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Tempered representations for classical p-adic groups

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1. Introductory material

1.1. Introduction

The purpose of this paper is to extend the Mœglin–Tadić classification of discrete series of classical p-adic groups to cover tempered representations of these groups.

Recall that in general, if G is the F-points of a connected reductive group defined over a p-adic field F, an irreducible tempered representation T of G embeds in a parabolically induced representation $\mathrm{Ind}_P^G(\delta)$, where δ is a discrete series representation of the Levi factor M of P. Note that δ is unique up to conjugacy. Further, every component of $\mathrm{Ind}_P^G(\delta)$ is tempered. For the classical groups considered in this paper, $\mathrm{Ind}_P^G(\delta)$ decomposes with multiplicity one. The components of $\mathrm{Ind}_P^G(\delta)$ are distinguished by the action of standard intertwining operators via the R-group. This is convenient for some purposes, but not well-suited for the application of Jacquet module methods.

For classical groups, an extension of the Mœglin–Tadić classification would characterize representations in the tempered dual based on parabolically induced representations into which they embed, a more convenient characterization for using Jacquet module methods. The Langlands classification for admissible representations is also well-suited for applying Jacquet module methods. However, it requires knowing the tempered duals of the Levi factors of standard parabolic subgroups—in this case, products of general linear groups and lower rank classical groups of the same type. By extending the Mœglin–Tadić classification to cover tempered representations, we may combine these to obtain a framework well-suited to Jacquet module methods. We note that Jacquet module methods have been quite effective in analyzing families of induced representations for classical groups in cases where the classification of tempered representations is not an issue (e.g., see [3,13,23–25,31], etc.). Thus, it is hoped that an extension of the Mœglin–Tadić classification will fill a gap and allow a broader application of these methods. However, our particular motivation for undertaking this project is to study duality for classical groups [1,26], where we expect Jacquet modules to play a key role (as in the case of general linear groups, e.g., see [22]).

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Our approach to extending the Mœglin–Tadić classification to tempered representations relies heavily on both the Mœglin–Tadić classification for discrete series and the R-group results of Goldberg. In order to describe our results, we first briefly review their results. For concreteness, we work with symplectic groups in the introduction. Recall that for $G = Sp(n, F)$, a parabolic subgroup has the form $P = MU$ with Levi factor $M \cong GL(n_1, F) \times \cdots \times GL(n_k, F) \times Sp(2n_0, F)$ with $n_0 + n_1 + \cdots + n_k = n$.

We start by reviewing the Mœglin–Tadić classification for discrete series. We must first review a bit of notation from [36]. Suppose ρ is an irreducible unitary supercuspidal representation of $GL(r, F)$. We let $\nu = |det|$ and, for a, b with $a - b \in \mathbb{Z}$ and $a \leq b$, we let $\delta([\nu^a\rho, \nu^b\rho])$ denote the generalized Steinberg representation associated to the segment $[\nu^a\rho, \nu^b\rho]$, i.e., the unique irreducible subrepresentation of $\mathrm{Ind}_P^G(\nu^b\rho \otimes \nu^{b-1}\rho \otimes \cdots \otimes \nu^a\rho)$, where $G = GL(r(b - a + 1), F)$ and P the standard parabolic subgroup having Levi factor $GL(r, F) \times \cdots \times GL(r, F)$. In [21], a discrete series representation δ of $G = Sp(2n, F)$ is embedded in a representation of the form

$$\delta \hookrightarrow \mathrm{Ind}_P^G \left(\delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_2-1}{2}}\rho_1\right]\right) \otimes \right.$$
$$\left. \cdots \otimes \delta\left(\left[\nu^{\frac{-a_{2k-1}+1}{2}}\rho_k, \nu^{\frac{a_{2k}-1}{2}}\rho_k\right]\right) \otimes \delta_{sp}\right),$$

with δ_{sp} a "strongly positive" discrete series and the ρ_i not necessarily distinct (see Sect. 1.3 for more; a precise definition of strongly positive discrete series is not crucial to the general discussion here in the introduction). Mœglin–Tadić then identify δ with a triple $(Jord, \sigma, \varepsilon)$ as follows: σ is the partial cuspidal support of δ, i.e., a supercuspidal representation of $Sp(2n_0, F)$ (possibly $n_0 = 0$) such that any term in the supercuspidal support of δ has the form $\cdots \otimes \sigma$. *Jord* consists of $(\rho_1, 2a_1 + 1), (\rho_1, 2a_2 + 1), \cdots, (\rho_k, 2a_{2k-1}), (\rho_k, a_{2k})$—essentially the ends of the segments appearing in the embedding above—plus additional elements contributed by δ_{sp}. The description of ε is more involved; it is a function defined on a subset of $Jord \cup (Jord \times Jord)$ with image in $\{\pm 1\}$. It essentially identifies representations of the above form into which δ may be embedded. To give a bit of the flavor, suppose $(\rho, a), (\rho, b) \in Jord$, $a < b$, and there is no c with $a < c < b$ and $(\rho, c) \in Jord$. Then ε is defined on the pair $((\rho, a), (\rho, b))$ and characterized by

$$\varepsilon\left((\rho, a), (\rho, b)\right) = 1 \Leftrightarrow \delta \hookrightarrow \mathrm{Ind}_P^G \left(\delta\left(\left[\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right) \otimes \theta\right) \tag{1.1}$$

for some irreducible θ.

We now discuss the results on tempered representations and R-groups which we need [6–8, 11]. If T is an irreducible tempered representation of $G = Sp(2n, F)$, it follows from the work of Goldberg and Herb that

$$T \cong \mathrm{Ind}_P^G \left(\delta\left(\left[\nu^{\frac{-b_1+1}{2}}\rho_1', \nu^{\frac{b_1-1}{2}}\rho_1'\right]\right) \otimes \cdots \otimes \delta\left(\left[\nu^{\frac{-b_j+1}{2}}\rho_j', \nu^{\frac{b_j-1}{2}}\rho_j'\right]\right) \otimes T_{ell}\right)$$

(irreducibly induced), with T_{ell} an elliptic tempered representation. (We retain the notation and terminology for the analogous representations of the non-connected group $O(2n, F)$; see Sects. 2.3, 2.4.) Thus, the key for us is understanding the elliptic tempered representations. Now,

$$T_{ell} \hookrightarrow \mathrm{Ind}_P^G \left(\delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1'', \nu^{\frac{c_1-1}{2}}\rho_1''\right]\right) \otimes \cdots \otimes \delta\left(\left[\nu^{\frac{-c_\ell+1}{2}}\rho_\ell'', \nu^{\frac{c_\ell-1}{2}}\rho_\ell''\right]\right) \otimes \delta\right),$$

with δ a discrete series representation and the induced representation having 2^ℓ (inequivalent) components. Note that this requires that $\mathrm{Ind}_{P_i}^{G_i}(\delta([\nu^{\frac{-c_i+1}{2}}\rho_i'', \nu^{\frac{c_i-1}{2}}\rho_i'']) \otimes \delta)$ all be reducible and, for $i \neq j$, $(\rho_i'', c_i) \not\cong (\rho_j'', c_j)$. The reducibility of $\mathrm{Ind}_{P_i}^{G_i}(\delta([\nu^{\frac{-c_i+1}{2}}\rho_i'', \nu^{\frac{c_i-1}{2}}\rho_i'']) \otimes \delta)$ is equivalent to $(\rho_i'', c_i) \notin Jord(\delta)$ and c_i of appropriate parity ($c_i \equiv 2red(\rho_i''; \sigma) + 1 \bmod 2$—see Sect. 1.3).

To extend the Mœglin–Tadić classification to tempered representations, let us first focus on the elliptic case. Suppose

$$T_{ell} \hookrightarrow \mathrm{Ind}_P^G \left(\delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1'', \nu^{\frac{c_1-1}{2}}\rho_1''\right]\right) \otimes \cdots \otimes \delta\left(\left[\nu^{\frac{-c_\ell+1}{2}}\rho_\ell'', \nu^{\frac{c_\ell-1}{2}}\rho_\ell''\right]\right) \otimes \delta\right)$$

as above. We would like to construct *Jord(T)* by adding two copies each of $(\rho_1'', c_1), \dots, (\rho_\ell'', c_\ell)$ (one for each end of $\delta([\nu^{\frac{-c_i+1}{2}}\rho_i'', \nu^{\frac{c_i-1}{2}}\rho_i''])$, even if $c_i = 1$). Thus we introduce a fourth datum, m_T—the multiplicity (which seems slightly more convenient than working with multisets), so have T associated to $(Jord(T), \sigma, \varepsilon_T, m_T)$. Thus,

$$Jord(T) = Jord(\delta) \cup \{(\rho_1'', c_1), \dots, (\rho_\ell'', c_\ell)\}$$

and

$$m_T(\rho, a) = \begin{cases} 1 & \text{if } (\rho, a) \in Jord(\delta), \\ 2 & \text{if } (\rho, a) \cong (\rho_i'', c_i) \text{ for some } i. \end{cases}$$

Again, we have $\varepsilon_T : S_T \longrightarrow \{\pm 1\}$, with the domain $S_T \subset Jord(T) \cup (Jord(T) \times Jord(T))$. We have $S_T \supset S_\delta$, and $\varepsilon_T|_{S_\delta} = \varepsilon_\delta$. The additional values of ε_T effectively distinguish the 2^ℓ components of $\mathrm{Ind}_P^G \Big(\delta([\nu^{\frac{-c_1+1}{2}}\rho_1'', \nu^{\frac{c_1-1}{2}}\rho_1'']) \otimes \cdots \otimes \delta([\nu^{\frac{-c_\ell+1}{2}}\rho_\ell'', \nu^{\frac{c_\ell-1}{2}}\rho_\ell'']) \otimes \delta\Big)$. In particular, we have the following extension of Eq. (1.1): if $(\rho, a), (\rho, b) \in Jord(T)$ with $a < b$ and no $a < c < b$ with $(\rho, c) \in Jord(T)$, then

(1) if $m(\rho, b) = 1$,

$$\varepsilon((\rho, a), (\rho, b)) = 1 \Leftrightarrow \delta \hookrightarrow \mathrm{Ind}_P^G \left(\delta\left(\left[\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right) \otimes \theta\right)$$

for some irreducible θ.

(2) if $m(\rho, b) = 2$,

$$\varepsilon((\rho, a), (\rho, b)) = 1 \Leftrightarrow \delta \hookrightarrow \mathrm{Ind}_P^G \left(\delta\left(\left[\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right) \otimes \delta\left(\left[\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right) \otimes \theta\right)$$

for some irreducible θ.

Other basic properties of the Mœglin–Tadić classification also have counterparts in the extension to the tempered case. For more general tempered representations—i.e., no longer assuming elliptic—we construct $Jord(T), m_T$ by similarly adding to $Jord(T_{ell}), m_{T_{ell}}$; S_T and ε_T match $S_{T_{ell}}$ and $\varepsilon_{T_{ell}}$ (noting that the corresponding induced representation is irreducible so we do not have components to distinguish).

At this point, we make a few remarks on the hypotheses. As we need the results of [21], we also assume their "Basic Assumption" (see Sect. 1.3). For purposes of the arguments made in this paper, we can work in the generality of [21]; no additional restrictions on char$F$ or the bilinear forms defining the groups are needed. While Goldberg's work is central to our results and assumes char$F = 0$ and the groups are split (or quasisplit in the unitary case), Theorem 13.1 of [21] is a suitable substitute and requires only the "Basic Assumption" (which we need in any case). Thus we may work in the same generality as [21].

We also remark that although $SO(2n, F)$ is not directly covered by the results of this paper, its tempered dual may be obtained from that of $O(2n, F)$ via restriction—see Lemma 2.3 and Proposition 9.1 of [18]. (We also take this opportunity to note a correction to [18]. In Definition 2.1, the statement "We remark that if $\alpha_{n-1}, \alpha_n \notin \Phi \dots$" is not correct—in this case, one has $c \circ i_{G,M}(\tau_1 \otimes \cdots \otimes \tau_{k-1} \otimes \tau_k) \cong i_{G,M}(\tau_1 \otimes \cdots \otimes \tau_{k-1} \otimes \tau_k^{-1})$, with τ_k a character. One should interpret $\tau_1 \times \cdots \times \tau_k \rtimes (1 \otimes c)$ as in the beginning of the definition: $\tau_1 \times \cdots \times \tau_k \rtimes (1 \otimes c) = c \circ i_{G,M}(\tau_1 \otimes \cdots \otimes \tau_k)$.)

We now describe the results section by section. Sections 1.2 and 1.3 review some general background and the Mœglin–Tadić classification, resp. The main results are summarized in Sect. 1.4. Sections 2.1–2.4 establish the basic classification of tempered representations. In Sect. 2.1, we give additional properties of the Mœglin–Tadić classification which are needed later. Section 2.2 looks at the case of $\mathrm{Ind}_P^G\left(\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \otimes \delta\right)$ while Sect. 2.3 uses this to build up to the case of general elliptic tempered representations. The extension to arbitrary tempered representations is given in Sect. 2.4. Sections 3.1–3.4 establish additional properties of the extension, key among them the following: Suppose $(\rho, a), (\rho, b) \in Jord(T)$ with $a < b$ and no $a < c < b$ with $(\rho, c) \in Jord(T)$, then

$$\begin{aligned}&\varepsilon_T\,((\rho, a), (\rho, b))\\ &= 1 \Rightarrow T \hookrightarrow \mathrm{Ind}_P^G\left(\underbrace{\delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right) \otimes \cdots \otimes \delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right)}_{m} \otimes T'\right),\end{aligned}$$

where $m = min\,(m(\rho, a), m(\rho, b))$ and T' is an irreducible tempered representation whose data is obtained by removing m copies each of $(\rho, a), (\rho, c)$; $\varepsilon_{T'}$ is then obtained by restriction of ε_T. Sections 3.1 and 3.2 establish some basic terminology and results on Jacquet modules of discrete series needed in the proofs which follow. Section 3.3 revisits the case of $\mathrm{Ind}(\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \otimes \delta)$; Sect. 3.4 uses this to prove the above result. The final section—Sect. 3.5—determines the action of normalized standard intertwining operators on the components of $\delta_1 \times \cdots \times \delta_k \rtimes \delta$ in the case where δ is generic. As a corollary, we also characterize the generic component; as in [10], it turns out to be the component with ε trivial (identically

1 on its domain). As we use Goldberg's results in Sect. 3.5, the classical groups considered there are restricted accordingly.

Before closing, we take a moment to note the recent paper of Tadić [34], which gives a similar classification of tempered representations of classical groups, but using somewhat different data. We also take the opportunity to thank the referee for a number of useful suggestions and corrections, as well as M. Tadić for explaining aspects of his work with C. Mœglin to us.

1.2. Notation and preliminaries

In what follows, we use $S(n, F)$ to denote one of the groups considered by Mœglin and Tadić [21]. In the symplectic case, we have $Sp(2n, F)$. In the orthogonal case, fix an anisotropic orthogonal space Y_0. If $\dim(Y_0)$ is odd and $n \geq \dim(Y_0)$, we take V_n to be the $(2n+1)$-dimensional space in the Witt tower over Y_0; $S(n, F)$ is then the special orthogonal group over V_n. If $\dim(Y_0)$ is even and $n \geq \dim(Y_0)$, we let V_n be the $2n$-dimensional space in the Witt tower over Y_0 and take $S(n, F)$ the orthogonal group over V_n. In the unitary case, we start with a fixed separable quadratic extension F' of F and an anisotropic unitary space Y_0 over F'. We take V_n the $2n$-dimensional (resp., $(2n+1)$-dimensional) space in the Witt tower over Y_0 and for $n \geq \dim(Y_0)$, let $S(n, F)$ be the unitary group over V_n.

We now discuss some structure theory from [2,30,36]. As in [4], for a p-adic group G with parabolic subgroup $P = MU$, we let $i_{G,M}$ and $r_{M,G}$ denote normalized parabolic induction and the normalized Jacquet functor, respectively. For $S(n, F)$ in one of the families of classical groups under consideration, set

$$R = \bigoplus_{n \geq 0} \mathcal{R}(GL(n, F)) \quad \text{and } R[S] = \bigoplus_{n \geq n_0} \mathcal{R}(S(n, F)),$$

where $\mathcal{R}(G)$ denotes the Grothendieck group of the category of smooth finite-length representations of G and n_0 is $\frac{1}{2}\dim(Y_0)$ or $\frac{1}{2}(\dim(Y_0) - 1)$, whichever is integral. We define multiplication on R as follows: suppose ρ_1, ρ_2 are representations of $GL(n_1, F), GL(n_2, F)$, resp. We have $M = GL(n_1, F) \times GL(n_2, F)$ is the Levi factor of a standard parabolic subgroup of $G = GL(n, F)$, where $n = n_1 + n_2$, and set $\tau_1 \times \tau_2 = i_{G,M}(\tau_1 \otimes \tau_2)$. This extends (after semisimplification) to give the multiplication $\times : R \times R \longrightarrow R$. To describe the comultiplication on R, let $M_{(i)}$ denote the standard Levi factor for $G = GL(n, F)$ having $M_{(i)} = GL(i, F) \times GL(n-i, F)$. For a representation τ of $GL(n, F)$, we define

$$m^*(\tau) = \sum_{i=0}^{n} r_{M_{(i)},G}(\tau),$$

the sum of semisimplified Jacquet modules (lying in $R \otimes R$). This extends to a map $m^* : R \longrightarrow R \otimes R$. We note that with this multiplication and comultiplication (and antipode map given by the Zelevinsky involution, a special case of the general duality operator of [1,26]), R is a Hopf algebra.

Similarly, suppose τ is a representation of $GL(n_1, F)$ and θ a representation of $S(n_2, F)$. We have $M = GL(n_1, F) \times S(n_2, F)$ the Levi factor of a standard parabolic subgroup of $G = S(n, F)$, with $n = n_1 + n_2$, and set $\tau \rtimes \theta = i_{G,M}(\tau \otimes \theta)$. If one extends $\rtimes$ to a map $\rtimes : R \otimes R[S] \longrightarrow R[S]$, we have $R[S]$ as a module over R. To describe its comodule structure, let $M_{(i)} = GL(i, F) \otimes S(n-i, F)$, a standard Levi factor for $G = S(n, F)$. For a representation π of $S(n, F)$, we define

$$\mu^*(\pi) = \sum_{i=0}^{n'} r_{M_{(i)},G}(\pi),$$

where n' denotes the Witt index (so is $n - \frac{1}{2}\dim(Y_0)$ or $n - \frac{1}{2}(\dim(Y_0) - 1)$) and the sum is of (normalized) semisimplified Jacquet modules (lying in $R \otimes R[S]$). This extends to a map $\mu^* : R[S] \longrightarrow R \otimes R[S]$.

For unitary groups, let σ denote the nontrivial element of the Galois group of the underlying quadratic extension. For a representation π of $S(n, F)$, we then define

$$\check{\pi} = \begin{cases} \tilde{\pi} \circ \sigma & \text{for unitary groups,} \\ \tilde{\pi} & \text{otherwise,} \end{cases}$$

where $\tilde{}$ denotes contragredient. Using this, we may give $R[S]$ the structure of an M^*-module over R (cf. [2,21,30]):

Theorem 1.2.1. *Define* $M^* : R \longrightarrow R \otimes R$ *by*

$$M^* = (m \otimes 1) \circ (\check{} \otimes m^*) \circ s \circ m^*,$$

where m *denotes the multiplication* $\times : R \otimes R \longrightarrow R$ *and* $s : R \otimes R \longrightarrow R \otimes R$ *the extension of the map defined on representations by* $s : \tau_1 \otimes \tau_2 \longmapsto \tau_2 \otimes \tau_1$. *Then*

$$\mu^*(\tau \rtimes \pi) = M^*(\tau) \rtimes \mu^*(\pi),$$

where $\rtimes$ *on the right hand side is determined by* $(\tau_1 \otimes \tau_2) \rtimes (\tau \otimes \theta) = (\tau_1 \times \tau) \otimes (\tau_2 \rtimes \theta)$.

In what follows, we frequently look at those terms in μ^*, m^*, M^* having a given first factor. In particular, if λ is an irreducible representation of a general linear group, we let μ^*_λ (resp., m^*_λ, M^*_λ) be the sum of everything in μ^* (resp., m^*, M^*) of the form $\lambda \otimes \theta$. More precisely, if $\mu^*(\pi) = \sum_i m_i \lambda_i \otimes \theta_i$ (m_i the multiplicity of $\lambda_i \otimes \theta_i$), we set $\mu^*_\lambda(\pi) = \sum_{i \in I_\lambda} m_i \lambda_i \otimes \theta_i$, where $I_\lambda = \{i \mid \lambda_i \cong \lambda\}$, and similarly for m^*_λ and M^*_λ. Similarly, if λ is an irreducible representation of a standard Levi factor M, we let r_λ be the sum of everything in $r_{M,G}$ of the form $\lambda \otimes \theta$. Formally, we let r_{min} the terms from minimal nonzero Jacquet modules (so the terms appearing in r_{min} are all supercuspidal).

We now take a moment to review cuspidal reducibility values. Suppose ρ is an irreducible unitary supercuspidal representation of a general linear group and σ an irreducible supercuspidal representation of a classical group. If $\rho \ncong \check{\rho}$, then $\nu^x \rho \rtimes \sigma$ is irreducible for all $x \in \mathbb{R}$; if $\rho \cong \check{\rho}$, then there is a unique nonnegative $x \in \mathbb{R}$ such that $\nu^x \rho \rtimes \sigma$ reduces (cf. [29] and Corollary 4.4 [3]), which we denote by $red(\rho; \sigma)$. The values for $red(\rho; \sigma)$ for $Sp(2n, F)$ and $SO(2n+1, F)$ have been determined (assuming certain conjectures) in [19,37]; in the generic case, it is known that they must lie in $\{0, \frac{1}{2}, 1\}$ (cf. [27,28]).

We next review the Casselman criterion for $S(n, F)$ (see [5,35], which extend easily to the non-connected group $O(2n, F)$). Suppose π is an irreducible representation of $S(n, F)$. Suppose $\nu^{x_1}\rho_1 \otimes \cdots \otimes \nu^{x_k}\rho_k \otimes \sigma \leq r_{M,G}\pi$ has ρ_i an irreducible unitary supercuspidal representation of $GL(m_i, F)$ for $i = 1, \dots, k$, σ an irreducible supercuspidal representation of $S(m, F)$, and $x_1, \dots, x_k \in \mathbb{R}$. The Casselman criterion tells us that if π is tempered, the following hold:

$$\begin{aligned} m_1x_1 &\geq 0 \\ m_1x_1 + m_2x_2 &\geq 0 \\ &\vdots \\ m_1x_1 + m_2x_2 + \cdots + m_kx_k &\geq 0. \end{aligned}$$

Conversely, if these inequalities hold for any such $\nu^{x_1}\rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma$ (i.e., ρ_i an irreducible unitary supercuspidal representation of $GL(m_i, F)$ and σ an irreducible supercuspidal representation of $S(m, F)$) appearing in a Jacquet module of π, then π is tempered. The criterion for square-integrability is the same except that the inequalities are strict.

The following lemmas are Lemma 3.1 of [21] and Lemma 5.5 of [14], respectively. The extension of Lemma 3.1 of [21] to the nonconnected group $O(2n, F)$ is not difficult and is included below.

Lemma 1.2.2. *Suppose π is an irreducible representation of G and σ a supercuspidal (not necessarily unitary) representation which appears as a subquotient of $r_{M,G}(\pi)$ for some parabolic subgroup $P = MN$ of G. Then, $\pi \hookrightarrow i_{G,M}(\sigma)$.*

Proof. The connected case is covered by Lemma 3.1 of [21]. Suppose $G = O(2n, F)$. Let $\sigma_0 \leq r_{M^0,M}(\sigma)$. Then $\sigma_0 \leq r_{M^0,G^0}(\pi_0)$ for some irreducible $\pi_0 \leq r_{G^0,G}(\pi)$. Further, $\pi \hookrightarrow i_{G,G^0}(\pi_0)$ (e.g., Lemma 2.13 of [9]). By the connected case, we have $\pi_0 \hookrightarrow i_{G^0,M^0}(\sigma_0)$. Therefore,

$$\pi \hookrightarrow i_{G,G^0}(\pi_0) \hookrightarrow i_{G,G^0} \circ i_{G^0,M^0}(\sigma_0) \cong i_{G,M} \circ i_{M,M^0}(\sigma_0).$$

Either $i_{M,M^0}(\sigma_0) \cong \sigma$ or $\sigma \oplus \hat{c}\sigma$, with $\hat{c}$ the (restriction to M of the) nontrivial character of $O(2n, F)$ which is trivial on $SO(2n, F)$ (e.g., Lemma 2.13 [9]). Therefore, $\pi \hookrightarrow i_{G,M}(\sigma)$ or $i_{G,M}(\hat{c}\sigma)$; by partial cuspidal support considerations, it must be the former. □

Lemma 1.2.3. *Suppose π is an irreducible representation of G, λ an irreducible representation of M and $\pi \hookrightarrow i_{G,M}(\lambda)$. If $L > M$, then there is an irreducible representation ρ of L such that*

(1) $\pi \hookrightarrow i_{G,L}(\rho)$
(2) *ρ is a subquotient of $i_{L,M}(\lambda)$.*

1.3. The Mœglin–Tadić classification

In this section, we review the construction of [21] of discrete series for classical groups. (This discussion also borrows freely from the review of the Mœglin–Tadić construction given in [24].) There is an alternate characterization of part of the construction, given in [32,33], which we also discuss. We then close with a lemma which characterizes part of the Mœglin–Tadić construction in terms of Jacquet modules.

Let π be an irreducible admissible representation of a classical group. If π is not supercuspidal, we may write

$$\pi \hookrightarrow \nu^{x_1}\rho_1 \times \cdots \times \nu^{x_\ell}\rho_\ell \rtimes \sigma,$$

with $x_1, \ldots, x_\ell \in \mathbb{R}$, $\rho_1, \ldots, \rho_\ell$ irreducible unitary supercuspidal representations of general linear groups, and σ an irreducible supercuspidal representation of a corresponding smaller classical group (possibly $\sigma = 1$, the trivial representation of the trivial group). Recall that in the case of $O(2n, F)$, Mœglin–Tadić do not treat representations of $O(2, F)$ as supercuspidal, so do not allow a representation of $O(2, F)$ to be the partial cuspidal support. Since $SO(2, F) \cong F^\times$, an irreducible representation of $O(2, F)$ can be embedded in a representation of the form $\text{Ind}\chi \cong \chi \rtimes 1$ with χ a character of $F^\times$. Thus we may take $\sigma = 1$ as the partial cuspidal support. The σ which appears is unique, and the partial cuspidal support of π is defined to be this σ.

Let δ be a discrete series representation for a classical group $S(n, F)$. *Jord*(δ) is defined to be the set of pairs (ρ, a), where ρ is an irreducible unitary supercuspidal representation of a general linear group having $\rho \cong \check{\rho}$ and $a \in \mathbb{N}$, which satisfy the following:

(1) a is even if and only if the L-function $L(\rho, R_d, s)$ has a pole at $s = 0$. Here, if ρ is a representation of $GL(d, F)$, we let $L(\rho, R_d, s)$ denote the L-function defined by Shahidi (cf. [27,28]), with R_d is the representation of $GL(d, \mathbb{C})$ on $\wedge^2\mathbb{C}^d$ in the symplectic and even-orthogonal cases, and on $\text{Sym}^2(\mathbb{C}^d)$ in the odd-orthogonal case. (For the unitary case, see appendix 2 of [20].)
(2) $\delta([\nu^{-\frac{(a-1)}{2}}\rho, \nu^{\frac{(a-1)}{2}}\rho]) \rtimes \delta$ is irreducible.

We note that, assuming the Basic Assumption of [21] (discussed later in this section), the first condition ensures the parity of a matches the parity of $2red(\rho; \sigma)+1$. (Notice that the parity does not depend on σ, though the particular reducibility value does.)

We remark that, for convenience, we use representations in the following description of admissible triples when we actually want equivalence classes of representations; the reader should interpret the discussion below accordingly. (Working this way saves us from having to make a somewhat awkward but obvious definition of equivalence of triples.)

Let *Trip* denote the collection of all triples $(Jord, \sigma, \varepsilon)$ which satisfy the following:

(1) *Jord* is a finite (possibly empty) set of pairs (ρ, a), where ρ is an irreducible unitary supercuspidal representation of a general linear group having $\check{\rho} \cong \rho$, and $a \in \mathbb{N}$ with a even if and only if $L(s, \rho, R_{d_\rho})$ has a pole at $s = 0$.

(2) σ is an irreducible supercuspidal representation of a classical group $S(n, F)$.
(3) $\varepsilon : S \longrightarrow \{\pm 1\}$ is a function on a subset $S \subset Jord \cup (Jord \times Jord)$ which satisfies certain conditions, which we discuss in more detail momentarily.

Let us start by describing the domain S of ε. S contains all $(\rho, a) \in Jord$ except those having a odd and $(\rho, a') \in Jord(\sigma)$ for some $a' \in \mathbb{N}$; S contains $\big((\rho, a), (\rho', a')\big) \in Jord \times Jord$ when $\rho \cong \rho'$ and $a \neq a'$. Several compatibility conditions must also be satisfied:

$$\begin{aligned} &\text{(i) if } (\rho, a), (\rho, a') \in S, \text{ we must have } \varepsilon\big((\rho, a), (\rho, a')\big) \\ &\qquad = \varepsilon(\rho, a)\varepsilon(\rho, a')^{-1}; \\ &\text{(ii) } \varepsilon\big((\rho, a), (\rho, a'')\big) \\ &\qquad = \varepsilon\big((\rho, a), (\rho, a')\big)\,\varepsilon\big((\rho, a'), (\rho, a'')\big) \text{ for all} (\rho, a), (\rho, a'), (\rho, a'') \\ &\qquad \in Jord \text{ having } a, a', a'' \text{ distinct;} \\ &\text{and (iii) } \varepsilon\big((\rho, a), (\rho, a')\big) = \varepsilon\big((\rho, a'), (\rho, a)\big) \text{ for all } \big((\rho, a), (\rho, a')\big) \in S. \end{aligned} \tag{1.2}$$

We follow the notation of [21] and, in light of (i) above, write $\varepsilon(\rho, a)\varepsilon(\rho, a')^{-1}$ for $\varepsilon\big((\rho, a), (\rho, a')\big)$ even when ε is undefined on (ρ, a) and (ρ, a') separately (i.e., even when (ρ, a) and (ρ, a') are not in S).

We now discuss triples of alternated type. Suppose $(\rho, a) \in Jord$. We define (ρ, a_-) by taking $a_- = max\{a' \in \mathbb{N} \,|\, (\rho, a') \in Jord \text{ and } a' < a\}$, noting that (ρ, a_-) may be undefined. Also, let us write $Jord_\rho = \{(\rho', a) \in Jord \,|\, \rho' \cong \rho\}$ and $Jord_\rho(\sigma) = \{(\rho', a) \in Jord(\sigma) \mid \rho' \cong \rho\}$. We call $(Jord, \sigma, \varepsilon) \in Trip$ a triple of alternated type if the following hold: (1) $\varepsilon(\rho, a)\varepsilon(\rho, a_-)^{-1} = -1$ whenever (ρ, a_-) is defined, and (2) $|Jord_\rho| = |Jord'_\rho(\sigma)|$, where

$$Jord'_\rho(\sigma) = \begin{cases} Jord_\rho(\sigma) \cup \{(\rho, 0)\} \text{ if } a \text{ is even and } \varepsilon(\rho, min\, Jord_\rho) = 1, \\ Jord_\rho(\sigma) \text{ otherwise.} \end{cases}$$

We write $Trip_{alt}$ for the subset of all alternated triples in $Trip$.

This brings us to admissible triples. First, suppose $(Jord, \sigma, \varepsilon) \in Trip$ has $(\rho, a) \in Jord$ with (ρ, a_-) defined and $\varepsilon(\rho, a)\varepsilon(\rho, a_-)^{-1} = 1$. Set $Jord' = Jord \setminus \{(\rho, a), (\rho, a_-)\}$ and let ε' be the restriction of ε to $S \cap [Jord' \cup (Jord' \times Jord')]$. One can check that $(Jord', \sigma, \varepsilon') \in Trip$. We say that $(Jord', \sigma, \varepsilon')$ is subordinated to $(Jord, \sigma, \varepsilon)$. We say the triple $(Jord, \sigma, \varepsilon)$ is admissible if there is a sequence of triples $(Jord_i, \sigma, \varepsilon_i)$, $1 \leq i \leq k$, such that (1) $(Jord_1, \sigma, \varepsilon_1) = (Jord, \sigma, \varepsilon)$, (2) $(Jord_{i+1}, \sigma, \varepsilon_{i+1})$ is subordinated to $(Jord_i, \sigma, \varepsilon_i)$ for all $1 \leq i \leq k-1$, and (3) $(Jord_k, \sigma, \varepsilon_k)$ is of alternated type. We write $Trip_{adm}$ for the set of admissible triples.

Mœglin–Tadić establish a bijection between the set of all equivalence classes of discrete series for all $S(n, F)$ (not including $O(2, F)$) and the set of all admissible triples. We now describe that correspondence. If δ is a discrete series representation for an orthogonal group, we write $(Jord(\delta), \sigma_\delta, \varepsilon_\delta)$ for the associated admissible triple. Here, $Jord(\delta)$ is as above and σ_δ is the partial cuspidal support of δ. It remains to describe ε_δ.

We first describe ε_δ on pairs. Suppose $(\rho, a) \in Jord_\delta$ with a_- defined. Then,

$$\varepsilon_\delta(\rho, a)\varepsilon_\delta^{-1}(\rho, a_-) = 1$$
$$\Updownarrow$$
$$\text{there is an irreducible representation } \theta \text{ such that } \delta \hookrightarrow \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \theta. \tag{1.3}$$

This property is sufficient to define ε_δ on that part of S contained in $Jord(\delta) \times Jord(\delta)$ (use property (3)(ii) from the definition of triple above). Now, suppose $(\rho, a) \in Jord(\delta)$ with a even. If a_{min} denotes the smallest value of a having $(\rho, a) \in Jord$, we have

$$\varepsilon_\delta(\rho, a_{min}) = 1$$
$$\Updownarrow$$
$$\text{there is an irreducible representation } \theta \text{ such that } \delta \hookrightarrow \delta\left(\left[\nu^{\frac{1}{2}}\rho, \nu^{\frac{a_{min}-1}{2}}\rho\right]\right) \rtimes \theta. \tag{1.4}$$

We can reformulate this by formally setting $\varepsilon_\delta(\rho, 0) = 1$; Eq. (1.3) is then sufficient to determine $\varepsilon_\delta(\rho, a)$ for all such $(\rho, a) \in Jord_\rho$. If $(\rho, a) \in S$ with a odd (in which case there is no b with $(\rho, b) \in Jord(\sigma_\delta)$), Mœglin–Tadić use normalized standard intertwining operators to define $\varepsilon_\delta(\rho, a)$ (cf. Proposition 6.1 [20]). In lieu of this approach, we follow that given in [33], which has a more representation-theoretic character.

In [33] (which provides proofs of results announced in [32]), Tadić gives another way of defining $\varepsilon_\delta(\rho, a)$ when $(\rho, a) \in S$ with a odd. In this case, the choice needed to fix ε on $Jord_\rho$ is a choice of components of $\rho \rtimes \sigma_\delta$. We next review this definition.

To start, we make a choice of components, writing $\rho \rtimes \sigma \cong \tau_1(\rho; \sigma) \oplus \tau_{-1}(\rho; \sigma)$. Then, for $a \in \mathbb{N}$ and $\eta \in \{\pm 1\}$, let $\delta([\nu\rho, \nu^a\rho]; \tau_\eta(\rho; \sigma))$ denote the unique irreducible subrepresentation of $\delta([\nu\rho, \nu^a\rho]) \rtimes \tau_\eta(\rho; \sigma)$. If a_{max} is the largest value of a such that $(\rho, a) \in Jord$, we define $\varepsilon_\delta(\rho, a_{max})$ as follows:

$$\begin{aligned} \varepsilon_\delta(\rho, a_{max}) \ = \ \eta \Leftrightarrow & \text{ there is an irreducible } \lambda \text{ such that } \delta \\ & \hookrightarrow \lambda \rtimes \delta\left([\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho]; \tau_\eta(\rho; \sigma)\right). \end{aligned} \tag{1.5}$$

Observe that once $\varepsilon_\delta(\rho, a_{max})$ is known, Eq. (1.3) is enough to determine ε_δ on S.

Before proceeding further, let us take a moment to recall the Basic Assumption under which the Mœglin–Tadić construction is done, and which we retain. Let ρ be an irreducible unitary supercuspidal representation of a general linear group having $\rho \cong \check{\rho}$ and σ an irreducible supercuspidal representation of some $S(n, F)$. The Basic Assumption is the following:

$$red(\rho; \sigma) = \begin{cases} \frac{a_{\rho,max}+1}{2} & \text{if } Jord_\rho(\sigma) \neq \emptyset, \\ \frac{1}{2} & \text{if } L(\rho, R_{d_\rho}, s) \text{ has a pole at } s = 0 \text{ and } Jord_\rho(\sigma) = \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where $a_{\rho,max}$ is the largest value of a for which $(\rho, a) \in Jord$. The reader is referred to section 12 of [21] for more on this assumption.

Note 1.3.1. *In the opposite direction, let $(Jord, \sigma, \varepsilon)$ be an admissible triple. Then the associated discrete series representation δ satisfies the following: if $(\rho, a), (\rho, a_-) \in Jord$ with $\varepsilon(\rho, a)\varepsilon(\rho, a_-)^{-1} = 1$, then*

$$\delta \hookrightarrow \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \delta',$$

where δ' has Mœglin–Tadić data $(Jord', \sigma, \varepsilon')$ with $Jord' = Jord \setminus \{(\rho, a), (\rho, a_-)\}$ and ε' is the restriction of ε to S' (the domain of ε'). Repeated application of this results in an embedding

$$\delta \hookrightarrow \delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_2}{2}}\rho_2\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-a_{2k-1}+1}{2}}\rho, \nu^{\frac{a_{2k}}{2}}\rho\right]\right) \rtimes \delta_{sp},$$

where δ_{sp} has its associated triple alternated (i.e., is a strongly positive discrete series). For $(Jord, \sigma, \varepsilon)$ alternated, we can construct the associated discrete series as follows: write $Jord_\rho = \{(\rho, a_1^\rho), \ldots, (\rho, a_{k_\rho}^\rho)\}$ with $a_1^\rho < \cdots < a_{k_\rho}^\rho$. Then,

$$\delta \hookrightarrow \left(\prod_\rho \prod_{i=1}^{k_\rho} \delta\left(\left[\nu^{(\phi_\rho(a_i^\rho)+1)/2}\rho, \nu^{(a_i^\rho-1)/2}\rho\right]\right)\right) \rtimes \sigma,$$

as unique irreducible subrepresentation, where $\phi_\rho : Jord_\rho \longrightarrow Jord'_\rho(\sigma)$ is an increasing bijection (noting that by the definition of alternated triple, these have the same cardinality).

Remark 1.3.2. It follows directly from the embeddings in Note 1.3.1 that if $\mu^*_{\nu^{\frac{a-1}{2}}\rho}(\delta) \neq 0$, then (ρ, a) must be in $Jord(\delta)$.

Lemma 1.3.3. *Let $\delta = \delta_{(Jord,\sigma,\varepsilon)}$ and $(\rho, a) \in Jord$ with (ρ, a_-) defined. Then*

$$\delta \hookrightarrow \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \theta \text{ for some irreducible } \theta$$
$$\Updownarrow$$
$$\mu^*(\delta) \geq \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \otimes \theta' \text{ for some irreducible } \theta'.$$

If a_- is not defined, we may replace a_- in the above statement with 0 or 1, whichever matches the parity of $2red(\rho; \sigma) + 1$.

Proof. The implication $(\Rightarrow)$ is immediate from Frobenius reciprocity (taking $\theta' = \theta$). For $(\Leftarrow)$, if a_- exists, observe that since $\mu^*(\delta) \geq \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \otimes \theta'$, we have

$$r_{min}(\delta) \geq \nu^{\frac{a-1}{2}}\rho \otimes \nu^{\frac{a-3}{2}}\rho \otimes \cdots \otimes \nu^{\frac{a_-+1}{2}}\rho \otimes \cdots$$
$$\Downarrow \text{ (Lemma 1.2.2)}$$
$$\delta \hookrightarrow \nu^{\frac{a-1}{2}}\rho \times \nu^{\frac{a-3}{2}}\rho \times \cdots \otimes \nu^{\frac{a_-+1}{2}}\rho \times \cdots$$
$$\Downarrow \text{ (Lemma 1.2.3)}$$
$$\delta \hookrightarrow \lambda \rtimes \theta''$$

for some irreducible $\lambda \leq \nu^{\frac{a-1}{2}}\rho \times \nu^{\frac{a-3}{2}}\rho \times \cdots \times \nu^{\frac{a_-+1}{2}}\rho$ and irreducible θ''. Any λ other than $\delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])$ would have $r_{\nu^x\rho}(\lambda) \neq 0$ for some $x \in \{\frac{a_-+1}{2}, \frac{a_-+3}{2}, \ldots, \frac{a-3}{2}\}$, hence $r_{\nu^x\rho}(\delta) \neq 0$. Since $r_{\nu^x\rho}(\delta) = 0$ for all such x by Remark 1.3.2, this is not the case, so $\lambda = \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])$, as needed. If a_- does not exist, the same argument still works, replacing a_- by 0 or 1, as appropriate. □

1.4. Summary

We close the first chapter by summarizing the main results. In particular, there is a bijection between irreducible admissible quadruples on the one hand, and irreducible tempered representations of a family of classical groups on the other. In what follows, we describe this more precisely.

An admissible quadruple is a quadruple of the form $(Jord, \sigma, \varepsilon, m)$ satisfying the following:

- $Jord$ consists of pairs (ρ, a) having ρ an irreducible unitary supercuspidal representation of a general linear group and $a \in \mathbb{N}$.
- σ is an irreducible supercuspidal representation of a classical group.
- $m : Jord \longrightarrow \mathbb{N}$ subject to
  (1) if $\rho \not\cong \check{\rho}$, then $m(\rho, a) = m(\check{\rho}, a)$, and
  (2) if $\rho \cong \check{\rho}$ but $a \not\equiv 2red(\rho; \sigma) + 1 \bmod 2$, then $m(\rho, a) \in 2\mathbb{N}$.
- For ε, we first define

$$Jord_{ell} = \{(\rho, a) \in Jord \mid \rho \cong \check{\rho} \text{ and } a \equiv 2red(\rho; \sigma) + 1 \bmod 2\}.$$

  We then define S_{ell} as for discrete series:

$$S_{ell} = \{(\rho, a) \in Jord_{ell} \mid red(\rho; \sigma) = 0 \text{ or } red(\rho; \sigma) \equiv \tfrac{1}{2} \bmod 1\} \\ \cup \left\{((\rho, a), (\rho', a')) \in Jord_{ell} \times Jord_{ell} \mid \rho \cong \rho' \text{ and } a \neq a'\right\}.$$

  The domain of ε is $S = S_{ell}$, and

$$\varepsilon : S \longrightarrow \{\pm 1\}$$

  subject to the following:

  (i) if $(\rho, a), (\rho, a') \in S$, we must have $\varepsilon\left((\rho, a), (\rho, a')\right) = \varepsilon(\rho, a)\varepsilon(\rho, a')^{-1}$;
  (ii) $\varepsilon\left((\rho, a), (\rho, a'')\right) = \varepsilon\left((\rho, a), (\rho, a')\right)\varepsilon\left((\rho, a'), (\rho, a'')\right)$ for all $(\rho, a), (\rho, a'), (\rho, a'') \in Jord$ having a, a', a'' distinct;
  and (iii) $\varepsilon\left((\rho, a), (\rho, a')\right) = \varepsilon\left((\rho, a'), (\rho, a)\right)$ for all $\left((\rho, a), (\rho, a')\right) \in S$.

  Further, let $Jord_{ds} = \{(\rho, a) \in Jord_{ell} \mid m(\rho, a) \text{ is odd}\}$. Then, $(Jord_{ds}, \sigma, \varepsilon_{ds})$ is an admissible triple in the sense of [21], where ε_{ds} the restriction of ε to S_{ds} (defined as in [21]).

The bijection is implemented in one direction as follows: let δ be the discrete series associated by Mœglin–Tadić to the triple $(Jord_{ds}, \sigma, \varepsilon_{ds})$. Define m_{ell} on $Jord_{ell}$ by

$$m_{ell}(\rho, a) = \begin{cases} 1 & \text{if} \quad m(\rho, a) \text{ is odd,} \\ 2 & \text{if} \quad m(\rho, a) \text{ is even.} \end{cases}$$

Associated to the quadruple $(Jord_{ell}, \sigma, \varepsilon, m_{ell})$ is an elliptic tempered representation

$$T_{ell} \hookrightarrow \left(\prod_{\{(\rho,a)\in Jord_{ell} \mid m(\rho,a)=2\}} \delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)\right) \rtimes \delta.$$

It is characterized inductively by the following:

(1) If $(\rho, a) \in Jord_{ell}$ with $m_{ell}(\rho, a) = 2$, then

$$T_{ell} \hookrightarrow \delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes T'_{ell},$$

where the data for T'_{ell} is obtained by removing both copies of (ρ, a) and restricting ε.

(2) If $\varepsilon(\rho, a)\varepsilon(\rho, a_-)^{-1} = 1$, let $m = min(m(\rho, a), m(\rho, a_-))$. Then,

$$T_{ell} \hookrightarrow \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)^m \rtimes T''_{ell},$$

where the data for T''_{ell} is obtained by removing m copies each of (ρ, a), (ρ, a_-) and restricting ε.

(3) If $Jord_\rho(T_{ell}) = \{(\rho, a)\}$ with $m(\rho, a) = 2$, we must have $red(\rho; \sigma) = 0$ or $\frac{1}{2}$. If $red(\rho; \sigma) = \frac{1}{2}$, we have $\varepsilon(\rho, a) = 1 \Leftrightarrow T \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])^2 \rtimes \theta$ for some irreducible θ. If $red(\rho; \sigma) = 0$, we have (for $\eta = \pm 1$) $\varepsilon(\rho, a) = \eta \Leftrightarrow T \hookrightarrow \lambda \rtimes T(\delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho])^2; \tau_\eta(\rho; \sigma))$ for some irreducible λ (see Lemma 2.2.2 for a description of $T(\delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho])^2; \tau_\eta(\rho; \sigma)))$.

Finally, let $Jord_{non} = Jord \setminus Jord_{ell}$. We choose a set X containing exactly one of (ρ, a), $(\check{\rho}, a)$ for those $(\rho, a), (\check{\rho}, a) \in Jord$ having $\check{\rho} \ncong \rho$. Then T is the irreducible induced representation

$$\begin{aligned} T = & \prod_{(\rho,a)\in X} \delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)^{m(\rho,a)} \\ & \times \prod_{\{(\rho,a)\in Jord_{non} \mid \rho\cong\check{\rho}\}} \delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)^{\frac{1}{2}m(\rho,a)} \\ & \times \prod_{(\rho,a)\in Jord_{ell}} \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)^{\frac{1}{2}[m(\rho,a)-m_{ell}(\rho,a)]} \rtimes T_{ell}. \end{aligned}$$

In the other direction, suppose T is an irreducible tempered representation of a classical group. In the data $(Jord, \sigma, \varepsilon, m)$, we have σ as the partial cuspidal support. Write

$$T \cong \delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-a_k+1}{2}}\rho_k, \nu^{\frac{a_k-1}{2}}\rho_k\right]\right) \rtimes T_{ell}$$

(irreducibly induced) with T_{ell} elliptic tempered. Then $Jord = Jord_{ell} \cup Jord_{non}$, where

$$\begin{aligned} Jord_{ell} = \{(\rho, a) \mid a \equiv 2red(\rho; \sigma) \\ +1 \bmod 2 \text{ and } \delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes T \text{ irreducible}\} \end{aligned}$$

and

$$Jord_{non} = \{(\rho_1, a_1), (\check{\rho}_1, a_1), \ldots, (\rho_k, a_k), (\check{\rho}_k, a_k)\}$$

(noting that there may be repetition among the elements listed in $Jord_{non}$). Let δ be the unique discrete series representation such that we may write

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-b_1+1}{2}}\rho_1', \nu^{\frac{b_1-1}{2}}\rho_1'\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-b_\ell+1}{2}}\rho_\ell', \nu^{\frac{b_\ell-1}{2}}\rho_\ell'\right]\right) \rtimes \delta.$$

Then,

$$m(\rho, a) = \begin{cases} 2|\{i \mid (\rho_i', b_i) = (\rho, a)\}| + 1 & \text{if } (\rho, a) \in Jord(\delta), \\ 2|\{i \mid (\rho_i', b_i) = (\rho, a)\}| + 2 & \text{if } (\rho, a) \in Jord_{ell} \setminus Jord(\delta), \\ 2|\{i \mid (\rho_i', b_i) = (\rho, a)\}| & \text{if } (\rho, a) \in Jord_{non} \text{ and } \check{\rho} \cong \rho, \\ |\{i \mid (\rho_i', b_i) = (\rho, a) \text{ or } (\check{\rho}, a)\}| & \text{if } \check{\rho} \not\cong \rho. \end{cases}$$

Alternatively, both *Jord* and m may be determined from Lemma 2.4.1. Finally, $\varepsilon = \varepsilon_{T_{ell}}$ is determined by the following:

(1)
$$\varepsilon(\rho, a)\varepsilon(\rho, a_-)^{-1} = 1 \Leftrightarrow T \hookrightarrow \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)^{m(\rho,a)} \rtimes \theta$$

for some irreducible θ,

(2) if $red(\rho; \sigma) \equiv \frac{1}{2} \bmod 1$,
$$\varepsilon(\rho, a_{min}) = 1 \Leftrightarrow T \hookrightarrow \delta\left(\left[\nu^{\frac{1}{2}}\rho, \nu^{\frac{a_{min}-1}{2}}\rho\right]\right)^{m(\rho,a_{min})} \rtimes \theta$$

for some irreducible θ (where a_{min} is the smallest value of b such that $(\rho, b) \in Jord_{ell}$), and

(3) if $red(\rho; \sigma) = 0$,

$$\varepsilon(\rho, a_{max}) = \eta$$
$$\Updownarrow$$
$$T \hookrightarrow \lambda \rtimes \left(\delta\left(\left[\nu^{\frac{-a_{max}+1}{2}}\rho, \nu^{\frac{a_{max}-1}{2}}\rho\right]\right)^{\frac{1}{2}(m(\rho,a_{max})-m_0)} \rtimes T(\delta([\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho])^{m_0}; \tau_\eta(\rho; \sigma))\right),$$

for some irreducible λ, where $m_0 = 1$ or 2, whichever matches the parity of $m(\rho, a_{max})$, and a_{max} is the largest value of b such that $(\rho, b) \in Jord_{ell}$ (see Definition 2.3.6 for a description of $T(\delta\left(\left[\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho\right]\right)^{m_0}; \tau_\eta(\rho; \sigma)))$.

2. Jacquet modules

2.1. Some lemmas on the Mœglin–Tadić classification

In this section, we give several results on the Mœglin–Tadić classification which are used later in the paper.

Let δ be a discrete series representation and M a standard Levi factor such that $r_{M,G}(\delta) \neq 0$ but $r_{L,G}(\delta) = 0$ for any standard Levi factor $L < M$. For $c \geq 0$, we let $n(\rho, c)$ denote the number of times $\nu^{\pm c}\rho$ appears in some $\chi \leq r_{M,G}(\delta)$. Note that this depends only on the supercuspidal support of δ and not on the particular M or χ used.

The following lemma tells us that $Jord(\delta)$ is determined by the supercuspidal support of δ.

Lemma 2.1.1. *Let $\delta = \delta_{(Jord,\sigma,\varepsilon)}$ be a discrete series representation. Suppose $c > 0$ with $c \equiv red(\rho;\sigma) \bmod 1$. Then, we have the following:*

(1) *If $c \geq red(\rho;\sigma)$,*

$$(\rho, 2c+1) \in Jord \Leftrightarrow n(\rho, c) - n(\rho, c+1) = 1.$$

(If $(\rho, 2c+1) \notin Jord$, then $n(\rho, c) - n(\rho, c+1) = 0$.)

(2) *If $c < red(\rho;\sigma)$,*

$$(\rho, 2c+1) \in Jord \Leftrightarrow n(\rho, c) - n(\rho, c+1) = 0.$$

(If $(\rho, 2c+1) \notin Jord$, then $n(\rho, c) - n(\rho, c+1) = -1$.)

The case $c = 0$ is the same except that $n(\rho, c) - n(\rho, c+1)$ is replaced by $2n(\rho, 0) - n(\rho, 1)$.

Proof. Let $\alpha = red(\rho;\sigma)$ and set $\alpha_0 = \begin{cases} 1 \text{ if } \alpha \equiv 0 \bmod 1 \\ \frac{1}{2} \text{ if } \alpha \equiv \frac{1}{2} \bmod 1 \end{cases}$. Write $Jord_\rho(\delta) = \{(\rho, a_1), \ldots, (\rho, a_k)\}$. For $c \geq 0$, we let $n_+(\rho, c)$ (resp., $n_-(\rho, c)$) denote the number of times c (resp., $-c$) appears in the set $\{\frac{a_1-1}{2}, \frac{a_2-1}{2}, \ldots, \frac{a_k-1}{2}, -\alpha_0, -\alpha_0-1, \ldots, -\alpha\}$. We remark that since *Jord* is multiplicity-free, $n_\pm(\rho, c) \leq 1$. Now, by Lemma 3.1 [15],

$$n_+(\rho, c) - n_-(\rho, c+1) = \begin{cases} n(\rho, c) - n(\rho, c+1) & \text{if} \quad c > 0, \\ 2n(\rho, 0) - n(\rho, 1) & \text{if} \quad c = 0. \end{cases}$$

The lemma now follows from the observations that

$$n_+(\rho, c) = \begin{cases} 1 & \text{if } (\rho, 2c+1) \in Jord, \\ 0 & \text{if not} \end{cases} \quad \text{and } n_-(\rho, c+1) = \begin{cases} 1 & \text{if } c < \alpha, \\ 0 & \text{if not.} \end{cases}$$

□

Proposition 2.1.2. *Let $\delta = \delta_{(Jord,\sigma.\varepsilon)}$ and $(\rho, b) \in Jord$. Suppose (ρ, b_-) exists and $b > b_- + 2$. Then,*

$$\mu^*_{\nu^{\frac{b-1}{2}}\rho}(\delta) = \nu^{\frac{b-1}{2}}\rho \otimes \delta',$$

where δ' is the discrete series representation with Mœglin–Tadić data described below. Here,

$$Jord(\delta') = (Jord(\delta) \setminus \{(\rho, b)\}) \cup \{(\rho, b-2)\}$$

and $\varepsilon_{\delta'}$ is defined by $\varepsilon_{\delta'} = \varepsilon$ on $S_{\delta'} \cap S$ ($S_{\delta'}$ the domain of $\varepsilon_{\delta'}$),

$$\varepsilon_{\delta'}(\rho, a)\varepsilon_{\delta'}(\rho, b-2)^{-1} = \varepsilon(\rho, a)\varepsilon(\rho, b)^{-1}$$

for $(\rho, a) \in Jord(\delta) \setminus \{(\rho, b)\}$, and

$$\varepsilon_{\delta'}(\rho, b-2) = \varepsilon(\rho, b)$$

if defined. Note that if b_- does not exist, we may replace $b_- + 2$ with 1 or 2 in the inequality $b_- + 2 < b$, whichever matches the parity of $2red(\rho; \sigma) + 1$.

Proof. We start by assuming b_- exists. We discuss the (largely similar) case where b_- does not exist at the end.

First we show $\delta \hookrightarrow \nu^{\frac{b-1}{2}}\rho \rtimes \pi'$ for some irreducible π', then show $\mu^*_{\nu^{\frac{b-1}{2}}\rho}(\delta) = \nu^{\frac{b-1}{2}}\rho \otimes \pi'$, and finally, $\pi' \cong \delta'$. From the results of Mœglin–Tadić (see Note 1.3.1), we have

$$\delta \hookrightarrow \delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho, \nu^{\frac{a_2-1}{2}}\rho\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-a_{2k-1}+1}{2}}\rho, \nu^{\frac{a_{2k}-1}{2}}\rho\right]\right) \rtimes \delta^*,$$

where $Jord(\delta^*) = Jord(\delta)\setminus\{(\rho, a_1), \ldots, (\rho, a_{2k})\}$ and the restriction of $\varepsilon^* = \varepsilon_{\delta^*}$ to $Jord_\rho(\delta^*) \cup (Jord_\rho(\delta^*) \times Jord_\rho(\delta^*))$ is alternated. We have three possibilities to consider: (i) $b \in \{a_2, a_4, \ldots, a_{2k}\}$, (ii) $b \in \{a_1, a_3, \ldots, a_{2k-1}\}$, and (iii) $(\rho, b) \in Jord(\delta^*)$.

First, suppose $b \in \{a_1, a_3, \ldots, a_{2k-1}\}$; for concreteness, $b = a_1$ (the hardest case). Observe that by irreducibility, $\nu^{\frac{-b+1}{2}}\rho \times \delta\left(\left[\nu^{\frac{-a_{2j-1}+1}{2}}\rho, \nu^{\frac{a_{2j}-1}{2}}\rho\right]\right) \cong \delta\left(\left[\nu^{\frac{-a_{2j-1}+1}{2}}\rho, \nu^{\frac{a_{2j}-1}{2}}\rho\right]\right) \times \nu^{\frac{-b+1}{2}}\rho$ when $j > 1$. Thus, "commuting" the $\nu^{\frac{-b+1}{2}}\rho$ back,

$$\begin{aligned}
\delta &\hookrightarrow \left(\delta\left(\left[\nu^{\frac{-b+3}{2}}\rho, \nu^{\frac{a_2-1}{2}}\rho\right]\right) \times \nu^{\frac{-b+1}{2}}\rho\right) \times \delta\left(\left[\nu^{\frac{-a_3+1}{2}}\rho, \nu^{\frac{a_5-1}{2}}\rho\right]\right) \times \cdots \\
&\quad \times \delta\left(\left[\nu^{\frac{-a_{2k-1}+1}{2}}\rho, \nu^{\frac{a_{2k}-1}{2}}\rho\right]\right) \rtimes \delta^* \\
&\cong \delta\left(\left[\nu^{\frac{-b+3}{2}}\rho, \nu^{\frac{a_2-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{-a_3+1}{2}}\rho, \nu^{\frac{a_5-1}{2}}\rho\right]\right) \times \nu^{\frac{-b+1}{2}}\rho \times \cdots \\
&\quad \times \delta\left(\left[\nu^{\frac{-a_{2k-1}+1}{2}}\rho, \nu^{\frac{a_{2k}-1}{2}}\rho\right]\right) \rtimes \delta^*
\end{aligned}$$

$$\cong \delta\left(\left[\nu^{\frac{-b+3}{2}}\rho, \nu^{\frac{a_2-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{-a_3+1}{2}}\rho, \nu^{\frac{a_5-1}{2}}\rho\right]\right) \times \cdots$$
$$\times \delta\left(\left[\nu^{\frac{-a_{2k-1}+1}{2}}\rho, \nu^{\frac{a_{2k}-1}{2}}\rho\right]\right) \times \nu^{\frac{-b+1}{2}}\rho \rtimes \delta^*.$$

By Proposition 3.1(ii) [23], we have $\nu^{\frac{b-1}{2}}\rho \rtimes \delta^*$ irreducible, hence $\nu^{\frac{-b+1}{2}}\rho \rtimes \delta^* \cong \nu^{\frac{b-1}{2}}\rho \rtimes \delta^*$. Now, noting that $\nu^{\frac{-b+1}{2}}\rho \times \delta([\nu^{\frac{-a_{2j-1}+1}{2}}\rho, \nu^{\frac{a_{2j}-1}{2}}\rho])$ is irreducible for all j, we continue

$$\delta \hookrightarrow \delta\left(\left[\nu^{\frac{-b+3}{2}}\rho, \nu^{\frac{a_2-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{-a_3+1}{2}}\rho, \nu^{\frac{a_5-1}{2}}\rho\right]\right) \times \cdots$$
$$\times \delta\left(\left[\nu^{\frac{-a_{2k-1}+1}{2}}\rho, \nu^{\frac{a_{2k}-1}{2}}\rho\right]\right) \times \nu^{\frac{b-1}{2}}\rho \rtimes \delta^*$$
$$\cong \nu^{\frac{b-1}{2}}\rho \times \delta\left(\left[\nu^{\frac{-b+3}{2}}\rho, \nu^{\frac{a_2-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{-a_3+1}{2}}\rho, \nu^{\frac{a_5-1}{2}}\rho\right]\right) \times \cdots$$
$$\times \delta\left(\left[\nu^{\frac{-a_{2k-1}+1}{2}}\rho, \nu^{\frac{a_{2k}-1}{2}}\rho\right]\right) \rtimes \delta^*$$
$$\Downarrow \text{ (Lemma 1.2.3)}$$
$$\delta \hookrightarrow \nu^{\frac{b-1}{2}}\rho \rtimes \pi'$$

for some irreducible $\pi' \leq \delta([\nu^{\frac{-b+3}{2}}\rho, \nu^{\frac{a_2-1}{2}}\rho]) \times \delta([\nu^{\frac{-a_3+1}{2}}\rho, \nu^{\frac{a_5-1}{2}}\rho]) \times \cdots \times \delta([\nu^{\frac{-a_{2k-1}+1}{2}}\rho, \nu^{\frac{a_{2k}-1}{2}}\rho]) \rtimes \delta^*$. The argument if $b \in \{a_2, a_4, \ldots, a_{2k}\}$ is similar but easier (as no inversion is required). Finally, if $(\rho, b) \in Jord(\delta^*)$, we have

$$\delta^* \hookrightarrow \delta\left(\left[\nu^{\alpha_0}\rho, \nu^{\frac{a_{2k+1}-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\alpha_0+1}\rho, \nu^{\frac{a_{2k+2}-1}{2}}\rho\right]\right) \times \cdots$$
$$\times \delta\left(\left[\nu^{\alpha}\rho, \nu^{\frac{a_{2k+\alpha}-1}{2}}\rho\right]\right) \rtimes \delta^{**},$$

with $\alpha = red(\rho;\sigma)$, $\alpha_0 = \begin{cases} 1 & \text{if } \alpha \equiv 0 \bmod 1 \\ \frac{1}{2} & \text{if } \alpha \equiv \frac{1}{2} \bmod 1 \end{cases}$ and $a_{2k+1} < a_{2k+2} < \cdots < a_{2k+\alpha-1}$. Again, for concreteness we focus on the hardest case and take $b = a_{2k+\alpha-1}$. Now, observe that since $b_- \geq a_{2k+\alpha-2}$ and $\frac{a_{2k+\alpha-2}-1}{2} \geq \alpha - 2$, we have

$$\frac{b-1}{2} \geq \frac{b_-+3}{2} \geq \frac{a_{2k+\alpha-2}+3}{2} \geq \alpha,$$

so $\delta([\nu^{\alpha}\rho, \nu^{\frac{b-1}{2}}\rho])$ is nondegenerate. Thus,

$$\delta^* \hookrightarrow \delta\left(\left[\nu^{\alpha_0}\rho, \nu^{\frac{a_{2k+1}-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\alpha_0+1}\rho, \nu^{\frac{a_{2k+2}-1}{2}}\rho\right]\right) \times \cdots$$
$$\times \nu^{\frac{b-1}{2}}\rho \times \delta\left(\left[\nu^{\alpha}\rho, \nu^{\frac{b-3}{2}}\rho\right]\right) \rtimes \delta^{**}$$

$$\cong \nu^{\frac{b-1}{2}}\rho \times \delta\left(\left[\nu^{\alpha_0}\rho, \nu^{\frac{a_{2k+1}-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\alpha_0+1}\rho, \nu^{\frac{a_{2k+2}-1}{2}}\rho\right]\right) \times \cdots$$
$$\times \times \delta\left(\left[\nu^{\alpha}\rho, \nu^{\frac{b-3}{2}}\rho\right]\right) \rtimes \delta^{**},$$

since $\delta([\nu^{\alpha_0+j}\rho, \nu^{\frac{a_{2k+1-j}-1}{2}}\rho]) \rtimes \nu^{\frac{b-1}{2}}\rho$ is irreducible (as $a_{2k+1} < \cdots < a_{2k+\alpha-2} \leq b_- < b-2$). Combining this with our original embedding for δ, we may argue as before to get

$$\delta \hookrightarrow \delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho, \nu^{\frac{a_2-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{-a_3+1}{2}}\rho, \nu^{\frac{a_4-1}{2}}\rho\right]\right) \times \cdots$$
$$\times \delta\left(\left[\nu^{\frac{-a_{2k-1}+1}{2}}\rho, \nu^{\frac{a_{2k}-1}{2}}\rho\right]\right)$$
$$\times \nu^{\frac{b-1}{2}}\rho \times \delta\left(\left[\nu^{\alpha_0}\rho, \nu^{\frac{a_{2k+1}-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\alpha_0+1}\rho, \nu^{\frac{a_{2k+2}-1}{2}}\rho\right]\right) \times \cdots$$
$$\times \delta\left(\left[\nu^{\alpha}\rho, \nu^{\frac{b-3}{2}}\rho\right]\right) \rtimes \delta^{**}$$
$$\cong \nu^{\frac{b-1}{2}}\rho \times \delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho, \nu^{\frac{a_2-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{-a_3+1}{2}}\rho, \nu^{\frac{a_4-1}{2}}\rho\right]\right) \times \cdots$$
$$\times \delta\left(\left[\nu^{\frac{-a_{2k-1}+1}{2}}\rho, \nu^{\frac{a_{2k}-1}{2}}\rho\right]\right)$$
$$\times \delta\left(\left[\nu^{\alpha_0}\rho, \nu^{\frac{a_{2k+1}-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\alpha_0+1}\rho, \nu^{\frac{a_{2k+2}-1}{2}}\rho\right]\right) \times \cdots$$
$$\times \delta\left(\left[\nu^{\alpha}\rho, \nu^{\frac{b-3}{2}}\rho\right]\right) \rtimes \delta^{**}$$

$\Downarrow$ (Lemma 1.2.3)

$$\delta \hookrightarrow \nu^{\frac{b-1}{2}}\rho \rtimes \pi'$$

for some irreducible $\pi' \leq \delta([\nu^{\frac{-a_1+1}{2}}\rho, \nu^{\frac{a_2-1}{2}}\rho]) \times \delta([\nu^{\frac{-a_3+1}{2}}\rho, \nu^{\frac{a_4-1}{2}}\rho]) \times \cdots \times \delta([\nu^{\frac{-a_{2k-1}+1}{2}}\rho, \nu^{\frac{a_{2k}-1}{2}}\rho]) \times \delta([\nu^{\alpha_0}\rho, \nu^{\frac{a_{2k+1}-1}{2}}\rho]) \times \delta([\nu^{\alpha_0+1}\rho, \nu^{\frac{a_{2k+2}-1}{2}}\rho]) \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{\frac{b-3}{2}}\rho]) \rtimes \delta^{**}$.

We now turn to the task of showing $\mu^*_{\nu^{\frac{b-1}{2}}\rho}(\delta) = \nu^{\frac{b-1}{2}}\rho \otimes \pi'$ and $\pi' \cong \delta'$. We first show π' is square-integrable. Now, were π' nontempered, we would have

$$\pi' \hookrightarrow \delta\left(\left[\nu^{-r}\rho', \nu^{s}\rho'\right]\right) \rtimes \pi''$$

with $r > s$. Then,

$$\delta \hookrightarrow \nu^{\frac{b-1}{2}}\rho \times \delta\left(\left[\nu^{-r}\rho', \nu^{s}\rho'\right]\right) \rtimes \pi''.$$

If $\nu^{\frac{b-1}{2}}\rho \times \delta([\nu^{-r}\rho', \nu^{s}\rho'])$ were irreducible, then we would get

$$\delta \hookrightarrow \delta\left(\left[\nu^{-r}\rho', \nu^{s}\rho'\right]\right) \times \nu^{\frac{b-1}{2}}\rho \rtimes \pi'',$$

which contradicts the Casselman criterion for the square-integrability of δ. Thus, $\nu^{\frac{b-1}{2}}\rho \times \delta([\nu^{-r}\rho', \nu^{s}\rho'])$ would have to be reducible. This requires $\rho' \cong \rho$ and $\frac{b-1}{2} = s+1$. However, since $r < s$,

$$\delta \hookrightarrow \nu^{s+1}\rho \times \delta\left(\left[\nu^{-r}\rho', \nu^{s}\rho'\right]\right) \rtimes \pi''$$

still contradicts the Casselman criterion for the square-integrability of δ. Therefore, π' must be tempered. Further, by Proposition 8.2 [21], since π' is tempered and has the same infinitesimal character as δ', π' must in fact be square-integrable as well.

Now, observe that the partial cuspidal support is σ for both π' and δ'. Further, since π' and δ' have the same infinitesimal character, it follows from Lemma 2.1.1 that $Jord(\pi') = Jord(\delta')$. We first use this fact to show $\mu^*_{\nu^{\frac{b-1}{2}}\rho}(\delta) = \nu^{\frac{b-1}{2}}\rho \otimes \pi'$. After that, we finish showing π' and δ' have the same Mœglin–Tadić data (so are equivalent) by checking $\varepsilon_{\pi'} = \varepsilon_{\delta'}$.

To show $\mu^*_{\nu^{\frac{b-1}{2}}\rho}(\delta) = \nu^{\frac{b-1}{2}}\rho \otimes \pi'$, write $\mu^*(\pi') = \sum_j \lambda_j \otimes \phi_j$. Since $M^*(\nu^{\frac{b-1}{2}}\rho) = \nu^{\frac{b-1}{2}}\rho \otimes 1 + 1 \otimes \nu^{\frac{b-1}{2}}\rho + \nu^{\frac{-b+1}{2}}\rho \otimes 1$ (noting $\rho \cong \check{\rho}$), we have

$$\mu^*(\nu^{\frac{b-1}{2}}\rho \rtimes \pi') = \sum_j \left(\left(\lambda_j \times \nu^{\frac{b-1}{2}}\rho \otimes \phi_j\right) + (\lambda_j \otimes \nu^{\frac{b-1}{2}}\rho \rtimes \phi_j) + \left(\lambda_j \times \nu^{\frac{-b+1}{2}}\rho \otimes \phi_j\right)\right). \tag{2.1}$$

Since $(\rho, b) \notin Jord(\pi')$, it follows from Remark 1.3.2 that $\mu^*_{\nu^{\frac{b-1}{2}}\rho}(\pi') = 0$. Therefore, the only way to obtain a term of the form $\nu^{\frac{b-1}{2}} \otimes \cdots$ in $\mu^*(\nu^{\frac{b-1}{2}}\rho \rtimes \pi')$ is to have $\lambda_j = 1$, hence $\phi_j = \pi'$. It then follows that $\mu^*_{\nu^{\frac{b-1}{2}}\rho}(\delta) = \nu^{\frac{b-1}{2}}\rho \otimes \pi'$, as claimed.

Let $\varepsilon' = \varepsilon_{\pi'}$. Suppose $(\rho', a), (\rho', a_-) \in Jord(\pi')$. We first show

$$\varepsilon'(\rho', a)\varepsilon'(\rho', a_-)^{-1} = \varepsilon(\rho', a)\varepsilon(\rho', a_-)^{-1}$$

when $(\rho', a), (\rho', a_-) \neq (\rho, b-2)$ (so $(\rho', a), (\rho', a_-)$ are also in $Jord(\delta)$). In this case, we have

$$\begin{gathered}
\varepsilon'(\rho', a)\varepsilon'(\rho', a_-)^{-1} = 1 \\
\Downarrow \\
\pi' \hookrightarrow \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho', \nu^{\frac{a-1}{2}}\rho'\right]\right) \rtimes \theta \\
\Downarrow \\
\delta \hookrightarrow \nu^{\frac{b-1}{2}}\rho \times \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho', \nu^{\frac{a-1}{2}}\rho'\right]\right) \rtimes \theta
\end{gathered}$$

Now, we claim $\nu^{\frac{b-1}{2}}\rho \times \delta([\nu^{\frac{a_-+1}{2}}\rho', \nu^{\frac{a-1}{2}}\rho'])$ is irreducible. In fact, the only way it could be reducible would be if $\rho' \cong \rho$ and $b = a_-$ (contradicting our assumption) or $b = a+2$ (which would imply $a_- = b$, again a contradiction). Then, a commuting argument implies

$$\begin{gathered}
\delta \hookrightarrow \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho', \nu^{\frac{a-1}{2}}\rho'\right]\right) \times \nu^{\frac{b-1}{2}}\rho \rtimes \theta \\
\Downarrow \text{ (Lemma 1.2.3)} \\
\delta \hookrightarrow \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho', \nu^{\frac{a-1}{2}}\rho'\right]\right) \rtimes \theta' \\
\Downarrow \\
\varepsilon(\rho', a)\varepsilon(\rho', a_-)^{-1} = 1
\end{gathered}$$

(for some irreducible $\theta' \leq \nu^{\frac{b-1}{2}}\rho \rtimes \theta$), as needed. In the converse direction,

$$\begin{gathered}
\varepsilon(\rho', a)\varepsilon(\rho', a_-)^{-1} = 1 \\
\Downarrow \\
\delta \hookrightarrow \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho', \nu^{\frac{a-1}{2}}\rho'\right]\right) \\
\rtimes\theta \\
\Downarrow \text{ (noting } \delta \hookrightarrow \nu^{\frac{b-1}{2}}\rho \rtimes \pi') \\
\mu^*\left(\nu^{\frac{b-1}{2}}\rho \rtimes \pi'\right) \geq \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho', \nu^{\frac{a-1}{2}}\rho'\right]\right) \otimes \theta.
\end{gathered}$$

Again, $\mu^*(\nu^{\frac{b-1}{2}}\rho \rtimes \pi')$ is given by Eq. (2.1). Since $\nu^{\frac{\pm(b-1)}{2}}\rho$ does not appear in $[\nu^{\frac{a_-+1}{2}}\rho', \nu^{\frac{a-1}{2}}\rho']$, the only way to pick up a copy of $\delta([\nu^{\frac{a_-+1}{2}}\rho', \nu^{\frac{a-1}{2}}\rho']) \otimes \theta$ is if it appears in $\lambda_j \otimes \nu^{\frac{b-1}{2}}\rho \rtimes \phi_j$. In this case, we would have to have $\lambda_j = \delta([\nu^{\frac{a_-+1}{2}}\rho', \nu^{\frac{a-1}{2}}\rho'])$. That is, $\mu^*(\pi') \geq \delta([\nu^{\frac{a_-+1}{2}}\rho', \nu^{\frac{a-1}{2}}\rho']) \otimes \phi_j$. By Lemma 1.3.3, this implies $\varepsilon'(\rho', a)\varepsilon'(\rho', a_-)^{-1} = 1$, as needed.

We now show $\varepsilon'(\rho, b-2)\varepsilon'(\rho, b_-)^{-1} = \varepsilon(\rho, b)\varepsilon(\rho, b_-)^{-1}$. Observe that

$$\begin{gathered}
\varepsilon'(\rho, b-2)\varepsilon'(\rho, b_-)^{-1} = 1 \\
\Downarrow \\
\pi' \hookrightarrow \delta\left(\left[\nu^{\frac{b_-+1}{2}}\rho, \nu^{\frac{b-3}{2}}\rho\right]\right) \rtimes \theta \\
\Downarrow \\
\delta \hookrightarrow \nu^{\frac{b-1}{2}}\rho \times \delta\left(\left[\nu^{\frac{b_-+1}{2}}\rho, \nu^{\frac{b-3}{2}}\rho\right]\right) \rtimes \theta \\
\Downarrow \text{ (Lemma 1.2.3)} \\
\delta \hookrightarrow \lambda \rtimes \theta
\end{gathered}$$

for some irreducible $\lambda \leq \nu^{\frac{b-1}{2}}\rho \times \delta([\nu^{\frac{b_-+1}{2}}\rho, \nu^{\frac{b-3}{2}}\rho])$. As in the proof of Lemma 1.3.3, we must have $\lambda = \delta([\nu^{\frac{b_-+1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho])$ (or else $\mu^*(\delta) \geq \nu^x\rho \otimes \theta'$ for some $x \in \{\frac{b_-+1}{2}, \ldots, \frac{b-3}{2}\}$). It then follows that $\varepsilon(\rho, b)\varepsilon(\rho, b_-)^{-1} = 1$. In the converse direction,

$$\begin{gathered}
\varepsilon(\rho, b)\varepsilon(\rho, b_-)^{-1} = 1 \\
\Downarrow \\
\delta \hookrightarrow \delta\left(\left[\nu^{\frac{b_-+1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right) \rtimes \theta \hookrightarrow \nu^{\frac{b-1}{2}}\rho \times \delta\left(\left[\nu^{\frac{b_-+1}{2}}\rho, \nu^{\frac{b-3}{2}}\rho\right]\right) \rtimes \theta \\
\Downarrow \\
r_{M,G}(\delta) \geq \nu^{\frac{b-1}{2}}\rho \otimes \delta\left(\left[\nu^{\frac{b_-+1}{2}}\rho, \nu^{\frac{b-3}{2}}\rho\right]\right) \otimes \theta \\
\Downarrow \\
r_{M,G}\left(\nu^{\frac{b-1}{2}}\rho \rtimes \pi'\right) \geq \nu^{\frac{b-1}{2}}\rho \otimes \delta\left(\left[\nu^{\frac{b_-+1}{2}}\rho, \nu^{\frac{b-3}{2}}\rho\right]\right) \otimes \theta
\end{gathered}$$

for the appropriate standard Levi factor M. Since $\mu^*_{\nu^{\frac{b-1}{2}}\rho}(\delta) = \nu^{\frac{b-1}{2}}\rho \otimes \pi'$, it follows that $\mu^*(\pi') \geq \delta([\nu^{\frac{b_-+1}{2}}\rho, \nu^{\frac{b-3}{2}}\rho]) \otimes \theta$. It now follows from Lemma 1.3.3 that $\varepsilon'(\rho, b-2)\varepsilon'(\rho, b_-)^{-1} = 1$, as needed.

We next show $\varepsilon'(\rho, b_+)\varepsilon'(\rho, b-2)^{-1} = \varepsilon(\rho, b_+)\varepsilon(\rho, b)^{-1}$ (assuming (ρ, b_+) exists). The argument that $\varepsilon'(\rho, b_+)\varepsilon'(\rho, b-2)^{-1} = 1 \Rightarrow \varepsilon(\rho, b_+)\varepsilon(\rho, b)^{-1} = 1$ is like that above for $(\rho', a), (\rho', a_-) \neq (\rho.b-2)$, noting that the irreducibility of $\nu^{\frac{b-1}{2}}\rho \times \delta([\nu^{\frac{b-1}{2}}\rho, \nu^{\frac{b_+-1}{2}}\rho])$ is immediate here. In the converse direction,

$$\begin{gathered}\varepsilon(\rho, b_+)\varepsilon(\rho, b)^{-1} = 1\\ \Downarrow \text{ (see Note 1.3.1)}\\ \delta \hookrightarrow \delta\left(\left[\nu^{\frac{-b+1}{2}}\rho, \nu^{\frac{b_+-1}{2}}\rho\right]\right) \rtimes \delta_1 \hookrightarrow \delta\left(\left[\nu^{\frac{b-1}{2}}\rho, \nu^{\frac{b_+-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{-b+1}{2}}\rho, \nu^{\frac{b-3}{2}}\rho\right]\right) \rtimes \delta_1\\ \Downarrow \text{ (Lemma 1.2.3)}\\ \delta \hookrightarrow \delta\left(\left[\nu^{\frac{b-1}{2}}\rho, \nu^{\frac{b_+-1}{2}}\rho\right]\right) \times \theta\end{gathered}$$

for some irreducible $\theta \leq \delta([\nu^{\frac{-b+1}{2}}\rho, \nu^{\frac{b-3}{2}}\rho]) \rtimes \delta_1$. Now, observe that since $\mu^*_{\nu^{\frac{b-1}{2}}\rho}(\delta) \neq 0$ and there are no terms of the form $\nu^{\frac{b-1}{2}}\rho \otimes \cdots$ in $M^*(\delta([\nu^{\frac{b-1}{2}}\rho, \nu^{\frac{b_+-1}{2}}\rho])$, we must have $\mu^*_{\nu^{\frac{b-1}{2}}\rho}(\theta) \neq 0$. Therefore, by Lemmas 1.2.2 and 1.2.3, we must have $\theta \hookrightarrow \nu^{\frac{b-1}{2}}\rho \rtimes \theta'$ for some irreducible θ'. Thus,

$$\begin{gathered}\delta \hookrightarrow \delta\left(\left[\nu^{\frac{b-1}{2}}\rho, \nu^{\frac{b_+-1}{2}}\rho\right]\right) \times \nu^{\frac{b-1}{2}}\rho \rtimes \theta' \cong \nu^{\frac{b-1}{2}}\rho \times \delta\left(\left[\nu^{\frac{b-1}{2}}\rho, \nu^{\frac{b_+-1}{2}}\rho\right]\right) \rtimes \theta'\\ \Downarrow \text{ (since } \mu^*_{\nu^{\frac{b-1}{2}}\rho}(\pi') = \nu^{\frac{b-1}{2}}\rho \otimes \pi')\\ \mu^*(\pi') \geq \delta\left(\left[\nu^{\frac{b-1}{2}}\rho, \nu^{\frac{b_+-1}{2}}\rho\right]\right) \otimes \theta'\\ \Downarrow \text{ (Lemma 1.3.3)}\\ \varepsilon'(\rho, b_+)\varepsilon'(\rho, b-2)^{-1} = 1,\end{gathered}$$

as needed.

The arguments above show ε' is as claimed on $S' \cap (Jord' \times Jord')$. It remains to show ε' is as claimed on $S' \cap Jord'$. Fix ρ' having $Jord'_{\rho'} \neq \emptyset$ and ε' defined on $Jord'_{\rho'}$. From the condition $\varepsilon'(\rho', a)\varepsilon'(\rho', a_-)^{-1} = 1 \Leftrightarrow \pi' \hookrightarrow \delta([\nu^{\frac{a_-+1}{2}}\rho', \nu^{\frac{a-1}{2}}\rho']) \rtimes \theta'$ (a consequence of the work above), it suffices to show that $\varepsilon'(\rho', a)$ is as claimed for a single value of a'. We break this into two cases: (1) $red(\rho'; \sigma) \equiv \frac{1}{2} \bmod 1$, and (2) $red(\rho'; \sigma) = 0$ (recalling that if $red(\rho'; \sigma) \in \mathbb{N}$, ε' is not defined on $Jord'_{\rho'}$).

First, suppose $red(\rho', \sigma) \equiv \frac{1}{2} \bmod 1$. Let a'_{min} be the smallest value of a having $(\rho', a) \in Jord'$. The argument that $\varepsilon'(\rho', a'_{min}) = 1 \Rightarrow \varepsilon(\rho', a'_{min}) = 1$ is like that for $(\rho', a), (\rho', a_-) \neq (\rho, b-2)$ above, replacing (ρ', a_-) with $(\rho, 0)$ in the argument. In this case, the irreducibility of $\nu^{\frac{b-1}{2}}\rho \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{a'_{min}-1}{2}}\rho])$ follows from $a'_{min} \leq b_- < b-2$. In the converse direction,

$$\begin{gathered}\varepsilon(\rho', a'_{min}) = 1\\ \Downarrow\\ \delta \hookrightarrow \delta\left(\left[\nu^{\frac{1}{2}}\rho', \nu^{\frac{a'_{min}-1}{2}}\rho'\right]\right) \rtimes \theta.\end{gathered}$$

Now, $\mu^*_{\nu^{\frac{b-1}{2}}\rho}(\delta([\nu^{\frac{1}{2}}\rho', \nu^{\frac{a'_{min}-1}{2}}\rho']) \rtimes \theta) \neq 0$ implies $\mu^*_{\nu^{\frac{b-1}{2}}\rho}(\theta) \neq 0$ (noting that if $\rho' \cong \rho$, $b > a'_{min}$). By Lemmas 1.2.2 and 1.2.3, we may then conclude $\theta \hookrightarrow \nu^{\frac{b-1}{2}}\rho \rtimes \theta'$ for some irreducible θ'. Thus,

$$\delta \hookrightarrow \delta\left(\left[\nu^{\frac{1}{2}}\rho', \nu^{\frac{a'_{min}-1}{2}}\rho'\right]\right) \times \nu^{\frac{b-1}{2}}\rho \rtimes \theta' \cong \nu^{\frac{b-1}{2}}\rho \times \delta\left(\left[\nu^{\frac{1}{2}}\rho', \nu^{\frac{a'_{min}-1}{2}}\rho'\right]\right) \rtimes \theta'$$

$$\Downarrow \left(\text{since } \mu^*_{\nu^{\frac{b-1}{2}}\rho}(\delta) = \nu^{\frac{b-1}{2}}\rho \otimes \pi'\right)$$

$$\mu^*(\pi') \geq \delta\left(\left[\nu^{\frac{1}{2}}\rho, \nu^{\frac{a'_{min}-1}{2}}\rho\right]\right) \otimes \theta'$$

$$\Downarrow$$

$$\varepsilon'(\rho', a'_{min}) = 1,$$

as needed.

Now, suppose $red(\rho'; \sigma) = 0$. Let a'_{max} be the largest value of a such that $(\rho', a) \in Jord'$. If $(\rho', a'_{max}) \neq (\rho, b-2)$, then for $\eta = \pm 1$, we have (see Sect. 1.3)

$$\varepsilon'(\rho', a'_{max}) = \eta$$

$$\Downarrow$$

$$\delta' \hookrightarrow \phi \rtimes \delta\left(\left[\nu\rho', \nu^{\frac{a'_{max}-1}{2}}\rho'\right]; \tau_\eta(\rho'; \sigma)\right)$$

$$\Downarrow$$

$$\delta \hookrightarrow \nu^{\frac{b-1}{2}}\rho \times \phi \rtimes \delta\left(\left[\nu\rho', \nu^{\frac{a'_{max}-1}{2}}\rho'\right]; \tau_\eta(\rho'; \sigma)\right)$$

$$\Downarrow \text{ (Lemma 1.2.3)}$$

$$\delta \hookrightarrow \phi' \rtimes \delta\left(\left[\nu\rho', \nu^{\frac{a'_{max}-1}{2}}\rho'\right]; \tau_\eta(\rho'; \sigma)\right)$$

$$\Downarrow$$

$$\varepsilon(\rho', a'_{max}) = \eta$$

(some irreducible $\phi' \leq \nu^{\frac{b-1}{2}}\rho \times \phi$), as needed. If $(\rho', a'_{max}) = (\rho, b-2)$, then

$$\varepsilon'(\rho, b-2) = \eta$$

$$\Downarrow \text{ (as above)}$$

$$\delta \hookrightarrow \nu^{\frac{b-1}{2}}\rho \times \phi \rtimes \delta\left(\left[\nu\rho, \nu^{\frac{b-3}{2}}\rho\right]; \tau_\eta(\rho; \sigma)\right) \cong \phi \times \nu^{\frac{b-1}{2}}\rho \rtimes \delta\left(\left[\nu\rho, \nu^{\frac{b-3}{2}}\rho\right]; \tau_\eta(\rho; \sigma)\right)$$

where the irreducibility of $\nu^{\frac{b-1}{2}}\rho \times \phi$ follows from the fact that any $\nu^x\rho$ in the supercuspidal support of ϕ has $x < \frac{b-1}{2} - 1$ (which follows from the assumption $b_- < b-2$). By Lemma 1.2.3, $\delta \hookrightarrow \phi \rtimes \theta$ for some irreducible $\theta \leq \nu^{\frac{b-1}{2}}\rho \rtimes \delta([\nu\rho, \nu^{\frac{b-3}{2}}\rho]; \tau_\eta(\rho; \sigma))$. Now, $\nu^{\frac{b-1}{2}}\rho \rtimes \delta([\nu\rho, \nu^{\frac{b-3}{2}}\rho]; \tau_\eta(\rho; \sigma))$ is dual to the generalized degenerate principal series of Proposition 3.2 [3] (noting $b > b_- + 2 \Rightarrow b > 3$, which translates to $\ell > 1$ in Proposition 3.2 [3]). In the notation of that paper (subrepresentation realization of the Langlands classification), one can see fairly easily that it has irreducible subquotients $L(\nu^{\frac{-b+1}{2}}\rho; \delta([\nu\rho, \nu^{\frac{b-3}{2}}\rho]; \tau_\eta(\rho; \sigma)))$ and $\delta([\nu\rho, \nu^{\frac{b-1}{2}}\rho]; \tau_\eta(\rho; \sigma))$. Thus, either

$$\delta \hookrightarrow \phi \rtimes L\left(\nu^{\frac{-b+1}{2}}\rho; \delta\left(\left[\nu\rho, \nu^{\frac{b-3}{2}}\rho\right]; \tau_\eta(\rho; \sigma)\right)\right)$$

or

$$\delta \hookrightarrow \phi \rtimes \delta\left(\left[\nu\rho, \nu^{\frac{b-1}{2}}\rho\right]; \tau_\eta(\rho; \sigma)\right).$$

In the former case, we would have

$$\begin{aligned}\delta &\hookrightarrow \phi \rtimes \nu^{\frac{-b+1}{2}}\rho \rtimes \delta\left(\left[\nu\rho, \nu^{\frac{b-3}{2}}\rho\right]; \tau_\eta(\rho; \sigma)\right)\\ &\cong \nu^{\frac{-b+1}{2}}\rho \times \phi \rtimes \delta\left(\left[\nu\rho, \nu^{\frac{b-3}{2}}\rho\right]; \tau_\eta(\rho; \sigma)\right),\end{aligned}$$

with the irreducibility of $\phi \times \nu^{\frac{b-1}{2}}\rho$ as above. However, this contradicts the Casselman criterion for δ. Thus,

$$\begin{gathered}\delta \hookrightarrow \phi \rtimes \delta\left(\left[\nu\rho, \nu^{\frac{b-1}{2}}\rho\right]; \tau_\eta(\rho; \sigma)\right)\\ \Downarrow\\ \varepsilon(\rho, b) = \eta,\end{gathered}$$

as needed. This finishes the proof that ε' is as claimed (i.e., $\pi' \cong \delta'$), and the case where b_- exists.

If b_- does not exist, the proof is largely similar; simplified in some places as the minimality of b then precludes certain possibilities. The exception is the argument that $\varepsilon'(\rho', a'_{min}) = \varepsilon(\rho', a'_{min})$ when $red(\rho'; \sigma) \equiv \frac{1}{2} \bmod 1$. In this case, if $\rho' \cong \rho$, a different argument is needed. Here, $a_{min} = b$, and we must show $\varepsilon'(\rho, b-2) = \varepsilon(\rho, b)$. We have

$$\begin{gathered}\varepsilon(\rho, b) = 1\\ \Downarrow\\ \delta \hookrightarrow \delta\left(\left[\nu^{\frac{1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right) \rtimes \theta\\ \Downarrow\\ \mu^*_{\delta\left(\left[\nu^{\frac{1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right)}\left(\nu^{\frac{b-1}{2}}\rho \rtimes \pi'\right) \neq 0\\ \Downarrow \text{ (noting } (\rho, b) \notin Jord(\pi'))\\ \mu^*_{\delta\left(\left[\nu^{\frac{1}{2}}\rho, \nu^{\frac{b-3}{2}}\rho\right]\right)}(\pi') \neq 0\\ \Downarrow\\ \varepsilon'(\rho, b-2) = 1.\end{gathered}$$

In the converse direction,

$$\begin{gathered}
\varepsilon'(\rho, b-2) = 1 \\
\Downarrow \\
\pi' \hookrightarrow \delta\left(\left[\nu^{\frac{1}{2}}\rho, \nu^{\frac{b-3}{2}}\rho\right]\right) \rtimes \theta' \text{ (some irreducible } \theta'\text{)} \\
\Downarrow \\
\delta \hookrightarrow \nu^{\frac{b-1}{2}}\rho \times \delta\left(\left[\nu^{\frac{1}{2}}\rho, \nu^{\frac{b-3}{2}}\rho\right]\right) \rtimes \theta' \\
\Downarrow \text{ (Lemma 1.2.3)} \\
\delta \hookrightarrow \delta\left(\left[\nu^{\frac{1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right) \rtimes \theta' \\
\text{or} \\
\delta \hookrightarrow \mathcal{L}(\delta\left(\left[\nu^{\frac{1}{2}}\rho, \nu^{\frac{b-3}{2}}\rho\right]\right), \nu^{\frac{b-1}{2}}\rho) \rtimes \theta'.
\end{gathered}$$

We can rule out the latter as it would give $\mu^*_{\nu^{\frac{b-3}{2}}\rho}(\delta) \neq 0$, contradicting $(\rho, b-2) \notin Jord(\delta)$. The former implies $\varepsilon(\rho, b) = 1$, as needed. □

Repeated application of this proposition gives the following:

Corollary 2.1.3. *Let $\delta = \delta_{(Jord,\sigma.\varepsilon)}$ and $(\rho, b) \in Jord$. Suppose (ρ, b_-) exists and $b > b_- + 2$. If $b_- + 2 < c < b$ with $c \equiv b \bmod 2$, then*

$$\mu^*_{\delta\left(\left[\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right)}(\delta) = \delta\left(\left[\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right) \otimes \delta',$$

where δ' is the discrete series representation with Mœglin–Tadić data described below. Here,

$$Jord(\delta') = (Jord(\delta) \setminus \{(\rho, b)\}) \cup \{(\rho, c)\}$$

and $\varepsilon_{\delta'}$ defined by $\varepsilon_{\delta'} = \varepsilon$ on $S_{\delta'} \cap S$,

$$\varepsilon_{\delta'}(\rho, a)\varepsilon_{\delta'}(\rho, c)^{-1} = \varepsilon(\rho, a)\varepsilon(\rho, b)^{-1}$$

for $(\rho, a) \in Jord(\delta) \setminus \{(\rho, b)\}$, and

$$\varepsilon_{\delta'}(\rho, c) = \varepsilon(\rho, b)$$

if defined. Note that if b_- does not exist, we may replace $b_- + 2$ with 1 or 2 in the inequality $b_- + 2 < c < b$, whichever matches the parity of $2red(\rho; \sigma) + 1$ (and $\varepsilon_{\delta'}$ just the restriction of ε).

Note 2.1.4. *The same argument as in the proof of Lemma 1.3.3 tells us*

$$\delta \hookrightarrow \delta\left(\left[\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right) \rtimes \delta'$$

in Corollary 2.1.3.

2.2. The case of $\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta$

In this section, we extend the Mœglin–Tadić classification to cover the components of the representation $\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta$ (assumed reducible). This serves as the basis for extending the classification to elliptic tempered representations in Sect. 2.3.

Lemma 2.2.1. *Suppose* $\delta = \delta_{(Jord,\sigma,\varepsilon)}$ *and* $(\rho, c) \notin Jord$ *but with* $c \equiv 2red(\rho;\sigma) + 1 \mod 2$.

(1) *Suppose there is some* $(\rho, x) \in Jord$ *with* $x > c$. *Let* a *be the minimal such* x. *Then, there is exactly one component* π' *of* $\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta$ *such that*

$$\mu^*_{\delta\left(\left[\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)}(\pi') \neq 0.$$

(2) *Suppose there is some* $(\rho, x) \in Jord$ *with* $x < c$. *Let* b *be the maximal such* x. *Then, there is exactly one component* π'' *(possibly* π'*) of* $\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta$ *such that*

$$\mu^*_{\left[\delta\left(\left[\nu^{\frac{b+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right)^2\right]}(\pi'') \neq 0.$$

Proof. Write $\mu^*(\delta) = \sum_\ell \tau_\ell \otimes \theta_\ell$ and observe that (noting $\check{\rho} \cong \rho$ here)

$$M^*\left(\delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right)\right) = \sum_{i=\frac{-c+1}{2}}^{\frac{c+1}{2}} \sum_{j=i}^{\frac{c+1}{2}} \delta\left(\left[\nu^{-i+1}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right)$$
$$\times \delta\left(\left[\nu^{j}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \otimes \delta\left(\left[\nu^{i}\rho, \nu^{j-1}\rho\right]\right).$$

Then, by Theorem 1.2.1,

$$\mu^*\left(\delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \delta\right) = \sum_\ell \sum_{i=\frac{-c+1}{2}}^{\frac{c+1}{2}} \sum_{j=i}^{\frac{c+1}{2}} \delta([\nu^{-i+1}\rho, \nu^{\frac{c-1}{2}}\rho])$$
$$\times \delta\left(\left[\nu^{j}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \times \tau_\ell \otimes \delta([\nu^{i}\rho, \nu^{j-1}\rho]) \rtimes \theta_\ell.$$

For (1), observe that to contribute to $\mu^*_{\delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])}$, we must have $j = -i + 1 = \frac{c+1}{2}$ and $\tau_\ell = \delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])$. By Corollary 2.1.3, this gives $\theta_\ell = \delta'$ (δ' as in the corollary) and

$$\mu^*_{\delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])}\left(\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta\right) = \delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])$$
$$\otimes \left(\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta'\right).$$

Since $(\rho, c) \in Jord(\delta')$ (again, see Corollary 2.1.3), we have $\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta'$ irreducible, from which (1) follows.

For (2), observe that it follows from Remark 1.3.2 that to have a term of the form $\nu^{\frac{x-1}{2}}\rho \otimes \lambda \leq m^*(\tau_\ell)$, we must have $(\rho, x) \in Jord$. In particular, since there is no $(\rho, x) \in Jord$ with $b < x \leq c$, we see that to contribute to $\mu^*_{[\delta([\nu^{\frac{b+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^2]}$, we must have τ_ℓ trivial. It then follows that $-i+1 = j = \frac{b+1}{2}$. Thus,

$$\mu^*_{[\delta([\nu^{\frac{b+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^2]}\left(\delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \delta\right) = \delta\left(\left[\nu^{\frac{b+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right)^2 \otimes \left(\delta\left(\left[\nu^{\frac{-b+1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right) \rtimes \delta\right).$$

Further, since $(\rho, b) \in Jord$, we have $\delta([\nu^{\frac{-b+1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho]) \rtimes \delta$ irreducible. (2) now follows immediately. □

Lemma 2.2.2. *Let $\delta = \delta_{(Jord,\sigma,\varepsilon)}$ with $Jord_\rho(\delta) = \emptyset$ (noting that this requires $red(\rho;\sigma) = 0$ or $\frac{1}{2}$) and suppose $\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta$ is reducible.*

(1) *If $red(\rho;\sigma) = \frac{1}{2}$, there is exactly one component π of $\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta$ such that*

$$\mu^*_{[\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^2]}(\pi) \neq 0$$

(2) *If $red(\rho;\sigma) = 0$, write*

$$\delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \sigma = \bigoplus_{\eta \in \{\pm 1\}} T(\delta([\nu\rho, \nu^{\frac{c-1}{2}}\rho])^2; \tau_\eta(\rho;\sigma))$$

where for $\eta = \pm 1$, $T(\delta([\nu\rho, \nu^{\frac{c-1}{2}}\rho])^2; \tau_\eta(\rho;\sigma))$ is characterized by

$$\mu^*\left(T(\delta([\nu\rho, \nu^{\frac{c-1}{2}}\rho])^2; \tau_\eta(\rho;\sigma))\right) \geq \delta([\nu\rho, \nu^{\frac{c-1}{2}}\rho])^2 \otimes \tau_\eta(\rho;\sigma)$$

with $\tau_\eta(\rho;\sigma)$ as in Sect. 1.3. Then, for $\eta = \pm 1$, there is exactly one component π of $\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta$ such that $\mu^*(\pi)$ contains a term of the form $\lambda \otimes T(\delta([\nu\rho, \nu^{\frac{c-1}{2}}\rho])^2; \tau_\eta(\rho;\sigma))$.

Proof. For (1), write $\mu^*(\delta) = \sum_\ell \tau_\ell \otimes \theta_\ell$. Then, using Theorem 1.2.1,

$$\mu^*\left(\delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \delta\right) = \sum_\ell \sum_{i=\frac{-c+1}{2}}^{\frac{c+1}{2}} \sum_{j=i}^{\frac{c+1}{2}} \delta([\nu^{-i+1}\rho, \nu^{\frac{c-1}{2}}\rho]) \times \delta\left(\left[\nu^j\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \times \tau_\ell \otimes \delta\left(\left[\nu^i\rho, \nu^{j-1}\rho\right]\right) \rtimes \theta_\ell.$$

Now, observe that to contribute to $\mu^*_{[\delta([\nu^{\frac{1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho])^2]}$, we must have $\tau_\ell = 1$ (by supercuspidal support considerations), so $\theta_\ell = \delta$. Therefore, $j = -i + 1 = \frac{c+1}{2}$ and $\tau_\ell = 1$, hence

$$\mu^*_{\left[\delta\left(\left[\nu^{\frac{1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho\right]\right)^2\right]}\left(\delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \delta\right) = \delta\left(\left[\nu^{\frac{1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right)^2 \otimes \delta,$$

from which (1) follows.

(2) is an immediate consequence of Corollary 7.5 and Definition 7.6 of [14] (whose proof uses only the μ^* structure and results on general linear groups, hence holds in the generality needed here). □

Lemma 2.2.3. *Let $\delta = \delta_{(Jord,\sigma,\varepsilon)}$ and suppose $(\rho, c) \notin Jord$. Further, suppose there are $(\rho, a), (\rho, a_-) \in Jord$ with $a > c > a_-$ and $c \equiv a \bmod 2$. Let $T \leq \delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta$.*

(1) *If $\varepsilon(\rho, a)\varepsilon(\rho, a_-)^{-1} = 1$, then*

$$\mu^*_{\delta([\nu^{\frac{c+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])}(T) \neq 0 \Leftrightarrow \mu^*_{[\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho])^2]}(T) \neq 0.$$

(2) *If $\varepsilon(\rho, a)\varepsilon(\rho, a_-)^{-1} = -1$, then*

$$\mu^*_{\delta([\nu^{\frac{c+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])}(T) \neq 0 \Leftrightarrow \mu^*_{[\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho])^2]}(T) = 0.$$

Proof. Write

$$\begin{aligned} I &= \delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \delta, \\ I' &= \delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \delta', \\ I'' &= \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \delta'', \end{aligned}$$

where $Jord(\delta') = (Jord \setminus \{(\rho, a)\}) \cup \{(\rho, c)\}$, $Jord(\delta'') = (Jord \setminus \{(\rho, a_-)\}) \cup \{(\rho, c)\}$, and $\varepsilon', \varepsilon''$ defined by

$$\varepsilon'|_{S\cap S'} = \varepsilon|_{S\cap S'}, \qquad \varepsilon'|_{S\cap S''} = \varepsilon|_{S\cap S''},$$

and the following (when defined):

$$\begin{aligned} &\varepsilon'(\rho, c) = \varepsilon(\rho, a), \qquad \varepsilon''(\rho, c) = \varepsilon(\rho, a_-), \\ &\varepsilon'(\rho, c)\varepsilon'(\rho, d)^{-1} = \varepsilon(\rho, a)\varepsilon(\rho, d)^{-1}, \\ &\varepsilon''(\rho, c)\varepsilon''(\rho, d)^{-1} = \varepsilon(\rho, a_-)\varepsilon(\rho, d)^{-1}. \end{aligned}$$

We first show that I, I' (resp., I, I'') have a common irreducible subquotient characterized by $\mu^*(T') \geq \delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \otimes \delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta'$ (resp., $\mu^*(T'') \geq \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^2 \otimes \delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho]) \rtimes \delta)$, noting that the right-hand factors in the tensor products are irreducible.

To show I and I' have a common irreducible subquotient with the property claimed, let

$$\mathcal{I}' = \delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \delta'.$$

By Note 2.1.4, one sees that $I, I' \leq \mathcal{I}'$. To prove the claim, it suffices to show that $\mu^*(I), \mu^*(I')$, and $\mu^*(\mathcal{I}')$ each contain $\delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \otimes \delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta'$ with multiplicity one—a straightforward μ^* calculation. Similarly, to show I and I'' have a common irreducible subquotient with the property claimed, let

$$\mathcal{I}'' = \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^2 \times \delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho]) \rtimes \delta.$$

Again, using Note 2.1.4, we see that $I'' \leq \mathcal{I}''$; to see that $I \leq \mathcal{I}''$, just observe that

$$\begin{aligned}
\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes\delta \hookrightarrow\ & \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \\
& \times\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{-a_--1}{2}}\rho\right]\right) \rtimes \delta \\
=\ & \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right) \\
& \times\delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \delta,
\end{aligned}$$

or $\mathcal{I}''$. To prove the claim, it suffices to show $\delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^2 \otimes \delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho]) \rtimes \delta$ appears with multiplicity one in $\mu^*(I)$, $\mu^*(I'')$, and $\mu^*(\mathcal{I}'')$. Again, this is a straightforward μ^* argument.

For (1), it suffices to show that $T' \cong T''$. To do so, it is enough to show $\mu^*_{[\delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^2]}(T') \neq 0$. A straightforward μ^* argument shows

$$\begin{aligned}
\mu^*_{\delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])}\left(\delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \delta'\right) =\ & \delta\left(\left[\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \\
& \otimes \left(\delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \delta'\right).
\end{aligned}$$

As noted above, we have $\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta'$ irreducible. It follows from a μ^* calculation and Lemma 1.2.2 that

$$\begin{aligned}
\delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \delta' \hookrightarrow\ & \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right)^2 \\
& \rtimes \left(\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right) \rtimes \delta'\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
T' \hookrightarrow\ & \delta\left(\left[\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right)^2 \\
& \rtimes \left(\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right) \rtimes \delta'\right).
\end{aligned}$$

Now, observe that by the description of δ' above—noting that $\varepsilon'(\rho, c)\varepsilon'(\rho, a_-)^{-1} = 1$, we have

$$\delta' \hookrightarrow \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \theta$$

for some irreducible θ. Since $\delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho]) \rtimes \delta'$ is irreducible, we see that for the appropriate standard Levi factor M,

$$r_{M,G}\left(\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right) \rtimes \delta'\right) \geq \nu^{\frac{c-1}{2}}\rho \otimes \nu^{\frac{c-3}{2}}\rho \otimes \cdots \otimes \nu^{\frac{a_-+1}{2}}\rho$$
$$\otimes\delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right) \rtimes \theta$$
$$\Downarrow \text{ (Lemmas 1.2.2 and 1.2.3)}$$
$$\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right) \rtimes \delta' \hookrightarrow \nu^{\frac{c-1}{2}}\rho \times \nu^{\frac{c-3}{2}}\rho \times \cdots \times \nu^{\frac{a_-+1}{2}}\rho \rtimes \theta'$$

for some irreducible θ'. By Lemma 1.2.3, it follows that

$$\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right) \rtimes \delta' \hookrightarrow \lambda \rtimes \theta'$$

for some irreducible $\lambda \leq \nu^{\frac{c-1}{2}}\rho \times \nu^{\frac{c-3}{2}}\rho \times \cdots \times \nu^{\frac{a_-+1}{2}}\rho$. Since $\mu^*_{\nu^x\rho}(\delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho]) \rtimes \delta') = 0$ for all $x \in \{\frac{c-3}{2}, \ldots, \frac{a_-+1}{2}\}$, we see that $\lambda = \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])$, i.e.,

$$\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right) \rtimes \delta' \hookrightarrow \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \theta'.$$

Therefore,

$$T' \hookrightarrow \delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \times \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^3 \rtimes \theta'$$
$$\Downarrow \text{ (Lemma 1.2.3)}$$
$$T' \hookrightarrow \lambda \rtimes \theta'$$

for some irreducible $\lambda \leq \delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \times \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^3$. It follows from Lemma 1.3.1 [17], e.g., that $\lambda = \mathcal{L}\left(\delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^3, \delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])\right)$ or $\delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^2 \times \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])$ (irreducible by Zelevinsky [36]). (In fact, one can show each appears with multiplicity one, but that is not needed in what follows.) Since $m^*_{[\delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^2]}(\lambda) \neq 0$ in either case, we get

$$\mu^*_{[\delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^2]}(T') \neq 0,$$

as needed.

For (2), it suffices to show $T' \ncong T''$. To do so, it is enough to show that $\mu^*_{\delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])}(T'') = 0$. For this, it is enough to show $\mu^*_{\delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])}(I'') = 0$. This follows directly if we show $\mu^*_{\delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])}(\delta'') = 0$. However, since $\varepsilon''(\rho, a)\varepsilon''(\rho, c)^{-1} = -1$, this holds. □

We now proceed to parameterize the components of $\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta$ in a manner which extends the Mœglin–Tadić classification of discrete series. In lieu of a triple $(Jord, \sigma, \varepsilon)$, we use a quadruple $(Jord, \sigma, \varepsilon, m)$, where $m : Jord \longrightarrow$

$\mathbb{N}$ represents the multiplicity (with the obvious interpretation of $m = 0$ when appropriate). (This seems to be a bit more convenient than allowing *Jord* to be a multiset.) This covers discrete series in the obvious way—one takes $m(\rho, a) = 1$ for all $(\rho, a) \in Jord$.

Let $\delta = \delta_{(Jord,\sigma,\varepsilon)}$ and write $\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta = T_1 \oplus T_{-1}$. If $(Jord_i, \sigma_i, \varepsilon_i, m_i)$ is the quadruple attached to T_i, we have

$$Jord_i = Jord \cup \{(\rho, c)\}$$

and

$$\sigma_i = \sigma.$$

The multiplicity is given by

$$m_i(\rho', x) = \begin{cases} 1 & \text{if } (\rho', x) \in Jord(\delta) \\ 2 & \text{if } (\rho', x) = (\rho, c). \end{cases}$$

It remains to describe ε_i for the two components.

The domain S_i of ε_i is like that for S (the domain for ε):

$$S_i = \{(\rho', x) \in Jord_i \mid red(\rho'; \sigma) \notin \mathbb{N}\} \cup \{((\rho', x), (\rho'', y)) \in Jord_i \times Jord_i \mid \rho' \cong \rho'' \text{ and } x \neq y\}.$$

Note that $S \subset S_i$. We then take ε_i to be the extension of ε defined as follows (retaining the convention of writing $\varepsilon((\rho, a), (\rho, b))$ as $\varepsilon(\rho, a)\varepsilon(\rho, b)^{-1}$):

(1) $Jord_\rho(\delta) \neq \emptyset$
Suppose (ρ, a) (resp., (ρ, b)) is the element in $Jord_\rho(\delta)$ with $a > c$ (resp., $b < c$) having a minimal (resp., b maximal), noting that not both need exist. We define (cf. Lemma 2.2.1)

$$\varepsilon_i(\rho, c)\varepsilon_i(\rho, a)^{-1} = 1 \Leftrightarrow \mu^*_{\delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])}(T_i) \neq 0$$

and

$$\varepsilon_i(\rho, c)\varepsilon_i(\rho, b)^{-1} = 1 \Leftrightarrow \mu^*_{[\delta([\nu^{\frac{b+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^2]}(T_i) \neq 0.$$

Note that by Lemma 2.2.3, if both (ρ, a) and (ρ, b) exist, this is well-defined. For $d \neq a, b, c$, we then set

$$\varepsilon_i(\rho, c)\varepsilon_i(\rho, d)^{-1} = \begin{cases} \varepsilon_i(\rho, c)\varepsilon_i(\rho, a)^{-1} \cdot \varepsilon_i(\rho, a)\varepsilon_i(\rho, d)^{-1} & \text{if } (\rho, a) \text{ defined;} \\ \varepsilon_i(\rho, c)\varepsilon_i(\rho, b)^{-1} \cdot \varepsilon_i(\rho, b)\varepsilon_i(\rho, d)^{-1} & \text{if } (\rho, b) \text{ defined,} \end{cases}$$

noting that well-definedness is a straightforward consequence of Lemma 2.2.3. For any d (i.e., including a and b), we then define

$$\varepsilon_i(\rho, d)\varepsilon_i(\rho, c)^{-1} = \varepsilon_i(\rho, c)\varepsilon_i(\rho, d)^{-1}.$$

If $(\rho, c) \in S_i$, we set

$$\varepsilon_i(\rho, c) = \begin{cases} \varepsilon_i(\rho, c)\varepsilon_i(\rho, a)^{-1} \cdot \varepsilon_i(\rho, a) & \text{if } (\rho, a) \text{ defined;} \\ \varepsilon_i(\rho, c)\varepsilon_i(\rho, b)^{-1} \cdot \varepsilon_i(\rho, b) & \text{if } (\rho, b) \text{ defined,} \end{cases}$$

again noting that well-definedness is a straightforward consequence of Lemma 2.2.3.

(2) $Jord_\rho(\delta) = \emptyset$
Note that this can occur only when $red(\rho; \sigma) = 0$ or $\frac{1}{2}$. If $red(\rho; \sigma) = \frac{1}{2}$, we define $\varepsilon_i(\rho, c) = 1$ if $\mu^*_{[\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^2]}(T_i) \neq 0$ (see Lemma 2.2.2). If $red(\rho; \sigma) = 0$, we define $\varepsilon_i(\rho, c) = 1$ if $\mu^*(T_i)$ contains a term of the form $\cdots \otimes T(\delta([\nu\rho, \nu^{\frac{c-1}{2}}\rho])^2; \tau_{+1}(\rho; \sigma))$ (cf. Lemma 2.2.2).

Remark 2.2.4. The definition in (2) when $red(\rho; \sigma) = \frac{1}{2}$ is equivalent to taking $\varepsilon(\rho, 0) = 1$ and using Eq. (1.1) (cf. p728 [21]).

We now observe the following:

Lemma 2.2.5. *With the definition above, properties (i)–(iii) of Eq. (1.2) hold.*

Proof. That (iii) holds is part of the definition. That (i) and (ii) hold follow from straightforward arguments using the corresponding properties for discrete series. E.g., if (ρ, b) exists, one can show (i) holds as follows:

$$\begin{aligned} \varepsilon_i(\rho, c)\varepsilon_i(\rho, d)^{-1} &= \varepsilon_i(\rho, c)\varepsilon_i(\rho, b)^{-1} \cdot \varepsilon_i(\rho, b)\varepsilon_i(\rho, d)^{-1} \\ &= \varepsilon_i(\rho, c)\varepsilon_i(\rho, b)^{-1} \cdot \varepsilon(\rho, b) \cdot \varepsilon(\rho, d)^{-1} \\ &= \varepsilon_i(\rho, c) \cdot \varepsilon_i(\rho, d)^{-1}, \end{aligned}$$

as needed. □

Note 2.2.6. *We can characterize $Jord_i$ as in [20]: it follows from [6,7] that if (ρ', x) has x of correct parity, then $(\rho', x) \in Jord_i$ if and only if $\delta([\nu^{\frac{-x+1}{2}}\rho', \nu^{\frac{x-1}{2}}\rho']) \rtimes T_i$ is irreducible.*

Corollary 2.2.7. *Suppose T is as above and $(\rho', a), (\rho', a_-) \in Jord$. Then,*

$$\mu^*_{\delta([\nu^{\frac{a_-+1}{2}}\rho', \nu^{\frac{a-1}{2}}\rho'])}(T) \neq 0 \Leftrightarrow T \hookrightarrow \delta([\nu^{\frac{a_-+1}{2}}\rho', \nu^{\frac{a-1}{2}}\rho']) \rtimes \theta$$

for some irreducible θ, and

$$\mu^*_{[\delta([\nu^{\frac{a_-+1}{2}}\rho', \nu^{\frac{a-1}{2}}\rho'])^2]}(T) \neq 0 \Leftrightarrow T \hookrightarrow \delta([\nu^{\frac{a_-+1}{2}}\rho', \nu^{\frac{a-1}{2}}\rho'])^2 \rtimes \theta'$$

for some irreducible θ'.

Proof. The argument is essentially the as in the proof of Lemma 1.3.3. We omit the details. □

We close by recording the following lemma, to be used later in the paper.

Lemma 2.2.8. *Suppose* $red(\rho;\sigma)=0$ *and* $Jord_\rho(\delta)=\emptyset$. *If* $\delta([\nu^{\frac{-c+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho])\rtimes\delta$ *reduces and* $T\le\delta([\nu^{\frac{-c+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho])\rtimes\delta$, *then* $\mu^*(T)$ *contains a term of the form* $\lambda\otimes T(\delta([\nu\rho,\nu^{\frac{c-1}{2}}\rho])^2;\tau_\eta(\rho;\sigma))$ *if and only if* $T\hookrightarrow\lambda'\rtimes T(\delta([\nu\rho,\nu^{\frac{c-1}{2}}\rho])^2;\tau_\eta(\rho;\sigma))$ *for some irreducible* λ'.

Proof. The implication ($\Leftarrow$) is immediate from Frobenius reciprocity. In the other direction, suppose μ^* contains a term of the form $\lambda\otimes T(\delta([\nu\rho,\nu^{\frac{c-1}{2}}\rho])^2;\tau_\eta(\rho;\sigma))$. Then, for the appropriate standard Levi M, we have

$$r_{M,G}(T)\ge\nu^{x_1}\rho_1\otimes\cdots\otimes\nu^{x_m}\rho_m\otimes(\nu^{\frac{c-1}{2}}\rho\otimes\nu^{\frac{c-1}{2}}\rho)\otimes(\nu^{\frac{c-3}{2}}\rho\otimes\nu^{\frac{c-3}{2}}\rho)\otimes\cdots\otimes(\nu\rho\otimes\nu\rho)\otimes\tau_\eta(\rho;\sigma)$$
$$\Downarrow\ \text{(Lemmas 1.2.2 and 1.2.3)}$$
$$T\hookrightarrow\nu^{x_1}\rho_1\times\cdots\times\nu^{x_m}\rho_m\rtimes\left(\nu^{\frac{c-1}{2}}\rho\times\nu^{\frac{c-1}{2}}\rho\times\nu^{\frac{c-3}{2}}\rho\times\nu^{\frac{c-3}{2}}\rho\times\cdots\times\nu\rho\times\nu\rho\rtimes\tau_{\eta'}(\rho;\sigma)\right),$$

with $\rho_i\not\cong\rho$ for all i. By Lemma 2.2.2, we must have $\eta'=\eta$. Therefore,

$$T\hookrightarrow\nu^{x_1}\rho_1\times\cdots\times\nu^{x_m}\rho_m\rtimes\left(\nu^{\frac{c-1}{2}}\rho\times\nu^{\frac{c-1}{2}}\rho\times\nu^{\frac{c-3}{2}}\rho\times\nu^{\frac{c-3}{2}}\rho\times\cdots\times\nu\rho\times\nu\rho\rtimes\tau_\eta(\rho;\sigma)\right)$$
$$\Downarrow\ \text{(Lemma 1.2.3)}$$
$$T\hookrightarrow\nu^{x_1}\rho_1\times\cdots\times\nu^{x_m}\rho_m\rtimes\theta$$

for some irreducible $\theta\le\nu^{\frac{c-1}{2}}\rho\times\nu^{\frac{c-1}{2}}\rho\times\nu^{\frac{c-3}{2}}\rho\times\nu^{\frac{c-3}{2}}\rho\times\cdots\times\nu\rho\otimes\nu\rho\rtimes\tau_\eta(\rho;\sigma)$. If $\theta\ne T(\delta([\nu\rho,\nu^{\frac{c-1}{2}}\rho])^2;\tau_\eta(\rho;\sigma))$, then $r_{min}(\theta)\not\ge\nu^{\frac{c-1}{2}}\rho\otimes\nu^{\frac{c-1}{2}}\rho\otimes\nu^{\frac{c-3}{2}}\rho\otimes\nu^{\frac{c-3}{2}}\rho\otimes\cdots\otimes\nu\rho\otimes\nu\rho\otimes\rho\otimes\sigma$. Since $\rho_i\not\cong\rho$ for all i, this would imply $r_{min}(T)\not\ge\nu^{x_1}\rho_1\otimes\cdots\otimes\nu^{x_m}\rho_m\otimes(\nu^{\frac{c-1}{2}}\rho\otimes\nu^{\frac{c-1}{2}}\rho)\otimes(\nu^{\frac{c-3}{2}}\rho\otimes\nu^{\frac{c-3}{2}}\rho)\otimes\cdots\otimes(\nu\rho\otimes\nu\rho)\otimes\rho\otimes\sigma$, a contradiction. Thus,

$$T\hookrightarrow\nu^{x_1}\rho_1\times\cdots\times\nu^{x_m}\rho_m\rtimes T(\delta([\nu\rho,\nu^{\frac{c-1}{2}}\rho])^2;\tau_\eta(\rho;\sigma))$$
$$\Downarrow\ \text{(Lemma 1.2.3)}$$
$$T\hookrightarrow\lambda'\rtimes T(\delta([\nu\rho,\nu^{\frac{c-1}{2}}\rho])^2;\tau_\eta(\rho;\sigma)),$$

as needed. □

2.3. Elliptic tempered representations

In this section, we define the quadruple associated to an irreducible elliptic tempered representation, building up from the results of the previous section. Key properties—the analogues of Eqs. (1.3), (1.4), and (1.5)—are also given, appearing as Proposition 2.3.2, Lemma 2.3.5, and Lemma 2.3.7, respectively. Once the Mœglin–Tadić classification has been extended to elliptic tempered representations, it is then a fairly straightforward matter to extend it to general tempered representations, which is done in Sect. 2.4.

Recall that if T is an elliptic tempered representation (in the symplectic, odd special orthogonal, or unitary case), we have

$$T\hookrightarrow\delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1,\nu^{\frac{a_1-1}{2}}\rho_1\right]\right)\times\cdots\times\delta\left(\left[\nu^{\frac{-a_k+1}{2}}\rho_k,\nu^{\frac{a_k-1}{2}}\rho_k\right]\right)\rtimes\delta$$

where $\delta([\nu^{\frac{-a_i+1}{2}}\rho_i, \nu^{\frac{a_i-1}{2}}\rho_i]) \rtimes \delta$ is reducible for all i, and $\delta([\nu^{\frac{-a_i+1}{2}}\rho_i, \nu^{\frac{a_i-1}{2}}\rho_i]) \not\cong \delta([\nu^{\frac{-a_j+1}{2}}\rho_j, \nu^{\frac{a_j-1}{2}}\rho_j])$ for $i \neq j$. Note that this requires $\check{\rho}_i \cong \rho_i$ for all i. For notational convenience, we also call a component of such an induced representation of $O(2n, F)$ elliptic.

Suppose $(Jord(\delta), \sigma, \varepsilon)$ is the Mœglin–Tadić triple for δ. We now give the data $(Jord(T), \sigma, \varepsilon_T, m_T)$ for T. Here,

$$Jord(T) = Jord(\delta) \cup \{(\rho_1, a_1), \ldots, (\rho_k, a_k)\},$$

and

$$m_T(\rho, a) = \begin{cases} 1 & \text{if } (\rho, a) \in Jord(\delta), \\ 2 & \text{if not, i.e., } (\rho, a) = (\rho_i, a_i) \text{ for some } i. \end{cases}$$

We note that *Jord(T)* could have been defined in terms of irreducibility as with discrete series—see Remark 2.3.4. It remains to define ε_T.

Let

$$S_T = \{(\rho, a) \in Jord(T) \mid red(\rho; \sigma) \notin \mathbb{N}\} \cup \{((\rho, a), (\rho', a')) \in Jord(T) \times Jord(T) \mid \rho \cong \rho' \text{ and } a \neq a'\},$$

the domain for ε_T. Also, let S_i be the domain needed for $\delta([\nu^{\frac{-a_i+1}{2}}\rho_i, \nu^{\frac{a_i-1}{2}}\rho_i]) \rtimes \delta$. Note that $S_i \subset S_T$ for all i. We define ε_T by first defining $\varepsilon_T|_{S_i}$ for all i.

Fix i and write $\delta([\nu^{\frac{-a_i+1}{2}}\rho_i, \nu^{\frac{a_i-1}{2}}\rho_i]) \rtimes \delta = T_{+1}(\rho_i, a_i; \delta) \oplus T_{-1}(\rho_i, a_i; \delta)$. Since $\delta([\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1]) \times \cdots \times \delta([\nu^{\frac{-a_k+1}{2}}\rho_k, \nu^{\frac{a_k-1}{2}}\rho_k]) \rtimes \delta$ decomposes with multiplicity one, we have

$$\begin{aligned} T \hookrightarrow\ & \delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1\right)\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-a_{i-1}+1}{2}}\rho_{i-1}, \nu^{\frac{a_{i-1}-1}{2}}\rho_{i-1}\right]\right) \\ & \times \delta\left(\left[\nu^{\frac{-a_{i+1}+1}{2}}\rho_{i+1}, \nu^{\frac{a_{i+1}-1}{2}}\rho_{i+1}\right]\right) \times \cdots \\ & \times \delta\left(\left[\nu^{\frac{-a_k+1}{2}}\rho_k, \nu^{\frac{a_k-1}{2}}\rho_k\right]\right) \rtimes T_\xi(\rho_i, a_i; \delta) \end{aligned}$$

for exactly one $\xi \in \{\pm 1\}$. We then define

$$\varepsilon_T|_{S_i} = \varepsilon_{T_\xi(\rho_i, a_i; \delta)},$$

(defined in Sect. 2.2). Since $S_i \cap S_j = S_\delta$ for $i \neq j$, we have ε_T well-defined on $\cup_i S_i$. It remains to define ε_T on the rest of S_T, i.e., define $\varepsilon_T(\rho, a)\varepsilon_T(\rho, b)^{-1}$ when $(\rho, a), (\rho, b) \in Jord(T) \backslash Jord(\delta)$ (again retaining the convention of writing $\varepsilon_T(\rho, a)\varepsilon_T(\rho, b)^{-1}$ for $\varepsilon_T((\rho, a), (\rho, b))$). There are two cases to consider:

(1) $Jord_\rho(\delta) \neq \emptyset$.

We choose $(\rho, a_{min}) \in Jord(\delta)$ having a_{min} minimal and define

$$\begin{aligned} \varepsilon_T(\rho, a)\varepsilon_T(\rho, b)^{-1} &= \varepsilon_T(\rho, b)\varepsilon_T(\rho, a)^{-1} \\ &= \varepsilon_T(\rho, a)\varepsilon(\rho, a_{min})^{-1} \cdot \varepsilon_T(\rho, a_{min})\varepsilon_T(\rho, b), \end{aligned}$$

noting that both terms on the right-hand side are defined (above).

(2) $Jord_\rho(\delta)=\emptyset$
In this case, $red(\rho;\sigma)=0$ or $\frac{1}{2}$, so $\varepsilon_T(\rho,a)$ and $\varepsilon_T(\rho,b)$ have already been defined (above). Then, we set

$$\varepsilon_T(\rho,a)\varepsilon_T(\rho,b)^{-1}=\varepsilon_T(\rho,a)\cdot\varepsilon_T(\rho,b)^{-1}.$$

Lemma 2.3.1. *Properties (i)–(iii) from Eq. (1.2) hold.*

Proof. The proof is a straightforward argument using Lemma 2.2.5 and the definition above. □

Proposition 2.3.2. *Let T be an elliptic tempered representation with associated data $(Jord,\sigma,\varepsilon,m)$ as above. Then, if $(\rho,a),(\rho,a_-)\in Jord$,*

$$\varepsilon(\rho,a)\varepsilon(\rho,a_-)^{-1}=1\Leftrightarrow T\hookrightarrow\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])^{m(\rho,a)}\rtimes\theta$$

for some irreducible θ.

Proof. We break the analysis into four cases based on the multiplicities of (ρ,a) and (ρ,a_-).

Case 1: $(\rho,a),(\rho,a_-)\in Jord(\delta)$ (i.e., $m(\rho,a)=m(\rho,a_-)=1$)
For $(\Rightarrow)$, observe that

$$\begin{aligned}T&\hookrightarrow\delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1,\nu^{\frac{a_1-1}{2}}\rho_1\right]\right)\times\cdots\times\delta([\nu^{\frac{-a_k+1}{2}}\rho_k,\nu^{\frac{a_k-1}{2}}\rho_k])\rtimes\delta\\&\hookrightarrow\delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1,\nu^{\frac{a_1-1}{2}}\rho_1\right]\right)\times\cdots\\&\quad\times\delta\left(\left[\nu^{\frac{-a_k+1}{2}}\rho_k,\nu^{\frac{a_k-1}{2}}\rho_k\right]\right)\times\delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho\right]\right)\rtimes\delta'\end{aligned}$$

since $\varepsilon_\delta(\rho,a)\varepsilon_\delta(\rho,a_-)^{-1}=\varepsilon(\rho,a)\varepsilon(\rho,a_-)^{-1}=1$. Observe that $\delta([\nu^{\frac{-a_i+1}{2}}\rho_i,\nu^{\frac{a_i-1}{2}}\rho_i])\times\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])$ is irreducible for all i—if $\rho_i\ncong\rho$, this is immediate. If $\rho_i\cong\rho$, we cannot have $a_-\le a_i<a$, so also holds. A commuting argument then gives

$$\begin{aligned}T\hookrightarrow\delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho\right]\right)\rtimes\delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1,\nu^{\frac{a_1-1}{2}}\rho_1\right]\right)\times\cdots\\\times\delta\left(\left[\nu^{\frac{-a_k+1}{2}}\rho_k,\nu^{\frac{a_k-1}{2}}\rho_k\right]\right)\rtimes\delta',\end{aligned}$$

and the result now follows from Lemma 1.2.3.
For $(\Leftarrow)$, observe that

$$\begin{gathered}T\hookrightarrow\delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho\right]\right)\rtimes\theta\\\Downarrow\\\mu^*_{\delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho\right]\right)}(T)\neq0\\\Downarrow\\\mu^*_{\delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho_1,\nu^{\frac{a-1}{2}}\rho\right]\right)}\left(\delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1,\nu^{\frac{a_1-1}{2}}\rho_1\right]\right)\times\cdots\right.\\\left.\times\delta\left(\left[\nu^{\frac{-a_k+1}{2}}\rho_k,\nu^{\frac{a_k-1}{2}}\rho_k\right]\right)\rtimes\delta\right)\neq0.\end{gathered}$$

Write $M^*\left(\delta([\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1]) \times \cdots \times \delta([\nu^{\frac{-a_k+1}{2}}\rho_k, \nu^{\frac{a_k-1}{2}}\rho_k])\right) = \sum_j \lambda_j \otimes \lambda'_j$ and $\mu^*(\delta) = \tau_i \otimes \theta_i$. Then,

$$\mu^*\left(\delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta([\nu^{\frac{-a_k+1}{2}}\rho_k, \nu^{\frac{a_k-1}{2}}\rho_k]) \rtimes \delta\right)$$
$$= \sum_i \sum_j \lambda_j \times \tau_i \otimes \lambda'_j \rtimes \theta_i.$$

Observe that since $\nu^x\rho' \otimes \cdots \leq r_{min}(\lambda_j)$ has $\nu^x\rho' \in \{\nu^{\frac{a_1-1}{2}}\rho_1, \ldots, \nu^{\frac{a_k-1}{2}}\rho_k\}$ when $\lambda_j \neq 1$, and $\nu^{\frac{a_i-1}{2}}\rho \notin \{\nu^{\frac{a_-+1}{2}}\rho, \ldots, \nu^{\frac{a-1}{2}}\rho\}$ for any i, we must have $\lambda_j = 1$. Thus to have $\mu^*_{\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])} \neq 0$, we must have $\tau_i = \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])$, i.e., $\mu^*_{\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])}(\delta) \neq 0$. By Lemma 1.3.3, this implies $\varepsilon_\delta(\rho, a)\varepsilon_\delta(\rho, a_-)^{-1} = 1$, hence $\varepsilon(\rho, a)\varepsilon(\rho, a_-)^{-1} = 1$, finishing this case.

Case 2: $(\rho, a) \in Jord(\delta)$, $(\rho, a_-) \notin Jord(\delta)$ (i.e., $m(\rho, a) = 1, m(\rho, a_-) = 2$)

In this case, $(\rho, a_-) \in \{(\rho_1, a_1), \ldots, (\rho_k, a_k)\}$; without loss of generality, suppose $(\rho, a_-) = (\rho_k, a_k)$. Let $T_k \leq \delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho]) \rtimes \delta$ be irreducible with

$$T \hookrightarrow \delta([\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1]) \times \cdots \times \delta([\nu^{\frac{-a_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{a_{k-1}-1}{2}}\rho_{k-1}]) \rtimes T_k.$$

Then, by definition and Corollary 2.2.7,

$$\varepsilon(\rho, a)\varepsilon(\rho, a_-)^{-1} = 1 \Leftrightarrow T_k \hookrightarrow \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \theta$$

for some irreducible θ.

For ($\Rightarrow$), we have

$$\begin{aligned} T &\hookrightarrow \delta([\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1]) \times \cdots \times \delta([\nu^{\frac{-a_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{a_{k-1}-1}{2}}\rho_{k-1}]) \rtimes T_k \\ &\hookrightarrow \delta([\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1]) \times \cdots \times \delta([\nu^{\frac{-a_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{a_{k-1}-1}{2}}\rho_{k-1}]) \\ &\quad \times \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \theta. \end{aligned}$$

Again, $\delta([\nu^{\frac{-a_i+1}{2}}\rho_i, \nu^{\frac{a_i-1}{2}}\rho_i]) \times \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])$ is irreducible for $i \neq k$ (since $\nu^{\frac{a_i-1}{2}}\rho_i \notin \{\nu^{\frac{a_--1}{2}}\rho, \ldots, \nu^{\frac{a-3}{2}}\rho\}$). Thus, a commuting argument gives

$$\begin{aligned} T \hookrightarrow\ & \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \times \delta([\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1]) \times \cdots \\ & \times \delta([\nu^{\frac{-a_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{a_{k-1}-1}{2}}\rho_{k-1}]) \rtimes \theta; \end{aligned}$$

the result now follows immediately from Lemma 1.2.3.

For ($\Leftarrow$), we note that the same argument as in Case 1 gives

$$\mu^*_{\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])}\Big(\delta([\nu^{\frac{-a_1+1}{2}}\rho_1,\nu^{\frac{a_1-1}{2}}\rho_1])\times\cdots \times\delta([\nu^{\frac{a_{k-1}+1}{2}}\rho_{k-1},\nu^{\frac{a_{k-1}-1}{2}}\rho_{k-1}])\rtimes T_k\Big)\neq 0$$
$$\Updownarrow$$
$$\mu^*_{\delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho\right]\right)}(T_k)\neq 0.$$

It now follows immediately that $T\hookrightarrow\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])\rtimes\theta\Rightarrow\mu^*_{\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])}(T_k)\neq 0$, hence $\varepsilon(\rho,a)\varepsilon(\rho,a_-)^{-1}=\varepsilon_k(\rho,a)\varepsilon_k(\rho,a_-)^{-1}=1$, as needed. This finishes Case 2.

Case 3: $(\rho,a_-)\in Jord(\delta)$, $(\rho,a)\notin Jord(\delta)$ (i.e., $m(\rho,a_-)=1, m(\rho,a)=2$)

In this case, $(\rho,a)\in\{(\rho_1,a_1),\ldots,(\rho_k,a_k)\}$; without loss of generality, suppose $(\rho,a)=(\rho_k,a_k)$. Let $T_k\leq\delta([\nu^{\frac{-a+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])\rtimes\delta$ be such that

$$T\hookrightarrow\delta\left(\left[\nu^{\frac{a_1+1}{2}}\rho_1,\nu^{\frac{a_1-1}{2}}\rho_1\right]\right)\times\cdots\times\delta([\nu^{\frac{a_{k-1}+1}{2}}\rho_{k-1},\nu^{\frac{a_{k-1}-1}{2}}\rho_{k-1}])\rtimes T_k.$$

Then, by definition,

$$\varepsilon(\rho,a)\varepsilon(\rho,a_-)^{-1}=1\Leftrightarrow T_k\hookrightarrow\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])^2\rtimes\theta.$$

At this point, the argument is essentially the same as in Case 2, except with $\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])^2$ used in place of $\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])$.

Case 4: $(\rho,a),(\rho,a_-)\notin Jord(\delta)$ (i.e., $m(\rho,a)=m(\rho,a_-)=2$)

In this case, we have $(\rho,a),(\rho,a_-)\in\{(\rho_1,a_1),\ldots,(\rho_k,a_k)\}$; without loss of generality, suppose $(\rho,a)=(\rho_{k-1},a_{k-1})$ and $(\rho,a_-)=(\rho_k,a_k)$. By Lemma 1.2.3, we have

$$T\hookrightarrow\delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1,\nu^{\frac{a_1-1}{2}}\rho_1\right]\right)\times\cdots\times\delta\left(\left[\nu^{\frac{-a_{k-2}+1}{2}}\rho_{k-2},\nu^{\frac{a_{k-2}-1}{2}}\rho_{k-2}\right]\right)\rtimes T'$$

for some irreducible $T'\leq\delta([\nu^{\frac{-a+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])\times\delta([\nu^{\frac{-a_-+1}{2}}\rho,\nu^{\frac{a_--1}{2}}\rho])\rtimes\delta$.

First, we claim $T\hookrightarrow\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])^2\rtimes\theta$ for some irreducible θ if and only if $T'\hookrightarrow\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])^2\rtimes\theta'$ for some irreducible θ'. For ($\Leftarrow$), observe that

$$T'\hookrightarrow\delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho\right]\right)^2\rtimes\theta'$$
$$\Downarrow$$
$$\begin{aligned}T&\hookrightarrow\delta([\nu^{\frac{-a_1+1}{2}}\rho_1,\nu^{\frac{a_1-1}{2}}\rho_1])\times\cdots\times\delta([\nu^{\frac{-a_{k-2}+1}{2}}\rho_{k-2},\nu^{\frac{a_{k-2}-1}{2}}\rho_{k-2}])\\&\quad\times\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])^2\rtimes\theta'\\&\cong\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])^2\times\delta([\nu^{\frac{-a_1+1}{2}}\rho_1,\nu^{\frac{a_1-1}{2}}\rho_1])\times\cdots\\&\quad\times\delta([\nu^{\frac{-a_{k-2}+1}{2}}\rho_{k-2},\nu^{\frac{a_{k-2}-1}{2}}\rho_{k-2}])\rtimes\theta',\end{aligned}$$

with the irreducibility of $\delta([\nu^{\frac{-a_i+1}{2}}\rho_i, \nu^{\frac{a_i-1}{2}}\rho_i]) \times \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])$ as above. The implication ($\Leftarrow$) now follows from Lemma 1.2.3. For ($\Rightarrow$), observe that

$$\begin{gathered} T \hookrightarrow \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])^2 \rtimes \theta \\ \Downarrow \\ \mu^*_{[\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])^2]}\Big(\delta([\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1]) \times \cdots \\ \times\delta([\nu^{\frac{-a_{k-2}+1}{2}}\rho_{k-2}, \nu^{\frac{a_{k-2}-1}{2}}\rho_{k-2}]) \rtimes T'\Big) \neq 0. \end{gathered}$$

The same considerations as above tell us $M^*(\delta([\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1]) \times \cdots \times \delta([\nu^{\frac{-a_{k-2}+1}{2}}\rho_{k-2}, \nu^{\frac{a_{k-2}-1}{2}}\rho_{k-2}]))$ cannot contribute, so $\mu^*_{[\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])^2]}(T') \neq 0$. That $T' \hookrightarrow \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])^2 \rtimes \theta$ for some irreducible θ now follows as in the proof of Corollary 2.2.7.

Next, we claim $\varepsilon_T(\rho, a)\varepsilon_T(\rho, a_-)^{-1} = 1$ if and only if $\varepsilon_{T'}(\rho, a)\varepsilon_{T'}(\rho, a_-)^{-1} = 1$. To see this, observe that for $T_a \leq \delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \delta$, we have

$$\begin{gathered} T \hookrightarrow \delta([\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1]) \times \cdots \\ \times\delta([\nu^{\frac{-a_{k-2}+1}{2}}\rho_{k-2}, \nu^{\frac{a_{k-2}-1}{2}}\rho_{k-2}]) \times \delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho]) \rtimes T_a \\ \Updownarrow \\ T' \hookrightarrow \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right) \rtimes T_a. \end{gathered}$$

If $Jord_\rho(\delta) \neq \emptyset$, it now follows from the definition that $\varepsilon_T(\rho, a)\varepsilon_T(\rho, b)^{-1} = \varepsilon_{T_a}(\rho, a)\varepsilon_{T_a}(\rho, b)^{-1}$ and $\varepsilon_{T'}(\rho, a)\varepsilon_{T'}(\rho, b)^{-1} = \varepsilon_{T_a}(\rho, a)\varepsilon_{T_a}(\rho, b)^{-1}$ for any $(\rho, b) \in Jord(\delta)$. A similar argument shows $\varepsilon_T(\rho, a_-)\varepsilon_T(\rho, b)^{-1} = \varepsilon_{T'}(\rho, a_-)\varepsilon_{T'}(\rho, b)^{-1}$. Coupled with the multiplicative properties of ε (Lemma 2.3.1), we now see $\varepsilon_T(\rho, a)\varepsilon_T(\rho, a_-)^{-1} = \varepsilon_{T'}(\rho, a)\varepsilon_{T'}(\rho, a_-)^{-1}$, as claimed. If $Jord_\rho(\delta) = \emptyset$, then $\varepsilon_T(\rho, a)$ and $\varepsilon_T(\rho, a_-)$ are defined. Again, one has $\varepsilon_T(\rho, a) = \varepsilon_{T_a}(\rho, a)$ and $\varepsilon_{T'}(\rho, a) = \varepsilon_{T_a}(\rho, a)$ as above (clear from the definition if $red(\rho; \sigma) \equiv \frac{1}{2} \bmod 1$ and a short, straightforward argument if $red(\rho; \sigma) = 0$). A similar argument applies to a_-, from which the claim follows.

Combining the claims above, it is enough to prove the proposition for T', which is covered by Lemma 2.3.3 below. □

Lemma 2.3.3. *Let* $T \leq \delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \times \delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho]) \rtimes \delta$ *be elliptic tempered as above. Then,*

$$\varepsilon_T(\rho, a)\varepsilon_T(\rho, a_-)^{-1} = 1 \Leftrightarrow \mu^*_{[\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])^2]}(T) \neq 0.$$

Proof. As a bookkeeping convenience, write (for $c = a$ or a_-)

$$\delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \delta = T_{+1}(\rho, c; \delta) \oplus T_{-1}(\rho, c; \delta),$$

where $T_\eta(\rho, c; \delta)$ is chosen as follows:

(1) If there is a $(\rho, b) \in Jord(\delta)$ with $b < a_-$, choose b maximal (so $b = (a_-)_-$ in *Jord(T)*) and define $T_\eta(\rho, c; \delta)$ by $\varepsilon_{T_\eta(\rho,c;\delta)}(\rho, b)\varepsilon_{T_\eta(\rho,c;\delta)}(\rho, c)^{-1} = \eta$.
(2) If (1) fails but there is a $(\rho, b) \in Jord(\delta)$ with $b > a$, choose b minimal (so $b = a_+$ in *Jord(T)*) and define $T_\eta(\rho, c; \delta)$ by $\varepsilon_{T_\eta(\rho,c;\delta)}(\rho, b)\varepsilon_{T_\eta(\rho,c;\delta)}(\rho, c)^{-1} = \eta$.
(3) If both (1) and (2) fail, then $Jord(\delta) = \emptyset$. We then make our choice so that $\varepsilon_{T_\eta(\rho,c;\delta)}(\rho, c) = \eta$ (cf. Sect. 2.2).

Observe that

$$\begin{aligned}
&\delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right) \rtimes \delta \\
&= \sum_{\eta\in\{\pm 1\}} \delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes T_\eta(\rho, a_-; \delta) \\
&= \sum_{\eta\in\{\pm 1\}} \delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho]) \rtimes T_\eta(\rho, a; \delta).
\end{aligned}$$

Now, observe that

$$\begin{aligned}
&\mu^*_{[\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])^2]}\left(\delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes T_\eta(\rho, a_-; \delta)\right) \\
&= \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])^2 \otimes \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right) \rtimes T_\eta(\rho, a_-; \delta)
\end{aligned}$$

(noting that there is no $(\rho, b) \in Jord(\delta)$ such that $a_- \leq b \leq a$). Next, observe that $\delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho]) \rtimes T_\eta(\rho, a_-; \delta)$ is irreducible (see Note 2.2.6), so exactly one component of $\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes T_\eta(\rho, a_-; \delta)$ has $\mu^*_{[\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])^2]} \neq 0$.

To prove the corresponding statement for $\delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho]) \rtimes T_\eta(\rho, a; \delta)$, we first claim that

$$\mu^*_{[\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])^2]}(T_\eta(\rho, a; \delta)) = \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])^2 \otimes T_\eta(\rho, a_-; \delta).$$

It follows directly from

$$\begin{aligned}
&\mu^*_{[\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])^2]}\left(\delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \delta\right) = \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)^2 \\
&\otimes \left(\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right) \rtimes \delta\right)
\end{aligned}$$

that

$$\mu^*_{[\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])^2]}(T_\zeta(\rho, a; \delta)) = \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])^2 \otimes T_\eta(\rho, a_-; \delta)$$

for some η. To show $\eta = \zeta$, we consider three cases corresponding to (1),(2),(3) above (noting that (3) gives rise to two subcases: $red(\rho; \sigma) = \frac{1}{2}$ and $red(\rho; \sigma) = 0$). These are fairly straightforward; the details are omitted.

Now, it follows from a straightforward μ^* argument that

$$\mu^*_{[\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])^2]}\left(\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho,\nu^{\frac{a_--1}{2}}\rho\right]\right)\rtimes T_\eta(\rho,a;\delta)\right)$$
$$=\delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho\right]\right)^2\otimes\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho,\nu^{\frac{a_--1}{2}}\rho\right]\right)\rtimes T_\eta(\rho,a_-;\delta).$$

It then follows that $\delta([\nu^{\frac{-a_-+1}{2}}\rho,\nu^{\frac{a_--1}{2}}\rho])\rtimes T_\eta(\rho,a;\delta)$ and $\delta([\nu^{\frac{-a+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])\rtimes T_\eta(\rho,a_-;\delta)$, $\eta=\pm1$, have a common component π_η characterized by having

$$\mu^*_{[\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])^2]}(\pi_\eta)$$
$$=\delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho\right]\right)^2\otimes\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho,\nu^{\frac{a_--1}{2}}\rho\right]\right)\rtimes T_\eta(\rho,a_-;\delta);$$

in particular, having $\mu^*_{[\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])^2]}\neq 0$. The common components of $\delta([\nu^{\frac{-a_-+1}{2}}\rho,\nu^{\frac{a_--1}{2}}\rho])\rtimes T_\eta(\rho,a;\delta)$ and $\delta([\nu^{\frac{-a+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])\rtimes T_{-\eta}(\rho,a_-;\delta)$ then have $\mu^*_{[\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])^2]}=0$. □

Remark 2.3.4. Note that it follows from [6,7] that we may characterize *Jord(T)* as in [20]: it consists of all pairs (ρ,a) with $\check{\rho}\cong\rho$ and $a\equiv 2red(\rho;\sigma)+1 \bmod 2$ such that $\delta([\nu^{\frac{-a+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])\rtimes T$ is irreducible.

Lemma 2.3.5. *Suppose T is elliptic tempered with $(\rho,a_{min})\in Jord(T)$ and a_{min} the smallest value of a having $(\rho,a)\in Jord(T)$. Further, suppose $red(\rho;\sigma)\equiv\frac{1}{2} \bmod 1$ (so that $\varepsilon(\rho,a_{min})$ is defined). Then,*

$$\varepsilon(\rho,a_{min})=1\Leftrightarrow T\hookrightarrow\delta([\nu^{\frac{1}{2}}\rho,\nu^{\frac{a_{min}-1}{2}}\rho])^{m(\rho,a_{min})}\rtimes\theta$$

for some irreducible θ.

Proof. First, suppose $m(\rho,a_{min})=1$. In this case, the proof is like that in Case 1 of the proof of Proposition 2.3.2 (if one takes $a_-=0$ and $\varepsilon(\rho,0)=1$).

Suppose $m(\rho,a_{min})=2$. If $Jord_\rho(\delta)=\emptyset$, the claim follows immediately from the definition of ε_T (see Sect. 2.3). Thus, we assume $Jord_\rho(\delta)\neq\emptyset$. By the same argument as in Corollary 2.2.7, we have

$$T\hookrightarrow\delta\left(\left[\nu^{\frac{1}{2}}\rho,\nu^{\frac{a_{min}-1}{2}}\rho\right]\right)^2\rtimes\theta\Leftrightarrow\mu^*_{[\delta([\nu^{\frac{1}{2}}\rho,\nu^{\frac{a_{min}-1}{2}}\rho])^2]}(T)\neq 0.$$

Therefore, it suffices to show

$$\varepsilon_T(\rho,a_{min})=1\Leftrightarrow\mu^*_{[\delta([\nu^{\frac{1}{2}}\rho,\nu^{\frac{a_{min}-1}{2}}\rho])^2]}(T)\neq 0.$$

Now, write

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-a_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{a_{k-1}-1}{2}}\rho_{k-1}\right]\right)$$
$$\times \delta([\nu^{\frac{-a_{min}+1}{2}}\rho, \nu^{\frac{a_{min}-1}{2}}\rho]) \rtimes \delta$$
$$\Downarrow$$
$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1\right]\right) \times \cdots$$
$$\times \delta([\nu^{\frac{-a_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{a_{k-1}-1}{2}}\rho_{k-1}]) \rtimes T'$$

for some irreducible $T' \leq \delta([\nu^{\frac{-a_{min}+1}{2}}\rho, \nu^{\frac{a_{min}-1}{2}}\rho]) \rtimes \delta$. By definition, $\varepsilon_T(\rho, a_{min}) = \varepsilon_{T'}(\rho, a_{min})$. A straightforward μ^* analysis and commuting argument tell us

$$\mu^*_{\left[\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{a_{min}-1}{2}}\rho])^2\right]}(T) \neq 0 \Leftrightarrow \mu^*_{[\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{a_{min}-1}{2}}\rho])^2]}(T') \neq 0.$$

Thus, we are reduced to showing

$$\varepsilon_{T'}(\rho, a_{min}) = 1 \Leftrightarrow \mu^*_{\left[\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{a_{min}-1}{2}}\rho])^2\right]}(T') \neq 0.$$

Now, since a_{min} is minimal in $Jord_\rho(T')$ and $Jord_\rho(\delta) \neq \emptyset$, we have $(\rho, a) \in Jord(\delta)$ with a minimal (so that in $Jord_\rho(T')$, $a_- = a_{min}$). Since $m(\rho, a) = 1$, it follows from Proposition 2.3.2 and Corollary 2.2.7

$$\varepsilon_{T'}(\rho, a_{min}) = \varepsilon_{T'}(\rho, a) \Leftrightarrow \mu^*_{\delta([\nu^{\frac{a_{min}+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])}(T') \neq 0.$$

At this point, we claim that if $\varepsilon_{T'}(\rho, a) = 1$,

$$\mu^*_{\left[\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{a_{min}-1}{2}}\rho])^2\right]}(T') \neq 0 \Leftrightarrow \mu^*_{\delta([\nu^{\frac{a_{min}+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])}(T') \neq 0$$

and if $\varepsilon_{T'}(\rho, a) = -1$,

$$\mu^*_{\left[\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{a_{min}-1}{2}}\rho])^2\right]}(T') \neq 0 \Leftrightarrow \mu^*_{\delta([\nu^{\frac{a_{min}+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])}(T') = 0,$$

from which the lemma then follows. The proof of this is essentially the same as that in Lemma 2.2.3—formally take $(\rho, 0)$ for (ρ, a_-) there, with $\varepsilon(\rho, a_-) = 1$. The degeneracy of $\delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho])$ simplifies part of the proof; the rest is the same. □

Definition 2.3.6. Suppose $red(\rho; \sigma) = 0$. For $m = 1$ or 2, set

$$T(\delta([\nu\rho, \nu^{\frac{b-1}{2}}\rho])^m; \tau_\eta(\rho; \sigma)) = \begin{cases} \delta([\nu\rho, \nu^{\frac{b-1}{2}}\rho]; \tau_\eta(\rho; \sigma)) & \text{if } m = 1, \\ T(\delta([\nu\rho, \nu^{\frac{b-1}{2}}\rho])^2; \tau_\eta(\rho; \sigma)) & \text{if } m = 2, \end{cases}$$

with the representations on the right defined in Sect. 1.3 and Lemma 2.2.2, resp.

Lemma 2.3.7. *Suppose $red(\rho;\sigma) = 0$. Let T be an elliptic tempered representation with $Jord_\rho(T) \neq 0$. Let a_{max} be the largest value of a such that $(\rho, a) \in Jord(T)$. Then,*

$$\varepsilon_T(\rho, a_{max}) = \eta \Leftrightarrow T \hookrightarrow \lambda \rtimes T(\delta([\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho])^{m(\rho,a_{max})}; \tau_\eta(\rho;\sigma))$$

for some irreducible λ.

Proof. First, suppose $m(\rho, a_{max}) = 2$ and $Jord_\rho(\delta) \neq \emptyset$. We have

$$T \hookrightarrow \underbrace{\delta([\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1]) \times \cdots \times \delta([\nu^{\frac{-a_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{a_{k-1}-1}{2}}\rho_{k-1}])}_{\lambda} \times \delta([\nu^{\frac{-a_{max}+1}{2}}\rho, \nu^{\frac{a_{max}-1}{2}}\rho]) \rtimes \delta$$
$$\Downarrow$$
$$T \hookrightarrow \lambda \rtimes T'$$

for some component $T' \leq \delta([\nu^{\frac{-a_{max}+1}{2}}\rho, \nu^{\frac{a_{max}-1}{2}}\rho]) \rtimes \delta$. By definition, $\varepsilon_T(\rho, a_{max}) = \varepsilon_{T'}(\rho, a_{max})$. Further, if c_{max} is the largest value of c such that $(\rho, c_{max}) \in Jord(\delta)$, we also have $\varepsilon_T(\rho, c_{max}) = \varepsilon_{T'}(\rho, c_{max})$ (both matching $\varepsilon_\delta(\rho, c_{max})$). Now, by Tadić's characterization, letting $\eta = \varepsilon_\delta(\rho, c_{max}) = \varepsilon_T(\rho, c_{max})$,

$$\begin{aligned}
&\delta \hookrightarrow \lambda' \rtimes \delta([\nu\rho, \nu^{\frac{c_{max}-1}{2}}\rho]); \tau_\eta(\rho;\sigma)) \\
&\Downarrow \\
T' &\hookrightarrow \delta([\nu^{\frac{-a_{max}+1}{2}}\rho, \nu^{\frac{a_{max}-1}{2}}\rho]) \times \lambda' \rtimes \delta([\nu\rho, \nu^{\frac{c_{max}-1}{2}}\rho]); \tau_\eta(\rho;\sigma)) \\
&\cong \lambda' \times \delta([\nu^{\frac{-a_{max}+1}{2}}\rho, \nu^{\frac{a_{max}-1}{2}}\rho]) \rtimes \delta([\nu\rho, \nu^{\frac{c_{max}-1}{2}}\rho]); \tau_\eta(\rho;\sigma)) \\
&\Downarrow \\
T' &\hookrightarrow \lambda' \rtimes \theta \text{ and } T \hookrightarrow \lambda \times \lambda' \rtimes \theta
\end{aligned}$$

for some $\theta \leq \delta([\nu^{\frac{-a_{max}+1}{2}}\rho, \nu^{\frac{a_{max}-1}{2}}\rho]) \rtimes \delta([\nu\rho, \nu^{\frac{c_{max}-1}{2}}\rho]); \tau_\eta(\rho;\sigma))$. Note that by construction, $\varepsilon_\theta(\rho, c_{max}) = \eta$. Also, as λ' commutes with $\delta([\nu^{\frac{c_{max}+1}{2}}\rho, \nu^{\frac{a_{max}-1}{2}}\rho])$, we have

$$\mu^*_{[\delta([\nu^{\frac{c_{max}+1}{2}}\rho, \nu^{\frac{a_{max}-1}{2}}\rho])^2]}(T') \neq 0 \Leftrightarrow \mu^*_{[\delta([\nu^{\frac{c_{max}+1}{2}}\rho, \nu^{\frac{a_{max}-1}{2}}\rho])^2]}(\theta) \neq 0$$
$$\Downarrow$$
$$\varepsilon_{T'}(\rho, a_{max})\varepsilon_{T'}(\rho, c_{max})^{-1} = \varepsilon_\theta(\rho, a_{max})\varepsilon_\theta(\rho, c_{max})^{-1}.$$

It now follows that $\varepsilon_\theta(\rho, a_{max}) = \varepsilon_T(\rho, a_{max})$ and $\varepsilon_\theta(\rho, c_{max}) = \varepsilon_T(\rho, c_{max})$.

In the notation of [3], we have

$$\begin{aligned}
&D_G\left(\delta([\nu^{\frac{-a_{max}+1}{2}}\rho, \nu^{\frac{a_{max}-1}{2}}\rho]) \rtimes \delta([\nu\rho, \nu^{\frac{c_{max}-1}{2}}\rho]); \tau_\eta(\rho;\sigma)\right) \\
&= \zeta([\nu^{\frac{-a_{max}+1}{2}}\rho, \nu^{\frac{a_{max}-1}{2}}\rho]) \rtimes \zeta_{-\eta}(\rho, \tfrac{c_{max}-1}{2} - 1; \sigma),
\end{aligned}$$

where D_G denotes the duality operator from [1,26] (extended to $O(2n, F)$ in [16]), noting $D_G(\tau_\eta(\rho;\sigma)) = \tau_{-\eta}(\rho;\sigma)$ (for the even-orthogonal groups, we make the choice of duality so that this holds—see Remark 6.6 [16]). This representation is analyzed in Theorem 3.4 of [3], the combinatorial arguments of which apply in the generality needed here. It then follows from Theorem 3.4 of [3] and duality that for $\zeta = \varepsilon_\theta(\rho, a_{max})\varepsilon_\theta(\rho, c_{max})^{-1}$, we have

$$\begin{aligned}
r_{\nu^{\frac{c_{max}-1}{2}}\rho\otimes\cdots\otimes\nu\rho\otimes\rho}(\theta) &= \nu^{\frac{c_{max}-1}{2}}\rho \otimes \cdots \otimes \nu\rho \otimes \rho \\
&\otimes T(\delta(([\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho])^2; \tau_{\zeta\eta}(\rho;\sigma)) \\
&\Downarrow \text{ (Lemmas 1.2.2 and 1.2.3)} \\
\theta \hookrightarrow \nu^{\frac{c_{max}-1}{2}}\rho \times \cdots \times \nu\rho \times \rho &\rtimes T(\delta([\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho])^2; \tau_{\zeta\eta}(\rho;\sigma)) \\
&\Downarrow \\
T \hookrightarrow \lambda \times \lambda' \times \nu^{\frac{c_{max}-1}{2}}\rho \times \cdots \times \nu\rho \times \rho &\rtimes T(\delta([\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho])^2; \tau_{\zeta\eta}(\rho;\sigma)) \\
&\Downarrow \text{ (Lemma 1.2.3)} \\
T \hookrightarrow \tau &\rtimes T(\delta([\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho])^2; \tau_{\zeta\eta}(\rho;\sigma)) \\
&\cong \tau \rtimes T(\delta([\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho])^2; \tau_{\xi}(\rho;\sigma))
\end{aligned}$$

where $\xi = \zeta\eta = \varepsilon_T(\rho, a_{max})$, as needed.

In the converse direction, it suffices to show that $\mu^*(T)$ contains no terms of the form $\lambda' \otimes T(\delta([\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho])^2; \tau_{-\eta}(\rho;\sigma))$. To this end, observe that

$$\begin{gathered}
T \hookrightarrow \lambda \rtimes T(\delta([\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho])^2; \tau_{\eta}(\rho;\sigma)) \\
\Downarrow \\
\mu^*(T) \le M^*(\lambda) \rtimes \mu^*\left(T(\delta([\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho])^2; \tau_{\eta}(\rho;\sigma))\right),
\end{gathered}$$

so it suffices to show $M^*(\lambda) \rtimes \mu^*\left(T(\delta([\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho])^2; \tau_{\eta}(\rho;\sigma))\right)$ contains no terms of the form $\lambda' \otimes T(\delta([\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho])^2; \tau_{-\eta}(\rho;\sigma))$. Write $M^*(\lambda) = \sum_i \lambda_i' \otimes \lambda_i''$ and $\mu^*\left(T(\delta([\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho])^2; \tau_{\eta}(\rho;\sigma))\right) = \sum_j \lambda_j \otimes \theta_j$. Then,

$$M^*(\lambda) \rtimes \mu^*\left(T(\delta([\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho])^2; \tau_{\eta}(\rho;\sigma))\right) = \sum_i \sum_j \lambda_i' \times \lambda_j \otimes \lambda_i'' \rtimes \theta_j.$$

Now, any term of the form $\nu^x\rho\otimes\cdots$ in $\mu^*\left(T(\delta([\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho])^2; \tau_{\eta}(\rho;\sigma))\right)$ must have $x = \frac{a_{max}-1}{2}$ (just consider $\mu^*\left(\delta([\nu^{\frac{-a_{max}+1}{2}}\rho, \nu^{\frac{a_{max}-1}{2}}\rho]) \rtimes \sigma\right)$). Therefore, if $\lambda_j \ne 1$, it must contain a $\nu^{\frac{a_{max}-1}{2}}\rho$ in its supercuspidal support. Therefore, θ_j can have at most one $\nu^{\frac{a_{max}-1}{2}}\rho$ in its supercuspidal support, hence the same for $\lambda_i''\rtimes\theta_j$ (as any $\nu^x\rho$ appearing in the supercuspidal support of λ_i'' must have $|x| < \frac{a_{max}-1}{2}$). In particular, $\lambda_i''\rtimes\theta_j$ cannot contain $T(\delta([\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho])^2; \tau_{\eta}(\rho;\sigma))$ as a subquotient. Thus we must have $\lambda_j = 1 \Rightarrow \theta_j = T(\delta([\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho])^2; \tau_{\eta}(\rho;\sigma))$, as needed. This finishes the case $m(\rho, a_{max}) = 2$ with $Jord_\rho(\delta) \ne \emptyset$.

Now, suppose $m(\rho, a_{max}) = 2$ with $Jord_\rho(\delta) = \emptyset$. We have

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-a_k+1}{2}}\rho_k, \nu^{\frac{a_k-1}{2}}\rho_k\right]\right) \rtimes \delta,$$

with $(\rho_i, a_i) = (\rho, a_{max})$ for some i; without loss of generality, suppose $i = k$. Then,

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-a_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{a_{k-1}-1}{2}}\rho_{k-1}\right]\right) \rtimes \left(\delta\left(\left[\nu^{\frac{-a_{max}+1}{2}}\rho, \nu^{\frac{a_{max}-1}{2}}\rho\right]\right) \rtimes \delta\right)$$
$$\Downarrow$$
$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-a_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{a_{k-1}-1}{2}}\rho_{k-1}\right]\right) \rtimes T_0$$

for exactly one irreducible $T_0 \leq \delta([\nu^{\frac{-a_{max}+1}{2}}\rho, \nu^{\frac{a_{max}-1}{2}}\rho]) \rtimes \delta$. By definition, $\varepsilon_T(\rho, a_{max}) = \varepsilon_{T_0}(\rho, a_{max})$. Thus,

$$\varepsilon_T(\rho, a_{max}) = \eta$$
$$\Downarrow$$
$$\varepsilon_{T_0}(\rho, a_{max}) = \eta$$
$$\Downarrow \text{ (definition—see Sect. 2.2)}$$
$$\mu^*(T_0) \geq \lambda_0' \otimes T(\delta\left(\left[\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho\right]\right)^2; \tau_\eta(\rho; \sigma))$$
$$\Downarrow \text{ (Lemma 2.2.8)}$$
$$T_0 \hookrightarrow \lambda_0 \rtimes T(\delta\left(\left[\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho\right]\right)^2; \tau_\eta(\rho; \sigma))$$
$$\Downarrow$$
$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-a_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{a_{k-1}-1}{2}}\rho_{k-1}\right]\right) \times \lambda_0 \rtimes T(\delta\left(\left[\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho\right]\right)^2; \tau_\eta(\rho; \sigma))$$
$$\Downarrow \text{ (Lemma 1.2.3)}$$
$$T \hookrightarrow \lambda \rtimes T(\delta\left(\left[\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho\right]\right)^2; \tau_\eta(\rho; \sigma)),$$

as needed. The converse direction is similar to the preceding case.

Finally, suppose $m(\rho, a_{max}) = 1$. Then, by definition, $\varepsilon_T(\rho, a_{max}) = \varepsilon_\delta(\rho, a_{max})$. An argument like that in the beginning of the case $m(\rho, a_{max}) = 2$ gives

$$\varepsilon_\delta(\rho, a_{max}) = \eta \Rightarrow T \hookrightarrow \lambda \rtimes \delta\left(\left[\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho\right]; \tau_\eta(\rho; \sigma)\right)$$

for some irreducible λ, as needed. The converse follows as in the case $m(\rho, a_{max}) = 2$. □

2.4. Tempered representations

In this section, we finish the process of extending the Mœglin–Tadić classification to general tempered representations. This is fairly straightforward given the extension to the elliptic case done in Sect. 2.3.

Recall that an arbitrary irreducible tempered representation T may be written

$$T = \delta\left(\left[\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-a_k+1}{2}}\rho_k, \nu^{\frac{a_k-1}{2}}\rho_k\right]\right) \rtimes T_{ell}, \quad (2.2)$$

with T_{ell} an elliptic tempered representation and $\delta([\nu^{\frac{-a_1+1}{2}}\rho_1, \nu^{\frac{a_1-1}{2}}\rho_1]) \otimes \cdots \otimes \delta([\nu^{\frac{-a_k+1}{2}}\rho_k, \nu^{\frac{a_k-1}{2}}\rho_k])$ unique up to conjugacy (i.e., permutations and sign changes). Note that by Goldberg [7], this is consistent with the use of elliptic in the case of $O(2n, F)$ from Sect. 2.3. Further, for each (ρ_i, a_i), exactly one of the following holds: (1) $(\rho_i, a_i) \in Jord(T_{ell})$, or (2) $a_i \not\equiv 2red(\rho_i; \sigma) + 1 \bmod 2$ (including the possibility $red(\rho_i; \sigma) = \infty$, i.e., $\rho \not\cong \check{\rho}$).

We now define $Jord(T) = Jord_{ell}(T) \cup Jord_{non}(T)$. Here, $Jord_{ell}(T) = Jord(T_{ell})$ and $Jord_{non}(T)$ consists of the (ρ_i, a_i) satisfying (2) above (including both (ρ, a) and $(\check{\rho}, a)$ if they are different). We define $m_T(\rho, a)$ by

$$m_T(\rho, a) = \begin{cases} m_{T_{ell}}(\rho, a) + 2|\{i \mid (\rho_i, a_i) \cong (\rho, a)\}| & \text{if } (\rho, a) \in Jord_{ell}(T) \\ 2|\{i \mid (\rho_i, a_i) \cong (\rho, a)\}| & \text{if } a \not\equiv 2red(\rho;\sigma) + 1 \bmod 2 \text{ with } red(\rho; \sigma) < \infty \\ |\{i \mid (\rho_i, a_i) \cong (\rho, a) \text{ or } (\check{\rho}, a)\}| & \text{if } \rho \not\cong \check{\rho}. \end{cases}$$

Of course, σ is the same as for T_{ell}. To finish the quadruple $(Jord, \sigma, \varepsilon, m)$ for T, we take $S_T = S_{T_{ell}}$ and $\varepsilon_T = \varepsilon_{T_{ell}}$. We have thus associated a quadruple $(Jord, \sigma, \varepsilon, m)$ to T.

We have the following analogue of Lemma 2.1.1. The proof is essentially the same; we omit the details.

Lemma 2.4.1. *Suppose T is tempered. With notation as in Lemma 2.1.1, if $c > 0$ we have the following:*

(1) *If $c \cong red(\rho; \sigma) \bmod 1$, then*

$$m(\rho, 2c+1) = \begin{cases} n(\rho, c) - n(\rho, c+1) & \text{if } c \geq red(\rho; \sigma) \\ n(\rho, c) - n(\rho, c+1) + 1 & \text{if } c < red(\rho; \sigma). \end{cases}$$

(2) *If $\rho \cong \check{\rho}$ and $c \not\equiv red(\rho; \sigma) \bmod 1$,*

$$m(\rho, 2c+1) = n(\rho, c) - n(\rho, c+1).$$

(3) *If $\rho \not\cong \check{\rho}$,*

$$\begin{aligned} m(\rho, 2c+1) &= m(\check{\rho}, 2c+1) \\ &= \frac{1}{2}[n(\rho, c) - n(\rho, c+1) + n(\check{\rho}, c) - n(\check{\rho}, c+1)]. \end{aligned}$$

If $c = 0$, we replace $n(\rho, c) - n(\rho, c+1)$ by $2n(\rho, 0) - n(\rho, 1)$ in the formulas above.

Note 2.4.2. *We have the following analogue of Lemma 1.3.3, proved using the same argument.*

Let $T = T_{(Jord,\sigma,\varepsilon,m)}$ be tempered and $(\rho, a) \in Jord$ with (ρ, a_-) defined. Then

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)^{m(\rho,a)} \rtimes \theta \text{ for some irreducible } \theta$$
$$\Updownarrow$$
$$\mu^*(T) \geq \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)^{m(\rho,a)} \otimes \theta' \text{ for some irreducible } \theta'.$$

If a_- is not defined, we may replace a_- in the above statement with 0 *or* 1*, whichever matches the parity of a.*

3. Tempered representations

3.1. The definition of f_π

The following definition and lemma are the analogues for classical groups of Definition 2.1.1 and Lemma 2.1.2 of [17].

Definition 3.1.1. Let π be an irreducible representation and X a set of (equivalence classes of) irreducible not-necessarily unitary supercuspidal representations of general linear groups. We define $f_\pi(X)$ to be the largest value of f such that $r_{min}(\pi)$ contains a term of the form $\nu^{x_1}\rho_1 \otimes \cdots \otimes \nu^{x_f}\rho_f \otimes \cdots$ with $\nu^{x_1}\rho_1, \ldots, \nu^{x_f}\rho_f \in X$. If $X = \{\nu^x\rho\}$, we write $f_\pi(\nu^x\rho)$ in lieu of $f_\pi(\{\nu^x\rho\})$. We let $\mu^*_X(\pi)$ denote the sum of everything in $\mu^*(\pi)$ of the form $\lambda \otimes \theta$ with λ, θ irreducible and $r_{min}(\lambda)$ containing a term of the form $\nu^{x_1}\rho_1 \otimes \cdots \otimes \nu^{x_f}\rho_f$ with $f = f_\pi(X)$ and $\nu^{x_1}\rho_1, \ldots, \nu^{x_f}\rho_f \in X$.

Remark 3.1.2. Note that $\mu^*_{\nu^x\rho}$ and $\mu^*_{\{\nu^x\rho\}}$ are not in general the same, so the distinction between $\nu^x\rho$ and $\{\nu^x\rho\}$ is otherwise maintained in what follows.

Lemma 3.1.3. *Let π be an irreducible representation and X a set of (equivalence classes of) irreducible not-necessarily unitary supercuspidal representations of general linear groups. Further, assume that if $\nu^x\rho \in X$, then $\nu^{-x}\check{\rho} \notin X$. Then, there are unique irreducible representations λ, θ and unique $f \in \mathbb{N} \cup \{0\}$ such that the following are all satisfied:*

(1) $\pi \hookrightarrow \lambda \rtimes \theta$.
(2) *If $\nu^{x_1}\rho_1 \otimes \cdots \otimes \nu^{x_f}\rho_f \leq r_{min}(\lambda)$, then $\nu^{x_1}\rho_1, \ldots, \nu^{x_f}\rho_f \in X$.*
(3) $r_{\nu^x\rho}(\theta) = 0$ *for all $\nu^x\rho \in X$.*

Furthermore, $f = f_\pi(X)$ and $\mu^*_X(\pi) = \lambda \otimes \theta$. In fact, $\pi \hookrightarrow \lambda \rtimes \theta$ as the unique irreducible subrepresentation. In particular, if π' is an irreducible representation with $\mu^*_X(\pi') = \mu^*_X(\pi)$, then $\pi' = \pi$.

Proof. The proof is similar to that of Lemma 2.1.2 of [17]. We omit the details. □

Remark 3.1.4. In what follows, our concern is with the case $|X| = 1$. If we have $X = \{\rho\}$ (with $\check{\rho} \cong \rho$), the above still holds except that $\lambda \otimes \theta$ can appear with multiplicity greater than one.

We close this section with the following extension of Remark 1.3.2:

Note 3.1.5. *Let T be an irreducible tempered representation with associated data $(Jord, \sigma, \varepsilon, m)$. It follows from the embedding*

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-a_1-1}{2}}\rho_1, \nu^{\frac{a_1+1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-a_k-1}{2}}\rho_k, \nu^{\frac{a_k+1}{2}}\rho_k\right]\right) \rtimes \delta$$

and Remark 1.3.2 (applied to δ) that $f_T(\nu^{\frac{a-1}{2}}\rho) \leq m(\rho, a)$

3.2. Jacquet modules for discrete series

The main result in this section is a characterization of terms of the form $\nu^x\rho \otimes T$ (ρ supercuspidal) which appear in the Jacquet module of a discrete series representation (Theorem 3.2.2). As a corollary, we may characterize the θ which appears in Eq. (1.3).

Lemma 3.2.1. *Suppose $\delta = \delta_{(Jord,\sigma,\varepsilon)}$ is a discrete series representation and $(\rho, 2) \in Jord_\rho$ (which implies $red(\rho;\sigma) \equiv \frac{1}{2} \bmod 1$). Then,*

$$f_\delta(\nu^{\frac{1}{2}}\rho) = \begin{cases} 1 & \textit{if } \varepsilon(\rho, 2) = 1, \\ 0 & \textit{if } \varepsilon(\rho, 2) = -1. \end{cases}$$

Further, when $f_\delta(\nu^{\frac{1}{2}}\rho) = 1$ we have

$$\mu^*_{\{\nu^{\frac{1}{2}}\rho\}}(\delta) = \nu^{\frac{1}{2}}\rho \otimes \delta',$$

where $\delta' = \delta_{(Jord',\sigma,\varepsilon')}$ has $Jord(\delta') = Jord(\delta) \setminus \{(\rho, 2)\}$ and $\varepsilon_{\delta'}$ the restriction of ε_δ

Proof. First, suppose that $\varepsilon(\rho, 2) = -1$. Then, it follows directly from the characterization $\varepsilon_\delta(\rho, a_{min}) = 1$ if and only if $\delta \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{a_{min}-1}{2}}\rho]) \rtimes \theta$ and Lemma 1.3.3 that $f_\delta(\nu^{\frac{1}{2}}\rho) = 0$.

Now, suppose $\varepsilon(\rho, 2) = 1$. If we show $\mu^*(\delta) \geq \nu^{\frac{1}{2}}\rho \otimes \delta'$, it follows that $f_\delta(\nu^{\frac{1}{2}}\rho) \geq 1$. On the other hand, by Note 3.1.5, we have $f_\delta(\nu^{\frac{1}{2}}\rho) \leq 1$. Thus, this would imply $\mu^*_{\{\nu^{\frac{1}{2}}\rho\}}(\delta) = \nu^{\frac{1}{2}}\rho \otimes \delta'$, as needed.

By [21], one has

$$\delta \hookrightarrow \nu^{\frac{1}{2}}\rho \rtimes \pi$$

for some irreducible π. We must show that $\pi = \delta'$ from the statement of the lemma. First, we claim that π is square-integrable—the argument is the same as in the proof of Proposition 2.1.2. That $Jord(\pi) = Jord(\delta')$ follows from Lemma 2.1.1. As the partial cuspidal supports match, all that remains to show $\varepsilon_\pi = \varepsilon_{\delta'}$.

If $|Jord_\rho| = 1$, that $\pi = \delta'$ follows immediately from the Mœglin–Tadić construction. If $|Jord_\rho| > 1$, let b be the smallest value greater than 2 having $(\rho, b) \in Jord$. By Lemma 2.3.1, it suffices to show $\varepsilon_\pi(\rho, b) = \varepsilon(\rho, b)$. This follows from an argument similar to that used in (the later part of) Proposition 2.1.2. □

Theorem 3.2.2. *Suppose δ is a discrete series representation with Mœglin–Tadić data $(Jord, \sigma, \varepsilon)$. We have $f_\delta(\nu^{\frac{a-1}{2}}\rho) = 0$ unless $(\rho, a) \in Jord$ and one of the following holds:*

(1) (ρ, a_-) *exists and either (a)* $a > a_- + 2$*, or (b)* $\varepsilon(\rho, a)\varepsilon(\rho, a_-)^{-1} = 1$.
(2) (ρ, a_-) *does not exist and either (a)* $a > 2$*, or (b)* $a = 2$ *and* $\varepsilon(\rho, a) = 1$.

In either of these cases, $f_\delta(\nu^{\frac{a-1}{2}}\rho) = 1$ *and* $\mu^*_{\nu^{\frac{a-1}{2}}\rho}(\delta) = \nu^{\frac{a-1}{2}}\rho \otimes T$*, where T is an elliptic tempered representation (noting that this includes the possibility of T being discrete series) with data $(Jord(T), \sigma, \varepsilon_T, m_T)$ as follows:*

$$Jord(T) = \begin{cases} (Jord \setminus \{(\rho, a)\}) \cup \{(\rho, a-2)\} & \text{if } a > 2, \\ Jord \setminus \{(\rho, a)\} & \text{if } a = 2 \end{cases}$$

and

$$m_T(\rho', b) = \begin{cases} m(\rho', b) & \text{if } (\rho', b) \neq (\rho, a) \text{ or } (\rho, a-2), \\ 0 & \text{if } (\rho', b) = (\rho, a), \\ m(\rho, a-2) + 1 & \text{if } (\rho', b) = (\rho, a-2) \text{ (with } a > 2), \end{cases}$$

and ε_T defined on its domain S_T by

$$\varepsilon_T(\rho', b) = \begin{cases} \varepsilon(\rho', b) & \text{if } (\rho', b) \neq (\rho, a-2), \\ \varepsilon(\rho, a) & \text{if } (\rho', b) = (\rho, a-2) \end{cases}$$

in those cases where $\varepsilon_T(\rho', b)$ is defined, and if $(\rho', b), (\rho', c) \in Jord(T)$,

$$\varepsilon_T(\rho', b)\varepsilon_T(\rho', c)^{-1} = \begin{cases} \varepsilon(\rho', b)\varepsilon(\rho', c)^{-1} & \text{if } (\rho', b), (\rho', c) \neq (\rho, a-2), \\ \varepsilon(\rho, a)\varepsilon(\rho, c)^{-1} & \text{if } (\rho', b) = (\rho, a-2), \\ \varepsilon(\rho, b)\varepsilon(\rho, a)^{-1} & \text{if } (\rho', c) = (\rho, a-2). \end{cases}$$

Proof. It follows from Remark 1.3.2 that $f_\delta(\nu^{\frac{a-1}{2}}\rho) > 0$ requires $(\rho, a) \in Jord$. Now, Cases 1(a) and 2(a) are covered by Proposition 2.1.2, noting (for 1(a)) that $f_\delta(\nu^{\frac{a-1}{2}}\rho) \leq 1$ follows from Note 3.1.5. Case 2(b) follows from Lemma 3.2.1. Thus, only Case 1(b) needs to be addressed. In particular, we may assume $(\rho, a), (\rho, a-2) \in Jord$. It is then immediate from Eq. (1.3) that $f_\delta(\nu^{\frac{a-1}{2}}\rho) = \begin{cases} 1 & \text{if } \varepsilon(\rho, a)\varepsilon(\rho, a-2)^{-1} = 1, \\ 0 & \text{if } \varepsilon(\rho, a)\varepsilon(\rho, a-2)^{-1} = -1 \end{cases}$ (noting that $f_\delta(\nu^{\frac{a-1}{2}}\rho) \leq 1$ follows from Note 3.1.5).

From [21], if $(\rho, a), (\rho, a-2) \in Jord$ with $\varepsilon(\rho, a)\varepsilon(\rho, a-2)^{-1} = 1$, we have

$$\delta \hookrightarrow \delta\left(\left[\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \delta_1 \hookrightarrow \nu^{\frac{a-1}{2}}\rho \rtimes \left(\delta\left(\left[\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho\right]\right) \rtimes \delta_1\right)$$

for the usual δ_1 (see Note 1.3.1). Note that since $(\rho, a-2) \notin Jord(\delta_1)$, we have $\delta([\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]) \rtimes \delta_1$ reducible. Let T be a component of $\delta([\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]) \rtimes \delta_1$ such that $\delta \hookrightarrow \nu^{\frac{a-1}{2}}\rho \rtimes T$. It follows from Lemma 3.1.3 and $f_\delta(\nu^{\frac{a-1}{2}}\rho) = 1$ that $\mu^*_{\nu^{\frac{a-1}{2}}\rho}(\delta) = \nu^{\frac{a-1}{2}}\rho \otimes T$, so that this holds for exactly one component of

$\delta([\nu^{\frac{-a+3}{2}}\rho, \nu^{\frac{a-3}{2}}\rho]) \rtimes \delta_1$. To see that this T has the data given in the statement of the theorem, observe that by the construction in Sect. 2.2, we have

$$\begin{aligned} Jord(T) &= Jord(\delta_1) \cup \{(\rho, a-2)\} \\ &= (Jord(\delta) \setminus \{(\rho, a), (\rho, a-2)\}) \cup \{(\rho, a-2)\} = Jord(\delta) \setminus \{(\rho, a)\} \end{aligned}$$

and

$$m_T(\rho', b) = \begin{cases} 1 & \text{if } (\rho', b) \in Jord(\delta_1), \\ 2 & \text{if } (\rho', b) = (\rho, a-2), \\ 0 & \text{otherwise,} \end{cases}$$

matching the description of *Jord(T)* and m_T in the statement of the theorem. Clearly, the partial cuspidal support is still σ. Thus, it remains to address ε_T.

First, if $(\rho', b), (\rho', c) \in Jord(\delta_1) = Jord(\delta) \setminus \{(\rho, a), (\rho, a-2)\}$, then the construction in Sect. 2.2 implies

$$\varepsilon_T(\rho', b)\varepsilon_T(\rho', c)^{-1} = \varepsilon_{\delta_1}(\rho', b)\varepsilon_{\delta_1}(\rho', c)^{-1} = \varepsilon_\delta(\rho', b)\varepsilon_\delta(\rho', c)^{-1}$$

and, if defined,

$$\varepsilon_T(\rho', b) = \varepsilon_{\delta_1}(\rho', b) = \varepsilon_\delta(\rho', b).$$

Again, this matches that part of the description of ε_T in the statement of the theorem. Thus, it remains to show $\varepsilon_T(\rho, b)\varepsilon(\rho, a-2)^{-1}$ and $\varepsilon_T(\rho, a-2)$ (when defined) have the values claimed.

For $\varepsilon_T(\rho, b)\varepsilon(\rho, a-2)^{-1}$, observe that by Lemma 2.3.1, it is enough to show this for some $b \neq a-2$. It therefore suffices to first address the case $b = a_+$, then (assuming a_+ does not exist) the case $b = a_-$. The arguments here are similar to those in (the later part of) Proposition 2.1.2. For $\varepsilon_T(\rho, a-2)$ (when defined), note that if $|Jord_\rho(\delta)| > 2$, then we have already argued above that $\varepsilon_T(\rho, c) = \varepsilon(\rho, c)$ for $c \neq a, a-2$, so are done by Lemma 2.3.1. Thus we may assume $Jord_\rho(\delta) = \{(\rho, a), (\rho, a-2)\}$; in particular, $a_{min} = a-2$. There are two cases, corresponding to $red(\rho; \sigma) \equiv \frac{1}{2} \bmod 1$ and $red(\rho; \sigma) = 0$. The proof of the former is also similar to the (later part of) the proof of Proposition 2.1.2; the proof of the latter to the proof of Lemma 2.3.7. Note that Proposition 2.3.2 and Note 2.4.2 serve the roles here that Eq. (1.3) and Lemma 1.3.3 play in the proof of Proposition 2.1.2. □

The following corollary identifies the θ in the characterization

$$\varepsilon_\delta(\rho, a)\varepsilon_\delta(\rho, a_-)^{-1} = 1 \Leftrightarrow \delta \hookrightarrow \delta([\nu^{\frac{a_- - 1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \theta.$$

Corollary 3.2.3. *Let $\delta = \delta_{(Jord,\sigma,\varepsilon)}$ be a discrete series representation. If $(\rho, a), (\rho, a_-) \in Jord$ with $\varepsilon(\rho, a)\varepsilon(\rho, a_-)^{-1} = 1$, then $\delta \hookrightarrow \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes T$, where T is elliptic tempered with data $(Jord(T), \sigma, \varepsilon_T, m_T)$ as follows: $Jord(T) = Jord \setminus \{(\rho, a)\}$, ε_T is the restriction of ε, and $m_T(\rho', b) = 1$ for all $(\rho', b) \in Jord(T)$ except for (ρ, a_-), which has multiplicity 2. Further, $\mu^*_{\delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])}(\delta) = \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \otimes T$.*

3.3. Jacquet modules for $\delta([\nu^{\frac{-c+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho])\rtimes\delta$

In this section, we establish the analogue of Note 1.3.1 for components of $\delta([\nu^{\frac{-c+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho])\rtimes\delta$ (assumed reducible). In Sect. 3.4, we extend this first to the elliptic case, then to general tempered representations.

Lemma 3.3.1. *Let δ be square-integrable and $T\leq\delta([\nu^{\frac{-c+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho])\rtimes\delta$ elliptic tempered. If $(\rho,a),(\rho,a_-)\in Jord(T)$ with $\varepsilon_T(\rho,a)\varepsilon_T(\rho,a_-)^{-1}=1$, then $T\hookrightarrow\delta([\nu^{\frac{-a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])\rtimes T'$, where T' is an elliptic tempered representation whose data is obtained as follows: one removes one copy each of (ρ,a) and (ρ,a_-) from Jord(T); $\varepsilon_{T'}$ is then the restriction of ε_T.*

Proof. We consider three cases: (1) $c=a$, (2) $c=a_-$, and (3) $c\neq a,a_-$.

Case 1: $c=a$

We have

$$\begin{gathered}T\hookrightarrow\delta\left(\left[\nu^{\frac{-c+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho\right]\right)\rtimes\delta\hookrightarrow\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho\right]\right)\\ \times\delta\left(\left[\nu^{\frac{-c+1}{2}}\rho,\nu^{\frac{-a_--1}{2}}\rho\right]\right)\rtimes\delta\\ \Downarrow\ (\text{Lemma } 1.2.3)\\ T\hookrightarrow\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho\right]\right)\rtimes\theta\end{gathered}$$

for some irreducible $\theta\leq\delta([\nu^{\frac{-c+1}{2}}\rho,\nu^{\frac{-a_--1}{2}}\rho])\rtimes\delta$. Note that since $\varepsilon_T(\rho,c)\varepsilon_T(\rho,a_-)^{-1}=1$, we have $\mu^*_{[\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho])^2]}(T)\neq0\Rightarrow\mu^*_{\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho])}(\theta)\neq0$. We claim that $\theta=T'$, with T' as in the statement of the lemma. Observe that in this case, T' is actually square-integrable. By Corollary 2.1.3 and Lemma 1.3.3, $T'\hookrightarrow\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho])\rtimes\delta$. Further, since

$$\mu^*_{\delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho\right]\right)}\left(\delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho\right]\right)\rtimes\delta\right)=\delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho\right]\right)\otimes\delta$$

(a straightforward μ^* argument), we see that T' is the only irreducible subquotient of $\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho])\rtimes\delta$ having $\mu^*_{\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho])}\neq0$. Therefore, $\theta=T'$, as claimed. This finishes Case 1.

Case 2: $c=a_-$

By Note 2.1.4, $\delta\hookrightarrow\delta([\nu^{\frac{c+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])\rtimes\delta'$. Note that δ' from Note 2.1.4 matches T' in the statement of the lemma. Thus,

$$\begin{gathered}T\hookrightarrow\delta\left(\left[\nu^{\frac{-c+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho\right]\right)\times\delta\left(\left[\nu^{\frac{c+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho\right]\right)\rtimes T'\\ \Downarrow\ (\text{Lemma } 1.2.3)\\ T\hookrightarrow\lambda\rtimes T'\end{gathered}$$

with $\lambda = \delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])$ or $\mathcal{L}(\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]), \delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]))$. To have $\mu^*_{\delta([\nu^{\frac{c+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])}(T) \neq 0$, we must have $M^*_{\delta([\nu^{\frac{c+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])}(\lambda) \neq 0$, which holds only for $\lambda = \delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])$. Thus,

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes T',$$

as needed.

Case 3: $c \neq a, a_-$ Since $(\rho, a), (\rho, a_-) \in Jord(\delta)$ and $\varepsilon_\delta(\rho, a)\varepsilon_\delta^{-1}(\rho, a_-) = \varepsilon_T(\rho, a)\varepsilon_T(\rho, a_-)^{-1} = 1$, we have

$$\delta \hookrightarrow \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \delta'$$

for δ' as in [21] (see Note 1.3.1). Therefore, noting $c > a$ or $c < a_-$,

$$\begin{aligned} T &\hookrightarrow \delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \delta' \\ &\cong \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \delta' \\ &\Downarrow \\ T &\hookrightarrow \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes T' \end{aligned}$$

for some $T' \leq \delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta'$. Clearly, $Jord(T')$, $m_{T'}$ and the partial cuspidal support are as claimed. It remains to show $\varepsilon_{T'}$ is also as claimed. For this, it is enough to show $\varepsilon_{T'}$ is as claimed on $S_{T'} \cap \big(Jord_\rho(T') \cup (Jord_\rho(T') \times Jord_\rho(T'))\big)$ (as the remaining values match those for $\varepsilon_{\delta'}$, hence ε_δ and so ε_T).

First, suppose $Jord_\rho(\delta') \neq \emptyset$. Then it suffices to show $\varepsilon_{T'}(\rho, c)\varepsilon_{T'}(\rho, b)^{-1} = \varepsilon_T(\rho, c)\varepsilon_T(\rho, b)^{-1}$ for some $(\rho, b) \in Jord(T')$. First, suppose we have $(\rho, b) \in Jord(T')$ with b minimal such that $b > c$. There are three possibilities to consider: (1) $a > a_- > b > c$, (2) $b > c > a > a_-$, and (3) $b > a > a_- > c$. If either $a > a_- > b > c$ or $b > c > a > a_-$, a straightforward argument using Corollary 2.2.7 and the definition in Sect. 2.2 shows that $\varepsilon_{T'}(\rho, c)\varepsilon_{T'}(\rho, b)^{-1} = 1 \Leftrightarrow \varepsilon_T(\rho, c)\varepsilon_T(\rho, b)^{-1} = 1$.

Now, suppose $b > a > a_- > c$. Suppose δ' is the discrete series representation having $Jord(\delta') = Jord(\delta)\backslash\{(\rho, a), (\rho, a_-)\}$, the same partial cuspidal support as δ, and $\varepsilon_{\delta'}$ given by restriction of ε_δ. By [21], we have $\delta \hookrightarrow \delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \delta'$.

Now, set

$$I_1 = \delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \delta$$

and

$$I_2^\xi = \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes T'_\xi,$$

for $\xi = \pm 1$, where T'_ξ is the component of $\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta'$ characterized by $\varepsilon_{T'_\xi}(\rho, b)\varepsilon_{T'_\xi}(\rho, c)^{-1} = \xi$, First, we claim I_1 and I_2^{ξ}, $\xi = \pm 1$, have exactly one irreducible subquotient in common. To see this, let

$$\mathcal{I} = \delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \delta'.$$

Observe that $I_1, I_2^{+1}, I_2^{-1} \leq \mathcal{I}$. It is not difficult to show that $\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \times \delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \otimes \delta'$ appears with multiplicity four in $\mu^*(\mathcal{I})$, multiplicity two in each $\mu^*(I_2^{\xi})$, and multiplicity two in $\mu^*(I_1)$—once in each component. Since $I_1 \not\leq I_2^{\xi}$ for either value of ξ (e.g., consider $\mu^*_{\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])}(I_2^{\xi})$), the claim follows. Now, let $\zeta = \varepsilon_\delta(\rho, b)\varepsilon_\delta(\rho, a)^{-1}$.

It is now a straightforward matter to show that $\delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho]) \times \delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \otimes \left(\delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes T'_{\zeta}\right)$ (irreducible) appears with multiplicity one in each of $\mu^*(I_1)$, $\mu^*(I_2^{\zeta})$, and $\mu^*(\mathcal{I})$, but not in $\mu^*(I_2^{-\zeta})$.

We now finish the case $b > a > a_- > c$. We first note that since $\mu^*_{\delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho])}$ is nonzero for exactly one component of I_1, we have $\mu^*_{\delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho])}(T) \neq 0 \Leftrightarrow \mu^*_{\delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho]) \times \delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])}(T) \neq 0$. Therefore,

$$\begin{aligned}
\varepsilon_T(\rho, b)\varepsilon_T(\rho, c)^{-1} = \zeta &\Rightarrow \varepsilon_T(\rho, a_-)\varepsilon_T(\rho, c)^{-1} = 1 \\
&\Rightarrow \mu^*_{\delta\left(\left[\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right)}(T) \neq 0 \text{ (definition)} \\
&\Rightarrow \mu^*_{\delta\left(\left[\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right) \times \delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])}(T) \neq 0 \\
&\Rightarrow T \leq I_1, I_2^{\zeta} \\
&\Rightarrow \varepsilon_{T'}(\rho, b)\varepsilon_{T'}(\rho, c)^{-1} = \zeta.
\end{aligned}$$

Conversely,

$$\begin{aligned}
\varepsilon_T(\rho, c)\varepsilon_T(\rho, c)^{-1} = -\zeta &\Rightarrow \varepsilon_T(\rho, a_-)\varepsilon_T(\rho, c)^{-1} = -1 \\
&\Rightarrow \mu^*_{\delta\left(\left[\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right)}(T) = 0 \\
&\Rightarrow T \not\leq I_2^{\zeta} \\
&\Rightarrow T \leq I_1, I_2^{-\zeta} \\
&\Rightarrow \varepsilon_{T'}(\rho, b)\varepsilon_{T'}(\rho, c)^{-1} = -\zeta,
\end{aligned}$$

as needed.

Now, consider the case where one has $(\rho, b) \in Jord(T')$ with b maximal having $b < c$. When $c > b > a > a_-$ or $a > a_- > c > b$, it is the same basic argument as for $b > c > a > a_-$ or $a > a_- > b > c$ above except $\delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho])$ is replaced by $\delta([\nu^{\frac{b+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^2$. When $c > a > a_- > b$, the argument is also similar, with the key difference being that one shows

$\mu^*_{[\delta([\nu^{\frac{a+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho])^2]}(I_1) = \delta([\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^2 \otimes \left(\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \delta\right)$ (irreducible), and $\delta([\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^2 \otimes \left(\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \delta\right)$ appears with multiplicity one in both $\mu^*(I_2^{\zeta})$, $\zeta = \varepsilon_\delta(\rho, a_-)\varepsilon_\delta(\rho, b)^{-1}$, and $\mu^*(\mathcal{I})$ (hence does not appear in $\mu^*(I_2^{-\zeta})$).

This finishes the case $Jord(\delta') \neq \emptyset$.

Finally, suppose $Jord_\rho(\delta') = \emptyset$. Then, there are two possibilities: $red(\rho;\sigma) = 0$ or $\frac{1}{2}$. If $red(\rho;\sigma) = \frac{1}{2}$, the argument is like that for $Jord_\rho(\delta') \neq \emptyset$ taking $b = 0$. If $red(\rho;\sigma) = 0$, the result follows from Lemma 3.3.2 below. This finishes the proof of the lemma. □

Lemma 3.3.2. *Suppose $red(\rho;\sigma) = 0$. Let δ be a discrete series with $Jord_\rho(\delta) = \{(\rho, a), (\rho, a_-)\}$ and $\varepsilon_\delta(\rho, a) = \varepsilon_\delta(\rho, a_-) = \zeta$. By [21], write $\delta \hookrightarrow \delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \delta'$. Suppose $c \equiv a \bmod 2$ with either $c > a$ or $0 < c < a_-$. Write*

$$\delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \delta' = \bigoplus_{\eta\in\{\pm1\}} T_\eta(\delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right); \delta'),$$

where $T_\eta(\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]); \delta')$ is the component satisfying $\varepsilon_{T_\eta(\delta([\nu^{\frac{-c+1}{2}}\rho,\nu^{\frac{c-1}{2}}\rho]);\delta')}(\rho, c) = \eta$. Set

$$I = \delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \delta$$

and

$$I'_\eta = \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes T_\eta(\delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right); \delta').$$

Then I and I'_η have a (unique) common irreducible subrepresentation T_η, with $\varepsilon_{T_\eta}(\rho, a) = \varepsilon_{T_\eta}(\rho, a_-) = \zeta$ and $\varepsilon_{T_\eta}(\rho, c) = \eta$.

Proof. Let

$$I' = \delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \delta'.$$

Note that $I, I'_\eta \hookrightarrow I'$ (for $\eta = \pm1$). Further,

$$\mu^*_{\delta\left(\left[\nu\rho,\nu^{\frac{c-1}{2}}\rho\right]\right)^2\times\delta\left(\left[\nu\rho,\nu^{\frac{a_--1}{2}}\rho\right]\right)\times\delta\left(\left[\nu\rho,\nu^{\frac{a-1}{2}}\rho\right]\right)}(I') = \delta\left(\left[\nu\rho, \nu^{\frac{c-1}{2}}\rho\right]\right)^2$$
$$\times\delta\left(\left[\nu\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right) \times \delta\left(\left[\nu\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \otimes \rho \times \rho \rtimes \delta';$$

in particular, contains $\delta([\nu\rho, \nu^{\frac{c-1}{2}}\rho])^2 \times \delta([\nu\rho, \nu^{\frac{a_--1}{2}}\rho]) \times \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho]) \otimes (\rho \rtimes T_\xi(\rho;\delta'))$ with multiplicity one (for $\xi = \pm1$). One can show that $\mu^*(I)$ and $\mu^*(I'_\zeta)$ both contain $\delta([\nu\rho, \nu^{\frac{c-1}{2}}\rho])^2 \times \delta([\nu\rho, \nu^{\frac{a_--1}{2}}\rho]) \times \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho]) \otimes$

$(\rho \rtimes \tau_\zeta(\rho;\sigma))$. Therefore, they have a common component T_ζ characterized by $\mu^*(T_\zeta) \geq \delta([\nu\rho, \nu^{\frac{c-1}{2}}\rho])^2 \times \delta([\nu\rho, \nu^{\frac{a_--1}{2}}\rho]) \times \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho]) \otimes (\rho \rtimes T_\zeta(\rho;\delta'))$. Since $T_\zeta \hookrightarrow \delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta$, by definition, $\varepsilon_{T_\zeta}(\rho, a) = \varepsilon_\delta(\rho, a) = \zeta$ and $\varepsilon_{T_\zeta}(\rho, a_-) = \varepsilon_\delta(\rho, a_-) = \zeta$. Further, it follows directly from $\mu^*(T_\zeta) \geq \delta([\nu\rho, \nu^{\frac{c-1}{2}}\rho])^2 \times \delta([\nu\rho, \nu^{\frac{a_--1}{2}}\rho]) \times \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho]) \otimes (\rho \rtimes T_\zeta(\rho;\delta'))$ that if $c > a$, we have $\mu^*_{[\delta([\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^2]}(T_\zeta) \neq 0$; if $c < a_-$, $\mu^*_{\delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho])}(T_\zeta) \neq 0$. In either case, it follows that $\varepsilon_{T_\zeta}(\rho, c) = \varepsilon_{T_\zeta}(\rho, a) = \varepsilon_{T_\zeta}(\rho, a_-) = \zeta$.

Let T'_ζ be the other component of I, so that $\varepsilon_{T'_\zeta}(\rho, a) = \varepsilon_{T'_\zeta}(\rho, a_-) = \zeta$ and $\varepsilon_{T'_\zeta}(\rho, c) = -\zeta$. Then,

$$\begin{gathered} T'_\zeta \hookrightarrow \delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \times \delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \delta' \\ \cong \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \left(\delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \delta'\right) \\ \Downarrow \\ T \hookrightarrow I'_\xi \end{gathered}$$

for some $\xi \in \{\pm 1\}$. Now, a straightforward μ^* argument shows that $\mu^*(I'_\xi)$ contains $\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \otimes \delta$ with multiplicity one. In particular, since $T_\zeta \leq I'_\zeta$ accounts for the only copy, we must have $T'_\zeta \not\leq I'_\zeta \Rightarrow T'_\zeta \leq I'_{-\zeta}$. The lemma follows. □

Lemma 3.3.3. *Suppose* $T \hookrightarrow \delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta$ *is elliptic tempered. If* $(\rho', a), (\rho', a_-) \in Jord(T)$ *and either (1)* $a_- < a-2$, *or (2)* $\varepsilon_T(\rho', a)\varepsilon_T(\rho', a_-)^{-1} = 1$, *then* $f_T(\nu^{\frac{a-1}{2}}\rho') = m_T(\rho', a)$ *and*

$$T \hookrightarrow \left(\nu^{\frac{a-1}{2}}\rho'\right)^{m_T(\rho',a)} \rtimes T',$$

where T' *is tempered with*

$$Jord(T') = \big(Jord(T) \setminus \{(\rho', a)\}\big) \cup \{(\rho', a-2)\},$$

$\sigma_{T'} = \sigma_T$, *and*

$$m_{T'}(\rho'', d) = \begin{cases} m_T(\rho'', d) & \textit{if } (\rho'', d) \neq (\rho', a), (\rho', a-2); \\ m_T(\rho', a-2) + m_T(\rho', a) & \textit{if } (\rho'', d) = (\rho', a-2); \\ 0 & \textit{if } (\rho'', d) = (\rho', a). \end{cases}$$

Further, $\varepsilon_{T'}$ *is determined by* $\varepsilon_{T'}|_{S_T \cap S_{T'}} = \varepsilon_T|_{S_T \cap S_{T'}}$,

$$\varepsilon_{T'}(\rho', a-2)\varepsilon_{T'}(\rho', b)^{-1} = \varepsilon_T(\rho', a)\varepsilon_T(\rho', b)^{-1}$$

and if applicable,

$$\varepsilon_{T'}(\rho', a-2) = \varepsilon_T(\rho', a).$$

Proof. First, we note that $f_\delta(\nu^{\frac{a-1}{2}}\rho') \le m_\delta(\rho', a)$, e.g., by Theorem 3.2.2. Then, $f_T(\nu^{\frac{a-1}{2}}\rho') \le m_T(\rho', a)$ follows directly from the embedding

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-c_k+1}{2}}\rho_k, \nu^{\frac{c_k-1}{2}}\rho_k\right]\right) \rtimes \delta$$

and a μ^* argument. That $f_T(\nu^{\frac{a-1}{2}}\rho') = m_T(\rho', a)$ then follows from Frobenius reciprocity once we show the embedding claimed. We break this into two cases based on $m(\rho', a)$.

Case 1: $m(\rho', a) = 2$

In this case, we have $(\rho', a) = (\rho, c)$. If $a_- = c_- = c - 2$ (so that $\varepsilon_T(\rho, c)\varepsilon_T(\rho, c-2)^{-1} = 1$), it follows from Proposition 2.3.2 that

$$T \hookrightarrow \nu^{\frac{c-1}{2}}\rho \times \nu^{\frac{c-1}{2}}\rho \rtimes \left(\delta\left(\left[\nu^{\frac{-c+3}{2}}\rho, \nu^{\frac{c-3}{2}}\rho\right]\right) \rtimes \delta\right)$$

(noting that one has $\mu^*_{\nu^{\frac{c-1}{2}}\rho\times\nu^{\frac{c-1}{2}}\rho}(T) = \nu^{\frac{c-1}{2}}\rho \times \nu^{\frac{c-1}{2}}\rho \otimes \left(\delta([\nu^{\frac{-c+3}{2}}\rho, \nu^{\frac{c-3}{2}}\rho]) \rtimes \delta\right)$, which is irreducible). Then, $T' \cong \delta([\nu^{\frac{-c+3}{2}}\rho, \nu^{\frac{c-3}{2}}\rho]) \rtimes \delta$, which is irreducible and matches the description in the statement. Thus, all that remains for Case 1 is $c_- < c - 2$. For this, the proof is similar to the first part of the proof of Proposition 2.1.2.

Case 2: $m_T(\rho', a) = 1$

In this case, $(\rho', a) \in Jord(\delta)$.

First, suppose $(\rho', a) = (\rho, c + 2)$. Then, by Theorem 3.2.2, $\mu^*_{\nu^{\frac{a-1}{2}}\rho}(\delta) = \nu^{\frac{a-1}{2}}\rho \otimes \delta'$. Therefore,

$$\mu^*_{\nu^{\frac{a-1}{2}}\rho}\left(\delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \delta\right) = \nu^{\frac{a-1}{2}}\rho \otimes \left(\delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \delta'\right).$$

Note that the right-hand side is irreducible as $(\rho, c) = (\rho, a - 2) \in Jord(\delta')$. Since $\varepsilon_T(\rho, a)\varepsilon_T(\rho, a-2)^{-1} = 1$, we have $\mu^*_{\nu^{\frac{a-1}{2}}\rho}(T) \ne 0$, so

$$T \hookrightarrow \nu^{\frac{a-1}{2}}\rho \rtimes \left(\delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \delta'\right),$$

and $T' \le \delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta'$. Noting that $(\rho, c) \in Jord(\delta')$, this matches the description of T' given.

We may now assume $(\rho', a) \ne (\rho, c + 2)$. By Theorem 3.2.2,

$$\delta \hookrightarrow \nu^{\frac{a-1}{2}}\rho' \rtimes \mathcal{T}'.$$

Since $\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \times \nu^{\frac{a-1}{2}}\rho'$ irreducible,

$$\begin{gathered} T \hookrightarrow \delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \\ \times \nu^{\frac{a-1}{2}}\rho' \rtimes \mathcal{T}' \cong \nu^{\frac{a-1}{2}}\rho' \times \delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \mathcal{T}' \\ \Downarrow \\ T \hookrightarrow \nu^{\frac{a-1}{2}}\rho' \rtimes T' \end{gathered}$$

for some irreducible $T' \leq \delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \mathcal{T}'$. Clearly, $Jord(T')$, $\sigma_{T'}$, and $m_{T'}$ are as claimed. Further, $\varepsilon_{T'}|_{S_{\mathcal{T}'}} = \varepsilon_{\mathcal{T}'}$, so only $\varepsilon_{T'}(\rho, c)\varepsilon_{T'}(\rho, b)^{-1}$ and (if defined) $\varepsilon_{T'}(\rho, c)$ are in question.

We first consider the possibility that (ρ, c_-) exists in $Jord(T')$. In this event, it again follows from Lemma 2.3.1 that it is enough to show $\varepsilon_{T'}(\rho, c)\varepsilon_{T'}(\rho, c_-)^{-1}$ is as claimed. Observe that if $(\rho', a) \neq (\rho, c_-)$, we have $\delta([\nu^{\frac{c_-+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \times \nu^{\frac{a-1}{2}}\rho'$ irreducible (since $(\rho', a) \neq (\rho, c+2)$). Then, a straightforward argument like that in (the later part of) the proof of Proposition 2.1.2 tells us $\varepsilon_{T'}(\rho, c)\varepsilon_{T'}(\rho, c_-)^{-1} = 1 \Leftrightarrow \varepsilon_T(\rho, c)\varepsilon_T(\rho, c_-)^{-1} = 1$.

If $(\rho, c_-) = (\rho', a)$, then (noting that in $Jord(T')$, we have $c_- = a-2$) we must show $\varepsilon_{T'}(\rho, c)\varepsilon_{T'}(\rho, a-2)^{-1} = 1$ if and only if $\varepsilon_T(\rho, c)\varepsilon_T(\rho, a)^{-1} = 1$. The implication ($\Rightarrow$) is essentially the same as for $(\rho', a) \neq (\rho, c_-)$. In the converse direction, a straightforward μ^* calculation gives

$$\begin{aligned} &\varepsilon_T(\rho, c)\varepsilon_T(\rho, a)^{-1} = 1\\ &\Downarrow\\ 0 &\neq \mu^*_{[\delta([\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^2]}(T)\\ &\leq \mu^*_{[\delta([\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^2]}\left(\delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \delta\right)\\ &= \delta\left(\left[\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right)^2 \otimes \left(\delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \delta\right). \end{aligned}$$

Now, since $(\rho, a) \in Jord(\delta)$, we have $\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \delta$ irreducible. In particular,

$$\mu^*_{[\delta([\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^2]}(T) = \delta\left(\left[\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right)^2 \otimes \left(\delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \delta\right).$$

As $(\rho, a), (\rho, a_-) \in Jord(\delta)$ and $\varepsilon(\rho, a)\varepsilon(\rho, a_-)^{-1} = 1$, we have $\mu^*_{\nu^{\frac{a-1}{2}}\rho}(\delta) \neq 0$. The irreducibility of $\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \delta$ then tells us $\mu^*\left(\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \delta\right) \geq (\nu^{\frac{a-1}{2}}\rho)^3 \otimes \theta''$ for some θ''. In particular, for the appropriate standard parabolic subgroup, we have

$$\begin{aligned} r_{M,G}(\nu^{\frac{a-1}{2}}\rho \rtimes T') &\geq \delta\left(\left[\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right)^2 \otimes (\nu^{\frac{a-1}{2}}\rho)^3 \otimes \theta''\\ &\Downarrow\\ &\mu^*_{[\delta([\nu^{\frac{a-1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^2]}(T') \neq 0 \end{aligned}$$

since $\delta([\nu^{\frac{a-1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^2 \times \nu^{\frac{a-1}{2}}\rho$ is the only irreducible representation having $m^* \geq \delta([\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho])^2 \otimes (\nu^{\frac{a-1}{2}}\rho)^3$. Thus, $\varepsilon_{T'}(\rho, c)\varepsilon_{T'}(\rho, a-2)^{-1} = 1$, as needed. This finishes the case where (ρ, c_-) exists.

When (ρ, c_+) exists but (ρ, c_-) does not, there are three cases to consider: (1) $(\rho', a) \neq (\rho, c_+)$ or $(\rho, c_+ + 2)$, (2) $(\rho', a) = (\rho, c_+)$ (with $c_+ > c + 2$), and (3) $(\rho', a) = (\rho, c_+ + 2)$. The arguments for all three cases are similar to those above, with the obvious change that one must use $\mu^*_{\delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{c_+-1}{2}}\rho])}(T)$ in place of $\mu^*_{[\delta([\nu^{\frac{c_-+1}{2}}\rho, \nu^{\frac{c+1}{2}}\rho])^2]}(T)$, etc. Also note the following: for (2), one shows

$$\varepsilon_T(\rho, c_+)\varepsilon_T(\rho, c)^{-1} = 1 \Leftrightarrow \varepsilon_{T'}(\rho, c_+ - 2)\varepsilon_{T'}(\rho, c)^{-1} = 1$$

and for (3), one has $m_{T'}(\rho, c_+) = 2$.

It remains to deal with the possibility $Jord_\rho(T') = \{(\rho, c)\}$, which implies $Jord_\rho(T) = \{(\rho, c)\}$. Note that this requires $Jord_\rho(\delta) = \emptyset$, hence $red(\rho; \sigma) = 0$ or $\frac{1}{2}$. For $red(\rho; \sigma) = \frac{1}{2}$, we can essentially think of $(\rho, c_-) = (\rho, 0)$ and use the same argument as for $(\rho, c_-) \neq (\rho', a)$ above to show $\varepsilon_{T'}(\rho, c) = 1$ if and only if $\varepsilon_T(\rho, c) = 1$. If $red(\rho; \sigma) = 0$, it suffices to show $\varepsilon_{T'}(\rho, c) = \eta \Rightarrow \varepsilon_T(\rho, c) = \eta$ for $\eta = \pm 1$, which may be done using an argument similar to those in the proof of Lemma 2.2.8. □

3.4. Tempered representations II

This section builds on the results of Sect. 3.3 to prove the tempered analogue of Note 1.3.1, first for elliptic tempered representations then, building on that, for general tempered representations.

Lemma 3.4.1. *Suppose T is elliptic tempered with*

$$T \leq \delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-c_k+1}{2}}\rho_k, \nu^{\frac{c_k-1}{2}}\rho_k\right]\right) \rtimes \delta.$$

Let $T_k \leq \delta([\nu^{\frac{-c_k+1}{2}}\rho_k, \nu^{\frac{c_k-1}{2}}\rho_k]) \rtimes \delta$ be such that

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-c_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{c_{k-1}-1}{2}}\rho_{k-1}\right]\right) \rtimes T_k.$$

Suppose $(\rho, a), (\rho, a_-) \in Jord(T)$ with $m(\rho, a) = m(\rho, a_-) = 1$ (so both are in $Jord(\delta)$) and $\varepsilon_T(\rho, a)\varepsilon_T(\rho, a_-)^{-1} = 1$. Write $\delta \hookrightarrow \delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \delta'$ as in [21]. Then,

$$\begin{aligned}
&\mu^*_{\delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1,\nu^{\frac{c_1-1}{2}}\rho_1\right]\right)\times\cdots\times\delta\left(\left[\nu^{\frac{-c_{k-1}+1}{2}}\rho_{k-1},\nu^{\frac{c_{k-1}-1}{2}}\rho_{k-1}\right]\right)\times\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho\right]\right)}(T) \\
&= \delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-c_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{c_{k-1}-1}{2}}\rho_{k-1}\right]\right) \\
&\times \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \otimes T'_k,
\end{aligned}$$

where $T'_k \leq \delta([\nu^{\frac{-c_k+1}{2}}\rho_k, \nu^{\frac{c_k-1}{2}}\rho_k]) \rtimes \delta'$ has $Jord(T'_k) = Jord(T_k) \setminus \{(\rho, a), (\rho, a_-)\}$ and $\varepsilon_{T'_k}$ given by restriction of ε_{T_k}.

Proof. For any of the 2^{k-1} components $T^{\#}$ of $\delta([\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1]) \times \cdots \times \delta([\nu^{\frac{-c_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{c_{k-1}-1}{2}}\rho_{k-1}]) \rtimes T_k$, we have (Lemma 3.3.1)

$$\mu^*_{\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho\right]\right)}(T_k) \neq 0$$
$$\Downarrow$$
$$\mu^*_{\delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1,\nu^{\frac{c_1-1}{2}}\rho_1\right]\right)\times\cdots\times\delta\left(\left[\nu^{\frac{-c_{k-1}+1}{2}}\rho_{k-1},\nu^{\frac{c_{k-1}-1}{2}}\rho_{k-1}\right]\right)\times\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho\right]\right)}(T^{\#}) \neq 0.$$

Therefore, it suffices to show for $\mathcal{I} = \delta([\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1]) \times \cdots \times \delta([\nu^{\frac{-c_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{c_{k-1}-1}{2}}\rho_{k-1}]) \rtimes T_k$ that

$$\mu^*_{\delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1,\nu^{\frac{c_1-1}{2}}\rho_1\right]\right)\times\cdots\times\delta\left(\left[\nu^{\frac{-c_{k-1}+1}{2}}\rho_{k-1},\nu^{\frac{c_{k-1}-1}{2}}\rho_{k-1}\right]\right)\times\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho\right]\right)}(\mathcal{I})$$
$$= 2^{k-1}\delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-c_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{c_{k-1}-1}{2}}\rho_{k-1}\right]\right)$$
$$\times\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \otimes T'_k.$$

Now,

$$\mu^*(\mathcal{I}) = M^*\left(\delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1\right]\right)\right) \times \cdots$$
$$\times M^*\left(\delta\left(\left[\nu^{\frac{-c_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{c_{k-1}-1}{2}}\rho_{k-1}\right]\right)\right) \rtimes \mu^*(T_k).$$

Further,

$$M^*\left(\delta\left(\left[\nu^{\frac{-c_\ell+1}{2}}\rho_\ell, \nu^{\frac{c_\ell-1}{2}}\rho_\ell\right]\right)\right) = \sum_{i_\ell=\frac{-c_\ell+1}{2}}^{\frac{c_\ell-1}{2}} \sum_{j_\ell=i_\ell}^{\frac{c_\ell+1}{2}} \delta\left(\left[\nu^{-i_\ell+1}\rho_\ell, \nu^{\frac{c_\ell-1}{2}}\rho_\ell\right]\right)$$
$$\times\delta\left(\left[\nu^{j_\ell}\rho_\ell, \nu^{\frac{c_\ell-1}{2}}\rho_\ell\right]\right) \otimes \delta\left(\left[\nu^{j_\ell-1}\rho_\ell, \nu^{i_\ell}\rho_\ell\right]\right).$$

Focusing on the copy of $\delta([\nu^{\frac{-c_\ell+1}{2}}\rho_\ell, \nu^{\frac{c_\ell-1}{2}}\rho_\ell])$ in $\delta([\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1]) \times \cdots \times \delta([\nu^{\frac{-c_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{c_{k-1}-1}{2}}\rho_{k-1}]) \times \delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])$, we claim either $j_\ell = i_\ell = \frac{-c_\ell+1}{2}$ or $j_\ell = i_\ell = \frac{c_\ell+1}{2}$. If not—and noting that $M^*_{\nu^{\frac{c_\ell-1}{2}}\rho_\ell}\left(\delta([\nu^{\frac{-c_m+1}{2}}\rho_m, \nu^{\frac{c_m-1}{2}}\rho_m])\right) = 0$ if $m \neq \ell$—we must have a nontrivial contribution of either $\delta([\nu^{\frac{-c_\ell+1}{2}}\rho_\ell, \nu^{-i_\ell}\rho_\ell])$ or $\delta([\nu^{\frac{-c_\ell+1}{2}}\rho_\ell, \nu^{-j_\ell-1}\rho_\ell])$ from

$$\left(\prod_{j\neq\ell} M^*\left(\delta\left(\left[\nu^{\frac{-c_j+1}{2}}\rho_j, \nu^{\frac{c_j-1}{2}}\rho_j\right]\right)\right)\right) \rtimes \mu^*(T_k)$$
$$= \mu^*\left(\left(\prod_{j\neq\ell} \delta\left(\left[\nu^{\frac{-c_j+1}{2}}\rho_j, \nu^{\frac{c_j-1}{2}}\rho_j\right]\right)\right) \rtimes T_k\right)$$

to δ. However, either of these possibilities have negative central exponent, so would contradict the Casselman criterion. Noting that $j_\ell = i_\ell = \frac{-c_\ell+1}{2}$ or $j_\ell = i_\ell + \frac{c_\ell-1}{2}$ both contribute $\delta\left(\left[\nu^{\frac{-c_\ell+1}{2}}\rho_\ell, \nu^{\frac{c_\ell-1}{2}}\rho_\ell\right]\right)$ to the product, it now follows that

$$\mu^*_{\delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1,\nu^{\frac{c_1-1}{2}}\rho_1\right]\right)\times\cdots\times\delta\left(\left[\nu^{\frac{-c_{k-1}+1}{2}}\rho_{k-1},\nu^{\frac{c_{k-1}-1}{2}}\rho_{k-1}\right]\right)\times\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho\right]\right)}(\mathcal{I})$$
$$= 2^{k-1}\left(\delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-c_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{c_{k-1}-1}{2}}\rho_{k-1}\right]\right) \otimes 1\right)$$
$$\rtimes \mu^*_{\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho\right]\right)}(T_k).$$

Thus, the lemma follows if we show $\mu^*_{\delta([\nu^{\frac{-a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])}(T_k) = \delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \otimes T'_k$. Since $\varepsilon_{T_k}(\rho, a)\varepsilon_{T_k}(\rho, a_-)^{-1} = \varepsilon_\delta(\rho, a)\varepsilon_\delta(\rho, a_-)^{-1} = 1$, this follows from Lemma 3.3.1 and a μ^* argument like that above. □

Lemma 3.4.2. *Let $T \leq \delta([\nu^{\frac{-c+1}{2}}\rho', \nu^{\frac{c-1}{2}}\rho']) \rtimes \delta$ be elliptic tempered and $(\rho, a), (\rho, a_-) \in Jord(T)$ with $m_T(\rho, a) = 1$ (so $(\rho, a) \in Jord(\delta)$). Suppose b satisfies $a_- < b < a$ and $b \equiv a \bmod 2$. Then,*

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{b+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes T',$$

where T' is elliptic tempered with the following data: $Jord(T') = (Jord(T)\setminus\{(\rho, a)\}) \cup \{(\rho, b)\}$, $\sigma_{T'} = \sigma_T$,

$$m_{T'}(\rho'', x) = \begin{cases} m_T(\rho'', x) & \text{if } (\rho'', x) \neq (\rho, a), (\rho, b), \\ 0 & \text{if } (\rho'', x) = (\rho, a), \\ 1 & \text{if } (\rho'', x) = (\rho, b), \end{cases}$$

and $\varepsilon_{T'}$ determined by $\varepsilon_{T'}|_{S_T \cap S_{T'}} = \varepsilon_T|_{S_T\cap S_{T'}}$, $\varepsilon_{T'}(\rho, b)\varepsilon_{T'}(\rho, x)^{-1} = \varepsilon_T(\rho, a)\varepsilon_T(\rho, x)^{-1}$, and if defined, $\varepsilon_{T'}(\rho, b) = \varepsilon_T(\rho, a)$. Note that if a_- does not exist, we may replace a_- with 1 or 2 in the inequality $a_- < b < a$, whichever matches the parity of $2red(\rho; \sigma) + 1$.

Proof. By Corollary 2.1.3, noting that $\rho' \cong \rho$ has either $c < b$ or $c > a$,

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-c+1}{2}}\rho', \nu^{\frac{c-1}{2}}\rho'\right]\right) \times \delta\left(\left[\nu^{\frac{b+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \delta' \cong \delta\left(\left[\nu^{\frac{b+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)$$
$$\times \delta\left(\left[\nu^{\frac{-c+1}{2}}\rho', \nu^{\frac{c-1}{2}}\rho'\right]\right) \rtimes \delta' \Downarrow \text{(Lemma 1.2.3)} T \hookrightarrow \delta\left(\left[\nu^{\frac{b+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes T'$$

for some irreducible $T' \leq \delta([\nu^{\frac{-c+1}{2}}\rho', \nu^{\frac{c-1}{2}}\rho']) \rtimes \delta'$ (δ' as in Corollary 2.1.3). It follows from Lemma 2.4.1 and partial cuspidal support considerations that the data $Jord(T')$, $\sigma_{T'}$, and $m_{T'}$ are as claimed. It remains to show $\varepsilon_{T'}$ is as claimed. The values of $\varepsilon_{T'}$ match those of $\varepsilon_{\delta'}$ on $S_{\delta'} \subset S_{T'}$. Thus, to finish, it is enough to show $\varepsilon_{T'}$ is as claimed when evaluated at elements of $S_{T'}$ which involve (ρ', c). By Lemma 2.3.1, it is enough to do show this for one such element of $S_{T'}$. We do this in three cases. Note that we assume a_- exists in the arguments below, but the arguments work if a_- does not exist and is replaced with 1 or 2 as in the statement of the lemma.

Case 1: $(\rho', c_-) \in Jord(T)$ exists

First, suppose either (1) $\rho' \not\cong \rho$, or (2) $\rho' \cong \rho$ but $a \neq c_-$, in which case either $c_- > a$ or $c < a$ (which implies $c < b$). Then, a straightforward μ^* argument tells us

$$\mu^*_{\delta\left(\left[\nu^{\frac{c_-+1}{2}}\rho', \nu^{\frac{c-1}{2}}\rho'\right]\right)}(T) \neq 0 \Leftrightarrow \mu^*_{\delta\left(\left[\nu^{\frac{c_-+1}{2}}\rho', \nu^{\frac{c-1}{2}}\rho'\right]\right)}(T') \neq 0.$$

As $(\rho', c_-) \neq (\rho, a), (\rho, b)$, it follows immediately that

$$\varepsilon_{T'}(\rho', c)\varepsilon_{T'}(\rho', c_-)^{-1} = 1 \Leftrightarrow \varepsilon_T(\rho', c)\varepsilon_T(\rho', c_-)^{-1} = 1,$$

which suffices.

Now, suppose $\rho' \cong \rho$ and $c_- = a$. It then follows (using Corollary 2.1.3 applied to δ) that

$$\begin{aligned} &\mu^*_{\delta\left(\left[\nu^{\frac{b+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)\times[\delta\left(\left[\nu^{\frac{b+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right)^2]}\left(\delta\left(\left[\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right) \rtimes \delta\right) \\ &= \delta\left(\left[\nu^{\frac{b+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{b+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right)^2 \otimes \left(\delta\left(\left[\nu^{\frac{-b+1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right) \rtimes \delta'\right), \end{aligned}$$

noting that $\delta([\nu^{\frac{-b+1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho]) \rtimes \delta'$ is irreducible. In particular,

$$\mu^*_{[\delta\left(\left[\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right)^2]}(T) \neq 0 \Leftrightarrow \mu^*_{\delta\left(\left[\nu^{\frac{b+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)\times[\delta\left(\left[\nu^{\frac{b+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho\right]\right)^2]}(T) \neq 0$$

(as ($\Leftarrow$) is clear and both hold for exactly one component of $\delta([\nu^{\frac{-c+1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \delta$). Using this observation, one can now argue as in (the later part of) the proof of Proposition 2.1.2 to show that $\varepsilon_T(\rho, c)\varepsilon_T(\rho, a)^{-1} = 1 \Leftrightarrow \varepsilon_{T'}(\rho, c)\varepsilon_{T'}(\rho, b)^{-1} = 1$, as needed.

Case 2: $(\rho', c_+) \in Jord(T)$ exists

If $\rho' \not\cong \rho$ or either $c_+ < a$ or $c > a$, the argument is similar to that given in the beginning of Case 1.

Suppose $\rho' \cong \rho$ and $c_+ = a$ (so $c = a_-$). In this case, we claim

$$\mu^*_{\delta\left(\left[\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)}(T) \neq 0 \Leftrightarrow \mu^*_{\delta\left(\left[\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho\right]\right)}(T') \neq 0.$$

The implication ($\Rightarrow$) is a standard μ^* argument on $\mu^*(\delta([\nu^{\frac{b+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes T')$. The implication ($\Leftarrow$) follows directly from $\mu^*_{\delta([\nu^{\frac{c+1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho])}(T') \neq 0$ and $\mu^*(T) \geq \delta([\nu^{\frac{b+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \otimes T'$. It is then immediate that

$$\varepsilon_{T'}(\rho, c)\varepsilon_{T'}(\rho, b)^{-1} = 1 \Leftrightarrow \varepsilon_T(\rho, c)\varepsilon_T(\rho, a)^{-1} = 1,$$

as needed.

Case 3: $Jord_{\rho'}(T)=\{(\rho',c)\}$ (so $Jord_{\rho'}(\delta)=\emptyset$)

Of course, $\rho'\not\cong\rho$ in this case. Further, since $Jord_{\rho'}(\delta)=\emptyset$, we must have $red(\rho';\sigma)=0$ or $\frac{1}{2}$.

If $red(\rho';\sigma)=0$ it follows from Corollary 7.5 and Definition 7.6 of [14] (whose proof uses only the μ^* structure and results on general linear groups, hence holds in the generality needed here) that

$$T\hookrightarrow\Lambda\rtimes T(\delta\left(\left[\nu\rho',\nu^{\frac{c-1}{2}}\rho'\right]\right)^2;\tau_\eta(\rho;\sigma))\text{ for some irreducible }\Lambda$$
$$\Updownarrow$$
$$T'\hookrightarrow\Lambda'\rtimes T(\delta\left(\left[\nu\rho',\nu^{\frac{c-1}{2}}\rho'\right]\right)^2;\tau_\eta(\rho;\sigma))\text{ for some irreducible }\Lambda'.$$

It then follows immediately from the definition in Sect. 2.2 that $\varepsilon_{T'}(\rho',c)=\varepsilon_T(\rho',c)$, as needed.

If $red(\rho';\sigma)=\frac{1}{2}$, an argument like that in (the later part of) the proof of Proposition 2.1.2 tells us $\varepsilon_{T'}(\rho',c)=1\Leftrightarrow\varepsilon_T(\rho',c)=1$, finishing the proof. □

Proposition 3.4.3. *Suppose T is elliptic tempered with* $(\rho,a),(\rho,a_-)\in Jord(T)$. *Let* $m=min(m(\rho,a),m(\rho,a_-))$. *If* $\varepsilon(\rho,a)\varepsilon(\rho,a_-)^{-1}=1$, *then* $T\hookrightarrow\delta([\nu^{\frac{-a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])^m\rtimes T'$, *where* T' *is an elliptic tempered representation whose data is obtained as follows: one removes m copies each of* (ρ,a) *and* (ρ,a_-) *from Jord(T);* $\varepsilon_{T'}$ *is then the restriction of* ε_T.

Proof. We have four cases.

Case 1: $m(\rho,a)=m(\rho,a_-)=1$

Since $\varepsilon_T(\rho,a)\varepsilon_T(\rho,a_-)^{-1}=\varepsilon_\delta(\rho,a)\varepsilon_\delta(\rho,a_-)^{-1}=1$, we have

$$\begin{aligned}T&\hookrightarrow\delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1,\nu^{\frac{c_1-1}{2}}\rho_1\right]\right)\times\cdots\times\delta\left(\left[\nu^{\frac{-c_k+1}{2}}\rho_k,\nu^{\frac{c_k-1}{2}}\rho_k\right]\right)\rtimes\delta\\&\hookrightarrow\delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1,\nu^{\frac{c_1-1}{2}}\rho_1\right]\right)\times\cdots\times\delta\left(\left[\nu^{\frac{-c_k+1}{2}}\rho_k,\nu^{\frac{c_k-1}{2}}\rho_k\right]\right)\\&\quad\times\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho\right]\right)\rtimes\delta'\\&\cong\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho\right]\right)\times\delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1,\nu^{\frac{c_1-1}{2}}\rho_1\right]\right)\times\cdots\\&\quad\times\delta\left(\left[\nu^{\frac{-c_k+1}{2}}\rho_k,\nu^{\frac{c_k-1}{2}}\rho_k\right]\right)\rtimes\delta'\end{aligned}$$
$$\Downarrow$$
$$T\hookrightarrow\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho\right]\right)\rtimes T^*$$

for some $T^*\leq\delta([\nu^{\frac{-c_1+1}{2}}\rho_1,\nu^{\frac{c_1-1}{2}}\rho_1])\times\cdots\times\delta([\nu^{\frac{-c_k+1}{2}}\rho_k,\nu^{\frac{c_k-1}{2}}\rho_k])\rtimes\delta'$. We must show $T^*\cong T'$. In particular, since $Jord(T^*)=Jord(T')$, $m_{T^*}=m_{T'}$, and the partial cuspidal supports match, we must show $\varepsilon_{T^*}=\varepsilon_{T'}$. For this, it suffices (from the definition) to show $\varepsilon_{T^*}|_{S'_\ell}=\varepsilon_{T'}|_{S'_\ell}$ for $\ell=1,\ldots,k$, where S'_ℓ is the domain associated to $\delta([\nu^{\frac{-c_\ell+1}{2}}\rho_\ell,\nu^{\frac{c_\ell-1}{2}}\rho_\ell])\rtimes\delta'$. Without loss of generality, we take $\ell=k$.

Write

$$T^* \hookrightarrow \delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-c_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{c_{k-1}-1}{2}}\rho_{k-1}\right]\right) \rtimes T_k^*$$

for some $T_k^* \leq \delta([\nu^{\frac{-c_k+1}{2}}\rho_k, \nu^{\frac{c_k-1}{2}}\rho_k]) \rtimes \delta'$. By definition, $\varepsilon_{T^*}|_{S_k'} = \varepsilon_{T_k^*}$. By Frobenius reciprocity,

$$\begin{aligned} T \hookrightarrow \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1\right]\right) \\ \times \cdots \times \delta\left(\left[\nu^{\frac{-c_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{c_{k-1}-1}{2}}\rho_{k-1}\right]\right) \rtimes T_k^* \\ \Downarrow \\ \mu^*(T) \geq \Bigg(\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \\ \times \delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-c_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{c_{k-1}-1}{2}}\rho_{k-1}\right]\right)\Bigg) \otimes T_k^*. \end{aligned}$$

In Lemma 3.4.1, we have $T_k' = T_k^* \Rightarrow \varepsilon_{T_k^*} = \varepsilon_{T_k'}$. From their descriptions, we have $\varepsilon_{T'}|_{S_k'} = \varepsilon_{T_k'}$. Combining the equalities, we get $\varepsilon_{T^*}|_{S_k'} = \varepsilon_{T_k^*} = \varepsilon_{T_k'} = \varepsilon_{T'}|_{S_k'}$, as needed.

Case 2: $m(\rho, a) = 1, m(\rho, a_-) = 2$

Without loss of generality, let $(\rho_k, c_k) = (\rho, a_-)$ and write

$$\begin{aligned} T \hookrightarrow \delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-c_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{c_{k-1}-1}{2}}\rho_{k-1}\right]\right) \times \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right) \rtimes \delta \\ \Downarrow \\ T \hookrightarrow \Lambda_i \times \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right) \rtimes T_i, \end{aligned}$$

where $\Lambda_i = \delta([\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1]) \times \cdots \times \delta([\nu^{\frac{-c_{i-1}+1}{2}}\rho_{i-1}, \nu^{\frac{c_{i-1}-1}{2}}\rho_{i-1}]) \times \delta([\nu^{\frac{-c_{i+1}+1}{2}}\rho_{i+1}, \nu^{\frac{c_{i+1}-1}{2}}\rho_{i+1}]) \times \cdots \times \delta([\nu^{\frac{-c_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{c_{k-1}-1}{2}}\rho_{k-1}])$ and $T_i \leq \delta([\nu^{\frac{-c_i+1}{2}}\rho_i, \nu^{\frac{c_i-1}{2}}\rho_i]) \rtimes \delta$. By definition, $\varepsilon_T|_{S_{T_i}} = \varepsilon_{T_i}$. By Lemma 3.4.2, we have $T_i \hookrightarrow \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes T_i'$ with T_i' as in Lemma 3.4.2. Then,

$$\begin{aligned} T &\hookrightarrow \Lambda_i \times \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes T_i' \\ &\cong \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \times \Lambda_i \rtimes T_i' \\ &\Downarrow \text{ (Lemma 1.2.3)} \\ T &\hookrightarrow \mathcal{L}\left(\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right), \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)\right) \times \Lambda_i \rtimes T_i' \\ &\text{or} \\ T &\hookrightarrow \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \times \Lambda_i \rtimes T_i'. \end{aligned}$$

As $\mu^*_{\delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])}\left(\mathcal{L}\left(\delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho]), \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])\right) \times \Lambda_i \rtimes T_i'\right) = 0$, we must have the latter. It follows directly that $T \hookrightarrow \delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}$

$\rho]) \rtimes T'$ for some T' having $Jord(T')$, $m_{T'}$ and partial cuspidal support as in the statement of the proposition. It remains to show $\varepsilon_{T'}$ is as in the statement of the proposition. To this end, a straightforward μ^* argument gives $\mu^*_{\delta([\nu^{\frac{-a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])}\left(\delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes T'\right) = 2\delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \otimes T'$. It then follows that $T' \leq \Lambda_i \rtimes T'_i$ for all i. Thus, by definition, $\varepsilon_{T'}|_{S_{T'_i}} = \varepsilon_{T'_i}$. From above, since $\varepsilon_{T_i}|_{S_{T'_i}} = \varepsilon_{T'_i}$, we get $\varepsilon_{T'}|_{S_{T'_i}} = \varepsilon_T|_{S_{T'_i}}$ for all i. As $\cup_i Jord(T_i) = Jord(T')$, this tells us $\varepsilon_{T'} = \varepsilon_T|_{S_{T'}}$, as needed.

Case 3: $m(\rho, a) = 2, m(\rho, a_-) = 1$

Without loss of generality, let $(\rho_k, c_k) = (\rho, a)$ and write

$$\begin{gathered} T \hookrightarrow \delta\left(\left[\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1\right]\right) \times \cdots \times \delta\left(\left[\nu^{\frac{-c_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{c_{k-1}-1}{2}}\rho_{k-1}\right]\right) \\ \times \delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \delta \\ \Downarrow \\ T \hookrightarrow \Lambda \rtimes T_a, \end{gathered}$$

with $\Lambda = \delta([\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1]) \times \cdots \times \delta([\nu^{\frac{-c_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{c_{k-1}-1}{2}}\rho_{k-1}])$ and $T_a \leq \delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \delta$. By definition, $\varepsilon_T|_{S_{T_a}} = \varepsilon_{T_a}$. In particular, $\varepsilon_{T_a}(\rho, a)\varepsilon_{T_a}(\rho, a_-)^{-1} = 1$. Therefore, by Lemma 3.3.1,

$$T_a \hookrightarrow \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \delta_a,$$

with δ_a as in Lemma 3.3.1. Thus,

$$\begin{gathered} T \hookrightarrow \Lambda \times \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \delta_a \\ \cong \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \times \Lambda \rtimes \delta_a \\ \Downarrow \\ T \hookrightarrow \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes T^* \end{gathered}$$

for some $T^* \leq \Lambda \rtimes \delta_a$. We must show $T^* \cong T'$, T' as in the statement of the proposition.

As in the previous case, let $\Lambda_i = \delta([\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1]) \times \cdots \times \delta([\nu^{\frac{-c_{i-1}+1}{2}}\rho_{i-1}, \nu^{\frac{c_{i-1}-1}{2}}\rho_{i-1}]) \times \delta([\nu^{\frac{-c_{i+1}+1}{2}}\rho_{i+1}, \nu^{\frac{c_{i+1}-1}{2}}\rho_{i+1}]) \times \cdots \times \delta([\nu^{\frac{-c_{k-1}+1}{2}}\rho_{k-1}, \nu^{\frac{c_{k-1}-1}{2}}\rho_{k-1}])$. Then,

$$\begin{gathered} T \hookrightarrow \delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \times \Lambda_i \times \delta\left(\left[\nu^{\frac{-c_i+1}{2}}\rho_i, \nu^{\frac{c_i-1}{2}}\rho_i\right]\right) \rtimes \delta \\ \Downarrow \\ T \hookrightarrow \delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \times \Lambda_i \rtimes T_i \end{gathered}$$

for some irreducible $T_i \leq \delta([\nu^{\frac{-c_i+1}{2}}\rho_i, \nu^{\frac{c_i-1}{2}}\rho_i]) \rtimes \delta$. By definition, $\varepsilon_T|_{S_{T_i}} = \varepsilon_{T_i}$. On the other hand,

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \times \Lambda_i \times \delta\left(\left[\nu^{\frac{-c_i+1}{2}}\rho_i, \nu^{\frac{c_i-1}{2}}\rho_i\right]\right) \rtimes \delta_a$$
$$\Downarrow$$
$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \times \Lambda_i \rtimes T_i^*$$

for some $T_i^* \leq \delta([\nu^{\frac{-c_i+1}{2}}\rho_i, \nu^{\frac{c_i-1}{2}}\rho_i]) \rtimes \delta_a$. One can show that T^* is the unique component of $\Lambda \rtimes \delta_a$ such that $T \hookrightarrow \delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes T^*$. Then, we have $T^* \leq \Lambda_i \rtimes T_i^* \Rightarrow \varepsilon_{T^*}|_{S_i^*} = \varepsilon_{T_i^*}$. Further, by Lemma 3.4.2, we know $Jord(T_i^*) = (Jord(T_i) \setminus \{(\rho, a_-)\}) \cup \{(\rho, a)\}$ and $\varepsilon_{T_i^*}$ matches ε_{T_i} if a_- is replaced by a. Since we have

$$\varepsilon_T|_{S_{T_i}} = \varepsilon_{T_i} \quad \text{and} \quad \varepsilon_{T^*}|_{S_{T_i^*}} = \varepsilon_{T_i^*},$$

for any i, we see that ε_{T^*} and ε_T match if a_- is replaced by a. That is, $T^* = T'$, as needed.

Case 4: $m(\rho, a) = m(\rho, a_-) = 2$

We have

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes T'$$

for a unique irreducible $T' \leq \delta([\nu^{\frac{-c_1+1}{2}}\rho_1, \nu^{\frac{c_1-1}{2}}\rho_1]) \times \cdots \times \delta([\nu^{\frac{-c_{k-2}+1}{2}}\rho_{k-2}, \nu^{\frac{c_{k-2}-1}{2}}\rho_{k-2}]) \rtimes \delta$; it follows easily from the definition that $\varepsilon_T|_{S_{T'}} = \varepsilon_{T'}$, so matches T' from the statement of the proposition. Now, from Case 4 in the proof of Proposition 2.3.2 and the proof of Lemma 2.3.3, we have

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)^2 \rtimes \left(\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right) \rtimes T''\right)$$

for some $T'' \leq \delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho]) \rtimes T'$. Since either possible T'' has $\delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho]) \rtimes T''$ irreducible, we see that

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)^2 \times \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right)^2 \rtimes T'$$
$$\Downarrow \text{(Lemma 1.2.3)}$$
$$T \hookrightarrow \lambda \rtimes T'$$

for some irreducible

$$\begin{aligned}\lambda &\leq \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)^2 \times \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right)^2 \\ &= \left(\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) + \mathcal{L}(\delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a_--1}{2}}\rho\right]\right), \\ &\quad \times \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right))\right)^2.\end{aligned}$$

Of the four products, only $\delta([\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])^2$ allows $\mu^*_{[\delta([\nu^{\frac{a_-+1}{2}}\rho,\nu^{\frac{a-1}{2}}\rho])^2]} \neq 0$. Thus,

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)^2 \rtimes T',$$

as needed. This finishes Case 4 and the proposition. □

Corollary 3.4.4. *Let T be an irreducible tempered representation with data $(Jord, \sigma, \varepsilon, m)$. For $(\rho, a) \in Jord_{ell}$, we define a_- as the largest value of $b < a$ such that $(\rho, b) \in Jord_{ell}$ (so $\varepsilon(\rho, a)\varepsilon(\rho, a_-)^{-1}$ is defined). If $\varepsilon(\rho, a)\varepsilon(\rho, a_-)^{-1} = 1$, we have*

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right)^m \rtimes T'$$

where $m = min(m(\rho, a), m(\rho, a_-))$ and T' is the irreducible tempered representation whose data is obtained as follows: one removes m copies each of (ρ, a) and (ρ, a_-) from Jord(T); $\varepsilon_{T'}$ is then the restriction of ε_T.

Proof. This follows from Proposition 3.4.3 and Eq. (2.2). □

Note 3.4.5. *From Lemmas 2.3.5 and 2.3.7 and Eq. 2.2, we have the following:*

(1) If $red(\rho; \sigma) \equiv \frac{1}{2}$ mod 1,

$$\varepsilon(\rho, a_{min}) = 1 \Leftrightarrow T \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{a_{min}-1}{2}}\rho])^{m(\rho,a_{min})} \rtimes \theta$$

for some irreducible θ (where a_{min} is the smallest value of b such that $(\rho, b) \in Jord_{ell}$).

(2) If $red(\rho; \sigma) = 0$,

$$\varepsilon(\rho, a_{max}) = \eta$$
$$\Updownarrow$$
$$T \hookrightarrow \lambda \rtimes \left(\delta\left(\left[\nu^{\frac{-a_{max}+1}{2}}\rho, \nu^{\frac{a_{max}-1}{2}}\rho\right]\right)^{\frac{1}{2}(m(\rho,a_{max})-m_0)} \rtimes T(\delta\left(\left[\nu\rho, \nu^{\frac{a_{max}-1}{2}}\rho\right]\right)^{m_0}; \tau_\eta(\rho; \sigma))\right),$$

for some irreducible λ, where $m_0 = 1$ or 2, whichever matches the parity of $m(\rho, a_{max})$ (and a_{max} is the largest value of b such that $(\rho, b) \in Jord_{ell}$).

3.5. Intertwining operators and generic representations

In this section, we consider the action of normalized standard intertwining operators on the components of $\delta_1 \times \cdots \times \delta_k \rtimes \delta$ in the case where δ is generic (i.e., admitting a Whittaker model with respect to some fixed character) and the components are elliptic tempered. As we use the results of Goldberg [6–8] here, we assume char $F = 0$ and restrict our discussion to the split orthogonal and symplectic groups and quasisplit unitary groups. In particular, Goldberg has determined the R-groups in this case, and we have

$$\mathbb{C}[R] \cong \mathrm{Hom}_G(\delta_1 \times \cdots \times \delta_k \rtimes \delta, \delta_1 \times \cdots \times \delta_k \rtimes \delta),$$

with the isomorphism implemented by $r \longmapsto A_r$, A_r the corresponding normalized standard intertwining operator. As the induced representation has $R \cong (\mathbb{Z}/2\mathbb{Z})^k$, the normalized standard intertwining operator acts on the isotypic subspaces as ± 1. For a given component T, we show that this is essentially given by ε_T.

Recall that discrete series of general linear groups are generic [12]. For δ to be generic, we must have the partial cuspidal support σ generic. It then follows from the results of Shahidi [27,28] that $red(\rho;\sigma) \in \{0, \frac{1}{2}, 1\}$. In the case where $red(\rho;\sigma) = 0$, the parameterization of discrete series requires a choice of components of $\rho \rtimes \sigma$. As is done in [32], we make the choice so that $\tau_{+1}(\rho;\sigma)$ is the generic component (also the choice needed in [10]—see Note 3.5.1 below).

Note 3.5.1. *Suppose δ is a discrete series representation with generic partial cuspidal support. Then, it follows from section 3 of* [10] *that δ is generic if and only if ε_δ is trivial (i.e., identically 1 on S_δ).*

Lemma 3.5.2. *Suppose δ is a generic discrete series, $\rho \cong \tilde{\rho}$ with $\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \delta$ reducible. A component $T \le \delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \delta$ is generic if and only if ε_T is trivial.*

Proof. The proof may be done in four cases: (ρ, a_-) exists (necessarily in $Jord(\delta)$), (ρ, a_+) exists (also necessarily in $Jord(\delta)$), $Jord(\delta) = \emptyset$ with $red(\rho;\sigma) = \frac{1}{2}$, and $Jord(\delta) = \emptyset$ with $red(\rho;\sigma) = 0$.

We start with the case $(\rho, a_-) \in Jord(\delta)$ exists. By Note 3.5.1, it suffices to show that T is generic if and only if $\varepsilon_T(\rho, a)\varepsilon_T(\rho, a_-) = 1$. We have

$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{-a_--1}{2}}\rho\right]\right) \rtimes \delta$$
$$\Downarrow \text{ (Lemma 5.5 [14])}$$
$$T \hookrightarrow \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \theta$$

for some irreducible $\theta \le \delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{-a_--1}{2}}\rho]) \rtimes \delta = \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \delta$. Let δ^* be the discrete series with the same partial cuspidal support as δ,

$$Jord(\delta^*) = (Jord(\delta) \setminus \{(\rho, a_-)\}) \cup \{(\rho, a)\},$$

and ε_{δ^*} trivial, noting that the admissibility of the data for δ^* is immediate from that for δ. By Note 3.5.1 and Corollary 2.1.3, δ^* is the generic irreducible subquotient of $\delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \delta$; by Note 2.1.4, $\delta^* \hookrightarrow \delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \delta$.

Now, suppose T_- is the component of $\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \delta$ having $\varepsilon_{T_-}(\rho, a)\varepsilon_{T_-}(\rho, a_-)^{-1} = -1$. In this case, we claim that $\theta \ne \delta^*$. Were that the case, we would have

$$\begin{aligned} T_- &\hookrightarrow \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \delta^* \\ &\hookrightarrow \delta\left(\left[\nu^{\frac{-a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \rtimes \delta \end{aligned}$$
$$\Downarrow$$
$$\mu^*_{[\delta([\nu^{\frac{a_-+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho])^2]}(T_-) \ne 0$$
$$\Downarrow$$
$$\varepsilon_{T_-}(\rho, a)\varepsilon_{T_-}(\rho, a_-)^{-1} = -1,$$

a contradiction. Thus $\theta \neq \delta^*$, from which it follows that T_- is not generic. The generic component must then be the component satisfying $\varepsilon_T(\rho, a)\varepsilon_T(\rho, a_-) = 1$, as needed.

The case $Jord(\delta) = \emptyset$ with $red(\rho; \sigma) = \frac{1}{2}$ is similar (replacing a_- by 0).

The case $(\rho, a_+) \in Jord(\delta)$ follows the same basic lines, but is a little easier. Here, we have

$$\begin{gathered} T \hookrightarrow \delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \\ \rtimes\delta \hookrightarrow \delta\left(\left[\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho\right]\right) \times \delta\left(\left[\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho\right]\right) \rtimes \delta' \\ \Downarrow \text{ (Lemma 5.5 [14])} \\ T \hookrightarrow \lambda \rtimes \delta' \end{gathered}$$

for some irreducible $\lambda \leq \delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \times \delta([\nu^{\frac{a+1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho])$. As above, we can show that for T_-, we must have $\lambda \neq \delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a_+-1}{2}}\rho])$ (the nongeneric subquotient), from which the result follows.

The last case—when $Jord(\delta) = \emptyset$ and $red(\rho; \sigma) = 0$—is a straightforward consequence of the definition in Sect. 2.2 once one notes that choosing $\tau_{+1}(\rho; \sigma)$ generic makes $T(\delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho])^2; \tau_{+1}(\rho; \sigma))$ the generic component of $\delta([\nu^{\frac{-a+1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \delta$. □

We now take up the action of normalized standard intertwining operators. We start with the elliptic case, where the reducibility actually occurs. We continue to assume δ generic.

Write $\delta_i = \delta([\nu^{\frac{-a_i+1}{2}}\rho_i, \nu^{\frac{a_i-1}{2}}\rho_i])$ (generic by Jacquet [12]) and assume $\delta_1 \times \cdots \times \delta_k \rtimes \delta$ is elliptic. we let $r_i \in W$ denote the ith block sign change—the element which inverts δ_i and leaves the rest of $\delta_1 \otimes \cdots \otimes \delta_k \otimes \delta$ unchanged. By Goldberg's work [6–8], we know that the R-group for $\delta_1 \times \cdots \times \delta_k \rtimes \delta$ has $R = \langle r_1, \dots, r_k \rangle$. Thus, to determine the action of normalized standard intertwining operators associated to R, it is enough to do so for $r_1, \dots, r_k$. We let A_{r_i} denote the normalized standard intertwining operator associated to r_i.

Recall that the normalization of intertwining operators is such that the action on generic components is trivial.

Proposition 3.5.3. *With notation as above, suppose δ is generic and $\delta_1 \times \cdots \times \delta_k \rtimes \delta$ with δ has elliptic tempered components. If T is a component of $\delta_1 \times \cdots \times \delta_k \rtimes \delta$, then the normalized standard intertwining operator A_{r_i} acts on the T-isotypic subspace as*

(1) $\varepsilon_T(\rho_i, a_i)$ if $red(\rho; \sigma) = 0$ or $\frac{1}{2}$,
(2) $\varepsilon_T(\rho_i, a_i)\varepsilon_T(\rho_i, b)^{-1}$ if $red(\rho; \sigma) = 1$, where $(\rho, b) \in Jord(\delta)$ (noting that ε_δ trivial ensures this does not depend on the choice of (ρ, b)).

Proof. We focus on the first case; the second is similar.

First, observe that if $i = k$, we have

$$T \hookrightarrow \delta_1 \times \cdots \times \delta_{k-1} \rtimes T_k$$

for some $T_k \leq \delta_k \rtimes \delta$. By definition, $\varepsilon_T(\rho_k, a_k) = 1 \Leftrightarrow \varepsilon_{T_k}(\rho_k, a_k) = 1$. The result now follows from Lemma 3.5.2 and induction in stages.

If $i < k$, it follows from Schur's lemma that there is a unique up to scalar intertwining map

$$E : \delta_1 \times \cdots \times \delta_{i-1} \times \delta_i \times \delta_{i+1} \times \cdots \times \delta_k \longrightarrow \delta_1 \times \cdots \times \delta_{i-1} \times \delta_{i+1} \times \cdots \times \delta_k \times \delta_i,$$

which induces to give

$$\mathcal{E} : \delta_1 \times \cdots \times \delta_{i-1} \times \delta_i \times \delta_{i+1} \times \cdots \times \delta_k \rtimes \delta \longrightarrow \delta_1 \times \cdots \times \delta_{i-1} \times \delta_{i+1} \times \cdots \times \delta_k \times \delta_i \rtimes \delta.$$

If $c_i \in W$ is the element which inverts the δ_i in $\delta_1 \otimes \cdots \otimes \delta_{i-1} \otimes \delta_{i+1} \otimes \cdots \otimes \delta_k \otimes \delta_i$, we have $A_{r_i} = \mathcal{E}^{-1} A_{c_i} \mathcal{E}$ up to a scalar which may easily be seen to be 1 (e.g., consider the action on a vector in the generic subspace). The result may then be deduced from the case $i = k$ just considered. □

We close by identifying the generic tempered representations in a manner analogous to that in [10].

Corollary 3.5.4. *Suppose T is an irreducible tempered (not necessarily elliptic) representation with $T \leq \delta_1 \times \cdots \times \delta_k \rtimes \delta$, where δ is generic. Then, T is generic if and only if ε_T is trivial (i.e., identically 1 on S_T).*

Proof. In the elliptic tempered case, this follows directly from Proposition 3.5.3. For the nonelliptic case, we have

$$T \cong \delta'_1 \times \cdots \times \delta'_\ell \rtimes T_{ell},$$

which is generic if and only if T_{ell} is. As $\varepsilon_T = \varepsilon_{T_{ell}}$, the corollary follows. □

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