho, \frac{a-3}{2}\right) - \left(n_T\left(\rho, \frac{a-1}{2}\right) + 1\right) = -1 \end{aligned}$$

a contradiction. The argument for $\frac{a-3}{2} < red(\rho;\sigma)$ is similar. The lemma now follows. □

Lemma 4.4. *Suppose $m_T(\rho, a-2) > 0$ and either (1) $m_T(\rho, a) = 0$, or (2) $\varepsilon_T(\rho, a)\varepsilon_T(\rho, a-2)^{-1} = 1$. Then, $\nu^{\frac{a-1}{2}}\rho \rtimes T$ is reducible.*

Proof. Let T' have data given by

$$m_{T'}(\rho', b) = \begin{cases} m_T(\rho, a-2) - 1 \text{ if } (\rho', b) = (\rho, a-2), \\ m_T(\rho, a) + 1 \text{ if } (\rho', b) = (\rho, a), \\ m_T(\rho', b) \text{ otherwise,} \end{cases}$$

and $\varepsilon_{T'}$ the restriction of ε_T to $S_{T'}$ (if $m_T(\rho, a) > 1$, we have $Jord(T') = Jord(T)$ and the restriction is an equality). We claim that such a T' exists, i.e., that the data given is admissible. Recall that for $(Jord(T'), \sigma, \varepsilon_{T'}, m_{T'})$ to be admissible, we need the underlying triple $(Jord(T')_{ds}, \sigma, (\varepsilon_{T'})_{ds})$ to be admissible in the [M-T] sense, where

$$\begin{aligned} Jord(T')_{ds} = \big\{&(\rho', b) \in Jord(T') \mid b \\ &\equiv 2red(\rho;\sigma) + 1 \bmod 2 \text{ and } m_{T'}(\rho', b) \text{ odd}\big\} \end{aligned}$$

and $(\varepsilon_{T'})_{ds}$ is given by restriction of $\varepsilon_{T'}$, which in turn is the restriction of ε_T. The admissibility may then be checked in cases based on the parity of $m_T(\rho, a)$ and $m_T(\rho, a-2)$. In all four cases, admissibility is immediate from Proposition A.1 in the appendix.

Now, observe that by Theorem 3.1,

$$\mu^*_{\{\nu^{\frac{a-1}{2}}\rho\}}(T') = \left(\nu^{\frac{a-1}{2}}\rho\right)^{m_T(\rho,a)+1} \otimes T'',$$

where $m_{T''}(\rho, a-2) = m_T(\rho, a-2) + m_T(\rho, a), m_{T''}(\rho, a) = 0$, and $\varepsilon_{T''}$ given by restriction. Observe that Theorem 3.1 also tells us $\mu^*_{\{\nu^{\frac{a-1}{2}}\rho\}}(T) = (\nu^{\frac{a-1}{2}}\rho)^{m_T(\rho,a)} \otimes T''$, so

$$\mu^*_{\{\nu^{\frac{a-1}{2}}\rho\}}\left(\nu^{\frac{a-1}{2}}\rho \rtimes T\right) = \left(\nu^{\frac{a-1}{2}}\rho\right)^{m_T(\rho,a)+1} \otimes T''.$$

By Lemma 2.2, $T' \hookrightarrow \nu^{\frac{a-1}{2}}\rho \rtimes ((\nu^{\frac{a-1}{2}}\rho)^{m_T(\rho,a)} \rtimes T'') \Rightarrow T' \hookrightarrow \nu^{\frac{a-1}{2}}\rho \rtimes \lambda$ for some irreducible $\lambda \leq (\nu^{\frac{a-1}{2}}\rho)^{m_T(\rho,a)} \rtimes T''$. Since $f_{T'}(\nu^{\frac{a-1}{2}}\rho) \leq 1 + f_\lambda(\nu^{\frac{a-1}{2}}\rho)$, we have $f_\lambda(\nu^{\frac{a-1}{2}}\rho) \geq m_T(\rho, a)$. Further, since $f_{T''}(\nu^{\frac{a-1}{2}}\rho) \leq m_{T''}(\rho, a) = 0$, we have $f_\lambda(\nu^{\frac{a-1}{2}}\rho) = m_T(\rho, a)$. Therefore, $\mu^*_{\{\nu^{\frac{a-1}{2}}\rho\}}(\lambda) = (\nu^{\frac{a-1}{2}}\rho)^{m_T(\rho,a)} \otimes T'' \Rightarrow \lambda = T$ by (1.1). As we also have $L(\nu^{\frac{-a+1}{2}}\rho; T) \leq \nu^{\frac{a-1}{2}}\rho \rtimes T$, reducibility follows. □

Lemma 4.5. *Suppose* $m_T(\rho, a-2) = 2$ *and* $(\rho, a) \in Jord$ *with* $\varepsilon_T(\rho, a)\varepsilon_T(\rho, a-2)^{-1} = -1$. *Then,* $\nu^{\frac{a-1}{2}}\rho \rtimes T$ *is reducible.*

Proof. Let $\pi = L(\nu^{\frac{-a+1}{2}}\rho; T)$. Observe that if $\nu^{\frac{a-1}{2}}\rho \rtimes T$ were irreducible, we would have $\pi \cong \nu^{\frac{a-1}{2}}\rho \rtimes T$, hence $f_\pi(\nu^{\frac{a-1}{2}}\rho) = 1 + f_T(\nu^{\frac{a-1}{2}}\rho)$. Thus, to show reducibility, it suffices to show $f_\pi(\nu^{\frac{a-1}{2}}\rho) = f_T(\nu^{\frac{a-1}{2}}\rho)$. Note that $f_T(\nu^{\frac{a-1}{2}}\rho) = m_T(\rho, a) - 1$ by Theorem 3.1.

As $m_T(\rho, a-2) = 2$, we have

$$\pi \hookrightarrow \nu^{\frac{-a+1}{2}}\rho \rtimes T \hookrightarrow \nu^{\frac{-a+1}{2}}\rho \times \delta\left(\left[\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right) \rtimes T',$$

where the data for T' is obtained by removing the two copies of $(\rho, a-2)$ and restricting ε_T (Lemma 2.4). By Lemma 2.2,

$$\pi \hookrightarrow \mathcal{L}\left(\nu^{\frac{-a+1}{2}}\rho, \delta\left(\left[\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right)\right) \rtimes T'$$

or

$$\pi \hookrightarrow \delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right) \rtimes T'.$$

As the latter has $L(\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]); T')$ as its unique irreducible subrepresentation, the former must hold.

Now, let $m = m_T(\rho, a) = m_{T'}(\rho, a)$. Since $(\rho, a-2) \notin Jord(T')$, we have $T' \hookrightarrow \left(\nu^{\frac{a-1}{2}}\rho\right)^m \rtimes T''$, T'' as in Theorem 3.1. Thus,

$$\pi \hookrightarrow \mathcal{L}\left(\nu^{\frac{-a+1}{2}}\rho, \delta\left(\left[\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right)\right) \times \left(\nu^{\frac{a-1}{2}}\rho\right)^m \rtimes T''.$$

Applying Lemma 2.2 and Lemma 3.6, we have

$$\pi \hookrightarrow \left(\nu^{\frac{a-1}{2}}\rho\right)^{m-1} \times \mathcal{L}\left(\nu^{\frac{-a+1}{2}}\rho, \delta([\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]), \nu^{\frac{a-1}{2}}\rho\right) \rtimes T''$$

or

$$\pi \hookrightarrow \left(\nu^{\frac{a-1}{2}}\rho\right)^{m-1} \times \mathcal{L}\left(\nu^{\frac{-a+1}{2}}\rho, \delta([\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])\right) \rtimes T''.$$

If the latter held, we would have

$$\begin{aligned}\pi &\hookrightarrow \left(\nu^{\frac{a-1}{2}}\rho\right)^{m-1} \times \nu^{\frac{-a+1}{2}}\rho \times \delta\left(\left[\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes T'' \\ &\cong \nu^{\frac{-a+1}{2}}\rho \times \left(\nu^{\frac{a-1}{2}}\rho\right)^{m-1} \times \delta\left(\left[\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes T'' \\ &\hookrightarrow \nu^{\frac{-a+1}{2}}\rho \times \left(\nu^{\frac{a-1}{2}}\rho\right)^{m} \times \delta\left(\left[\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right) \rtimes T''.\end{aligned}$$

As $\mu^*_{\{\nu^{\frac{-a+1}{2}}\rho\}}(\pi) = \nu^{\frac{-a+1}{2}}\rho \otimes T$, this would imply $r_{M,G}(T) \geq \left(\nu^{\frac{a-1}{2}}\rho\right)^m \otimes \delta([\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]) \otimes T''$, contradicting $f_T(\nu^{\frac{a-1}{2}}) = m-1$. Thus, the former holds. However, as $(\rho, a) \notin Jord(T'')$ and $M^*\left(\mathcal{L}\left(\nu^{\frac{-a+1}{2}}\rho, \delta([\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]), \nu^{\frac{a-1}{2}}\rho\right)\right)$ contains no terms of the form $\nu^{\frac{a-1}{2}}\rho \otimes \cdots$, the former tells us $f_\pi(\nu^{\frac{a-1}{2}}\rho) = m-1$, i.e., $f_\pi(\nu^{\frac{a-1}{2}}\rho) = f_T(\nu^{\frac{a-1}{2}}\rho)$, as needed. □

Lemma 4.6. *Suppose* $m_T(\rho, a-2) = 1$ *and* $(\rho, a) \in Jord$ *with* $\varepsilon_T(\rho, a)\varepsilon_T(\rho, a-2)^{-1} = -1$. *Then,* $\nu^{\frac{a-1}{2}}\rho \rtimes T$ *is irreducible.*

Proof. From Proposition 4.1, there are three possible forms for an irreducible subquotient of $\nu^{\frac{a-1}{2}}\rho \rtimes T$: $L(\nu^{\frac{-a+1}{2}}\rho; T)$, $L(\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]); T')$, and T''. The first appears with multiplicity one by the Langlands classification. Thus if we show the second and third do not appear, irreducibility follows.

First, consider T''. On one hand,

$$\begin{gathered} T'' \leq \nu^{\frac{a-1}{2}}\rho \rtimes T \\ \Downarrow \\ f_{T''}\left(\nu^{\frac{a-1}{2}}\rho\right) \leq 1 + f_T(\nu^{\frac{a-1}{2}}\rho) = m_T(\rho, a). \end{gathered}$$

On the other hand, it follows from Lemma 2.4.1 [12] that $m_{T''}(\rho, a) = m_T(\rho, a) + 1$ and $m_{T''}(\rho, a-2) = m_T(\rho, a-2) - 1 = 0$. This implies $f_{T''}(\nu^{\frac{a-1}{2}}\rho) = m_{T''}(\rho, a) = m_T(\rho, a) + 1$, a contradiction. Thus, we cannot have T''.

We now consider $L(\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]); T')$. Were this to occur, we would have

$$\delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right) \otimes T' \leq M^*\left(\nu^{\frac{a-1}{2}}\rho\right) \rtimes \mu^*(T).$$

If $\mu^*(T) = \sum_i \lambda_i \otimes \xi_i$, we have (Theorem 2.1)

$$\begin{aligned} &M^*\left(\nu^{\frac{a-1}{2}}\rho\right) \rtimes \mu^*(T) \\ &= \sum_i \left(\nu^{\frac{-a+1}{2}}\rho \times \lambda_i \otimes \xi_i + \nu^{\frac{a-1}{2}}\rho \times \lambda_i \otimes \xi_i + \lambda_i \otimes \nu^{\frac{a-1}{2}}\rho \rtimes \xi_i\right). \end{aligned}$$

As the Casselman criterion tells us λ_i has nonnegative central exponent, the only possibility is

$$\delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right) \otimes T' \leq \nu^{\frac{-a+1}{2}}\rho \times \lambda_i \otimes \xi_i.$$

For λ_i irreducible and $\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]) \leq \nu^{\frac{-a+1}{2}}\rho \times \lambda_i$, we must have $\lambda_i = \delta([\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho])$. Then ξ_i must be tempered (as any violation of the Casselman criterion for ξ_i immediately produces a violation for $\lambda_i \otimes \xi_i = \delta([\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]) \otimes \xi_i$, contradicting the temperedness of T). It then follows from Lemma 2.4.1 [12] that $m_T(\rho, a-2) = 2 + m_{\xi_i}(\rho, a-2) \geq 2$, contradicting $m_T(\rho, a-2) = 1$. Thus, $L(\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]); T')$ does not occur either, and we have the needed irreducibility. □

Theorem 4.7. *Consider $\nu^{\frac{a-1}{2}}\rho \rtimes T$, where $a \equiv 2red(\rho;\sigma) + 1 \bmod 2$ and $a > 2$.*

(1) *If $m_T(\rho, a-2) = 0$, we have $\nu^{\frac{a-1}{2}}\rho \rtimes T$ irreducible.*
(2) *If $m_T(\rho, a-2) = 1$, we have*
   (a) *$\nu^{\frac{a-1}{2}}\rho \rtimes T$ reducible if $m(\rho, a) = 0$ or $\varepsilon(\rho, a)\varepsilon(\rho, a-2)^{-1} = 1$,*
   (b) *$\nu^{\frac{a-1}{2}}\rho \rtimes T$ irreducible if $\varepsilon(\rho, a)\varepsilon(\rho, a-2) = -1$.*
(3) *If $m_T(\rho, a-2) \geq 2$, we have $\nu^{\frac{a-1}{2}}\rho \rtimes T$ reducible.*

In the case $a = 2$, we have $\nu^{\frac{1}{2}}\rho \rtimes T$ reducible if and only if $m_T(\rho, 2) = 0$ or $\varepsilon_T(\rho, 2) = 1$. (Recall that for $a = 1$, we have $\rho \rtimes T$ irreducible if and only if $(\rho, 1) \in Jord(T)$.)

Proof. The case $a > 2$ is covered by the preceding lemmas.

For $a = 2$, one irreducible subquotient of $\nu^{\frac{1}{2}}\rho \rtimes T$ is $L(\nu^{-\frac{1}{2}}\rho; T)$, and it appears with multiplicity one. The issue is therefore to determine whether or not there is another irreducible subquotient. Note that by Proposition 4.1, such an irreducible subquotient would have to be tempered.

If $m_T(\rho, 2) = 0$, we construct T' by adding a single copy of $(\rho, 2)$ to $Jord(T)$ and taking $\varepsilon_{T'}(\rho, 2) = 1$, with all other data matching that for T. That the data is admissible may be seen from the appendix (noting $\alpha_0 = \frac{1}{2}$); from Theorem 3.1 we have $\mu^*_{\{\nu^{\frac{1}{2}}\rho\}}(T') = \nu^{\frac{1}{2}}\rho \otimes T$. Then $T' \hookrightarrow \nu^{\frac{1}{2}}\rho \rtimes T$, implying reducibility.

If $\varepsilon_T(\rho, 2) = 1$, a similar argument works. In this case, we obtain T' by adding a single copy of $(\rho, 2)$, i.e., taking $m_{T'}(\rho, 2) = m_T(\rho, 2) + 1$, and letting all remaining data remain the same. Again, that the data so obtained is admissible follows from the appendix. One can also see from the data that if $\mu^*_{\{\nu^{\frac{1}{2}}\rho\}}(T) = (\nu^{\frac{1}{2}}\rho)^f \otimes \theta$, then $\mu^*_{\{\nu^{\frac{1}{2}}\rho\}}(T') = (\nu^{\frac{1}{2}}\rho)^{f+1} \otimes \theta$. From Lemma 2.2, we then have $T' \hookrightarrow \nu^{\frac{1}{2}}\rho \rtimes \lambda$ for some irreducible $\lambda \leq (\nu^{\frac{1}{2}}\rho)^f \rtimes \theta$. Further, it follows from this embedding that in order to have $\mu^*_{\{\nu^{\frac{1}{2}}\rho\}}(T') = (\nu^{\frac{1}{2}}\rho)^{f+1} \otimes \theta$, we must have $\mu^*_{\{\nu^{\frac{1}{2}}\rho\}}(\lambda) = (\nu^{\frac{1}{2}}\rho)^f \otimes \theta$, implying $\lambda = T$. Thus $T' \hookrightarrow \nu^{\frac{1}{2}}\rho \rtimes T$, again giving reducibility.

Now, consider the case $\varepsilon_T(\rho, 2) = -1$. Suppose we did have $\nu^{\frac{1}{2}}\rho \rtimes T$ reducible. As $L(\nu^{-\frac{1}{2}}\rho; T)$ is the Langlands quotient in $\nu^{\frac{1}{2}}\rho \rtimes T$, it cannot also appear as a subrepresentation by multiplicity one. So, let $T' \hookrightarrow \nu^{\frac{1}{2}}\rho \rtimes T$ be a subrepresentation (necessarily tempered as above). Note that from this embedding, we see that if $\mu^*_{\{\nu^{\frac{1}{2}}\rho\}}(T) = (\nu^{\frac{1}{2}}\rho)^f \otimes \theta$, then $\mu^*_{\{\nu^{\frac{1}{2}}\rho\}}(T') = (\nu^{\frac{1}{2}}\rho)^{f+1} \otimes \theta$. On the other hand, by supercuspidal support considerations (Lemma 2.4.1 [12]), we see that $Jord(T') = Jord(T)$, $m_{T'}(\rho, 2) = m_T(\rho, 2) + 1$, and remaining multiplicities the same. Further, it follows from Lemma 3.3 that since $m_{T'}(\rho, 2) = m_T(\rho, 2) + 1$ and $f_{T'}(\nu^{\frac{1}{2}}\rho) = f_T(\nu^{\frac{1}{2}}\rho) + 1$, we must have $\varepsilon_{T'}(\rho, 2) = \varepsilon_T(\rho, 2)$. However, since $m_{T'}(\rho, 2)$ and $m_T(\rho, 2)$ have different parities, Theorem 3.1(8) tells us one of them would require θ tempered and one would require θ nontempered, a contradiction. Thus we have irreducibility in this case. □

Note 4.8. *In the case where $a \not\equiv 2red(\rho; \sigma) + 1 \mod 2$, we have the following: $\nu^{\frac{a-1}{2}}\rho \rtimes T$ is reducible if and only if $(\rho, a-2) \in Jord(T)$. In particular, in the context of (2.1) and (2.3), we have $\nu^{\frac{a-1}{2}}\rho \rtimes \delta$ irreducible, with any reducibility which arises then coming from the reducibility of $\nu^{\frac{a-1}{2}}\rho \times \delta([\nu^{\frac{-c_i+1}{2}}\rho_i, \nu^{\frac{c_i-1}{2}}\rho_i])$ or $\nu^{\frac{a-1}{2}}\rho \times \delta([\nu^{\frac{-d_i+1}{2}}\rho'_i, \nu^{\frac{d_i-1}{2}}\rho'_i])$ (or with $\nu^{\frac{-a+1}{2}}\check{\rho}$ in place of $\nu^{\frac{a-1}{2}}\rho$).*

Appendix A. Characterization of admissible triples

There are a number of places in this paper where it is claimed that a particular tempered representation exists, i.e., that the data given is indeed admissible. At the core, this corresponds to the claim that the underlying discrete series data corresponds to an admissible triple. The purpose of this appendix is to give a characterization of admissibility in the discrete series case which simplifies the task of determining whether a particular ε does indeed correspond to an admissible triple, thereby making the admissibility claims in this paper essentially obvious.

In what follows, we use a variation of the Mœglin–Tadić classification as in [16]. In particular, rather than taking the domain of ε to be in $Jord \cup (Jord \times Jord)$, we simply take the domain to be $Jord$. For the case of $(\rho; \sigma)$ having half-integrable reducibility, this makes no difference as the values of ε on $Jord_\rho$ determine those on $Jord_\rho \times Jord_\rho$. The same holds when $red(\rho; \sigma) = 0$.

When $red(\rho; \sigma) \in \mathbb{N}$, write $Jord_\rho = \{a_1, \dots, a_k\}$ with $a_1 < \cdots < a_k$. If ε_0 denotes the Mœglin–Tadić datum, we define ε by arbitrarily choosing $\varepsilon(\rho, a_1) \in \{\pm 1\}$. We then take

$$\begin{gathered}\varepsilon(\rho, a_2) = \varepsilon(\rho, a_1)[\varepsilon_0(\rho, a_2)\varepsilon_0(\rho, a_1)^{-1}] \\ \varepsilon(\rho, a_3) = \varepsilon(\rho, a_2)[\varepsilon_0(\rho, a_3)\varepsilon_0(\rho, a_2)^{-1}] \\ \vdots\end{gathered}$$

(noting the compatibility conditions on p. 729 [20]). Now, we may identify $\varepsilon_0|_{Jord_\rho \cup (Jord_\rho \times Jord_\rho)}$ with both $\varepsilon|_{Jord_\rho}$ and $-\varepsilon|_{Jord_\rho}$, so we no longer have a bijection between admissible triples and equivalence classes of discrete series. We must therefore introduce an equivalence relation $\sim$ on triples, where $(Jord_1, \sigma_1, \varepsilon_1) \sim (Jord_2, \sigma_2, \varepsilon_2)$ if $Jord_1 = Jord_2$, $\sigma_1 = \sigma_2 = \sigma$, and for each ρ having $red(\rho;\sigma) \in \mathbb{N}$,

$$\varepsilon_1|_{Jord_{1,\rho}} = \pm\varepsilon_2|_{Jord_{2,\rho}}.$$

Now, fix a ρ and consider $Jord_\rho$. Again, write $Jord_\rho = \{a_1, \ldots, a_k\}$ with $a_1 < \cdots < a_k$. We may then identify $\varepsilon|_{Jord_\rho}$ with the k-tuple

$$\varepsilon_\rho = (\varepsilon(\rho, a_1), \varepsilon(\rho, a_2), \ldots, \varepsilon(\rho, a_k)).$$

Also, we set $\varepsilon_{alt} = (1, -1, 1, -1, \ldots, (-1)^k)$ (with k implicit from context) and

$$\alpha_0 = \alpha_0(\rho;\sigma) = \begin{cases} 0 & \text{if } \alpha = red(\rho;\sigma) \in \mathbb{Z}, \\ \frac{1}{2} & \text{if } \alpha = red(\rho;\sigma) \in \frac{1}{2} + \mathbb{Z}. \end{cases}$$

Proposition A.1. *ε is admissible if and only if for each ρ we have*

$$\langle \varepsilon_\rho, \varepsilon_{alt} \rangle \in \{\alpha_0 \pm \alpha\},$$

where $\langle\ ,\ \rangle$ denotes the usual inner product.

Proof. ($\Rightarrow$) Fix ρ and consider ε_ρ. Recall that if ε_ρ is not alternated, there exist a_i, a_{i+1} with $\varepsilon(\rho, a_i) = \varepsilon(\rho, a_{i+1})$. Removing (ρ, a_i), (ρ, a_{i+1}) from $Jord$ and restricting ε produces the subordinated triple $(Jord', \sigma, \varepsilon')$. To be admissible means this process may be repeated until one arrives at an alternated triple. One clearly has $\langle \varepsilon_\rho, \varepsilon_{alt} \rangle = \langle \varepsilon'_\rho, \varepsilon_{alt} \rangle$, so it suffices to address the alternated case.

In the case $\alpha = red(\rho;\sigma) \in \mathbb{N} \cup \{0\}$, a strongly positive discrete series representation has $|Jord_\rho| = |Jord_\rho(\sigma)| = |\{1, 3, 5, \ldots, 2\alpha - 1\}| = \alpha$ (e.g., see 14.2 [20]). Thus $\varepsilon = \pm(1, -1, 1, -1, \ldots, (-1)^\alpha)$ and $\langle \varepsilon_\rho, \varepsilon_{alt} \rangle = \pm\alpha$. As $\alpha_0 = 0$, this matches the description in the proposition. (Note that while $\alpha = 0$ is vacuous, the proposition has already been done in this case in Lemma 6.1 [11].)

In the case $\alpha = red(\rho;\sigma) \in -\frac{1}{2} + \mathbb{N}$, there are two possibilities for a strongly positive discrete series:

$$|Jord_\rho| = \begin{cases} |Jord_\rho(\sigma)| \text{ if } \varepsilon(\rho, a_{min}) = -1, \\ |Jord_\rho(\sigma)| + 1 \text{ if } \varepsilon(\rho, a_{min}) = 1. \end{cases}$$

As $|Jord_\rho(\sigma)| = |\{2, 4, 6, \ldots, 2\alpha - 1\}| = \alpha - \frac{1}{2}$, the possible alternating ε_ρ are

$$\varepsilon_1 = \left(-1, 1, -1, 1, \ldots, (-1)^{\alpha - \frac{1}{2}}\right)$$

and

$$\varepsilon_2 = \left(1, -1, 1, -1, \ldots, (-1)^{\alpha + \frac{1}{2}}\right).$$

We then observe that

$$\langle \varepsilon_1, \varepsilon_{alt} \rangle = -\alpha + \frac{1}{2}$$

and

$$\langle \varepsilon_2, \varepsilon_{alt} \rangle = \alpha + \frac{1}{2},$$

matching the claim in the proposition. Note that in the case $\alpha = \frac{1}{2}$, we have $\varepsilon_1 = \emptyset$ corresponding to $|Jord_\rho| = 0$ and σ as the strongly positive discrete series. In lieu of building from the vacuous case, we may simply note that when $|Jord_\rho| = 2$, we have $\varepsilon_\rho = (1, 1)$ or $(-1, -1)$ (Example 14.1.2 [20]), which satisfies $\langle \varepsilon_\rho, \varepsilon_{alt} \rangle = 0$.
($\Leftarrow$): Fix ρ and consider $\varepsilon_\rho = \varepsilon|_{Jord_\rho}$. It suffices to show that each such ε_ρ satisfies the conditions for admissibility.

First, suppose $\alpha = red(\rho; \sigma) \in \mathbb{N} \cup \{0\}$. As the case $\alpha = 0$ is covered by Lemma 6.1 [11], we may assume $\alpha \in \mathbb{N}$. Observe that in order to have $|\langle \varepsilon_\rho, \varepsilon_{alt} \rangle| = \alpha$, we must have the number of entries in ε_ρ—call it n–satisfying $n \equiv \alpha \bmod 2$. Further, note that $|\langle \varepsilon_\rho, \varepsilon_{alt} \rangle| \leq n$, implying $n \geq \alpha$. If $n = \alpha$, we have $|\langle \varepsilon_\rho, \varepsilon_{alt} \rangle| = \alpha$ implying $\varepsilon_\rho = \pm\varepsilon_{alt}$, and $\pm\varepsilon_{alt}$ is known to be admissible in this case (alternating, of course). Thus, we suppose $n > \alpha$.

Now, as $|\langle \varepsilon_\rho, \varepsilon_{alt} \rangle| = \alpha < n$, there must be a pair of consecutive entries satisfying $\varepsilon(\rho, a_i) = \varepsilon(\rho, a_{i+1})$ (or else ε_ρ is alternating and $|\langle \varepsilon_\rho, \varepsilon_{alt} \rangle| = n > \alpha$). Let $Jord^{(1)}, \varepsilon^{(1)}$ be obtained by removing $(\rho, a_i), (\rho, a_{i+1})$ from $Jord$ and restricting ε. Note that $(Jord, \sigma, \varepsilon)$ is admissible if and only if $(Jord^{(1)}, \sigma, \varepsilon^{(1)})$ is admissible. We have $|\langle \varepsilon_\rho^{(1)}, \varepsilon_{alt} \rangle| = \alpha$ but $n^{(1)} = n-2$. If $n-2 = \alpha$ we are done: $\varepsilon^{(1)}$ is alternating and known to be admissible in this case as above. Otherwise, $n-2 > \alpha$. We iterate this argument and eventually obtain $\varepsilon_\rho^{(\frac{n-\alpha}{2})} = \pm\varepsilon_{alt}$ (alternating and known to be admissible), finishing this case.

The argument in the half-integral case is similar. The difference comes from the fact that in this case, the known alternating ε's have $\langle \varepsilon_\rho, \varepsilon_{alt} \rangle = \frac{1}{2} \pm \alpha$ (noting that the case $\langle \varepsilon_\rho, \varepsilon_{alt} \rangle = \frac{1}{2} - \frac{1}{2} = 0$ corresponds to $Jord_\rho = \emptyset$).

As before, we have $|\langle \varepsilon_\rho, \varepsilon_{alt} \rangle| \leq n$. Therefore, to have $|\langle \varepsilon_\rho, \varepsilon_{alt} \rangle| = \alpha \pm \frac{1}{2}$, we must have $n \geq \alpha - \frac{1}{2}$. If $n = \alpha \pm \frac{1}{2}$, then $\langle \varepsilon_\rho, \varepsilon_{alt} \rangle = \frac{1}{2} \pm \alpha$ implies $\varepsilon_\rho = \begin{cases} \varepsilon_{alt} \text{ if } n = \alpha + \frac{1}{2}, \\ -\varepsilon_{alt} \text{ if } n = \alpha - \frac{1}{2}, \end{cases}$ corresponding to known admissible (alternating) cases. As before, if $n > \alpha + \frac{1}{2}$, we must have $\varepsilon(\rho, a_i) = \varepsilon(\rho, a_{i+1})$ for some i (or else $\varepsilon_\rho = \pm\varepsilon_{alt}$ and $|\langle \varepsilon_\rho, \varepsilon_{alt} \rangle| = n > \alpha + \frac{1}{2}$). Let $(Jord^{(1)}, \sigma, \varepsilon^{(1)})$ be obtained by removing $(\rho, a_i), (\rho, a_{i+1})$ and restricting ε. We have $(Jord, \sigma, \varepsilon)$ admissible if and only if $(Jord^{(1)}, \sigma, \varepsilon^{(1)})$ admissible. Now, $\varepsilon_\rho^{(1)}$ has $n^{(1)} = n-2$: if $n-2 = \alpha \pm \frac{1}{2}$, we have $\varepsilon^{(1)}$ admissible (alternating). If not, we iterate the argument, eventually reaching the alternating case □

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