Canad. J. Math. Vol. **57** (1), 2005 pp. 159–179

Duality and Supports of Induced Representations for Orthogonal Groups

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Abstract. In this paper, we construct a duality for p-adic orthogonal groups.

1 Introduction

The aim of this paper is to construct a duality operator for representations of p-adic orthogonal groups (not necessarily split) along the lines of the constructions of [Au1, Au2, S-S]. As a consequence, we are also able to establish a decomposition for these groups (as well as certain special orthogonal groups) analogous to that given in [J3] for $Sp(2n, F)$ and $SO(2n+1, F)$, F p-adic.

The sorts of duality operators we are interested in have been studied for years in a number of different contexts; we note the work of [Cu, Al, Kw] for finite groups of Lie type (Curtis-Alvis duality), [I-M] for Hecke algebras (the Iwahori-Matsumoto involution), and [Ze] for p-adic (general linear) groups (the Zelevinsky involution). Our interest here is in producing an alternating sum formula and showing it takes irreducible representations to irreducible representations, as done in [D-L1, D-L2] (finite groups of Lie type), [Kt] (Hecke algebras), and [Au1, Au2, S-S] (p-adic groups). We remark that duality questions have been looked at in the non-connected case. In [D-M], duality for non-connected finite groups of Lie type is addressed; in [J-K], the Iwahori-Matsumoto involution is discussed for Iwahori-spherical Hecke algebras associated to non-connected p-adic groups. However, neither of these provides the sort of alternating sum formula we want (though *cf.* [D-M, p. 377]).

Duality for p-adic groups has been quite useful in the study of induced representations (*e.g.,* [J2, Mu, T2]), owing at least in part to its ability to relate reducibility questions for inducing representations with different asymptotic properties—*e.g.,* representations induced from (twists of) the trivial and Steinberg representations are related by duality. The application of duality to a conjecture of Arthur (*cf.* [Ba2, Ba3, Ba-Zh]) is a source of recent interest. However, our immediate interest here is in extending the structural decomposition of [J3] to cover orthogonal groups.

In what follows, we will work with semisimplified representations. To this end, if G is a p-adic group, let $\mathcal{R}(G)$ denote the Grothendieck group of the category of smooth, finite-length representations of G. Recall that this means $\pi_1 = \pi_2$ in $\mathcal{R}(G)$ if $m(\tau, \pi_1) = m(\tau, \pi_2)$ for every smooth, irreducible representation τ of G, where $m(\tau, \pi) =$ multiplicity of τ in π. Similarly, $\pi_1 + \pi_2$ is defined by $m(\tau, \pi_1 + \pi_2) = m(\tau, \pi_1) + m(\tau, \pi_2)$.

Received by the editors February 14, 2003; revised September 8, 2003.
AMS subject classification: 22E50.

Let us take a moment to discuss the orthogonal groups. Let J denote the $n \times n$ antidiagonal matrix

$$J_n = \begin{pmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ \cdot & & & \\ 1 & & & \end{pmatrix}.$$

Fix a $q \times q$ matrix Q for a nonisotropic orthogonal form as in [Br, chapter 2, section 3]. Note that $q \in \{0, 1, 2, 3, 4\}$. Then,

$$X_n = \begin{pmatrix} & & J_n \\ & Q & \\ J_n & & \end{pmatrix}$$

is the matrix of an orthogonal form, and any orthogonal form is equivalent to one of these. We let $O(X_n, F)$ denote the group of matrices (with entries in F) preserving this form. It is a non-connected group with two components (except if $n = q = 0$, in which case we have the trivial group). If $q = 0$, the root system is of type D_n; if $q > 0$, it is of type B_n (*cf.* [Bo, section 6] for more details). Let C be a set of representatives for G/G^0 in W (Weyl group), chosen as in section 2. Then, C acts on the simple roots, Π. When $q > 0$, this action is trivial and we have the direct product decomposition $W_G = W_{G^0} \times C$ ($W_{G^0} \subset W$ the Weyl group for G^0). When $q = 0$, C interchanges the last two simple roots; we have $W_G = W_{G^0} \rtimes C$. In the case $q = 0$, we use the structural similarity between $O(2n, F)$ and $Sp(2n, F)$, $SO(2n+1, F)$ in proving our results; when $q > 0$, the fact that $W_G = W_{G^0} \times C$ will make the arguments easier. (In fact, if q is odd, we have $O(X_n, F) = SO(X_n, F) \times \{\pm I\}$, which can be used to simplify matters greatly.)

For the purpose of extending the results of [J3], we need to have a duality operator which has essentially the same form as that for $Sp(2n, F)$ or $SO(2n + 1, F)$. We take a moment to describe how to construct such a duality operator. First, we consider the case $q = 0$ (*i.e.,* $SO(2n, F)$). Let $\alpha_1, \ldots, \alpha_n$ denote the simple roots for $SO(2n, F)$. Also, let c denote the nth sign change for $O(2n, F)$ (the usual generator for $O(2n, F)/SO(2n, F)$). We let $S = \{s_{\alpha_1}, \ldots, s_{\alpha_{n-1}}, c\}$, with s_α denoting the corresponding root reflection. (We note that for $Sp(2n, F)$ and $SO(2n + 1, F)$, the nth sign change is a simple reflection, so that S corresponds to the set of simple root reflections for these groups.) If $I \subset S$, we let $P_I = M_I U_I = \langle P_{min}, I \rangle$. In the case $q > 0$, we take $S = \{s_{\alpha_1}, \ldots, s_{\alpha_n}\}$ (simple reflections). For $I \subset S$, we let $P_{I,C} = M_{I,C} U_I = \langle P_{min}, I, C \rangle$, where C denotes an appropriate set of representatives for $O(X_n, F)/SO(X_n, F)$ (*cf.* section 2). We may now define the duality operators D_G as follows:

Definition 1.1

(1) *For* $q = 0$,

$$D_G = \sum_{I \subset S} (-1)^{|I|} i_{G,M_I} \circ r_{M_I,G}.$$

(2) *For* $q > 0$,

$$D_G = \sum_{I \subset S} (-1)^{|I|} i_{G,M_{I,C}} \circ r_{M_{I,C},G}.$$

We remark that the duality operator for $q = 0$ includes both connected and non-connected parabolic subgroups, whereas the duality operator for $q > 0$ uses only non-connected parabolic subgroups (however, see Remark 6.6).

The proof of the properties of duality given in [Au1, Au2] relies on four key properties of induction and Jacquet modules (*cf.* [Au1, (1.1)–(1.4)]), three of which were proven only in the connected case. Thus, in order to extend duality to the orthogonal groups, we first need to extend these results. We remark that even though our main application will be to orthogonal groups, because of their general usefulness in the connected case, we verify these results for non-connected groups in greater generality.

The first result that needs to be generalized is the description of composition factors for $r_{N,G} \circ i_{G,M}$ given in [B-Z, Lemma 2.12] and [Ca, Theorem 6.3.5]. We deal with this in section 3 (*cf.* Proposition 3.5). Another result that needs to be generalized is [BDK, Lemma 5.4(iii)] which says if M is a standard Levi and $w \in W$ (Weyl group) is such that $w(M)$ is also a standard Levi, then $i_{G,w(M)} \circ Ad(w) = i_{G,M}$. We do this in section 4 (*cf.* Proposition 4.1). Finally, we need to generalize the characterization of the contragredient of a Jacquet module given in [Ca, Corollary 4.2.5]. We address this in section 5 (*cf.* Proposition 5.5).

In the sixth section, we give the results on duality (*cf.* Theorems 6.1 and 6.5; also Proposition 6.3). The duality operator for $O(X_n, F)$ is the last ingredient needed to extend the results of [J3] to the case of orthogonal groups. This extension is given in the seventh section.

The assumptions on the characteristic of F which are needed vary from section to section. Therefore, we make no assumptions on the characteristic of F a priori, but will indicate the necessary assumptions in each section.

2 Notation and Preliminaries

In this section, we introduce notation and give some background results. In sections 3–5, we work with non-connected groups more general than orthogonal groups; we take a moment to discuss the non-connected groups considered.

Let F be a p-adic field and G the group of F-points of a quasi-split reductive algebraic group defined over F. Let G^0 denote the connected component of the identity in G. We assume that $C = G/G^0$ is a finite abelian group (with finiteness automatic).

In the group G^0, fix a Borel subgroup $P_\varnothing \subset G^0$. We let Δ^+ denote the corresponding set of positive roots; $\Pi \subset \Delta^+$ the simple roots. For $\Phi \subset \Pi$, we let $P_\Phi = M_\Phi U_\Phi$ denote the standard parabolic subgroup determined by Φ.

Before we go into notation and basic definitions for G, we need to do a couple of things. First, we fix a choice of representatives for G/G^0 which stabilize the Borel subgroup, hence act on the simple roots. By abuse of notation, we use C for both the component groups and the image of these representatives in the Weyl group (they correspond to the elements of the Weyl group having length 0). If $c \in C$, we use $\bar{c}$ to denote its chosen representative in G; we use $\bar{C}$ for the chosen set of representatives. If π is an irreducible representation of G^0 and $c \in C$, we define $c \cdot \pi$ by

$$c \cdot \pi(g) = \pi((\bar{c})^{-1} g \bar{c}),$$

for all $g \in G^0$. The equivalence class of $c \cdot \pi_1$ does not depend on the choice of representative $\bar{c}$.

Let $P_\Phi = M_\Phi U_\Phi \subset G^0$ be the standard parabolic subgroup of G^0 corresponding to $\Phi \subset \Pi$. Let

$$C(\Phi) = \{c \in C \mid c \cdot \Phi = \Phi\}.$$

We let

$$M_{\Phi,C(\Phi)} = \langle M_\Phi, \overline{C(\Phi)} \rangle.$$

More generally, if $D \subset C(\Phi)$, we let

$$M_{\Phi,D} = \langle M_\Phi, \bar{D} \rangle.$$

(Note that $M_{\Phi,1} = M_\Phi$). We note that $M_{\Phi,D}$ does not depend on the choice of representatives $\bar{D}$. Suppose that M satisfies

$$M_\Phi \leq M \leq M_{\Phi,C(\Phi)}$$

(such an M has the form $M_{\Phi,D}$). We will consider subgroups of the form $P = MU = M_{\Phi,D} U_\Phi$. We write $P_{\Phi,D} = M_{\Phi,D} U_\Phi$. Since M normalizes U, we can define functors $i_{G,M}$ and $r_{M,G}$ as in [B-Z]. The standard properties of the functors $i_{G,M}$ and $r_{M,G}$ are described in [B-Z, Proposition 1.9].

In G^0, the standard parabolic subgroups are non-conjugate. We arrange this for G as in [B-J1]. Observe that for any $c \in C$, we have

$$\bar{c} M_\Phi (\bar{c})^{-1} = M_{c \cdot \Phi},$$

$$c C(\Phi) c^{-1} = C(c \cdot \Phi),$$

so the groups $M_{\Phi,C(\Phi)}$ and $M_{c\cdot\Phi,C(c\cdot\Phi)}$ are conjugate. Similarly, if $M_\Phi \leq M \leq M_{\Phi,C(\Phi)}$, then $M = M_{\Phi,D}$, where $D \leq C(\Phi)$, and

$$\bar{c} M (\bar{c})^{-1} = M_{c\cdot\Phi, cDc^{-1}} \leq M_{c\cdot\Phi,C(c\cdot\Phi)}.$$

To arrange standard parabolic subgroups for G to be non-conjugate, we need to choose one group from among $\{M_{c\cdot\Phi}\}_{c\in C}$, *i.e.*, a representative of the set $\{c \cdot \Phi\}_{c \in C}$.

Choose an ordering on the elements of Π. Then, one has a lexicographic order on the subsets of Π. (To be precise, if $\Phi_1 = \{\beta_1, \dots, \beta_k\}$ and $\Phi_2 = \{\gamma_1, \dots, \gamma_l\}$ with $\beta_1 > \cdots > \beta_k$ and $\gamma_1 > \cdots > \gamma_l$, we write $\Phi_1 \succ \Phi_2$ if $\beta_1 > \gamma_1$ or $\beta_1 = \gamma_1$ and $\beta_2 > \gamma_2$, etc. The absence of a root is lower than a root, so $\varnothing$ is minimal.) We define

$$X_C = \{\Phi \subset \Pi \mid \Phi \text{ is maximal among } \{c \cdot \Phi\}_{c \in C}\}.$$

In particular, any $\Phi \subset \Pi$ is conjugate in G to an element of X_C. We take as standard parabolic subgroups those subgroups of the form $P = MU_\Phi$ with $M_\Phi \leq M \leq M_{\Phi,C(\Phi)}$ and $\Phi \in X_C$.

3 A Generalization of a Result of Bernstein-Zelevinsky/Casselman to Non-Connected Groups

In this section, our aim is to extend [B-Z, Lemma 2.12] and [Ca, Theorem 6.3.5] (which are essentially the same result) to the non-connected groups of section 2. We remark that since the results in [B-Z, section 5] apply to the non-connected groups we consider, our task consists primarily of proving the Weyl group results necessary to formulate this like [B-Z, Lemma 2.12] and [Ca, Theorem 6.3.5]. For this section, we make no restrictions on the characteristic of F except in Lemmas 3.6 and 3.7, where we assume char $F \neq 2$.

First, recall that if P_{Φ_1}, P_{Φ_2} are standard parabolic subgroups of G^0, then

$$W_{G^0}^{M_{\Phi_1}M_{\Phi_2}} = \{w \in W_{G^0} \mid w \cdot \Phi_1 \subset \Delta^+,\ w^{-1} \cdot \Phi_2 \subset \Delta^+\},$$

where $W_{G^0} \subset W$ denotes the Weyl group of G^0. This subset plays a key role in [B-Z, Lemma 2.12] and [Ca, Theorem 6.3.5]. Now, suppose $P_1 = M_{\Phi_1,C_1}U_{\Phi_1}$ and $P_2 = M_{\Phi_2,C_2}U_{\Phi_2}$ are two standard parabolic subgroups of G. Our first goal is to define a suitable subset $W^{M_{\Phi_1,C_1}M_{\Phi_2,C_2}} \subset W$ of double-coset representatives for

$$W_{\Phi_2,C_2} \backslash W / W_{\Phi_1,C_1}$$

(with $W_{\Phi_1,C_1} \subset W$ the Weyl group of M_{Φ_1,C_1}). We start by defining $W^{M_{\Phi_1}M_{\Phi_2}} \subset W$ and showing it has certain useful properties.

Definition 3.1 *We let $W^{M_{\Phi_1}M_{\Phi_2}} \subset W$ be the subset $W^{M_{\Phi_1}M_{\Phi_2}} = \bigcup_{c \in C}(W_{G^0}^{M_{c(\Phi_1)}M_{\Phi_2}} c)$.*

Since $W = \bigcup_{c \in C}(W_{G^0}c)$, it is clear that $W^{M_{\Phi_1}M_{\Phi_2}}$ is a set of double-coset representatives for $W_{\Phi_2} \backslash W / W_{\Phi_1}$. It also has the following useful properties:

Lemma 3.2 *Suppose $w \in W^{M_{\Phi_1}M_{\Phi_2}}$. Then,*

(1) *$w \cdot \Phi_1 \subset \Delta^+$ and $w^{-1} \cdot \Phi_2 \subset \Delta^+$.*
(2) *w is the (unique) element of $W_{\Phi_1}wW_{\Phi_2}$ of minimal length.*

Conversely, if $w \in W$ satisfies either (1) *or* (2) *above, then $w \in W^{M_{\Phi_1}M_{\Phi_2}}$.*

Proof For (1), write $w = w'c$ with $w' \in W_{G^0}^{M_{c(\Phi_1)}M_{\Phi_2}}$. We check both containments: To see $w \cdot \Phi_1 \subset \Delta^+$, observe that

$$w \cdot \Phi_1 = w'c \cdot \Phi_1 = w' \cdot (c(\Phi_1)) \subset \Delta^+$$

since $w' \in W_{G^0}^{M_{c(\Phi_1)}M_{\Phi_2}}$. We now check $w^{-1} \cdot \Phi_2 \subset \Delta^+$. Since $w' \in W_{G^0}^{M_{c(\Phi_1)}M_{\Phi_2}}$, we have $(w')^{-1} \cdot \Phi_2 \subset \Delta^+$. Therefore,

$$w^{-1} \cdot \Phi_2 = (w'c)^{-1} \cdot \Phi_2 = c^{-1}(w')^{-1} \cdot \Phi_2 \subset c^{-1} \cdot \Delta^+ = \Delta^+$$

as needed. This finishes (1).

For (2), we again write $w = w'c$ as above. Suppose $x \in W_{\Phi_2}wW_{\Phi_1}$ with $x \neq w$ and $\ell(x) \leq \ell(w)$. Observe that

$$W_{\Phi_2}wW_{\Phi_1} = W_{\Phi_2}w'cW_{\Phi_1} = W_{\Phi_2}w'W_{c(\Phi_1)}c.$$

Therefore, $xc^{-1} \in W_{\Phi_2}w'W_{c(\Phi_1)}$, forcing $\ell(xc^{-1}) > \ell(w')$ (since $w' \in W_{G^0}^{M_{c(\Phi_1)}M_{\Phi_2}}$). However, since $\ell(x) = \ell(xc^{-1})$ and $\ell(w) = \ell(w')$, we get $\ell(x) > \ell(w)$, a contradiction. Therefore, w is the (unique) element of $W_{\Phi_2}wW_{\Phi_1}$ of minimal length.

We now address the converses. The converse to (2) follows from the uniqueness in (2). For (1), write $w = w'c$ with $w' \in W_{G^0}$ and $c \in C$. We have

$$w'c \cdot \Phi_1 \subset \Delta^+ \Rightarrow w' \cdot (c(\Phi_1)) \subset \Delta^+$$

and

$$(w'c)^{-1} \cdot \Phi_2 \subset \Delta^+ \Rightarrow c^{-1}(w') \cdot \Phi_2 \subset \Delta^+ \Rightarrow (w')^{-1} \cdot \Phi_2 \subset c \cdot \Delta^+ = \Delta^+.$$

Therefore, $w' \in W_{G^0}^{M_{c(\Phi_1)}M_{\Phi_2}}$. The result follows. ∎

We now turn to constructing $W^{M_{\Phi_1,C_1}M_{\Phi_2,C_2}}$. Recall that

$$W = \bigcup_{w \in W^{M_{\Phi_1}M_{\Phi_2}}} (W_{\Phi_2}wW_{\Phi_1}).$$

Now, consider $w \in W^{M_{\Phi_1}M_{\Phi_2}}$. We have $W_{\Phi_2,C_2}wW_{\Phi_1,C_1} = W_{\Phi_2}C_2wC_1W_{\Phi_1}$. We can choose any element of C_2wC_1 as a representative for this double-coset. (Of course, in general C_2wC_1 can contain elements of $W^{M_{\Phi_1}M_{\Phi_2}}$ other than w.) This choice needs to be made only once for each double-coset $C_2wC_1 \subset W^{M_{\Phi_1}M_{\Phi_2}}$. Fix such a set $W^{M_{\Phi_1,C_1}M_{\Phi_2,C_2}}$.

Lemma 3.3 *Suppose $w \in W^{M_{\Phi_1,C_1}M_{\Phi_2,C_2}}$. Then,*

(1) *$w \cdot \Phi_1 \subset \Delta^+$ and $w^{-1} \cdot \Phi_2 \subset \Delta^+$.*
(2) *w is of minimal length in $W_{\Phi_2,C_2}wW_{\Phi_1,C_1}$ (though need not be unique of minimal length).*

Conversely, suppose $w \in W$ satisfies either (1) or (2) above. Then, $w = c_2w'c_1$ for some $w' \in W^{M_{\Phi_1,C_1}M_{\Phi_2,C_2}}$ and $c_i \in C_i$ (so w could have been chosen as the representative of $W_{\Phi_2}(C_2)wW_{\Phi_1}(C_1)$ in $W^{M_{\Phi_1,C_1}M_{\Phi_2,C_2}}$).

Proof (1) follows from $W^{M_{\Phi_1,C_1}M_{\Phi_2,C_2}} \subset W^{M_{\Phi_1}M_{\Phi_2}}$ and part (1) of the preceding lemma.

For (2), suppose $x \in W_{\Phi_2,C_2}wW_{\Phi_1,C_1}$ with $\ell(x) < \ell(w)$. Write $x = c_2w_2ww_1c_1$, with $w_i \in W_{\Phi_i}$ and $c_i \in C_i$. Then, $c_2^{-1}xc_1^{-1} = w_2ww_1$. Therefore, $c_2^{-1}xc_1^{-1} \in W_{\Phi_2}wW_{\Phi_1}$ with $\ell(c_2^{-1}xc_1^{-1}) = \ell(x) < \ell(w)$, contradicting w of minimal length in $W_{\Phi_2}wW_{\Phi_1}$ (which holds by (2) of the preceding lemma). Thus (2) holds.

We now look at the converses. For our w satisfying (1) or (2), let $w' \in W^{M_{\Phi_1,C_1}M_{\Phi_2,C_2}}$ be the chosen double-coset representative. From (2) (just proven), we know $\ell(w') \leq \ell(w)$.

By the preceding lemma, w satisfying (1) or (2) above is enough to tell us $w \in W^{M_{\Phi_1}M_{\Phi_2}}$. Write $w = w_2 c_2 w' c_1 w_1$. Then $c_2 w' c_1 \in W_{\Phi_2} w W_{\Phi_1}$ and $\ell(c_2 w' c_1) = \ell(w')$. Since $w \in W^{M_{\Phi_1}M_{\Phi_2}}$, the preceding lemma implies $\ell(c_2 w' c_1) \geq \ell(w)$. Thus, $\ell(c_2 w' c_1) = \ell(w') = \ell(w)$. By (2) of the preceding lemma, $c_2 w' c_1 = w$, as needed. ■

Lemma 3.4 *Let $H \subset G$ be a subgroup. Set $H_0 = H \cap G^0$ (H_0 is normal in H) and $H/H_0 = C_H$. (N.B. C_H is a subgroup of C). Suppose that representatives for C_H may be chosen from $\bar{C}$. If $P_2 = M_{\Phi_2,C_2}U_{\Phi_2}$ is a standard parabolic subgroup of G and H_0 is decomposable with respect to $M_{\Phi_2}U_{\Phi_2}$ (*cf. [B-Z, p. 460]*), then H is decomposable with respect to $M_{\Phi_2,C_2}U_{\Phi_2}$.*

Proof We need to show

$$H \cap (M_{\Phi_2,C_2}U_{\Phi_2}) = (H \cap M_{\Phi_2,C_2})(H \cap U_{\Phi_2}).$$

An element of $H \cap (M_{\Phi_2,C_2}U_{\Phi_2})$ has the form

$$h = \bar{c}_0' h_0 = \bar{c}_2' m_2 u_2,\ c_0 \in C_H,\ h_0 \in H_0,\ c_2 \in C_2,\ m_2 \in M_{\Phi_2},\ u_2 \in U_{\Phi_2},$$

where $\bar{c}_0', \bar{c}_2'$ are representatives in H, M_{Φ_2,C_2}, resp., of c_0, c_2. For $\bar{c}_0' h_0$ and $\bar{c}_2' m_2 u_2$ to even lie in the same component of G, we must have $c_0 = c_2$. By hypothesis, we may without loss of generality assume $\bar{c}_0' = \bar{c}_2'$. Therefore, $h_0 = m_2 u_2$. Since H_0 is decomposable with respect to $M_{\Phi_2}U_{\Phi_2}$, we may assume $m_2 \in H_0 \cap M_{\Phi_2}$ and $u_2 \in H_0 \cap U_{\Phi_2}$. Therefore, $h = \bar{c}_2' m_2 u_2$ has $\bar{c}_2' m_2 \in H \cap M_{\Phi_2,C_2}$ and $u_2 \in H_0 \cap U_{\Phi_2}$, as needed. ■

Proposition 3.5 *With notation as above, let τ be an admissible representation of M_{Φ_1,C_1}. Then, $r_{M_{\Phi_2,C_2}G} \circ i_{GM_{\Phi_1,C_1}}(\tau)$ has a composition series with factors*

$$i_{M_{\Phi_2,C_2},M_2'} \circ w \circ r_{M_1',M_{\Phi_1,C_1}}(\tau), \qquad w \in W^{M_{\Phi_1,C_1}M_{\Phi_2,C_2}},$$

where $M_1' = M_{\Phi_1,C_1} \cap w^{-1}(M_{\Phi_2,C_2})$, $M_2' = w(M_{\Phi_1,C_1}) \cap M_{\Phi_2,C_2}$.

Proof The result follows from [B-Z, Theorem 5.2] once a few facts have been established.

First, we need to know that (1)–(4) from the hypotheses of [B-Z, Theorem 5.2] hold. For our situation, (1)–(3) are clear; we address (4) next.

For condition (4), we show that for $w \in W^{M_{\Phi_1,C_1}M_{\Phi_2,C_2}}$, $w(M_{\Phi_1,C_1})$ is decomposable with respect to $M_{\Phi_2,C_2}U_{\Phi_2}$. The remaining decomposability conditions may be handled similarly. By Lemma 3.4, it is enough to show that $w(M_{\Phi_1})$ is decomposable with respect to $M_{\Phi_2}U_{\Phi_2}$. That is, we want

$$w(M_{\Phi_1}) \cap (M_{\Phi_2}U_{\Phi_2}) = (w(M_{\Phi_1}) \cap M_{\Phi_2})(w(M_{\Phi_1}) \cap U_{\Phi_2}).$$

Write $w = w_1 c$ with $w_1 \in W_{G^0}$. Then $w(M_{\Phi_1}) = w_1(M_{c(\Phi_1)})$. Thus, we want to show that

$$w_1(M_{c(\Phi_1)}) \cap (M_{\Phi_2} U_{\Phi_2}) = (w_1(M_{c(\Phi_1)}) \cap M_{\Phi_2})(w_1(M_{c(\Phi_1)}) \cap U_{\Phi_2}).$$

From the connected case, it is enough to check that $w_1 \in W^{M_{c(\Phi_1)} M_{\Phi_2}}$, or more explicitly, that $w_1(c(\Phi_1)) \subset \Delta^+$ and $w_1^{-1}(\Phi_2) \subset \Delta^+$. The first of these is trivial: $w_1(c(\Phi_1)) = w(\Phi_1) \subset \Delta^+$ since $w \in W^{M_{\Phi_1,C_1} M_{\Phi_2,C_2}}$. For the second,

$$w_1^{-1}(\Phi_2) = cw^{-1}(\Phi_2) \subset c \cdot \Delta^+ = \Delta^+$$

since $w \in W^{M_{\Phi_1,C_1} M_{\Phi_2,C_2}}$. Thus, $w_1 \in W^{M_{c(\Phi_1)} M_{\Phi_2}}$ and we have the decomposability claimed.

Finally, we need to check that $\varepsilon_1 w^{-1}(\varepsilon_2)$ is the trivial character of M_1', where ε_i is as in [B-Z, section 5.1]. Recall that the modular function for $M_\Phi(D)$ is just that for M_Φ extended trivially to D. Thus, it is enough to show that $w(\varepsilon_1')\varepsilon_2'$ is trivial, where $\varepsilon_1', \varepsilon_2'$ are the restrictions to the connected components. Let $L_1 = (M_1)^0 = M_{\Phi_1} \cap w^{-1}(M_{\Phi_2})$ and $L_2 = (M_2)^0 = w(M_{\Phi_1}) \cap M_{\Phi_2}$. Write $\varepsilon_i' = \delta_{L_i, M_{\Phi_i}}^{1/2}$ (essentially letting $\delta_{L,M}$ denote the modular function for $L < M$). Note that by considering the half-sum of positive roots, one gets $c \circ \delta_{L,M} = \delta_{c(L),c(M)}$. Now, if we write $w = w_1 c$ as above,

$$\begin{aligned} w(\varepsilon_1')\varepsilon_2' &= w_1 c(\delta_{L_1, M_{\Phi_1}})\delta_{L_2, M_{\Phi_2}} \\ &= w_1(\delta_{c(L_1), M_{c(\Phi_1)}})\delta_{L_2, M_{\Phi_2}}. \end{aligned}$$

Since $w_1 \in W^{M_{c(\Phi_1)} M_{\Phi_2}}$ as above, we can now conclude that $w(\varepsilon_1')\varepsilon_2'$ is trivial from the connected case. ■

We now consider these double-coset representatives for $O(X_n, F)$. For this discussion, we want to assume char $F \neq 2$. We begin with $O(2n, F)$ (*i.e.,* $q = 0$).

In order to use the analogy between $O(2n, F)$ and $Sp(2n, F)$, $SO(2n+1, F)$, let W denote the Weyl group for all of these, viewed as permutations and sign changes. Let $I, J \subset S$, S as in section 1. For each of the groups $Sp(2n, F)$, $SO(2n+1, F)$, $O(2n, F)$, there are corresponding parabolic subgroups $P_I = M_I U_I$ and $P_J = M_J U_J$. Considering just $Sp(2n, F)$ or $SO(2n+1, F)$ for the moment, we can let $W^{M_I M_J} \subset W$ denote the usual (minimal length) set of double-coset representatives for $W_{M_I} \backslash W / W_{M_J}$. When W is viewed in terms of permutations and sign changes (*i.e.,* as a Coxeter group), let $\mathcal{D}(I, J)$ denote the corresponding subset of W. The following lemma tells us this is also a suitable set of double-coset representatives (in the sense of Lemma 3.6) for $O(2n, F)$.

Lemma 3.6 *Let $G = O(2n, F)$ and $I, J \subset S$. Then, we may take $W^{M_I M_J} = \mathcal{D}(I, J)$.*

Proof Let Δ_C^+ (resp., Π_C) denote the positive (resp., simple) roots for $Sp(2n, F)$ and Δ_D^+ (resp., Π_D) the positive (resp., simple) roots for $O(2n, F)$. For $I \subset S$, let $\Pi_C(M_I)$

(resp., $\Pi_D(M_I)$) denote the simple roots of $M_I \subset Sp(2n,F)$ (resp., $M_I \subset O(2n,F)$). By Lemma 3.3, it suffices to show that if $w \in \mathcal{D}(I, J)$, then $w \cdot \Pi_D(M_I) \subset \Delta_D^+$ and $w^{-1} \cdot \Pi_D(M_J) \subset \Delta_D^+$. We focus on showing $w \cdot \Pi_D(M_I) \subset \Delta_D^+$ below; the argument for $w^{-1} \cdot \Pi_D(M_J) \subset \Delta_D^+$ is done the same way.

Now, if we write $\Pi_C = \{e_1 - e_2, \dots, e_{n-1} - e_n, 2e_n\}$ and $\Pi_D = \{e_1 - e_2, \dots, e_{n-1} - e_n, e_{n-1} + e_n\}$, we can identify the roots in a manner consistent with the action of W on each. Observe that in this context,

$$\Delta_D^+ = \{\alpha \in \Delta_C^+ \mid \text{length}(\alpha) = 2\}.$$

First, suppose $\alpha = e_i - e_{i+1} \in \Pi_D(M_I)$. Then, $e_i - e_{i+1} \in \Pi_C(M_I)$. Since the action of W preserves lengths, we have $w \cdot \alpha \in \Delta_C^+$ and $\text{length}(w \cdot \alpha) = 2$. Therefore, $w \cdot \alpha \in \Delta_D^+$. Now, suppose $\alpha = e_{n-1} + e_n \in \Pi_D(M_I)$. Then, we must have $s_{n-1}, c \in I$. Therefore, $e_{n-1} - e_n, 2e_n \in \Pi_C(M_I)$. Since $w \in \mathcal{D}(I, J)$, we have

$$w \cdot \alpha = w \cdot \big((e_{n-1} - e_n) + 2e_n\big) = w \cdot (e_{n-1} - e_n) + w \cdot (2e_n) \in \Delta_C^+.$$

Since $\text{length}(\alpha) = \text{length}(w \cdot \alpha) = 2$, we again have $w \cdot \alpha \in \Delta_D^+$, as needed. ■

In the case of $q > 0$, we have the following:

Lemma 3.7 *If $q > 0$ and $I, J \subset S$, we may take*

$$W^{M_{I,C} M_{J,C}} = W_{G^0}^{M_I M_J}.$$

Proof In this case, we have $W = W_{G^0} \times C$, which implies

$$W_{M_{J,C}} w W_{M_{I,C}} = W_{M_J} w W_{M_I} \cup W_{M_J} wc W_{M_I}$$

for all $w \in W$. The lemma follows easily from this. ■

4 A Generalization of a Result of Bernstein-Deligne-Kazhdan to Non-Connected Groups

In this section, our aim is to extend Lemma 5.4(iii) of [BDK] to the non-connected groups of section 2. For this section, we make no restriction on the characteristic of F.

Proposition 4.1 *Suppose $w \in W$ and L, M are Levi factors of standard parabolic subgroups of G such that $M = wLw^{-1}$. Then,*

$$i_{G,M} \circ w(\tau) = i_{G,L}(\tau)$$

for any smooth, finite-length representation τ of L. (Note that this is an equality in the Grothendieck group.)

Proof The proof of this in [BDK] relies on three results: the linear independence of characters, the Langlands classification, and [B-Z, Theorem 2.12] (equivalently, [Ca, Theorem 6.5]). The linear independence of characters is general, and holds for the non-connected groups we are considering (*cf.* [Si2, Lemma 1.13.1]). The Langlands classification for the groups under consideration is done in [B-J1]; along with [B-J3], we have the necessary results on the Langlands classification. The extension of Theorem 2.12 [B-Z] is Proposition 3.5 of this paper. With these observations, the proof from [BDK] extends to cover the non-connected groups under consideration. ■

5 A Generalization of a Result of Casselman to Non-Connected Groups

In this section, our aim is to extend [Ca, Corollary 4.2.5] to non-connected groups. Here, we assume $G = G^0 \rtimes C$, *i.e.*, representatives of C may be chosen which form a group. We assume $\bar{C}$ consists of such representatives. We make no assumptions on the characteristic of F.

Let $P = M_{\Phi,D}U_\Phi$ be a standard parabolic subgroup of G. We remind the reader that the Jacquet modules $r_{M_\Phi,G}(\pi)$ and $r_{M_{\Phi,D},G}(\pi)$ have the same space, denoted $V_U = V_{U_\Phi}$. We also use $\pi_{U,D}$ for the Jacquet module $r_{M_{\Phi,D},G}(\pi)$.

Lemma 5.1 *Let $K_1, K_2 \subset G^0$ be open compact subgroups having Iwahori factorizations with respect to P. Then $K_1 \cap K_2$ is also an open compact subgroup having an Iwahori factorization with respect to P.*

Proof Clearly, $K_1 \cap K_2$ is an open compact subgroup. If $K_1 = U_1^- M_1 U_1$ and $K_2 = U_2^- M_2 U_2$ are the respective Iwahori factorizations, we claim that $K_1 \cap K_2$ has Iwahori factorization $(U_1^- \cap U_2^-)(M_1 \cap M_2)(U_1 \cap U_2)$. This is straightforward to check; we omit the details. ■

Corollary 5.2 *Let $K_0 \subset G^0$ be an open compact subgroup having an Iwahori factorization with respect to P. Then, $\bigcap_{c\in C(\Phi)}(\bar{c}K_0(\bar{c})^{-1}) \subset K_0$ is an open compact subgroup which* (1) *has an Iwahori factorization with respect to P, and* (2) *is normalized by $\overline{C(\Phi)}$.*

Proof It is a straightforward matter to check that $\bar{c}K_0(\bar{c})^{-1}$ has Iwahori factorization $(\bar{c}U_0^-(\bar{c})^{-1})(\bar{c}M_0(\bar{c})^{-1})(\bar{c}U_0(\bar{c})^{-1})$ with respect to P. The result then follows from Lemma 5.1. ■

Lemma 5.3 *Suppose that U_1 is an open compact subgroup of U. Then there is an open compact subgroup $U_2 \supset U_1$ of U such that U_2 is normalized by all $\bar{c} \in \overline{C(\Phi)}$.*

Proof Choose U_2' such that $\bar{c}U_1(\bar{c})^{-1} \subset U_2'$ for all $c \in C(\Phi)$. Let

$$U_2 = \bigcap_{c\in C(\Phi)} \bar{c}U_2'(\bar{c})^{-1}.$$

It is straightforward to check that U_2 has the desired properties. ■

In order to deal with the non-connectedness, we wish to be a bit more restrictive than [Ca] in our choice of canonical lifts. Since this involves modifications to the results of [Ca, Section 4], we freely use notation from there.

Suppose $x \in V_U$. Choose M_0, K_0 such that $K_0 = U_0^- M_0 U_0$ (Iwahori factorization) and $x \in V_U^{M_0}$. Replacing K_0 by $\bigcap_{c \in C(\Phi)} (\bar{c} K_0 (\bar{c})^{-1})$ and M_0 by $\bigcap_{c \in C(\Phi)} (\bar{c} M_0 (\bar{c})^{-1})$ (*cf.* Corollary 5.2), we may assume that M_0, K_0 are normalized by $\overline{C(\Phi)}$. Choose U_1 such that $V^{K_0} \cap V(U) \subset V(U_1)$. By Lemma 5.3, replacing U_1 if needed, we may assume U_1 is normalized by $\overline{C(\Phi)}$. Finally, choose $a \in A^-$ such that $aU_1 a^{-1} \subset U_0$. We take $v \in V_a^{K_0}$ for our canonical lift (*cf.* [Ca, section 4.1]). We call this a $\overline{C(\Phi)}$-canonical lift to avoid ambiguity.

Lemma 5.4 *Let $M_{\Phi,D}U_\Phi$ be a standard parabolic subgroup of G. Suppose $x \in V_U$ and v a $\overline{C(\Phi)}$-canonical lift of x. Then $\pi(\bar{d})v$ is a $\overline{C(\Phi)}$-canonical lift of $\pi_{U,D}(\bar{d})x$ for any $d \in D \subset C(\Phi)$.*

Proof Since v is a $\overline{C(\Phi)}$-canonical lift of x, we may write $v = \pi(\mathrm{char}_{K_0 a K_0})v'$, with $K_0 = U_0^- M_0 U_0$, U_1, and a satisfying the conditions above.

Now, since $\bar{d} \in \overline{C(\Phi)}$ normalizes K_0, we have

$$\pi(\bar{d})\pi(\mathrm{char}_{K_0 a K_0})v' = \pi(\mathrm{char}_{K_0 a' K_0})v',$$

where $a' = \bar{d} a \bar{d}^{-1}$ (noting that conjugation by elements of C preserves Haar measures, *cf.* [B-J1, Lemma 2.2]). Observe that since $d \cdot \Pi = \Pi$, we have $a' \in A^-$. Also, since $\bar{d}$ normalizes U_1, we have $a' U_1 a'^{-1} \subset U_0$. With the trivial observation that $\pi(\bar{d})v$ maps to $\pi_{U,D}(\bar{d})x$ under the canonical projection, we can now conclude that $\pi(\bar{d})v$ is a $\overline{C(\Phi)}$-canonical lift of $\pi_{U,D}(\bar{d})x$, as needed. ■

Let $\langle \cdot , \cdot \rangle$ be the pairing of V with $\tilde{V}$ and $\langle \cdot , \cdot \rangle_U$ be the canonical pairing of V_U with $\tilde{V}_{U^-}$ from [Ca, Section 4.2]. We define the pairing $\langle \cdot , \cdot \rangle_{U,\bar{D}}$ by

$$\langle x, \tilde{x} \rangle_{U,\bar{D}} = \sum_{d \in D} \left\langle \pi_{U,D}(\bar{d})x, \tilde{\pi}_{U^-,D}(\bar{d})\tilde{x} \right\rangle_U .$$

It is a straightforward matter to verify that $\langle \cdot , \cdot \rangle_{U,\bar{D}}$ is $M_{\Phi,D}$-invariant.

We check that $\langle \cdot , \cdot \rangle_{U,\bar{D}}$ is non-degenerate. Let $v, \tilde{v}$ be $\overline{C(\Phi)}$-canonical lifts of $x, \tilde{x}$. Then, by Lemma 5.4,

$$\begin{aligned} \langle x, \tilde{x} \rangle_{U,\bar{D}} &= \sum_{d \in D} \left\langle \pi_{U,D}(\bar{d})x, \tilde{\pi}_{U^-,D}(\bar{d})\tilde{x} \right\rangle_U \\ &= \sum_{d \in D} \left\langle \pi(\bar{d})v, \tilde{\pi}(\bar{d})\tilde{v} \right\rangle \\ &= |D| \langle v, \tilde{v} \rangle . \end{aligned}$$

Since $\langle \cdot , \cdot \rangle$ is non-degenerate, the claim follows. We may now conclude the following:

Proposition 5.5 *The contragredient of $(\pi_{U,D}, V_U)$ is isomorphic to $(\pi_{U^-,D}, \tilde{V}_{U^-})$ (representations of $M_{\Phi,D}$).*

6 Duality for Orthogonal Groups

In this section, we consider the duality operators for orthogonal groups given in Definition 1.1. We establish the basic properties for this operator given in [Au1, Au2]. Since we deal with orthogonal groups, we make the assumption $\operatorname{char} F \neq 2$ in this section.

Let us begin by introducing one piece of notation which will be useful in this section. We let *sgn* denote the nontrivial one-dimensional representation of $G = O(X_n, F)$ (*i.e., sgn* is 1 on G^0 and -1 on $G \setminus G^0$). It is an easy consequence of the results of [G-K, Section 2](*cf.* [B-J1, Lemma 2.1], *e.g.,* for a convenient formulation) that twisting by *sgn* may be accomplished using the involution $i_{G,G^0} \circ r_{G^0,G} - Id$.

The following is [Au1 Théorème 1.7] for $G = O(X_n, F)$. As in [Au1], for $q = 0$ (resp., $q > 0$), we let $w_J \in W^{M_J M_\varnothing}$ (resp., $w_J \in W^{M_{J,C} M_{\varnothing,C}}$) be the element of maximal length (*cf.* Lemmas 3.6 and 3.7).

Theorem 6.1 *The operator D_G for $O(X_n, F)$ (*cf.* Definition 1.1) has the following properties:*

(1) *If $\tilde{}$ denotes contragredient, we have $\tilde{} \circ D_G = D_G \circ \tilde{}$.*
(2) *For q=0 and any $J \subset S$, we have*

$$D_G \circ i_{G,M_J} = i_{G,M_J} \circ D_{M_J}$$

and

$$r_{M_J,G} \circ D_G = w_J \circ D_{M_{J'}} \circ r_{M_{J'},G},$$

with w_J as above and $J' = w_J^{-1} \cdot J$. For $q > 0$ and any $J \subset S$, we have

$$D_G \circ i_{G,M_{J,C}} = i_{G,M_{J,C}} \circ D_{M_{J,C}}$$

and

$$r_{M_{J,C},G} \circ D_G = w_J \circ D_{M_{J',C}} \circ r_{M_{J',C},G}.$$

(3) *$D_G^2 = Id$.*
(4) *If π is supercuspidal in the sense of* [B-J1] *(*i.e., *$r_{G^0,G}\pi$ has supercuspidal components;* cf. [B-J1, Definition 2.5]*), then $D_G(\pi) = (-1)^{|S|}\pi$ unless $G = O(2, F)$, in which case D_G acts by twisting by sgn.*

Proof The calculations needed to prove (1)–(3) here are identical to those in the proof of [Au1 Théorème 1.7]. We note that the properties (1.1)–(1.4) [Au1, p. 2121] have been established, so we are free to use them here. The only other issue is the combinatorial identity of Solomon used in the proof of (2). Here, it is quite helpful that we have formulated duality for $O(2n, F)$ (*i.e.,* $q = 0$) in the same way as for $Sp(2n, F)$ and $SO(2n+1, F)$. In view of Lemma 3.6, it is the same combinatorial identity whether viewed as occuring in $Sp(2n, F)$, $SO(2n + 1, F)$, or $O(2n, F)$, so remains valid. In the case of $q > 0$, Lemma 3.7 allows us to reduce the combinatorial identity to that for G^0. The proof of (4) is trivial. ∎

We note that the preceding proposition deals with $D_G \circ i_{G,M}$ and $r_{M,G} \circ D_G$ only for the parabolic subgroups which appear in the definition of D_G. While this is sufficient for the purpose of generalizing [J3], we would like to be able to deal with arbitrary standard parabolic subgroups. The following proposition allows us to do that. We begin with an easy lemma.

Lemma 6.2 *Let G be a non-connected group and $C = G/G^0$. Let $P = MU$ be a standard parabolic subgroup of G. Then, for $c \in C$, we have the following equivalences:*

$$c \circ i_{G,M} \cong i_{G,c(M)} \circ c, \quad c \circ r_{M,G} \cong r_{c(M),G} \circ c,$$

where $c(M)$ denotes the Levi factor of $c(P)$ (not necessarily standard in the sense of [B-J1]*).*

Proof Straightforward. ∎

Proposition 6.3 *Let $G = O(X_n, F)$. Then,*

$$D_G \circ i_{G,G^0} = i_{G,G^0} \circ D_{G^0} \text{ and } r_{G^0,G} \circ D_G = D_{G^0} \circ r_{G,G^0}.$$

Proof First, let $G = O(2n, F)$ (*i.e.*, $q = 0$). We focus on the induction claim. We have

$$D_G \circ i_{G,G^0} = \sum_{I \subset S} (-1)^{|I|} i_{G,M_I} \circ r_{M_I,G} \circ i_{G,G^0}.$$

By Proposition 3.5 (noting that $c = c^{-1}$),

$$r_{M_I,G} \circ i_{G,G^0} = \begin{cases} r_{M_I,G^0} + c \circ r_{M_{c(I)},G^0} & \text{if } c \notin I \\ i_{M_I,M_I^0} \circ r_{M_I^0,G^0} & \text{if } c \in I. \end{cases}$$

By the preceding lemma and induction in stages, we can break the sum up and rewrite it as follows:

$$\begin{aligned} D_G \circ i_{G,G^0} = & \sum_{\substack{I \subset S \\ s_{n-1}, c \notin I}} (-1)^{|I|} (i_{G,M_I} \circ r_{M_I,G^0} + i_{G,M_{c(I)}} \circ r_{M_{c(I)},G^0}) \\ & + \sum_{\substack{I \subset S \\ c \notin I \\ s_{n-1} \in I}} (-1)^{|I|} (i_{G,M_I} \circ r_{M_I,G^0} + i_{G,M_{c(I)}} \circ r_{M_{c(I)},G^0}) \\ & + \sum_{\substack{I \subset S \\ c \in I \\ s_{n-1} \notin I}} (-1)^{|I|} i_{G,M_I^0} \circ r_{M_I^0,G^0} + \sum_{\substack{I \subset S \\ s_{n-1}, c \in I}} (-1)^{|I|} i_{G,M_I^0} \circ r_{M_I^0,G^0}. \end{aligned}$$

Let Π denote the set of simple roots for G^0. The above sums may then be rewritten as follows:

$$\sum_{\substack{I \subset S \\ s_{n-1}, c \notin I}} (-1)^{|I|} (i_{G,M_I} \circ r_{M_I,G^0} + i_{G,M_{c(I)}} \circ r_{M_{c(I)},G^0}) = 2 \sum_{\substack{I \subset \Pi \\ \alpha_{n-1}, \alpha_n \notin I}} (-1)^{|I|} i_{G,M_I} \circ r_{M_I,G^0},$$

$$\sum_{\substack{I\subset S\\ c\notin I\\ s_{n-1}\in I}} (-1)^{|I|}(i_{G,M_I}\circ r_{M_I,G^0} + i_{G,M_{c(I)}}\circ r_{M_{c(I)},G^0})$$
$$= \sum_{\substack{I\subset\Pi\\ \alpha_{n-1}\notin I\\ \alpha_n\in I}} (-1)^{|I|} i_{G,M_I}\circ r_{M_I,G^0} + \sum_{\substack{I\subset\Pi\\ \alpha_{n-1}\in I\\ \alpha_n\notin I}} (-1)^{|I|} i_{G,M_I}\circ r_{M_I,G^0},$$
$$\sum_{\substack{I\subset\Pi\\ c\in I\\ s_{n-1}\notin I}} (-1)^{|I|} i_{G,M_I^0}\circ r_{M_I^0,G^0} = \sum_{\substack{I\subset\Pi\\ \alpha_{n-1},\alpha_n\notin I}} (-1)^{|I|+1} i_{G,M_I}\circ r_{M_I,G^0},$$
$$\sum_{\substack{I\subset S\\ s_{n-1},c\in I}} (-1)^{|I|} i_{G,M_I^0}\circ r_{M_I^0,G^0} = \sum_{\substack{I\subset\Pi\\ \alpha_{n-1},\alpha_n\in I}} (-1)^{|I|} i_{G,M_I}\circ r_{M_I,G^0}.$$

If we use the above equalities, along with writing $i_{G,M_I} = i_{G,G^0}\circ i_{G^0,M_I}$, we are reduced to the following:

$$\begin{aligned} D_G\circ i_{G,G^0} &= \sum_{I\subset\Pi}(-1)^{|I|} i_{G,G^0}\circ i_{G^0,M_I}\circ r_{M_I,G^0}\\ &= i_{G,G^0}\circ D_{G^0}, \end{aligned}$$

as needed.

The Jacquet module claim for $O(2n,F)$ is done similarly. The proof for $q>0$ is similar, but easier. ■

Corollary 6.4 *$D_G(\pi\otimes sgn) = D_G(\pi)\otimes sgn$.*

Proof It suffices to check that

$$D_{G^0}\circ(i_{G,G^0}\circ r_{G^0,G} - Id) = (i_{G,G^0}\circ r_{G^0,G} - Id)\circ D_G,$$

which is immediate from the preceding proposition. (Alternatively, one can show this using [B-Z, Proposition 1.9(f).) ■

Theorem 6.5 *The duality operators D_G for $O(X_n,F)$ take irreducible representations to irreducible representations (up to sign).*

Proof The proof is essentially the same as that from [Au2, Théorème] and [Au1, Corollaire 3.9]. We give a brief sketch for $G = O(2n,F)$ (*i.e.,* $q=0$), in order to note where changes or additional arguments are needed. The argument for $q>0$ is similar, but with fewer complications. Since we are following Aubert's proof, we freely use notation from [Au1, Au2] below.

As in [Au1, Corollaire 3.9], it is enough to show the exactness of

$$0\longrightarrow E\longrightarrow \bigoplus_{|J|=|S|-1}\tilde{E}_J\longrightarrow \bigoplus_{|J|=|S|-2}\tilde{E}_J\longrightarrow\cdots\longrightarrow \bigoplus_{|J|=|I|}\tilde{E}_J.$$

As in [Au2], this follows if we can show that

$$0 \longrightarrow E \longrightarrow \bigoplus_{|J|=|S|-1} \tilde{E}_{J,I} \longrightarrow \bigoplus_{|J|=|S|-2} \tilde{E}_{J,I} \longrightarrow \cdots \longrightarrow \bigoplus_{|J|=|I|} \tilde{E}_{J,I}$$

is exact.

We take Θ as in [Au2]. In particular, Θ consists of all subsets of W having the property that if $w \in \theta$ and $\ell(w') > \ell(w)$, then $w' \in \theta$. For $\theta \in \Theta$, let

$$G_\theta = \bigcup_{w \in \theta} BwB,$$

where B is the Borel subgroup. For I, J given and $\theta \in \Theta$, let $\theta' \in \Theta$ denote the largest left-W_J and right-W_I invariant subset of θ. We let $E^{\theta}_{J,I}$ denote the subspace of $E_{J,I}$ consisting of functions supported on G_θ. Note that an element of $E_{J,I}$ which is supported on G_θ is then supported on $G_{\theta'}$ (since an element of $E_{J,I}$ is determined by its values on a set of representatives for $P_J \backslash G/P_I$). We remark that this definition of $E^{\theta}_{J,I}$ represents a minor correction to [Au2].

Next, we fix a filtration as in [Au2]:

$$W = \theta_1 \supset \theta_2 \supset \cdots \supset \theta_{t+1} = \varnothing$$

with $\theta_i \in \Theta$ and $\theta_i \setminus \theta_{i+1} = \{w_i\}$. Let $\mathcal{F}$ denote this filtration. Note that in general, we cannot use the same filtration (*i.e.,* the same ordering of the w_i) for $O(2n, F)$ that is used for $SO(2n+1, F)$ or $Sp(2n, F)$ (*e.g.,* consider $s_1 s_2$ and $cs_{n-1}c$ when $n > 2$). Let $W^{M_I M_J}(\mathcal{F})$ denote the following set of double-coset representatives: for the double-coset $W_J w W_I$, we choose $w_i \in W_J w W_I$ having i maximal. We remark that w_i will be of minimal length in $W_J w W_I$. While we cannot use the same filtration as for $Sp(2n, F)$ or $SO(2n+1, F)$, Lemma 3.6 ensures that we can choose $\mathcal{F}$ so that $W^{M_I M_J}(\mathcal{F}) = \mathcal{D}(I, J)$. (*E.g.,* suppose $w \in W_J w W_I$ is of minimal length but cw, wc, cwc are distinct. While all four have the same length in the Weyl group for $O(2n, F)$, in $SO(2n+1, F)$ and $Sp(2n, F)$, one is of minimal length. We choose the filtration so that this element is the last of the four to appear as a $\theta_i \setminus \theta_{i+1}$. Note that even if different I, J are considered, we still want this to be the last of the four to appear.) We fix such a filtration. We note that in the case $q > 0$, Lemma 3.7 tells us we can use the same filtration as for $SO(X_n, F)$, thereby simplifying this part of the argument.

Fix i and let $w = w_i$. As in [Au2], the exactness of

$$0 \longrightarrow E \longrightarrow \bigoplus_{|J|=|S|-1} \tilde{E}_{J,I} \longrightarrow \bigoplus_{|J|=|S|-2} \tilde{E}_{J,I} \longrightarrow \cdots \longrightarrow \bigoplus_{|J|=|I|} \tilde{E}_{J,I}$$

follows from the exactness of

$$0 \longrightarrow E \longrightarrow \bigoplus_{|J|=|S|-1} \tilde{E}^{\theta_i}_{J,I}/\tilde{E}^{\theta_{i+1}}_{J,I} \longrightarrow \bigoplus_{|J|=|S|-2} \tilde{E}^{\theta_i}_{J,I}/\tilde{E}^{\theta_{i+1}}_{J,I} \longrightarrow \cdots \longrightarrow \bigoplus_{|J|=|I|} \tilde{E}^{\theta_i}_{J,I}/\tilde{E}^{\theta_{i+1}}_{J,I}.$$

With our results from section 3, the same argument as in [Au2] reduces this to showing the exactness of

$$\cdots \longrightarrow \bigoplus_{\substack{|J|=k \\ w(I)\subset J\subset S^w}} Ad(w^{-1})(r^G_{wM_Iw^{-1}}(E)) \xrightarrow{\delta_k} \bigoplus_{\substack{|J|=k-1 \\ w(I)\subset J\subset S^w}} Ad(w^{-1})(r^G_{wM_Iw^{-1}}(E)) \xrightarrow{\delta_{k-1}} \cdots .$$

As in [Au2], if we consider the maps used to reduce us to showing the exactness of the preceding complex, we see that δ_k is the identity tensored with the appropriate sign. (Note that whereas Aubert uses [Ca] to calculate δ_k, we must use the results from [B-Z, Section 5] since $O(2n, F)$ is not connected.) This still reduces to well-known results, finishing the proof. ■

Remark 6.6 It is worth taking a moment to discuss duality in more generality. One property a duality operator should have is to send the trivial representation to the Steinberg representation (or, in the case of non-connected groups, something which might reasonably be called a Steinberg representation). We also want our duality operator to have the form

$$(*) \qquad D_G = \sum a_M i_{G,M} \circ r_{M,G},$$

where the sum is over (the Levi factors of) all standard parabolic subgroups. For a given group, one can essentially solve $D_G(\mathit{trivial}) = \mathit{Steinberg}$ to find the coefficients a_M.

We now discuss Steinberg representations for $O(2n, F)$ (*i.e.*, $q = 0$). In this case, there are two representations which might reasonably be called Steinberg representations. In particular, the representation

$$i_{G,M_\varnothing}\delta^{\frac{1}{2}} = i_{G,M_\varnothing}(|\cdot|^{n-1} \otimes \cdots \otimes |\cdot|^1 \otimes |\cdot|^0)$$

has two irreducible subrepresentations which we call St_{triv} and St_{sgn}. They are the unique irreducible subrepresentations of $i_{G,M_{\{c\}}}(|\cdot|^{n-1} \otimes \cdots \otimes |\cdot|^1 \otimes triv_{O(2)})$ and $i_{G,M_{\{c\}}}(|\cdot|^{n-1} \otimes \cdots \otimes |\cdot|^1 \otimes sgn_{O(2)})$, respectively. We have $St_{sgn} = St_{triv} \otimes sgn$. The situation for $q > 0$ is similar.

Examples show that there are only two reasonable duality operators for orthogonal groups: D_G from Definition 1.1 and its twist by sgn. We note that for $q = 0$, $D_G(triv) = St_{sgn}$; its twist by sgn sends the trivial representation to St_{triv}. For $q > 0$, $D_G(triv) = St_{triv}$ and its twist by sgn sends $triv$ to St_{sgn}. To obtain an explicit realization for $D_G \otimes sgn$, one can rewrite this as $D_G \circ (i_{G,G^0} \circ r_{G^0,G} - Id)$ to find the coefficients a_M. For $q > 0$, this operator may be written as

$$D_G \otimes sgn = \sum_{I \subset S'} (-1)^{|I|} i_{G,M_I} \circ r_{M_I,G},$$

with $S' = \{s_{\alpha_1}, \ldots, s_{\alpha_n}, c\}$; for $q = 0$ there is no particularly nice description.

The question naturally arises as to generalizing this to other non-connected groups. It is not clear to the author at this point whether a duality operator of the form $(*)$ exists in general or not.

7 On Supports of Induced Representations

In this section, we extend the results of [J3]—originally done for $Sp(2n, F)$ and $SO(2n+1, F)$—to certain other families of groups having similar structural properties. In particular, we consider the families $O(X_n, F)$ and $SO(X_n, F)$ with $q > 1$ ($q = 1$ is just $SO(2n+1, F)$, already covered). For this section, we can assume char $F = 0$, though there is some flexibility possible (*cf.* Remark 3.1).

The results in [J3] are proven using knowledge about induced representations for general linear groups ([Ze]) and a number of results for $Sp(2n, F)$, $SO(2n+1, F)$:

(1) $R(S)$ comodule structure [T1],
(2) The Langlands classification/Cassleman criterion [B-W, Si1, Ca],
(3) Duality [Au1, Au2, S-S],
(4) R-groups results [Go1].

We now discuss these results for the families of groups under consideration. The $R(S)$ comodule structure of [T1] was extended to $O(2n, F)$ in [Ba1]; the modifications of [T1] necessary for $O(X_n, F)$ with $q > 0$ and $SO(X_n, F)$ with $q > 1$ are given in [M-T]. The Langlands classification of [B-W, Si1] and Cassleman criterion cover the connected groups $SO(X_n, F)$. For the non-connected groups $O(X_n, F)$, the Langlands classification is covered by [B-J1, B-J3]; the Casselman criterion is an easy consequence of the definition of tempered being used (*i.e.,* restriction to G^0 having tempered components—*cf.* [B-J1, Definition 2.5]). The duality results of [Au1, Au2, S-S] also cover the connected groups $SO(X_n, F)$; the non-connected groups $O(X_n, F)$ are covered by the results of section 6 above. The R-group results analogous to [Go1] are given in [Go2] for $O(X_n, F)$ with $q = 0, 1$. These results have not been verified for $O(X_n)$ with $q > 1$ or $SO(X_n, F)$ with $q > 1$, though there is every reason to believe they hold there as well. In fact, [M-T, Theorem 13.1] which covers all the groups in question, is a suitable substitute. However, these results require assuming certain conjectures of Arthur. (On the other hand, it is the use of Goldberg's results that imposes the hypothesis char $F = 0$.) We summarize:

Remark 7.1 For $O(X_n, F)$ with $q > 1$ and $SO(X_n, F)$ with $q > 1$, we need to assume the results of Goldberg hold. Alternatively, we may assume the conjectures necessary for [M-T], in which case [M-T, Theorem 13.1] serves as a substitute.

To make matters more precise, we first consider general linear groups. Let

$$R = \bigoplus_{n \geq 0} \mathcal{R}(GL(n, F)).$$

This has the structure of a Hopf algebra, which we now describe (*cf.* [Ze]). Recall that a parabolic subgroup of $GL(n, F)$ has the form $P = MU$, with $M = GL(n_1, F) \times \cdots \times GL(n_k, F)$ and $n_1 + \cdots + n_k = n$. Let $i_{G,M}$ and $r_{M,G}$ denote the (normalized) induction and (normalized) Jacquet functors (*cf.* [B-Z]). If $\pi_1 \in Irr(GL(n_1, F))$, $\pi_2 \in Irr(GL(n_2, F))$, we define $\pi_1 \times \pi_2$ as the semisimplification of $i_{G,M}(\pi_1 \otimes \pi_2)$. This extends to give

$$\times \colon R \otimes R \longrightarrow R,$$

the Hopf algebra multiplication. The comultiplication m^* is defined via Jacquet modules. Let $M_{(n_1,\ldots,n_k)} = GL(n_1,F) \times \cdots \times GL(n_k,F)$, the Levi factor of a standard parabolic subgroup of $G = GL(n,F)$, $n = n_1 + \cdots + n_k$. On $\mathcal{R}(GL(n,F))$, we take

$$m^* = \sum_{i=0}^{n} r_{M_{(i,n-i)},G}.$$

This extends to give the comultiplication $m^*\colon R \longrightarrow R \otimes R$.

Let $S(n,F)$ denote one of the following families of groups: $O(X_n,F)$, $SO(X_n,F)$ with $q > 1$. Let

$$R(S) = \bigoplus_{n\geq 0} \mathcal{R}(S(n,F)).$$

This has the structure of an M^*-Hopf module over R, which we now describe (*cf.* [Ba1] for $O(2n,F)$ and [M-T] for the extension of [T1] to the remaining families). Recall that a parabolic subgroup of $S(n,F)$ has the form $P = MU$, with $M = GL(n_1,F) \times \cdots \times GL(n_k,F) \times S(n_0,F)$ and $n_1 + \cdots + n_k + n_0 = n$. If $\pi \in Irr(GL(n_1,F))$ and $\theta \in Irr(S(n_0,F))$, we define $\pi \rtimes \theta$ as the semisimplification of $i_{G,M}(\pi \otimes \theta)$. This can be extended to give

$$\rtimes\colon R \otimes R(S) \longrightarrow R(S),$$

the module structure for $R(S)$ over R. The comodule structure is defined using Jacquet modules. Let $M_{(n_1,\ldots,n_k;n_0)} = GL(n_1,F) \times \cdots \times GL(n_k,F) \times S(n_0,F)$, the Levi factor of a standard parabolic subgroup of $G = S(n,F)$, $n = n_1 + \cdots + n_k + n_0$. Then, on $\mathcal{R}(S)$, we take

$$\mu^* = \sum_{i=0}^{n} r_{M_{(i;n-i)},G}.$$

This extends to give $\mu^*\colon R(S) \longrightarrow R \otimes R(S)$. Let $M^* = (m \otimes 1) \circ (\tilde{\ } \otimes m^*) \circ s \circ m^*$, where m denotes the multiplication $\times$ for general linear groups, s is defined by $s\colon \pi_1 \otimes \pi_2 \mapsto \pi_2 \otimes \pi_1$, and $\tilde{\ }$ denotes contragredient. Then, μ^* gives $R(S)$ the structure of an M^*-Hopf module over R, that is, $\mu^* = M^* \rtimes \mu^*$. Here, $(\tau_1 \otimes \tau_2) \rtimes (\tau \otimes \pi)$ is defined to be $(\tau_1 \times \tau) \otimes (\tau_2 \rtimes \pi)$. We refer the reader to [T1, Ba1, M-T] for more details.

We pause to remark that the Hopf algebra/M^*-Hopf module structures described above have been very useful in using Jacquet module techniques to study the representation theory of classical groups (*cf.* [T2, J1, T3, J4, J5, M-T, B-J2], *etc.*). However, our present interest is in the M^*-Hopf module itself. Buried in this structure is a great deal of information on the representation theory of classical groups.

First, consider R. Suppose $\rho_1, \ldots, \rho_k$ are irreducible, unitary, supercuspidal representations of $GL(n_1,F), \ldots, GL(n_k,F)$. Assume no $\rho_i \cong \rho_j$ for $i \neq j$. We let $R(\rho_1, \ldots, \rho_k) \subset R$ denote the subalgebra generated by representations whose supercuspidal support lies in $\{|\det|^\alpha \rho_1\}_{\alpha\in\mathbb{R}} \cup \cdots \cup \{|\det|^\alpha \rho_k\}_{\alpha\in\mathbb{R}}$. Then,

$$R(\rho_1, \ldots, \rho_k) \cong R(\rho_1) \otimes \cdots \otimes R(\rho_k)$$

with the isomorphism in one direction defined by $\pi_1 \otimes \cdots \otimes \pi_k \mapsto \pi_1 \times \cdots \times \pi_k$ (*cf.* [Ze]) for $\pi_i \in R(\rho_i)$ irreducible. The isomorphism in the other direction may be described using Jacquet modules.

Now, consider $R(S)$. Suppose $\rho_1, \ldots, \rho_k$ are irreducible, unitary, supercuspidal representations of $GL(n_1, F), \ldots, GL(n_k, F)$; σ an irreducible, supercuspidal representation of $S(n_0, F)$. Assume no pair has $\rho_i \cong \rho_j$ or $\tilde{\rho}_j$. Let $R(\rho_1, \ldots, \rho_k; \sigma) \subset R(S)$ denote the submodule generated by representations with supercuspidal support on $\{|\det|^\alpha \rho_1, |\det|^{-\alpha}\tilde{\rho}_1\}_{\alpha\in\mathbb{R}} \cup \cdots \cup \{|\det|^\alpha \rho_k, |\det|^{-\alpha}\tilde{\rho}_k\}_{\alpha\in\mathbb{R}} \cup \{\sigma\}$. For $Sp(2n, F)$, $SO(2n+1, F)$, the following result is [J3, Proposition 9.8]. We claim that it also holds for $O(X_n, F)$ and $SO(X_n, F)$ with $q > 1$.

Theorem 7.2 *With notation as above (and assuming Remark 7.1 where appropriate), we have*

$$R(\rho_1, \ldots, \rho_k; \sigma) \cong R(\rho_1; \sigma) \otimes \cdots \otimes R(\rho_k; \sigma)$$

as M^*-Hopf modules over

$$R(\rho_1, \tilde{\rho}_1, \ldots, \rho_k, \tilde{\rho}_k) \cong R(\rho_1, \tilde{\rho}_1) \otimes \cdots \otimes R(\rho_k, \tilde{\rho}_k).$$

*Further, the isomorphism respects contragredience, duality, temperedness, square-integrability, and data for the Langlands classification (*cf. [J3, Theorem 9.3] *for a more precise statement).*

Proof We begin by describing the isomorphism. Suppose π is an irreducible representation with $\pi \in R(\rho_1, \ldots, \rho_k; \sigma)$. By [J3, Lemma 5.7], which is essentially a corollary of Frobenius reciprocity, there exist irreducible representations $\tau_1, \ldots, \tau_{k-1}$ and θ_k with $\tau_i \in R(\rho_i, \tilde{\rho}_i)$ and $\theta_k \in R(\rho_k; \sigma)$ such that

$$\pi \hookrightarrow \tau_1 \times \cdots \times \tau_{k-1} \rtimes \theta_k.$$

Further, θ_k is unique (*cf.* [J3, Corollary 7.5 and Definition 7.6]). We note that since [J3, Corollary 7.5] is an easy consequence of the structure theory of [T1]; by [Ba1, M-T], it also holds for the families under consideration. In a similar fashion, we could single out $\rho_1, \ldots, \rho_{k-1}$, resp., to produce $\theta_1, \ldots, \theta_{k-1}$, resp. The isomorphism is then given in one direction by

$$\pi \longmapsto \theta_1 \otimes \cdots \otimes \theta_k.$$

The proof that this map gives an isomorphism with the desired properties is identical to that in [J3]—the Langlands classification, Casselman criterion, duality, and R-group structures have the same forms for the families under consideration as for $SO(2n+1, F)$ and $Sp(2n, F)$, so the proofs in [J3] go through verbatim. In particular, [J3, Propositions 8.1 and 8.4] show that the above map is a bijection; [J3, Theorem 9.3] shows it is an isomorphism of modules (as well as other properties claimed); [J3, Lemma 9.9] shows it also respects the comodule structure. ∎

Remark 7.3 The refinements to [J3, Proposition 9.8] given in [J3, Proposition 10.10] are also valid for $O(X_n, F)$ and $SO(X_n, F)$ with $q > 1$.

Acknowledgements We would like to take the opportunity to thank A.-M. Aubert, D. Ban, and D. Goldberg for conversations valuable to this work. My thanks also go out to the referees for valuable comments and corrections.

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