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ON SUPPORTS OF INDUCED REPRESENTATIONS FOR SYMPLECTIC AND ODD-ORTHOGONAL GROUPS

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Abstract. Let G be $Sp(2n,F)$ (resp. $SO(2n+1,F)$), where F is a p-adic field of characteristic zero. In this paper, we give a correspondence which associates to an irreducible representation π of G an m-tuple of irreducible representations of lower rank symplectic (resp. orthogonal) groups based on the supercuspidal support of π. We show that this correspondence respects the induction and Jacquet module functors (in a sense to be made precise), as well as verifying a number of other useful properties. In essence, this correspondence allows one to isolate the effects of the different families of supercuspidal representations of general linear groups which appear in the support of π.

1. Introduction. Let F be a p-adic field of characteristic zero and $S_n(F) = Sp_{2n}(F)$ or $SO_{2n+1}(F)$. Suppose $\psi_1,\dots,\psi_m$ are inequivalent irreducible unitary supercuspidal representations of $GL_{r_1}(F),\dots,GL_{r_m}(F)$ with $\psi_i \not\cong \tilde{\psi}_j$ for $i \neq j$, where $\tilde{}$ denotes the contragredient. Let

$$\mathcal{S}(\psi_i) = \{\nu^\alpha\psi_i, \nu^\alpha\tilde{\psi}_i\}_{\alpha\in\mathbb{R}},$$

where ν denotes $|det|$ on $GL(F)$. Let ψ be an irreducible supercuspidal representation of $S_r(F)$. Set $\mathcal{S}(\psi_1,\dots,\psi_m;\psi) = \mathcal{S}(\psi_1)\cup\cdots\cup\mathcal{S}(\psi_m)\cup\{\psi\}$. In this paper, we give a correspondence between representations π supported on $\mathcal{S}(\psi_1,\dots,\psi_m;\psi)$ and m-tuples $(\sigma_1,\dots,\sigma_m)$ of representations supported on $\mathcal{S}(\psi_1;\psi),\dots,\mathcal{S}(\psi_m;\psi)$. In particular, suppose π is an irreducible representation of $S_n(F)$ supported on $\mathcal{S}(\psi_1,\dots,\psi_m;\psi)$. We define maps ψ_i, $i=1,\dots,m$ such that $\psi_i(\pi)$ is an irreducible representation of $S_{k_i}(F)$ supported on $\mathcal{S}(\psi_i;\psi)$. Further, if π' is an irreducible representation supported on $\mathcal{S}(\psi_1,\dots,\psi_m;\psi)$ with $\psi_i(\pi')=\psi_i(\pi)$ for $i=1,\dots,m$, then $\pi=\pi'$. We also show that this correspondence behaves reasonably with respect to induction and Jacquet modules. The basic properties are summarized in Theorem 9.3. Properly interpreted, this correspondence may be viewed as an extension of Goldberg's results on R-groups, at least as they apply to supercuspidal inducing representations.

A similar decomposition holds for representations of $GL(F)$ by the work of Zelevinsky. In particular, we get a correspondence between irreducible represen-

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tations supported on $\cup_{i=1}^{m}\{\nu^{\alpha}\psi_i\}_{\alpha\in\mathbb{R}}$ and m-tuples of representations supported on $\{\nu^{\alpha}\psi_1\}_{\alpha\in\mathbb{R}},\ldots,\{\nu^{\alpha}\psi_m\}_{\alpha\in\mathbb{R}}$. The correspondence is given by

$$\mathrm{Ind}(\tau(\psi_1)\otimes\cdots\otimes\tau(\psi_m))\longleftrightarrow(\tau(\psi_1),\ldots,\tau(\psi_m)),$$

where $\tau(\psi_i)$ is an irreducible representation supported on $\{\nu^{\alpha}\psi_i\}_{\alpha\in\mathbb{R}}$. That this correspondence is well-defined is an immediate result of Lemma 5.6.

We now describe the contents section by section. In the next section, we review notation and basic results to be used later. The third section is a discussion of the Langlands classification, the Casselman criteria for square-integrability/temperedness, and some of their properties. In the fourth section, we review the duality operator of Aubert. We also give a corollary to this which tells us that either (1) π is nontempered, (2) π is tempered with $\hat{\pi}$ nontempered (where $\hat{\pi}$ denotes the dual of π defined by Aubert), or (3) π is a component of $i_{GM}\rho$ with ρ unitary supercuspidal. The proofs of some of the main results will be broken into cases along these lines. Roughly speaking, (1) will be dealt with using the Langlands data, (2) by dualizing the results from (1), and (3) using R-groups. In the fifth section, we discuss Tadić's μ^*. The sixth section discusses Goldberg's R-group calculations and some consequences. In the seventh section, we define a variation $\mu^*_{\psi_1,\ldots,\psi_k}$ of μ^* and discuss some of its properties. This allows us to define $\psi_i(\pi)$. It will also be needed to verify the existence and properties of the correspondence. In the eighth section, we show that if $\psi_i(\pi)=\psi_i(\pi')$ for all i, then $\pi=\pi'$, establishing the correspondence. Further, we show that the correspondence respects temperedness/square-integrability and determine its effect on the Langlands data. In the ninth section, we verify that the correspondence behaves as one would expect with regard to induction and Jacquet modules. Let $R=\oplus_{n\geq 0}R(GL_n(F))$ and $R[S]=\oplus_{n\geq 0}R(S_n(F))$, where $R(GL_n(F))$ (resp. $R(S_n(F))$) denotes the Grothendieck group of the category of smooth finite-length representations of $GL_n(F)$ (resp. $S_n(F)$). Then, $R[S]$ is an M_S^*-Hopf module over the Hopf algebra R. We close section nine by interpreting the correspondence as a tensor product decomposition on $R[S]$. In the final section, we refine the results from section nine: if $\beta_j\in\mathbb{R}$ and ψ_i as above, set $\mathcal{S}_{\beta_j}(\psi_i)=\{\nu^{z+\beta_j}\psi_i,\nu^{-z-\beta_j}\tilde{\psi}_i\}_{z\in\mathbb{Z}}$. If $\beta_1,\ldots,\beta_n\in\mathbb{R}$ are such that the corresponding $\mathcal{S}_{\beta_j}(\psi_i)$, $j=1,\ldots,n$, are pairwise disjoint, set $\mathcal{S}_{\beta_1,\ldots,\beta_n}(\psi_i;\psi)=\mathcal{S}_{\beta_1}(\psi_i)\cup\cdots\cup\mathcal{S}_{\beta_n}(\psi_i)\cup\psi$ with ψ as above. We then get a similar correspondence between irreducible representations supported on $\mathcal{S}_{\beta_1,\ldots,\beta_n}(\psi_i;\psi)$ and n-tuples of irreducible representations supported on $\mathcal{S}_{\beta_1}(\psi_i;\psi),\ldots,\mathcal{S}_{\beta_n}(\psi_i;\psi)$. This may also be interpreted in terms of a tensor product decomposition.

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2. Notation and preliminaries. In this section, we introduce notation and recall some results that will be needed in the rest of the paper. This largely follows the setup used in [Tad1].

Let F be a p-adic field with char$F$=0. Let $|\cdot|$ denote the absolute value on F, normalized so that $|\varpi| = q^{-1}$, ϖ a uniformizer. As in [Zel], we let $\nu = |det|$ on $GL_n(F)$ (with the value of n clear from context). Define $\times$ on $GL(F)$ as in [Zel]: if $\rho_1, \ldots, \rho_k$ are representations of $GL_{n_1}(F), \ldots, GL_{n_k}(F)$, let $\rho_1 \times \cdots \times \rho_k$ denote the representation of $GL_{n_1+\cdots+n_k}(F)$ obtained by inducing $\rho_1 \otimes \cdots \otimes \rho_k$ from the standard parabolic subgroup of $GL_{n_1+\cdots+n_k}(F)$ with Levi factor $GL_{n_1}(F) \times \cdots \times GL_{n_k}(F)$.

In most of this paper, we work with the components (irreducible composition factors) of a representation rather than with the actual composition series. That is, we usually work with the semisimplified representation. So, for any representation π and irreducible representation ρ, let $m(\rho, \pi)$ denote the multiplicity of ρ in π. We write $\pi = \pi_1 + \cdots + \pi_k$ if $m(\rho, \pi) = m(\rho, \pi_1) + \cdots + m(\rho, \pi_k)$ for every irreducible ρ. Similarly, we write $\pi \geq \pi_0$ if $m(\rho, \pi) \geq m(\rho, \pi_0)$ for every such ρ. We write $\pi \cong \pi_0$ if we mean that they are actually equivalent.

We now turn to symplectic and odd-orthogonal groups. Let

$$J_n = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}$$

denote the $n \times n$ antidiagonal matrix above. Then,

$$SO_{2n+1}(F) = \{X \in SL_{2n+1}(F) | {}^T X J_{2n+1} X = J_{2n+1}\},$$

$$Sp_{2n}(F) = \left\{ X \in GL_{2n}(F) | {}^T X \begin{pmatrix} & -J \\ J & \end{pmatrix} X = \begin{pmatrix} & -J \\ J & \end{pmatrix} \right\}.$$

We use $S_n(F)$ to denote either $SO_{2n+1}(F)$ or $Sp_{2n}(F)$. In either case, the Weyl group is $W = \{$ permutations and sign changes on n letters $\}$.

We take as minimal parabolic subgroup in $S_n(F)$ the subgroup P_{min} consisting of upper triangular matrices. Let $\alpha = (n_1, \ldots, n_k)$ be an ordered partition of a nonnegative integer $m \leq n$ into positive integers. Let $M_\alpha \subset S_n(F)$ be the subgroup

$$M_\alpha = \left\{ \begin{pmatrix} X_1 & & & & & & \\ & \ddots & & & & & \\ & & X_k & & & & \\ & & & X & & & \\ & & & & {}^\tau X_k & & \\ & & & & & \ddots & \\ & & & & & & {}^\tau X_1 \end{pmatrix} \middle| \, X_i \in GL_{n_i}(F),\ X \in S_{n-m}(F) \right\},$$

where ${}^\tau X = J^T X^{-1} J$. Then $P_\alpha = M_\alpha P_{min}$ is a parabolic subgroup of S_n and every parabolic subgroup is of this form (up to conjugation). For $\alpha = (n_1, \ldots, n_k)$, let $\rho_1, \ldots, \rho_k$ be representations of $GL_{n_1}(F), \ldots, GL_{n_k}(F)$, respectively, and ρ a representation of $S_{n-m}(F)$. Let $\rho_1 \times \cdots \times \rho_k \rtimes \rho$ denote the representation of $S_n(F)$ obtained by inducing the representation $\rho_1 \otimes \cdots \otimes \rho_k \otimes \rho$ of M_α (extended trivially to P_α). If $m = n$, we write $\rho_1 \times \cdots \times \rho_k \rtimes 1$, where 1 denotes the trivial representation of $S_0(F)$.

We recall some structures which will be useful later (cf. Section 1 of [Zel] and Section 4 of [Tad3]). Let $R(GL_n(F))$ (resp. $R(S_n(F))$) denote the Grothendieck group of the category of all smooth finite-length $GL_n(F)$-modules (resp. $S_n(F)$-modules). Set $R = \oplus_{n\geq 0} R(GL_n(F))$ and $R[S] = \oplus_{n\geq 0} R(S_n(F))$. The operators $\times$ and $\rtimes$ lift naturally to

$$\times \colon R \otimes R \longrightarrow R \qquad \text{and} \qquad \rtimes \colon R \otimes R[S] \longrightarrow R[S].$$

With these multiplications, R becomes an algebra and $R[S]$ a module over R.

Let π be an irreducible representation of $S_n(F)$. Then, there is a standard Levi M and an irreducible supercuspidal representation $\rho_1 \otimes \cdots \otimes \rho_k \otimes \rho$ of M (with ρ_i an irreducible supercuspidal representation of $GL_{n_i}(F)$ and ρ an irreducible supercuspidal representation of $S_{n-m}(F)$) such that π is a subquotient of $i_{GM}(\rho_1 \otimes \cdots \otimes \rho_k \otimes \rho)$. We say that the multiset $\{\rho_1, \ldots, \rho_k; \rho\}$ is in the support of π. Further, M and $\rho_1 \otimes \cdots \otimes \rho_k \otimes \rho$ are unique up to conjugation (cf. Theorem 2.9, [B-Z]). By Propositions 4.1 and 4.2 of [Tad3],

$$\rho_1 \times \cdots \times \rho_{i-1} \times \rho_i \times \rho_{i+1} \times \cdots \times \rho_k \rtimes \rho = \rho_1 \times \cdots \times \rho_{i-1} \times \tilde{\rho}_i \times \rho_{i+1} \times \cdots \times \rho_k \rtimes \rho,$$

where $\tilde{}$ denotes contragredient. Thus, if $\{\rho_1, \ldots, \rho_{i-1}, \rho_i, \rho_{i+1}, \ldots, \rho_k; \rho\}$ is in the support of π, so is $\{\rho_1, \ldots, \rho_{i-1}, \tilde{\rho}_i, \rho_{i+1}, \ldots, \rho_k; \rho\}$. Therefore, every $\{\rho'_1, \ldots, \rho'_k; \rho\}$, with $\rho'_i = \rho_i$ or $\tilde{\rho}_i$, is in the support of π. Further, these exhaust the support of π. More generally, we extend the definition of support as in [Tad5]: if π is a finite-length representation and $\{\rho_1, \ldots, \rho_k; \rho\}$ is in the support of π' for every irreducible subquotient π' of π, we say that $\{\rho_1, \ldots, \rho_k; \rho\}$ is in the support of π.

If $\psi_1, \ldots, \psi_m$ are inequivalent irreducible unitary supercuspidal representations of $GL_{r_1}(F), \ldots, GL_{r_m}(F)$ with $\psi_i \not\cong \tilde{\psi}_j$ for $i \neq j$ and ψ an irreducible cuspidal representation of $S_r(F)$, we define the following multiset:

$$\mathcal{S}(\psi_1, \ldots, \psi_m; \psi) = \{\nu^\alpha \psi_1, \nu^\alpha \tilde{\psi}_1\}_{\alpha \in \mathbb{R}} \cup \cdots \cup \{\nu^\alpha \psi_m, \nu^\alpha \tilde{\psi}_m\}_{\alpha \in \mathbb{R}} \cup \{\psi\}.$$

(Note that if ψ_0 is an irreducible unitary supercuspidal representation of $GL_{r_0}(F)$ with $\psi_0 \not\cong \tilde{\psi}_0$, then $\nu^\alpha \psi_0 \rtimes \psi$ is irreducible for all $\alpha \in \mathbb{R}$.)

The following fact about induced representations for $GL_n(F)$ will be needed later.

THEOREM 2.1. (Zelevinsky) *Let $\psi_1, \ldots, \psi_m$ be inequivalent irreducible unitary supercuspidal representations of $GL_{r_1}(F), \ldots, GL_{r_m}(F)$, respectively. Let $\tau(\psi_1), \ldots, \tau(\psi_m)$ be irreducible $GL(F)$-representations supported on $\{\nu^\alpha \psi_1\}_{\alpha \in \mathbb{R}}$, $\ldots, \{\nu^\alpha \psi_m\}_{\alpha \in \mathbb{R}}$, respectively. Then, $\tau(\psi_1) \times \cdots \times \tau(\psi_m)$ is irreducible.*

Proof. See Proposition 8.5 of [Zel]. □

Next, we introduce some notation for Jacquet modules. If π is a representation of some $S_n(F)$ and α is a partition of $m \leq n$, let $s_\alpha(\pi)$ denote the Jacquet module with respect to M_α. Note that, by abuse of notation, we also allow s_α to be applied to representations of M_β if $M_\beta > M_\alpha$ (cf. Section 2.1, [B-Z]). Further, we define s_{GL} as in [Tad1]: for $\pi \leq \rho_1 \times \cdots \times \rho_k \rtimes \sigma$ with ρ_i a supercuspidal representation of $GL_{n_i}(F)$ and σ a supercuspidal representation of $S_{n-m}(F)$, we set $s_{GL}(\pi) = s_{(n_1+\cdots+n_k)}(\pi)$. We will occasionally use similar notation for representations of $GL_n(F)$. If $\alpha = (n_1, \ldots, n_k)$ is a partition of $m \leq n$, $GL_n(F)$ has a standard parabolic subgroup with Levi factor $L_\alpha \cong GL_{n_1}(F) \times \cdots \times GL_{n_k}(F) \times GL_{n-m}(F)$ (L_α consists of block-diagonal matrices; the corresponding parabolic subgroup of block upper triangular matrices). If π is a representation of $GL_n(F)$, we let $r_\alpha(\pi)$ denote the Jacquet module of π with respect to L_α.

We now give two theorems on Jacquet modules. Here, the notation is as in [B-Z]. If L is the Levi factor of a standard parabolic subgroup $P_L \subset G$, let i_{GL} denote induction from P_L to G; r_{LG} the functor taking the Jacquet module with respect to P_L.

THEOREM 2.2. (Frobenius reciprocity) *Let G be a connected reductive p-adic group, $P = MU$ a parabolic subgroup, ρ an admissible representation of M, π an admissible representation of G. Then*

$$Hom_M(r_{MG}(\pi), \rho) \cong Hom_G(\pi, i_{GM}(\rho)).$$

THEOREM 2.3. (Bernstein-Zelevinsky/Casselman) *Let G be a connected reductive p-adic group, MU and NV standard parabolic subgroups. Let ρ be an* *admissible representation of M. Then, $r_{NG} \circ i_{GM}(\rho)$ has a composition series with factors*

$$i_{NN'} \circ w \circ r_{M'M}(\rho)$$

where $M' = M \cap w^{-1}(N), N' = w(M) \cap N$, and $w \in W^{MN} = \{w \in W \mid w(P_{min} \cap M) \subset P_{min},\ w^{-1}(P_{min} \cap N) \subset P_{min}\}$.

Proof. See [B-Z] or [Cas]. □

3. The Langlands classification and the Casselman criteria. In this section, we review the Langlands classification ([B-W], [Sil]) and the Casselman criteria for square-integrability/temperedness ([Cas]) in the context of $S_n(F)$. This discussion is largely based on those in [Tad3], [Tad1]. We also give some consequences which will be needed later.

We begin by giving the Langlands classification for $S_n(F)$ (cf. [Tad3]). Suppose that δ is an irreducible essentially tempered representation of $GL_n(F)$. Then, there is an $\varepsilon(\delta) \in \mathbb{R}$ such that $\nu^{-\varepsilon(\delta)}\delta$ is unitarizable. Let $\delta_1, \ldots, \delta_k$ be irreducible essentially tempered representations of $GL_{n_1}(F), \ldots, GL_{n_k}(F)$ satisfying $\varepsilon(\delta_1) < \cdots < \varepsilon(\delta_k) < 0$ and τ a tempered representation of $S_{n-m}(F)$. Then, $\delta_1 \times \cdots \times \delta_k \rtimes \tau$ has a unique irreducible subrepresentation which we denote by $L(\delta_1, \ldots, \delta_k; \tau)$. At times, it will be convenient not to worry about listing $\delta_1, \ldots, \delta_k$ in increasing order. So, if $\delta_1, \ldots, \delta_k$ satisfy $\varepsilon(\delta_i) < 0$ and $\varepsilon(\delta_i) \neq \varepsilon(\delta_j)$ for $i \neq j$, then there is some permutation $\delta_{\sigma_1}, \ldots, \delta_{\sigma_k}$ which satisfies $\varepsilon(\delta_{\sigma_1}) < \cdots < \varepsilon(\delta_{\sigma_k}) < 0$. Then, by $L(\delta_1, \ldots, \delta_k; \tau)$ we mean $L(\delta_{\sigma_1}, \ldots, \delta_{\sigma_k}; \tau)$. Note that we use the Langlands classification in the subrepresentation setting rather than the quotient setting for the following reason: in the subrepresentation setting, $\delta_1 \otimes \cdots \otimes \delta_k \otimes \tau \leq s_{(n_1,\ldots,n_k)}(L(\delta_1, \ldots, \delta_k; \tau))$ (by Frobenius reciprocity).

The Langlands classification for $GL_n(F)$ is similar. If $\delta_1, \ldots, \delta_k$ are irreducible essentially tempered representations satisfying $\varepsilon(\delta_1) < \cdots < \varepsilon(\delta_k)$, then $\delta_1 \times \cdots \times \delta_k$ has a unique irreducible subrepresentation which we denote by $\mathcal{L}(\delta_1, \ldots, \delta_k)$. As with $S_n(F)$, there will be times when it is convenient not to worry about listing $\delta_1, \ldots, \delta_k$ in increasing order, so we adopt the same convention for $GL_n(F)$: if $\varepsilon(\delta_i) \neq \varepsilon(\delta_j)$ for $i \neq j$, we let $\mathcal{L}(\delta_1, \ldots, \delta_k) = \mathcal{L}(\delta_{\sigma_1}, \ldots, \delta_{\sigma_k})$ for the permutation satisfying $\varepsilon(\delta_{\sigma_1}) < \cdots < \varepsilon(\delta_{\sigma_k})$.

Next, by analogy with [K-R], we make the following definition.

Definition 3.1. Consider a representation of the form

$$\chi = (\psi_1^{(1)} \otimes \cdots \otimes \psi_1^{(k_1)}) \otimes (\psi_2^{(1)} \otimes \cdots \otimes \psi_2^{(k_2)}) \otimes \cdots \otimes (\psi_m^{(1)} \otimes \cdots \otimes \psi_m^{(k_m)})$$

with $\psi_i^{(j)}$ irreducible $GL(F)$-representations. By a shuffle of χ, we mean any permutation on χ such that for all i, $\psi_i^{(1)}, \ldots, \psi_i^{(k_i)}$ appear in that order. (That is, the relative orders in the parenthesized pieces are preserved.) If ψ is an irreducible representation of $S(F)$ and $\chi' = \chi \otimes \psi$, we define a shuffle of χ' to be anything of the form $sh(\chi) \otimes \psi$, where $sh(\chi)$ is a shuffle of χ.

If χ is a representation of a standard Levi M of $GL(F)$ and $sh(\chi)$ is a shuffle of χ, we let $sh(M)$ denote the Levi subgroup of $GL(F)$ for $sh(\chi)$. Similarly, if χ' is a representation of a standard Levi M' of $S(F)$ and $sh(\chi')$ a shuffle of χ', we let $sh(M')$ denote the corresponding Levi subgroup of $S(F)$.

Also, let us make the following definition:

Definition 3.2. Suppose π is a representation of $S_n(F)$. Consider

$$\mathcal{M}_{min} = \{M \text{ standard Levi} \mid r_{MG}(\pi) \neq 0 \text{ but } r_{LG}(\pi) = 0 \text{ for all } L < M\}.$$

Note that these are all conjugate. Then, formally set

$$s_{min}(\pi) = \sum_{M \in \mathcal{M}_{min}} r_{MG}(\pi).$$

We now briefly review the Casselman criteria for temperedness/square-integrability. Let π be an irreducible representation of $S_n(F)$ and $\nu^{\alpha_1}\rho_1 \otimes \cdots \otimes \nu^{\alpha_k}\rho_k \otimes \sigma \leq s_{min}(\pi)$, with ρ_i an irreducible unitary supercuspidal representation of $GL_{m_i}(F)$, σ an irreducible supercuspidal representation of $S_m(F)$ and $\alpha_i \in \mathbb{R}$. Then, if π is tempered,

$$\begin{cases} m_1\alpha_1 \geq 0, \\ m_1\alpha_1 + m_2\alpha_2 \geq 0, \\ \vdots \\ m_1\alpha_1 + m_2\alpha_2 + \cdots + m_k\alpha_k \geq 0. \end{cases}$$

Conversely, if the corresponding inequalities hold for every component of $s_{min}(\pi)$, then π is tempered. The criteria for square-integrability is the same except that the weak inequalities are replaced by strict inequalities.

The criteria for $GL_n(F)$ are similar. If π is an irreducible tempered representation of $GL_n(F)$ and $\nu^{\alpha_1}\rho_1 \otimes \cdots \otimes \nu^{\alpha_k}\rho_k \leq r_{MG}(\pi)$ for some standard Levi M which is minimal for π, then

$$\begin{cases} m_1\alpha_1 \geq 0, \\ m_1\alpha_1 + m_2\alpha_2 \geq 0, \\ \vdots \\ m_1\alpha_1 + m_2\alpha_2 + \cdots + m_{k-1}\alpha_{k-1} \geq 0, \\ m_1\alpha_1 + m_2\alpha_2 + \cdots + m_k\alpha_k = 0. \end{cases}$$

Conversely, if these inequalities hold for all such $\nu^{\alpha_1}\rho_1 \otimes \cdots \otimes \nu^{\alpha_k}\rho_k$, then π is tempered. Again, the criteria for square-integrability are the same except that the weak inequalities are replaced by strict inequalities.

We record the following corollary for future use.

COROLLARY 3.3. *Suppose T is an irreducible tempered $S(F)$-representation. Write $s_{GL}(T) = \sum_i \mathcal{L}(\Delta_i) \otimes \sigma$ (possibly $\mathcal{L}(\Delta_i) = \Delta_i$ essentially tempered). If $\Delta_i = \nu^{\gamma_1^{(i)}}\delta_1^{(i)} \otimes \cdots \otimes \nu^{\gamma_\ell^{(i)}}\delta_\ell^{(i)}$, with $\delta_1^{(i)}, \ldots, \delta_\ell^{(i)}$ tempered and $\gamma_j^{(i)} \in \mathbb{R}$, then $\gamma_1^{(i)} \geq 0$.*

Proof. This is an easy consequence of the Casselman criteria. If $\delta_1^{(i)}$ is a representation of $GL_{n_1^{(i)}}(F)$ and $\gamma_1^{(i)} < 0$, we can use $n_1^{(i)}$ to violate the Casselman criteria. Let $\nu^{x_1}\rho_1^{(i)} \otimes \cdots \otimes \nu^{x_{k_1^{(i)}}}\rho_{k_1^{(i)}}^{(i)}$ be a component of a minimal Jacquet module for $\delta_1^{(i)}$, with $\rho_j^{(i)}$ an irreducible unitary supercuspidal representation of $GL_{m_j^{(i)}}(F)$ and $x_1, \ldots, x_{k_1^{(i)}} \in \mathbb{R}$. Since $\delta_1^{(i)}$ is unitary, $m_1^{(i)}x_1 + \cdots + m_{k_1^{(i)}}^{(i)}x_{k_1^{(i)}} = 0$. Now, $\nu^{x_1+\gamma_1^{(i)}}\rho_1^{(i)} \otimes \cdots \otimes \nu^{x_{k_1^{(i)}}+\gamma_1^{(i)}}\rho_{k_1^{(i)}}^{(i)}$ is the corresponding component of a minimal Jacquet module for $\nu^{\gamma_1^{(i)}}\delta_1^{(i)}$. However,

$$m_1^{(i)}(x_1 + \gamma_1^{(i)}) + \cdots + m_{k_1^{(i)}}^{(i)}(x_{k_1^{(i)}} + \gamma_1^{(i)}) = n_1^{(i)}\gamma_1^{(i)} < 0,$$

contradicting the temperedness of T. □

The following results will also be needed later.

LEMMA 3.4. *Suppose $\nu^{\alpha_1}\delta_1 \otimes \cdots \otimes \nu^{\alpha_d}\delta_d \otimes T$ is Langlands data for $S_n(F)$. Let M denote the Levi factor for $\nu^{\alpha_1}\delta_1 \otimes \cdots \otimes \nu^{\alpha_d}\delta_d \otimes T$. Then,*

$$m(\nu^{\alpha_1}\delta_1 \otimes \cdots \otimes \nu^{\alpha_d}\delta_d \otimes T, r_{MG}(\nu^{\alpha_1}\delta_1 \times \cdots \times \nu^{\alpha_d}\delta_d \rtimes T)) = 1.$$

Further,

$$m(\nu^{\alpha_1}\delta_1 \otimes \cdots \otimes \nu^{\alpha_d}\delta_d \otimes S, r_{MG}(\nu^{\alpha_1}\delta_1 \times \cdots \times \nu^{\alpha_d}\delta_d \rtimes T)) = 0$$

for any irreducible $S \neq T$.

Proof. For convenience, we shall do the case $d = 2$; the general case is similar. Let $M \cong GL_m(F) \times GL_n(F) \times S_r(F)$ be the standard Levi for $\nu^{\alpha_1}\delta_1 \otimes \nu^{\alpha_2}\delta_2 \otimes T$.

Consider another term $\nu^{\beta_1}\tau_1 \otimes \nu^{\beta_2}\tau_2 \otimes S \leq r_{MG}(\nu^{\alpha_1}\delta_1 \times \nu^{\alpha_2}\delta_2 \rtimes T)$. By Theorem 2.3, it has the form $i_{MM'} \circ w \circ r_{M''M}(\nu^{\alpha_1}\delta_1 \otimes \nu^{\alpha_2}\delta_2 \otimes T)$ for $w \in W^{MM}$. We claim that if $w \neq 1$, then $\beta_1 > \alpha_1$ or $\beta_1 = \alpha_1$ and $\beta_2 > \alpha_2$. We do this by analyzing the components of $s_{min}(\nu^{\beta_1}\tau_1 \otimes \nu^{\beta_2}\tau_2 \otimes S)$. Note that a component of $r_{M_{min}M}(\nu^{\beta_1}\tau_1 \otimes \nu^{\beta_2}\tau_2 \otimes S)$, for $M_{min} \in \mathcal{M}_{min}$ with $M_{min} < M$, has the form $w \cdot \chi'$ for some $\chi' \leq r_{M'_{min}M}(\nu^{\alpha_1}\delta_1 \otimes \nu^{\alpha_2}\delta_2 \otimes T)$, where $w \in W^{MM_{min}}$ and $M'_{min} = w^{-1}(M_{min}) < M$.

Let $(\nu^{x_1}\rho_1 \otimes \cdots \otimes \nu^{x_j}\rho_j) \otimes (\nu^{y_1}\rho'_1 \otimes \cdots \otimes \nu^{y_k}\rho'_k) \otimes (\nu^{z_1}\rho''_1 \otimes \cdots \otimes \nu^{z_\ell}\rho''_\ell \otimes \sigma)$ be in $s_{min}(\nu^{\alpha_1}\delta_1 \otimes \nu^{\alpha_2}\delta_2 \otimes T)$. Here, $\nu^{x_1}\rho_1 \otimes \ldots \otimes \nu^{x_j}\rho_j$ comes from $\nu^{\alpha_1}\delta_1$, etc., and $\rho_1, \ldots, \rho_j, \rho'_1, \ldots, \rho'_k, \rho''_1, \ldots, \rho''_\ell$ are irreducible unitary supercuspidal representations of $GL_{m_1}(F), \ldots, GL_{m_j}(F), GL_{n_1}(F), \ldots, GL_{n_k}(F), GL_{r_1}(F), \ldots, GL_{r_\ell}(F)$, σ an irreducible supercuspidal representation of $S_{r_0}(F)$, and $x_1, \ldots, x_j, y_1, \ldots, y_k, z_1, \ldots, z_\ell \in \mathbb{R}$.

Consider a component $\chi \leq s_{min}(\nu^{\alpha_1}\delta_1 \times \nu^{\alpha_2}\delta_2 \rtimes T)$ associated to a Weyl conjugate of $(\nu^{x_1}\rho_1 \otimes \cdots \otimes \nu^{x_j}\rho_j) \otimes (\nu^{y_1}\rho'_1 \otimes \cdots \otimes \nu^{y_k}\rho'_k) \otimes (\nu^{z_1}\rho''_1 \otimes \cdots \otimes \nu^{z_\ell}\rho''_\ell \otimes \sigma)$. In the first t terms in the tensor product decomposition of χ, there must be a shuffle of

(1) $\nu^{x_1}\rho_1 \otimes \nu^{x_2}\rho_2 \otimes \cdots \otimes \nu^{x_{i_1}}\rho_{i_1}$

(2) $\nu^{-x_j}\tilde{\rho}_j \otimes \nu^{-x_{j-1}}\tilde{\rho}_{j-1} \otimes \cdots \otimes \nu^{-x_{i_2}}\tilde{\rho}_{i_2}$

(3) $\nu^{y_1}\rho'_1 \otimes \nu^{y_2}\rho'_2 \otimes \cdots \otimes \nu^{y_{i_3}}\rho'_{i_3}$

(4) $\nu^{-y_k}\widetilde{\rho'_k} \otimes \nu^{-y_{k-1}}\widetilde{\rho'_{k-1}} \otimes \cdots \otimes \nu^{-y_{i_4}}\widetilde{\rho'_{i_4}}$

(5) $\nu^{z_1}\rho''_1 \otimes \nu^{z_2}\rho''_2 \otimes \cdots \otimes \nu^{z_{i_5}}\rho''_{i_5}$,

with $i_1 + (j - i_2 + 1) + i_3 + (k - i_4 + 1) + i_5 = t$. If $\chi \leq s_{min}(\nu^{\beta_1}\tau_1 \otimes \nu^{\beta_2}\tau_2 \otimes S)$, we must have

$$(m_1 + \cdots + m_{i_1}) + (m_j + \cdots + m_{i_2}) + (n_1 + \cdots + n_{i_3}) + (n_k + \cdots n_{i_4}) + (r_1 + \cdots + r_{i_5}) = m.$$

We claim $\beta_1 > \alpha_1$, except for the trivial case (i.e., $i_1 = 1$, $i_2 = i_3 = i_4 = i_5 = 0$).

To see this, first observe that by considering the central character, we have

$$m\alpha_1 = m_1x_1 + \cdots + m_jx_j$$

and

$$\begin{aligned} m\beta_1 = {} & m_1x_1 + m_2x_2 + \cdots + m_{i_1}x_{i_1} + (-m_jx_j - m_{j-1}x_{j-1} - \cdots - m_{i_2}x_{i_2}) \\ & + n_1y_1 + n_2y_2 + \cdots + n_{i_3}y_{i_3} + (-n_ky_k - n_{k-1}y_{k-1} - \cdots - n_{i_4}y_{i_4}) \\ & + r_1z_1 + r_2z_2 + \cdots + r_{i_5}z_{i_5}. \end{aligned}$$

Also, the Casselman criteria for $GL(F)$ tell us

$$m_1x_1 + m_2x_2 + \cdots + m_{i_1}x_{i_1} \geq (m_1 + m_2 + \cdots + m_{i_1})\alpha_1.$$

Since $m_1x_1 + \cdots + m_jx_j = m\alpha_1$, we also get that $m_jx_j + m_{j-1}x_{j-1} + \cdots + m_{i_2}x_{i_2} \leq$

$(m_j + m_{j-1} + \cdots + m_{i_2})\alpha_1$, or

$$\begin{aligned} -m_jx_j - m_{j-1}x_{j-1} - \cdots - m_{i_2}x_{i_2} &\geq -(m_j + m_{j-1} + \cdots + m_{i_2})\alpha_1 \geq 0 \\ &\geq (m_j + m_{j-1} + \cdots + m_{i_2})\alpha_1. \end{aligned}$$

Similarly, we have

$$n_1y_1 + n_2y_2 + \cdots + n_{i_3}y_{i_3} \geq (n_1 + n_2 + \cdots + n_{i_3})\alpha_2 \geq (n_1 + n_2 + \cdots + n_{i_3})\alpha_1$$

and

$$\begin{aligned} -n_ky_k - n_{k-1}y_{k-1} - \cdots - n_{i_4}y_{i_4} &\geq -(n_k - n_{k-1} - \cdots - n_{i_4})\alpha_2 \geq 0 \\ &\geq (n_k + n_{k-1} + \cdots + n_{i_4})\alpha_1. \end{aligned}$$

Finally, the Casselman criteria for $S(F)$ tell us

$$r_1z_1 + r_2z_2 + \cdots + r_{i_5}z_{i_5} \geq 0 \geq (r_1 + r_2 + \cdots + r_{i_5})\alpha_1.$$

Adding, we get

$$\begin{aligned} &m_1x_1 + m_2x_2 + \cdots + m_{i_1}x_{i_1} + (- m_jx_j - m_{j-1}x_{j-1} - \cdots - m_{i_2}x_{i_2}) + n_1y_1 + n_2y_2 \\ &+ \cdots + n_{i_3}y_{i_3} + (- n_ky_k - n_{k-1}y_{k-1} - \cdots - n_{i_4}y_{i_4}) + r_1z_1 + r_2z_2 + \cdots + r_{i_5}z_{i_5} \\ &\quad \geq [(m_1 + m_2 + \cdots + m_{i_1}) + (m_j + m_{j-1} + \cdots + m_{i_2}) + (n_1 + n_2 + \cdots + n_{i_3}) \\ &\qquad + (n_k + n_{k-1} + \cdots + n_{i_4}) + (r_1 + r_2 + \cdots + r_{i_5})]\alpha_1 \\ &\quad = m\alpha_1, \end{aligned}$$

as claimed. Further, the inequality is strict unless $i_1 = j$ and $i_2 = i_3 = i_4 = i_5 = 0$.

We claimed that $\beta_1 > \alpha_1$ or $\beta_1 = \alpha_1$ and $\beta_2 > \alpha_2$ for $w \neq 1$. We now have $\beta_1 \geq \alpha_1$. If $\beta_1 > \alpha_1$, the claim holds. Suppose $\beta_1 = \alpha_1$. Then the above calculation shows $i_1 = j$ and $i_2 = i_3 = i_4 = i_5 = 0$, i.e., w acts trivially on the first $GL_m(F)$. A similar argument shows that in this case, $\beta_2 \geq \alpha_2$ with equality strict unless w acts trivially on the second $GL_n(F)$. However, the only way for such a w to act trivially on both the first $GL_m(F)$ and second $GL_n(F)$ is if $w = 1$, as claimed. This finishes the proof. □

COROLLARY 3.5. *Suppose* $\Delta = \nu^{\alpha_1}\delta_1 \otimes \cdots \otimes \nu^{\alpha_d}\delta_d$ *with* $\delta_1, \ldots, \delta_d$ *tempered and* $\alpha_1 < \cdots < \alpha_d < 0$. *Let T be a tempered $S(F)$ representation; M the standard Levi for* $\mathcal{L}(\Delta) \otimes T$. *Then,*

$$m(\mathcal{L}(\Delta) \otimes T, r_{MG}(\mathcal{L}(\Delta) \rtimes T)) = 1$$

and

$$m(\mathcal{L}(\Delta)\otimes\mu, r_{MG}(\mathcal{L}(\Delta)\rtimes T)) = 0$$

for any irreducible $\mu \not\cong T$.

Proof. Observe that $r_{LM}(\mathcal{L}(\Delta)\otimes T) \geq \nu^{\alpha_1}\delta_1\otimes\cdots\otimes\nu^{\alpha_d}\delta_d\otimes T$, where L is the standard Levi for $\nu^{\alpha_1}\delta_1\otimes\cdots\otimes\nu^{\alpha_d}\delta_d\otimes T$. Since

$$\mathcal{L}(\Delta)\rtimes T \hookrightarrow \nu^{\alpha_1}\delta_1\times\cdots\times\nu^{\alpha_d}\delta_d\rtimes T,$$

the corollary follows immediately from the preceding lemma. □

4. Duality. In this section, we review Aubert's generalization of the Iwahori-Matsumoto involution. We also give a corollary which will have important consequences later. In particular, it tells us that for an irreducible representation π, one of the following holds:

(1) π is nontempered.

(2) π is tempered with $\hat{\pi}$ nontempered (where $\hat{\pi} = D_G(\pi)$ denotes the dual of π defined by Aubert).

(3) π is a component of $i_{GM}\rho$ for some irreducible unitary supercuspidal ρ.

Some of the main results in this paper will be done by breaking the proof into these three cases (using Langlands data to verify the first case, dualizing that to verify the second, and using R-groups to get the third).

THEOREM 4.1. (Aubert) *Define the operator D_G on the Grothendieck group $R(G)$ by*

$$D_G = \sum_{\Phi\subset\Pi}(-1)^{|\Phi|}i_{GL_\Phi}\circ r_{L_\Phi G},$$

where Π denotes the set of simple roots and for $\Phi\subset\Pi$, L_Φ is the Levi of the standard parabolic obtained by adjoining the simple reflections from Φ to the minimal parabolic. D_G has the following properties:

(1) $D_G\circ\tilde{} = \tilde{}\circ D_G$.

(2) $D_G\circ i_{GL_\Phi} = i_{GL_\Phi}\circ D_{L_\Phi}$.

(3) $r_{L_\Phi G}\circ D_G = Ad(w_\Phi)\circ D_{L_{\Phi'}}\circ r_{L_{\Phi'}G}$, *where w_Φ is the longest element of $W^{AL_\Phi} = \{w\in W \mid w^{-1}(P_{min}\cap L_\Phi)\subset P_{min}\}$ and $\Phi' = w_\Phi^{-1}(\Phi)$.*

(4) $D_G^2 = identity$.

(5) $D_G(\pi) = \pm\pi$ *for supercuspidal π.*

(6) D_G *takes irreducible representations to irreducible representations.*

For convenience, we write $\hat{\pi}$ for $D_G(\pi)$.

Proof. See [Aub]. □

COROLLARY 4.2. *Let π be an irreducible representation of $S_n(F)$. If both π and $\hat{\pi}$ are tempered, then π is a component of $i_{GM}(\rho)$ for some irreducible unitary supercuspidal ρ.*

Proof. Let $\nu^{\alpha_1}\rho_1 \otimes \cdots \otimes \nu^{\alpha_k}\rho_k \otimes \sigma \leq r_{MG}(\pi)$ for some $M \in \mathcal{M}_{min}$, where ρ_i is an irreducible unitary supercuspidal representation of $GL_{r_i}(F)$, $\alpha_i \in \mathbb{R}$, and σ is an irreducible unitary supercuspidal representation of $S_r(F)$. By the Casselman criteria, π tempered implies

$$\begin{aligned} r_1\alpha_1 &\geq 0, \\ r_1\alpha_1 + r_2\alpha_2 &\geq 0, \\ &\vdots \\ r_1\alpha_1 + \cdots + r_k\alpha_k &\geq 0. \end{aligned}$$

On the other hand, we note that (cf. p. 130, [Tad3])

$$\begin{aligned} w_\Phi(\nu^{\alpha_1}\rho_1 \otimes \cdots \otimes \nu^{\alpha_k}\rho_k \otimes \sigma) &= {}^\tau(\nu^{\alpha_1}\rho_1)^{-1} \otimes \cdots \otimes {}^\tau(\nu^{\alpha_k}\rho_k)^{-1} \otimes \sigma \\ &= \nu^{-\alpha_1}\tilde{\rho}_1 \otimes \cdots \otimes \nu^{-\alpha_k}\tilde{\rho}_k \otimes \sigma. \end{aligned}$$

Therefore, by duality, $\nu^{\alpha_1}\rho_1 \otimes \cdots \otimes \nu^{\alpha_k}\rho_k \otimes \sigma \leq r_{MG}(\pi)$ if and only if $\nu^{-\alpha_1}\tilde{\rho}_1 \otimes \cdots \otimes \nu^{-\alpha_k}\tilde{\rho}_k \otimes \sigma \leq r_{MG}(\hat{\pi})$. By the Casselman criteria, this forces

$$\begin{aligned} r_1(-\alpha_1) &\geq 0, \\ r_1(-\alpha_1) + r_2(-\alpha_2) &\geq 0, \\ &\vdots \\ r_1(-\alpha_1) + \cdots + r_k(-\alpha_k) &\geq 0. \end{aligned}$$

Combining these inequalities gives $\alpha_1 = \cdots = \alpha_k = 0$. The corollary follows. □

5. Structure of induced representations. In this section, we recall the μ^* structure of Tadić ([Tad4]). We also use some of the ideas in [Tad3] to give some additional properties of Jacquet modules.

We begin by recalling the μ^* structure of Tadić.

Definition 5.1. (1) If τ is a representation of $GL_k(F)$, set

$$m^*(\tau) = \sum_{i=0}^{k} r_{(i)}(\tau).$$

(2) If π is a representation of $S_n(F)$, set

$$\mu^*(\pi) = \sum_{i=0}^{n} s_{(i)}(\pi).$$

Note that $m^*\colon R \longrightarrow R \otimes R$ and $\mu^*\colon R[S] \longrightarrow R \otimes R[S]$.

If τ_1 and τ_2 are representations of $GL_{m_1}(F), GL_{m_2}(F)$, respectively, let $s(\tau_1 \otimes \tau_2) = \tau_2 \otimes \tau_1$ and $m(\tau_1 \otimes \tau_2) = \tau_1 \times \tau_2$. If τ is a representation of $GL_m(F)$ and ϑ is a representation of $S_n(F)$, let $(\tau_1 \otimes \tau_2) \rtimes (\tau \otimes \vartheta) = (\tau_1 \times \tau) \otimes (\tau_2 \rtimes \vartheta)$. Define $M_S^*\colon R \longrightarrow R \otimes R$ by $M_S^* = (m \otimes 1) \circ (\,\tilde{}\, \otimes m^*) \circ s \circ m^*$ ($\tilde{}$ denotes contragredient).

THEOREM 5.2. (Tadić) *If τ is a representation of $GL_m(F)$ and ϑ a representation of $S_n(F)$, then*

$$\mu^*(\tau \rtimes \vartheta) = M_S^*(\tau) \rtimes \mu^*(\vartheta).$$

Proof. See [Tad4]. □

We mention that this has a counterpart for general linear groups. If we let $(\tau_1 \otimes \tau_2) \times (\tau_1' \otimes \tau_2') = (\tau_1 \times \tau_1') \otimes (\tau_2 \times \tau_2')$, then $m^*(\pi_1 \times \pi_2) = m^*(\pi_1) \times m^*(\pi_2)$. In particular, with multiplication defined by $\times$ and comultiplication by m^*, R becomes a Hopf algebra (cf. Section 1.7, [Zel]). With $\rtimes$ and μ^*, $R[S]$ then acquires the structure of an M_S^*-Hopf module over R (cf. Theorem 7.2, [Tad4]).

COROLLARY 5.3. *Let ρ be a representation of $GL_m(F)$, σ a representation of $S_n(F)$. If*

$$s_{GL}(\rho \rtimes 1) = \sum_i \rho_i \otimes 1$$

and

$$s_{GL}(\sigma) = \sum_j \tau_j \otimes \psi,$$

then

$$s_{GL}(\rho \rtimes \sigma) = \sum_{i,j} (\rho_i \times \tau_j) \otimes \psi.$$

Note that for supercuspidal ψ, $s_{GL}(\rho \rtimes 1) = \sum_i \rho_i \otimes 1$ if and only if $s_{GL}(\rho \rtimes \psi) = \sum_i \rho_i \otimes \psi$.

LEMMA 5.4. (shuffling)

(1) *Suppose π is an irreducible representation of $S_n(F)$ such that $r_{MG}(\pi) \geq \chi$, where χ has the form*

$$\begin{aligned}\chi \;=\; &(\nu^{\alpha_{1,1}}\psi_1 \otimes \cdots \otimes \nu^{\alpha_{1,j_1}}\psi_1) \otimes (\nu^{\alpha_{2,1}}\psi_2 \otimes \cdots \otimes \nu^{\alpha_{2,j_2}}\psi_2) \otimes \cdots \\ &\otimes (\nu^{\alpha_{m,1}}\psi_m \otimes \cdots \otimes \nu^{\alpha_{m,j_m}}\psi_m) \otimes \psi,\end{aligned}$$

with $\psi_1, \dots, \psi_m$ inequivalent irreducible unitary supercuspidal $GL(F)$-representations, ψ an irreducible supercuspidal $S(F)$-representation, and $\alpha_{i,j} \in \mathbb{R}$. Then, for every shuffle $sh(\chi)$ of χ (cf. Definition 3.1), we have $r_{sh(M)G}(\pi) \geq sh(\chi)$. Further, if $r_{sh(M)G}(\pi) \geq sh(\chi)$ for any such shuffle, we necessarily have $r_{MG}(\pi) \geq \chi$, and therefore $r_{sh(M)G}(\pi) \geq sh(\chi)$ for every such shuffle.

(2) *$i_{GM}(\chi) \cong i_{Gsh(M)}(sh(\chi))$ for any shuffle of χ.*

A similar result holds for representations of $GL_n(F)$.

Proof. Recall that for $i \neq j$, $\nu^{\alpha_i}\psi_i \times \nu^{\alpha_j}\psi_j$ is irreducible, so that $\nu^{\alpha_i}\psi_i \times \nu^{\alpha_j}\psi_j \cong \nu^{\alpha_j}\psi_j \times \nu^{\alpha_i}\psi_i$.

For the first claim, we begin by separating out the key observation. Suppose

$$\lambda_1 \otimes \cdots \otimes \lambda_{k-1} \otimes (\nu^{\alpha_i}\psi_i \otimes \nu^{\alpha_j}\psi_j) \otimes \lambda_{k+2} \otimes \cdots \otimes \lambda_t \otimes \psi \leq r_{M'G}(\pi),$$

with $i \neq j$, λ_ℓ cuspidal for all ℓ and $M' \in \mathcal{M}_{min}$ the Levi factor of the appropriate parabolic subgroup. Then, we claim

$$\begin{gathered}
\lambda_1 \otimes \cdots \otimes \lambda_{k-1} \otimes (\nu^{\alpha_i}\psi_i \otimes \nu^{\alpha_j}\psi_j) \otimes \lambda_{k+2} \otimes \cdots \otimes \lambda_t \otimes \psi \leq r_{M'G}(\pi) \\
\Downarrow \\
\lambda_1 \otimes \cdots \otimes \lambda_{k-1} \otimes (\nu^{\alpha_i}\psi_i \times \nu^{\alpha_j}\psi_j) \otimes \lambda_{k+2} \otimes \cdots \otimes \lambda_t \otimes \psi \leq r_{M''G}(\pi) \\
\Downarrow \\
\lambda_1 \otimes \cdots \otimes \lambda_{k-1} \otimes (\nu^{\alpha_j}\psi_j \otimes \nu^{\alpha_i}\psi_i) \otimes \lambda_{k+2} \otimes \cdots \otimes \lambda_t \otimes \psi \leq r_{M'''G}(\pi),
\end{gathered}$$

where M'' is obtained by fusing the $k^{\underline{th}}$ and $k+1^{\underline{st}}$ blocks of M'; M''' by switching them. The first implication follows from the fact that the only representation of $GL_{r_i+r_j}(F)$ which has $\nu^{\alpha_i}\psi_i \otimes \nu^{\alpha_j}\psi_j$ as a component of its Jacquet module is $\nu^{\alpha_i}\psi_i \times \nu^{\alpha_j}\psi_j$. The second implication follows from the fact that $\nu^{\alpha_i}\psi_i \times \nu^{\alpha_j}\psi_j \cong \nu^{\alpha_j}\psi_j \times \nu^{\alpha_i}\psi_i$ has $\nu^{\alpha_j}\psi_j \otimes \nu^{\alpha_i}\psi_i$ as a component of its Jacquet module.

At this point, the first part follows immediately from the observation that every shuffle of χ may be obtained by a sequence of transpositions like that above.

As for the second part—the proof is similar to that of the first. The key observation is that

$$\begin{aligned}
&\lambda_1 \times \cdots \times \lambda_{k-1} \times (\nu^{\alpha_i}\psi_i \times \nu^{\alpha_j}\psi_j) \times \lambda_{k+2} \times \cdots \times \lambda_t \rtimes \psi \\
&\quad \cong \lambda_1 \times \cdots \times \lambda_{k-1} \times (\nu^{\alpha_j}\psi_j \times \nu^{\alpha_i}\psi_i) \times \lambda_{k+2} \times \cdots \times \lambda_t \rtimes \psi. \qquad \square
\end{aligned}$$

We now give a simple but useful lemma.

LEMMA 5.5. *Suppose π is an irreducible representation of G, λ an irreducible representation of M and $\pi \hookrightarrow i_{GM}(\lambda)$. If $L > M$, then there is an irreducible* *representation ρ of L such that*

(1) $\pi \hookrightarrow i_{GL}(\rho)$.

(2) *ρ is a subquotient of $i_{LM}(\lambda)$.*

Proof. Since $\pi \hookrightarrow i_{GM}(\lambda) \cong i_{GL}(i_{LM}(\lambda))$, Frobenius reciprocity tells us

$$0 \neq \mathrm{Hom}_G(\pi, i_{GM}(\lambda)) \cong \mathrm{Hom}_L(r_{LG}(\pi), i_{LM}(\lambda)).$$

Therefore, there exists an irreducible quotient ρ of $r_{LG}(\pi)$ which is a subquotient of $i_{LM}(\lambda)$. By Frobenius reciprocity,

$$0 \neq \mathrm{Hom}_L(r_{LG}(\pi), \rho) \cong \mathrm{Hom}_G(\pi, i_{GL}(\rho)).$$

Therefore, $\pi \hookrightarrow i_{GL}(\rho)$, as needed. □

COROLLARY 5.6. *Let $\psi_1, \ldots, \psi_m$ be inequivalent irreducible unitary supercuspidal representations of $GL_{r_1}(F), \ldots, GL_{r_m}(F)$. Let $\tau(\psi_i)$ be an irreducible $GL(F)$-representation supported on $\{\nu^\alpha \psi_i\}_{\alpha\in\mathbb{R}}$. Let M be the standard Levi of $GL_n(F)$ corresponding to $\tau(\psi_1) \otimes \cdots \otimes \tau(\psi_m)$.*

(1) *$\tau(\psi_1) \times \cdots \times \tau(\psi_m)$ is irreducible.*

(2) *$m(\tau(\psi_1) \otimes \cdots \otimes \tau(\psi_m), r_{MG}(\tau(\psi_1) \times \cdots \times \tau(\psi_m))) = 1.$*

Further, if $\tau(\psi_i)$ is an irreducible $GL(F)$-representation supported on $\{\nu^\alpha \psi_i\}_{\alpha\in\mathbb{R}}$, then

$$m(\tau'(\psi_1) \otimes \cdots \otimes \tau'(\psi_m), r_{MG}(\tau(\psi_1) \times \cdots \times \tau(\psi_m))) = 0$$

if $\tau'(\psi_1) \otimes \cdots \otimes \tau'(\psi_m) \neq \tau(\psi_1) \otimes \cdots \otimes \tau(\psi_m)$.

(3) *If π is an irreducible representation of $GL_n(F)$ and $r_{MG}(\pi) \geq \tau(\psi_1) \otimes \cdots \otimes \tau(\psi_m)$, then $\pi = \tau(\psi_1) \times \cdots \times \tau(\psi_m)$.*

Proof. (1) follows immediately from the work of Zelevinsky (cf. Theorem 2.1). For (2), we use the Bernstein-Zelevinsky/Casselman characterization of Jacquet modules (cf. Theorem 2.3). Observe that $1 \in W^{MM}$ gives rise to $\tau(\psi_1)\otimes\cdots\otimes\tau(\psi_m) \leq r_{MG}(\tau(\psi_1)\times\cdots\times\tau(\psi_m))$. Any $w \neq 1$ in W^{MM} (or more generally, W^{MM_0}, where $M_0 = GL_{r_1}(F)\times\cdots\times GL_{r_1}(F)\times\cdots\times GL_{r_m}(F)\times\cdots\times GL_{r_m}(F)$) gives rise to a nontrivial shuffle. Therefore, $\tau(\psi_1) \otimes \cdots \otimes \tau(\psi_m)$ appears with multiplicity one and no other $\tau'(\psi_1) \otimes \cdots \otimes \tau'(\psi_m)$ appears. For (3), suppose $r_{MG}(\pi) \geq \tau(\psi_1) \otimes \cdots \otimes \tau(\psi_m)$ for some irreducible π. By the subrepresentation theorem and shuffling, we have

$$\pi \hookrightarrow \nu^{\alpha_{1,1}}\psi_1 \times \cdots \times \nu^{\alpha_{1,j_1}}\psi_1 \times \cdots \times \nu^{\alpha_{m,1}}\psi_m \times \cdots \times \nu^{\alpha_{m,j_m}}\psi_m$$

for some $\alpha_{i,j} \in \mathbb{R}$. By the preceding lemma, this forces

$$\pi \hookrightarrow \tau'(\psi_1) \times \cdots \times \tau'(\psi_m)$$

for some irreducible $\tau'(\psi_i)$. Therefore, by irreducibility, $\pi = \tau'(\psi_1) \times \cdots \times \tau'(\psi_m)$. By (2), this forces $\tau'(\psi_i) = \tau(\psi_i)$ for $i = 1, \ldots, m$, as needed. □

LEMMA 5.7. *Let $\psi_1, \ldots, \psi_m$ be inequivalent irreducible unitary supercuspidal representations of $GL_{r_1}(F), \ldots, GL_{r_m}(F)$ with $\psi_i \not\cong \tilde{\psi}_j$ for $i \neq j$ and ψ an irreducible supercuspidal representation of $S_r(F)$. Suppose π is an irreducible representation of $S_n(F)$ with support contained in $\mathcal{S}(\psi_1, \ldots, \psi_m; \psi)$. Then, there are irreducible $GL(F)$-representations $\rho_1, \ldots, \rho_m$ with supports contained in $\mathcal{S}(\psi_1), \ldots, \mathcal{S}(\psi_m)$, respectively, such that*

$$\pi \hookrightarrow \rho_1 \times \cdots \times \rho_m \rtimes \psi.$$

Further, for any $i \in \{1, \ldots m\}$, there is an irreducible subquotient σ_i of $\rho_i \rtimes \psi$ such that

$$\pi \hookrightarrow \rho_1 \times \cdots \times \rho_{i-1} \times \rho_{i+1} \times \cdots \times \rho_m \rtimes \sigma_i.$$

Proof. First, by the subrepresentation theorem, there exist supercuspidal $\lambda_1, \ldots, \lambda_t \in \mathcal{S}(\psi_1, \ldots, \psi_m)$ such that

$$\pi \hookrightarrow \lambda_1 \times \cdots \times \lambda_t \rtimes \psi.$$

Shuffling, we get

$$\pi \hookrightarrow (\lambda_1^{(1)} \times \cdots \times \lambda_{j_1}^{(1)}) \times \cdots \times (\lambda_1^{(m)} \times \cdots \times \lambda_{j_m}^{(m)}) \rtimes \psi,$$

with $\lambda_i^{(k)} \in \mathcal{S}(\psi_k)$. Rewrite this as $\pi \hookrightarrow i_{GM}(\lambda')$. If L denotes the standard Levi corresponding to $(\lambda_1^{(1)} \times \cdots \times \lambda_{j_1}^{(1)}) \otimes \cdots \otimes (\lambda_1^{(m)} \times \cdots \times \lambda_{j_m}^{(m)}) \otimes \psi$, then Lemma 5.5 finishes the first claim.

For the second claim, observe that a shuffling argument (which holds since $\rho_i \times \rho_j \cong \rho_j \times \rho_i$ is irreducible for $i \neq j$, cf. Theorem 2.1) gives us

$$\pi \hookrightarrow \rho_1 \times \cdots \times \rho_m \rtimes \psi \cong \rho_1 \times \cdots \times \rho_{i-1} \times \rho_{i+1} \times \cdots \times \rho_m \rtimes (\rho_i \rtimes \psi).$$

Then, we can use Lemma 5.5 to see that there is an irreducible subquotient σ_i of $\rho_i \rtimes \psi$ such that

$$\pi \hookrightarrow \rho_1 \times \cdots \times \rho_{i-1} \times \rho_{i+1} \times \cdots \times \rho_m \rtimes \sigma_i. \qquad \square$$

6. R-groups. As mentioned in Section 4, the main results in this paper will be done in three cases. In order to address the eventual third case, we explicitly examine representations of the form

$$\underbrace{\psi_1 \times \cdots \times \psi_1}_{n_1} \times \underbrace{\psi_2 \times \cdots \times \psi_2}_{n_2} \times \cdots \times \underbrace{\psi_m \times \cdots \times \psi_m}_{n_m} \rtimes \psi,$$

where $\psi_1, \dots, \psi_m$ are inequivalent irreducible unitary supercuspidal representations of $GL_{r_1}(F), \dots, GL_{r_m}(F)$, ψ an irreducible supercuspidal representation of $S_r(F)$. The results we need follow fairly easily from the work of Goldberg ([Gol]). (We will say a bit more about these in Case 3 of the proof of Proposition 8.4.)

PROPOSITION 6.1. *Let ψ_0 be an irreducible unitary supercuspidal representation of $GL_{r_0}(F)$, ψ an irreducible supercuspidal representation of $S_r(F)$.*

(1) *If $\psi_0 \rtimes \psi$ is irreducible, then*

$$\underbrace{\psi_0 \times \cdots \times \psi_0}_{n} \rtimes \psi$$

is irreducible as well. Further,

$$s_{GL}(\underbrace{\psi_0 \times \cdots \times \psi_0}_{n} \rtimes \psi) = \underbrace{(\psi_0 + \tilde{\psi}_0) \times \cdots \times (\psi_0 + \tilde{\psi}_0)}_{n} \otimes \psi.$$

(2) *If $\psi_0 \rtimes \psi$ is reducible, then*

$$\underbrace{\psi_0 \times \cdots \times \psi_0}_{n} \rtimes \psi = T_1^{(n)}(\psi_0; \psi) \oplus T_2^{(n)}(\psi_0; \psi),$$

with $T_1^{(n)}(\psi_0; \psi) \not\cong T_2^{(n)}(\psi_0; \psi)$ both irreducible. Further,

$$s_{GL}(T_i^{(n)}(\psi_0; \psi)) = 2^{n-1} \cdot \psi_0 \times \cdots \times \psi_0 \otimes \psi$$

for $i = 1, 2$. (Recall that $\psi_0 \rtimes \psi$ reducible implies $\psi_0 \cong \tilde{\psi}_0$.)

Proof. All but the last claim follow immediately from [Gol] (in particular, Theorems 6.4 and 6.5 (1)).

To show $s_{GL}(T_i^{(n)}(\psi_0; \psi)) = 2^{n-1} \cdot \psi_0 \times \cdots \times \psi_0 \otimes \psi$, we induct on n. If $n = 1$, note that

$$s_{GL}(T_1^{(1)}(\psi_0; \psi)) = s_{GL}(T_2^{(1)}(\psi_0; \psi)) = \psi_0 \otimes \psi,$$

as needed. Suppose $n > 1$ and $s_{GL}(T_i^{(n)}(\psi_0;\psi)) = 2^{n-1}\cdot\psi_0\times\cdots\times\psi_0\otimes\psi$. Observe that

$$\underbrace{\psi_0\times\cdots\times\psi_0}_{n+1}\rtimes\psi = \psi_0\rtimes T_1^{(n)}(\psi_0;\psi) + \psi_0\rtimes T_2^{(n)}(\psi_0;\psi).$$

Since $\psi_0\times\cdots\times\psi_0\rtimes\psi$ has only two components, both $\psi_0\rtimes T_1^{(n)}(\psi_0;\psi)$ and $\psi_0\rtimes T_2^{(n)}(\psi_0;\psi)$ are irreducible. So, we can write $T_i^{(n+1)}(\psi_0;\psi) = \psi_0\rtimes T_i^{(n)}(\psi_0;\psi)$. Since $|s_{min}(T_1^{(n)}(\psi_0;\psi))| = |s_{min}(T_2^{(n)}(\psi_0;\psi))|$, this shows $|s_{min}(T_1^{(n+1)}(\psi_0;\psi))| = |s_{min}(T_2^{(n+1)}(\psi_0;\psi))|$ (using $|\cdot|$ to denote the length of the representation). As $s_{GL}(\underbrace{\psi_0\times\cdots\times\psi_0}_{n+1}\rtimes\psi) = 2^{n+1}\cdot\underbrace{\psi_0\times\cdots\times\psi_0}_{n+1}\otimes\psi$, this forces

$$s_{GL}(T_i^{(n+1)}(\psi_0;\psi)) = 2^n\cdot\underbrace{\psi_0\times\cdots\times\psi_0}_{n+1}\otimes\psi,$$

as needed. □

PROPOSITION 6.2. *Let $\psi_1,\dots,\psi_m$ be inequivalent irreducible unitary supercuspidal representations of $GL_{r_1}(F),\dots,GL_{r_m}(F)$, ψ an irreducible supercuspidal representation of $S_r(F)$. Without loss of generality, suppose that $\psi_1\rtimes\psi,\dots,\psi_k\rtimes\psi$ are reducible and $\psi_{k+1}\rtimes\psi,\dots,\psi_m\rtimes\psi$ are irreducible. Then, $\underbrace{\psi_1\times\cdots\times\psi_1}_{n_1}\times\cdots\times\underbrace{\psi_m\times\cdots\times\psi_m}_{n_m}\rtimes\psi$ has 2^k inequivalent components. Write*

$$\underbrace{\psi_1\times\cdots\times\psi_1}_{n_1}\times\cdots\times\underbrace{\psi_m\times\cdots\times\psi_m}_{n_m}\rtimes\psi = \sum_{(i_1,\dots,i_k)} T_{(i_1,\dots,i_k)}^{(n_1,\dots,n_m)}(\psi_1,\dots,\psi_m;\psi),$$

where $(i_1,\dots,i_k)$ runs over all k-tuples with $i_j\in\{1,2\}$. Further,

$$\begin{aligned}s_{GL}(T_{(i_1,\dots,i_k)}^{(n_1,\dots,n_m)}(\psi_1,\dots,\psi_m)) = {} & (2^{n_1-1}\cdot\underbrace{\psi_1\times\cdots\times\psi_1}_{n_1})\times\cdots\times(2^{n_k-1}\cdot\underbrace{\psi_k\times\cdots\times\psi_k}_{n_k})\\ &\times[\underbrace{(\psi_{k+1}+\tilde{\psi}_{k+1})\times\cdots\times(\psi_{k+1}+\tilde{\psi}_{k+1})}_{n_{k+1}}]\times\cdots\\ &\times[\underbrace{(\psi_m+\tilde{\psi}_m)\times\cdots\times(\psi_m+\tilde{\psi}_m)}_{n_m}]\otimes\psi.\end{aligned}$$

Proof. We induct on $(n_1,\dots,n_m)$.

First, suppose $(n_1,\dots,n_m) = (1,\dots,1)$. Then,

$$s_{GL}(\psi_1\times\cdots\times\psi_m\rtimes\psi) = 2^k\cdot\psi_1\times\cdots\times\psi_k\times(\psi_{k+1}+\tilde{\psi}_{k+1})\times\cdots\times(\psi_m+\tilde{\psi}_m)\otimes\psi.$$

Note that ψ_i may or may not be equivalent to $\tilde{\psi}_i$ for $i > k$ (if so, $\psi_i+\tilde{\psi}_i = 2\cdot\psi_i$), but in any case, the right-hand side above is a sum of 2^m irreducible representations.

Let π_0 be a component of $\psi_1 \times \cdots \times \psi_m \rtimes \psi$. Then,

$$\begin{aligned}\pi_0 &\hookrightarrow \psi_1 \times \cdots \times \psi_k \rtimes (\psi_{k+1} \times \cdots \times \psi_m \rtimes \psi)\\ &\cong \psi_1 \times \cdots \times \psi_k \rtimes T^{(1,\ldots,1)}(\psi_{k+1}, \ldots, \psi_m; \psi)\end{aligned}$$

by the irreducibility of $\psi_{k+1} \times \cdots \times \psi_m \rtimes \psi$. Further, irreducibility tells us

$$s_{GL}(T^{(1,\ldots,1)}(\psi_{k+1}, \ldots, \psi_m; \psi)) = (\psi_{k+1} + \tilde{\psi}_{k+1}) \times \cdots \times (\psi_m + \tilde{\psi}_m) \otimes \psi.$$

Thus,

$$s_{GL}(\pi_0) \geq \psi_1 \times \cdots \times \psi_k \times (\psi_{k+1} + \tilde{\psi}_{k+1}) \times \cdots \times (\psi_m + \tilde{\psi}_m) \otimes \psi.$$

On the other hand, by [Gol], we know that $\psi_1 \times \cdots \times \psi_m \rtimes \psi$ has exactly 2^k components. As noted above,

$$s_{GL}(\psi_1 \times \cdots \times \psi_m \rtimes \psi) = 2^k \cdot \psi_1 \times \cdots \times \psi_k \times (\psi_{k+1} + \tilde{\psi}_{k+1}) \times \cdots \times (\psi_m + \tilde{\psi}_m) \otimes \psi.$$

Thus, the only possibility is

$$s_{GL}(\pi_0) = \psi_1 \times \cdots \times \psi_k \times (\psi_{k+1} + \tilde{\psi}_{k+1}) \times \cdots \times (\psi_m + \tilde{\psi}_m) \otimes \psi$$

for every component π_0 of $\psi_1 \times \cdots \times \psi_m \rtimes \psi$. This finishes the case $(n_1, \ldots, n_m) = (1, \ldots, 1)$.

We now turn to the inductive step. By the same argument as in Proposition 6.1, we can write

$$T^{(n_1,\ldots,n_{j-1},n_j+1,n_{j+1},\ldots,n_m)}_{(i_1,\ldots,i_k)}(\psi_1, \ldots, \psi_m; \psi) = \psi_j \rtimes T^{(n_1,\ldots,n_m)}_{(i_1,\ldots,i_k)}(\psi_1, \ldots, \psi_m; \psi).$$

Therefore, by Corollary 5.3,

$$\begin{aligned}&s_{GL}(T^{(n_1,\ldots,n_{j-1},n_j+1,n_{j+1},\ldots,n_m)}_{(i_1,\ldots,i_k)}(\psi_1, \ldots, \psi_m; \psi))\\ &\quad = (\psi_j + \tilde{\psi}_j) \times s_{GL}(T^{(n_1,\ldots,n_m)}_{(i_1,\ldots,i_k)}(\psi_1, \ldots, \psi_m; \psi)).\end{aligned}$$

The inductive step then follows immediately. □

7. Modified Jacquet modules. In this section, we introduce a variation Tadić's μ^* operator (cf. Section 5) which will be useful in dealing with support questions. We then give a number of its basic properties. We close with a definition which will play a key role in the correspondence set up in the next section.

Definition 7.1. Let $\psi_1, \dots, \psi_m$ be inequivalent irreducible unitary supercuspidal representations of $GL_{r_1}(F), \dots, GL_{r_m}(F)$ with $\psi_i \ncong \tilde{\psi_j}$ for $i \neq j$; ψ an irreducible supercuspidal representation of $S_r(F)$. Suppose π is a representation of $S_n(F)$ supported in $\mathcal{S}(\psi_1, \dots, \psi_m; \psi)$. Write $\mu^*(\pi) = \sum_i \rho_i \otimes \sigma_i$, a sum of irreducible representations in $R \otimes R[S]$. Then, let $\mu^*_{\psi_1,\dots,\psi_k}(\pi)$ denote the sum of every $\rho_i \otimes \sigma_i$ in $\mu^*(\pi)$ such that the support of ρ_i is contained in $\mathcal{S}(\psi_1, \dots, \psi_k)$ and the support of σ_i is contained in $\mathcal{S}(\psi_{k+1}, \dots, \psi_m; \psi)$.

LEMMA 7.2. *If π has support contained in $\mathcal{S}(\psi_1, \dots, \psi_m; \psi)$, then $\mu^*_{\psi_1,\dots,\psi_k}(\pi)$ is nonzero.*

Proof. By Lemma 5.7, we can find $\rho_1, \dots, \rho_m$ irreducible and supported on $\mathcal{S}(\psi_1), \dots, \mathcal{S}(\psi_m)$ such that

$$\pi \hookrightarrow \rho_1 \times \cdots \times \rho_m \rtimes \psi.$$

Using Lemma 5.5, we can find an irreducible subquotient σ of $\rho_{k+1} \times \cdots \times \rho_m \rtimes \psi$ such that

$$\pi \hookrightarrow \rho \rtimes \sigma,$$

where $\rho = \rho_1 \times \cdots \times \rho_k$ (irreducible by Theorem 2.1). By Frobenius reciprocity, $\mu^*_{\psi_1,\dots,\psi_k}(\pi) \geq \rho \otimes \sigma$. □

Definition 7.3. Suppose ρ is a representation of $GL_r(F)$ supported in $\mathcal{S}(\psi_1, \dots, \psi_m)$. Write $M^*_S(\rho) = \sum_i \tau_i \otimes \tau_i'$, a sum of irreducible representations in $R \otimes R$. Let $M^*_{\psi_1,\dots,\psi_k}(\rho)$ denote the sum of every summand $\tau_i \otimes \tau_i'$ in $M^*_S(\rho)$ such that the support of τ_i is contained in $\mathcal{S}(\psi_1, \dots, \psi_k)$ and the support of τ_i' is contained in $\mathcal{S}(\psi_{k+1}, \dots, \psi_m)$.

PROPOSITION 7.4. *Suppose ρ is a representation of $GL(F)$ with support contained in $\mathcal{S}(\psi_1, \dots, \psi_m)$ and σ a representation of $S(F)$ with support contained in $\mathcal{S}(\psi_1, \dots, \psi_m; \psi)$. Then,*

$$\mu^*_{\psi_1,\dots,\psi_k}(\rho \rtimes \sigma) = M^*_{\psi_1,\dots,\psi_k}(\rho) \rtimes \mu^*_{\psi_1,\dots,\psi_k}(\sigma).$$

Proof. Recall that

$$\mu^*(\rho \rtimes \sigma) = M^*_S(\rho) \rtimes \mu^*(\sigma)$$

(cf. Theorem 5.2). Write

$$M_S^*(\rho) = \sum_i \rho_i' \otimes \rho_i'',$$

$$\mu^*(\sigma) = \sum_j \rho_j \otimes \sigma_j.$$

Then,

$$\mu^*(\rho \rtimes \sigma) = \sum_{i,j} (\rho_i' \times \rho_j) \otimes (\rho_i'' \rtimes \sigma_j).$$

Now, in order for $(\rho_i' \times \rho_j) \otimes (\rho_i'' \rtimes \sigma_j)$ to contribute to $\mu^*_{\psi_1,\ldots,\psi_k}(\rho \rtimes \sigma)$, we must have ρ_i', ρ_j with support contained in $\mathcal{S}(\psi_1, \ldots, \psi_k)$, ρ_i'' with support contained in $\mathcal{S}(\psi_{k+1}, \ldots, \psi_m)$, and σ_j with support contained in $\mathcal{S}(\psi_{k+1}, \ldots, \psi_m; \psi)$. This means $\rho_i' \otimes \rho_i''$ is in $M^*_{\psi_1,\ldots,\psi_k}(\rho)$ and $\rho_j \otimes \sigma_j$ is in $\mu^*_{\psi_1,\ldots,\psi_k}(\sigma)$. □

COROLLARY 7.5. *Suppose ρ has support contained in $\mathcal{S}(\psi_1, \ldots, \psi_k)$ and σ has support contained in $\mathcal{S}(\psi_{k+1}, \ldots, \psi_m; \psi)$.*

(1) *Suppose*

$$s_{GL}(\rho \rtimes 1) = \sum_i \tau_i \otimes 1.$$

Then,

$$\mu^*_{\psi_1,\ldots,\psi_k}(\rho \rtimes \sigma) = \sum_i \tau_i \otimes \sigma.$$

We note that $s_{GL}(\rho \rtimes 1) = \sum_i \tau_i \otimes 1$ if and only if $s_{GL}(\rho \rtimes \psi) = \sum_i \tau_i \otimes \psi$ (cf. Corollary 5.3).

(2) *Suppose*

$$\rho \rtimes \psi = \sum_i \mu_i$$

and

$$s_{GL}(\sigma) = \sum_j \xi_j \otimes \psi.$$

Then,

$$\mu^*_{\psi_{k+1},\ldots,\psi_m}(\rho \rtimes \sigma) = \sum_{i,j} \xi_j \otimes \mu_i.$$

Proof. Part 1 of the corollary follows from the proposition once we establish that

$$M^*_{\psi_1,\dots,\psi_k}(\rho) = \sum_i \tau_i \otimes 1$$

and

$$\mu^*_{\psi_1,\dots,\psi_k}(\sigma) = 1 \otimes \sigma.$$

The second of these is trivial.

Since ρ has support contained in $\mathcal{S}(\psi_1, \dots, \psi_k)$, $M^*_{\psi_1,\dots,\psi_k}(\rho)$ consists of everything of the form $\tau \otimes 1$ in $M^*_S(\rho)$. Now, suppose

$$M^*_S(\rho) = \sum_i \tau'_i \otimes \tau''_i.$$

Then,

$$\mu^*(\rho \rtimes 1) = \sum_i \tau'_i \otimes (\tau''_i \rtimes 1).$$

Now, $\tau'_i \otimes (\tau''_i \rtimes 1)$ contributes to $s_{GL}(\rho \rtimes 1)$ if and only if τ''_i is the trivial representation of $GL_0(F)$, i.e., $\tau'_i \otimes \tau''_i$ is of the form $\tau \otimes 1$. So, if we write $s_{GL}(\rho \rtimes 1) = \sum_i \tau_i \otimes 1$, we get

$$M^*_{\psi_1,\dots,\psi_k}(\rho) = \sum_i \tau_i \otimes 1.$$

For the second part, we need to check that

$$M^*_{\psi_{k+1},\dots,\psi_m}(\rho) = 1 \otimes \rho$$

and

$$\mu^*_{\psi_{k+1},\dots,\psi_m}(\sigma) = \sum_j \xi_j \otimes \psi.$$

Both are easy. □

Suppose π is an irreducible representation of $S_n(F)$ supported on $\mathcal{S}(\psi_1, \dots, \psi_m; \psi)$. Fix $i \in \{1, \dots, m\}$. By Lemma 5.7, there is an irreducible $\rho_i \otimes \sigma_i$ with ρ_i supported on $\mathcal{S}(\psi_1, \dots, \psi_{i-1}, \psi_{i+1}, \dots, \psi_m)$ and σ_i supported on $\mathcal{S}(\psi_i; \psi)$ such that

$$\pi \hookrightarrow \rho_i \rtimes \sigma_i.$$

Further, by the preceding corollary,

$$\begin{aligned}\mu^*_{\psi_1,\ldots,\psi_{i-1},\psi_{i+1},\ldots,\psi_m}(\pi) &\le \mu^*_{\psi_1,\ldots,\psi_{i-1},\psi_{i+1},\ldots,\psi_m}(\rho_i \rtimes \sigma_i)\\ &= \sum_j (\tau_i)_j \otimes \sigma_i.\end{aligned}$$

Thus, the following definition makes sense:

Definition 7.6. $\psi_i(\pi) = \sigma_i$.

One direction of the correspondence we are after sends π to $(\psi_1(\pi), \ldots, \psi_m(\pi))$.

8. Correspondence, part I. In this section, we establish the correspondence that is the subject of this paper and give a few of its basic properties. The two key results are Propositions 8.1 and 8.4. Again, let $\psi_1, \ldots, \psi_m$ be inequivalent irreducible unitary supercuspidal representations of $GL_{r_1}(F), \ldots, GL_{r_m}(F)$ with $\psi_i \not\cong \tilde{\psi}_j$ for $i \neq j$, ψ an irreducible supercuspidal representation of $S_r(F)$. Suppose that $\sigma_1, \ldots, \sigma_m$ are irreducible $S(F)$-representations supported on $\mathcal{S}(\psi_1; \psi), \ldots, \mathcal{S}(\psi_m; \psi)$, respectively. Proposition 8.1 shows the existence of an irreducible π supported on on $\mathcal{S}(\psi_1, \ldots, \psi_m; \psi)$ with $\psi_i(\pi) = \sigma_i$ for $i = 1, \ldots, m$ (cf. Definition 7.6). Proposition 8.4 then shows that this π is unique. This establishes the basic correspondence.

PROPOSITION 8.1. *Suppose $\sigma_1, \ldots, \sigma_m$ are irreducible $S(F)$-representations supported on $\mathcal{S}(\psi_1; \psi), \ldots, \mathcal{S}(\psi_m; \psi)$, respectively. Choose $\rho_1, \ldots, \rho_m$ irreducible $GL(F)$-representations supported on $\mathcal{S}(\psi_1), \ldots, \mathcal{S}(\psi_m)$, respectively, such that $\sigma_i \hookrightarrow \rho_i \rtimes \psi$. Then, there is a component π of $\rho_1 \times \cdots \times \rho_m \rtimes \psi$ such that $\psi_i(\pi) = \sigma_i$ for $i = 1, \ldots, m$.*

Proof. The proof is by induction on m. If $m = 1$, the result is trivial.

Suppose the result holds for $m-1$. Then, there is an irreducible representation θ which is a component of $\rho_1 \times \cdots \times \rho_{m-1} \rtimes \psi$ such that $\psi_1(\theta) = \sigma_1, \ldots, \psi_{m-1}(\theta) = \sigma_{m-1}$. Consider $\rho_m \rtimes \theta$. Write

$$s_{GL}(\rho_m \rtimes \psi) = \sum_i \tau_i \otimes \psi.$$

Then, by Corollary 7.5,

$$\mu^*_{\psi_m}(\rho_m \rtimes \theta) = \sum_i \tau_i \otimes \theta.$$

In particular, this forces any component π' of $\rho_m \rtimes \theta$ to have $\psi_1(\pi') = \sigma_1, \ldots, \psi_{m-1}(\pi') = \sigma_{m-1}$.

On the other hand, write $\rho_m \rtimes \psi = \sum_i \mu_i$ and $\mu^*_{\psi_1,\ldots,\psi_{m-1}}(\theta) = \sum_j \xi_j \otimes \psi$. Then, by Corollary 7.5,

$$\mu^*_{\psi_1,\ldots,\psi_{m-1}}(\rho_m \rtimes \theta) = \sum_{i,j} \xi_j \otimes \mu_i.$$

Since $\mu_i = \sigma_m$ for some i, we can choose a component π of $\rho_m \rtimes \theta$ such that $\mu^*_{\psi_1,\ldots,\psi_{m-1}}(\pi) \geq \xi_j \otimes \sigma_m$. In particular, $\psi_m(\pi) = \sigma_m$, as needed. □

The following lemma will be enough for now. A more precise version is given in the next section (cf. Proposition 9.1).

LEMMA 8.2. *Suppose π is an irreducible representation of $S_n(F)$ supported on $\mathcal{S}(\psi_1, \ldots, \psi_m; \psi)$. Write $\psi_i(\pi) = \sigma_i$ for $i = 1, \ldots, m$. Suppose*

$$s_{GL}(\sigma_i) = \sum_j c_j(\psi_i)\tau_j(\psi_i) \otimes \psi,$$

where $\tau_j(\psi_i)$ is an irreducible representation of $GL(F)$ supported on $\mathcal{S}(\psi_i)$ and $c_j(\psi_i)$ is its multiplicity. Then,

$$s_{GL}(\pi) = \sum_{j_1,\ldots,j_m} a_{j_1,\ldots,j_m}\tau_{j_1}(\psi_1) \times \cdots \times \tau_{j_m}(\psi_m) \otimes \psi,$$

with $a_{j_1,\ldots,j_m} \geq 1$.

Proof. First, we show that $\tau_{j_1}(\psi_1)\times\cdots\times\tau_{j_m}(\psi_m)\otimes\psi$ are the only terms which can appear. Suppose there were another term, say $\tau(\psi_1) \times \cdots \times \tau(\psi_m) \otimes \psi$ with $\tau(\psi_1) \not\cong \tau_j(\psi_1)$ for any j. Since $\tau(\psi_1)\times\cdots\times\tau(\psi_m) \cong \tau(\psi_2)\times\cdots\times\tau(\psi_m)\times\tau(\psi_1)$ (irreducible), $r_{M_\alpha G}(\pi) \geq (\tau(\psi_2)\times\cdots\times\tau(\psi_m))\otimes\tau(\psi_1)\otimes\psi$ for the appropriate M_α. There must be some term $(\tau(\psi_2) \times \cdots \times \tau(\psi_m)) \otimes \sigma(\psi_1; \psi)$ in $\mu^*_{\psi_2,\ldots,\psi_m}(\pi)$ with $s_\alpha((\tau(\psi_2)\times\cdots\times\tau(\psi_m))\otimes\sigma(\psi_1; \psi)) \geq (\tau(\psi_2)\times\cdots\times\tau(\psi_m))\otimes\tau(\psi_1)\otimes\psi$. However, since $\tau(\psi_1)\otimes\psi \not\leq s_{GL}(\sigma_1)$, we must have $\sigma(\psi_1; \psi) \not\cong \sigma_1$. This contradicts $\psi_1(\pi) = \sigma_1$ (Corollary 7.5).

Next, we show that every $\tau_{j_1}(\psi_1) \times \cdots \times \tau_{j_m}(\psi_m) \otimes \psi$ occurs at least once. The proof is by induction on m. The case $m = 1$ is trivial. Suppose it holds for

$m-1$. Choose ρ irreducible such that $\pi \hookrightarrow \rho \rtimes \sigma_m$ (cf. Lemma 5.7). Then,

$$\begin{gathered}
\mu^*_{\psi_1,\dots,\psi_{m-1}}(\pi) \geq \rho \otimes \sigma_m \\
\Downarrow \\
s_{\alpha_1}(\pi) \geq \sum_j \rho \otimes \tau_j(\psi_m) \otimes \psi \\
\Downarrow \\
s_{\alpha_2}(\pi) \geq \sum_j \tau_j(\psi_m) \otimes \rho \otimes \psi \\
\Downarrow \\
\mu^*_{\psi_m}(\pi) \geq \sum_j \tau_j(\psi_m) \otimes \theta_j,
\end{gathered}$$

where M_{α_1} is the standard Levi for $\rho \otimes \tau_j(\psi_m) \otimes \psi$, M_{α_2} the standard Levi for $\tau_j(\psi_m) \otimes \rho \otimes \psi$, and θ_j some $S(F)$-representation supported on $\mathcal{S}(\psi_1, \dots, \psi_{m-1}; \psi)$. Now, $\psi_i(\theta_j) = \psi_i(\pi) = \sigma_i$ for $i = 1, \dots, m-1$. So, by the inductive hypothesis,

$$s_{GL}(\theta_j) \geq \sum_{j_1,\dots,j_{m-1}} \tau_{j_1}(\psi_1) \times \cdots \times \tau_{j_{m-1}}(\psi_{m-1}) \otimes \psi.$$

Therefore, if M_α denotes the standard Levi for $\tau_{j_m}(\psi_m) \otimes (\tau_{j_1}(\psi_1) \times \cdots \times \tau_{j_{m-1}}(\psi_{m-1})) \otimes \psi$,

$$\begin{aligned}
s_\alpha(\pi) &\geq \sum_{j_m} \sum_{j_1,\dots,j_{m-1}} \tau_{j_m}(\psi_m) \otimes \tau_{j_1}(\psi_1) \times \cdots \times \tau_{j_{m-1}}(\psi_{m-1}) \otimes \psi \\
&\Downarrow \quad \text{(irreducibility)} \\
s_{GL}(\pi) &\geq \sum_{j_1,\dots,j_m} \tau_{j_m}(\psi_m) \times \tau_{j_1}(\psi_1) \times \cdots \times \tau_{j_{m-1}}(\psi_{m-1}) \otimes \psi \\
&= \sum_{j_1,\dots,j_m} \tau_{j_1}(\psi_1) \times \cdots \times \tau_{j_m}(\psi_m) \otimes \psi,
\end{aligned}$$

as needed. □

COROLLARY 8.3. *Suppose π is an irreducible representation of $S_n(F)$ supported on $\mathcal{S}(\psi_1, \dots, \psi_m; \psi)$. Write $\psi_i(\pi) = \sigma_i$ for $i = 1, \dots, m$. Then, π is tempered (resp. square-integrable) if and only if $\sigma_1, \dots, \sigma_m$ are all tempered (resp. square-integrable).*

Proof. Write

$$s_{min}(\sigma_i) = \sum_j d_j(\psi_i)[\chi_j(\psi_i) \otimes \psi],$$

with $d_j(\psi_i)$ the multiplicity of $\chi_j(\psi_i) \otimes \psi$. Further, write

$$\chi_j(\psi_i) = \nu^{\alpha_{i,j}^{(1)}} \psi_i^{(j,1)} \otimes \cdots \otimes \nu^{\alpha_{i,j}^{(k_j)}} \psi_i^{(j,k_j)},$$

where $\psi_i^{(j,\ell)}$ is ψ_i or $\tilde{\psi}_i$ for all (j, ℓ) (only the exponents $\alpha_{i,j}^{(\ell)}$ actually matter in the proof).

π tempered $\Rightarrow$ σ_i tempered for $i = 1, \ldots, m$. To show σ_i tempered, we show $\chi_j(\psi_i) \otimes \psi$ satisfies the Casselman criteria inequalities for all j. By the preceding lemma,

$$s_{min}(\pi) \geq \chi_j(\psi_i) \otimes \chi_1(\psi_1) \otimes \cdots \otimes \chi_1(\psi_{i-1}) \otimes \chi_1(\psi_{i+1}) \otimes \cdots \otimes \chi_1(\psi_m) \otimes \psi.$$

Applying the Casselman criteria to this term, we get

$$\begin{aligned} r_i\alpha_{i,j}^{(1)} &\geq 0, \\ r_i\alpha_{i,j}^{(1)} + r_i\alpha_{i,j}^{(2)} &\geq 0, \\ &\vdots \\ r_i\alpha_{i,j}^{(1)} + \cdots + r_i\alpha_{i,j}^{(k_i)} &\geq 0, \\ r_i\alpha_{i,j}^{(1)} + \cdots + r_i\alpha_{i,j}^{(k_i)} + r_1\alpha_{1,1}^{(1)} &\geq 0, \\ &\vdots . \end{aligned}$$

The first k_i of these are exactly the inequalities the Casselman criteria requires of $\chi_j(\psi_i) \otimes \psi$ for temperedness of σ_i. Since this argument works for all i, j, we have π tempered implies $\sigma_1, \ldots, \sigma_m$ all tempered.

$\sigma_1, \ldots, \sigma_m$ all tempered $\Rightarrow$ π tempered. First, the Casselman criteria for σ_i tells us that for all $\chi_j(\psi_i) \otimes \psi$,

$$\begin{aligned} r_i\alpha_{i,j}^{(1)} &\geq 0, \\ r_i\alpha_{i,j}^{(1)} + r_i\alpha_{i,j}^{(2)} &\geq 0, \\ &\vdots \\ r_i\alpha_{i,j}^{(1)} + \cdots + r_i\alpha_{i,j}^{(k_i)} &\geq 0. \end{aligned}$$

Now, consider a $\chi \otimes \psi \leq s_{min}(\pi)$. Write $\chi \otimes \psi = \nu^{\alpha_1}\psi_{i_1} \otimes \cdots \otimes \nu^{\alpha_k}\psi_{i_k} \otimes \psi$ where $k = k_1 + \cdots + k_m$. Then, we need

$$\begin{aligned} r_{i_1}\alpha_1 &\geq 0, \\ r_{i_1}\alpha_1 + r_{i_2}\alpha_2 &\geq 0, \\ &\vdots \\ r_{i_1}\alpha_1 + \cdots + r_{i_\ell}\alpha_\ell &\geq 0, \\ &\vdots \\ r_{i_1}\alpha_1 + \cdots + r_{i_k}\alpha_k &\geq 0. \end{aligned}$$

We show the general inequality $r_{i_1}\alpha_1+\cdots+r_{i_\ell}\alpha_\ell \geq 0$. By the preceding lemma, we know that χ is a shuffle of some $\chi_{j_1}(\psi_1) \otimes \cdots \otimes \chi_{j_m}(\psi_m)$. Therefore, rearranging, we get

$$\begin{aligned} r_{i_1}\alpha_1 + \ldots + r_{i_\ell}\alpha_\ell \quad &= (r_1\alpha_{1,j_1}^{(1)} + \cdots + r_1\alpha_{1,j_1}^{(\ell_1)}) \\ &+(r_2\alpha_{2,j_2}^{(1)} + \cdots + r_2\alpha_{2,j_2}^{(\ell_2)}) \\ &+\cdots \\ &+(r_m\alpha_{m,j_m}^{(1)} + \cdots + r_m\alpha_{m,j_m}^{(\ell_m)}). \end{aligned}$$

But, by the Casselman criteria inequalities for $\sigma_1, \ldots, \sigma_m$, each of the parenthesized terms is ≥ 0. Thus, we have

$$r_{i_1}\alpha_1 + \cdots + r_{i_\ell}\alpha_\ell \geq 0,$$

as needed.

The proof for square integrability works the same way. □

PROPOSITION 8.4. *Suppose π and π' are irreducible representations of $S_n(F)$ supported on $\mathcal{S}(\psi_1, \ldots, \psi_m; \psi)$. If $\psi_i(\pi) = \psi_i(\pi')$ for $i = 1, \ldots, m$, then $\pi \cong \pi'$.*

Proof. The proof is by induction on k, where k is the parabolic rank of an element of $\mathcal{M}_{min}$. The case $k = 1$ is trivial. We consider three cases. (We note that the third case is done directly, and does not require either of the first two cases or the inductive hypothesis.)

Case 1. *π nontempered.* In this case, there is Langlands (subrepresentation) data $\lambda(\pi)$ and $\pi \hookrightarrow i_{GM}(\lambda(\pi))$. Write

$$\begin{aligned} \lambda(\pi) \ = \ &(\nu^{\alpha_1}\tau_1(\psi_1) \times \cdots \times \nu^{\alpha_1}\tau_1(\psi_m)) \otimes \cdots \otimes (\nu^{\alpha_\ell}\tau_\ell(\psi_1) \times \cdots \times \nu^{\alpha_\ell}\tau_\ell(\psi_m)) \\ &\otimes T(\psi_1, \ldots, \psi_m; \psi), \end{aligned}$$

where $\tau_i(\psi_j)$ is tempered and has support contained in $\mathcal{S}(\psi_j)$ (n.b.: $\tau_i(\psi_j)$ may be the trivial representation of $GL_0(F)$), $T(\psi_1, \dots, \psi_m; \psi)$ is a tempered representation of some $S_{n'}(F)$, and $\alpha_1 < \cdots < \alpha_\ell < 0$. Write

$$T(\psi_i; \psi) = \psi_i(T(\psi_1, \dots, \psi_m; \psi)).$$

By the preceding corollary, $T(\psi_i; \psi)$ is tempered. Write

$$\lambda(\sigma(\psi_i; \psi)) = \nu^{\alpha_1}\tau_1(\psi_i) \otimes \cdots \otimes \nu^{\alpha_\ell}\tau_\ell(\psi_i) \otimes T(\psi_i; \psi).$$

Then, $\lambda(\sigma(\psi_i; \psi))$ is the Langlands data for some representation which we call $\sigma(\psi_i; \psi)$. We begin by showing that $\sigma(\psi_i; \psi) = \psi_i(\pi)$.

By Lemma 5.7, there are irreducible representations $\rho(\psi_1), \dots, \rho(\psi_{m-1})$ such that

$$T(\psi_1, \dots, \psi_m; \psi) \hookrightarrow \rho(\psi_1) \times \cdots \times \rho(\psi_{m-1}) \rtimes T(\psi_m; \psi).$$

Therefore,

$$\begin{aligned}
\pi &\hookrightarrow (\nu^{\alpha_1}\tau_1(\psi_1) \times \cdots \times \nu^{\alpha_1}\tau_1(\psi_m)) \\
&\qquad\qquad \times \cdots \times (\nu^{\alpha_\ell}\tau_\ell(\psi_1) \times \cdots \times \nu^{\alpha_\ell}\tau_\ell(\psi_m)) \rtimes T(\psi_1, \dots, \psi_m; \psi) \\
&\Downarrow \\
\pi &\hookrightarrow (\nu^{\alpha_1}\tau_1(\psi_1) \times \cdots \times \nu^{\alpha_1}\tau_1(\psi_m)) \times \cdots \times (\nu^{\alpha_\ell}\tau_\ell(\psi_1) \times \cdots \times \nu^{\alpha_\ell}\tau_\ell(\psi_m)) \\
&\qquad\qquad \times \rho(\psi_1) \times \cdots \times \rho(\psi_{m-1}) \rtimes T(\psi_m; \psi) \\
&\Downarrow \\
\pi &\hookrightarrow (\nu^{\alpha_1}\tau_1(\psi_1) \times \cdots \times \nu^{\alpha_\ell}\tau_\ell(\psi_1) \times \rho(\psi_1)) \\
&\qquad \times \cdots \times (\nu^{\alpha_1}\tau_1(\psi_{m-1}) \times \cdots \times \nu^{\alpha_\ell}\tau_\ell(\psi_{m-1}) \times \rho(\psi_{m-1})) \\
&\qquad\qquad \rtimes (\nu^{\alpha_1}\tau_1(\psi_m) \times \cdots \times \nu^{\alpha_\ell}\tau_\ell(\psi_m) \rtimes T(\psi_m; \psi)).
\end{aligned}$$

Therefore, by Frobenius reciprocity,

$$\begin{aligned}
r_{M'G}(\pi) \geq\ & \nu^{\alpha_1}\tau_1(\psi_1) \otimes \cdots \otimes \nu^{\alpha_\ell}\tau_\ell(\psi_1) \otimes \rho(\psi_1) \\
& \otimes \cdots \otimes \nu^{\alpha_1}\tau_1(\psi_{m-1}) \otimes \cdots \otimes \nu^{\alpha_\ell}\tau_\ell(\psi_{m-1}) \otimes \rho(\psi_{m-1}) \\
& \otimes \nu^{\alpha_1}\tau_1(\psi_m) \otimes \cdots \otimes \nu^{\alpha_\ell}\tau_\ell(\psi_m) \otimes T(\psi_m; \psi).
\end{aligned}$$

Thus, there is an irreducible $R(\psi_1, \dots, \psi_{m-1}) \otimes S(\psi_m; \psi)$ with $R(\psi_1, \dots, \psi_{m-1})$ a $GL(F)$-representation supported on $\mathcal{S}(\psi_1, \dots, \psi_{m-1})$ and $S(\psi_m; \psi)$ an $S(F)$-representation supported on $\mathcal{S}(\psi_m; \psi)$ such that $R(\psi_1, \dots, \psi_{m-1}) \otimes S(\psi_m; \psi) \leq \mu^*_{\psi_m}(\pi)$ and

$$\begin{aligned}
& r_{M'M}(R(\psi_1, \dots, \psi_{m-1}) \otimes S(\psi_m; \psi)) \\
& \quad \geq \nu^{\alpha_1}\tau_1(\psi_1) \otimes \cdots \otimes \nu^{\alpha_\ell}\tau_\ell(\psi_1) \otimes \rho(\psi_1) \\
& \quad\quad \otimes \cdots \otimes \nu^{\alpha_1}\tau_1(\psi_{m-1}) \otimes \cdots \otimes \nu^{\alpha_\ell}\tau_\ell(\psi_{m-1}) \otimes \rho(\psi_{m-1}) \\
& \quad\quad \otimes \nu^{\alpha_1}\tau_1(\psi_m) \otimes \cdots \otimes \nu^{\alpha_\ell}\tau_\ell(\psi_m) \otimes T(\psi_m; \psi).
\end{aligned}$$

Further,

$$\begin{aligned}\mu^*_{\psi_m}(\pi) \leq \mu^*_{\psi_m}([\nu^{\alpha_1}\tau_1(\psi_1) \times \cdots \times \nu^{\alpha_\ell}\tau_\ell(\psi_1) \times \rho(\psi_1) \\ \times \cdots \times \nu^{\alpha_1}\tau_1(\psi_{m-1}) \times \cdots \times \nu^{\alpha_\ell}\tau_\ell(\psi_{m-1}) \times \rho(\psi_{m-1})] \\ \rtimes[\nu^{\alpha_1}\tau_1(\psi_m) \times \cdots \times \nu^{\alpha_\ell}\tau_\ell(\psi_m) \rtimes T(\psi_m;\psi)]).\end{aligned}$$

Therefore, by Corollary 7.5 we see that $S(\psi_m;\psi)$ must be a component of $\nu^{\alpha_1}\tau_1(\psi_m)\times\cdots\times\nu^{\alpha_\ell}\tau_\ell(\psi_m)\rtimes T(\psi_m;\psi)$. By the Langlands classification, the only component of $\nu^{\alpha_1}\tau_1(\psi_m) \times \cdots \times \nu^{\alpha_\ell}\tau_\ell(\psi_m) \rtimes T(\psi_m;\psi)$ containing $\nu^{\alpha_1}\tau_1(\psi_m) \otimes \cdots \otimes \nu^{\alpha_\ell}\tau_\ell(\psi_m) \otimes T(\psi_m;\psi)$ in its Jacquet module is $\sigma(\psi_m;\psi)$. Thus, $S(\psi_m;\psi) = \sigma(\psi_m;\psi)$. On the other hand, $R(\psi_1,\dots,\psi_{m-1}) \otimes S(\psi_m;\psi) \leq \mu^*_{\psi_1,\dots,\psi_{m-1}}(\pi)$, so we must have $S(\psi_m;\psi) = \psi_m(\pi)$. Thus, $\sigma(\psi_m,\psi) = \psi_m(\pi)$, as claimed. The same argument works for $\psi_1,\dots,\psi_{m-1}$.

Next, by Corollary 8.3, we also know that π' is nontempered. Therefore, π' has Langlands data

$$\begin{aligned}\lambda(\pi') \;=\; &(\nu^{\alpha'_1}\tau'_1(\psi_1) \times \cdots \times \nu^{\alpha'_1}\tau'_1(\psi_m)) \otimes \cdots \otimes (\nu^{\alpha'_{\ell'}}\tau'_{\ell'}(\psi_1) \times \cdots \times \nu^{\alpha'_{\ell'}}\tau'_{\ell'}(\psi_m)) \\ &\otimes T'(\psi_1,\dots,\psi_m;\psi).\end{aligned}$$

As above, we can produce Langlands data

$$\lambda(\sigma'(\psi_i;\psi)) = \nu^{\alpha'_1}\tau'_1(\psi_i) \otimes \cdots \otimes \nu^{\alpha'_{\ell'}}\tau'_{\ell'}(\psi_i) \otimes T'(\psi_i;\psi).$$

As above, this is the Langlands data for $\sigma'(\psi_i;\psi) = \psi_i(\pi') = \psi_i(\pi)$. Since the Langlands data is unique, we must have $\ell = \ell'$ and

$$\nu^{\alpha'_j}\tau'_j(\psi_i) = \nu^{\alpha_j}\tau_j(\psi_i),$$

$$T'(\psi_i;\psi) = T(\psi_i;\psi).$$

By the inductive hypothesis, $T'(\psi_1,\dots,\psi_m;\psi) = T(\psi_1,\dots,\psi_m;\psi)$. Therefore, $\lambda(\pi) = \lambda(\pi')$, so $\pi \cong \pi'$. This finishes Case 1.

Case 2. π *tempered*, $\hat{\pi}$ *nontempered.* First, as a consequence of Aubert's work—in particular, Theorem 4.1 (3)—we have $\psi_i(\hat{\pi}) = \widehat{\psi_i(\pi)}$. Therefore, $\psi_i(\pi) = \psi_i(\pi')$ implies $\psi_i(\hat{\pi}) = \psi_i(\widehat{\pi'})$. By Case 1, this means $\hat{\pi} \cong \widehat{\pi'}$, forcing $\pi \cong \pi'$, as needed. This finishes Case 2.

Case 3. π, $\hat{\pi}$ *both tempered.* From Corollary 4.2, we have

$$\pi \hookrightarrow \underbrace{\psi_1 \times \cdots \times \psi_1}_{n_1} \times \cdots \times \underbrace{\psi_m \times \cdots \times \psi_m}_{n_m} \rtimes \psi.$$

Without loss of generality, let us assume $\psi_1 \rtimes \psi, \ldots, \psi_k \rtimes \psi$ are reducible and $\psi_{k+1} \rtimes \psi, \ldots, \psi_m \rtimes \psi$ irreducible.

We now refine the results of Proposition 6.2 a bit. For notational convenience, if $\psi_i \rtimes \psi$ is irreducible, let

$$T_1^{(n_i)}(\psi_i; \psi) = \underbrace{\psi_i \times \cdots \times \psi_i}_{n_i} \rtimes \psi.$$

Now, by Proposition 8.1 (using $\rho_i = \underbrace{\psi_i \times \cdots \times \psi_i}_{n_i}$), we see that there is an irreducible representation $\pi^{(n_1,\ldots,n_m)}_{(j_1,\ldots,j_m)}$ of $S_n(F)$ with

$$\pi^{(n_1,\ldots,n_m)}_{(j_1,\ldots,j_m)} \hookrightarrow \underbrace{\psi_1 \times \cdots \times \psi_1}_{n_1} \times \cdots \times \underbrace{\psi_m \times \cdots \times \psi_m}_{n_m} \rtimes \psi$$

such that $\psi_i(\pi^{(n_1,\ldots,n_m)}_{(j_1,\ldots,j_m)}) = T_{j_i}^{(n_i)}(\psi_i; \psi)$ for $i = 1, \ldots, m$. Since $j_1, \ldots, j_k$ can be either 1 or 2, there are 2^k distinct m-tuples

$$\left(\psi_1(\pi^{(n_1,\ldots,n_m)}_{(j_1,\ldots,j_m)}), \ldots, \psi_m(\pi^{(n_1,\ldots,n_m)}_{(j_1,\ldots,j_m)})\right).$$

As the maps $\psi_1, \ldots, \psi_m$ of Definition 7.6 are well-defined, the $\pi^{(n_1,\ldots,n_m)}_{(j_1,\ldots,j_m)}$ must be distinct and therefore exhaust the 2^k components of

$$\underbrace{\psi_1 \times \cdots \times \psi_1}_{n_1} \times \cdots \times \underbrace{\psi_m \times \cdots \times \psi_m}_{n_m} \rtimes \psi.$$

If we write

$$T^{(n_1,\ldots,n_m)}_{(j_1,\ldots,j_m)}(\psi_1, \ldots, \psi_m; \psi) = \pi^{(n_1,\ldots,n_m)}_{(j_1,\ldots,j_m)},$$

we have

$$\underbrace{\psi_1 \times \cdots \times \psi_1}_{n_1} \times \cdots \times \underbrace{\psi_m \times \cdots \times \psi_m}_{n_m} \rtimes \psi = \bigoplus_{(j_1,\ldots,j_m)} T^{(n_1,\ldots,n_m)}_{(j_1,\ldots,j_m)}(\psi_1, \ldots, \psi_m; \psi)$$

and $T^{(n_1,\ldots,n_m)}_{(j_1,\ldots,j_m)}(\psi_1, \ldots, \psi_m; \psi)$ is characterized by

$$\psi_i(T^{(n_1,\ldots,n_m)}_{(j_1,\ldots,j_m)}(\psi_1, \ldots, \psi_m; \psi)) = T_{j_i}^{(n_i)}(\psi_i; \psi)$$

for all $i = 1, \ldots, m$. We can take this as our definition of $T^{(n_1,\ldots,n_m)}_{(j_1,\ldots,j_m)}(\psi_1, \ldots, \psi_m; \psi)$, but we note that it is consistent with the inductive property used in the proof of Proposition 6.2.

Now, returning to π, suppose $\pi = T^{(n_1,\dots,n_m)}_{(j_1,\dots,j_m)}(\psi_1,\dots,\psi_m;\psi)$. Then,

$$\psi_i(\pi') = \psi_i(\pi) = T^{(n_i)}_{j_i}(\psi_i;\psi)$$

for $i = 1,\dots,m$. By Lemma 8.2, π' must be supported on $\{\psi_1,\dots,\psi_k,\psi_{k+1},\tilde{\psi}_{k+1},\dots,\psi_m,\tilde{\psi}_m\}$. In particular, we can unshuffle (using $\tilde{\psi}_i \rtimes \psi \cong \psi_i \rtimes \psi$ irreducible for $i > k$) to get that

$$\pi' \hookrightarrow \underbrace{\psi_1 \times \cdots \times \psi_1}_{n_1} \times \cdots \times \underbrace{\psi_m \times \cdots \times \psi_m}_{n_m} \rtimes \psi.$$

However, this then forces

$$\pi' = T^{(n_1,\dots,n_m)}_{(j_1,\dots,j_m)}(\psi_1,\dots,\psi_m;\psi) = \pi,$$

as needed. This finishes Case 3 and the Proposition. □

We now make the following definition.

Definition 8.5. Suppose $\sigma_1,\dots,\sigma_m$ are irreducible $S(F)$-representations supported on $\mathcal{S}(\psi_1;\psi),\dots,\mathcal{S}(\psi_m;\psi)$, respectively. Let $\Psi(\sigma_1,\dots,\sigma_m)$ denote the irreducible representation supported on $\mathcal{S}(\psi_1,\dots,\psi_m;\psi)$ which satisfies

$$\psi_i(\Psi(\sigma_1,\dots,\sigma_m)) = \sigma_i$$

for $i = 1,\dots,m$.

COROLLARY 8.6. *Suppose*

$$\sigma_i = L(\nu^{\alpha_1}\tau_1(\psi_i),\dots,\nu^{\alpha_\ell}\tau_\ell(\psi_i); T(\psi_i;\psi))$$

for $i = 1,\dots,m$ (n.b.: recall that $\tau_j(\psi_i)$ may be the trivial representation of $GL_0(F)$). Then,

$$\begin{aligned}\Psi(\sigma_1,\dots,\sigma_m) \;&= L(\nu^{\alpha_1}\tau_1(\psi_1) \times \cdots \times \nu^{\alpha_1}\tau_1(\psi_m),\\ &\quad \dots,\nu^{\alpha_\ell}\tau_\ell(\psi_1) \times \cdots \times \nu^{\alpha_\ell}\tau_\ell(\psi_m); \Psi(T(\psi_1;\psi),\dots,T(\psi_m;\psi))).\end{aligned}$$

(In the quotient setting of the Langlands classification, the same result holds.)

Proof. In the subrepresentation setting, the corollary is an immediate consequence of the proof of Proposition 8.4. For the quotient setting, just observe that if π has subrepresentation data $\nu^{\alpha_1}\tau_1 \otimes \cdots \otimes \nu^{\alpha_\ell}\tau_\ell \otimes T$, then it has quotient data $\nu^{-\alpha_1}\tilde{\tau}_1 \otimes \cdots \otimes \nu^{-\alpha_\ell}\tilde{\tau}_\ell \otimes T$. □

Remark 8.7. It is worth noting that

$$\Psi(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_m) = \Psi(\widetilde{\sigma_1, \ldots}, \sigma_m)$$

(an easy consequence of the fact that ˜ respects induction).

9. Correspondence II. In this section, we give some important properties of the correspondence set up in the preceding section. We begin by giving a refinement to Lemma 8.2, which gives the exact multiplicities which appear in $\mu^*_{\psi_1,\ldots,\psi_k}(\Psi(\sigma_1, \ldots, \sigma_m))$, which in turn allows us to determine how the correspondence behaves with respect to induction. We summarize the basic properties of the correspondence in Theorem 9.3. We close by interpreting the correspondence in terms of tensor product decompositions of R and $R[S]$.

PROPOSITION 9.1. *Suppose that $\sigma(\psi_1;\psi), \ldots, \sigma(\psi_m;\psi)$ are irreducible $S(F)$-representations supported on $\mathcal{S}(\psi_1;\psi), \ldots, \mathcal{S}(\psi_m;\psi)$. Suppose that*

$$s_{GL}(\sigma(\psi_i;\psi)) = \sum_j c_j(\psi_i)\tau_j(\psi_i) \otimes \psi,$$

where $\tau_j(\psi_i)$ is an irreducible $GL(F)$-representation supported on $\mathcal{S}(\psi_i)$ and $c_j(\psi_i)$ is its multiplicity. Then,

$$\begin{aligned} &s_{GL}(\Psi(\sigma(\psi_1;\psi), \ldots, \sigma(\psi_m;\psi))) \\ &\quad = \sum_{j_1,\ldots,j_m} (c_{j_1}(\psi_1)\cdots c_{j_m}(\psi_m))\tau_{j_1}(\psi_1) \times \cdots \times \tau_{j_m}(\psi_m) \otimes \psi. \end{aligned}$$

Further, in general

$$\begin{aligned} &\mu^*_{\psi_1,\ldots,\psi_k}(\Psi(\sigma(\psi_1;\psi), \ldots, \sigma(\psi_m;\psi))) \\ &\quad = \sum_{j_1,\ldots,j_k} (c_{j_1}(\psi_1)\cdots c_{j_k}(\psi_k))\tau_{j_1}(\psi_1) \times \cdots \times \tau_{j_k}(\psi_k) \\ &\qquad \otimes \Psi(\sigma(\psi_{k+1};\psi), \ldots, \sigma(\psi_m;\psi)). \end{aligned}$$

Proof. Write $\pi = \Psi(\sigma(\psi_1;\psi), \ldots, \sigma(\psi_m;\psi))$. The proof follows the same basic lines as Proposition 8.4. Again, the proof is by induction on the parabolic rank of an element of $\mathcal{M}_{min}$. We first look at s_{GL}, then move on to the general claim.

Case 1. *π nontempered.* By Corollary 8.3, at least one of $\sigma(\psi_1;\psi), \ldots,$ $\sigma(\psi_m;\psi)$ is nontempered. Without loss of generality, assume $\sigma(\psi_1;\psi)$ is nontempered. In general, write

$$\sigma(\psi_i;\psi) = L(\Delta(\psi_i); T(\psi_i;\psi)) = L(\nu^{\alpha_1}\delta_1(\psi_i), \dots, \nu^{\alpha_\ell}\delta_\ell(\psi_i); T(\psi_i;\psi))$$

(using the same $\alpha_1, \dots, \alpha_\ell$ for all i, but allowing trivial representation of $GL_0(F)$ to occur). Let $T(\psi_1, \dots, \psi_m; \psi) = \Psi(T(\psi_1;\psi), \dots, T(\psi_m;\psi))$. By Corollary 8.6,

$$\pi = L(\Delta(\psi_1), \dots, \Delta(\psi_m); T(\psi_1, \dots, \psi_m; \psi)).$$

By the inductive hypothesis,

$$\begin{aligned}&\mu^*_{\psi_2,\dots,\psi_m}(L(\Delta(\psi_2), \dots, \Delta(\psi_m); T(\psi_1, \dots, \psi_m; \psi)))\\ &\qquad = \sum_{j_2,\dots,j_m} (c_{j_2}(\psi_2)\cdots c_{j_m}(\psi_m))\tau_{j_2}(\psi_2) \times \cdots \times \tau_{j_m}(\psi_m) \otimes T(\psi_1;\psi).\end{aligned}$$

We proceed by first calculating $\mu^*_{\psi_2,\dots,\psi_m}(\pi)$, then use this to verify the $s_{GL}(\pi)$ claim.

First, if Δ is a set of Langlands data, let $I(\Delta)$ (resp. $\mathcal{I}(\Delta)$ for $GL(F)$) denote the corresponding induced representation. In particular, $L(\Delta) \hookrightarrow I(\Delta)$ (resp. $\mathcal{L}(\Delta) \hookrightarrow \mathcal{I}(\Delta)$ for $GL(F)$). (As with $L(\Delta)$, $\mathcal{L}(\Delta)$, if Δ is not in the proper order for Langlands data, we permute it into order.) Note that (shuffling arguments)

$$\mathcal{I}(\Delta(\psi_1)) \times \cdots \times \mathcal{I}(\Delta(\psi_m)) \cong \mathcal{I}(\Delta(\psi_1), \dots, \Delta(\psi_m)).$$

Thus,

$$\begin{aligned}&\mathcal{L}(\Delta(\psi_1)) \rtimes L(\Delta(\psi_2), \dots, \Delta(\psi_m); T(\psi_1, \dots, \psi_m; \psi))\\ &\qquad \hookrightarrow \mathcal{L}(\Delta(\psi_1)) \rtimes I(\Delta(\psi_2), \dots, \Delta(\psi_m); T(\psi_1, \dots, \psi_m; \psi))\\ &\qquad \cong \mathcal{L}(\Delta(\psi_1)) \times \mathcal{I}(\Delta(\psi_2), \dots, \Delta(\psi_m)) \rtimes T(\psi_1, \dots, \psi_m; \psi))\\ &\qquad \hookrightarrow \mathcal{I}(\Delta(\psi_1)) \times \mathcal{I}(\Delta(\psi_2), \dots, \Delta(\psi_m)) \rtimes T(\psi_1, \dots, \psi_m; \psi))\\ &\qquad \cong \mathcal{I}(\Delta(\psi_1), \Delta(\psi_2), \dots, \Delta(\psi_m)) \rtimes T(\psi_1, \dots, \psi_m; \psi))\\ &\qquad \cong I(\Delta(\psi_1), \dots, \Delta(\psi_m); T(\psi_1, \dots, \psi_m; \psi)).\end{aligned}$$

Since π is the unique irreducible subrepresentation of $I(\Delta(\psi_1), \dots, \Delta(\psi_m); T(\psi_1, \dots, \psi_m; \psi))$, we get

$$(*) \qquad \pi \hookrightarrow \mathcal{L}(\Delta(\psi_1)) \rtimes L(\Delta(\psi_2), \dots, \Delta(\psi_m); T(\psi_1, \dots, \psi_m; \psi)).$$

Further, π appears with multiplicity one in $\mathcal{L}(\Delta(\psi_1)) \rtimes L(\Delta(\psi_2), \dots, \Delta(\psi_m); T(\psi_1, \dots, \psi_m; \psi))$.

Next, by the inductive hypothesis and Proposition 7.4,

$$\mu^*_{\psi_2,\dots,\psi_m}(L(\Delta(\psi_2), \dots, \Delta(\psi_m); T(\psi_1, \dots, \psi_m; \psi)))$$

$$= \sum_{j_2,\dots,j_m} (c_{j_2}(\psi_2) \cdots c_{j_m}(\psi_m))\tau_{j_2}(\psi_2) \times \cdots \times \tau_{j_m}(\psi_m) \otimes T(\psi_1; \psi)$$

$$\Downarrow$$

$$\mu^*_{\psi_2,\dots,\psi_m}(\mathcal{L}(\Delta(\psi_1)) \rtimes L(\Delta(\psi_2), \dots, \Delta(\psi_m); T(\psi_1, \dots, \psi_m; \psi)))$$

$$= \sum_{j_2,\dots,j_m} (c_{j_2}(\psi_2) \cdots c_{j_m}(\psi_m))\tau_{j_2}(\psi_2) \times \cdots \times \tau_{j_m}(\psi_m) \otimes \mathcal{L}(\Delta(\psi_1)) \rtimes T(\psi_1; \psi).$$

Consequently,

$$\begin{gathered} m(\tau_{j_2}(\psi_2) \times \cdots \times \tau_{j_m}(\psi_m) \otimes L(\Delta(\psi_1); T(\psi_1; \psi)), \\ \mu^*_{\psi_2,\dots,\psi_m}(\mathcal{L}(\Delta(\psi_1)) \rtimes L(\Delta(\psi_2), \dots, \Delta(\psi_m); \\ T(\psi_1, \dots, \psi_m; \psi)))) = c_{j_2}(\psi_2) \dots c_{j_m}(\psi_m) \end{gathered}$$

(cf. Corollary 3.5). Any component

$$\pi' \leq \mathcal{L}(\Delta(\psi_1)) \rtimes L(\Delta(\psi_2), \dots, \Delta(\psi_m); T(\psi_1, \dots, \psi_m; \psi))$$

has

$$\psi_i(\pi') = \psi_i(L(\Delta(\psi_2), \dots, \Delta(\psi_m); T(\psi_1, \dots, \psi_m; \psi))) = \sigma(\psi_i; \psi)$$

for $i = 2, \dots, m$. Thus,

$$m(\tau_{j_2}(\psi_2) \times \cdots \times \tau_{j_m}(\psi_m) \otimes L(\Delta(\psi_1); T(\psi_1; \psi)), \mu^*_{\psi_2,\dots,\psi_m}(\pi')) = 0$$

for any component $\pi' \leq \mathcal{L}(\Delta(\psi_1)) \rtimes L(\Delta(\psi_2), \dots, \Delta(\psi_m); T(\psi_1, \dots, \psi_m; \psi))$ with $\pi' \not\cong \pi$ (if not, $\psi_i(\pi') = \psi_i(\pi)$ for all i, contradicting Proposition 8.4). Consequently, as π appears with multiplicity one in

$$\mathcal{L}(\Delta(\psi_1)) \rtimes L(\Delta(\psi_2), \dots, \Delta(\psi_m); T(\psi_1, \dots, \psi_m; \psi)),$$

we have

$$m(\tau_{j_2}(\psi_2) \times \cdots \times \tau_{j_m}(\psi_m) \otimes L(\Delta(\psi_1); T(\psi_1; \psi)), \mu^*_{\psi_2,\dots,\psi_m}(\pi)) = c_{j_2}(\psi_2) \cdots c_{j_m}(\psi_m).$$

Further, by $\psi_1(\pi)$ considerations,

$$m(\tau_{j_2}(\psi_2) \times \cdots \times \tau_{j_m}(\psi_m) \otimes \sigma'(\psi_1; \psi), \mu^*_{\psi_2,\ldots,\psi_m}(\pi)) = 0$$

if $\sigma'(\psi_1; \psi)$ is irreducible and $\sigma'(\psi_1; \psi) \not\cong L(\Delta(\psi_1); T(\psi_1; \psi))$. Thus,

$$\mu^*_{\psi_2,\ldots,\psi_m}(\pi) = \sum_{j_2,\ldots,j_m} (c_{j_2}(\psi_2) \cdots c_{j_m}(\psi_m)) \tau_{j_2}(\psi_2) \times \cdots \times \tau_{j_m}(\psi_m) \otimes L(\Delta(\psi_1); T(\psi_1; \psi)),$$

as needed.

From here, it is easy to finish Case 1. The preceding paragraph tells us

$$\begin{gathered} \mu^*_{\psi_2,\ldots,\psi_m}(\pi) = \sum_{j_2,\ldots,j_m} (c_{j_2}(\psi_2) \cdots c_{j_m}(\psi_m)) \tau_{j_2}(\psi_2) \times \cdots \times \tau_{j_m}(\psi_m) \otimes \sigma(\psi_1; \psi) \\ \Downarrow \\ \tau_{j_2}(\psi_2) \times \cdots \times \tau_{j_m}(\psi_m) \otimes \tau_{j_1}(\psi_1) \otimes \psi \text{ appears with multiplicity} \\ c_{j_2}(\psi_2) \ldots c_{j_m}(\psi_m) c_{j_1}(\psi_1) \text{ in } s_\alpha(\pi) \\ \Downarrow \\ s_{GL}(\pi) = \sum_{j_1,\ldots,j_m} (c_{j_1}(\psi_1) \cdots c_{j_m}(\psi_m)) \tau_{j_1}(\psi_1) \times \cdots \times \tau_{j_m}(\psi_m) \otimes \psi, \end{gathered}$$

where M_α is the standard Levi for $\tau_{j_2}(\psi_2) \times \cdots \times \tau_{j_m}(\psi_m) \otimes \tau_{j_1}(\psi_1) \otimes \psi$. This finishes Case 1.

Case 2. π *tempered,* $\hat{\pi}$ *nontempered.* First, by Theorem 4.1,

$$\begin{aligned} s_{GL}(\widehat{\sigma(\psi_i; \psi)}) &= Ad(w_{\Phi_i}) \circ \hat{} \circ s_{GL}(\sigma(\psi_i; \psi)) \\ &= Ad(w_{\Phi_i}) \circ \hat{} \left(\sum_j c_j(\psi_i) \tau_j(\psi_i) \otimes \psi \right) \\ &= Ad(w_{\Phi_i}) \left(\sum_j c_j(\psi_i) \widehat{\tau_j(\psi_i)} \otimes \psi \right) \\ &= \sum_j c_j(\psi_i) \widetilde{\widehat{\tau_j(\psi_i)}} \otimes \psi. \end{aligned}$$

Note that, as in Case 2 of the proof of Proposition 8.4, $\hat{\pi} = \Psi(\widehat{\sigma(\psi_1; \psi)}, \ldots, \widehat{\sigma(\psi_m; \psi)})$. So, by Case 1,

$$s_{GL}(\hat{\pi}) = \sum_{j_1,\ldots,j_m} (c_{j_1}(\psi_1) \cdots c_{j_m}(\psi_m)) \widetilde{\widehat{\tau_{j_1}(\psi_1)}} \times \cdots \times \widetilde{\widehat{\tau_{j_m}(\psi_m)}} \otimes \psi.$$

Since ˆ and ˜ commute and both respect induction, we see that

$$s_{GL}(\hat{\pi}) = \sum_{j_1,\dots,j_m} c_{j_1}(\psi_1)\cdots c_{j_m}(\psi_m)(\widehat{\widetilde{\tau_{j_1}(\psi_1)\times\cdots\times\tau_{j_m}(\psi_m)}})\otimes\psi.$$

Next, again by Theorem 4.1,

$$\begin{aligned} s_{GL}(\pi) &= Ad(w_\Phi)\circ\hat{}\circ s_{GL}(\hat{\pi}) \\ &= Ad(w_\Phi)\circ\hat{}\left(\sum_{j_1,\dots,j_m} c_{j_1}(\psi_1)\cdots c_{j_m}(\psi_m)(\widehat{\widetilde{\tau_{j_1}(\psi_1)\times\cdots\times\tau_{j_m}(\psi_m)}})\otimes\psi\right) \\ &= Ad(w_\Phi)\left(\sum_{j_1,\dots,j_m} c_{j_1}(\psi_1)\cdots c_{j_m}(\psi_m)(\widetilde{\tau_{j_1}(\psi_1)\times\cdots\times\tau_{j_m}(\psi_m)})\otimes\psi\right) \\ &= \sum_{j_1,\dots,j_m}(c_{j_1}(\psi_1)\cdots c_{j_m}(\psi_m))\tau_{j_1}(\psi_1)\times\cdots\times\tau_{j_m}(\psi_m)\otimes\psi, \end{aligned}$$

as needed. This finishes Case 2.

Case 3. *π, $\hat{\pi}$ both tempered.* From Corollary 4.2, this means

$$\pi\hookrightarrow\underbrace{\psi_1\times\cdots\times\psi_1}_{n_1}\times\cdots\times\underbrace{\psi_m\times\cdots\times\psi_m}_{n_m}\rtimes\psi.$$

The claim then follows immediately from Propositions 6.1 and 6.2. This finishes Case 3 and the $s_{GL}(\pi)$ proof.

We now turn to $\mu^*_{\psi_1,\dots,\psi_k}(\pi)$. An immediate consequence of the $s_{GL}(\pi)$ result is that

$$m(\tau_{j_1}(\psi_1)\otimes\cdots\otimes\tau_{j_m}(\psi_m)\otimes\psi, r_{M_\alpha G}(\pi)) = c_{j_1}(\psi_1)\cdots c_{j_m}(\psi_m),$$

where M_α is the standard Levi for $\tau_{j_1}(\psi_1)\otimes\cdots\otimes\tau_{j_m}(\psi_m)\otimes\psi$ (cf. Corollary 5.6). Observe that

$$\begin{aligned} &m(\tau_{j_1}(\psi_1)\otimes\cdots\otimes\tau_{j_m}(\psi_m)\otimes\psi, r_{M_\alpha G}(\pi)) \\ &\quad= \sum_{\Psi'}[m(\tau_{j_1}(\psi_1)\otimes\cdots\otimes\tau_{j_k}(\psi_k)\otimes\Psi', r_{M_\beta G}(\pi)) \\ &\qquad\cdot m(\tau_{j_1}(\psi_1)\otimes\cdots\otimes\tau_{j_m}(\psi_m)\otimes\psi, r_{M_\alpha M_\beta}(\tau_{j_1}(\psi_1)\otimes\cdots\otimes\tau_{j_k}(\psi_k)\otimes\Psi'))], \end{aligned}$$

where M_β is the standard Levi for

$$\tau_{j_1}(\psi_1)\otimes\cdots\otimes\tau_{j_k}(\psi_k)\otimes\Psi(\sigma(\psi_{k+1};\psi),\dots,\sigma(\psi_m;\psi))$$

and the sum is over all irreducible Ψ' with $\tau_{j_1}(\psi_1)\otimes\cdots\otimes\tau_{j_k}(\psi_k)\otimes\Psi' \leq r_{M_\beta G}(\pi)$. Now, by $\psi_{k+1}(\pi),\ldots,\psi_m(\pi)$ considerations, we see that

$$m(\tau_{j_1}(\psi_1)\otimes\cdots\otimes\tau_{j_k}(\psi_k)\otimes\Psi', r_{M_\beta,G}(\pi)) = 0$$

for any irreducible $\Psi' \not\cong \Psi(\sigma(\psi_{k+1};\psi),\ldots,\sigma(\psi_m;\psi))$. Therefore,

$$\begin{aligned} &m(\tau_{j_1}(\psi_1)\otimes\cdots\otimes\tau_{j_m}(\psi_m)\otimes\psi, r_{M_\alpha G}(\pi)) \\ &\quad = m(\tau_{j_1}(\psi_1)\otimes\cdots\otimes\tau_{j_k}(\psi_k)\otimes\Psi(\sigma(\psi_{k+1};\psi),\ldots,\sigma(\psi_m;\psi)), r_{M_\beta G}(\pi)) \\ &\qquad \cdot m(\tau_{j_1}(\psi_1)\otimes\cdots\otimes\tau_{j_m}(\psi_m)\otimes\psi, r_{M_\alpha M_\beta}(\tau_{j_1}(\psi_1)\otimes\cdots\otimes\tau_{j_k}(\psi_k) \\ &\qquad \otimes\Psi(\sigma(\psi_{k+1};\psi),\ldots,\sigma(\psi_m;\psi)))). \end{aligned}$$

The s_{GL}-result applied to $\Psi(\sigma(\psi_{k+1};\psi),\ldots,\sigma(\psi_m;\psi))$ tells us that

$$\begin{aligned} &m(\tau_{j_1}(\psi_1)\otimes\cdots\otimes\tau_{j_m}(\psi_m)\otimes\psi, r_{M_\alpha M_\beta}(\tau_{j_1}(\psi_1)\otimes\cdots\otimes\tau_{j_k}(\psi_k) \\ &\quad \otimes\Psi(\sigma(\psi_{k+1};\psi),\ldots,\sigma(\psi_m;\psi)))) = c_{j_{k+1}}(\psi_{k+1})\cdots c_{j_m}(\psi_m). \end{aligned}$$

Thus,

$$\begin{aligned} &m(\tau_{j_1}(\psi_1)\otimes\cdots\otimes\tau_{j_k}(\psi_k)\otimes\Psi(\sigma(\psi_{k+1};\psi),\ldots,\sigma(\psi_m;\psi)), r_{M_\beta G}(\pi)) \\ &\quad = c_{j_1}(\psi_1)\cdots c_{j_k}(\psi_k). \end{aligned}$$

As a consequence, $\tau_{j_1}(\psi_1)\times\cdots\times\tau_{j_k}(\psi_k)\otimes\Psi(\sigma(\psi_1;\psi),\ldots,\sigma(\psi_m;\psi))$ has multiplicity $c_{j_1}(\psi_1)\cdots c_{j_k}(\psi_k)$ in $\mu^*_{\psi_1,\ldots,\psi_k}(\pi)$. This finishes the proof. □

COROLLARY 9.2. *Let $\rho(\psi_1),\ldots,\rho(\psi_m)$ be irreducible $GL(F)$ representations with supports contained in $\mathcal{S}(\psi_1),\ldots,\mathcal{S}(\psi_m)$ and $\sigma(\psi_1;\psi),\ldots,\sigma(\psi_m;\psi)$ irreducible $S(F)$ representations with supports contained in $\mathcal{S}(\psi_1;\psi),\ldots,\mathcal{S}(\psi_m;\psi)$. (We allow the possibility that $\rho(\psi_i)=1$ or $\sigma(\psi_i;\psi)=\psi$.) Suppose*

$$\rho(\psi_i)\rtimes\sigma(\psi_i;\psi) = \sum_j m_j(\psi_i)\sigma_j(\psi_i;\psi),$$

with $\sigma_j(\psi_i;\psi)$ irreducible and $m_j(\psi_i)$ its multiplicity. Then,

$$\begin{aligned} &(\rho(\psi_1)\times\cdots\times\rho(\psi_m))\rtimes\Psi(\sigma(\psi_1;\psi),\ldots,\sigma(\psi_m;\psi)) \\ &\quad = \sum_{j_1,\ldots,j_m}(m_{j_1}(\psi_1)\cdots m_{j_m}(\psi_m))\Psi(\sigma_{j_1}(\psi_1;\psi),\ldots,\sigma_{j_m}(\psi_m;\psi)). \end{aligned}$$

Proof. Write Ψ for $\Psi(\sigma(\psi_1;\psi),\dots,\sigma(\psi_m;\psi))$. First, we consider $\rho(\psi_m)\rtimes\Psi$. By Propositions 7.4 and 9.1, we have

$$\begin{aligned}
&\mu^*_{\psi_1,\dots,\psi_{m-1}}(\rho(\psi_m)\rtimes\Psi)\\
&\quad=\sum_{j_1,\dots,j_{m-1}}(c_{j_1}(\psi_1)\cdots c_{j_{m-1}}(\psi_{m-1}))\tau_{j_1}(\psi_1)\times\cdots\times\tau_{j_{m-1}}(\psi_{m-1})\\
&\quad\quad\otimes\rho(\psi_m)\rtimes\sigma(\psi_m;\psi)\\
&\quad=\sum_{i,j_1,\dots,j_{m-1}}(m_i(\psi_m)c_{j_1}(\psi_1)\cdots c_{j_{m-1}}(\psi_{m-1}))\tau_{j_1}(\psi_1)\times\cdots\times\tau_{j_{m-1}}(\psi_{m-1})\\
&\quad\quad\otimes\sigma_i(\psi_m;\psi).
\end{aligned}$$

For any component π of $\rho(\psi_m)\rtimes\Psi$, we have $\psi_i(\pi)=\sigma(\psi_i;\psi)$ for $i=1,\dots,m-1$. Thus,

$$\rho(\psi_m)\rtimes\Psi=\sum_i m_i\Psi(\sigma(\psi_1;\psi),\dots,\sigma(\psi_{m-1};\psi),\sigma_i(\psi_m;\psi))$$

for multiplicities m_i. We claim that $m_i=m_i(\psi_m)$. This is straightforward. Write $\Psi_i=\Psi(\sigma(\psi_1;\psi),\dots,\sigma(\psi_{m-1};\psi),\sigma_i(\psi_m;\psi))$. By Proposition 9.1,

$$\begin{aligned}
\mu^*_{\psi_1,\dots\psi_{m-1}}(\Psi_i)&=\sum_{j_1,\dots,j_{m-1}}(c_{j_1}(\psi_1)\cdots c_{j_{m-1}}(\psi_{m-1}))\tau_{j_1}(\psi_1)\times\cdots\\
&\quad\times\tau_{j_{m-1}}(\psi_{m-1})\otimes\sigma_i(\psi_m;\psi).
\end{aligned}$$

Now, if M_α is the standard Levi for $\tau_{j_1}(\psi_1)\times\cdots\times\tau_{j_{m-1}}(\psi_{m-1})\otimes\sigma_i(\psi_m;\psi)$,

$$m(\tau_{j_1}(\psi_1)\times\cdots\times\tau_{j_{m-1}}(\psi_{m-1})\otimes\sigma_i(\psi_m;\psi),s_\alpha(\Psi'))=0$$

for any component $\Psi'\le\rho(\psi_m)\rtimes\Psi$ with $\Psi'\ne\Psi_i$ (or else $\psi_j(\Psi')=\psi_j(\Psi_i)$ for $j=1,\dots,m$, contradicting Proposition 8.4). Thus, we get $m_i=m_i(\psi_m)$, as needed.

To get the general result, we iterate. The same argument tells us

$$\begin{aligned}
\rho(\psi_{m-1})\rtimes\Psi_{j_m}&=\sum_{j_{m-1}}m_{j_{m-1}}(\psi_{m-1})\Psi(\sigma(\psi_1;\psi),\dots,\sigma(\psi_{m-2};\psi),\\
&\qquad\qquad\sigma_{j_{m-1}}(\psi_{m-1};\psi),\sigma_{j_m}(\psi_m;\psi)).
\end{aligned}$$

Summing over j_m gives

$$\begin{aligned}
&(\rho(\psi_{m-1})\times\rho(\psi_m))\rtimes\Psi\\
&\quad=\sum_{j_{m-1},j_m}m_{j_{m-1}}(\psi_{m-1})m_{j_m}(\psi_m)\Psi(\sigma(\psi_1;\psi),\dots,\sigma(\psi_{m-2};\psi),\\
&\qquad\qquad\sigma_{j_{m-1}}(\psi_{m-1};\psi),\sigma_{j_m}(\psi_m;\psi)).
\end{aligned}$$

Repeating the argument for $m-2, m-3, \dots, 1$ gives the corollary. □

We summarize the main results in the following theorem.

THEOREM 9.3. *Suppose $\psi_1, \dots, \psi_m$ are inequivalent, irreducible, unitary, supercuspidal representations of $GL_{r_1}(F), \dots, GL_{r_m}(F)$ with $\psi_i \not\cong \tilde{\psi}_j$ for $i \neq j$ and ψ an irreducible supercuspidal representation of $S_r(F)$. Let $Irr(\psi_1, \dots, \psi_m; \psi)$ denote the set of all irreducible representations of all $S_n(F)$, $n \geq 0$, supported on $\mathcal{S}(\psi_1, \dots, \psi_m; \psi)$, and similarly for $Irr(\psi_1; \psi), \dots, Irr(\psi_m; \psi)$. Then the maps (cf. Definitions 7.6 and 8.5)*

$$Irr(\psi_1, \dots, \psi_m; \psi) \underset{\Psi}{\overset{(\psi_1, \dots, \psi_m)}{\rightleftarrows}} Irr(\psi_1; \psi) \times \cdots \times Irr(\psi_m; \psi)$$

give a bijective correspondence with the following properties:

(1) Ψ *and* $(\psi_1, \dots, \psi_m)$ *are inverses of each other.*

(2) *If $\sigma(\psi_i; \psi) \in Irr(\psi_i; \psi)$ is a representation of $S_{n_i+r}(F)$, then $\pi = \Psi(\sigma(\psi_1; \psi), \dots, \sigma(\psi_m; \psi))$ is a representation of $S_{n_1+\dots+n_m+r}(F)$.*

(3) $\widetilde{\Psi(\sigma(\psi_1; \psi), \dots, \sigma(\psi_m; \psi))} = \Psi(\widetilde{\sigma(\psi_1; \psi)}, \dots, \widetilde{\sigma(\psi_m; \psi)})$ *and* $\psi_i(\tilde{\pi}) = \widetilde{\psi_i(\pi)}$, *where* ˜ *denotes contragredient.*

(4) $\widehat{\Psi(\sigma(\psi_1; \psi), \dots, \sigma(\psi_m; \psi))} = \Psi(\widehat{\sigma(\psi_1; \psi)}, \dots, \widehat{\sigma(\psi_m; \psi)})$ *and* $\psi_i(\hat{\pi}) = \widehat{\psi_i(\pi)}$, *where* ˆ *denotes the involution of Aubert (cf. Theorem 4.1).*

(5) *Suppose that*

$$s_{GL}(\sigma(\psi_i; \psi)) = \sum_j c_j(\psi_i)\tau_j(\psi_i) \otimes \psi,$$

where $\tau_j(\psi_i)$ is an irreducible representation and $c_j(\psi_i)$ its multiplicity. With $\mu^*_{\psi_1,\dots,\psi_k}$ as in Definition 7.1,

$$\begin{aligned}
&\mu^*_{\psi_1,\dots,\psi_k}(\Psi(\sigma(\psi_1; \psi), \dots, \sigma(\psi_m; \psi))) \\
&\quad = \sum_{j_1,\dots,j_k} (c_{j_1}(\psi_1) \cdots c_{j_k}(\psi_k))\tau_{j_1}(\psi_1) \times \cdots \times \tau_{j_k}(\psi_k) \\
&\qquad \otimes \Psi(\sigma(\psi_{k+1}; \psi), \dots, \sigma(\psi_m; \psi)).
\end{aligned}$$

(6) *Let $\rho = \rho(\psi_1) \times \cdots \times \rho(\psi_m)$ be an irreducible representation of $GL(F)$ with support contained in $\mathcal{S}(\psi_1, \dots, \psi_m)$ and $\Psi = \Psi(\sigma(\psi_1; \psi), \dots, \sigma(\psi_m; \psi))$ an irreducible representation of $S(F)$ with support contained in $\mathcal{S}(\psi_1, \dots, \psi_m; \psi)$. (We*

allow the possibility that $\rho(\psi_i) = 1$ *or* $\sigma(\psi_i;\psi) = \psi$.*) Suppose*

$$\rho(\psi_i) \rtimes \sigma(\psi_i;\psi) = \sum_j m_j(\psi_i)\sigma_j(\psi_i;\psi),$$

with $\sigma_j(\psi_i)$ *irreducible and* $m_j(\psi_i)$ *its multiplicity. Then,*

$$\rho \rtimes \Psi = \sum_{j_1,\dots,j_m} (m_{j_1}(\psi_1)\cdots m_{j_m}(\psi_m))\Psi(\sigma_{j_1}(\psi_1;\psi),\dots,\sigma_{j_m}(\psi_m;\psi)).$$

(7) $\Psi(\sigma(\psi_1;\psi),\dots,\sigma(\psi_m;\psi))$ *is tempered (resp. square-integrable) if and only if* $\sigma(\psi_1;\psi),\dots,\sigma(\psi_m;\psi)$ *are all tempered (resp. square-integrable).*

(8) *Suppose, in the subrepresentation setting of the Langlands classification,*

$$\sigma(\psi_i;\psi) = L(\nu^{\alpha_1}\tau_1(\psi_i),\dots,\nu^{\alpha_\ell}\tau_\ell(\psi_i);T(\psi_i;\psi))$$

for $i = 1,\dots,m$ *(n.b.: recall that* $\tau_j(\psi_i)$ *may be the trivial representation of* $GL_0(F)$*;* $T(\psi_i;\psi)$ *may just be* ψ*). Then,*

$$\begin{aligned}&\Psi(\sigma(\psi_1;\psi),\dots,\sigma(\psi_m;\psi))\\ &= L(\nu^{\alpha_1}\tau_1(\psi_1)\times\cdots\times\nu^{\alpha_1}\tau_1(\psi_m),\dots,\nu^{\alpha_\ell}\tau_\ell(\psi_1)\times\cdots\times\nu^{\alpha_\ell}\tau_\ell(\psi_m);\\ &\qquad\qquad \Psi(T(\psi_1;\psi),\dots,T(\psi_m;\psi))).\end{aligned}$$

In the other direction, if

$$\begin{aligned}\pi \;=\; & L(\nu^{\alpha_1}\tau_1(\psi_1)\times\cdots\times\nu^{\alpha_1}\tau_1(\psi_m),\dots,\nu^{\alpha_\ell}\tau_\ell(\psi_1)\\ &\times\cdots\times\nu^{\alpha_\ell}\tau_\ell(\psi_m);T(\psi_1,\dots,\psi_m;\psi)),\end{aligned}$$

then

$$\psi_i(\pi) = L(\nu^{\alpha_1}\tau_1(\psi_i),\dots,\nu^{\alpha_\ell}\tau_\ell(\psi_i);\psi_i(T(\psi_1,\dots,\psi_m;\psi))).$$

(In the quotient setting of the Langlands classification, the same results hold.)

Remark 9.4. Proposition 5.3 in [Jan] contains a special case of this result.

Recall, from Sections 2 and 5, that using $\times\colon R\otimes R \longrightarrow R$ and $m^*\colon R \longrightarrow R\otimes R$ to define multiplication and comultiplication, R becomes a Hopf algerba. Further, using $\rtimes\colon R\otimes R[S] \longrightarrow R[S]$ and $\mu^*\colon R[S] \longrightarrow R\otimes R[S]$, $R[S]$ becomes an M_S^*-Hopf module over R. In what follows, we interpret the correspondence of Theorem 9.3 in terms of these structures. Thanks go to Marko Tadić and Guy Henniart for pointing out this interpretation to me.

Definition 9.5. Let $R(\psi_1, \ldots, \psi_m)$ denote the subalgebra of R generated by representations supported on $\mathcal{S}(\psi_1, \ldots, \psi_m)$.

PROPOSITION 9.6. *$R(\psi_1, \ldots, \psi_m) \cong R(\psi_1) \otimes \cdots \otimes R(\psi_m)$ as Hopf subalgebras of R.*

Proof. This is an immediate consequence of Remark 8.7 of [Zel]. ◻

Let Ψ^{GL}: $R(\psi_1) \otimes \cdots \otimes R(\psi_m) \longrightarrow R(\psi_1, \ldots, \psi_m)$ denote the isomorphism from Proposition 9.6. It is defined by

$$\Psi^{GL}(\tau(\psi_1) \otimes \cdots \otimes \tau(\psi_m)) \longmapsto \tau(\psi_1) \times \cdots \times \tau(\psi_m).$$

If we use m to denote multiplication in the following diagrams (and this is the only place where we use m for multiplication), then Proposition 9.6 tells us that the following diagrams commute:

$$\begin{tikzcd}
{[R(\psi_1) \otimes R(\psi_1)] \otimes \cdots \otimes [R(\psi_m) \otimes R(\psi_m)]} \arrow[r, "m \otimes \cdots \otimes m"] \arrow[d, "\Psi^{GL} \otimes \Psi^{GL}"] & R(\psi_1) \otimes \cdots \otimes R(\psi_m) \arrow[d, "\Psi^{GL}"] \\
R(\psi_1, \ldots, \psi_m) \otimes R(\psi_1, \ldots, \psi_m) \arrow[r, "m"] & R(\psi_1, \ldots, \psi_m)
\end{tikzcd}$$

and

$$\begin{tikzcd}
R(\psi_1) \otimes \cdots \otimes R(\psi_m) \arrow[r, "m^* \otimes \cdots \otimes m^*"] \arrow[d, "\Psi^{GL}"] & {[R(\psi_1) \otimes R(\psi_1)] \otimes \cdots \otimes [R(\psi_m) \otimes R(\psi_m)]} \arrow[d, "\Psi^{GL} \otimes \Psi^{GL}"] \\
R(\psi_1, \ldots, \psi_m) \arrow[r, "m^*"] & R(\psi_1, \ldots, \psi_m) \otimes R(\psi_1, \ldots, \psi_m),
\end{tikzcd}$$

where $\Psi^{GL} \otimes \Psi^{GL}$ sends $[\tau'(\psi_1) \otimes \tau''(\psi_1)] \otimes \cdots \otimes [\tau'(\psi_m) \otimes \tau''(\psi_m)]$ to $\Psi^{GL}(\tau'(\psi_1) \otimes \cdots \otimes \tau'(\psi_m)) \otimes \Psi^{GL}(\tau''(\psi_1) \otimes \cdots \otimes \tau''(\psi_m))$.

Definition 9.7. Let $R(\psi_1, \ldots, \psi_m; \psi)$ denote the (additive) subgroup of $R[S]$ generated by $Irr(\psi_1, \ldots, \psi_m; \psi)$.

PROPOSITION 9.8. *$R(\psi_1,\dots,\psi_m;\psi) \cong R(\psi_1;\psi)\otimes\cdots\otimes R(\psi_m;\psi)$ as $R(\psi_1,\dots,\psi_m)\cong R(\psi_1)\otimes\cdots\otimes R(\psi_m)$ M_S^*-Hopf modules.*

Proof. Write $\mu(\rho\otimes\sigma)=\rho\rtimes\sigma$ for $\rho\in R$ and $\sigma\in R[S]$. Then, we need to check that the following diagrams commute:

$$\begin{tikzcd}
{[R(\psi_1)\otimes R(\psi_1;\psi)]\otimes\cdots\otimes[R(\psi_m)\otimes R(\psi_m;\psi)]} \arrow[r, "\mu\otimes\cdots\otimes\mu"] \arrow[d, "\Psi^{GL}\otimes\Psi"] & R(\psi_1;\psi)\otimes\cdots\otimes R(\psi_m;\psi) \arrow[d, "\Psi"] \\
R(\psi_1,\dots,\psi_m)\otimes R(\psi_1,\dots,\psi_m;\psi) \arrow[r, "\mu"] & R(\psi_1,\dots,\psi_m;\psi)
\end{tikzcd}$$

and

$$\begin{tikzcd}
R(\psi_1;\psi)\otimes\cdots\otimes R(\psi_m;\psi) \arrow[r, "\mu^*\otimes\cdots\otimes\mu^*"] \arrow[d, "\Psi"] & {[R(\psi_1)\otimes R(\psi_1;\psi)]\otimes\cdots\otimes[R(\psi_m)\otimes R(\psi_m;\psi)]} \arrow[d, "\Psi^{GL}\otimes\Psi"] \\
R(\psi_1,\dots,\psi_m;\psi) \arrow[r, "\mu^*"] & R(\psi_1,\dots,\psi_m)\otimes R(\psi_1,\dots,\psi_m;\psi),
\end{tikzcd}$$

where $\Psi^{GL}\otimes\Psi$ takes $[\tau(\psi_1)\otimes\sigma(\psi_1;\psi)]\otimes\cdots\otimes[\tau(\psi_m)\otimes\sigma(\psi_m;\psi)]$ to $\Psi^{GL}(\tau(\psi_1)\otimes\cdots\otimes\tau(\psi_m))\otimes\Psi(\sigma(\psi_1;\psi),\dots,\sigma(\psi_m;\psi))$, with Ψ extended to $R(\psi_1;\psi)\otimes\cdots\otimes R(\psi_m;\psi)$ multilinearly.

That the first diagram commutes follows immediately from Theorem 9.3(6). The commutativity of the second diagram follows from Lemma 9.9 below. $\square$

LEMMA 9.9. *Suppose $\psi_1,\dots,\psi_m$ are as in Theorem 9.3. Let $\sigma(\psi_i;\psi)\in Irr(\psi_i;\psi)$ for $i=1,\dots,m$. Suppose,*

$$\mu^*(\sigma(\psi_i;\psi))=\sum_{j_i} n_{j_i}(\psi_i)\eta_{j_i}(\psi_i)\otimes\theta_{j_i}(\psi_i;\psi),$$

with $\eta_{j_i}(\psi_i)\otimes\theta_{j_i}(\psi_i;\psi)$ irreducible and $n_{j_i}(\psi_i)$ its multiplicity. Then,

$$\mu^*(\Psi(\sigma(\psi_1;\psi),\dots,\sigma(\psi_m;\psi)))$$

$$= \sum_{j_1,\dots,j_m} (n_{j_1}(\psi_1)\cdots n_{j_m}(\psi_m))(\eta_{j_1}(\psi_1)\times\cdots\times\eta_{j_m}(\psi_m))$$
$$\otimes \Psi(\theta_{j_1}(\psi_1;\psi),\dots,\theta_{j_m}(\psi_m;\psi)).$$

Proof. Write $\sigma = \Psi(\sigma(\psi_1;\psi),\dots,\sigma(\psi_m;\psi))$. Suppose $\eta_{j_i}(\psi_i)$ is a representation of $GL_{k^{(i)}_{j_i}}(F)$. We begin by showing

$$\eta_{j_m}(\psi_m)\otimes\Psi(\sigma(\psi_1;\psi),\dots,\sigma(\psi_{m-1};\psi),\theta_{j_m}(\psi_m;\psi))$$

appears with multiplicity $n_{j_m}(\psi_m)$ in $s_{(k^{(m)}_{j_m})}(\sigma)$. From Theorem 9.3(5),

$$\mu^*_{\psi_1,\dots,\psi_{m-1}}(\sigma) = \sum_{j_1,\dots,j_{m-1}} (c_{j_1}(\psi_1)\cdots c_{j_{m-1}}(\psi_{m-1}))\tau_{j_1}(\psi_1)\times\cdots\times\tau_{j_{m-1}}(\psi_{m-1})\otimes\sigma(\psi_m;\psi)$$
$$\Downarrow$$
$$m(\tau_{j_1}(\psi_1)\times\cdots\times\tau_{j_{m-1}}(\psi_{m-1})\otimes\eta_{j_m}(\psi_m)\otimes\theta_{j_m}(\psi_m;\psi), s_{\alpha_1}(\sigma)) = c_{j_1}(\psi_1)\cdots c_{j_{m-1}}(\psi_{m-1})n_{j_m}(\psi_m)$$
$$\Downarrow$$
$$m(\eta_{j_m}(\psi_m)\otimes\tau_{j_1}(\psi_1)\times\cdots\times\tau_{j_{m-1}}(\psi_{m-1})\otimes\theta_{j_m}(\psi_m;\psi), s_{\alpha_2}(\sigma)) = c_{j_1}(\psi_1)\cdots c_{j_{m-1}}(\psi_{m-1})n_{j_m}(\psi_m)$$

where M_{α_1} is the standard Levi for $\tau_{j_1}(\psi_1)\times\cdots\times\tau_{j_{m-1}}(\psi_{m-1})\otimes\eta_{j_m}(\psi_m)\otimes\theta_{j_m}(\psi_m;\psi)$ and M_{α_2} the standard Levi for $\eta_{j_m}(\psi_m)\otimes\tau_{j_1}(\psi_1)\times\cdots\times\tau_{j_{m-1}}(\psi_{m-1})\otimes\theta_{j_m}(\psi_m;\psi)$. Observe that

$$m(\eta_{j_m}(\psi_m)\otimes\tau_{j_1}(\psi_1)\times\cdots\times\tau_{j_{m-1}}(\psi_{m-1})\otimes\theta_{j_m}(\psi_m;\psi), s_{\alpha_2}(\sigma))$$
$$= \sum_{\Psi'}[m(\eta_{j_m}(\psi_m)\otimes\Psi', \mu^*(\sigma))$$
$$\cdot\, m(\eta_{j_m}(\psi_m)\otimes\tau_{j_1}(\psi_1)\times\cdots\times\tau_{j_{m-1}}(\psi_{m-1})\otimes\theta_{j_m}(\psi_m;\psi), s_{\alpha_2}(\eta_{j_m}(\psi_m)\otimes\Psi'))],$$

where the sum is over all irreducible Ψ' such that $\eta_{j_m}(\psi_m)\otimes\Psi' \le \mu^*(\sigma)$. Now, consider any irreducible $\eta_{j_m}(\psi_m)\otimes\Psi' \le \mu^*(\sigma)$ such that

$$\eta_{j_m}(\psi_m)\otimes\tau_{j_1}(\psi_1)\times\cdots\times\tau_{j_{m-1}}(\psi_{m-1})\otimes\theta_{j_m}(\psi_m;\psi) \le s_{\alpha_2}(\eta_{j_m}(\psi_m)\otimes\Psi').$$

By $\psi_1(\Psi'),\dots,\psi_m(\Psi')$ considerations,

$$\Psi' = \Psi(\sigma(\psi_1;\psi),\dots,\sigma(\psi_{m-1};\psi),\theta_{j_m}(\psi_m;\psi)).$$

From Theorem 9.3(5), we have

$$m(\eta_{j_m}(\psi_m)\otimes\tau_{j_1}(\psi_1)\times\cdots\times\tau_{j_{m-1}}(\psi_{m-1})\otimes\theta_{j_m}(\psi_m;\psi),$$
$$s_{\alpha_2}(\eta_{j_m}(\psi_m)\otimes\Psi(\sigma(\psi_1;\psi),\dots,\sigma(\psi_{m-1};\psi),\theta_{j_m}(\psi_m;\psi))))$$
$$= c_{j_1}(\psi_1)\cdots c_{j_{m-1}}(\psi_{m-1}).$$

Thus,

$$m(\eta_{j_m}(\psi_m)\otimes\Psi(\sigma(\psi_1;\psi),\dots,\sigma(\psi_{m-1};\psi),\theta_{j_m}(\psi_m)),s_{(k_{j_m}^{(m)})}(\sigma))=n_{j_m}(\psi_m),$$

as claimed. More generally,

$$\eta_{j_i}(\psi_i)\otimes\Psi(\sigma(\psi_1;\psi),\dots,\sigma(\psi_{i-1};\psi),\theta_{j_i}(\psi_i;\psi),\sigma(\psi_{i+1};\psi),\dots,\sigma(\psi_m;\psi))$$

appears with multiplicity $n_{j_i}(\psi_i)$ in $s_{(k_{j_i}^{(i)})}(\sigma)$.

We now turn to the lemma itself. First, we show that if $\eta(\psi_1)\times\cdots\times\eta(\psi_m)\otimes\Psi(\theta(\psi_1;\psi),\dots,\theta(\psi_m;\psi))\leq\mu^*(\sigma)$, then $\eta(\psi_j)\otimes\theta(\psi_j;\psi)\leq\mu^*(\sigma(\psi_j;\psi))$, so only those terms in the statement of the lemma can appear in $\mu^*(\sigma)$. Now,

$$\begin{gathered}\mu^*(\sigma)\geq\eta(\psi_1)\times\cdots\times\eta(\psi_m)\otimes\Psi(\theta(\psi_1;\psi),\dots,\theta(\psi_m;\psi))\\ \Downarrow\\ s_{\beta_1}(\sigma)\geq\eta(\psi_1)\times\cdots\times\eta(\psi_m)\otimes\eta'(\psi_1)\times\cdots\times\eta'(\psi_{m-1})\otimes\theta(\psi_m;\psi)\\ \Downarrow\\ s_{\beta_2}(\sigma)\geq\eta(\psi_1)\times\cdots\times\eta(\psi_{m-1})\otimes\eta'(\psi_1)\times\cdots\times\eta'(\psi_{m-1})\otimes\eta(\psi_m)\otimes\theta(\psi_m;\psi),\end{gathered}$$

where $\eta'(\psi_1)\times\cdots\times\eta'(\psi_{m-1})\otimes\theta(\psi_m;\psi)\leq\mu^*_{\psi_1,\dots,\psi_{m-1}}(\Psi(\theta(\psi_1;\psi),\dots,\theta(\psi_m;\psi)))$, M_{β_1} is the standard Levi for $\eta(\psi_1)\times\cdots\times\eta(\psi_m)\otimes\eta'(\psi_1)\times\cdots\times\eta'(\psi_{m-1})\otimes\theta(\psi_m;\psi)$ and M_{β_2} the standard Levi for $\eta(\psi_1)\times\cdots\times\eta(\psi_{m-1})\otimes\eta'(\psi_1)\times\cdots\times\eta'(\psi_{m-1})\otimes\eta(\psi_m)\otimes\theta(\psi_m;\psi)$. Therefore, there must be some irreducible $\tau(\psi_1,\dots,\psi_{m-1})\otimes\sigma(\psi_m;\psi)\leq\mu^*_{\psi_1,\dots,\psi_{m-1}}(\sigma)$ such that

$$\begin{aligned}&[\eta(\psi_1)\times\cdots\times\eta(\psi_{m-1})\otimes\eta'(\psi_1)\times\cdots\times\eta'(\psi_{m-1})]\otimes[\eta(\psi_m)\otimes\theta(\psi_m;\psi)]\\ &\qquad\leq s_{\beta_2}(\tau(\psi_1,\dots,\psi_{m-1})\otimes\sigma(\psi_m;\psi)).\end{aligned}$$

In particular, this forces $\eta(\psi_m)\otimes\theta(\psi_m;\psi)\leq\mu^*(\sigma(\psi_m;\psi))$. A similar argument shows $\eta(\psi_j)\otimes\theta(\psi_j;\psi)\leq\mu^*(\sigma(\psi_j;\psi))$, as claimed.

The final step is to show that the terms which appear in the statement of the lemma have the multiplicities claimed. We do this by induction on the parabolic rank of an element of $\mathcal{M}_{min}$, as we did in Proposition 8.4 and Proposition 9.1. First, observe that $1\otimes\sigma=1\otimes\Psi(\sigma(\psi_1;\psi),\dots,\sigma(\psi_m;\psi))$ appears with multiplicity one in $\mu^*(\sigma)$. Now, consider a term $\eta_{j_1}(\psi_1)\times\cdots\times\eta_{j_m}(\psi_m)\otimes\Psi(\theta_{j_1}(\psi_1;\psi),\dots,\theta_{j_m}(\psi_m;\psi))$ with $\eta_{j_1}(\psi_1)\times\cdots\times\eta_{j_m}(\psi_m)\not\cong 1$. Then, $\theta_{j_i}(\psi_i;\psi)\not\cong\sigma(\psi_i;\psi)$ for some i; without loss of generality, assume $\theta_{j_m}(\psi_m;\psi)\not\cong\sigma(\psi_m;\psi)$. By the inductive hypothesis, we have

$$\begin{aligned}&\eta_{j_1}(\psi_1)\times\cdots\times\eta_{j_{m-1}}(\psi_{m-1})\otimes\Psi(\theta_{j_1}(\psi_1;\psi),\dots,\theta_{j_{m-1}}(\psi_{m-1};\psi),\theta_{j_m}(\psi_m;\psi))\\ &\text{appears with multiplicity } n_{j_1}(\psi_1)\cdots n_{j_{m-1}}(\psi_{m-1})\\ &\text{in } \mu^*(\Psi(\sigma(\psi_1;\psi),\dots,\sigma(\psi_{m-1};\psi),\theta_{j_m}(\psi_m;\psi))).\end{aligned}$$

Now, observe that

$$
\begin{aligned}
&m(\eta_{j_m}(\psi_m)\otimes\eta_{j_1}(\psi_1)\times\cdots\times\eta_{j_{m-1}}(\psi_{m-1})\otimes\Psi(\theta_{j_1}(\psi_1;\psi),\ldots,\theta_{j_m}(\psi_m;\psi)),s_\gamma(\sigma))\\
&\quad=\sum_{\Psi''}[m(\eta_{j_m}(\psi_m)\otimes\Psi'',\mu^*(\sigma))\cdot m(\eta_{j_m}(\psi_m)\otimes\eta_{j_1}(\psi_1)\times\cdots\times\eta_{j_{m-1}}(\psi_{m-1})\\
&\qquad\otimes\Psi(\theta_{j_1}(\psi_1;\psi),\ldots,\theta_{j_m}(\psi_m;\psi)),s_\gamma(\eta_{j_m}(\psi_m)\otimes\Psi''))],
\end{aligned}
$$

where M_γ is the obvious standard Levi and the sum is over all irreducible Ψ'' such that $\eta_{j_m}(\psi_m)\otimes\Psi''\leq\mu^*(\sigma)$. Note that for such a Ψ'', $\psi_i(\Psi'')=\sigma(\psi_i;\psi)$ for $i=1,\ldots,m-1$. Further, if

$$
\begin{aligned}
&\eta_{j_m}(\psi_m)\otimes\eta_{j_1}(\psi_1)\times\cdots\times\eta_{j_{m-1}}(\psi_{m-1})\otimes\Psi(\theta_{j_1}(\psi_1;\psi),\ldots,\theta_{j_m}(\psi_m;\psi))\\
&\qquad\leq s_\gamma(\eta_{j_m}(\psi_m)\otimes\Psi''),
\end{aligned}
$$

then $\psi_m(\Psi'')=\theta_{j_m}(\psi_m;\psi)$. Thus, we must have

$$\Psi''=\Psi(\sigma(\psi_1;\psi),\ldots,\sigma(\psi_{m-1};\psi),\theta_{j_m}(\psi_m)).$$

Therefore, we can argue as follows:

$$
\begin{gathered}
\eta_{j_m}(\psi_m)\otimes\Psi(\sigma(\psi_1;\psi),\ldots,\sigma(\psi_{m-1};\psi),\theta_{j_m}(\psi_m;\psi))\\
\text{appears with multiplicity } n_{j_m}(\psi_m) \text{ in } \mu^*(\sigma)\\
\Downarrow\\
\eta_{j_m}(\psi_m)\otimes\eta_{j_1}(\psi_1)\times\cdots\times\eta_{j_{m-1}}(\psi_{m-1})\otimes\Psi(\theta_{j_1}(\psi_1;\psi),\ldots,\theta_{j_{m-1}}(\psi_{m-1};\psi),\\
\theta_{j_m}(\psi_m;\psi)) \text{ appears with multiplicity } n_{j_m}(\psi_m)n_{j_1}(\psi_1)\cdots n_{j_{m-1}}(\psi_{m-1}) \text{ in } s_\gamma(\sigma)\\
\Downarrow\\
\eta_{j_1}(\psi_1)\times\cdots\times\eta_{j_m}(\psi_m)\otimes\Psi(\theta_{j_1}(\psi_1;\psi),\ldots,\theta_{j_m}(\psi_m;\psi))\\
\text{appears with multiplicity } n_{j_1}(\psi_1)\cdots n_{j_m}(\psi_m) \text{ in } \mu^*(\sigma),
\end{gathered}
$$

as needed. This finishes the proof of the lemma (and that of Proposition 9.6). □

10. Refinements. In this section, we give a refinement of Theorem 9.3 and a corresponding interpretation in terms of a tensor product decomposition. I would like to thank Marko Tadić and Guy Henniart for pointing it out to me.

Fix ψ_i,ψ from Theorem 9.3. For $\beta\in\mathbb{R}$, set

$$\mathcal{S}_\beta(\psi_i)=\{\nu^{\beta+z}\psi_i,\nu^{-\beta-z}\tilde{\psi}_i\}_{z\in\mathbb{Z}}.$$

If $\psi_i\cong\tilde{\psi}_i$, we assume $0\leq\beta\leq 1/2$; if $\psi_i\ncong\tilde{\psi}_i$, we assume $0\leq\beta<1$. Then, if $\beta_1\neq\beta_2$, $\mathcal{S}_{\beta_1}(\psi_i)\cap\mathcal{S}_{\beta_2}(\psi_i)=\emptyset$. For $\beta_1,\ldots,\beta_n$ distinct, we set

$$\mathcal{S}_{\beta_1,\ldots,\beta_n}(\psi_i)=\mathcal{S}_{\beta_1}(\psi_i)\cup\cdots\cup\mathcal{S}_{\beta_n}(\psi_i)$$

and

$$\mathcal{S}_{\beta_1,\ldots,\beta_n}(\psi_i;\psi) = \mathcal{S}_{\beta_1}(\psi_i) \cup \cdots \cup \mathcal{S}_{\beta_n}(\psi_i) \cup \{\psi\}.$$

We are going to produce a correspondence between representations supported on $\mathcal{S}_{\beta_1,\ldots,\beta_n}(\psi_i;\psi)$ and n-tuples of representations supported on $\mathcal{S}_{\beta_1}(\psi_i;\psi),\ldots,\mathcal{S}_{\beta_n}(\psi_i;\psi)$ analgous to Theorem 9.3. The proof is the same with the exception of one simplification noted below. Therefore, we will give a precise statement of the main result—the analogue of Theorem 9.3—as well as the required definitions, but be less detailed in discussing the results that go into the proof of the theorem. (In any case, most of the main propositions, etc., that go into the proof of the theorem are more-or-less repeated in the statement of the theorem.)

First, we note that if we replace the sets $\{\nu^{\alpha}\psi_1\}_{\alpha\in\mathbb{R}},\ldots,\{\nu^{\alpha}\psi_m\}_{\alpha\in\mathbb{R}}$ with the sets $\{\nu^{\beta_1+z}\psi_i\}_{z\in\mathbb{Z}},\ldots,\{\nu^{\beta_n+z}\psi_i\}_{z\in\mathbb{Z}}$, Lemma 5.4 and Corollary 5.6 hold, and for the same reasons. Similarly, replacing $\mathcal{S}(\psi_1,\ldots,\psi_m;\psi)$ and $\mathcal{S}(\psi_1;\psi),\ldots,\mathcal{S}(\psi_m;\psi)$ with $\mathcal{S}_{\beta_1,\ldots,\beta_n}(\psi_i;\psi)$ and $\mathcal{S}_{\beta_1}(\psi_i;\psi),\ldots,\mathcal{S}_{\beta_n}(\psi_i;\psi)$, Lemma 5.7 holds, also with the same proof. (As Theorem 2.1 is repeated in Corollary 5.6, these are the only results before Section 7 which need any changes.)

We need the following analogues to Definitions 7.1 and 7.3:

Definition 10.1. Suppose π is a representation of $S_s(F)$ supported in $\mathcal{S}_{\beta_1,\ldots,\beta_n}(\psi_i;\psi)$. Let $\mu^*_{\beta_1,\ldots,\beta_k}(\pi)$ denote the sum of every $\rho\otimes\sigma$ in $\mu^*(\pi)$ such that the support of ρ is contained in $\mathcal{S}_{\beta_1,\ldots,\beta_k}(\psi_i)$ and the support of σ is contained in $\mathcal{S}_{\beta_{k+1},\ldots,\beta_n}(\psi_i;\psi)$.

The obvious analogue of Lemma 7.2 holds; same proof.

Definition 10.2. Suppose ρ is a representation of $GL_r(F)$ supported in $\mathcal{S}_{\beta_1,\ldots,\beta_n}(\psi_i)$. Let $M^*_{\beta_1,\ldots,\beta_k}(\rho)$ denote the sum of every $\tau\otimes\tau'$ in $M^*_S(\rho)$ such that the support of τ is contained in $\mathcal{S}_{\beta_1,\ldots,\beta_k}(\psi_i)$ and the support of τ' is contained in $\mathcal{S}_{\beta_{k+1},\ldots,\beta_n}(\psi_i)$.

Now, $\mu^*_{\beta_1,\ldots,\beta_k}(\rho\rtimes\sigma) = M^*_{\beta_1,\ldots,\beta_k}(\rho)\rtimes\mu^*_{\beta_1,\ldots,\beta_k}(\sigma)$, as in Proposition 7.4. Therefore, we can obtain the analogue to Corollary 7.5 with the same proof. In particular, if ρ is a representation of $GL_r(F)$ supported on $\mathcal{S}_{\beta_1,\ldots,\beta_k}(\psi_i)$ and σ is a representation of $S_s(F)$ supported on $\mathcal{S}_{\beta_{k+1},\ldots,\beta_n}(\psi_i;\psi)$, then $\mu^*_{\beta_1,\ldots,\beta_k}(\rho\rtimes\sigma) = \sum_j \tau_j\otimes\sigma$, where τ_j is defined by $s_{GL}(\rho\rtimes 1) = \sum_j \tau_j\otimes 1$. Suppose π is an irreducible $S(F)$ representation supported on $\mathcal{S}_{\beta_1,\ldots,\beta_n}(\psi_i;\psi)$. Choose ρ_j an irreducible $GL(F)$ representation supported on $\mathcal{S}_{\beta_1,\ldots,\beta_{j-1},\beta_{j+1},\ldots,\beta_n}(\psi_i)$ and σ_j an irreducible $S(F)$ representation supported on $\mathcal{S}_{\beta_j}(\psi_i;\psi)$ such that $\pi\hookrightarrow\rho_j\rtimes\sigma_j$ (by the analogue of Lemma 5.7). Then, we have

$$\begin{aligned}\mu^*_{\beta_1,\ldots,\beta_{j-1},\beta_{j+1},\ldots,\beta_n}(\pi) &\leq \mu^*_{\beta_1,\ldots,\beta_{j-1},\beta_{j+1},\ldots,\beta_n}(\rho_j\rtimes\sigma_j)\\ &\leq \sum_{\ell}(\tau_j)_\ell\otimes\sigma_j\end{aligned}$$

and the following definition makes sense:

Definition 10.3. $\psi_{\beta_j}(\pi) = \sigma_j$.

Next, if we replace $\mathcal{S}(\psi_1, \ldots, \psi_m)$ and $\mathcal{S}(\psi_1, \ldots, \psi_m; \psi)$ by $\mathcal{S}_{\beta_1,\ldots,\beta_n}(\psi_i)$ and $\mathcal{S}_{\beta_1,\ldots,\beta_n}(\psi_i; \psi)$, the analogues to Proposition 8.1, Lemma 8.2, and Corollary 8.3 all hold; same proofs. Proposition 8.4 also holds. It has the same proof, except that Case 3 becomes trivial (since only $\beta = 0$ can contribute). Thus, we are led to the following definition:

Definition 10.4. Suppose $\sigma(\beta_1), \ldots, \sigma(\beta_n)$ are irreducible $S(F)$ representations supported on $\mathcal{S}_{\beta_1}(\psi_i; \psi), \ldots, \mathcal{S}_{\beta_n}(\psi_i; \psi)$, respectively. Let $\boldsymbol{\Psi}_{\beta_1,\ldots,\beta_n}(\sigma(\beta_1), \ldots, \sigma(\beta_n))$ denote the the irreducible representation supported on $\mathcal{S}_{\beta_1,\ldots,\beta_n}(\psi_i; \psi)$ which satisfies

$$\psi_{\beta_j}(\boldsymbol{\Psi}_{\beta_1,\ldots,\beta_n}(\sigma(\beta_1), \ldots, \sigma(\beta_n))) = \sigma(\beta_j)$$

for $j = 1, \ldots, n$.

Finally, the analogues of Proposition 9.1 and Corollary 9.2 also hold in this setting. The proofs are the same, except that like Proposition 8.4, Case 3 in Proposition 9.1 is now trivial. Thus, we get the analogue to Theorem 9.3, which we now state precisely.

THEOREM 10.5. *Suppose ψ_i, ψ are as in Theorem 9.3 and $\beta_1, \ldots, \beta_n$ as above. Let $Irr(\beta_1, \ldots, \beta_n)$ denote the set of all irreducible representations of all $S_s(F)$, $s \geq 0$, supported on $\mathcal{S}_{\beta_1,\ldots,\beta_n}(\psi_i; \psi)$, and similarly for $Irr(\beta_1), \ldots, Irr(\beta_n)$. Then the maps (cf. Definitions 10.3 and 10.4)*

$$Irr(\beta_1, \ldots, \beta_n) \underset{\boldsymbol{\Psi}_{\beta_1,\ldots,\beta_n}}{\overset{(\psi_{\beta_1}, \ldots, \psi_{\beta_n})}{\rightleftarrows}} Irr(\beta_1) \times \cdots \times Irr(\beta_n)$$

give a bijective correspondence with the following properties:

(1) $\boldsymbol{\Psi}_{\beta_1,\ldots,\beta_n}$ *and* $(\psi_{\beta_1}, \ldots, \psi_{\beta_n})$ *are inverses of each other.*

(2) *If $\sigma(\beta_\ell) \in Irr(\beta_\ell)$ is a representation of $S_{n_\ell+r}(F)$, then $\pi = \boldsymbol{\Psi}_{\beta_1,\ldots,\beta_n}(\sigma(\psi_1; \psi), \ldots, \sigma(\psi_m; \psi))$ is a representation of $S_{n_1+\cdots+n_n+r}(F)$ (recall ψ is a representation of $S_r(F)$).*

(3) $\widetilde{\boldsymbol{\Psi}_{\beta_1,\ldots,\beta_n}}(\sigma(\beta_1), \ldots, \sigma(\beta_n)) = \boldsymbol{\Psi}_{\beta_1,\ldots,\beta_n}(\widetilde{\sigma(\beta_1)}, \ldots, \widetilde{\sigma(\beta_n)})$ *and* $\psi_{\beta_\ell}(\tilde{\pi}) = \widetilde{\psi_{\beta_\ell}}(\pi)$, *where* ˜ *denotes contragredient.*

(4) $\widehat{\boldsymbol{\Psi}_{\beta_1,\ldots,\beta_n}}(\sigma(\beta_1), \ldots, \sigma(\beta_n)) = \boldsymbol{\Psi}_{\beta_1,\ldots,\beta_n}(\widehat{\sigma(\beta_1)}, \ldots, \widehat{\sigma(\beta_n)})$ *and* $\psi_{\beta_\ell}(\hat{\pi}) = \widehat{\psi_{\beta_\ell}(\pi)}$, *where* ˆ *denotes the involution of Aubert (cf. Theorem 4.1).*

(5) *Suppose that*

$$s_{GL}(\sigma(\beta_\ell)) = \sum_j c_j(\beta_\ell)\tau_j(\beta_\ell) \otimes \psi,$$

where $\tau_j(\beta_\ell)$ *is an irreducible representation and* $c_j(\beta_\ell)$ *its multiplicity. With* $\mu^*_{\beta_1,\dots,\beta_k}$ *as in Definition 10.1,*

$$\begin{aligned}\mu^*_{\beta_1,\dots,\beta_k}&(\Psi_{\beta_1,\dots,\beta_n}(\sigma(\beta_1),\dots,\sigma(\beta_n)))\\ &= \sum_{j_1,\dots,j_k}(c_{j_1}(\beta_1)\cdots c_{j_k}(\beta_k))\tau_{j_1}(\beta_1)\times\cdots\times\tau_{j_k}(\beta_k)\\ &\qquad \otimes \Psi_{\beta_{k+1},\dots,\beta_n}(\sigma(\beta_{k+1}),\dots,\sigma(\beta_n)).\end{aligned}$$

(6) *Let* $\rho = \rho(\beta_1)\times\cdots\times\rho(\beta_n)$ *be an irreducible representation of* $GL(F)$ *with support contained in* $\mathcal{S}_{\beta_1,\dots,\beta_n}(\psi_i)$ *and* $\Psi = \Psi_{\beta_1,\dots,\beta_n}(\sigma(\beta_1),\dots,\sigma(\beta_n))$ *an irreducible representation of* $S(F)$ *with support contained in* $\mathcal{S}_{\beta_1,\dots,\beta_n}(\psi_i;\psi)$*. (We allow the possibility that* $\rho(\beta_\ell)=1$ *or* $\sigma(\beta_\ell)=\psi$*.) Suppose*

$$\rho(\beta_\ell)\rtimes\sigma(\beta_\ell) = \sum_j m_j(\beta_\ell)\sigma_j(\beta_\ell),$$

with $\sigma_j(\beta_\ell)$ *irreducible and* $m_j(\beta_\ell)$ *its multiplicity. Then,*

$$\rho\rtimes\Psi = \sum_{j_1,\dots,j_n}(m_{j_1}(\beta_1)\cdots m_{j_n}(\beta_n))\Psi_{\beta_1,\dots,\beta_n}(\sigma_{j_1}(\beta_1),\dots,\sigma_{j_n}(\beta_n)).$$

(7) $\Psi_{\beta_1,\dots,\beta_n}(\sigma(\beta_1),\dots,\sigma(\beta_n))$ *is tempered (resp. square-integrable) if and only if* $\sigma(\beta_1),\dots,\sigma(\beta_n)$ *are all tempered (resp. square-integrable)*

(8) *Suppose, in the subrepresentation setting of the Langlands classification,*

$$\sigma(\beta_\ell) = L(\nu^{\alpha_1}\tau_1(\beta_\ell),\dots,\nu^{\alpha_k}\tau_k(\beta_\ell);T(\beta_\ell))$$

for $i=1,\dots,n$ *(n.b.* $\tau_j(\beta_\ell)$ *may be the trivial representation of* $GL_0(F)$*;* $T(\beta_\ell)$ *may just be* ψ*). Then,*

$$\begin{aligned}&\Psi_{\beta_1,\dots,\beta_n}(\sigma(\beta_1),\dots,\sigma(\beta_n))\\ &\quad= L(\nu^{\alpha_1}\tau_1(\beta_1)\times\cdots\times\nu^{\alpha_1}\tau_1(\beta_n),\dots,\nu^{\alpha_k}\tau_k(\beta_1)\times\cdots\times\nu^{\alpha_k}\tau_k(\beta_n);\\ &\qquad \Psi_{\beta_1,\dots,\beta_n}(T(\beta_1),\dots,T(\beta_n))).\end{aligned}$$

In the other direction, if

$$\pi = L(\nu^{\alpha_1}\tau_1(\beta_1)\times\cdots\times\nu^{\alpha_1}\tau_1(\beta_n),\dots,\nu^{\alpha_k}\tau_k(\beta_1)\times\cdots\times\nu^{\alpha_k}\tau_k(\beta_n);T(\beta_1,\dots,\beta_n)),$$

then

$$\psi_{\beta_\ell}(\pi) = L(\nu^{\alpha_1}\tau_1(\beta_1), \dots, \nu^{\alpha_k}\tau_k(\beta_\ell); \psi_{\beta_\ell}(T(\beta_1, \dots, \beta_n))).$$

(In the quotient setting of the Langlands classification, the same results hold.)

Remark 10.6. Special cases of this correspondence may be seen in Proposition 3.5 of [Tad2] and Theorem 9.8 of [Tad5].

This refinement has an interpretation in terms of tensor products analogous to that of Proposition 9.6.

Definition 10.7. Let $\psi_i; \psi$ and $\beta_1, \dots, \beta_n$ be as in Theorem 10.5. Let $R_{\beta_1,\dots,\beta_n}(\psi_i)$ denote the subalgebra of $R(\psi_i)$ supported on $\mathcal{S}_{\beta_1,\dots,\beta_n}(\psi_i)$. Similarly, let $R_{\beta_1,\dots,\beta_n}(\psi_i;\psi)$ denote the (additive) subgroup of $R(\psi_i;\psi)$ generated by $Irr(\beta_i)$.

Again, an immediate consequence of Remark 8.7 ([Zel]) is the following:

PROPOSITION 10.8. *$R_{\beta_1,\dots,\beta_n}(\psi_i) \cong R_{\beta_1}(\psi_i)\otimes\cdots\otimes R_{\beta_n}(\psi_i)$ as Hopf subalgebras of $R(\psi_i)$.*

The same argument as in Lemma 9.9 allows us to show the following:

LEMMA 10.9. *Suppose ψ_i, ψ and $\beta_1, \dots, \beta_n$ are as in Theorem 10.5. Let $\sigma(\beta_j) \in Irr(\beta_j)$ for $j = 1, \dots, n$. Suppose that*

$$\mu^*(\sigma(\beta_j)) = \sum_{\ell_j} n_{\ell_j}(\beta_j)\eta_{\ell_j}(\beta_j) \otimes \theta_{\ell_j}(\beta_j),$$

with $\eta_{\ell_j}(\beta_j) \otimes \theta_{\ell_j}(\beta_j)$ irreducible and $n_{\ell_j}(\beta_j)$ its multiplicity. Then,

$$\begin{aligned}\mu^*(&\Psi_{\beta_1,\dots,\beta_n}(\sigma(\beta_1), \dots, \sigma(\beta_n)))\\ &= \sum_{\ell_1,\dots,\ell_n} (n_{\ell_1}(\beta_1)\cdots n_{\ell_n}(\beta_n))(\eta_{\ell_1}(\beta_1) \times \cdots \times \eta_{\ell_n}(\beta_n))\\ &\qquad \otimes \Psi_{\beta_1,\dots,\beta_n}(\theta_{\ell_1}(\beta_1), \dots, \theta_{\ell_n}(\beta_n)).\end{aligned}$$

From this lemma and Theorem 10.5, we obtain the following result:

PROPOSITION 10.10. *Let ψ_i, ψ and $\beta_1, \dots, \beta_n$ be as in Theorem 10.5. Then,*

$$R_{\beta_1,\dots,\beta_n}(\psi_i;\psi) \cong R_{\beta_1}(\psi_i;\psi) \otimes \cdots \otimes R_{\beta_n}(\psi_i;\psi)$$

as $R_{\beta_1,\dots,\beta_n}(\psi_i) \cong R_{\beta_1}(\psi_i) \otimes \cdots \otimes R_{\beta_n}(\psi_i)$ M_S^*-Hopf modules.

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