Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 305 (2006) 802–819
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Jacquet modules of induced representations for p-adic special orthogonal groups ☆

Chris Jantzen

Department of Mathematics, East Carolina University, Greenville, NC 27858, USA

Received 5 October 2005
Available online 9 February 2006
Communicated by Dihua Jiang

1. Introduction

In recent years, Jacquet module methods have taken on an increasingly important role in the analysis of parabolically induced representations (or subquotients thereof), especially for the classical p-adic groups $Sp(2n, F)$, $SO(2n+1, F)$, and $O(2n, F)$ (cf. [T2,J2,M-T, B-J,M1,M2], etc.). Central to this is the ability to efficiently calculate Jacquet modules of induced representations. In the case of $Sp(2n, F)$ and $SO(2n+1, F)$, this is provided by the results of [T1]; these were extended in [B] to cover $O(2n, F)$ (also, cf. [M-T] for the extension to the odd-unitary groups and non-split odd-orthogonal groups). We remark that such results were previously given for $GL(n, F)$ in [Z], though did not play a central role in the analysis of induced representations for general linear groups owing to the availability of other techniques for those groups. The purpose of this paper is to provide an analogous result for $SO(2n, F)$. While one can study representations for $SO(2n, F)$ by using $O(2n, F)$ and restricting (cf. [B-J]), having such a result for $SO(2n, F)$ would simplify matters and make it possible to work with $SO(2n, F)$ directly.

The results in [T1] for $Sp(2n, F)$ and $SO(2n+1, F)$ are based on the geometric lemma of [B-Z] (also, cf. [C, Section 6]); these are recalled as Theorem 2.1 of this paper. For the (non-connected) groups $O(2n, F)$, the more general formulation in [B-Z, Section 5] is used (cf. [B]). The results for $Sp(2n, F)$, $SO(2n+1, F)$, and $O(2n, F)$ are essentially the same. This is not too surprising as all three have the same Weyl group and further, the double-

☆ Research supported in part by NSA grant H98230-04-1-0029.
E-mail addresses: jantzenc@mail.ecu.edu, jantzen@mail.ecu.edu.

0021-8693/$ – see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.jalgebra.2005.12.026

coset representatives required in [B-Z,C] for all three families are essentially the same (cf. [J3, Lemma 3.6]). In fact, one typically addresses all three families simultaneously as the resulting combinatorics do not depend on the particular type of group. However, $SO(2n, F)$ is of a different nature, so there are certain issues involved with producing an analogous result.

The technical issues may be seen at the Weyl group level. $Sp(2n, F)$, $SO(2n+1, F)$, and $O(2n, F)$ all have the same Weyl group—consisting of permutations and sign changes on n letters. However, $SO(2n, F)$ allows only permutations and even sign changes on n letters. Further, the maximal parabolic subgroups of $SO(2n, F)$ are not quite as convenient as those of the other groups, e.g., there are two non-conjugate parabolic subgroups which one might reasonably call Siegel. Thus there are two technical issues in shifting to special even orthogonal groups: (1) keeping track of the number of sign changes (at least mod 2), and (2) choosing the correct parabolic subgroups to use. In terms of the results of [T1,B], the former amounts to adapting M^* to keep track of sign changes; the latter amounts to choosing a suitable definition of μ^*. These are addressed in Definitions 3.1 and 3.3, respectively.

With suitable definitions in place, the analogue to the results of [T1,B] is Theorem 3.4, the main result of this paper. This allows one to calculate Jacquet modules of induced representations for $SO(2n, F)$ in much the same way that [T1] allows for $Sp(2n, F)$ and $SO(2n+1, F)$ (and by [B,Z] for $O(2n, F)$, $GL(n, F)$, respectively). As with these results, the proof is essentially a calculation using the results of [B-Z,C]. However, things are a bit easier for us—the necessary double-coset representatives have already been worked out in [B], so we are saved that step. We remark that these results do give an M_D^*-Hopf module structure similar to that in [T1], though a bit more involved to set up.

We now discuss the results section by section. The next section introduces notation and reviews background material. In the third section, we give the main result (cf. Theorem 3.4) and a short example of its application to the calculation of Jacquet modules. The proof of Theorem 3.4 is a straightforward calculation; as it is not particularly short or enlightening, its proof is deferred until Section 4.

Let me close the introduction by thanking the referee, whose comments helped significantly improve the exposition of this paper.

2. Notation and preliminaries

In this section, we review some background material and notation which is needed in the rest of the paper.

Let F be a p-adic field with $\operatorname{char} F \neq 2$.

First, in general, suppose G is the F-points of a split connected reductive group defined over F. Let W_G denote the Weyl group for G. We fix a Borel subgroup of G. Suppose $P = MU$ is the Levi factorization of a standard parabolic subgroup of G. If $\mathrm{Alg}_0(G)$ denotes the category of smooth finite-length representations of G, we let $i_{G,M} : \mathrm{Alg}_0(M) \to \mathrm{Alg}_0(G)$ and $r_{M,G} : \mathrm{Alg}_0(G) \to \mathrm{Alg}_0(M)$ denote the (normalized) induction and Jacquet functors, respectively. Let $R(G)$ denote the Grothendieck group for $\mathrm{Alg}_0(G)$. We also use $i_{G,M} : R(M) \to R(G)$ and $r_{M,G} : R(G) \to R(M)$ for the semisimplified maps. Note that in what follows, we use = when working in the Grothendieck group setting; actual equivalences of representations are denoted by $\cong$.

We begin by recalling a few things about general linear groups (cf. [B-Z,Z]). Let B denote the Borel subgroup of $GL(n,F)$ consisting of the upper triangular matrices in $GL(n,F)$. This has maximal split torus consisting of the diagonal matrices in $GL(n,F)$. We have $W_{GL(n,F)} \cong \{\text{permutations on } n \text{ letters}\}$, which acts on the maximal split torus by permuting the entries. If $m_1,\dots,m_k$ are positive integers satisfying $m_1+\cdots+m_k=n$, we let $P_{(m_1,\dots,m_k)}$ denote the standard parabolic subgroup having Levi factor $M_{(m_1,\dots,m_k)} = GL(m_1,F)\times\cdots\times GL(m_k,F)$. Now, let $R=\bigoplus_{n\geqslant 0} R(GL(n,F))$. This is a $\mathbb{Z}_+$-graded Hopf algebra over $\mathbb{Z}$ (cf. [Sw] for the definition of Hopf algebra); with multiplication defined for representations by

$$m: R\otimes R\to R,$$
$$m:\tau_1\otimes\tau_2\mapsto i_{G,M}(\tau_1\otimes\tau_2),$$

where $G=GL(n_1+n_2,F)$ and $M=M_{(n_1,n_2)}$ (τ_i a representation of $GL(n_i,F)$); comultiplication is defined for representations by

$$m^*: R\to R\otimes R,$$
$$m^*:\tau\mapsto\sum_{k=0}^{n} r_{M_{(k,n-k)},G}\,\tau,$$

where τ is a representation of $GL(n,F)$. These are then extended $\mathbb{Z}$-bilinearly and $\mathbb{Z}$-linearly to obtain the bialgebra structure. Note that we will often use $\times$ to denote multiplication: $\tau_1\times\tau_2=m(\tau_1\otimes\tau_2)$. For completeness, we remark that the antipode map is given by the Zelevinsky involution (which is a special case of the duality of [Au,S-S]), though this will not play a significant role in what follows.

We now discuss orthogonal groups (cf. [B]). The special orthogonal group $SO(2n,F)$, $n\geqslant 1$, is the group

$$SO(2n,F)=\left\{X\in SL(2n,F)\mid {}^{\tau}XX=I_{2n}\right\},$$

where ${}^{\tau}X$ denotes the matrix of X transposed with respect to the second diagonal. For $n=1$, we get

$$SO(2,F)=\left\{\begin{bmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{bmatrix}\,\middle|\,\lambda\in F^{\times}\right\}\cong F^{\times}.$$

$SO(0,F)$ is defined to be the trivial group. We let

$$c=\begin{bmatrix} I & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I\end{bmatrix}.$$

This is a representative for the non-trivial element of $C = O(2n, F)/SO(2n, F)$ and acts on $SO(2n, F)$ by conjugation. We denote the trivial element of C by e (and use 1 as its representative).

We fix the Borel subgroup for $SO(2n, F)$ consisting of the upper triangular matrices in $SO(2n, F)$. This has maximal split torus consisting of the diagonal matrices in $SO(2n, F)$, which have the form $\operatorname{diag}(a_1, \ldots, a_n, a_n^{-1}, \ldots, a_1)$ with $a_1, \ldots, a_n \in F^\times$. We have $W_{SO(2n,F)} \cong \{\text{permutations and even sign changes}\}$, which acts on the maximal split torus by permuting and inverting the entries. The simple roots for $SO(2n, F)$, $n \geqslant 2$, are $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ with

$$\alpha_i = \begin{cases} e_i - e_{i+1} & \text{for } i < n, \\ e_{n-1} + e_n & \text{for } i = n. \end{cases}$$

Note that

$$c \cdot \alpha_i = \begin{cases} \alpha_i & \text{for } i < n-1, \\ \alpha_{2n-1-i} & \text{for } i = n-1, n. \end{cases}$$

The standard parabolic subgroups $P = MU$ have $M = M_{(n_1,\ldots,n_k;n_0)}$ of the form $M_{(n_1,\ldots,n_k;n_0)} = GL(n_1, F) \times \cdots \times GL(n_k, F) \times SO(2n_0, F)$, where $n_1 + \cdots + n_k + n_0 = n$. We note that in the case $n_0 = 0$, $n_k > 1$ (i.e., the corresponding subset of simple roots contains exactly one of α_{n-1}, α_n), there are two non-conjugate standard parabolic subgroups of this form. We use $M_{(n_1,\ldots,n_k;0)}$ to denote the Levi factor for the standard parabolic subgroup having α_{n-1} in the corresponding subset of simple roots; the other is then $c(M_{(n_1,\ldots,n_k;0)})$, and so denoted. If $\tau_1 \otimes \cdots \otimes \tau_k \otimes \sigma$ is a representation of $M = M_{(n_1,\ldots,n_k;n_0)}$, we write $\tau_1 \times \cdots \times \tau_k \rtimes \sigma$ for the induced representation $i_{G,M}(\tau_1 \otimes \cdots \otimes \tau_k \otimes \sigma)$ (or its image in the Grothendieck group). If $M = c(M_{(n_1,\ldots,n_k;0)})$, then $i_{G,M}(\tau_1 \otimes \cdots \otimes \tau_k \otimes 1) \cong c(\tau_1 \times \cdots \times \tau_k \rtimes 1)$ (cf. Lemma 3.2) and we write the representation as $c(\tau_1 \times \cdots \times \tau_k \rtimes 1)$.

We close by recalling the following result of [B-Z,C]:

Theorem 2.1. *Let G be the F-points of a connected reductive p-adic group defined over F, MU and NV the Levi factorizations standard parabolic subgroups. Let π be a smooth representation of M. Then, $r_{N,G} \circ i_{G,M}\pi$ has a composition series with factors*

$$i_{N,N'} \circ w \circ r_{M',M}\pi, \quad w \in [W_N \backslash W / W_M],$$

where $M' = M \cap w^{-1}(N)$, $N' = w(M) \cap N$, and $[W_N \backslash W / W_M]$ consists of the double-coset representatives of $W_N \backslash W_G / W_M$ of minimal length.

3. Calculation of Jacquet modules

In this section, we give the main result of this paper (cf. Theorem 3.4) and a short example of its application to the calculation of Jacquet modules. Note that the proof of Theorem 3.4 is deferred to the next section.

We first define an analogue to the M^* of [T1], modifying M^* to keep track of the number of sign changes. As in [T1], we let $s : R \otimes R \to R \otimes R$ be defined by $s(\tau_1 \otimes \tau_2) = \tau_2 \otimes \tau_1$ (which defines s on a basis).

Definition 3.1. Let $M_D^* : R \to R \otimes R \otimes \mathbb{Z}[C]$ be defined by

$$M_D^* = (m \otimes 1)_D \circ (\tilde{} \otimes m^*)_D \circ s \circ m^*,$$

where $(\tilde{} \otimes m^*)_D : R \otimes R \to R \otimes R \otimes R \otimes \mathbb{Z}[C]$ is given by

$$(\tilde{} \otimes m^*)_D(\tau_1 \otimes \tau_2) = \begin{cases} \tilde{\tau}_1 \otimes m^*(\tau_2) \otimes e \\ \quad \text{if } \tau_1 \text{ is a representation of } GL(n_1, F) \text{ with } n_1 \text{ even}, \\ \tilde{\tau}_1 \otimes m^*(\tau_2) \otimes c \\ \quad \text{if } \tau_1 \text{ is a representation of } GL(n_1, F) \text{ with } n_1 \text{ odd}, \end{cases}$$

and $(m \otimes 1)_D : R \otimes R \otimes R \otimes \mathbb{Z}[C] \to R \otimes R \otimes \mathbb{Z}[C]$ is given by

$$(m \otimes 1)_D(\tau_1 \otimes \tau_2 \otimes \tau_3 \otimes c') = (\tau_1 \times \tau_2) \otimes \tau_3 \otimes c'.$$

Recall that in [T1], Tadić observes that $R[S] = \bigoplus_{n \geqslant 0} R(SO(2n+1, F))$ is a module over R (and similarly for symplectic groups). It is not a Hopf module, but rather what Tadić refers to as an M^*-Hopf module over R. We give a corresponding interpretation here, though it requires a bit more to set up.

First, let $R[D] = \bigoplus_{n \geqslant 0} R(SO(2n, F))$. We consider $R[D] \otimes \mathbb{Z}[C]$. Let

$$\mathcal{K} = \operatorname{span}_{\mathbb{Z}}\{\theta \otimes c - c\theta \otimes e\},$$

where θ runs over irreducible representations of $SO(2n, F)$ for all $n > 0$. We then set

$$R_D = \big(R[D] \otimes \mathbb{Z}[C]\big)/\mathcal{K}.$$

It is R_D which will carry the structure analogous to that given in [T1]—that of an M_D^*-Hopf module.

We now define the R-module structure on R_D. First, let

$$\mu : R \otimes R[D] \otimes \mathbb{Z}[C] \to R[D] \otimes \mathbb{Z}[C]$$

be defined by

$$\mu : \tau \otimes \theta \otimes c' \mapsto (\tau \rtimes \theta) \otimes c',$$

noting that it is enough to define μ for $\tau \otimes \theta \otimes c'$ with $\tau \otimes \theta$ irreducible. An easy calculation shows that $\mu : R \otimes \mathcal{K} \to \mathcal{K}$. Therefore, μ descends to a well-defined map

$$\mu_D : R \otimes R_D \to R_D.$$

This gives R_D the structure of a module over R. As in [T1], it is a $\mathbb{Z}_+$-graded module. We also use $\rtimes$ to denote μ_D.

Lemma 3.2. *Let M be a standard Levi subgroup for $SO(2n, F)$. Then*

$$c \circ r_{M,G} \cong r_{c(M),G} \circ c,$$
$$c \circ i_{G,M} \cong i_{G,c(M)} \circ c.$$

Proof. Straightforward. □

We now define the comodule structure. As in [B], we let

$$\Omega_k = \begin{cases} \Pi \setminus \{\alpha_k\} & \text{if } k \leqslant n-2, \\ \Pi \setminus \{\alpha_{n-1}, \alpha_n\} & \text{if } k = n-1, \\ \Pi \setminus \{\alpha_n\} & \text{if } k = n; \end{cases}$$

letting $\bar{\Omega}_n = \Pi \setminus \{\alpha_{n-1}\} = c(\Omega_n)$. We let $\Omega_0 = \Pi$, so $M_{\Omega_0} = G$.

We first define

$$\mu^* : R[D] \otimes \mathbb{Z}[C] \to R \otimes R[D] \otimes \mathbb{Z}[C]$$

as follows: For π an irreducible representation of $SO(2n, F)$, with $n \geqslant 2$, and $0 \leqslant k \leqslant n$, write $r_{M_{\Omega_k},G}(\pi) = \sum_{i \in I_k} \tau_i(k) \otimes \theta_i(k)$ and $r_{M_{\bar{\Omega}_n},G}(\theta) = \sum_{j \in J} c(\tau_j \otimes 1)$. We set

$$\mu^*(\pi \otimes c') = \sum_{k=0}^{n} \left(\sum_{i \in I_k} \tau_i(k) \otimes \theta_i(k) \otimes c' \right) + \sum_{j \in J} (\tau_j \otimes 1 \otimes cc').$$

For $n = 0$, the only irreducible representation of $SO(0, F)$ (trivial group) is 1, and we set

$$\mu^*(1 \otimes c') = 1 \otimes 1 \otimes c'.$$

For $n = 1$, an irreducible representation of $SO(2, F)$ has the form $\chi \rtimes 1$ for χ a (quasi)character of $F^\times$ (noting that under $SO(2, F) \cong F^\times$, this corresponds to the character χ), and we set

$$\mu^*(\chi \rtimes 1 \otimes c') = 1 \otimes (\chi \rtimes 1) \otimes c' + \chi \otimes 1 \otimes c' + \chi^{-1} \otimes 1 \otimes cc'.$$

We then extend μ^* to $R[D] \otimes \mathbb{Z}[C]$ $\mathbb{Z}$-bilinearly.

Definition 3.3. A straightforward calculation (using Lemma 3.2) shows $\mu^* : \mathcal{K} \to R \otimes \mathcal{K}$. Therefore, μ^* descends to a well-defined map

$$\mu_D^* : R_D \to \big(R \otimes R[D] \otimes \mathbb{Z}[C]\big)/(R \otimes \mathcal{K}) \cong R \otimes R_D$$

under the obvious isomorphism (i.e., $\tau \otimes \theta \otimes c' + R \otimes \mathcal{K} \leftrightarrow \tau \otimes (\theta \otimes c' + \mathcal{K})$).

Theorem 3.4. *R_D is an M^*_D-Hopf module over R. In particular, for τ, θ representations of $GL(n_1, F), SO(2n, F)$, respectively, and $c' \in C$, we have*

$$\mu^*_D\big(\tau \rtimes (\theta \otimes c' + \mathcal{K})\big) = M^*_D(\tau) \rtimes \mu^*_D(\theta \otimes c' + \mathcal{K}).$$

Note that on the right-hand side, $\rtimes$ is defined as follows: let

$$\rtimes : \big(R \otimes R \otimes \mathbb{Z}[C]\big) \otimes \big(R \otimes R[D] \otimes \mathbb{Z}[C]\big) \to R \otimes R[D] \otimes \mathbb{Z}[C]$$

be defined on representations by

$$(\tau_1 \otimes \tau_2 \otimes c_1) \rtimes (\tau \otimes \theta \otimes c_2 + \mathcal{K}) \mapsto (\tau_1 \times \tau) \otimes (\tau_2 \rtimes \theta) \otimes c_1 c_2 + \mathcal{K}.$$

This then descends to a well-defined map $\rtimes : (R \otimes R \otimes \mathbb{Z}[C]) \otimes (R \otimes R_D) \to R \otimes R_D$ used on the right-hand side.

Proof. See Section 4. $\square$

As in [T1], this is a $\mathbb{Z}_+$-graded comodule.

We give an example to show how this may be used to calculate Jacquet modules of induced representations:

Example 3.5. Let χ be a (quasi)character of $F^\times$. Then $\chi \circ \det_{GL(2,F)} \rtimes 1$ is a representation of $SO(4, F)$ whose Jacquet modules are calculated below using Theorem 3.4. Note that 1 is used below for both the trivial representation of $GL(0, F)$ and $SO(0, F)$ (both trivial groups); the particular interpretations being clear from context. Note that $1 \rtimes 1$ also appears; this is also the trivial representation of $SO(0, F)$.

First, we have

$$\begin{aligned} &M^*_D(\chi \circ \det_{GL(2,F)}) \\ &\quad = \chi \circ \det_{GL(2,F)} \otimes 1 \otimes e + |\cdot|^{-\frac{1}{2}}\chi \otimes |\cdot|^{\frac{1}{2}}\chi \otimes e + 1 \otimes \chi \circ \det_{GL(2,F)} \otimes e \\ &\qquad + |\cdot|^{-\frac{1}{2}}\chi^{-1} \times |\cdot|^{-\frac{1}{2}}\chi \otimes 1 \otimes c + |\cdot|^{-\frac{1}{2}}\chi^{-1} \otimes |\cdot|^{-\frac{1}{2}}\chi \otimes c \\ &\qquad + \chi^{-1} \circ \det_{GL(2,F)} \otimes 1 \otimes e \end{aligned}$$

and

$$\mu^*_D(1 \otimes e + \mathcal{K}) = 1 \otimes 1 \otimes e + \mathcal{K}.$$

Therefore, by Theorem 3.4,

$$\begin{aligned} &\mu^*_D\big(\chi \circ \det_{GL(2,F)} \rtimes (1 \otimes e + \mathcal{K})\big) \\ &\quad = \chi \circ \det_{GL(2,F)} \otimes 1 \rtimes 1 \otimes e + |\cdot|^{-\frac{1}{2}}\chi \otimes |\cdot|^{\frac{1}{2}}\chi \rtimes 1 \otimes e \end{aligned}$$

$$+ 1 \otimes \chi \circ \det_{GL(2,F)} \rtimes 1 \otimes e + |\cdot|^{-\frac{1}{2}}\chi^{-1} \times |\cdot|^{-\frac{1}{2}}\chi \otimes 1 \rtimes 1 \otimes c$$
$$+ |\cdot|^{-\frac{1}{2}}\chi^{-1} \otimes |\cdot|^{-\frac{1}{2}}\chi \rtimes 1 \otimes c + \chi^{-1} \circ \det_{GL(2,F)} \otimes 1 \rtimes 1 \otimes e + \mathcal{K}.$$

Thus,

$$r_{M_{\Omega_1},G}(\chi \circ \det_{GL(2,F)} \rtimes 1) = |\cdot|^{-\frac{1}{2}}\chi \otimes |\cdot|^{\frac{1}{2}}\chi + |\cdot|^{-\frac{1}{2}}\chi^{-1} \otimes |\cdot|^{\frac{1}{2}}\chi^{-1},$$
$$r_{M_{\Omega_2},G}(\chi \circ \det_{GL(2,F)} \rtimes 1) = \chi \circ \det_{GL(2,F)} + \chi^{-1} \circ \det_{GL(2,F)},$$
$$r_{M_{\bar{\Omega}_2},G}(\chi \circ \det_{GL(2,F)} \rtimes 1) = c\big(|\cdot|^{-\frac{1}{2}}\chi^{-1} \times |\cdot|^{-\frac{1}{2}}\chi\big).$$

Note that this matches the results calculated in [J1, Section 4.2], done using the results of [B-Z,C].

4. Proof of main theorem

In this section, we give the proof of Theorem 3.4. The proof is essentially a long but straightforward calculation. In particular, we show that $M_D^*(\tau) \rtimes \mu_D^*(\theta \otimes e + \mathcal{K}) = \mu_D^*(\tau \rtimes (\theta \otimes e + \mathcal{K}))$ for τ and θ representations, which suffices. For convenience, we suppress the $\mathcal{K}$ in what follows. The calculation of $M_D^*(\tau) \rtimes \mu_D^*(\theta \otimes e)$ is done from the definition in Section 3. We calculate $\mu_D^*(\tau \rtimes \theta \otimes e)$ using the results of Bernstein–Zelevinsky and Casselman (cf. Theorem 2.1), along with the double-coset representatives given by Ban in [B, Section 5].

Suppose $\tau \rtimes \theta$ is a representation of $SO(2n, F)$. The cases $n = 0$ and $n = 1$ are covered by the definition of μ_D^*, so we assume $n \geqslant 2$ below. Also, if $\tau \otimes \theta$ is a representation of $GL(i_2, F) \times SO(2(n-i_2), F)$, we treat $n - i_2 = 0$ and $n - i_2 = 1$ as special cases, dealing with them at the end. Thus, we assume $i_2 < n - 1$ for now.

We start by calculating $M_D^*(\tau) \rtimes \mu_D^*(\theta)$. To this end, write $r_{M_{\Omega_\ell},G}(\theta) = \sum_{t\in T_\ell} \lambda_t(\ell) \otimes \theta_t(n - i_2 - \ell)$ and $r_{M_{\bar{\Omega}_n},G}(\theta) = \sum_{u\in U} c(\lambda_u(n - i_2) \otimes 1)$, where T_ℓ and U are the appropriate indexing sets. To add clarity, we have, e.g., written $\lambda_t(\ell)$ rather than just λ_t, indicating the rank of the underlying group as an argument. Then,

$$\mu_D^*(\theta \otimes e) \sum_{\ell=0}^{n-i_2} \sum_{t\in T_\ell} \lambda_t(\ell) \otimes \theta_t(n - i_2 - \ell) \otimes e + \sum_{u\in U} \lambda_u(n - i_2) \otimes 1 \otimes c.$$

Also,

$$\begin{aligned}
M_D^*(\tau) &= (m \otimes 1)_D \circ (\tilde{} \otimes m^*)_D \circ s \circ m^*(\tau) \\
&= (m \otimes 1)_D \circ (\tilde{} \otimes m^*)_D \circ s \left(\sum_{j=0}^{i_2} \sum_{r\in R_j} \rho_r(j) \otimes \sigma_r(i_2 - j) \right) \\
&= (m \otimes 1)_D \circ (\tilde{} \otimes m^*)_D \left(\sum_{j=0}^{i_2} \sum_{r\in R_j} \sigma_r(i_2 - j) \otimes \rho_r(j) \right),
\end{aligned}$$

where R_j is the appropriate indexing set. Continuing,

$$\begin{aligned} M_D^*(\tau) &= (m\otimes 1)_D\left(\sum_{j=0}^{i_2}\sum_{r\in R_j}\tilde{\sigma}_r(i_2-j)\right.\\ &\quad\left.\otimes\left(\sum_{k=0}^{j}\sum_{s\in S(r,j,k)}\rho_{r,s}^{(1)}(k)\otimes\rho_{r,s}^{(2)}(j-k)\right)\otimes c^{i_2-j}\right)\\ &= \sum_{j=0}^{i_2}\sum_{k=0}^{j}\sum_{r\in R_j}\sum_{s\in S(r,j,k)}\big(\tilde{\sigma}_r(i_2-j)\times\rho_{r,s}^{(1)}(k)\big)\otimes\rho_{r,s}^{(2)}(j-k)\otimes c^{i_2-j}, \qquad (1) \end{aligned}$$

where $S(r,j,k)$ is the appropriate indexing set. It now follows that

$$\begin{aligned} M_D^*(\tau)\rtimes\mu_D^*(\theta\otimes e) &= \sum_{j=0}^{i_2}\sum_{k=0}^{j}\sum_{r\in R_j}\sum_{\ell=0}^{n-i_2}\sum_{s\in S(r,j,k)}\sum_{t\in T_\ell}\tilde{\sigma}_r(i_2-j)\times\rho_{r,s}^{(1)}(k)\times\lambda_t(\ell)\\ &\quad\otimes\rho_{r,s}^{(2)}(j-k)\rtimes\theta_t(n-i_2-\ell)\otimes c^{i_2-j}\\ &\quad+\sum_{j=0}^{i_2}\sum_{k=0}^{j}\sum_{r\in R_j}\sum_{s\in S(r,j,k)}\sum_{u\in U}\tilde{\sigma}_r(i_2-j)\times\rho_{r,s}^{(1)}(k)\times\lambda_u(n-i_2)\\ &\quad\otimes\rho_{r,s}^{(2)}(j-k)\rtimes 1\otimes c^{i_2-j+1}. \end{aligned}$$

Now, to match this result with what we get for $\mu_D^*(\tau\rtimes(\theta\otimes e))$ below, let $j=i_2-d$ and $\ell=i_1-d-k$ (which defines d and i_1). We then have

$$\begin{aligned} &M_D^*(\tau)\rtimes\mu_D^*(\theta\otimes e)\\ &\quad=\sum_{d=0}^{i_2}\sum_{k=0}^{i_2-d}\sum_{i_1=d+k}^{n-i_2+d+k}\sum_{r\in R_{i_2-d}}\sum_{s\in S(r,i_2-d,k)}\sum_{t\in T_{i_1-d-k}}\tilde{\sigma}_r(d)\times\rho_{r,s}^{(1)}(k)\times\lambda_t(i_1-d-k)\\ &\quad\otimes\rho_{r,s}^{(2)}(i_2-d-k)\rtimes\theta_t(n+d+k-i_1-i_2)\otimes c^d\\ &\quad+\sum_{d=0}^{i_2}\sum_{k=0}^{i_2-d}\sum_{r\in R_{i_2-d}}\sum_{s\in S(r,i_2-d,k)}\sum_{u\in U}\tilde{\sigma}_r(d)\times\rho_{r,s}^{(1)}(k)\times\lambda_u(n-i_2)\\ &\quad\otimes\rho_{r,s}^{(2)}(i_2-d-k)\rtimes 1\otimes c^{d+1}. \end{aligned}$$

Using $n-i_2=i_1-d-k$, so $k=i_1+i_2-n-d$, we may rewrite the second sum as

$$\begin{aligned} &\sum_{d=0}^{i_2}\sum_{i_1=n+d-i_2}^{n}\sum_{r\in R_{i_2-d}}\sum_{s\in S(r,i_2-d,i_1+i_2-n-d)}\sum_{u\in U}\tilde{\sigma}_r(d)\times\rho_{r,s}^{(1)}(i_1+i_2-n-d)\\ &\quad\times\lambda_u(n-i_2)\otimes\rho_{r,s}^{(2)}(n-i_1)\rtimes 1\otimes c^{d+1}. \end{aligned}$$

Interchanging the order of the d, i_1, and (in the first sum) k summations, we get

$$\begin{aligned}
&M_D^*(\tau) \rtimes \mu_D^*(\theta \otimes e) \\
&\quad = \sum_{i_1=0}^{n} \sum_{d=0}^{\min\{i_1,i_2\}} \sum_{k=\max\{0,i_1+i_2-n-d\}}^{\min\{i_1,i_2\}-d} \sum_{r \in R_{i_2-d}} \sum_{s \in S(r,i_2-d,k)} \sum_{t \in T_{i_1-d-k}} \tilde{\sigma}_r(d) \\
&\qquad \times \rho_{r,s}^{(1)}(k) \times \lambda_t(i_1-d-k) \otimes \rho_{r,s}^{(2)}(i_2-d-k) \rtimes \theta_t(n+d+k-i_1-i_2) \otimes c^d \\
&\qquad + \sum_{i_1=n-i_2}^{n} \sum_{d=0}^{i_1+i_2-n} \sum_{r \in R_{i_2-d}} \sum_{s \in S(r,i_2-di_1+i_2-n-d)} \sum_{u \in U} \tilde{\sigma}_r(d) \\
&\qquad \times \rho_{r,s}^{(1)}(i_1+i_2-n-d) \times \lambda_u(n-i_2) \otimes \rho_{r,s}^{(2)}(n-i_1) \rtimes 1 \otimes c^{d+1}. \qquad (2)
\end{aligned}$$

We now turn to the calculation of $\mu_D^*(\tau \rtimes (\theta \otimes e))$. As in [B], let $[W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}]$ (respectively $[W_{\bar{\Omega}_n} \backslash W / W_{\Omega_{i_2}}]$) denote the double-coset representatives of shortest length, where $W = W_{SO(2n,F)}$ and $W_{\Omega_i} = W_{M_{\Omega_i}}$. By the results of Bernstein–Zelevinsky and Casselman (cf. Theorem 2.1), we have

$$\begin{aligned}
\mu_D^*\bigl(\tau \rtimes (\theta \otimes e)\bigr) = {} & \sum_{i_1=0}^{n} \sum_{w \in [W_{\Omega_{i_1}} \backslash W / W_{\Omega_{i_2}}]} F_{i_1,i_2}(w)(\tau \otimes \theta) \otimes e \\
& + \sum_{w \in [W_{\bar{\Omega}_n} \backslash W / W_{\Omega_{i_2}}]} \bar{F}_{n,i_2}(w)(\tau \otimes \theta) \otimes c,
\end{aligned}$$

where

$$F_{i_1,i_2}(w) : \tau \otimes \theta \mapsto i_{N,N'} \circ w \circ r_{M',M}(\tau \otimes \theta),$$

with $M = M_{\Omega_{i_2}}$, $N = N_{\Omega_{i_1}}$, $N' = N \cap w(M)$, and $M' = M \cap w^{-1}(N)$; $\bar{F}(n, i_2)(w)$ is defined the same way except using $N = N_{\bar{\Omega}_n}$. Using the double-coset representatives worked out in [B, Section 5] (and retaining the notation in [B]), we may write the sum more explicitly as

$$\begin{aligned}
\mu_D^*\bigl(\tau \rtimes (\theta \otimes e)\bigr) = {} & \sum_{i_1=0}^{n} \sum_{\substack{d=0 \\ d \text{ even}}}^{\min\{i_1,i_2\}} \sum_{k=\max\{0,i_1+i_2-n-d\}}^{\min\{i_1,i_2\}-d} F_{i_1,i_2}\bigl(q_n(d,k)_{i_1,i_2}^{(0,0)}\bigr)(\tau \otimes \theta) \otimes e \\
& + \sum_{i_1=0}^{n-1} \sum_{\substack{d=0 \\ d \text{ even} \\ i_1+i_2-n-d \geqslant 0}}^{\min\{i_1,i_2\}} F_{i_1,i_2}\bigl(q_n(d, i_1+i_2-n-d)_{i_1,i_2}^{(1,1)}\bigr)(\tau \otimes \theta) \otimes e
\end{aligned}$$

$$+\sum_{i_1=0}^{n-1}\sum_{\substack{d=0\\ d\text{ odd}}}^{\min\{i_1,i_2\}}\sum_{k=\max\{0,i_1+i_2-n-d+1\}}^{\min\{i_1,i_2\}-d} F_{i_1,i_2}\big(q_n(d,k)^{(0,0)}_{i_1,i_2}\big)(\tau\otimes\theta)\otimes e$$

$$+\sum_{i_1=0}^{n}\sum_{\substack{d=0\\ d\text{ odd}\\ i_1+i_2-n-d\geqslant 0}}^{\min\{i_1,i_2\}} F_{i_1,i_2}\big(q_n(d,i_1+i_2-n-d)^{(1,0)}_{i_1,i_2}\big)(\tau\otimes\theta)\otimes e$$

$$+\sum_{i_1=0}^{n-1}\sum_{\substack{d=0\\ d\text{ odd}\\ i_1+i_2-n-d\geqslant 0}}^{\min\{i_1,i_2\}} F_{i_1,i_2}\big(q_n(d,i_1+i_2-n-d)^{(0,1)}_{i_1,i_2}\big)(\tau\otimes\theta)\otimes e$$

$$+\sum_{\substack{d=0\\ d\text{ odd}}}^{i_2} \bar F_{n,i_2}\big(q_n(d,i_2-d)^{(-1,-1)}_{n,i_2}\big)(\tau\otimes\theta)\otimes c$$

$$+\sum_{\substack{d=0\\ d\text{ even}}}^{i_2} \bar F_{n,i_2}\big(q_n(d,i_2-d)^{(\pm 1,-1)}_{n,i_2}\big)(\tau\otimes\theta)\otimes c, \tag{3}$$

noting that the upper bounds of $n-1$ (instead of n) in the second, third, and fifth sums arise from the conditions in the definition of $q_n(d,k)^{(0,0)}_{i_1,i_2}$, etc., on [B, pp. 160–161]. Now, let us write

$$r_{M_\Omega,M}(\tau\otimes\theta)=\sum_{s\in S(d,k)_{i_1,i_2}}\tau_s^{(1)}(k)\otimes\tau_s^{(2)}(i_2-d-k)\otimes\tau_s^{(3)}(d)$$
$$\otimes\lambda_s(i_1-d-k)\otimes\theta_s(n+d+k-i_1-i_2)$$

for $\Omega=\Omega_k\cap\Omega_{i_2-d}\cap\Omega_{i_2}\cap\Omega_{i_1+i_2-d-k}$ (defining the indexing set $S(d,k)_{i_1,i_2}$); we write

$$r_{M_{\Omega'},M}(\tau\otimes\theta)=\sum_{s\in\bar S(d,k)_{i_1,i_2}}c\big(\tau_s^{(1)}(k)\otimes\tau_s^{(2)}(i_2-d-k)\otimes\tau_s^{(3)}(d)\otimes\lambda_s(n-i_2)\otimes 1\big)$$

for $\Omega'=\Omega_k\cap\Omega_{i_2-d}\cap\Omega_{i_2}\cap\bar\Omega_n$ with $i_1+i_2-d-k=n$ (defining the indexing set $\bar S(d,k)_{i_1,i_2}$). Therefore (noting that conjugations by q_n's produce the contragredient of $\tau_s^{(3)}(d)$ as in [T1]; cf. [G-K]) we have

$$\mu_D^*\big(\tau\rtimes(\theta\otimes e)\big)$$
$$=\sum_{i_1=0}^{n}\sum_{\substack{d=0\\ d\text{ even}}}^{\min\{i_1,i_2\}}\sum_{k=\max\{0,i_1+i_2-n-d\}}^{\min\{i_1,i_2\}-d}\sum_{s\in S(d,k)_{i_1,i_2}}\tau_s^{(1)}(k)\times\lambda_s(i_1-d-k)\times\tilde\tau_s^{(3)}(d)$$
$$\otimes\tau_s^{(2)}(i_2-d-k)\rtimes\theta_s(n+d+k-i_1-i_2)\otimes e$$

$$+\sum_{i_1=0}^{n-1}\sum_{\substack{d=0\\ d\text{ even}\\ i_1+i_2-n-d\geqslant 0}}^{\min\{i_1,i_2\}}\sum_{s\in\bar{S}(d,i_1+i_2-n-d)_{i_1,i_2}}\tau_s^{(1)}(i_1+i_2-n-d)\times\lambda_s(n-i_2)\times\tilde{\tau}_s^{(3)}(d)$$

$$\otimes\tau_s^{(2)}(n-i_1)\rtimes 1\otimes c$$

$$+\sum_{i_1=0}^{n-1}\sum_{\substack{d=0\\ d\text{ odd}}}^{\min\{i_1,i_2\}}\sum_{k=\max\{0,i_1+i_2-n-d+1\}}^{\min\{i_1,i_2\}-d}\sum_{s\in S(d,k)_{i_1,i_2}}\tau_s^{(1)}(k)\times\lambda_s(i_1-d-k)\times\tilde{\tau}_s^{(3)}(d)$$

$$\otimes\tau_s^{(2)}(i_2-d-k)\rtimes\theta_s(n+d+k-i_1-i_2)\otimes c$$

$$+\sum_{i_1=0}^{n}\sum_{\substack{d=0\\ d\text{ odd}\\ i_1+i_2-n-d\geqslant 0}}^{\min\{i_1,i_2\}}\sum_{s\in\bar{S}(d,i_1+i_2-n-d)_{i_1,i_2}}\tau_s^{(1)}(i_1+i_2-n-d)\times\lambda_s(n-i_2)\times\tilde{\tau}_s^{(3)}(d)$$

$$\otimes\tau_s^{(2)}(n-i_1)\rtimes 1\otimes e$$

$$+\sum_{i_1=0}^{n-1}\sum_{\substack{d=0\\ d\text{ odd}\\ i_1+i_2-n-d\geqslant 0}}^{\min\{i_1,i_2\}}\sum_{s\in S(d,i_1+i_2-n-d)_{i_1,i_2}}\tau_s^{(1)}(i_1+i_2-n-d)\times\lambda_s(n-i_2)\times\tilde{\tau}_s^{(3)}(d)$$

$$\otimes\tau_s^{(2)}(n-i_1)\rtimes 1\otimes c$$

$$+\sum_{\substack{d=0\\ d\text{ odd}}}^{i_2}\sum_{s\in S(d,i_2-d)_{n,i_2}}\tau_s^{(1)}(i_2-d)\times\lambda_s(n-i_2)\times\tilde{\tau}_s^{(3)}(d)\otimes 1\otimes c$$

$$+\sum_{\substack{d=0\\ d\text{ even}}}^{i_2}\sum_{s\in\bar{S}(d,i_2-d)_{n,i_2}}\tau_s^{(1)}(i_2-d)\times\lambda_s(n-i_2)\times\tilde{\tau}_s^{(3)}(d)\otimes 1\otimes c, \tag{4}$$

noting that $q_n(d,i_2-d)^{(\pm 1,-1)}$ are defined differently for $d=0$ but can be combined into one sum. We also note that the inducing subgroups in the sums above match what is given in [B, Lemmas 5.11 and 5.12]. We now combine the $i_1=n$ terms from the first and fourth sums with the sixth and seventh sums, respectively, to get

$$\mu_D^*\bigl(\tau\rtimes(\theta\otimes e)\bigr)$$

$$=\sum_{i_1=0}^{n-1}\sum_{\substack{d=0\\ d\text{ even}}}^{\min\{i_1,i_2\}}\sum_{k=\max\{0,i_1+i_2-n-d\}}^{\min\{i_1,i_2\}-d}\sum_{s\in S(d,k)_{i_1,i_2}}\tau_s^{(1)}(k)\times\lambda_s(i_1-d-k)\times\tilde{\tau}_s^{(3)}(d)$$

$$\otimes\tau_s^{(2)}(i_2-d-k)\rtimes\theta_s(n+d+k-i_1-i_2)\otimes e$$

$$+\sum_{i_1=0}^{n-1}\sum_{\substack{d=0\\ d\text{ even}\\ i_1+i_2-n-d\geqslant 0}}^{\min\{i_1,i_2\}}\sum_{s\in\bar{S}(d,i_1+i_2-n-d)_{i_1,i_2}}\tau_s^{(1)}(i_1+i_2-n-d)\times\lambda_s(n-i_2)\times\tilde{\tau}_s^{(3)}(d)$$

$$\otimes\tau_s^{(2)}(n-i_1)\rtimes 1\otimes c$$

$$+\sum_{i_1=0}^{n-1}\sum_{\substack{d=0\\ d\text{ odd}}}^{\min\{i_1,i_2\}}\sum_{k=\max\{0,i_1+i_2-n-d+1\}}^{\min\{i_1,i_2\}-d}\sum_{s\in S(d,k)_{i_1,i_2}}\tau_s^{(1)}(k)\times\lambda_s(i_1-d-k)\times\tilde{\tau}_s^{(3)}(d)$$

$$\otimes\tau_s^{(2)}(i_2-d-k)\rtimes\theta_s(n+d+k-i_1-i_2)\otimes c$$

$$+\sum_{i_1=0}^{n-1}\sum_{\substack{d=0\\ d\text{ odd}\\ i_1+i_2-n-d\geqslant 0}}^{\min\{i_1,i_2\}}\sum_{s\in\bar{S}(d,i_1+i_2-n-d)_{i_1,i_2}}\tau_s^{(1)}(i_1+i_2-n-d)\times\lambda_s(n-i_2)\times\tilde{\tau}_s^{(3)}(d)$$

$$\otimes\tau_s^{(2)}(n-i_1)\rtimes 1\otimes e$$

$$+\sum_{i_1=0}^{n-1}\sum_{\substack{d=0\\ d\text{ odd}\\ i_1+i_2-n-d\geqslant 0}}^{\min\{i_1,i_2\}}\sum_{s\in S(d,i_1+i_2-n-d)_{i_1,i_2}}\tau_s^{(1)}(i_1+i_2-n-d)\times\lambda_s(n-i_2)\times\tilde{\tau}_s^{(3)}(d)$$

$$\otimes\tau_s^{(2)}(n-i_1)\rtimes 1\otimes c$$

$$+\sum_{d=0}^{i_2}\sum_{s\in S(d,i_2-d)_{n,i_2}}\tau_s^{(1)}(i_2-d)\times\lambda_s(n-i_2)\times\tilde{\tau}_s^{(3)}(d)\otimes 1\otimes c^d$$

$$+\sum_{d=0}^{i_2}\sum_{s\in\bar{S}(d,i_2-d)_{n,i_2}}\tau_s^{(1)}(i_2-d)\times\lambda_s(n-i_2)\times\tilde{\tau}_s^{(3)}(d)\otimes 1\otimes c^{d+1}.$$

We now combine the fourth sum with the second and the fifth sum with the third (noting that the fifth sum corresponds to $k=i_1+i_2-n-d$ when combined with the third) to get

$$\mu_D^*\big(\tau\rtimes(\theta\otimes e)\big)$$

$$=\sum_{i_1=0}^{n-1}\sum_{\substack{d=0\\ d\text{ even}}}^{\min\{i_1,i_2\}}\sum_{k=\max\{0,i_1+i_2-n-d\}}^{\min\{i_1,i_2\}-d}\sum_{s\in S(d,k)_{i_1,i_2}}\tau_s^{(1)}(k)\times\lambda_s(i_1-d-k)\times\tilde{\tau}_s^{(3)}(d)$$

$$\otimes\tau_s^{(2)}(i_2-d-k)\rtimes\theta_s(n+d+k-i_1-i_2)\otimes e$$

$$+\sum_{i_1=0}^{n-1}\sum_{\substack{d=0\\ i_1+i_2-n-d\geqslant 0}}^{\min\{i_1,i_2\}}\sum_{s\in\bar{S}(d,i_1+i_2-n-d)_{i_1,i_2}}\tau_s^{(1)}(i_1+i_2-n-d)\times\lambda_s(n-i_2)\times\tilde{\tau}_s^{(3)}(d)$$

$$\otimes\tau_s^{(2)}(n-i_1)\rtimes 1\otimes c^{d+1}$$

$$+\sum_{i_1=0}^{n-1}\sum_{\substack{d=0\\ d\text{ odd}}}^{\min\{i_1,i_2\}}\sum_{k=\max\{0,i_1+i_2-n-d\}}^{\min\{i_1,i_2\}-d}\sum_{s\in S(d,k)_{i_1,i_2}}\tau_s^{(1)}(k)\times\lambda_s(i_1-d-k)\times\tilde{\tau}_s^{(3)}(d)$$
$$\otimes\,\tau_s^{(2)}(i_2-d-k)\rtimes\theta_s(n+d+k-i_1-i_2)\otimes c$$
$$+\sum_{d=0}^{i_2}\sum_{s\in S(d,i_2-d)_{n,i_2}}\tau_s^{(1)}(i_2-d)\times\lambda_s(n-i_2)\times\tilde{\tau}_s^{(3)}(d)\otimes 1\otimes c^d$$
$$+\sum_{d=0}^{i_2}\sum_{s\in\bar{S}(d,i_2-d)_{n,i_2}}\tau_s^{(1)}(i_2-d)\times\lambda_s(n-i_2)\times\tilde{\tau}_s^{(3)}(d)\otimes 1\otimes c^{d+1}.$$

Finally, combining the fifth sum above with the second, and the third and fourth and sums with the first (noting that the fourth and fifth sums correspond to $i_1=n$), we get

$$\mu_D^*\big(\tau\rtimes(\theta\otimes e)\big)$$
$$=\sum_{i_1=0}^{n}\sum_{d=0}^{\min\{i_1,i_2\}}\sum_{k=\max\{0,i_1+i_2-n-d\}}^{\min\{i_1,i_2\}-d}\sum_{s\in S(d,k)_{i_1,i_2}}\tau_s^{(1)}(k)\times\lambda_s(i_1-d-k)\times\tilde{\tau}_s^{(3)}(d)$$
$$\otimes\,\tau_s^{(2)}(i_2-d-k)\rtimes\theta_s(n+d+k-i_1-i_2)\otimes c^d$$
$$+\sum_{i_1=0}^{n}\sum_{\substack{d=0\\ i_1+i_2-n-d\geqslant 0}}^{\min\{i_1,i_2\}}\sum_{s\in\bar{S}(d,i_1+i_2-n-d)_{i_1,i_2}}\tau_s^{(1)}(i_1+i_2-n-d)\times\lambda_s(n-i_2)\times\tilde{\tau}_s^{(3)}(d)$$
$$\otimes\,\tau_s^{(2)}(n-i_1)\rtimes 1\otimes c^{d+1}. \tag{5}$$

To finish the case $i_2<n-1$, it remains to show Eqs. (2) and (5) are the same. In particular, we show that the first sum in (2) matches the first sum in (5) and the second sum in (2) matches the second sum in (2). For the first sum, the i_1,d,k domains of summation are the same, so it suffices to show that for fixed i_1,d,k we have

$$\sum_{s\in S(d,k)_{i_1,i_2}}\tau_s^{(1)}(k)\times\lambda_s(i_1-d-k)\times\tilde{\tau}_s^{(3)}(d)$$
$$\otimes\,\tau_s^{(2)}(i_2-d-k)\rtimes\theta_s(n+d+k-i_1-i_2)\otimes c^d$$
$$=\sum_{r\in R_{i_2-d}}\sum_{s\in S(r,i_2-d,k)}\sum_{t\in T_{i_1-d-k}}\tilde{\sigma}_r(d)\times\rho_{r,s}^{(1)}(k)\times\lambda_t(i_1-d-k)$$
$$\otimes\,\rho_{r,s}^{(2)}(i_2-d-k)\rtimes\theta_t(n+d+k-i_1-i_2)\otimes c^d.$$

For this, it is sufficient to check that

$$\sum_{s\in S(d,k)_{i_1,i_2}} \tau_s^{(1)}(k)\otimes\tau_s^{(2)}(i_2-d-k)\otimes\tau_s^{(3)}(d)\otimes\lambda_s(i_1-d-k)$$
$$\otimes\theta_s(n+d+k-i_1-i_2)$$
$$=\sum_{r\in R_{i_2-d}}\ \sum_{s\in S(r,i_2-d,k)}\ \sum_{t\in T_{i_1-d-k}} \rho_{r,s}^{(1)}(k)\otimes\rho_{r,s}^{(2)}(i_2-d-k)\otimes\sigma_r(d)\otimes\lambda_t(i_1-d-k)$$
$$\otimes\theta_t(n+d+k-i_1-i_2). \tag{6}$$

However, both sides of (6) easily reduce to $r_{M_\Omega,G}(\tau\otimes\theta)$, where $\Omega=\Omega_k\cap\Omega_{i_2-d}\cap\Omega_{i_2}\cap\Omega_{i_1+i_2-d-k}$. We now show the second sums match up. We claim the i_1, d domains of summation are the same in (2) and (5). In particular, consider the second sum for $\mu_D^*(\tau\rtimes(\theta\otimes e))$ in (5). Since $i_1, i_2\geqslant i_1+i_2-n$, we have $\min\{i_1,i_2\}\geqslant i_1+i_2-n$. The condition $i_1+i_2-n-d\geqslant 0$ implies $d\leqslant i_1+i_2-n$; as this is lower than $\min\{i_1,i_2\}$, it is effectively the upper limit of summation. Further, this means that if $i_1<n-i_2$, the sum degenerates. Thus the i_1 summation effectively has lower limit $i_1=n-i_2$. Therefore the domains of summation match. The rest of the argument is essentially the same as for the first sums. This finishes the case $i_2<n-1$.

The cases $i_2=n-1, n$ remain. We first look at the case $i_2=n$. In (3), the 2nd, 3rd, 4th, and 7th sums do not occur if $i_2=n$ (from the conditions in the definition of $q_n(d,k)_{i_1,i_2}^{(0,0)}$, etc., [B, pp. 160–161 and 164]). Therefore, the 2nd, 3rd, 4th, and 7th sums in (4) do not occur if $i_2=n$. If we delete these sums and set $i_2=n$ in the rest, we get

$$\mu_D^*\big(\tau\rtimes(\theta\otimes e)\big)$$
$$=\sum_{i_1=0}^{n}\ \sum_{\substack{d=0\\ d\text{ even}}}^{i_1}\ \sum_{s\in S(d,i_1-d)_{i_1,n}} \tau_s^{(1)}(i_1-d)\times\tilde{\tau}_s^{(3)}(d)\otimes\tau_s^{(2)}(n-i_1)\rtimes 1\otimes e$$
$$+\sum_{i_1=0}^{n-1}\ \sum_{\substack{d=0\\ d\text{ odd}}}^{i_1}\ \sum_{s\in S(d,i_1-d)_{i_1,n}} \tau_s^{(1)}(i_1-d)\times\tilde{\tau}_s^{(3)}(d)\otimes\tau_s^{(2)}(n-i_1)\rtimes 1\otimes c$$
$$+\sum_{\substack{d=0\\ d\text{ odd}}}^{n}\ \sum_{s\in S(d,n-d)_{n,n}} \tau_s^{(1)}(n-d)\times\tilde{\tau}_s^{(3)}(d)\otimes 1\otimes c.$$

Now, combining these, we get

$$\mu_D^*\big(\tau\rtimes(\theta\otimes e)\big)=\sum_{i_1=0}^{n}\sum_{d=0}^{i_1}\sum_{s\in S(d,i_1-d)_{i_1,n}} \tau_s^{(1)}(i_1-d)\times\tilde{\tau}_s^{(3)}(d)\otimes\tau_s^{(2)}(n-i_1)\rtimes 1\otimes c^d.$$

On the other hand, we note that the second sum in (2) is absent if $i_2=n$ (as $\mu_D^*(1\otimes e)$ is just $1\otimes 1\otimes e$). Therefore, we get

$$M_D^*(\tau)\rtimes\mu_D^*(\theta\otimes e)=\sum_{i_1=0}^{n}\sum_{d=0}^{i_1}\sum_{r\in R_{n-d}}\sum_{s\in S(r,i_2-d)}\tilde{\sigma}_r(d)\times\rho_{r,s}^{(1)}(i_1-d)$$
$$\otimes\rho_{r,s}^{(2)}(n-i_1)\rtimes 1\otimes c^d,$$

which match up as in the case $i_2<n-1$ (showing (6) holds).

We now turn to the case $i_2=n-1$. We begin by looking at $M_D^*(\tau)\rtimes\mu_D^*(\theta\otimes e)$, which is where most of the work is for this case. Here, $M_D^*(\tau)$ is as in (1) and

$$\mu_D^*(\theta\otimes e)=1\otimes\theta\otimes e+\chi\otimes 1\otimes e+\chi^{-1}\otimes 1\otimes c,$$

where $\theta=\chi\rtimes 1$. Therefore (using $i_2=n-1$)

$$\begin{aligned}
&M_D^*(\tau)\rtimes\mu_D^*(\theta\otimes e)\\
&=\sum_{j=0}^{n-1}\sum_{k=0}^{j}\sum_{r\in R_j}\sum_{s\in S(r,j,k)}\big[\tilde{\sigma}_r(n-1-j)\times\rho_{r,s}^{(1)}(k)\otimes\rho_{r,s}^{(2)}(j-k)\rtimes\theta\otimes c^{n-1-j}\\
&\quad+\tilde{\sigma}_r(n-1-j)\times\rho_{r,s}^{(1)}(k)\times\chi\otimes\rho_{r,s}^{(2)}(j-k)\rtimes 1\otimes c^{n-1-j}\\
&\quad+\tilde{\sigma}_r(n-1-j)\times\rho_{r,s}^{(1)}(k)\times\chi^{-1}\otimes\rho_{r,s}^{(2)}(j-k)\rtimes 1\otimes c^{n-1-j+1}\big].
\end{aligned}$$

As earlier when we used $j=i_2-d$, $\ell=i_1-d-k$, we now let $d=n-1-j$ and $\ell=i_1-d-k$ (so $\ell=0$ for the first term and $\ell=1$ for the second and third terms). We have

$$\begin{aligned}
&M_D^*(\tau)\rtimes\mu_D^*(\theta\otimes e)\\
&=\sum_{d=0}^{n-1}\sum_{i_1=d}^{n-1}\sum_{r\in R_{n-1-d}}\sum_{s\in S(r,n-1-d,i_1-d)}\tilde{\sigma}_r(d)\times\rho_{r,s}^{(1)}(i_1-d)\otimes\rho_{r,s}^{(2)}(n-1-i_1)\rtimes\theta\otimes c^d\\
&\quad+\sum_{d=0}^{n-1}\sum_{i_1=d+1}^{n}\sum_{r\in R_{n-1-d}}\sum_{s\in S(r,n-1-d,i_1-d)}\big[\tilde{\sigma}_r(d)\times\rho_{r,s}^{(1)}(i_1-d-1)\times\chi\\
&\quad\otimes\rho_{r,s}^{(2)}(n-i_1)\rtimes 1\otimes c^d\\
&\quad+\tilde{\sigma}_r(d)\times\rho_{r,s}^{(1)}(i_1-d-1)\times\chi^{-1}\otimes\rho_{r,s}^{(2)}(n-i_1)\rtimes 1\otimes c^{d+1}\big].
\end{aligned}$$

Now, reversing the d and i_1 summations gives

$$\begin{aligned}
&M_D^*(\tau)\rtimes\mu_D^*(\theta\otimes e)\\
&=\sum_{i_1=0}^{n-1}\sum_{d=0}^{i_1}\sum_{r\in R_{n-1-d}}\sum_{s\in S(r,n-1-d,i_1-d)}\tilde{\sigma}_r(d)\times\rho_{r,s}^{(1)}(i_1-d)\\
&\quad\otimes\rho_{r,s}^{(2)}(n-1-i_1)\rtimes\theta\otimes c^d
\end{aligned}$$

$$+\sum_{i_1=1}^{n}\sum_{d=0}^{i_1-1}\sum_{r\in R_{n-1-d}}\sum_{s\in S(r,n-1-d,i_1-d)}\tilde{\sigma}_r(d)\times\rho_{r,s}^{(1)}(i_1-d-1)\times\chi$$
$$\otimes\rho_{r,s}^{(2)}(n-i_1)\rtimes 1\otimes c^d$$
$$+\sum_{i_1=1}^{n}\sum_{d=0}^{i_1-1}\sum_{r\in R_{n-1-d}}\sum_{s\in S(r,n-1-d,i_1-d)}\tilde{\sigma}_r(d)\times\rho_{r,s}^{(1)}(i_1-d-1)\times\chi^{-1}$$
$$\otimes\rho_{r,s}^{(2)}(n-i_1)\rtimes 1\otimes c^{d+1}. \tag{7}$$

On the other hand, from (5) (which is valid for $i_2=n-1$), we have

$$\mu_D^*\big(\tau\rtimes(\theta\otimes e)\big)$$
$$=\sum_{i_1=0}^{n}\sum_{d=0}^{\min\{i_1,n-1\}}\sum_{k=\max\{0,i_1-d-1\}}^{\min\{i_1,n-1\}-d}\sum_{s\in S(d,k)_{i_1,n-1}}\tau_s^{(1)}(k)\times\lambda_s(i_1-d-k)\times\tilde{\tau}_s^{(3)}(d)$$
$$\otimes\tau_s^{(2)}(n-1-d-k)\rtimes\theta_s(d+k+1-i_1)\otimes c^{d+1}$$
$$+\sum_{i_1=0}^{n}\sum_{\substack{d=0\\ i_1-d-1\geqslant 0}}^{\min\{i_1,n-1\}}\sum_{s\in\bar{S}(d,i_1-d-1)_{i_1,n-1}}\tau_s^{(1)}(i_1-d-1)\times\lambda_s(1)\times\tilde{\tau}_s^{(3)}(d)$$
$$\otimes\tau_s^{(2)}(n-i_1)\rtimes 1\otimes c^{d+1}.$$

Notice that in the first sum, we must have $k=i_1-d-1$ or i_1-d, and we break it into two sums accordingly. Note that to ensure $k,d\geqslant 0$, the case $k=i_1-d-1$ requires $1\leqslant i_1$ and $d\leqslant i_1-1\leqslant\min\{i_1,n-1\}$ (so we change the bounds of summation). The case $k=i_1-d$ requires $i_1\leqslant n-1$ (to ensure $\tau_s^{(2)}$ is associated to a group of non-negative rank), giving $\min\{i_1,n-1\}=i_1$ (so we change the bounds accordingly). We get

$$\mu_D^*\big(\tau\rtimes(\theta\otimes e)\big)$$
$$=\sum_{i_1=1}^{n}\sum_{d=0}^{i_1-1}\sum_{s\in S(d,i_1-d-1)_{i_1,n-1}}\tau_s^{(1)}(i_1-d-1)\times\lambda_s(1)\times\tilde{\tau}_s^{(3)}(d)$$
$$\otimes\tau_s^{(2)}(n-i_1)\rtimes 1\otimes c^d$$
$$+\sum_{i_1=0}^{n-1}\sum_{d=0}^{i_1}\sum_{s\in S(d,i_1-d)_{i_1,n-1}}\tau_s^{(1)}(i_1-d)\times\tilde{\tau}_s^{(3)}(d)\otimes\tau_s^{(2)}(n-i_1-1)\rtimes\theta_s(1)\otimes c^d$$
$$+\sum_{i_1=1}^{n}\sum_{d=0}^{i_1-1}\sum_{s\in\bar{S}(d,i_1-d-1)_{i_1,n-1}}\tau_s^{(1)}(i_1-d-1)\times\lambda_s(1)\times\tilde{\tau}_s^{(3)}(d)$$
$$\otimes\tau_s^{(2)}(n-i_1)\rtimes 1\otimes c^{d+1}. \tag{8}$$

We now claim the first (respectively second, third) sum in (7) is equal to the second (respectively first, third) sum in (8). This follows from the same considerations used at the end of the case $i_2 < n - 1$ (showing (6) holds). This finishes the case $i_2 = n - 1$ and the proof.

References

[Au] A.-M. Aubert, Dualité dans le groupe de Grothendieck de la catégorie des représentations lisses de longueur finie d'un groupe réductif p-adique, Trans. Amer. Math. Soc. 347 (1995) 2179–2189; Erratum: Trans. Amer. Math. Soc. 348 (1996) 4687–4690.

[B] D. Ban, Parabolic induction and Jacquet modules of representations of $O(2n, F)$, Glasnik. Mat. 34 (54) (1999) 147–185.

[B-J] D. Ban, C. Jantzen, Degenerate principal series for even orthogonal groups, Represent. Theory 7 (2003) 440–480.

[B-Z] I. Bernstein, A. Zelevinsky, Induced representations of reductive p-adic groups I, Ann. Sci. École Norm. Sup. 10 (1977) 441–472.

[C] W. Casselman, Introduction to the theory of admissible representations of p-adic reductive groups, preprint, available online at http://www.math.ubc.ca/people/faculty/cass/research.html, as "The p-adic notes".

[G-K] I. Gelfand, D. Kazhdan, Representations of $GL(n, K)$ where K is a local field, in: I.M. Gelfand (Ed.), Lie Groups and Their Representations, Halsted Press, New York, 1975, pp. 95–118.

[J1] C. Jantzen, Degenerate principal series for orthogonal groups, J. Reine Angew. Math. 441 (1993) 61–98.

[J2] C. Jantzen, Degenerate principal series for symplectic and odd-orthogonal groups, Mem. Amer. Math. Soc. 590 (1996) 1–100.

[J3] C. Jantzen, Duality and supports of induced representations for orthogonal groups, Canad. J. Math. 57 (2005) 159–179.

[M-T] C. Mœglin, M. Tadić, Construction of discrete series for classical p-adic groups, J. Amer. Math. Soc. 15 (2002) 715–786.

[M1] G. Muić, Composition series of generalized principal series; the case of strongly positive discrete series, Israel J. Math. 140 (2004) 157–202.

[M2] G. Muić, Reducibility of generalized principal series, Canad. J. Math., in press, available as prepublication on journal website.

[S-S] P. Schneider, U. Stuhler, Representation theory and sheaves on the Bruhat–Tits building, Publ. Math. Inst. Hautes Études Sci. 85 (1997) 97–191.

[Sw] M. Sweedler, Hopf Algebras, Benjamin, New York, 1969.

[T1] M. Tadić, Structure arising from induction and Jacquet modules of representations of classical p-adic groups, J. Algebra 177 (1995) 1–33.

[T2] M. Tadić, On reducibility of parabolic induction, Israel J. Math. 107 (1998) 29–91.

[Z] A. Zelevinsky, Induced representations of reductive p-adic groups, II. On irreducible representations of $GL(n)$, Ann. Sci. École Norm. Sup. 13 (1980) 165–210.