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Degenerate principal series for orthogonal groups

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Introduction

We begin with a short description of the problem. Let F be a p-adic field with char $F = 0$ and odd residual characteristic. Set $G = SO_m(F)$. Let $P = MU$ be a maximal parabolic subgroup of G. If $m = 2n+1$ is odd, then $M \cong GL_k(F) \times SO_{2(n-k)+1}(F)$, for some k, $1 \leqq k \leqq n$. If $m = 2n$ is even, then $M \cong GL_k(F) \times SO_{2(n-k)}(F)$, with $k < n-1$, or $M \cong GL_n(F)$ (n.b. there are two nonconjugate parabolic subgroups with $M \cong GL_n(F)$). A character on M is of the form $\chi \circ \det$, with χ a (quasi)character of $F^\times$ – not necessarily unitary – and may be extended trivially to P. We also denote the resulting characters as χ. The question we investigate is whether $\pi = \mathrm{Ind}_P^G \chi$ is irreducible or not (where induction is normalized so that unitary representations induce to unitary representations). If it is reducible, we determine the components and give their Langlands data and Jacquet modules.

We now briefly outline the contents of this paper. The first chapter consists of notation and preliminaries.

In the second chapter, we give certain results that are needed later, which we summarize in Proposition 2.10. These results cannot be obtained using the Jacquet module techniques of the later chapters, but are needed to finish the analysis for the nonregular cases for $SO_5(F)$ and $SO_7(F)$. These results are obtained using Hecke algebra arguments, similar to those in [Gus] or chapter 2 of [Jan].

Chapter 3 contains the results for the "regular" case, i.e., when the Jacquet module of the inducing representation is not fixed by anything in the Weyl group. The results here are based on the Jacquet module techniques of Tadić [Tad3], principally in the guise of Theorem 3.1 (Theorem 3.1.2 from [Jan]). The results for the odd orthogonal groups are summarized in Theorem 3.7; for the even ones, in Theorem 3.10.

In chapter 4, we finish the analysis for the low-rank orthogonal groups, that is, we do the nonregular cases. The arguments are of the same general flavor as those above, but more involved because of the nonregularity. Essentially, a case-by-case analysis is used to finish the nonregular cases for $SO_n(F)$, $n = 4, 5, 6, 7$.

1. Notation and preliminaries

1.1. $SO_m(F)$. The purpose of this chapter is to introduce notation and review some standard facts that will be of use in the rest of this paper.

Let F be a nonarchimedean local field of characteristic zero and odd residual characteristic. Let $\mathcal{O}$ denote the ring of integers, $\mathcal{P}$ the prime ideal in $\mathcal{O}$, and ϖ a uniformizer. Then, $\mathcal{O}/\mathcal{P}$ is a finite field. Let q denote the number of elements in the residual field, $\mathbb{F}_q = \mathcal{O}/\mathcal{P}$. We normalize the Haar measure on F so that $\operatorname{vol}(\mathcal{O}) = 1$ and the absolute value so that $|\varpi| = q^{-1}$.

Suppose that χ is a (quasi)character of $F^\times$ (i.e., a multiplicative homomorphism from $F^\times$ to $\mathbb{C}^\times$ – not necessarily unitary). If $x \in F^\times$, x may be decomposed as $x = \varpi^k x_0$, with $x_0 \in \mathcal{O}^\times$. We can then decompose χ as $\chi = |\cdot|^s \chi_u$ by $\chi(x) = |\varpi^k|^s \chi_u(x_0)$, where χ_u is a character of $\mathcal{O}^\times$ and $0 \leqq \operatorname{Im} s < \frac{2\pi}{\ln q}$. It may be convenient, at times, to view χ_u as a character on $F^\times$ by $\chi_u(\varpi^k x_0) = \chi_u(x_0)$. We shall use 1 for the trivial character and sgn to denote a nontrivial character satisfying $\operatorname{sgn}^2 = 1$. In addition, we use $\nu = |\cdot|$.

As most of this paper concerns induced representations for $SO_m(F)$, we next discuss $SO_m(F)$ and induced representations. In this section, we review some of the structure theory for $SO_m(F)$.

Recall that we may take

$$SO_m(F) = \{x \in SL_m(F) \mid {}^T XJX = J\},$$

where

$$J = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$$

(entries left vacant are zeros). Let $K = SO_m(\mathcal{O})$. This is a maximal compact subgroup of $SO_m(F)$. It has a filtration of open compact normal subgroups

$$\ldots \lhd K_2 \lhd K_1 \lhd K,$$

where

$$K_i = \{X \in K \mid X \equiv I \bmod \mathcal{P}^i\}.$$

Let

$$A = \left\{ \begin{pmatrix} a_1 & & & & & & & \\ & a_2 & & & & & & \\ & & \ddots & & & & & \\ & & & a_n & & & & \\ & & & & a_n^{-1} & & & \\ & & & & & \ddots & & \\ & & & & & & a_2^{-1} & \\ & & & & & & & a_1^{-1} \end{pmatrix} \right\}$$

and

$$A = \left\{ \begin{pmatrix} a_1 & & & & & & \\ & \ddots & & & & & \\ & & a_n & & & & \\ & & & 1 & & & \\ & & & & a_n^{-1} & & \\ & & & & & \ddots & \\ & & & & & & a_1^{-1} \end{pmatrix} \right\}$$

be maximal split tori in $SO_m(F)$ for m even and m odd, respectively.

The Weyl group of $SO_m(F)$ is

$$W = N_G(A)/A,$$

where $N_G(A)$ denotes the normalizer of A in G. The Weyl group of $SO_{2n+1}(F)$ may be viewed as $W = \{$permutations and sign changes of $\{e_1, \ldots, e_n\}\}$, where one can view e_i as the linear functional on the Lie algebra of A which associates to an element of the Lie algebra of A the value of its ith (diagonal) entry. W is generated by the simple root reflections $\{s_1, \ldots, s_n\}$, where $s_1, \ldots, s_{n-1}, s_n$ denote the reflections corresponding to the simple roots $e_1 - e_2, \ldots, e_{n-1} - e_n, e_n$, respectively. We also use $s_1, \ldots, s_n$ to denote representatives for these elements.

Similarly, the Weyl group for $SO_{2k}(F)$ may be viewed as $W = \{$permutations and even sign changes of $\{e_1, \ldots, e_n\}\}$. W is generated by the simple root reflections $\{s_1, \ldots, s_n\}$, where $s_1, \ldots, s_{n-1}, s_n$ denote the reflections corresponding to the simple roots $e_1 - e_2, \ldots, e_{n-1} - e_n$, $e_{n-1} + e_n$, respectively. Again, we also use $s_1, \ldots, s_n$ to denote representatives for these.

We now review the parabolic and parahoric subgroups for $SO_m(F)$. A standard minimal parabolic subgroup for $SO_m(F)$, $P_{\min} = AU_{\min}$, consists of the upper triangular elements in $SO_m(F)$ (A as above). Let $\Phi \subset \{s_1, \ldots, s_n\}$. The standard parabolic subgroups of $SO_m(F)$ are parameterized by such subsets of the simple reflections. Associated to Φ is the parabolic subgroup $P_\Phi = \langle P_{\min}, \Phi \rangle$.

In $SO_{2n+1}(F)$, if $P = MN$ is the Levi factorization of P then M takes one of the following two forms:

1. if $s_n \in \Phi$, $M \cong GL_{k_1}(F) \times GL_{k_2}(F) \times \ldots \times GL_{k_{l-1}}(F) \times SO_{2k_l+1}(F)$,

2. if $s_n \notin \Phi$, $M \cong GL_{k_1}(F) \times GL_{k_2}(F) \times \ldots \times GL_{k_{l-1}}(F) \times GL_{k_l}(F)$, with $k_i > 0$ and $k_1 + k_2 + \ldots + k_l = n$.

In $SO_{2n}(F)$, if $P = MN$ is the Levi factorization of P then M takes one of the following forms:

1. if $s_{n-1}, s_n \in \Phi$, $M \cong GL_{k_1}(F) \times GL_{k_2}(F) \times \ldots \times GL_{k_{l-1}}(F) \times SO_{2k_l}(F)$,

2. if exactly one of s_{n-1}, s_n is in Φ,

$$M \cong GL_{k_1}(F) \times GL_{k_2}(F) \times \ldots \times GL_{k_{l-1}}(F) \times GL_{k_l}(F), \quad \text{with } k_l > 1,$$

3. if $s_{n-1}, s_n \notin \Phi$, $M \cong GL_{k_1}(F) \times GL_{k_2}(F) \times \ldots \times GL_{k_{l-1}}(F) \times F^{\times}$, with $k_i > 0$ and $k_1 + k_2 + \ldots + k_l = n$, or in the third case, $k_1 + k_2 + \ldots + k_{l-1} + 1 = n$.

The parahoric subgroups may be constructed in a similar fashion. A standard Iwahori subgroup for $SO_m(F)$ may be constructed as follows: let $\Psi : K \to SO_m(\mathbb{F}_q)$ be the reduction mod $\mathscr{P}$ homomorphism. Then, take $I = \Psi^{-1}(P_{\min}(\mathbb{F}_q))$ as the standard Iwahori subgroup. If we let $\{s_0, s_1, \ldots, s_n\}$ denote the generators of the affine Weyl group (where s_i for $i \geqq 1$ are as above), then the standard parahoric subgroups are in bijective correspondence with the subsets Φ of $\{s_0, s_1, \ldots, s_n\}$, the correspondence being given by

$$B_\Phi = \langle I, \Phi \rangle .$$

Note that if $\Phi \subset \{s_1, \ldots, s_n\}$, then

$$B_\Phi = \Psi^{-1}(P_\Phi(\mathbb{F}_q)) .$$

These are the parahoric subgroups we are most interested in.

1.2. Induced representations and Jacquet modules. We now review the construction of induced representations and Jacquet modules.

We start by reviewing the construction of induced representations. Let G be a reductive p-adic group and $P = MU$ a parabolic subgroup of G. Let (ϱ, X) be an admissible representation of M. Then ϱ may be extended trivially to U to get a representation of P, $\varrho \otimes 1$. The induced representation $\mathrm{Ind}_P^G \varrho \otimes 1$ acts by right translations on the space

$$V = \left\{ f : G \to X \,\middle|\, \begin{array}{l} f \text{ smooth}, \\ f(mug) = \delta^{\frac{1}{2}}(m)\,\varrho(m) f(g) \quad \forall m \in M,\, u \in U,\, g \in G \end{array} \right\},$$

where δ denotes the modular function for P. We give more convenient notation for induced representations after we discuss Jacquet modules.

We now describe the construction of Jacquet modules. Again, suppose G is a reductive p-adic group and $P = MU$ a parabolic subgroup of G. Let (π, V) be a representation of G. The Jacquet module of π with respect to P, denoted π_U, is a representation of M on the space

$$V_U = V / V(U) ,$$

where $V(U) = \text{span}\{\pi(u)v - v \mid v \in V,\, u \in U\}$. The action of M is given by

$$\pi_U(m)\big(v + V(U)\big) = \delta^{-\frac{1}{2}}(m)\,\pi(m)v + V(U)\,.$$

One checks that this defines a representation.

We shall frequently use the notation of Bernstein-Zelevinsky [B-Z] for induced representations and Jacquet modules. If $P = MU$ is a parabolic subgroup of G (containing a fixed minimal parabolic subgroup), (ϱ, X) an admissible representation of M, (π, V) an admissible representation of G, then set

$$i_{GM}\,\varrho = \text{Ind}_P^G\,\varrho \otimes 1$$

and

$$r_{MG}\,\pi = \pi_U\,.$$

Generally speaking, we are more interested in the components (irreducible composition factors) of a representation than in the actual structure of composition series. Therefore, if π has $\pi_1, \ldots, \pi_m$ as components, we write $\pi = \pi_1 + \ldots + \pi_m$. In the same spirit, let us write $\pi_0 \leqq \pi$ if the multiplicity of any irreducible admissible representation in π_0 is less than or equal to its multiplicity in π.

The following notation for induced representations in $SO_m(F)$ will also be convenient. It is just an extension (cf. [Tad1]) of the shorthand notation of Bernstein-Zelevinsky for induced representations in $GL_n(F)$. First, suppose $P = MU$ is a standard parabolic subgroup of $SO_{2n+1}(F)$, with

$$M \cong GL_{k_1}(F) \times \ldots \times GL_{k_{l-1}}(F) \times SO_{2k_l+1}(F)\,.$$

Let $\varrho_1, \ldots, \varrho_{l-1}$ be admissible representations of $GL_{k_1}(F), \ldots, GL_{k_{l-1}}(F)$, and τ an admissible representation of $SO_{2k_l+1}(F)$. Then, let

$$\varrho_1 \times \ldots \times \varrho_{l+1} \propto \tau = i_{GM}\,\varrho_1 \otimes \ldots \otimes \varrho_{l-1} \otimes \tau\,.$$

The sign $\propto$ is used only to indicate that the last factor comes from an orthogonal group. If $k_l = 0$, the representation will be denoted

$$\varrho_1 \times \ldots \times \varrho_{l-1} \propto 1\,.$$

The trivial representation of $SO_{2k+1}(F)$, $k > 0$, will be denoted tr_k, so there will be no confusion. Also, we let $[\chi]_k$ denote the character $\chi \circ \det$ on $GL_k(F)$.

We use a similar notation for $SO_{2n}(F)$. There is one additional complication, however. There are two different ways that the group

$$M \cong GL_{k_1}(F) \times \ldots \times GL_{k_l}(F)$$

with $k_l > 1$ can occur as the Levi factor of a parabolic subgroup. Essentially, such an M occurs if Φ contains one of s_{n-1} or s_n, but not both. (If both are present, the last factor is orthogonal.) We let

$$\varrho_1 \times \ldots \times \varrho_l = i_{GM}\varrho_1 \otimes \ldots \otimes \varrho_l$$

if s_{n-1} is in Φ, and

$$\varrho_1 \times \ldots \times \{\varrho_l\} = i_{GM}\varrho_1 \otimes \ldots \otimes \varrho_l$$

if s_n is in Φ instead.

We next give four theorems on induced representations and Jacquet modules.

Theorem 1.1 (Frobenius Reciprocity). *Let G be a connected reductive p-adic group, $P = MU$ a parabolic subgroup, ϱ an admissible representation of M, π an admissible representation of G. Then,*

$$\mathrm{Hom}_M(r_{MG}(\pi), \varrho) \cong \mathrm{Hom}_G(\pi, i_{GM}\varrho).$$

Proof. cf. [B-Z].

Theorem 1.2. *Let (τ, G, L) be an irreducible subquotient of $\mathrm{Ind}_{P_{\min}}^G \psi$ (ψ a character, not necessarily unitary). Then, there is a $w \in W$ so that τ embeds in $\mathrm{Ind}_{P_{\min}}^G w\psi$. Moreover, all $\mathrm{Ind}_{P_{\min}}^G w\psi$ for $w \in W$ have the same components.*

Proof. See [Cas2] for the first part and [B-Z] for the second.

We next recall the following theorem, which says the constructions of induced representations and Jacquet modules may be done in stages.

Theorem 1.3. *Let $L < M$ be standard Levis for G. Then*

1. $i_{GL} = i_{GM} \circ i_{ML}$,

2. $r_{LG} = r_{LM} \circ r_{MG}$.

Proof. cf. [B-Z].

Finally, we give a theorem of Bernstein-Zelevinsky, Casselman. Let M, N be standard Levis for a connected reductive p-adic group G. Set

$$W^{MN} = \{w \in W \mid w(P_{\min} \cap M) \subset P_{\min},\ w^{-1}(P_{\min} \cap N) \subset P_{\min}\}.$$

We remark that these correspond to the elements of shortest length in the double-cosets $W_N \backslash W / W_M$ (W_M = Weyl group of M, etc.).

Theorem 1.4 (Bernstein-Zelevinsky, Casselman). *Let ϱ be an admissible representation of M. Then, $r_{NG} \circ i_{GM}\varrho$ has a composition series with factors*

$$i_{NN'} \circ w \circ r_{M'M}\varrho, \quad w \in W^{MN},$$

where $M' = M \cap w^{-1}(N)$, $N' = w(M) \cap N$.

Proof. See [Cas2], chapter 6 or [B-Z] "geometrical lemma" (Lemma 2.12), proved in chapter 6.

The composition factors from the preceding theorem play an important role later in this paper. For convenience, we refer to them as B-Z composition factors.

It is worth pointing out that by the theorems above, if π is a reducible degenerate principal series representation, say $\pi = \pi_1 + \ldots + \pi_k$, then

1. $r_{LG}\pi = r_{LG}\pi_1 + \ldots + r_{LG}\pi_k$,

2. $r_{LG}\pi_k \neq 0$.

1.3. Langlands classification. We now turn to the Langlands classification. Before going any further, it is worth making the following remark: we set things up so that inducing a set of Langlands data gives rise to a unique (Langlands) subrepresentation, rather than a quotient. The reason is that by setting it up this way, it forces the Langlands data to lie in the Jacquet module with respect to the corresponding parabolic subgroup. This makes things a bit easier. We do not give the general statement of Langlands classification here; instead giving more explicit descriptions for GL_n and SO_n below. The reader is referred to [B-W] for the general result.

We start with GL_n. If ϱ is a representation of GL_m, we call ϱ essentially tempered if there is a real number $\varepsilon(\varrho)$ such that $|\det|^{-\varepsilon(\varrho)}\varrho$ is tempered. Suppose that $\varrho_1 \otimes \ldots \otimes \varrho_j$ is a representation of the standard Levi $GL_{k_1} \otimes \ldots \otimes GL_{k_j}$ with all of the ϱ's essentially tempered. If $\varepsilon(\varrho_1) < \ldots < \varepsilon(\varrho_j)$, this gives us a set of Langlands data. In particular, $\varrho_1 \times \ldots \times \varrho_j$ has a unique irreducible subrepresentation which we denote $\mathscr{L}(\varrho_1 \otimes \ldots \otimes \varrho_j)$. The use of $\mathscr{L}$ distinguishes this from the case of orthogonal groups, where L is used.

We now look at odd orthogonal groups. Suppose that $\varrho_1 \otimes \ldots \otimes \varrho_j \otimes \tau$ is a representation of the standard Levi $GL_{k_1} \otimes \ldots \otimes GL_{k_j} \otimes SO_{2k_{j+1}+1}$ with all of the ϱ's essentially tempered and τ tempered. If $\varepsilon(\varrho_1) < \ldots < \varepsilon(\varrho_j) < 0$, this gives us a set of Langlands data. In particular, $\varrho_1 \times \ldots \times \varrho_j \propto \tau$ has a unique irreducible subrepresentation which we denote $L(\varrho_1 \otimes \ldots \otimes \varrho_j \otimes \tau)$.

We finally consider even orthogonal groups. Things are slightly messier here. Suppose that $\varrho_1, \ldots, \varrho_j$ are essentially tempered representations of $GL_{k_1}, \ldots, GL_{k_j}$, respectively, and τ is a tempered representation of $SO_{2k_{j+1}}$. Let $n = k_1 + \ldots + k_j + k_{j+1}$.

1. If $e_{n-1} - e_n, e_{n-1} + e_n \notin \Phi$ (so that $\tau = 1$ and $k_j = 1$) has

$$\varepsilon(\varrho_1) < \ldots < \varepsilon(\varrho_{j-1}) < -|\varepsilon(\varrho_j)|,$$

then $\varrho_1 \otimes \ldots \otimes \varrho_{j-1} \otimes \varrho_j$ gives us a set of Langlands data.

2. If $e_{n-1} - e_n \in \Phi$ and $e_{n-1} + e_n \notin \Phi$ (so that $\tau = 1$ and $k_j > 1$) has $\varepsilon(\varrho_1) < \ldots < \varepsilon(\varrho_j) < 0$, then $\varrho_1 \otimes \ldots \otimes \varrho_{j-1} \otimes \varrho_j$ gives us a set of Langlands data.

3. If $e_{n-1} - e_n \notin \Phi$ and $e_{n-1} + e_n \in \Phi$ (so that $\tau = 1$ and $k_j > 1$) has $\varepsilon(\varrho_1) < \ldots < \varepsilon(\varrho_j) < 0$, then $\varrho_1 \otimes \ldots \otimes \varrho_{j-1} \otimes \{\varrho_j\}$ gives us a set of Langlands data.

4. If $e_{n-1} - e_n, e_{n-1} + e_n \in \Phi$ (so that there is a τ) has $\varepsilon(\varrho_1) < \ldots < \varepsilon(\varrho_j) < 0$, then $\varrho_1 \otimes \ldots \otimes \varrho_j \otimes \tau$ gives us a set of Langlands data.

In any case, $\varrho_1 \times \ldots \times \varrho_j \propto \tau$ (or $\varrho_1 \times \ldots \times \varrho_{j-1} \times \{\varrho_j\} \propto 1$ in the third case) has a unique irreducible subrepresentation which we denote $L(\varrho_1 \otimes \ldots \otimes \varrho_j \otimes \tau)$ or $L(\varrho_1 \otimes \ldots \otimes \varrho_j)$ if $\tau = 1$ (in the third case, $L(\varrho_1 \otimes \ldots \otimes \varrho_{j-1} \otimes \{\varrho_j\})$). Note that there is no confusion in using the same notation for both the odd and even orthogonal groups – the two cases are not discussed together in the remainder of this paper.

We close with the following remark, an easy consequence of Lemma 5.4 of [BDK] and the fact that $[\chi^{-1}]_k \otimes \mathrm{tr}_{n-k}$ is $w_0 \circ ([\chi]_k \otimes \mathrm{tr}_{n-k})$ for $w_0 \in W^{MA}$ of maximal length, M the corresponding standard Levi. It holds for both the even and odd orthogonal groups.

Remark 1.5. The representations $[\chi]_k \propto \mathrm{tr}_{n-k}$ and $[\chi^{-1}]_k \propto \mathrm{tr}_{n-k}$ have the same components (also true for $\{[\chi]_n\} \propto 1$ and $\{[\chi^{-1}]_n\} \propto 1$).

2. Some Hecke algebra results

2.1. General theorems. The principal goal of this chapter is to provide certain facts which will be needed later. In particular, most of the results in this paper will be based on the Jacquet module techniques of Tadić. However, in order to get a thorough picture, some additional information will be required for the odd orthogonal groups. For example, the Jacquet module techniques do not answer the question of whether the representation $[\chi]_2 \propto 1$ of $SO_5(F)$ is reducible when $\chi = \nu^{\frac{1}{2}}$. In this chapter, we give the results that will be needed. They are summarized in Proposition 2.10. Throughout this chapter, we are really only interested in the representation $\pi = [\chi]_2 \propto 1$ of $G = SO_5(F)$.

The extra information is obtained by Hecke algebra methods (cf. [Gus] or [Jan], chapter 2). We start by briefly reviewing how these Hecke algebra methods may be used in the study of degenerate principal series, then summarize the necessary theorems and definitions. First, for our degenerate principal series representation π, we construct an open compact subgroup B_χ (which depends only on χ_u). To the representation (π, G, V), we can associate a representation of the algebra $H(G//B_\chi)$ of compactly supported B_χ-biinvariant functions on the finite-dimensional space V^{B_χ}. This representation, denoted $(\pi, H(G//B_\chi), V^{B_\chi})$, has the property that the subquotients of (π, G, V) and those of $(\pi, H(G//B_\chi), V^{B_\chi})$ are in bijective correspondence, the bijection being implemented by $X \subset V$ G-invariant goes to $X^{B_\chi} \subset V^{B_\chi}$ $H(G//B_\chi)$-invariant. Once this correspondence is established, we can do our analysis on the Hecke algebra side, where the finite-dimensionality of the representation facilitates computations. Here, we just summarize the pertinent definitions, theorems, etc. – they are given in detail in [Jan], e.g.

Let $P \subset G$ be a parabolic subgroup, with $P = MU$ its Levi decomposition. Let χ be a character of M and set $\pi = i_{GM}\chi$. The subgroup B_χ which we use may be described by its Iwahori factorization with respect to P. In particular,

$$B_\chi = U_h^- M_\chi U_0$$

where $U_0 = U \cap K$, $U_h^- = U^- \cap K_h$, with h the smallest positive integer for which $\chi|_{M_h}$ is trivial (with $M_h = M \cap K_h$) and $M_\chi = M_0 \cap \ker\chi$. We normalize our measures so that $|B_\chi| = |U_h^-| = |M_\chi| = |U_0| = 1$. It is a group.

We now describe the representation $(\pi, H(G//B_\chi), V^{B_\chi})$. By definition, $H(G//B_\chi)$ is the (convolution) algebra of compactly supported B_χ-biinvariant functions. V^{B_χ} is the space of B_χ-invariant elements of V; finite-dimensional since B_χ is open compact. The action is given by

$$\pi(h)v = \int_G h(g)\pi(g)v\,dg$$

for

$$h \in H(G//B_\chi), \quad v \in V^{B_\chi}.$$

The following theorems, etc., are stated without proof. Proofs and references may be found in section 2.1 of [Jan].

Proposition 2.1. *Any subquotient of* (π, G, V) *has a (nonzero)* B_χ*-fixed vector.*

Lemma 2.2. *Let V be a smooth finite length representation of G, B an open compact subgroup of G. Suppose that every subquotient of V has a nonzero B-fixed vector. Then, every subquotient of V is generated by its B-fixed vectors.*

Theorem 2.3. *The map* $W \to W^{B_\chi}$ *gives a bijective correspondence between subquotients of* (π, G, V) *and subquotients of* $(\pi, H(G//B_\chi), V^{B_\chi})$.

2.2. Selected computations. The purpose of this section is to explicitly compute how certain elements of the Hecke algebra act (on V^{B_χ}). When coupled with the theorems from the preceding section, we will be able to draw certain conclusions about π – such as the fact that $[\chi]_2 \propto 1$ is irreducible for $\chi = \nu^{\frac{1}{2}}$. We summarize these results in Proposition 2.10.

The operators we compute in this section are $\pi(\mathscr{A}_y)$ and $\pi(\mathscr{A}_{s_2})$, where $\mathscr{A}_y, \mathscr{A}_{s_2} \in H(G//B_\chi)$ are described below. We do these computations for both the unramified case ($\chi_u = 1$) and the case χ_u order 2. As the computations here parallel those given in the second chapter of [Jan] for $Sp_{2_n}(F)$, we are content to outline the results.

The first step in this process is to determine a basis for V^{B_χ}. The process of determining a basis for V^{B_χ} goes roughly as follows. First, using the Bruhat decomposition as a starting point, we obtain a decomposition $K = \bigcup_{\alpha \in S} P_0 \alpha B_\chi$, so that $G = PK = \bigcup_{\alpha \in S} P\alpha B_\chi$ (S finite). Thus, $f \in V^{B_\chi}$ is determined by its values on $\alpha \in S$. The natural first attempt at a basis would then be $\{f_\alpha\}_{\alpha \in S}$, where

$$f_\alpha(g) = \begin{cases} \delta^{1/2}\chi(p) & \text{if } g = p\alpha b \in P\alpha B_\chi, \\ 0 & \text{if } g \notin P\alpha B_\chi. \end{cases}$$

Although these need not always be well-defined (cf. chapter 2 of [Jan]), for the cases we do here, that is not a problem – they turn out to be well-defined. We now determine such a decomposition.

First, let B denote the parahoric $\Psi^{-1}(P(\mathbb{F}_q))$ (cf. section 1.1). We start by giving the following Bruhat decomposition:

$$K = \bigcup_{i=0}^{2} Bw_iB$$

where $w_0 = I$, $w_1 = s_2$, and $w_2 = s_2s_1s_2$. Since $G = PK$, from this we can easily obtain

$$G = \bigcup_{i=0}^{2} Pw_iB.$$

In the unramified case, it is this decomposition which we use. In the ramified case, we modify it slightly. In either case, one can show that the prospective basis of V^{B_χ} described above is actually a basis.

In the case where χ is unramified, we have $B_\chi = B$. The decomposition above gives rise to the following basis for V^B:

Lemma 2.4. *A basis for V^B consists of*

$$f_i(g) = \begin{cases} \delta^{1/2}\chi(p) & \text{if } g = pw_ib \in Pw_iB, \\ 0 & \text{if } g \notin Pw_iB \end{cases}$$

for $i = 0, 1, 2$.

For the next step, we compute $\mathscr{A}_\gamma$, where $\mathscr{A}_\gamma = (|B_\chi\gamma B_\chi|)^{-1}\,\mathrm{char}_{B_\chi\gamma B_\chi}$ with

$$\gamma = \begin{pmatrix} \varpi & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \varpi^{-1} \end{pmatrix}.$$

Also, let $\mathscr{A}_{s_2} = (|B_\chi s_2 B_\chi|)^{-1}\,\mathrm{char}_{B_\chi s_2 B_\chi}$ (we write B_χ instead of B so that both of these definitions may be applied to the case χ_u of order two as well). The sort of computations used in [Gus] or chapter 2 of [Jan] give us the following, which we shall use in Proposition 2.10:

Lemma 2.5. *With respect to the basis $\{f_0, f_1, f_2\}$,*

$$\pi(\mathscr{A}_\gamma) = \begin{pmatrix} q^{-s-1} & 0 & 0 \\ \dfrac{1}{q+1}(1-q^{-1})q^{-s-1} & \dfrac{qq^{-s-1}+q^{-1}q^{s+1}}{q+1} & 0 \\ q^{-1}(1-q^{-1})q^{-s-1} & (1-q^{-1})q^{-s-1} & q^{-2}q^{s+1} \end{pmatrix}.$$

Lemma 2.6. *With respect to the basis* $\{f_0, f_1, f_2\}$,

$$\pi(\mathscr{A}_{s_2}) = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{q+1} q^{-1} & \frac{1}{q+1}(1-q^{-1}) & \frac{q}{q+1} \\ 0 & q^{-1} & (1-q^{-1}) \end{pmatrix}.$$

We now turn to the case where χ_u has order two. In this case, one can show

$$B = B_\chi \cup \varepsilon B_\chi,$$

where $\varepsilon \in A$ and $\chi(\varepsilon) = -1$. Then, it is easy to see that

$$G = \bigcup_{i=0}^{2} P w_i B_\chi.$$

From this decomposition, we can obtain a basis for V^{B_χ}.

Lemma 2.7. *A basis for* V^{B_χ} *consists of*

$$f_i(g) = \begin{cases} \delta^{1/2}\chi(p) & \text{if } g = p w_i b \in P w_i B_\chi, \\ 0 & \text{if } g \notin P w_i B_\chi \end{cases}$$

for $i = 0, 1, 2$.

The following pair of operators serve the same role for the case $\chi_u^2 = 1$ in Proposition 2.10 as their counterparts from the unramified case do.

Lemma 2.8. *With respect to the basis* $\{f_0, f_1, f_2\}$,

$$\pi(\mathscr{A}_\gamma) = \begin{pmatrix} q^{-s-1} & 0 & 0 \\ \frac{1}{q+1}(1-q^{-1})q^{-s-1}\chi(-2) & \frac{qq^{-s-1}+q^{-1}q^{s+1}}{q+1} & 0 \\ q^{-1}(1-q^{-1})q^{-s-1} & (1-q^{-1})q^{-s-1}\chi(-2) & q^{-2}q^{s+1} \end{pmatrix}.$$

Lemma 2.9. *With respect to the basis* $\{f_0, f_1, f_2\}$,

$$\pi(\mathscr{A}_{s_2}) = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{q+1} q^{-1} & \frac{1}{q+1}(1-q^{-1})\chi(-2) & \frac{q}{q+1} \\ 0 & q^{-1} & (1-q^{-1})\chi(-2) \end{pmatrix}.$$

Proposition 2.10. *If* $\chi = \nu^{\pm\frac{1}{2}}, \nu^{\pm\frac{1}{2}}\,\mathrm{sgn}$, *then* π *is irreducible. Also, if* $\chi = 1, \mathrm{sgn}$, *then* π *can have at most two components.*

Proof. First, in the case where $\chi = \nu^{\pm\frac{1}{2}}, \nu^{\pm\frac{1}{2}}\,\mathrm{sgn}$, we can check that $\pi(\mathscr{A}_\gamma)$ and $\pi(\mathscr{A}_{s_2})$ have no common invariant subspace. This gives us the irreducibility of $(\pi, H(G//B_\chi), V^{B_\chi})$, hence the result we were after (by Theorem 2.3). Note that we could also obtain irreducibility for other χ's, but we do not need them – they are covered by the results in chapter 4.

Similarly, when $\chi = 1$, sgn, it is easy to check that the operators admit exactly two proper invariant subspaces, one one-dimensional and the other two-dimensional, complementary to the first. Therefore, π can have at most two components at the Hecke algebra level, hence at the group level, as needed.

3. The regular case

3.1. A general theorem on reducibility/irreducibility. In the first section of this chapter, we recall some necessary and sufficient conditions for irreducibility of an induced representation. In the second and third sections, we apply this to degenerate principal series for orthogonal groups. Much of this section follows chapter 3 of [Jan], which is based on the Jacquet module techniques of Tadić [Tad3].

The general result we use in the second and third section is Theorem 3.1. To set this up, let G denote a split connected reductive p-adic group, $P = MU$ a parabolic subgroup of G, and ϱ an irreducible admissible representation of M. Set $\pi = i_{GM}\varrho$. Next, let $s_1, \ldots, s_n$ be the simple reflections in W. Let $P_{\min} = AU_{\min}$ denote the minimal parabolic, and set $P_i = \langle P_{\min}, s_i \rangle = M_i U_i$. For notational convenience, we let $BZ_N(\pi)$ denote the collection of representations $i_{NN'} \circ w \circ \tau$ as τ runs over the components of $r_{M'M}\varrho$ and w runs over W^{MN}. These are the representations from Theorem 1.4, i.e., the composition factors of $r_{NG} \circ i_{GM}\varrho$ as computed by Bernstein-Zelevinsky and Casselman.

We now proceed to set up Theorem 3.1, which serves as a starting point for the regular case, giving necessary and sufficient conditions for reducibility of $i_{GM}\varrho$. We require three things of ϱ. First, we want ϱ irreducible. Second, we want $r_{AM}(\varrho) \neq 0$. Finally, we require a regularity condition on ϱ. Let ψ be a character in $r_{AM}(\varrho)$. We require that ψ be regular with respect to W (not just W_M). Note that if this is true for one character in $r_{AM}(\varrho)$, it will be true for all of them.

We now associate a graph to π as follows:

vertices: the vertices are the elements of $r_{AG}(\pi)$,

edges: two vertices ψ_1, ψ_2 are connected by an edge if there is some Levi N and some $\tau \leqq r_{NG}(\pi)$ such that the following hold:

1. τ is an irreducible representation of N.

2. $\psi_1, \psi_2 \leqq r_{AN}(\tau)$.

This brings us to Theorem 3.1.

Theorem 3.1. *Under the conditions above, the following are equivalent:*

1. *π is irreducible,*

2. *the graph of π is connected,*

3. *$\tau \in BZ_{M_i}(\pi) \Rightarrow \tau$ is irreducible (for any i and τ).*

Proof. This is Theorem 3.1.2 of [Jan]. The proof is based on the Jacquet module techniques of Tadić. Although we do not include the proof of this theorem, there is a discussion in section 3.2 which describes what happens in the case of degenerate principal series for $SO_{2n+1}(F)$. We also give the following corollary of the proof of the theorem.

Corollary 3.2. *Under the same hypotheses, the number of components of π is less than or equal to the number of components in the graph of π.*

Remark 3.3. *If one drops the hypothesis of regularity, the theorem no longer holds. Counterexamples are discussed in Remarks* 4.2 *and* 4.7.

3.2. Applications to degenerate principal series for $SO_{2n+1}(F)$ in the regular case. In this section, we apply Theorem 3.1 to the case of degenerate principal series for $G = SO_{2n+1}(F)$.

We start by checking when $r_{AM}([\chi]_k \otimes \mathrm{tr}_{n-k})$ is regular (so

$$M \cong GL_k(F) \times SO_{2(n-k)+1}(F)).$$

Lemma 3.4. *Let*

$$S_1 = \left\{-n + \frac{k}{2} + 1, -n + \frac{k}{2} + 2, \ldots, n - \frac{k}{2} - 1\right\},$$

$$S_2 = \left\{\frac{-k+1}{2}, \frac{-k+2}{2}, \ldots, \frac{k-1}{2}\right\}.$$

Then, $r_{AM}([\chi]_k \otimes \mathrm{tr}_{n-k})$ is nonregular precisely when

$$\chi \in \{\nu^\alpha | \alpha \in S_1\} \cup \{\nu^\alpha \psi | \alpha \in S_2\},$$

where $\psi^2 = 1$ (i.e., $\psi = 1$ or sgn).

Proof. First, we observe that

$$r_{AM}([\chi]_k \otimes \mathrm{tr}_{n-k})$$
$$= \nu^{\frac{-k+1}{2}}\chi \otimes \nu^{\frac{-k+3}{2}}\chi \otimes \ldots \otimes \nu^{\frac{k-1}{2}}\chi \otimes \nu^{-n+k+\frac{1}{2}} \otimes \ldots \otimes \nu^{-n+k+\frac{3}{2}} \otimes \ldots \otimes \nu^{-\frac{1}{2}}.$$

Note that if $k = n$, then all the terms have a χ. In general, the character $\chi_1 \otimes \chi_2 \otimes \ldots \otimes \chi_n$ will be nonregular for $SO_{2n+1}(F)$ if either of the following conditions holds:

i) $\chi_i = \chi_j$ for some i, j with $i \neq j$

or

ii) $\chi_i = \chi_j^{-1}$ for some i, j.

In our situation, these become

1) $\nu^{\frac{-k+1}{2}+l}\chi = \nu^{-n+k+\frac{1}{2}+m}$ for some l, m with $0 \leqq l \leqq k-1, 0 \leqq m \leqq n-k-1$,

2) $(\nu^{\frac{-k+1}{2}+l}\chi)^{-1} = \nu^{-n+k+\frac{1}{2}+m}$ for some l, m with $0 \leqq l \leqq k-1, 0 \leqq m \leqq n-k-1$,

3) $\nu^{\frac{-k+1}{2}+l}\chi = (\nu^{\frac{-k+1}{2}+m}\chi)^{-1}$ for some l, m with $0 \leqq l, m \leqq k-1$.

Note that cases 1) and 2) do not occur when $n = k$. This is accounted for in the lemma by the fact that if $n = k$, $S_1 \subset S_2$.

Finally, if we work these out, we see that i) and ii) together give nonregularity for $\chi \in \{\nu^\alpha | \alpha \in S_1\}$. Case 3) contributes $\chi \in \{\nu^\alpha \psi | \alpha \in S_2\}$.

At this point it is useful to give a concrete description of the action of W^{MA} and the Jacquet module of π, and then interpret the results in terms of Tadić's approach.

Let $G = SO_{2n+1}$, $M = GL_k \times SO_{2(n-k)+1}$. We now describe how W^{MA} acts on $\mu = \chi_1 \otimes \ldots \otimes \chi_k \otimes \psi_1 \otimes \ldots \otimes \psi_{n-k}$. If $w \in W^{MA}$, we claim that the entries of $w\mu$ are a permutation of $\chi_1, \ldots, \chi_\ell, \chi_{\ell+1}^{-1}, \ldots, \chi_k^{-1}, \psi_1, \ldots, \psi_{n-k}$ (any ℓ), subject to the following constraints:

1. $\chi_1, \ldots, \chi_\ell$ occur in that order,

2. $\chi_{\ell+1}^{-1}, \ldots, \chi_k^{-1}$ appear in reverse order,

3. $\psi_1, \ldots, \psi_{n-k}$ appear in that order.

Furthermore, the action of W^{MA} on μ gives every character satisfying the conditions above.

Next, suppose that

$$\ldots \otimes \underbrace{\chi_i \otimes \psi_j}_{t,t+1} \otimes \ldots \in W^{MA}\mu .$$

By the description of $W^{MA}\mu$ above, $\ldots \otimes \psi_j \otimes \chi_i \otimes \ldots \in W^{MA}\mu$ as well, where χ_i, ψ_j are switched and everything else is the same. Then, $\sigma = \ldots \otimes (\chi_i \times \psi_j) \otimes \ldots$ is in BZ_{M_t}. If σ is irreducible, then the characters above share an edge and one can "commute" them. More precisely, if π_0 is a component of π with $\ldots \otimes \chi_i \otimes \psi_j \otimes \ldots \leqq r_{AG}\pi_0$, then we get the following implications:

$$\ldots \otimes \chi_i \otimes \psi_j \otimes \ldots \leqq r_{AG}\pi_0$$
$$\Downarrow$$
$$\ldots \otimes (\chi_i \otimes \psi_j) \otimes \ldots \leqq r_{M_t G}\pi_0$$
$$\Downarrow$$
$$\ldots \otimes \psi_j \otimes \chi_i \otimes \ldots \leqq r_{AG}\pi_0 .$$

One can do the same thing for $\chi_i^{-1} \times \psi_j$, and $\chi_i^{-1} \times \chi_j$. A similar argument works for $\chi_i \propto 1$:

$$\ldots \otimes \chi_i \leqq r_{AG}\pi_0$$
$$\Downarrow$$
$$\ldots \otimes (\chi_i \propto 1) \leqq r_{M_n G}\pi_0$$
$$\Downarrow$$
$$\ldots \otimes \chi_i^{-1} \leqq r_{AG}\pi_0 .$$

In summary, assuming the irreducibility of the appropriate representation of GL_2 or SO_3, for a character in $r_{AG}\pi$, we can

1. commute adjacent χ_i and ψ_j,
2. invert χ_i if it is at the end,
3. commute adjacent ψ_j and χ_i^{-1},
4. commute adjacent χ_j and χ_i^{-1},

and the resulting character will still be in the same component of the graph of π, hence in the Jacquet module of the same component of π. We comment that if all of the BZ_{M_i} are irreducible, then it is possible to generate all of $W^{MA}\mu$ by starting with μ and successively applying the commuting/inverting processes described above. This gives a connected graph and irreducible representation, as in the theorem. If these representations are not all irreducible, the resulting characters are still in $r_{AG}\pi$, but do not need to come from the same component of the graph of π.

We use this description to tell us when condition 3 of Theorem 3.1 is satisfied. Before doing this, we pause to note the following:

Lemma 3.5. 1. *The representation* $\chi_1 \times \chi_2$ *of* $GL_2(F)$ *is reducible if and only if* $\chi_1 = \nu^{\pm 1}\chi_2$. *We have* $\nu^{-\frac{1}{2}}\chi \times \nu^{\frac{1}{2}}\chi = [\chi]_2 + St_{GL(2)}(\chi)$.

2. *The representation* $\chi \propto 1$ *of* $SO_3(F)$ *is reducible if and only if* $\chi^2 = \nu^{\pm 1}$. *We have* $\nu^{-\frac{1}{2}} \propto 1 = tr_1 + St_{SO(3)}$ *and write* $\nu^{-\frac{1}{2}}\,\mathrm{sgn} \propto 1 = L(\nu^{-\frac{1}{2}}\,\mathrm{sgn}) + St_{SO(3)}(\mathrm{sgn})$ *(thereby giving the second component a name).*

Proof. See [B-Z] for the first claim. The second follows from [Mul].

Lemma 3.6. *All the* B-Z *composition factors of the Jacquet modules* $r_{M_i G}(\pi)$, $1 \leqq i \leqq n$, *are irreducible* (*i.e.*, *condition* 3 *in Theorem* 3.1 *holds*) *except when* χ *has one of the following forms*:

$$\chi \in \{\nu^{-n+\frac{k}{2}}, \nu^{-n+\frac{k}{2}+1}, \ldots, \nu^{n-\frac{k}{2}}\}$$

or

$$\chi^2 \in \{\nu^{-k}, \nu^{-k+1}, \ldots, \nu^{k}\} .$$

Exception: *if* $k = 1$, *then* $\chi^2 = 1$ *does not actually give a reducible* B-Z *composition factor*.

Proof. By the preceding discussion, it is enough to check whether the following representations are reducible or irreducible.

(i) $(\nu^{\frac{-k+1}{2}+l}\chi) \propto 1$ for $0 \leqq l \leqq k-1$,

(ii) $(\nu^{\frac{-k+1}{2}+l}\chi) \times \nu^{-m-\frac{1}{2}}$ for $0 \leqq l \leqq k-1, 0 \leqq m \leqq n-k-1$,

(iii) $(\nu^{\frac{-k+1}{2}+l}\chi) \times \nu^{m+\frac{1}{2}}$ for $0 \leqq l \leqq k-1, 0 \leqq m \leqq n-k-1$,

(iv) $(\nu^{\frac{-k+1}{2}+l}\chi) \times (\nu^{\frac{-k+1}{2}+m}\chi)^{-1}$ for $0 \leqq l, m \leqq k-1, l \neq m$.

Note that (ii) and (iii) do not occur if $k = n$.

These are reducible at the following points:

(i) $\chi^2 \in \{\nu^{-k}, \nu^{-k+2}, \ldots, \nu^{k}\}$,

(ii) $\chi \in \{\nu^{-n+\frac{k}{2}}, \nu^{-n+\frac{k}{2}+1}, \ldots, \nu^{\frac{k}{2}}\}$,

(iii) $\chi \in \{\nu^{-\frac{k}{2}}, \nu^{-\frac{k}{2}+1}, \ldots, \nu^{n-\frac{k}{2}}\}$,

(iv) $\chi^2 \in \{\nu^{-k+1}, \nu^{-k+2}, \ldots, \nu^{k-1}\}$,

except that (iv) does not occur if $k = 1$, and if $k = 2$, is not reducible at $\chi^2 = 1$ (but that is included in (i) anyway). Combining these gives the lemma.

Theorem 3.7. *The only values of χ for which $\pi = [\chi]_k \propto \mathrm{tr}_{n-k}$ have $r_{AM}([\chi]_k \otimes \mathrm{tr}_{n-k})$ regular and π reducible are*

$$\chi \in \{\nu^{-n+\frac{k}{2}}, \nu^{-\frac{k}{2}}\,\mathrm{sgn}, \nu^{\frac{k}{2}}\,\mathrm{sgn}, \nu^{n-\frac{k}{2}}\}.$$

In each case, the representation has two components. The components are given below:

(1) $\chi = \nu^{-n+\frac{k}{2}}$. *In this case,*

$$\pi = L(\nu^{-n+\frac{1}{2}} \otimes \nu^{-n+\frac{3}{2}} \otimes \ldots \otimes \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}})$$

$$+ L(\nu^{-n+\frac{1}{2}} \otimes \nu^{-n+\frac{k}{2}+1} \otimes \ldots \otimes \nu^{-n+k-\frac{3}{2}} \otimes St_{GL(2)}(\nu^{-n+k}) \otimes \nu^{-n+k+\frac{3}{2}} \otimes \ldots \otimes \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}}).$$

The former is just the trivial representation, and $r_{AG}\,\mathrm{tr}_n$ contains only

$$\nu^{-n+\frac{1}{2}} \otimes \nu^{-n+\frac{3}{2}} \otimes \ldots \otimes \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}}.$$

The Jaquet module of the second contains all of the remaining components of $r_{AG}\,\pi$.

(2) $\chi = \nu^{-\frac{k}{2}}\operatorname{sgn}$. *Let* $\psi = \operatorname{sgn}$. *Then, if* $k \leqq n/2$, *we have*

$$\pi = L\big(\nu^{-n+k+\frac{1}{2}} \otimes \ldots \otimes \nu^{-k-\frac{1}{2}} \otimes (\nu^{-k+\frac{1}{2}} \times \nu^{-k+\frac{1}{2}}\psi)$$
$$\otimes \ldots \otimes (\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}\psi)\big) + L\big(\nu^{-n+k+\frac{1}{2}} \otimes \ldots \otimes \nu^{-k-\frac{1}{2}} \otimes (\nu^{-k+\frac{1}{2}} \times \nu^{-k+\frac{1}{2}}\psi)$$
$$\otimes \ldots \otimes (\nu^{-\frac{3}{2}} \times \nu^{-\frac{3}{2}}\psi) \otimes \nu^{-\frac{1}{2}} \otimes St_{SO(3)}(\psi)\big).$$

They split up the Jacquet module of π *as follows: if* $\mu \leqq r_{AG}\pi$ *has* $\nu^{-\frac{1}{2}}\psi$ *in it, then* μ *is in the Jacquet module of the first component. On the other hand, if* μ *has* $\nu^{\frac{1}{2}}\psi$ *in it, then it is in the Jacquet module of the second. The case* $k > n/2$ *is similar – here*

$$\pi = L\big(\nu^{-k+\frac{1}{2}}\psi \otimes \ldots \otimes \nu^{-n+k-\frac{1}{2}}\psi \otimes (\nu^{-n+k+\frac{1}{2}} \times \nu^{-n+k+\frac{1}{2}}\psi) \otimes \ldots \otimes (\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}\psi)\big)$$
$$+ L\big(\nu^{-k+\frac{1}{2}}\psi \otimes \ldots \otimes \nu^{-n+k-\frac{1}{2}}\psi \otimes (\nu^{-n+k+\frac{1}{2}} \times \nu^{-n+k+\frac{1}{2}}\psi)$$
$$\otimes \ldots \otimes (\nu^{-\frac{3}{2}} \times \nu^{-\frac{3}{2}}\psi) \otimes \nu^{-\frac{1}{2}} \otimes St_{SO(3)}(\psi)\big).$$

The Jacquet module splits up the same way.

By Remark 1.5, *the cases* $\chi = \nu^{n-\frac{k}{2}}, \nu^{\frac{k}{2}}\operatorname{sgn}$ *have the same components as* (1) *and* (2) *above.*

Proof. Combine Lemmas 3.4 and 3.6 to see that these are exactly the reducibility points (for the regular case).

We can use Jacquet module considerations to show that the representations given in the statement of the theorem are components of π. For example, for $\pi = [\nu^{-n+\frac{k}{2}}]_k \propto \mathrm{tr}_{n-k}$, π is a subrepresentation of $\tau = \nu^{-n+\frac{1}{2}} \times \ldots \times \nu^{-\frac{3}{2}} \times \nu^{-\frac{1}{2}} \propto 1$. Since $\nu^{-n+\frac{1}{2}} \otimes \ldots \otimes \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}}$ is regular, every character in $r_{AG}\tau$ appears exactly once. Now,

$$L(\nu^{-n+\frac{k}{2}} \otimes \nu^{-n+\frac{k}{2}+1} \otimes \ldots \otimes \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}}) \text{ and } L(\nu^{-n+\frac{k}{2}} \otimes \nu^{-n+\frac{k}{2}+1} \otimes \ldots \otimes \nu^{-\frac{3}{2}} \otimes St_{SO(3)})$$

are components of τ. Observe that the only copy of $\nu^{-n+\frac{k}{2}} \otimes \ldots \otimes \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}}$ in $r_{AG}\tau$ lies in $r_{AG}L(\nu^{-n+\frac{k}{2}} \otimes \ldots \otimes \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}})$ and the only copy of $\nu^{-n+\frac{k}{2}} \otimes \ldots \otimes \nu^{-\frac{3}{2}} \otimes \nu^{\frac{1}{2}}$ lies in $r_{AG}L(\nu^{-n+\frac{k}{2}} \otimes \ldots \otimes \nu^{-\frac{3}{2}} \otimes St_{SO(3)})$. Since $r_{AG}\pi$ contains both $\nu^{-n+\frac{k}{2}} \otimes \ldots \otimes \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}}$ and $\nu^{-n+\frac{k}{2}} \otimes \ldots \otimes \nu^{-\frac{3}{2}} \otimes \nu^{\frac{1}{2}}$, we must have that π contains both $L(\nu^{-n+\frac{k}{2}} \otimes \ldots \otimes \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}})$ and $L(\nu^{-n+\frac{k}{2}} \otimes \ldots \otimes \nu^{-\frac{3}{2}} \otimes St_{SO(3)})$. The case $\chi = \nu^{\pm\frac{k}{2}}\operatorname{sgn}$ is similar.

Thus, it remains to argue that they are the only components. We concentrate on the second (harder) case. By Corollary 3.2, it suffices to show that the graph of π has only two components. The characterization of the Jacquet modules tells us how to characterize the components: one has characters containing $\nu^{-\frac{1}{2}}\psi$; the other, those with $\nu^{\frac{1}{2}}\psi$. Observe that the only reducible BZ_{M_i} which occur are of the form $\ldots \otimes (\nu^{-\frac{1}{2}}\psi \propto 1)$. Then, given the description of the action of Jacquet modules preceding Lemma 3.5, it is not hard to see that these components are connected. This finishes the second case. The first is easier since the only reducible BZ_{M_i} is $\nu^{-n+\frac{k}{2}} \otimes \dots \otimes \nu^{-\frac{k}{2}-2} \otimes (\nu^{-\frac{k}{2}-1} \times \nu^{-\frac{k}{2}}) \otimes \nu^{-\frac{k}{2}+1} \otimes \dots \otimes \nu^{-\frac{1}{2}}$, so only one edge is missing. Thus, the representation can have at most two components.

3.3. Applications to degenerate principal series for $SO_{2n}(F)$ (in the regular case). In this section, we apply Theorem 3.1 to the case of degenerate principal series for $G = SO_{2n}(F)$.

Again, we start by checking when $r_{AM}([\chi]_k \otimes \mathrm{tr}_{n-k})$ with $k \leqq n-2$, $r_{AM}([\chi]_n)$, and $r_{AM}(\{[\chi]_n\})$ are regular.

Lemma 3.8. *Let*

$$S_1 = \left\{-n + \frac{k+3}{2}, -n + \frac{k+3}{2} + 1, \dots, n - \frac{k+3}{2}\right\},$$

$$S_2 = \left\{\frac{-k+1}{2}, \frac{-k+2}{2}, \dots, \frac{k-1}{2}\right\},$$

$$S = \left\{\frac{-n+2}{2}, \frac{-n+3}{2}, \dots, \frac{n-2}{2}\right\}.$$

Case 1: $k < n-1$. *$r_{AM}([\chi]_k \otimes \mathrm{tr}_{n-k})$ is nonregular precisely when*

$$\chi \in \{\nu^\alpha \mid \alpha \in S_1\} \cup \{\nu^\alpha \psi \mid \alpha \in S_2\},$$

where $\psi^2 \equiv 1$.

Case 2: $k = n$. *$r_{AM}(\{[\chi]_n\})$ is nonregular for $\chi \in \{\nu^\alpha \psi \mid \alpha \in S\}$, where $\psi^2 \equiv 1$. $r_{AM}([\chi]_n)$ has the same regularity points as $r_{AM}(\{[\chi]_n\})$.*

Proof. First, we observe that

1. $r_{AM}([\chi]_k \otimes \mathrm{tr}_{n-k})$
$$= \nu^{\frac{-k+1}{2}}\chi \otimes \nu^{\frac{-k+3}{2}}\chi \otimes \dots \otimes \nu^{\frac{k-1}{2}}\chi \otimes \nu^{-n+k+1} \otimes \nu^{-n+k+2} \otimes \dots \otimes \nu^{-1} \otimes 1,$$

2. $r_{AM}(\{[\chi]_n\}) = \nu^{\frac{-n+1}{2}}\chi \otimes \nu^{\frac{-n+3}{2}}\chi \otimes \dots \otimes \nu^{\frac{n-3}{2}}\chi \otimes \nu^{\frac{-n+1}{2}}\chi^{-1}$.

In general, the character $\chi_1 \otimes \chi_2 \otimes \dots \otimes \chi_n$ will be nonregular for $SO_{2n}(F)$ if either of the following conditions holds:

(i) $\chi_i = \chi_j$ for some i, j with $i \neq j$,

ii) $\chi_i = \chi_j^{-1}$ for some $i \neq j$,

or

iii) $\chi_i = \chi_i^{-1}, \chi_j = \chi_j^{-1}$ for some $i \neq j$.

We remark that since the Weyl group inverts χ_j's in pairs, there must actually be a pair of statements in case iii). For $k = n$, case iii) contributes nothing. If $k \neq n$, then this case just requires $\chi_i = \chi_i^{-1}$ since one can use $\chi_j = 1$ for the other. The computations required to verify the lemma are straightforward and are omitted.

Lemma 3.9. *All the B-Z composition factors of the Jacquet modules $r_{M_iG}(\pi)$, $1 \leqq i \leqq n$, are irreducible except when χ has one of the following forms:*

$$\chi \in \{\nu^{-n+\frac{k+1}{2}}, \nu^{-n+\frac{k+1}{2}+1}, \ldots, \nu^{n-\frac{k+1}{2}}\} \quad or \quad \chi^2 \in \{\nu^{-k+1}, \nu^{-k+2}, \ldots, \nu^{k-1}\}$$

with the exception that if $k = 1$, $\chi = \operatorname{sgn}$ does not occur and if $k = 2$, $\chi^2 = 1$ does not occur.

Proof. As in Lemma 3.6, it is enough to check whether the following representations are reducible or irreducible.

(i) $(\nu^{\frac{-k+1}{2}+l}\chi) \times \nu^{-m-\frac{1}{2}}$ for $0 \leqq l \leqq k-1, 0 \leqq m \leqq n-k-1$,

(ii) $(\nu^{\frac{-k+1}{2}+l}\chi) \times \nu^{m+\frac{1}{2}}$ for $0 \leqq l \leqq k-1, 0 \leqq m \leqq n-k-1$,

(iii) $(\nu^{\frac{-k+1}{2}+l}\chi) \times (\nu^{\frac{-k+1}{2}+m}\chi)^{-1}$ for $0 \leqq l, m \leqq k-1, l \neq m$.

Note that (i) and (ii) do not occur if $k = n$; (iii) does not occur if $k = 1$.

Cases (i) and (ii) combine to give reducibility for $\chi \in \{\nu^{-n+\frac{k+1}{2}}, \nu^{-n+\frac{k+1}{2}+1}, \ldots, \nu^{n-\frac{k+1}{2}}\}$. The third gives reducibility for $\chi^2 \in \{\nu^{-k+1}, \nu^{-k+2}, \ldots, \nu^{k-1}\}$. Observe that if $k = n$, then this contains the first set, so there is no problem with the lemma (even though (i) and (ii) do not occur). Also, note that (iii) does not occur if $k = 1$, and if $k = 2$, is not reducible at $\chi^2 = 1$.

Theorem 3.10. 1. *The representation $[\chi]_k \propto \operatorname{tr}_{n-k}$ with $k < n-1$ is reducible in the regular case when $\chi = \nu^{-n+\frac{k+1}{2}}$ or $\nu^{n-\frac{k+1}{2}}$, and irreducible for all other regular χ. For $\chi = \nu^{-n+\frac{k+1}{2}}$, we have*

$$\pi = L(\nu^{-n+1} \otimes \ldots \otimes \nu^{-2} \otimes \nu^{-1} \otimes 1)$$
$$+ L(\nu^{-n+1} \otimes \ldots \otimes \nu^{-n+k-1} \otimes St_{GL(2)}(\nu^{-n+k+\frac{1}{2}}) \otimes \nu^{-n+k+2} \otimes \ldots \nu^{-1} \otimes 1).$$

The first of these is just the trivial representation, and its Jacquet module contains only the character $\nu^{-n+1} \otimes \ldots \otimes \nu^{-1} \otimes 1$. The Jacquet module of the second contains all the remaining components of $r_{AG}\pi$. By Remark 1.5, the case $\chi = \nu^{n-\frac{k+1}{2}}$ has the same components.

2. *The representation $[\chi]_n \propto 1$ is reducible in the regular case when $\chi = \nu^{\pm(\frac{n-1}{2})}$ or $\nu^{\pm(\frac{n-1}{2})}\operatorname{sgn}$, and irreducible for all other regular χ. If $\chi = \nu^{-\frac{n-1}{2}}\operatorname{sgn}$, then, letting $\psi = \operatorname{sgn}$*

$$\pi = L(\nu^{-n+1}\psi \otimes \ldots \otimes \nu^{-2}\psi \otimes \nu^{-1}\psi \otimes \psi) + L(\nu^{-n+1}\psi \otimes \ldots \otimes \nu^{-2}\psi \otimes \{St_{GL(2)}(\nu^{-\frac{1}{2}}\psi)\}).$$

If $\chi = \nu^{-\frac{n-1}{2}}$, *the result is the same, except with* ψ *replaced by* 1. *Again, the cases* $\chi = \nu^{(\frac{n-1}{2})}$ sgn, $\nu^{(\frac{n-1}{2})}$ *have the same components as those above.*

3. $\{[\chi]_n\} \propto 1$ *has the same regular reducibility points as* $[\chi]_n \propto 1$. *If* $\chi = \nu^{-\frac{n-1}{2}}$ sgn, *then, letting* $\psi =$ sgn

$$\pi = L(\nu^{-n+1}\psi \otimes \ldots \otimes \nu^{-2}\psi \otimes \nu^{-1}\psi \otimes \psi) + L(\nu^{-n+1}\psi \otimes \ldots \otimes \nu^{-2}\psi \otimes St_{GL(2)}(\nu^{-\frac{1}{2}}\psi)).$$

Again, if $\chi = \nu^{-\frac{n-1}{2}}$, *the result is the same, except with* ψ *replaced by* 1. *Also, the cases* $\chi = \nu^{(\frac{n-1}{2})}$ sgn, $\nu^{(\frac{n-1}{2})}$ *have the same components as those above.*

Proof. Combining Lemmas 3.8 and 3.9 gives the claims about points of reducibility.

By Jacquet module considerations like those in Theorem 3.7, the representations given in the statement of the theorem are components of π. Thus, it remains to argue that they are the only components. In each case, there is only one reducible BZ_{M_i}, hence the graph has at most two components.

4. Reducibility for the low-rank cases

4.1. The case of SO_5. In this section, we analyze degenerate principal series for $SO_5(F)$. We rely mainly on the Jacquet module techniques of Tadić, but also need the results of chapter 2.

Let $L \cong F^\times \times SO_3(F) \subset \langle P_{\min}, s_2\rangle$ and $M \cong GL_2(F) \subset \langle P_{\min}, s_1\rangle$ be the Levi factors of the standard maximal parabolic subgroups of $G = SO_5(F)$. Then, the subsets of W which are used in Theorem 1.4 are

$$W^{MA} = \{1, s_2, s_1 s_2, s_2 s_1 s_2\},$$
$$W^{LA} = \{1, s_1, s_2 s_1, s_1 s_2 s_1\}.$$

The Jacquet modules and graphs of $\chi \propto \mathrm{tr}_1$ and $[\chi]_2 \propto 1$ are given in Figures 1-2. Note that the tables are set up to respect taking Jacquet modules in stages. For example, for a representation under $r_{MG}\pi$, the characters in $r_{AG}\pi$ obtained by applying r_{AM} to it are connected to it by lines. Although, we have defined the graph only for the regular case, by labeling vertices with elements of the Weyl group instead of characters in the Jacquet module, the notion can be extended to the nonregular case (cf. section 4.1 of [Jan] for details). The edges are "colored" according to which of r_{MG} or r_{LG} gives rise to them. The values of χ listed in parentheses by an edge indicate the values of χ for which that edge disappears, i.e., for which the corresponding B-Z composition factor reduces.

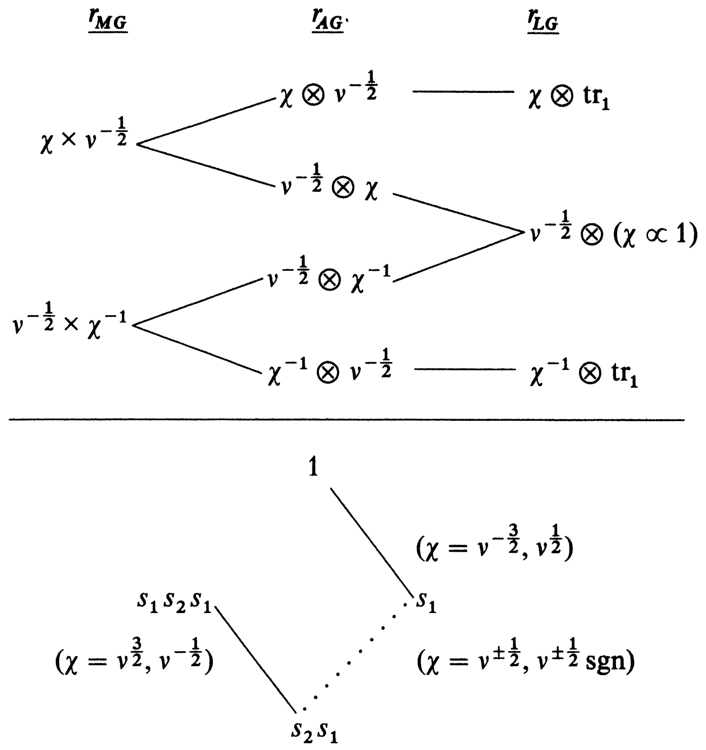

Figure 1. Jacquet modules and graph of $\chi \propto \mathrm{tr}_1$ in $SO_5(F)$

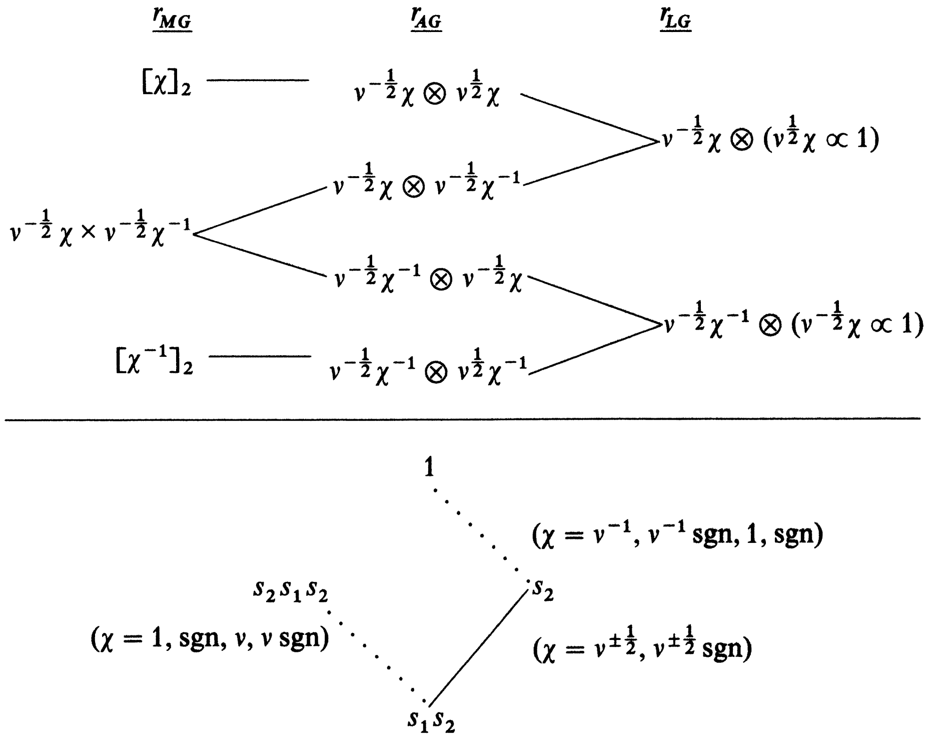

Figure 2. Jacquet modules and graph for $[\chi]_2 \propto 1$ in $SO_5(F)$

We now give the main result of this section (in fact, the only one).

Theorem 4.1. 1. $\pi = [\chi]_2 \rtimes 1$ *is reducible if and only if* $\chi \in \{\nu^{\pm 1}, \nu^{\pm 1}\,\mathrm{sgn}, 1, \mathrm{sgn}\}$. *The components are described below*:

(a) $\chi = 1$.

$$\pi = L(\nu^{-\frac{1}{2}} \otimes St_{SO(3)}) + L(\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}),$$

$$r_{AG}\, L(\nu^{-\frac{1}{2}} \otimes St_{SO(3)}) = \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}},$$

$$r_{AG}\, L(\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}) = 2(\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}) + \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}.$$

(b) $\chi = \mathrm{sgn}$. *Let* $\psi = \mathrm{sgn}$. *Then,*

$$\pi = L(\nu^{-\frac{1}{2}} \otimes St_{SO(3)}(\psi)) + L(\nu^{-\frac{1}{2}}\psi \times \nu^{-\frac{1}{2}}\psi),$$

$$r_{AG}\, L(\nu^{-\frac{1}{2}}\psi \otimes St_{SO(3)}(\psi)) = \nu^{-\frac{1}{2}}\psi \otimes \nu^{\frac{1}{2}}\psi,$$

$$r_{AG}\, L(\nu^{-\frac{1}{2}}\psi \times \nu^{-\frac{1}{2}}\psi) = 2(\nu^{-\frac{1}{2}}\psi \otimes \nu^{-\frac{1}{2}}\psi) + \nu^{-\frac{1}{2}}\psi \otimes \nu^{\frac{1}{2}}\psi.$$

(c) $\chi = \nu^{-1}$.

$$\pi = L(\nu^{-\frac{3}{2}} \otimes St_{SO(3)}) + L(\nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}}),$$

$$r_{AG}\, L(\nu^{-\frac{3}{2}} \otimes St_{SO(3)}) = \nu^{-\frac{3}{2}} \otimes \nu^{\frac{1}{2}} + \nu^{\frac{1}{2}} \otimes \nu^{-\frac{3}{2}} + \nu^{\frac{1}{2}} \otimes \nu^{\frac{3}{2}},$$

$$r_{AG}\, L(\nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}}) = \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}}.$$

Note that $L(\nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}}) = \mathrm{tr}_2$.

(d) $\chi = \nu^{-1}\,\mathrm{sgn}$. *Again, letting* $\psi = \mathrm{sgn}$,

$$\pi = L(\nu^{-\frac{3}{2}}\psi \otimes St_{SO(3)}(\psi)) + L(\nu^{-\frac{3}{2}}\psi \otimes \nu^{-\frac{1}{2}}\psi),$$

$$r_{AG}\, L(\nu^{-\frac{3}{2}}\psi \otimes St_{SO(3)}(\psi)) = \nu^{-\frac{3}{2}}\psi \otimes \nu^{\frac{1}{2}}\psi + \nu^{\frac{1}{2}}\psi \otimes \nu^{-\frac{3}{2}}\psi + \nu^{\frac{1}{2}}\psi \otimes \nu^{\frac{3}{2}}\psi,$$

$$r_{AG}\, L(\nu^{-\frac{3}{2}}\psi \otimes \nu^{-\frac{1}{2}}\psi) = \nu^{-\frac{3}{2}}\psi \otimes \nu^{-\frac{1}{2}}\psi.$$

By Remark 1.5, *the cases* $\chi = \nu$, $\nu\,\mathrm{sgn}$ *have the same components as* (c) *and* (d), *respectively.*

2. $\pi = \chi \rtimes \mathrm{tr}_1$ *is reducible if and only if* $\chi \in \{\nu^{\pm\frac{3}{2}}, \nu^{\pm\frac{1}{2}}, \nu^{\pm\frac{1}{2}}\,\mathrm{sgn}\}$. *The components are described below*:

(a) $\chi = \nu^{-\frac{1}{2}}$.

$$\pi = L(\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}) + T_1,$$

where T_1 is a tempered representation; a component of $St_{GL(2)}(1) \propto 1$, with $r_{AG} T_1 = \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}$. We have described $r_{AG} L(\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}})$ in 1.(a).

(b) $\chi = \nu^{-\frac{1}{2}} \mathrm{sgn}$. *Let $\psi = \mathrm{sgn}$. Then,*

$$\pi = L(\nu^{-\frac{1}{2}} \otimes St_{SO(3)}(\psi)) + L(\nu^{-\frac{1}{2}}\psi \times \nu^{-\frac{1}{2}}),$$

$$r_{AG} L(\nu^{-\frac{1}{2}} \otimes St_{SO(3)}(\psi)) = \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}\psi + \nu^{\frac{1}{2}}\psi \otimes \nu^{-\frac{1}{2}},$$

$$r_{AG} L(\nu^{-\frac{1}{2}}\psi \times \nu^{-\frac{1}{2}}) = \nu^{-\frac{1}{2}}\psi \otimes \nu^{-\frac{1}{2}} + \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}\psi .$$

(c) $\chi = \nu^{-\frac{3}{2}}$.

$$\pi = L(\nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}}) + L(St_{GL(2)}(\nu^{-1})),$$

$$r_{AG} L(St_{GL(2)}(\nu^{-1})) = \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{3}{2}} + \nu^{-\frac{1}{2}} \otimes \nu^{\frac{3}{2}} + \nu^{\frac{3}{2}} \otimes \nu^{-\frac{1}{2}}.$$

Again, $L(\nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}}) = \mathrm{tr}_2$.

Again, the cases $\chi = \nu^{\frac{1}{2}}, \nu^{\frac{1}{2}} \mathrm{sgn}, \nu^{\frac{3}{2}}$ have the same components as (a), (b), (c) *above.*

Proof. The regular case was done in the previous chapter (cf. Theorem 3.7), so we need only consider the nonregular case. For $[\chi]_2 \propto 1$, nonregularity occurs for $\chi \in \{\nu^{\pm\frac{1}{2}}, \nu^{\pm\frac{1}{2}} \mathrm{sgn}, 1, \mathrm{sgn}\}$; for $\chi \propto \mathrm{tr}_2$, when $\chi \in \{\nu^{\pm\frac{1}{2}}, 1, \mathrm{sgn}\}$ (cf. Lemma 3.4).

(i) $[\chi]_2 \propto 1$, $\chi = \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}\mathrm{sgn}$. In this case, Proposition 2.10 tells us that the representation is irreducible (see Remark 4.2).

(ii) $[\chi]_2 \propto 1$, $\chi = 1$, sgn. We do the case $\chi = 1$ – the case $\chi = \mathrm{sgn}$ is similar.

First, we show that π is reducible. For $\chi = 1$, one of the missing edges corresponds to the reducibility of $\nu^{-\frac{1}{2}} \otimes (\nu^{\frac{1}{2}} \propto 1)$. Let τ_1 and τ_2 be the components of $\nu^{-\frac{1}{2}} \otimes (\nu^{\frac{1}{2}} \propto 1)$. Then, $r_{AG}\tau_1 = \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}$ and $r_{AG}\tau_2 = \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}$ Set $\pi_i = i_{GL}\tau_i$. Then, by Theorem 1.4,

$$r_{AG}\pi_1 = \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} + \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} + \nu^{\frac{1}{2}} \otimes \nu^{\frac{1}{2}} + \nu^{\frac{1}{2}} \otimes \nu^{\frac{1}{2}}$$

and

$$r_{AG}\pi_2 = \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} + \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} + \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} + \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}.$$

Now, if $\pi = [1]_2 \propto 1$ were irreducible, it would have to be a subquotient of π_1 or π_2 since together they contain all the components of $\nu^{-\frac{1}{2}} \times \nu^{\frac{1}{2}} \propto 1$ (which contains π as a subrepresentation). However, $r_{AG}\pi$ contains two copies of $\nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}$, and neither $r_{AG}\pi_1$ nor $r_{AG}\pi_2$ does, so this is not the case. Thus, π is reducible.

Next, we claim that π has two components. Although the graph of π suggests that there might be three components, Proposition 2.10 shows that there are only two. We now must identify the components and determine their Jacquet modules.

To identify the components, consider $\nu^{-\frac{1}{2}} \times \nu^{\frac{1}{2}} \propto 1$. π is a subrepresentation of $\nu^{-\frac{1}{2}} \times \nu^{\frac{1}{2}} \propto 1$ and π accounts for the two copies each of $\nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}$ and $\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}$ which occur in $r_{AG}(\nu^{-\frac{1}{2}} \times \nu^{\frac{1}{2}} \propto 1)$. Therefore, any component of $\nu^{-\frac{1}{2}} \times \nu^{\frac{1}{2}} \propto 1$ which contains a copy of either $\nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}$ or $\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}$ in its Jacquet module must be a component of π. Thus, $L(\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}})$ and $L(\nu^{-\frac{1}{2}} \otimes St_{SO(3)})$ are the components of π. Since

$$r_{AG}(\nu^{-\frac{1}{2}} \propto St_{SO(3)}) = \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} + \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} + 2(\nu^{\frac{1}{2}} \otimes \nu^{\frac{1}{2}}),$$

the fact that $0 \neq r_{AG} L(\nu^{-\frac{1}{2}} \otimes St_{SO(3)}) \leqq r_{AG}\pi \cap r_{AG}(\nu^{-\frac{1}{2}} \propto St_{SO(3)})$ tells us that we must have $r_{AG} L(\nu^{-\frac{1}{2}} \otimes St_{SO(3)}) = \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}$, and therefore,

$$r_{AG} L(\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}) = \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} + 2(\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}).$$

(iii) $\chi \propto \mathrm{tr}_1$, $\chi = \nu^{-\frac{1}{2}}$. First, observe that $\pi = \nu^{-\frac{1}{2}} \propto \mathrm{tr}_1$ is a subrepresentation of $\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \propto 1$, which has the same components as $\nu^{-\frac{1}{2}} \times \nu^{\frac{1}{2}} \propto 1$. Therefore, it is clear from the same Jacquet module considerations as above that $L(\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}})$ must be a component of π, and therefore that π is reducible. Since we know $r_{AG}\pi$ and $r_{AG} L(\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}})$, we can conclude that the other component of π has Jacquet module $\nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}$ (so there can, in fact, be only one other component). Now, observe that $r_{AG}(St_{GL(2)}(1) \propto 1) = 2(\nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}) + 2(\nu^{\frac{1}{2}} \otimes \nu^{\frac{1}{2}})$ and the components of $St_{GL(2)}(1) \propto 1$ are components of $\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \propto 1$. Therefore, by Jacquet, module considerations, the other component of π must also be a component of $St_{GL(2)}(1) \propto 1$, hence tempered. We denote this representation as T_1.

(iv) $\chi \propto \mathrm{tr}_1$, $\chi = 1, \mathrm{sgn}$. We take $\chi = 1$; $\chi = \mathrm{sgn}$ is done similarly. We show $\pi = 1 \propto \mathrm{tr}_1$ is irreducible, essentially by using the irreducibility argument from the proof of Theorem 3.1.

First, note that $r_{AG}\pi = 2(1 \otimes \nu^{-\frac{1}{2}}) + 2(\nu^{-\frac{1}{2}} \otimes 1)$. Let π_0 be a subquotient of π such that $r_{AG}\pi_0$ contains a copy of $1 \otimes \nu^{-\frac{1}{2}}$. Then, since $1 \times \nu^{-\frac{1}{2}} = \nu^{-\frac{1}{2}} \times 1$ is irreducible, $r_{MG}\pi_0$ must contain a copy of $1 \times \nu^{-\frac{1}{2}}$. Thus, we also get

$$\nu^{-\frac{1}{2}} \otimes 1 \leqq r_{AM}(1 \times \nu^{-\frac{1}{2}}) \leqq r_{AM} \circ r_{MG}\pi_0 = r_{AG}\pi_0.$$

Next, since $r_{AG}\pi_0$ contains a copy of $\nu^{-\frac{1}{2}} \otimes 1$, we get that $r_{LG}\pi_0$ must contain $\nu^{-\frac{1}{2}} \otimes (1 \propto 1)$ (since $1 \propto 1$ is irreducible). Thus, since $r_{AL}(\nu^{-\frac{1}{2}} \otimes (1 \propto 1)) \leqq r_{AG}\pi_0$, we see that $r_{AG}\pi_0$ will contain both copies of $\nu^{-\frac{1}{2}} \otimes 1$. Finally, if $r_{AG}\pi_0$ contains both copies of $\nu^{-\frac{1}{2}} \otimes 1$, then $r_{MG}\pi_0$ must contain both copies of $1 \times \nu^{-\frac{1}{2}}$. Thus, we conclude that $r_{AG}\pi_0$ contains both copies of $1 \otimes \nu^{-\frac{1}{2}}$ as well. Since we now have $r_{AG}\pi_0 = r_{AG}\pi$, π_0 cannot be a proper subquotient. This tells us π must be irreducible.

Remark 4.2. $[\chi]_2 \propto 1$ for $\chi = \nu^{-\frac{1}{2}}$ is irreducible. However, one of the B-Z composition factors is $\nu^{-1} \times 1$, which is reducible. This shows that if one drops the regularity hypothesis in Theorem 3.1, then (1) $\Rightarrow$ (3) no longer holds. (A similar phenomenon occurs in $GL_3(F)$ – cf. Lemma 4.4).

4.2. The case of SO_4. In this section, we look at degenerate principal series for $SO_4(F)$ using the techniques of the preceding section.

Let $L \cong GL_2(F) \subset \langle P_{\min}, s_2\rangle$ and $M \cong GL_2(F) \subset \langle P_{\min}, s_1\rangle$ be the Levi factors of the standard maximal parabolic subgroups of $G = SO_4(F)$. Then, the subsets of W which are used in Theorem 1.4 are

$$W^{LA} = \{1, s_1\}, \quad W^{MA} = \{1, s_2\}.$$

We can use these and Theorem 1.4 to construct the composition factors of the Jacquet modules for $SO_4(F)$. The results are summarized in Figures 3 and 4.

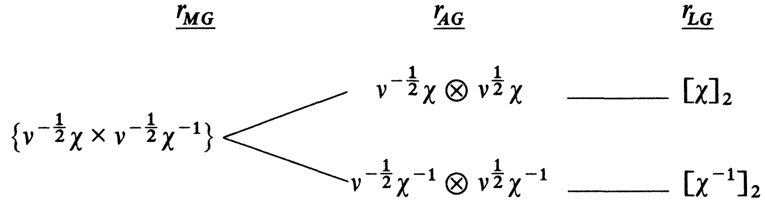

Figure 3. Jacquet modules of $[\chi]_2 \propto 1$ in $SO_4(F)$

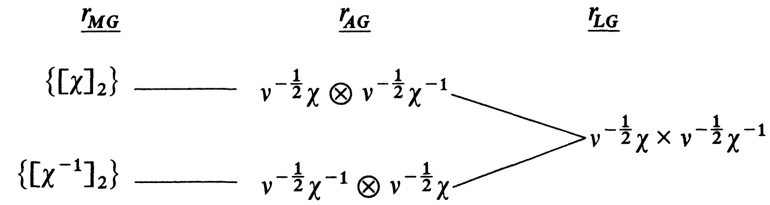

Figure 4. Jacquet modules of $\{[\chi]_2 \propto 1\}$ in $SO_4(F)$

Recall from Lemma 3.8 that nonregularity for $[\chi]_2 \propto 1$ and for $\{[\chi]_2\} \propto 1$ occurs at $\chi = 1$, sgn. At these points, the same sort of argument used in the preceding section tells us that the representations are irreducible. The regular case is covered by Theorem 3.10.

Theorem 4.3. 1. $[\chi]_2 \propto 1$ *is reducible if and only if* $\chi \in \{\nu^{\pm\frac{1}{2}}, \nu^{\pm\frac{1}{2}}\,\mathrm{sgn}\}$. *We have* $[\nu^{-\frac{1}{2}}]_2 \propto 1 = L(\nu^{-1} \otimes 1) + L(\{St_{GL(2)}(\nu^{-\frac{1}{2}})\})$.

$$r_{AG}\, L(\nu^{-1} \otimes 1) = \nu^{-1} \otimes 1\,,$$

$$r_{AG}\, L(\{St_{GL(2)}(\nu^{-\frac{1}{2}})\}) = 1 \otimes \nu\,.$$

Similarly, letting $\psi = \text{sgn}$, $[\nu^{-\frac{1}{2}}\psi]_2 \propto 1 = L(\nu^{-1}\psi \otimes \psi) + L(\{St_{GL(2)}(\nu^{-\frac{1}{2}}\psi)\})$.

$$r_{AG} L(\nu^{-1}\psi \otimes \psi) = \nu^{-1}\psi \otimes \psi,$$

$$r_{AG} L(\{St_{GL(2)}(\nu^{-\frac{1}{2}}\psi)\}) = \psi \otimes \nu\psi.$$

The cases $\chi = \nu^{\frac{1}{2}}, \nu^{\frac{1}{2}}\text{sgn}$ *have the same components as those above by Remark* 1.5.

2. $\{[\chi]_2\} \propto 1$ *is reducible if and only if* $\chi \in \{\nu^{\pm\frac{1}{2}}, \nu^{\pm\frac{1}{2}}\text{sgn}\}$. *We have*

$$\{[\nu^{-\frac{1}{2}}]_2\} \propto 1 = L(\nu^{-1} \otimes 1) + L(St_{GL(2)}(\nu^{-\frac{1}{2}})).$$

$$r_{AG} L(\nu^{-1} \otimes 1) = \nu^{-1} \otimes 1,$$

$$r_{AG} L(St_{GL(2)}(\nu^{-\frac{1}{2}})) = 1 \otimes \nu^{-1}.$$

Similarly, letting $\psi = \text{sgn}$, $\{[\nu^{-\frac{1}{2}}\psi]_2\} \propto 1 = L(\nu^{-1}\psi \otimes \psi) + L(St_{GL(2)}(\nu^{-\frac{1}{2}}\psi))$.

$$r_{AG} L(\nu^{-1}\psi \otimes \psi) = \nu^{-1}\psi \otimes \psi,$$

$$r_{AG} L(St_{GL(2)}(\nu^{-\frac{1}{2}}\psi)) = \psi \otimes \nu^{-1}\psi.$$

Again, the cases $\chi = \nu^{\frac{1}{2}}, \nu^{\frac{1}{2}}\text{sgn}$ *have the same components as those above.*

4.3. The case of SO_7. First, let the Levi factors of the standard parabolics be denoted as follows:

$$\begin{aligned} L &\subset \langle P_{\min}, s_2, s_3\rangle, & L_1 &= M_1 \subset \langle P_{\min}, s_3\rangle, \\ M &\subset \langle P_{\min}, s_1, s_3\rangle, & L_2 &= N_1 \subset \langle P_{\min}, s_2\rangle, \\ N &\subset \langle P_{\min}, s_1, s_2\rangle, & M_2 &= N_2 \subset \langle P_{\min}, s_1\rangle. \end{aligned}$$

The Jacquet modules for degenerate principal series for $SO_7(F)$ are tabulated in the appendix. For completeness, we list the subsets of the Weyl group used to compute $r_{AG}\pi$.

$$W^{LA} = \{1, s_1, s_2s_1, s_3s_2s_1, s_2s_3s_2s_1, s_1s_2s_3s_2s_1\},$$

$$W^{MA} = \left\{\begin{array}{l} 1, s_2, s_1s_2, s_3s_2, s_1s_3s_2, s_2s_3s_2, s_2s_1s_3s_2, \\ s_1s_2s_3s_2, s_3s_2s_1s_3s_2, s_1s_2s_1s_3s_2, \\ s_3s_1s_2s_1s_3s_2, s_2s_3s_1s_2s_1s_3s_2 \end{array}\right\},$$

$$W^{NA} = \{1, s_3, s_2s_3, s_3s_2s_3, s_1s_2s_3, s_1s_3s_2s_3, s_2s_1s_3s_2s_3, s_3s_2s_1s_3s_2s_3\}.$$

Lemma 4.4. *The representation* $[\chi_1]_2 \times \chi_2$ *of* $GL_3(F)$ *is reducible if and only if* $\chi_2 = \nu^{-\frac{3}{2}}\chi$ *or* $\nu^{\frac{3}{2}}\chi$. *Furthermore,*

$$[\chi]_2 \times \nu^{-\frac{3}{2}}\chi = \mathscr{L}(\nu^{-\frac{3}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi) + \mathscr{L}(St_{GL(2)}(\nu^{-1}\chi) \otimes \nu^{\frac{1}{2}}\chi).$$

The first has one-dimensional Jacquet module consisting of $\nu^{-\frac{3}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi$; *the second has Jacquet module* $\nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi \otimes \nu^{-\frac{3}{2}}\chi + \nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{3}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi$. *The case* $\chi_2 = \nu^{\frac{3}{2}}\chi_1$ *is similar to the first.*

Proof. The reducibility points follow from [Zel]. The decomposition in the reducible cases follows easily from Jacquet module considerations.

Theorem 4.5. 1. $\pi = \chi \propto \mathrm{tr}_2$ *is reducible if and only if* $\chi \in \{\nu^{\pm\frac{5}{2}}, \nu^{\pm\frac{1}{2}}, \nu^{\pm\frac{1}{2}}\,\mathrm{sgn}\}$. *The cases* $\chi = \nu^{\pm\frac{5}{2}}, \nu^{\pm\frac{1}{2}}\,\mathrm{sgn}$ *are regular, hence covered by Theorem* 3.7. *The case* $\chi = \nu^{-\frac{1}{2}}$ *is described below*:

$$\pi = L(\nu^{-\frac{3}{2}} \otimes (\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}})) + L(\nu^{-\frac{3}{2}} \otimes T_1),$$

$$r_{AG} L(\nu^{-\frac{3}{2}} \otimes (\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}})) = \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}} + 2(\nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}) + \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}},$$

$$r_{AG} L(\nu^{-\frac{3}{2}} \otimes T_1) = \nu^{-\frac{3}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} + \nu^{\frac{1}{2}} \otimes \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}},$$

where T_1 *is described in Theorem* 4.1.2 (a). *By Remark* 1.5, *the case* $\chi = \nu^{\frac{1}{2}}$ *has the same components as the case* $\chi = \nu^{-\frac{1}{2}}$.

2. $\pi = [\chi]_2 \propto \mathrm{tr}_1$ *is reducible if and only if* $\chi \in \{\nu^{\pm 2}, \nu^{\pm 1}, \nu^{\pm 1}\,\mathrm{sgn}, \mathrm{sgn}\}$. *The cases* $\chi = \nu^{\pm 2}, \nu^{\pm 1}\,\mathrm{sgn}$ *are regular, hence covered by the results of Theorem* 3.7. *For the nonregular cases, the components are described below*:

(a) $\chi = \mathrm{sgn}$. *Let* $\psi = \mathrm{sgn}$. *Then*

$$\pi = L((\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}\psi) \otimes St_{SO(3)}(\psi)) + L(\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}\psi \times \nu^{-\frac{1}{2}}\psi),$$

$$r_{AG}\ L((\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}\psi) \otimes St_{SO(3)}(\psi)) = \nu^{-\frac{1}{2}}\psi \otimes \nu^{\frac{1}{2}}\psi \otimes \nu^{-\frac{1}{2}} + \nu^{-\frac{1}{2}}\psi \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}\psi + \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}\psi \otimes \nu^{\frac{1}{2}}\psi,$$

$$r_{AG} L(\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}\psi \times \nu^{-\frac{1}{2}}\psi) = 2(\nu^{-\frac{1}{2}}\psi \otimes \nu^{-\frac{1}{2}}\psi \otimes \nu^{-\frac{1}{2}}) + 2(\nu^{-\frac{1}{2}}\psi \otimes \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}\psi) + 2(\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}\psi \otimes \nu^{-\frac{1}{2}}\psi) + \nu^{-\frac{1}{2}}\psi \otimes \nu^{\frac{1}{2}}\psi \otimes \nu^{-\frac{1}{2}} + \nu^{-\frac{1}{2}}\psi \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}\psi + \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}\psi \otimes \nu^{\frac{1}{2}}\psi.$$

(b) $\chi = \nu^{-1}$.

$$\pi = L(\nu^{-\frac{3}{2}} \otimes (\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}})) + L(St_{GL(2)}(\nu^{-1}) \otimes St_{SO(3)}) + L(\nu^{-\frac{3}{2}} \otimes T_1) + L(St_{GL(3)}(\nu^{-\frac{1}{2}})),$$

$$r_{AG} L(St_{GL(2)}(\nu^{-1}) \otimes St_{SO(3)}) = \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{3}{2}} \otimes \nu^{\frac{1}{2}} + \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{-\frac{3}{2}} + \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{\frac{3}{2}},$$

$$r_{AG} L(St_{GL(3)}(\nu^{-\frac{1}{2}})) = \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{3}{2}} + \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{3}{2}} + \nu^{\frac{1}{2}} \otimes \nu^{\frac{3}{2}} \otimes \nu^{-\frac{1}{2}},$$

and $r_{AG} L(\nu^{-\frac{3}{2}} \otimes (\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}))$ *and* $r_{AG} L(\nu^{-\frac{3}{2}} \otimes T_1)$ *as in part* 1.

Again, the case $\chi = \nu$ *has the same components as* (b) *above.*

3. $\pi = [\chi]_3 \propto 1$ *is reducible if and only if* $\chi \in \{\nu^{\pm\frac{3}{2}}, \nu^{\pm\frac{3}{2}}\,\mathrm{sgn}, \nu^{\pm\frac{1}{2}}, \nu^{\pm\frac{1}{2}}\,\mathrm{sgn}\}$. *The cases* $\chi = \nu^{\pm\frac{3}{2}}, \nu^{\pm\frac{3}{2}}\,\mathrm{sgn}$ *are regular, hence covered by Theorem* 3.7. *For the nonregular cases, the components are described below*:

(a) $\chi = \nu^{-\frac{1}{2}}$.

$$\pi = L(\nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}} \otimes St_{SO(3)}) + L(\nu^{-\frac{3}{2}} \otimes (\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}})) + L(St_{GL(2)}(\nu^{-1}) \otimes St_{SO(3)}),$$

$$r_{AG}\, L(\nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}} \otimes St_{SO(3)}) = \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}},$$

$$r_{AG}\, L(St_{GL(2)}(\nu^{-1}) \otimes St_{SO(3)}) = \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{3}{2}} \otimes \nu^{\frac{1}{2}} + \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{-\frac{3}{2}} + \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{\frac{3}{2}}$$

with $r_{AG}\, L(\nu^{-\frac{3}{2}} \otimes (\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}))$ *as in part* 1.

(b) $\chi = \nu^{-\frac{1}{2}}\,\mathrm{sgn}$. *Let* $\psi = \mathrm{sgn}$. *Then,*

$$\pi = L(\nu^{-\frac{3}{2}}\psi \otimes \nu^{-\frac{1}{2}}\psi \otimes St_{SO(3)}(\psi)) + L(\nu^{-\frac{3}{2}}\psi \otimes (\nu^{-\frac{1}{2}}\psi \times \nu^{-\frac{1}{2}}\psi)) + L(St_{GL(2)}(\nu^{-1}\psi) \otimes St_{SO(3)}(\psi)),$$

$$r_{AG}\, L(\nu^{-\frac{3}{2}}\psi \otimes \nu^{-\frac{1}{2}}\psi \otimes St_{SO(3)}(\psi)) = \nu^{-\frac{3}{2}}\psi \otimes \nu^{-\frac{1}{2}}\psi \otimes \nu^{\frac{1}{2}}\psi,$$

$$r_{AG}\, L(\nu^{-\frac{3}{2}}\psi \otimes (\nu^{-\frac{1}{2}}\psi \times \nu^{-\frac{1}{2}}\psi)) = 2(\nu^{-\frac{3}{2}}\psi \otimes \nu^{-\frac{1}{2}}\psi \otimes \nu^{-\frac{1}{2}}\psi) + \nu^{-\frac{1}{2}}\psi \otimes \nu^{-\frac{3}{2}}\psi \otimes \nu^{-\frac{1}{2}}\psi + \nu^{-\frac{3}{2}}\psi \otimes \nu^{-\frac{1}{2}}\psi \otimes \nu^{\frac{1}{2}}\psi,$$

$$r_{AG}\, L(St_{GL(2)}(\nu^{-1}\psi) \otimes St_{SO(3)}(\psi)) = \nu^{-\frac{1}{2}}\psi \otimes \nu^{-\frac{3}{2}}\psi \otimes \nu^{\frac{1}{2}}\psi + \nu^{-\frac{1}{2}}\psi \otimes \nu^{\frac{1}{2}}\psi \otimes \nu^{-\frac{3}{2}}\psi + \nu^{-\frac{1}{2}}\psi \otimes \nu^{\frac{1}{2}}\psi \otimes \nu^{\frac{3}{2}}\psi.$$

Again, the cases $\chi = \nu^{\frac{1}{2}}, \nu^{\frac{1}{2}}\,\mathrm{sgn}$ *have the same components as* (a) *and* (b), *respectively.*

Proof. The regular case was done in the preceding chapter, so we only need to consider the nonregular case. By Lemma 3.4, for $\chi \propto \mathrm{tr}_2$, nonregularity occurs for $\chi \in \{\nu^{\pm\frac{3}{2}}, \nu^{\pm\frac{1}{2}}, 1, \mathrm{sgn}\}$. For $[\chi]_2 \propto \mathrm{tr}_1$, nonregularity occurs for $\chi \in \{\nu^{\pm 1}, \nu^{\pm\frac{1}{2}}, \nu^{\pm\frac{1}{2}}\,\mathrm{sgn}, 1, \mathrm{sgn}\}$. Finally, for $[\chi]_3 \propto 1$, there is nonregularity when $\chi \in \{\nu^{\pm 1}, \nu^{\pm 1}\,\mathrm{sgn}, \nu^{\pm\frac{1}{2}}, \nu^{\pm\frac{1}{2}}\,\mathrm{sgn}, 1, \mathrm{sgn}\}$.

First note that $\nu^{-\frac{1}{2}} \propto \mathrm{tr}_2$, $[\nu^{-1}]_2 \propto 1$, and $[\nu^{-\frac{1}{2}}]_3 \propto 1$ all have their semisimplifications contained in that of $\tau = \nu^{-\frac{3}{2}} \times \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \propto 1$. Also, $r_{AG}\tau$ consists of 24 distinct characters, each occurring with multiplicity two. We use this implicitly in (i)–(iii) below.

(i) $\chi \propto \mathrm{tr}_2, \chi = \nu^{-\frac{1}{2}}$. First, an argument like that used for $[\chi]_2 \propto 1$ in $SO_5(F)$ can be used to see that π is reducible: $[\nu^{-1}]_2 \times \nu^{\frac{1}{2}} = \sigma_1 + \sigma_2$ is a reducible B-Z composition factor. If

π were irreducible, it would have to be a component of either $i_{GN}\sigma_1$ or $i_{GN}\sigma_2$. However, since $r_{AG}\pi \nleq r_{AG}i_{GN}\sigma_1$ and $r_{AG}\pi \nleq r_{AG}i_{GN}\sigma_2$, this cannot be the case (cf. Theorem 1.4).

Next, one of the B-Z composition factors for π is

$$\nu^{-\frac{3}{2}} \otimes (\nu^{-\frac{1}{2}} \propto \mathrm{tr}_1) = \nu^{-\frac{3}{2}} \otimes L(\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}) + \nu^{-\frac{3}{2}} \otimes T_1.$$

Let π_1 be the component of π such that $\nu^{-\frac{3}{2}} \otimes L(\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}) \leqq r_{LG}\pi_1$. Then,

$$r_{AG}\pi_1 \geqq 2(\nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}) + \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}.$$

Therefore, since $r_{AG}\pi_1$ contains both copies of $\nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}$, we must have $\nu^{-\frac{1}{2}} \times [\nu^{-1}]_2 \leqq r_{NG}\pi_1$ and consequently must also have $\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}} \leqq r_{AG}\pi_1$. Clearly, since

$$\nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \leqq r_{AG}L\big(\nu^{-\frac{3}{2}} \otimes (\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}})\big),$$

we must have $\pi_1 = L\big(\nu^{-\frac{3}{2}} \otimes (\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}})\big)$. That $r_{AG}\pi_1$ contains nothing else follows from the next paragraph.

Let π_2 denote the other component of π (there is only one other, and this will become obvious). Now, $(\nu^{-\frac{3}{2}} \times \nu^{\frac{1}{2}}) \otimes \mathrm{tr}_1 \leqq r_{MG}\pi_2$ (or else $\pi_1 = \pi$). Thus,

$$r_{AG}\pi_2 = \nu^{-\frac{3}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} + \nu^{\frac{1}{2}} \otimes \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}}.$$

We claim $\pi_2 = L(\nu^{-\frac{3}{2}} \otimes T_1)$. Observe that $\nu^{-\frac{3}{2}} \times St_{GL(2)}(1) \propto 1 = \nu^{-\frac{3}{2}} \propto T_1 + \nu^{-\frac{3}{2}} \propto T_2$ also has its semisimplification contained in τ. We recall that $r_{AG}T_1$ has one component, so $r_{AG}T_2$ must have three (by Bruhat theory, there are only two components of $St_{GL(2)}(1) \propto 1$). Both copies of $\nu^{-\frac{3}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}$ are in $r_{AG}(\nu^{-\frac{3}{2}} \times St_{GL(2)} \propto 1)$, one in each of $r_{AG}(\nu^{-\frac{3}{2}} \propto T_1)$, $r_{AG}(\nu^{-\frac{3}{2}} \propto T_2)$ – in fact, they must be in $r_{AG}L(\nu^{-\frac{3}{2}} \otimes T_1)$ and $r_{AG}L(\nu^{-\frac{3}{2}} \otimes T_2)$. Now, π_2 cannot be $L(\nu^{-\frac{3}{2}} \otimes T_2)$ since $r_{AG}L(\nu^{-\frac{3}{2}} \otimes T_2) \geqq r_{AL}(\nu^{-\frac{3}{2}} \otimes T_2)$ has at least three components. Therefore, $\pi_2 = L(\nu^{-\frac{3}{2}} \otimes T_1)$.

(ii) $[\chi]_3 \propto 1$, $\chi = \nu^{-\frac{1}{2}}$. First, by Jacquet module considerations, it is clear that $L\big(\nu^{-\frac{3}{2}} \otimes (\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}})\big)$ must be a component of π. Also, there are two copies of $\nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}$ in $r_{AG}\tau$, one in $L\big(\nu^{-\frac{3}{2}} \otimes (\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}})\big)$, the other in

$$r_{AG}L(\nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}} \otimes St_{SO(3)}).$$

Since $r_{AG}\pi$ has both copies, $L(\nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}} \otimes St_{SO(3)})$ must also be a component of π.

Next, observe that $L(\nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}} \otimes St_{SO(3)})$ is the unique irreducible subrepresentation of $\nu^{-\frac{3}{2}} \times \nu^{-\frac{1}{2}} \propto St_{SO(3)}$, hence a subrepresentation of $[\nu^{-1}]_2 \propto St_{SO(3)}$. None of the three components of $r_{AG}\pi$ still unaccounted for can be in $r_{AG}([\nu^{-1}]_2 \propto St_{SO(3)})$, hence cannot be in $r_{AG}L(\nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}} \otimes St_{SO(3)})$. Therefore, we have

$$r_{AG}L(\nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}} \otimes St_{SO(3)}) = \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}},$$

and there must be at least one more component of π.

One reducible B-Z composition factor for $r_{LG}\pi$ is

$$\nu^{-\frac{1}{2}} \otimes ([\nu^{-1}]_2 \propto 1) = \nu^{-\frac{1}{2}} \otimes \mathrm{tr}_2 + \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{3}{2}} \otimes St_{SO(3)}).$$

Observe that the remaining components of $r_{AG}\pi$ are exactly $r_{AL}(\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{3}{2}} \otimes St_{SO(3)}))$. Thus, there is only one more component, π_3, and $r_{AG}\pi_3$ contains the remaining three components of $r_{AG}\pi$. By Frobenius reciprocity, π_3 is a subrepresentation of $\nu^{-\frac{1}{2}} \propto L(\nu^{-\frac{3}{2}} \otimes St_{SO(3)})$, hence of $\nu^{-\frac{1}{2}} \times \nu^{-\frac{3}{2}} \propto St_{SO(3)}$. Now,

$$\nu^{-\frac{1}{2}} \times \nu^{-\frac{3}{2}} \propto St_{SO(3)} = St_{GL(2)}(\nu^{-1}) \propto St_{SO(3)} + [\nu^{-1}]_2 \propto St_{SO(3)}.$$

A comparison of Jacquet modules forces π_3 to be in $St_{GL(2)}(\nu^{-1}) \propto St_{SO(3)}$. Since, $r_{AG}L(St_{GL(2)}(\nu^{-1}) \otimes St_{SO(3)})$ contains the only copy of $\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{3}{2}} \otimes \nu^{\frac{1}{2}}$ in

$$r_{AG}(St_{GL(2)}(\nu^{-1}) \propto St_{SO(3)}),$$

we see that $\pi_3 = L(St_{GL(2)}(\nu^{-1}) \otimes St_{SO(3)})$.

(iii) $[\chi]_2 \propto \mathrm{tr}_1$, $\chi = \nu^{-1}$. First, Jacquet module considerations tell us that $L(\nu^{-\frac{3}{2}} \otimes (\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}))$ is a component of π. The same argument as in (i) tells us that one of $L(\nu^{-\frac{3}{2}} \otimes T_i)$ is also a component of π, and again, it must be $L(\nu^{-\frac{3}{2}} \otimes T_1)$.

The remaining components of π contribute

$$(\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{3}{2}} \otimes \nu^{\frac{1}{2}} + \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{-\frac{3}{2}} + \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{\frac{3}{2}})$$
$$+ (\nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{3}{2}} + \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{3}{2}} + \nu^{\frac{1}{2}} \otimes \nu^{\frac{3}{2}} \otimes \nu^{-\frac{1}{2}}).$$

By considering $\nu^{\frac{1}{2}} \otimes L(St_{GL(2)}(\nu^{-1}))$, a component of the reducible B-Z composition factor $\nu^{\frac{1}{2}} \otimes (\nu^{-\frac{3}{2}} \propto \mathrm{tr}_1)$, we see that the last three come from the same component. Similarly, $\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{3}{2}} \otimes St_{SO(3)})$, a component of the reducible B-Z composition factor $\nu^{-\frac{1}{2}} \otimes ([\nu^{-1}]_2 \propto 1)$, tells us that the first three come from the same component. Therefore, there are at most two more components of π.

It is not difficult to show that there are two more components. Consider $\nu^{-\frac{1}{2}} \times \nu^{\frac{1}{2}} \times \nu^{-\frac{3}{2}} \propto 1 = [1]_2 \propto (\nu^{-\frac{3}{2}} \propto 1) + St_{GL(2)}(1) \propto (\nu^{-\frac{3}{2}} \propto 1)$, which has the same components as τ. It is easy to check that $r_{AG}([1]_2 \propto (\nu^{-\frac{3}{2}} \propto 1))$ and $r_{AG}(St_{GL(2)}(1) \propto (\nu^{-\frac{3}{2}} \propto 1))$ are disjoint and

$$(\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{3}{2}} \otimes \nu^{\frac{1}{2}} + \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{-\frac{3}{2}} + \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{\frac{3}{2}}) \leqq r_{AG}([1]_2 \propto (\nu^{-\frac{3}{2}} \propto 1)),$$

$$(\nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{3}{2}} + \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{3}{2}} + \nu^{\frac{1}{2}} \otimes \nu^{\frac{3}{2}} \otimes \nu^{-\frac{1}{2}}) \leqq r_{AG}(St_{GL(2)}(1) \propto (\nu^{-\frac{3}{2}} \propto 1)).$$

Thus, there are two more components, with Jacquet modules as above. Let π_3 denote the first, π_4 the second.

The same sort of argument as in (ii) shows that $\pi_3 = L(St_{GL(2)}(\nu^{-1}) \otimes St_{SO(3)})$. Next, observe that

$$\nu^{\frac{1}{2}} \times St_{GL(2)}(\nu^{-1}) = St_{GL(3)}(\nu^{-\frac{1}{2}}) + \mathscr{L}(St_{GL(2)}(\nu^{-1}) \otimes \nu^{\frac{1}{2}}).$$

By Frobenius reciprocity, since $r_{LG}\pi_4 = \nu^{\frac{1}{2}} \otimes L(St_{GL(2)}(\nu^{-1}))$, π_4 is a component of $\nu^{\frac{1}{2}} \times St_{GL(2)}(\nu^{-1}) \propto 1$, therefore of either $St_{GL(3)}(\nu^{-\frac{1}{2}}) \propto 1$ or $\mathscr{L}(St_{GL(2)}(\nu^{-1}) \otimes \nu^{\frac{1}{2}}) \propto 1$. A comparison of Jacquet modules shows that π_4 is not a component of the latter, hence must be a component of $St_{GL(3)}(\nu^{-\frac{1}{2}}) \propto 1$. Since the copy of $\nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{3}{2}}$ in $r_{AG}(St_{GL(3)}(\nu^{-\frac{1}{2}}) \propto 1)$ lies in $r_{AG}L(St_{GL(3)}(\nu^{-\frac{1}{2}}))$, we see that $\pi_4 = L(St_{GL(3)}(\nu^{-\frac{1}{2}}))$.

(iv) $[\chi]_2 \propto \mathrm{tr}_1$, $\chi = \mathrm{sgn}$. Let $\psi = \mathrm{sgn}$. Note that π is a subrepresentation of $\tau' = \nu^{-\frac{1}{2}}\psi \times \nu^{\frac{1}{2}}\psi \times \nu^{-\frac{1}{2}} \propto 1$ and that $r_{AG}(\nu^{-\frac{1}{2}}\psi \times \nu^{\frac{1}{2}}\psi \times \nu^{-\frac{1}{2}} \propto 1)$ contains 24 distinct characters, each occurring with multiplicity 2.

Observe that $L((\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}\psi) \otimes St_{SO(3)}(\psi))$ is a component of τ' and has $\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}\psi \otimes \nu^{\frac{1}{2}}\psi$ in its Jacquet module. Since $r_{AG}\pi$ contains both copies of $\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}\psi \otimes \nu^{\frac{1}{2}}\psi$, we must have that $L((\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}\psi) \otimes St_{SO(3)}(\psi))$ is a component of π. A similar argument tells us that $L(\nu^{-\frac{1}{2}}\psi \times \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}\psi)$ is also a component of π.

Next, we note that we must have $[\psi]_2 \times \nu^{-\frac{1}{2}} \leqq r_{NG}L((\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}\psi) \otimes St_{SO(3)}(\psi))$. Therefore, we can conclude that

$$r_{AG}L((\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}\psi) \otimes St_{SO(3)}(\psi)) \geqq \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}\psi \otimes \nu^{\frac{1}{2}}\psi + \nu^{-\frac{1}{2}}\psi \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}\psi + \nu^{-\frac{1}{2}}\psi \otimes \nu^{\frac{1}{2}}\psi \otimes \nu^{-\frac{1}{2}}.$$

Similarly, we observe that $\nu^{-\frac{1}{2}}\psi \times \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}\psi \leqq r_{NG}L(\nu^{-\frac{1}{2}}\psi \times \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}\psi)$, and therefore

$$r_{AG}L(\nu^{-\frac{1}{2}}\psi \times \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}\psi) \geqq 2(\nu^{-\frac{1}{2}}\psi \otimes \nu^{-\frac{1}{2}}\psi \otimes \nu^{-\frac{1}{2}}) + 2(\nu^{-\frac{1}{2}}\psi \otimes \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}\psi) + 2(\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}\psi \otimes \nu^{-\frac{1}{2}}\psi).$$

This leaves $\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}\psi \otimes \nu^{\frac{1}{2}}\psi + \nu^{-\frac{1}{2}}\psi \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}\psi + \nu^{-\frac{1}{2}}\psi \otimes \nu^{\frac{1}{2}}\psi \otimes \nu^{-\frac{1}{2}}$ unaccounted for. The same argument as above shows that they come from the same component of π. Now, the B-Z composition factor

$$\nu^{-\frac{1}{2}} \otimes ([\psi]_2 \propto 1) = \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}\psi \otimes St_{SO(3)}(\psi)) + \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}\psi \times \nu^{-\frac{1}{2}}\psi).$$

By Jacquet module considerations, we see that

$$\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}\psi \times \nu^{-\frac{1}{2}}\psi) \leqq r_{LG} L(\nu^{-\frac{1}{2}}\psi \times \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}\psi),$$

therefore $\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}\psi \otimes \nu^{\frac{1}{2}}\psi \leqq r_{AG} L(\nu^{-\frac{1}{2}}\psi \times \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}\psi)$. Therefore, the remaining characters are in $r_{AG} L(\nu^{-\frac{1}{2}}\psi \times \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}\psi)$.

(v) $[\chi]_2 \propto \mathrm{tr}_1$, $\chi = 1$. In this case, we need to show irreducibility. First, note that

$$r_{LG}\pi = 3(\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}})) + 2(\nu^{-\frac{1}{2}} \otimes T_1) + \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}} \otimes St_{SO(3)}).$$

Now, suppose π_0 is a component of π such that $\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \leqq r_{AG}\pi_0$. Then, $\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \leqq r_{NG}\pi_0$ and this implies $6(\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}) \leqq r_{AG}\pi_0$. From this, we can deduce that $3(\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}})) \leqq r_{LG}\pi_0$, hence $3(\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}) \leqq r_{AG}\pi_0$. This forces $2([1]_2 \times \nu^{-\frac{1}{2}}) \leqq r_{NG}\pi_0$, which is enough to tell us that $r_{NG}\pi_0 = r_{NG}\pi$. Thus, π_0 cannot be a proper subquotient of π.

(vi) Remaining cases. There is one remaining case where there is reducibility, namely $[\chi]_3 \propto 1$ for $\chi = \nu^{\pm\frac{1}{2}}\,\mathrm{sgn}$. This case is essentially the same as (ii).

The remaining cases are $\chi = \nu^{\pm\frac{3}{2}}, 1, \mathrm{sgn}$ for $\chi \propto \mathrm{tr}_2$, $\chi = \nu^{\pm\frac{1}{2}}, \nu^{\pm\frac{1}{2}}\mathrm{sgn}$ for $[\chi]_2 \propto \mathrm{tr}_1$, and $\chi = \nu^{\pm 1}, \nu^{\pm 1}\mathrm{sgn}, 1, \mathrm{sgn}$ for $[\chi]_3 \propto 1$. All of these give rise to irreducible representations. We do not go into detail on these – the arguments are similar to that used in case (v) above (or those for the $SO(5)$ results).

4.4. The case of SO_6. Let the parabolic subgroups be denoted as in the preceding section (based on which of the simple reflections are used in generating them). We now give the subsets of the Weyl group which serve as a starting point. The Jacquet modules are listed in the appendix.

$$\begin{aligned} W^{LA} &= \{1, s_1, s_2s_1, s_3s_1, s_2s_3s_1, s_1s_2s_3s_1\}, \\ W^{MA} &= \{1, s_2, s_1s_2, s_3s_1s_2\}, \\ W^{NA} &= \{1, s_3, s_1s_3, s_2s_1s_3\}. \end{aligned}$$

We start with a lemma giving certain reducibility results for $SO_4(F)$ which will be needed in Theorem 4.8.

Lemma 4.6. *In SO_4, the following hold:*

1. *The representation $1 \times 1 \propto 1$ is irreducible.*

2. *Let $\psi = \mathrm{sgn}$ denote an order 2 character. The representation $\psi \times 1 \propto 1$ is the direct sum of two inequivalent tempered representations, which we denote T_1 and T_2. For $i = 1, 2$, we have $r_{AG} T_i = \psi \otimes 1 + 1 \otimes \psi$.*

3. *The representation* $v^{-1} \times 1 \propto 1$ *reduces as follows*:

$$v^{-1} \times 1 \propto 1 = L(v^{-1} \otimes 1) + L\big(St_{GL(2)}(v^{-\frac{1}{2}})\big) + L\big(\{St_{GL(2)}(v^{-\frac{1}{2}})\}\big) + St_{SO(4)}.$$

Each of the components has a one-dimensional Jacquet module. In order, they are $v^{-1} \otimes 1$, $1 \otimes v^{-1}$, $1 \otimes v$, *and* $v \otimes 1$.

Proof. The fact that (1) is irreducible and (2) has two inequivalent components follows from the R-group results of [Keys]. The Jacquet module claims for (2) are straightforward. Next, observe that

$$v^{-1} \times 1 \propto 1 = [v^{-\frac{1}{2}}]_2 \propto 1 + St_{GL(2)}(v^{-\frac{1}{2}}) \propto 1$$

and

$$v^{-1} \times 1 \propto 1 = \{[v^{-\frac{1}{2}}]_2\} \propto 1 + \{St_{GL(2)}(v^{-\frac{1}{2}})\} \propto 1.$$

Then, it is easy to show from Jacquet module considerations and the reducibility results in Theorem 4.3 that there are four components, and to identify those that occur in Theorem 4.3. The fact that the fourth component is the Steinberg representation follows from [Cas1].

Remark 4.7. It is worth pointing out that the B-Z composition factors for $\psi \times 1 \propto 1$ are all irreducible, yet the representation itself is reducible. This provides a counterexample to (3) $\Rightarrow$ (1) in Theorem 3.1 if one drops the regularity hypothesis.

Theorem 4.8. *Degenerate principal series for* $SO_6(F)$ *have the following reducibility*:

1. $\pi = \chi \propto \mathrm{tr}_2$ *is reducible if and only if* $\chi \in \{v^{\pm 2}, v^{\pm 1}, \mathrm{sgn}\}$. *The case* $\chi = v^{\pm 2}$ *is regular, hence covered by Theorem* 3.10.

(a) $\chi = v^{-1}$.

$$\pi = L\big((v^{-1} \times v^{-1}) \otimes 1\big) + L(v^{-1} \otimes St_{SO(4)}),$$

$$r_{AG} L\big((v^{-1} \times v^{-1}) \otimes 1\big) = 2(v^{-1} \otimes v^{-1} \otimes 1) + v^{-1} \otimes 1 \otimes v^{-1} + v^{-1} \otimes 1 \otimes v,$$

$$r_{AG} L(v^{-1} \otimes St_{SO(4)}) = v^{-1} \otimes v \otimes 1 + v \otimes v^{-1} \otimes 1.$$

(b) $\chi = \mathrm{sgn}$. *Let* $\psi = \mathrm{sgn}$. *Then,*

$$\pi = L(v^{-1} \otimes T_1) + L(v^{-1} \otimes T_2)$$

where T_1 *and* T_2 *are described in Lemma* 4.6.

$$r_{AG} L(v^{-1} \otimes T_i) = \psi \otimes v^{-1} \otimes 1 + v^{-1} \otimes \psi \otimes 1 + v^{-1} \otimes 1 \otimes \psi$$

for $i = 1, 2$.

By Remark 1.5, *the case* $\chi = v$ *has the same components as* (a) *above.*

2. $\{[\chi]_3\} \propto 1$ *is reducible if and only if* $\chi \in \{v^{\pm 1}, v^{\pm 1}\,\mathrm{sgn}\}$. *Both cases are regular, hence covered by Theorem* 3.10.

3. $[\chi]_3 \propto 1$ *is reducible if and only if* $\chi \in \{\nu^{\pm 1}, \nu^{\pm 1}\,\mathrm{sgn}\}$. *Both cases are regular, hence covered by Theorem* 3.10.

Proof. Again, the regular case was done in the preceding chapter, so here we are only concerned with the nonregular case. For $\chi \propto \mathrm{tr}_2$, nonregularity occurs for $\chi \in \{\nu^{\pm 1}, 1, \mathrm{sgn}\}$. For $\{[\chi]_3\} \propto 1$, nonregularity occurs for $\chi \in \{\nu^{\pm\frac{1}{2}}, \nu^{\pm\frac{1}{2}}\,\mathrm{sgn}, 1, \mathrm{sgn}\}$. $[\chi]_3 \propto 1$ has the same nonregularity points as $\{[\chi]_3\} \propto 1$.

First, we note that the nonregular cases for $\{[\chi]_3\} \propto 1$ and $[\chi]_3 \propto 1$ all correspond to points of irreducibility. The arguments used to show irreducibility are the same basic irreducibility arguments that have appeared throughout – one shows that all the elements of $r_{AG}\pi$ lie in the Jacquet module for the same component of π by using components of intermediate Jacquet modules to connect them. Thus, we are reduced to the following cases.

(i) $\chi \propto \mathrm{tr}_2$ for $\chi = 1$. Again, this is the same basic irreducibility argument that has been used throughout this paper. However, it should be pointed out that Lemma 4.6 (1) is required in order to make this argument work.

(ii) $\chi \propto \mathrm{tr}_2$ for $\chi = \mathrm{sgn}$. Let $\psi = \mathrm{sgn}$. First, observe that the irreducibility of $\psi \times [\nu^{-\frac{1}{2}}]_2 \leqq r_{NG}\pi$ implies that there are at most two components of π. Furthermore, if $\pi = \pi_1 + \pi_2$, then

$$r_{AG}\pi_i = \psi \otimes \nu^{-1} \otimes 1 + \nu^{-1} \otimes \psi \otimes 1 + \nu^{-1} \otimes 1 \otimes \psi$$

for $i = 1, 2$. Now, observe that by Lemma 4.6 (2), $\psi \times 1 \propto 1 = T_1 + T_2$ with $r_{AG}T_i = \psi \otimes 1 + 1 \otimes \psi$, for $i = 1, 2$. There are two copies of $\nu^{-1} \otimes \psi \otimes 1$ in $r_{AG}(\nu^{-1} \times \psi \times 1 \propto 1)$ – one in $r_{AG}L(\nu^{-1} \otimes T_1)$, the other in $r_{AG}L(\nu^{-1} \otimes T_2)$. Since $r_{AG}\pi$ contains both copies of $\nu^{-1} \otimes \psi \otimes 1$, we get $\pi = L(\nu^{-1} \otimes T_1) + L(\nu^{-1} \otimes T_2)$, as needed.

(iii) $\chi \propto \mathrm{tr}_2$ for $\chi = \nu^{-1}$. First, the irreducibility of $\nu^{-1} \times [\nu^{-\frac{1}{2}}]_2 \leqq r_{NG}\pi$, $\{\nu^{-1} \times [\nu^{-\frac{1}{2}}]_2\} \leqq r_{MG}\pi$, and $(\nu^{-1} \times \nu) \otimes 1 \leqq r_{M_2G}\pi$ and the usual Jacquet module arguments show that π has at most two components, and further, that if $\pi = \pi_1 + \pi_2$, we must have

$$r_{AG}\pi_1 = 2(\nu^{-1} \otimes \nu^{-1} \otimes 1) + \nu^{-1} \otimes 1 \otimes \nu^{-1} + \nu^{-1} \otimes 1 \otimes \nu\,,$$

$$r_{AG}\pi_2 = \nu^{-1} \otimes \nu \otimes 1 + \nu \otimes \nu^{-1} \otimes 1\,.$$

Now, since there are only two copies of $\nu^{-1} \otimes \nu^{-1} \otimes 1$ in $r_{AG}(\nu^{-1} \times \nu^{-1} \times 1 \propto 1)$ and π_1 contains both, we must have $\pi_1 = L\big((\nu^{-1} \times \nu^{-1}) \otimes 1\big)$. Note that this representation is just $[1]_3 \propto 1$, which has Jacquet module exactly that of π_1 above (and so π is reducible).

Finally, we identify π_2. Observe that by Lemma 4.4,

$$[\nu^{-\frac{1}{2}}]_2 \times \nu = \mathscr{L}(\nu^{-1} \otimes 1 \otimes \nu) + \mathscr{L}\big(\nu^{-1} \otimes St_{GL(2)}(\nu^{\frac{1}{2}})\big).$$

Now, $r_{NG}\pi_2 = \mathscr{L}\big(\nu^{-1} \otimes St_{GL(2)}(\nu^{\frac{1}{2}})\big)$. Therefore, by Frobenius reciprocity, π_2 must be a subrepresentation of $\mathscr{L}\big(\nu^{-1} \otimes St_{GL(2)}(\nu^{\frac{1}{2}})\big) \propto 1$, hence of $\nu^{-1} \times St_{GL(2)}(\nu^{\frac{1}{2}}) \propto 1$. Now, since $St_{GL(2)}(\nu^{\frac{1}{2}}) \propto 1 = St_{SO(4)} + L(St_{GL(2)}(\nu^{-\frac{1}{2}}))$, we see that π_2 must be a component of $\nu^{-1} \propto St_{SO(4)}$ or $\nu^{-1} \propto L(St_{GL(2)}(\nu^{-\frac{1}{2}}))$. By Jacquet module considerations, it must be the former. Since the only copy of $\nu^{-1} \otimes \nu \otimes 1$ in $r_{AG}(\nu^{-1} \propto St_{SO(4)})$ is in $L(\nu^{-1} \otimes St_{SO(4)})$, the fact that $\nu^{-1} \otimes \nu \otimes 1 \leqq r_{AG}\pi_2$ forces $\pi_2 = L(\nu^{-1} \otimes St_{SO(4)})$.

Appendix

Below, we give the B-Z/Casselman composition factors for degenerate principal series for $SO_7(F)$.

$\pi = \chi \propto \mathrm{tr}_2$.

$$\begin{aligned} r_{AG}\pi &= \chi \otimes \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}} + \nu^{-\frac{3}{2}} \otimes \chi \otimes \nu^{-\frac{1}{2}} + \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \chi \\ &\quad + \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \chi^{-1} + \nu^{-\frac{3}{2}} \otimes \chi^{-1} \otimes \nu^{-\frac{1}{2}} + \chi^{-1} \otimes \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} r_{L_1G}\pi &= \chi \otimes \nu^{-\frac{3}{2}} \otimes \mathrm{tr}_1 + \nu^{-\frac{3}{2}} \otimes \chi \otimes \mathrm{tr}_1 + \nu^{-\frac{3}{2}} \otimes \nu^{-\frac{1}{2}} \otimes (\chi \propto 1) \\ &\quad + \nu^{-\frac{3}{2}} \otimes \chi^{-1} \otimes \mathrm{tr}_1 + \chi^{-1} \otimes \nu^{-\frac{3}{2}} \otimes \mathrm{tr}_1, \end{aligned}$$

$$\begin{aligned} r_{L_2G}\pi &= \chi \otimes [\nu^{-1}]_2 + \nu^{-\frac{3}{2}} \otimes (\chi \times \nu^{-\frac{1}{2}}) + \nu^{-\frac{3}{2}} \otimes (\nu^{-\frac{1}{2}} \times \chi^{-1}) \\ &\quad + \chi^{-1} \otimes [\nu^{-1}]_2, \end{aligned}$$

$$\begin{aligned} r_{M_2G}\pi &= (\chi \times \nu^{-\frac{3}{2}}) \otimes \nu^{-\frac{1}{2}} + [\nu^{-1}]_2 \otimes \chi + [\nu^{-1}]_2 \otimes \chi^{-1} \\ &\quad + (\nu^{-\frac{3}{2}} \times \chi^{-1}) \otimes \nu^{-\frac{1}{2}}, \end{aligned}$$

$$r_{LG}\pi = \chi \otimes \mathrm{tr}_2 + \nu^{-\frac{3}{2}} \otimes (\chi \propto \mathrm{tr}_1) + \chi^{-1} \otimes \mathrm{tr}_2,$$

$$r_{MG}\pi = (\chi \times \nu^{-\frac{3}{2}}) \otimes \mathrm{tr}_1 + [\nu^{-1}]_2 \otimes (\chi \propto 1) + (\nu^{-\frac{3}{2}} \times \chi^{-1}) \otimes \mathrm{tr}_1,$$

$$r_{NG}\pi = \chi \times [\nu^{-1}]_2 + [\nu^{-1}]_2 \times \chi^{-1}.$$

$\pi = [\chi]_2 \propto \mathrm{tr}_1$.

$$\begin{aligned} r_{AG}\pi &= \nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}} + \nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}\chi \\ &\quad + \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi + \nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}\chi^{-1} \\ &\quad + \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi^{-1} + \nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}} \\ &\quad + \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}}\chi + \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}} \\ &\quad + \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1} + \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}\chi \\ &\quad + \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}\chi^{-1} + \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}} \end{aligned}$$

$$r_{L_1G}\pi = \nu^{-\frac{1}{2}}\chi \otimes \nu^{\frac{1}{2}}\chi \otimes \mathrm{tr}_1 + \nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}} \otimes (\nu^{\frac{1}{2}}\chi \propto 1) + \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}\chi \otimes (\nu^{\frac{1}{2}}\chi \propto 1) + \nu^{-\frac{1}{2}}\chi \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes \mathrm{tr}_1 + \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}}\chi \otimes \mathrm{tr}_1 + \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}\chi^{-1} \otimes (\nu^{-\frac{1}{2}}\chi \propto 1) + \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{-\frac{1}{2}} \otimes (\nu^{-\frac{1}{2}}\chi \propto 1) + \nu^{-\frac{1}{2}}\chi^{-1} \otimes \nu^{\frac{1}{2}}\chi^{-1} \otimes \mathrm{tr}_1,$$

$$r_{L_2G}\pi = \nu^{-\frac{1}{2}}\chi \otimes (\nu^{\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}) + \nu^{-\frac{1}{2}} \otimes [\chi]_2 + \nu^{-\frac{1}{2}}\chi \otimes (\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}\chi^{-1}) + \nu^{-\frac{1}{2}} \otimes (\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi^{-1}) + \nu^{-\frac{1}{2}}\chi^{-1} \otimes (\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}) + \nu^{-\frac{1}{2}} \otimes [\chi^{-1}]_2 + \nu^{-\frac{1}{2}}\chi^{-1} \otimes (\nu^{-\frac{1}{2}} \times \nu^{\frac{1}{2}}\chi^{-1}),$$

$$r_{M_2G}\pi = [\chi]_2 \otimes \nu^{-\frac{1}{2}} + (\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}) \otimes \nu^{\frac{1}{2}}\chi + (\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}) \otimes \nu^{-\frac{1}{2}}\chi^{-1} + (\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi^{-1}) \otimes \nu^{-\frac{1}{2}} + (\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}\chi^{-1}) \otimes \nu^{-\frac{1}{2}}\chi + (\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}\chi^{-1}) \otimes \nu^{\frac{1}{2}}\chi^{-1} + [\chi^{-1}]_2 \otimes \nu^{-\frac{1}{2}},$$

$$r_{LG}\pi = \nu^{-\frac{1}{2}}\chi \otimes (\nu^{\frac{1}{2}}\chi \propto \mathrm{tr}_1) + \nu^{-\frac{1}{2}} \otimes ([\chi]_2 \propto 1) + \nu^{-\frac{1}{2}}\chi^{-1} \otimes (\nu^{-\frac{1}{2}}\chi \propto \mathrm{tr}_1),$$

$$r_{MG}\pi = [\chi]_2 \otimes \mathrm{tr}_1 + (\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}) \otimes (\nu^{\frac{1}{2}}\chi \propto 1) + (\nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi^{-1}) \otimes \mathrm{tr}_1 + (\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}\chi^{-1}) \otimes (\nu^{-\frac{1}{2}}\chi \propto 1) + [\chi^{-1}]_2 \otimes \mathrm{tr}_1,$$

$$r_{NG}\pi = [\chi]_2 \times \nu^{-\frac{1}{2}} + \nu^{-\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}}\chi^{-1} + \nu^{-\frac{1}{2}} \times [\chi^{-1}]_2.$$

$\pi = [\chi]_3 \propto 1.$

$$r_{AG}\pi = \nu^{-1}\chi \otimes \chi \otimes \nu\chi + \nu^{-1}\chi \otimes \chi \otimes \nu^{-1}\chi^{-1} + \nu^{-1}\chi \otimes \nu^{-1}\chi^{-1} \otimes \chi + \nu^{-1}\chi \otimes \nu^{-1}\chi^{-1} \otimes \chi^{-1} + \nu^{-1}\chi^{-1} \otimes \nu^{-1}\chi \otimes \chi + \nu^{-1}\chi^{-1} \otimes \nu^{-1}\chi \otimes \chi^{-1} + \nu^{-1}\chi^{-1} \otimes \chi^{-1} \otimes \nu^{-1}\chi + \nu^{-1}\chi^{-1} \otimes \chi^{-1} \otimes \nu\chi^{-1},$$

$$r_{L_1G}\pi = \nu^{-1}\chi \otimes \chi \otimes (\nu\chi \propto 1) + \nu^{-1}\chi \otimes \nu^{-1}\chi^{-1} \otimes (\chi \propto 1) + \nu^{-1}\chi^{-1} \otimes \nu^{-1}\chi \otimes (\chi \propto 1) + \nu^{-1}\chi^{-1} \otimes \chi^{-1} \otimes (\nu^{-1}\chi \propto 1),$$

$$r_{L_2G}\pi = \nu^{-1}\chi \otimes [\nu^{\frac{1}{2}}\chi]_2 + \nu^{-1}\chi \otimes (\chi \times \nu^{-1}\chi^{-1}) + \nu^{-1}\chi \otimes [\nu^{-\frac{1}{2}}\chi^{-1}]_2 + \nu^{-1}\chi^{-1} \otimes [\nu^{-\frac{1}{2}}\chi]_2 + \nu^{-1}\chi^{-1} \otimes (\nu^{-1}\chi \times \chi^{-1}) + \nu^{-1}\chi^{-1} \otimes [\nu^{\frac{1}{2}}\chi^{-1}]_2,$$

$$r_{M_2G}\pi = [\nu^{-\frac{1}{2}}\chi]_2 \otimes \nu\chi + [\nu^{-\frac{1}{2}}\chi]_2 \otimes \nu^{-1}\chi^{-1} + (\nu^{-1}\chi \times \nu^{-1}\chi^{-1}) \otimes \chi + (\nu^{-1}\chi \times \nu^{-1}\chi^{-1}) \otimes \chi^{-1} + [\nu^{-\frac{1}{2}}\chi^{-1}]_2 \otimes \nu^{-1}\chi + [\nu^{-\frac{1}{2}}\chi^{-1}]_2 \otimes \nu\chi^{-1},$$

$$r_{LG}\pi = \nu^{-1}\chi \otimes ([\nu^{\frac{1}{2}}\chi]_2 \propto 1) + \nu^{-1}\chi^{-1} \otimes ([\nu^{-\frac{1}{2}}\chi]_2 \propto 1),$$

$$r_{MG}\pi = [\nu^{-\frac{1}{2}}\chi]_2 \otimes (\nu\chi \propto 1) + (\nu^{-1}\chi \times \nu^{-1}\chi^{-1}) \otimes (\chi \propto 1)$$
$$+ [\nu^{-\frac{1}{2}}\chi^{-1}]_2 \otimes (\nu^{-1}\chi \propto 1),$$

$$r_{NG}\pi = [\chi]_3 + [\nu^{-\frac{1}{2}}\chi]_2 \times \nu^{-1}\chi^{-1} + \nu^{-1}\chi \times [\nu^{-\frac{1}{2}}\chi^{-1}]_2 + [\chi^{-1}]_3.$$

Next, we give the B-Z/Casselman composition factors for degenerate principal series for $SO_6(F)$.

$\pi = \chi \propto \mathrm{tr}_2$.

$$r_{AG}\pi = \chi \otimes \nu^{-1} \otimes 1 + \nu^{-1} \otimes \chi \otimes 1 + \nu^{-1} \otimes 1 \otimes \chi^{-1}$$
$$+ \nu^{-1} \otimes 1 \otimes \chi + \nu^{-1} \otimes \chi^{-1} \otimes 1 + \chi^{-1} \otimes \nu^{-1} \otimes 1,$$

$$r_{L_1G}\pi = \chi \otimes \{[\nu^{-\frac{1}{2}}]_2\} + \nu^{-1} \otimes \{\chi \times 1\} + \nu^{-1} \otimes \{1 \times \chi^{-1}\}$$
$$+ \chi^{-1} \otimes \{[\nu^{-\frac{1}{2}}]_2\},$$

$$r_{L_2G}\pi = \chi \otimes [\nu^{-\frac{1}{2}}]_2 + \nu^{-1} \otimes (\chi \times 1) + \nu^{-1} \otimes (\chi^{-1} \times 1)$$
$$+ \chi^{-1} \otimes [\nu^{-\frac{1}{2}}]_2,$$

$$r_{M_2G}\pi = (\chi \times \nu^{-1}) \otimes 1 + [\nu^{-\frac{1}{2}}]_2 \otimes \chi^{-1} + [\nu^{-\frac{1}{2}}]_2 \otimes \chi$$
$$+ (\nu^{-1} \times \chi^{-1}) \otimes 1,$$

$$r_{LG}\pi = \chi \otimes \mathrm{tr}_2 + \nu^{-1} \otimes (\chi \times 1 \propto 1) + \chi^{-1} \otimes \mathrm{tr}_2,$$

$$r_{MG}\pi = \{\chi \times [\nu^{-\frac{1}{2}}]_2\} + \{[\nu^{-\frac{1}{2}}]_2 \times \chi^{-1}\},$$

$$r_{NG}\pi = \chi \times [\nu^{-\frac{1}{2}}]_2 + [\nu^{-\frac{1}{2}}]_2 \times \chi^{-1}.$$

$\pi = \{[\chi]_3 \propto 1\}$.

$$r_{AG}\pi = \nu^{-1}\chi \otimes \chi \otimes \nu^{-1}\chi^{-1} + \nu^{-1}\chi \otimes \nu^{-1}\chi^{-1} \otimes \chi + \nu^{-1}\chi^{-1} \otimes \nu^{-1}\chi \otimes \chi$$
$$+ \nu^{-1}\chi^{-1} \otimes \chi^{-1} \otimes \nu\chi^{-1},$$

$$r_{L_1G}\pi = \nu^{-1}\chi \otimes \{[\nu^{\frac{1}{2}}\chi]_2\} + \nu^{-1}\chi \otimes \{[\nu^{-\frac{1}{2}}\chi^{-1}]_2\} + \nu^{-1}\chi^{-1} \otimes \{\nu^{-1}\chi \times \chi^{-1}\},$$

$$r_{L_2G}\pi = \nu^{-1}\chi \otimes (\chi \times \nu^{-1}\chi^{-1}) + \nu^{-1}\chi^{-1} \otimes [\nu^{-\frac{1}{2}}\chi]_2 + \nu^{-1}\chi^{-1} \otimes [\nu^{\frac{1}{2}}\chi]_2,$$

$$r_{M_2G}\pi = [\nu^{-\frac{1}{2}}\chi]_2 \otimes \nu^{-1}\chi^{-1} + (\nu^{-1}\chi \times \nu^{-1}\chi^{-1}) \otimes \chi + [\nu^{-\frac{1}{2}}\chi^{-1}]_2 \otimes \nu\chi^{-1},$$

$$r_{LG}\pi = \nu^{-1}\chi \otimes (\{[\nu^{\frac{1}{2}}\chi]_2\} \propto 1) + \nu^{-1}\chi^{-1} \otimes ([\nu^{-\frac{1}{2}}\chi]_2 \propto 1),$$

$$r_{MG}\pi = \{[\chi]_3\} + \{\nu^{-1}\chi \times [\nu^{-\frac{1}{2}}\chi^{-1}]_2\},$$

$$r_{NG}\pi = [\nu^{-\frac{1}{2}}\chi]_2 \times \nu^{-1}\chi^{-1} + [\chi^{-1}]_3.$$

$\pi = [\chi]_3 \propto 1.$

$$r_{AG}\pi = \nu^{-1}\chi \otimes \chi \otimes \nu\chi + \nu^{-1}\chi \otimes \nu^{-1}\chi^{-1} \otimes \chi^{-1} + \nu^{-1}\chi^{-1} \otimes \nu^{-1}\chi \otimes \chi^{-1} + \nu^{-1}\chi^{-1} \otimes \chi^{-1} \otimes \nu^{-1}\chi,$$

$$r_{L_1G}\pi = \nu^{-1}\chi \otimes \{\chi \times \nu^{-1}\chi^{-1}\} + \nu^{-1}\chi^{-1} \otimes \{[\nu^{-\frac{1}{2}}\chi]_2\} + \nu^{-1}\chi^{-1} \otimes \{[\nu^{\frac{1}{2}}\chi]_2\},$$

$$r_{L_2G}\pi = \nu^{-1}\chi \otimes [\nu^{\frac{1}{2}}\chi]_2 + \nu^{-1}\chi \otimes [\nu^{-\frac{1}{2}}\chi^{-1}]_2 + \nu^{-1}\chi^{-1} \otimes (\nu^{-1}\chi \times \chi^{-1}),$$

$$r_{M_2G}\pi = [\nu^{-\frac{1}{2}}\chi]_2 \otimes \nu\chi + (\nu^{-1}\chi \times \nu^{-1}\chi^{-1}) \otimes \chi^{-1} + [\nu^{-\frac{1}{2}}\chi^{-1}]_2 \otimes \nu^{-1}\chi,$$

$$r_{LG}\pi = \nu^{-1}\chi \otimes ([\nu^{\frac{1}{2}}\chi]_2 \propto 1) + \nu^{-1}\chi^{-1} \otimes (\{[\nu^{-\frac{1}{2}}\chi]_2\} \propto 1),$$

$$r_{MG}\pi = \{\nu^{-1}\chi^{-1} \times [\nu^{-\frac{1}{2}}\chi]_2\} + \{[\chi^{-1}]_3\},$$

$$r_{NG}\pi = [\chi]_3 + \nu^{-1}\chi \times [\nu^{-\frac{1}{2}}\chi^{-1}]_2.$$

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