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ON THE IWAHORI-MATSUMOTO INVOLUTION AND APPLICATIONS

BY CHRIS JANTZEN

ABSTRACT. – We begin by collecting some results on the Iwahori-Matsumoto involution. These are then used to verify a special case of a conjecture of Arthur on R-groups.

1. Introductory material

1.1. Introduction

The purpose of this paper is to address a special case of a conjecture of Arthur. Suppose G is a split, connected, reductive p-adic group, M the Levi factor of a standard parabolic subgroup of G, and σ a square-integrable representation of M. Then, the reducibility of the induced representation $i_{GM}\sigma$ (in the notation of [B-Z]; *see* section 1.2) is governed by the R-group. The R-group is a subgroup of the Weyl group which gives a basis of the intertwining algebra consisting of (normalized) standard intertwining operators. Classically, the R-group is defined in terms of the Plancherel measure, hence requires that σ be square-integrable. There is an alternate description of the R-group, in terms of the L-group and the Langlands correspondence. Arthur conjectured that in this context, one should be able to define an R-group, with the right basic properties, for certain cases of nontempered σ. For example, if σ is one-dimensional unitary, then the R-group for $i_{GM}\sigma$ should be the same as the R-group obtained if σ is replaced by the corresponding Steinberg representation.

In what follows, we restrict ourselves to the case where the representations are generated by their Iwahori-fixed vectors. This allows us to work with the corresponding representations of the Hecke algebra $\mathcal{H}(G//B)$ (B=Iwahori subgroup). Our main tool is the Iwahori-Matsumoto involution.

We now describe the contents of this paper section by section. The next section introduces some notation and basic results that are needed in the rest of the paper.

In section 2.1, we recall that the study of representations of G generated by their Iwahori-fixed vectors may be reduced to studying the associated representation of $\mathcal{H} = \mathcal{H}(G//B)$. We also review two ways of constructing induced representations for $\mathcal{H}$–taking Iwahori-fixed vectors of induced representations of G and tensoring up from subalgebras–and how they are related. Both settings will be used–it is preferable to discuss the Iwahori-

Matsumoto involution in terms of tensoring, while standard intertwining operators are best dealt with in the Iwahori-fixed vector setting.

In section 2.2, we discuss the Iwahori-Matsumoto involution. In the tensor product setting for induced representations of $\mathcal{H}$, it is not difficult to show that

$$\widehat{\mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}\sigma} \cong \mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}\hat{\sigma}.$$

Therefore, the corresponding fact holds for representations of $\mathcal{H}$ in the Iwahori-fixed vector setting, hence for representations of G. If $\pi = i_{GA}\chi$ (A=split torus), set $\pi_0 = i_{GA}\chi^{-1}$. We will also need to know something about the map $\mathcal{M} : V_\pi^B \longrightarrow V_{\pi_0}^B$ giving the equivalence $\hat{\pi} \cong \pi_0$. So, we close section 2.2 with a characterization of $\mathcal{M}$ which will suffice for the purposes of chapter 3.

Section 3.1 reviews the construction and properties of standard intertwining operators and R-groups.

In section 3.2, we look at unnormalized standard intertwining operators. If s is a simple reflection, the operator $\mathcal{A}_s(\chi)$ intertwines $i_{GA}\chi$ and $i_{GA}s\chi$. We give $\mathcal{A}_s(\chi) : V_{i_{GA}\chi}^B \longrightarrow V_{i_{GA}s\chi}^B$ in Proposition 3.2.1. Now, $\mathcal{A}_s(\chi)$ must also intertwine $\widehat{i_{GA}\chi}$ and $\widehat{i_{GA}s\chi}$. In particular, it gives rise to

$$\mathcal{M}\mathcal{A}_s(\chi)\mathcal{M}^{-1} : V_{i_{GA}\chi^{-1}}^B \longrightarrow V_{i_{GA}s\chi^{-1}}^B$$

By comparing this with $\mathcal{A}_s(\chi^{-1}) : V_{i_{GA}\chi^{-1}}^B \longrightarrow V_{i_{GA}s\chi^{-1}}^B$–using the results on $\mathcal{A}_s(\chi)$ and the results on $\mathcal{M}$ from section 2.2–we show that $\mathcal{M}\mathcal{A}_s(\chi)\mathcal{M}^{-1} = -\mathcal{A}_s(\chi^{-1})$. More generally, for $w \in W$

$$\mathcal{M}\mathcal{A}_w(\chi)\mathcal{M}^{-1} = (-1)^{\ell(w)}\mathcal{A}_w(\chi^{-1})$$

Finally, in section 3.3, we bring normalizing factors into the picture. If we take σ discrete series (with suitable properties), we get a relation similar to that above for normalized intertwining operators: suitably interpreted,

$$\mathcal{M}A_w(\sigma)\mathcal{M}^{-1} = \lambda(w,\sigma)A_w(\hat{\sigma})$$

where $A_w(\sigma)$ (resp. $A_w(\hat{\sigma})$) denote the normalized intertwining operators for $\pi = i_{GM}\sigma$ (resp. $\hat{\pi} \cong i_{GM}\hat{\sigma}$) and $\lambda(w,\sigma)$ is a scalar. Further, we have that $\lambda(r,\sigma) = \rho_{St}(r)$ for $r \in R$ =R-group of π, where ρ_{St} is a certain one-dimensional representation of R. Then we can easily show that if we set $(\hat{\pi})_\rho = \widehat{\pi_{\rho_{St}\rho}}$ for all $\rho \in \hat{R}$ (dual of R), then R will have the properties we want for an R-group of $\hat{\pi}$.

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1.2. Notation and preliminaries

In this section, we introduce some notation and basic facts that will be needed in the rest of the paper.

In this paper, G will denote a split connected reductive p-adic group. We also assume that G has a reduced root system.

Let F be a p-adic field with $charF = 0$. We let $\mathcal{O}$ denote the ring of integers and $\mathcal{P}$ the prime ideal. Then, $\mathcal{O}/\mathcal{P}$ is a finite field with q elements; we write $\mathbb{F}_q = \mathcal{O}/\mathcal{P}$. Let ϖ be a uniformizer. We normalize absolute value so that $|\varpi| = q^{-1}$. A character of $F^\times$ is called unramified if its restriction to $\mathcal{O}^\times$ is trivial (for our purposes, a character does not need to be unitary). Such a character necessarily has the form $|\cdot|^s$. These are the characters of $F^\times$ that will be needed later.

Fix P_{min}, a minimal parabolic subgroup. It has Levi factorization $P_{min} = AU_{min}$, where A denotes a maximal split torus. We let Δ denote the roots of G with respect to A, Δ^+ the positive roots corresponding to P_{min}, and Π the simple roots. For $\alpha \in \Delta$, let s_α denote the corresponding reflection and $h_\alpha : F^\times \longrightarrow A$ the associated one-parameter subgroup. The Weyl group of G is $W = Norm_G(A)/A$. We will not be too careful about distinguishing between elements of W and their representatives, but their representatives will always be taken in $K = G(\mathcal{O})$. Note that if $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, then $W = \langle s_{\alpha_1}, \ldots, s_{\alpha_n}\rangle$. For $w \in W$, let $\ell(w)$ denote the length of w, *i.e.*, the smallest ℓ such that w may be written $w = s_{\alpha_{i_1}} \ldots s_{\alpha_{i_\ell}}$. There is a unique element of W having maximal length; we denote it by w_0. Note that $w_0^2 = 1$.

We now discuss parabolic subgroups. The standard parabolic subgroups may be parameterized by subsets Φ of Π. In particular,

$$P = \langle P_{min}, \{s_\alpha\}_{\alpha\in\Phi}\rangle.$$

is the parabolic subgroup of G associated to Φ. P has a Levi factorization $P = MU$, with M reductive and U unipotent. We let δ_P denote the modular function for P; we just use δ if $P = P_{min}$. For L, M Levis of standard parabolics, let

$$W^{LM} = \{w \in W | w \cdot (P_{min} \cap L) \subset P_{min},\ w^{-1} \cdot (P_{min} \cap M) \subset P_{min}\}.$$

We note that these correspond to the elements of shortest length in the double cosets $W_M \backslash W / W_L$ (W_L = Weyl group of L, etc.).

We now discuss induced representations. Suppose $P = MU$ is a standard parabolic subgroup and (σ, M, V_σ) is an (admissible) representation of M. The induced representation $i_{GM}\sigma$ has space

$$V = \{f : G \longrightarrow V_\sigma | f(pg) = \delta_P^{\frac{1}{2}}(m)\sigma(m)f(g) \text{ for all } p = mu \in P,\ g \in G\}.$$

(more precisely, we want the smooth vectors in V). The action is right regular, *i.e.*, $(\pi(g)f)(x) = f(xg)$. One important property that is used later is induction in stages. Note that if $L < M$ are the Levis of standard parabolics, then L is the Levi of a standard parabolic of M (in particular, $(LU_{min}) \cap M$ is a parabolic subgroup of M). Then,

$$i_{GL} = i_{GM} \circ i_{ML}.$$

We also mention that for $P = P_{min}$, $i_{GA}\delta^{-\frac{1}{2}}$ has the trivial representation as a subrepresentation (an easy computation) and the Steinberg representation as a quotient (*cf.* [Cas2]).

Finally, we mention a few compact subgroups which will be needed later. Let $K = G(\mathcal{O})$. Also, let $K_1 = \{k \in K | k \equiv I \bmod \mathcal{P}\}$. Note that $K_1 \triangleleft K$. The Iwahori subgroup, which we denote by B, plays an important role in this paper. Let

$$\Psi : K \longrightarrow G(\mathbb{F}_q)$$

denote the reduction mod $\mathcal{P}$ homomorphism. Then, the Iwahori subgroup is

$$B = \Psi^{-1}(P_{min}(\mathbb{F}_q)).$$

We normalize Haar measure so that B has measure = 1.

2. On the Iwahori-Matsumoto involution

2.1. Hecke algebra representations

We begin this section by recalling the results of Borel and Casselman that allow one to reduce the study of certain questions about an unramified principal series representation (π, G, V) (or subquotients thereof) to the corresponding questions about the (finite-dimensional) representation $(\pi, \mathcal{H}, V^B)$ of the Hecke algebra $\mathcal{H} = \mathcal{H}(G//B)$ (B=Iwahori subgroup). We then review the structure of $\mathcal{H}$. After this, we discuss another way of constructing induced representations for $\mathcal{H}$: tensoring up from subalgebras (the first method being taking Iwahori-fixed vectors of induced representations of G). We close by relating the two ways of inducing.

We start by recalling the definition of unramified principal series. A character χ of A is called unramified if $\chi|_{A(\mathcal{O})}$ is trivial. (For $A \cong (F^\times)^n$, χ is the product of n unramified characters of $F^\times$.) An unramified principal series representation is a representation of the form $i_{GA}\chi$ with χ unramified.

We now recall some results of Borel and Casselman (*cf.* [Bor], [Cas1]). Let G be as in chapter 1, and B its Iwahori subgroup. Let $\mathcal{H}$ denote the Hecke algebra of smooth, compactly supported, B-biinvariant functions on G, *i.e.*,

$$\mathcal{H} = \{f \in C_c^\infty(G) | f(b_1 g b_2) = f(g) \text{ for all } b_1, b_2 \in B \text{ and } g \in G\}.$$

It is an algebra under convolution. If (π, G, V) is an admissible representation of G, then $(\pi, \mathcal{H}, V^B)$is a finite-dimensional representation of $\mathcal{H}$, where $V^B = \{v \in V | \pi(b)v = v \text{ for all } b \in B\}$ and $\mathcal{H}$ acts on V^B according to

$$\pi(h)v = \int_G h(g)\pi(g)v dg.$$

This is of interest in the case where π is generated by its B-fixed vectors. We call such a π B-unramified. In particular, we are interested in the case where π is an unramified principal series representation or a subquotient thereof. Borel and Casselman have shown that such a π is B-unramified. Moreover, they have shown that any irreducible B-unramified representation is necessarily a component of an unramified principal series representation. Further, we have the following:

THEOREM 2.1.1. – *The category of admissible B-unramified representations of G and the category of finite-dimensional representations of $\mathcal{H}$ are equivalent. The equivalence is given by* $(\pi, G, V) \longmapsto (\pi, \mathcal{H}, V^B)$. *Note that*

$$\mathrm{Hom}_{\mathcal{H}}(\pi, \pi) \cong \mathrm{Hom}_G(\pi, \pi).$$

Thus, if we wish to study subquotients of unramified principal series, it is enough to work with the corresponding representations of $\mathcal{H}$ on the B-fixed vectors, which is what we do. When working with the unramified principal series $i_{GA}\chi$, we use the basis $\{f_w\}_{w\in W}$ of V_π^B, where

$$f_w(g) = \begin{cases} \delta^{\frac{1}{2}}\chi(p) & \text{if } g = pwb \in PwB \\ 0 & \text{if not.} \end{cases}$$

That these constitute a basis follows easily from the decomposition $G = \cup_{w\in W} PwB$ (*cf.* [Cas3]).

At this point, it is natural to review the structure of $\mathcal{H}$. Here, we favor the description of Bernstein-Zelevinsky (*cf.* [Lus]; the classic description may be found in [I-M]). For $x \in G$, let T_x denote the characteristic function of the double-coset BxB. Then, as a vector space, we have $\mathcal{H} \cong \mathcal{H}_W \otimes \Theta$. Here, $\mathcal{H}_W$ is the finite Hecke algebra $\mathcal{H}(K//B)$. It has basis $\{T_w\}_{w\in W}$. Further,

$$\begin{aligned} T_s^2 &= (q-1)T_s + q \quad \text{for } s \text{ simple} \\ T_{w_1}T_{w_2} &= T_{w_1w_2} \quad \text{if } \ell(w_1) + \ell(w_2) = \ell(w_1w_2). \end{aligned}$$

Θ is an abelian subalgebra with basis $\{\theta_a | a \in A/A \cap K\}$. For $a \in A$, choose $a_1, a_2 \in A^- = \{a \in A|\ |\alpha(a)| \leq 1\ \forall\ \alpha \in \Pi\}$ such that $a = a_1a_2^{-1}$. Then $\theta_a = \delta^{\frac{1}{2}}(a)T_{a_1}T_{a_2}^{-1}$. Note that for $s = s_\alpha$,

$$\theta_a T_s = T_s\theta_{sas} + (q-1)\frac{\theta_a - \theta_{sas}}{1 - \theta_{-\alpha}}$$

where $\theta_\alpha = \theta_{h_\alpha(\varpi)}$.

There is also a notion of induced representations for Hecke algebras. Let $M \subset G$ be a standard Levi and $\mathcal{H}_M \subset \mathcal{H}$ its Hecke algebra. More precisely, $\mathcal{H}_M$ is generated by $\{T_w\}_{w\in W_M}$ and Θ. Then, if σ is a representation of $\mathcal{H}_M$, we define the induced representation $\pi = \mathrm{Ind}_{\mathcal{H}_M}^{\mathcal{H}}\sigma$ as follows:

space: $V_\pi = \mathcal{H} \otimes_{\mathcal{H}_M} V_\sigma = \mathcal{H} \otimes_{\mathbb{C}} V_\sigma/\mathcal{K}(\sigma)$
where $\mathcal{K}(\sigma) =$ subspace generated by
$\{hh_M \otimes v - h \otimes \sigma(h_M)v|\ h \in \mathcal{H},\ h_M \in \mathcal{H}_M,\ v \in V_\sigma\}$

action: for $h_1,\ h_2 \in \mathcal{H}$,
$\pi(h_1)[h_2 \otimes v + \mathcal{K}(\sigma)] = h_1h_2 \otimes v + \mathcal{K}(\sigma).$

When working with $\pi = \mathrm{Ind}_{\Theta}^{\mathcal{H}}\chi$, we use the basis $\{T_w + \mathcal{K}(\chi)\}_{w\in W}$ for $\mathcal{H} \otimes_{\Theta} \mathbb{C}_{\chi}$.

We now relate this notion of induction to the induction for groups described in section 1.2.

PROPOSITION 2.1.2. – *Suppose σ is a B-unramified representation of a standard Levi M. Let $w_0' \in W^{MA}$ denote the longest element and set $M' = w_0 M w_0 = w_0' M w_0'^{-1}$ (a standard Levi). Then, as representations of $\mathcal{H}$,*

$$(i_{GM}\sigma)^B \cong \mathrm{Ind}_{\mathcal{H}_{M'}}^{\mathcal{H}} w_0' \cdot \sigma^{B\cap M}$$

First, in case $\mathcal{H}_M = \Theta$, the equivalence is in [Re2]. The map $\mathcal{E} : \mathcal{H}\otimes_{\Theta}\mathbb{C}[w_0\chi] \longrightarrow V^B_{i_{GA}\chi}$ has

$$\mathcal{E} : 1 + \mathcal{K}(w_0\chi) \longmapsto f_{w_0}$$

If we apply T_w to each side, equivariance gives

$$\mathcal{E} : T_w + \mathcal{K}(w_0\chi) \longmapsto \pi_1(T_w) f_{w_0}$$

where $\pi_1 = (i_{GA}\chi)^B$

The proof of our proposition goes roughly as follows: if $\sigma \hookrightarrow i_{MA}\chi$, we can find the subspace of V_{π_1} corresponding to $(i_{GM}\sigma)^B$. Similarly, we can find the subspace of $V_{\pi_2}(\pi_2 = \mathrm{Ind}_{\Theta}^{\mathcal{H}} w_0\chi)$ corresponding to $\mathrm{Ind}_{\mathcal{H}'_M}^{\mathcal{H}} w_0' \cdot \sigma^{B\cap M}$. We then show that these subspaces match up under $\mathcal{E}$.

The following two lemmas will help us deal with the π_1 side.

LEMMA 2.1.3. – *Let $\pi = i_{GA}\chi$ and s a simple reflection. Then,*

$$\pi(T_s) f_w = \begin{cases} f_{ws} & if\ w \in W^{MA} \\ q f_{ws} + (q-1) f_w & if\ w \notin W^{MA} \end{cases}$$

where M is the Levi factor of $\langle P_{min}, s\rangle$.

Proof. – *See* [Re1]. p. 325. □

We note that an immediate consequence of this is that for $w \in W$, $f_w = \pi(T_{w^{-1}}) f_1$.

LEMMA 2.1.4. – *Suppose $\{\tilde{f}_i\}_{i=1,\dots,k}$ is a basis for $V_{\sigma}^{B\cap M} \subset V_{i_{MA}\chi}^{B\cap M}$. If $\tilde{f}_i = \sum_{x\in W_M} b_x \tilde{f}_x \in V_{i_{MA}\chi}^{B\cap M}$, then let $f_i = \sum_{x\in W_M} b_x f_x \in V^B_{i_{GA}\chi}$. Then, a basis for $V^B_{i_{GM}\sigma} \subset V^B_{i_{GA}\chi}$ is*

$$\{\pi_1(T_w) f_i\}_{\substack{i=1,\dots,k \\ w\in W^{MA}}}$$

where $\pi_1 = i_{GA}\chi$.

Proof. – We begin by following the proof that $\{f_w\}_{w\in W}$ constitutes a basis for $V^B_{i_{GA}\chi}$. First, we note that we have a decomposition $G = \cup_{w\in W^{AM}} PwB$ $(P = MU)$. Therefore, we can define Ψ by

$$\Psi(w, \tilde{f})(g) = \begin{cases} 0 & \text{if } g \notin PwB \\ \delta_P^{\frac{1}{2}}(m)\sigma(m)\tilde{f} & \text{if } g = muwb \in PwB \end{cases}$$

for $w \in W^{AM}$, $\tilde{f} \in V_{\sigma}^{B\cap M} \subset V_{i_{MA}\chi}^{B\cap M}$. It is not difficult to see that $\Psi(w, \tilde{f})$ is well defined and $\{\Psi(w, \tilde{f})\}_{\substack{i=1,\ldots,k \\ w\in W^{AM}}}$ constitute a basis for $V_{i_{GM}\sigma}^{B}$.

Next, using the equivalence $i_{GM} \circ i_{MA}\chi \cong i_{GA}\chi$ (implemented by $f \in V_{i_{GM}(i_{MA}\chi)}$ maps to $f(\cdot)(1) \in V_{i_{GA}\chi}$), we see that $\Psi(w, \tilde{f}_x) = f_{xw} \in V_{i_{GA}\chi}^{B}$. Since $\ell(xw) = \ell(x) + \ell(w)$, the preceding lemma shows that $\Psi(w, \tilde{f}_x) = \pi_1(T_{w^{-1}})f_x$. The lemma follows. □

Next, let $C_{w_0'}$ denote conjugation by w_0'. Then, $C_{w_0'} : M \longrightarrow M'$. Further, since $C_{w_0'} : B \cap M \longrightarrow B \cap M'$, we get a corresponding isomorphism $C_{w_0'} : \mathcal{H}_M \longrightarrow \mathcal{H}_{M'}$. Note that for $x \in W_M$, $a \in A$,

$$C_{w_0'}(\tilde{T}_x) = \tilde{T}_{w_0' x w_0'^{-1}}$$
$$C_{w_0'}(\tilde{\theta}_a) = \tilde{\theta}_{w_0' a w_0'^{-1}}$$

where $\tilde{T}_x$, $\tilde{\theta}_a \in \mathcal{H}_M$ and $\tilde{T}_{w_0' x w_0'^{-1}}$, $\tilde{\theta}_{w_0' a w_0'^{-1}} \in \mathcal{H}_{M'}$ (the tildes are to emphasize that we are viewing these as elements of $\mathcal{H}_M$, $\mathcal{H}_{M'}$ rather than $\mathcal{H}$).

LEMMA 2.1.5. – *$w_0' \cdot \operatorname{Ind}_{\Theta}^{\mathcal{H}_M} \chi \cong \operatorname{Ind}_{\Theta}^{\mathcal{H}_{M'}} w_0'\chi$ with the equivalence given by*

$$\tilde{T}_w + \mathcal{K}_M(\chi) \longmapsto \tilde{T}_{w_0' w w_0'^{-1}} + \mathcal{K}_{M'}(w_0'\chi).$$

Proof. – Straightforward. □

LEMMA 2.1.6. – *Let σ be a representation of $\mathcal{H}_M$. Suppose $\{\tilde{T}_i + \mathcal{K}_M(\chi)\}_{i=1,\ldots,k}$ is a basis for $V_\sigma \subset \mathcal{H}_M \otimes_\Theta \mathbb{C}_\chi$. Then a basis for $\mathcal{H} \otimes_{\mathcal{H}_M} V_\sigma \subset \mathcal{H} \otimes_\Theta \mathbb{C}_\chi$ consists of*

$$\{\pi_2(T_w)T_i + \mathcal{K}(\chi)\}_{\substack{i=1,\ldots,k \\ w\in W^{MA}}}$$

where $\pi_2 = \operatorname{Ind}_{\Theta}^{\mathcal{H}} \chi$.

Proof. – Consider $\pi(T_w)T_i + \mathcal{K}(\chi)$. Under the equivalence $\mathcal{H} \otimes_\Theta \mathbb{C}_\chi \cong \mathcal{H} \otimes_{\mathcal{H}_M} (\mathcal{H}_M \otimes_\Theta \mathbb{C}_\chi)$ (which is implemented by $h + \mathcal{K}(\chi) \in \mathcal{H} \otimes_\Theta \mathbb{C}_\chi$ maps to $h \otimes [1 + \mathcal{K}_M(\chi)] + \mathcal{K}(\operatorname{Ind}_{\Theta}^{\mathcal{H}_M} \chi) \in \mathcal{H} \otimes_{\mathcal{H}_M} (\mathcal{H}_M \otimes_\Theta \mathbb{C}_\chi)$), this corresponds to

$$T_w T_i \otimes [1 + \mathcal{K}_M(\chi)] + \mathcal{K}(\operatorname{Ind}_{\Theta}^{\mathcal{H}_M} \chi) = T_w \otimes [\tilde{T}_i + \mathcal{K}_M(\chi)] + \mathcal{K}(\operatorname{Ind}_{\Theta}^{\mathcal{H}_M} \chi) \in \mathcal{H} \otimes_{\mathcal{H}_M} V_\sigma.$$

Since $sp\{\pi_2(T_w)T_i + \mathcal{K}(\chi)\}$ has the same dimension as $\mathcal{H} \otimes_{\mathcal{H}_M} V_\sigma$, the lemma holds. □

We now return to the proof of the theorem. Suppose $\sigma \hookrightarrow i_{MA}\chi$. Then $\sigma^B \hookrightarrow \operatorname{Ind}_{\Theta}^{\mathcal{H}_M} w_0''\chi$ by the case of minimal P described above, where $w_0'' \in W_M$ is the element of maximal length. Suppose that $\{\tilde{T}_i + \mathcal{K}_M(w_0''\chi)\}_{i=1,\ldots,k}$ constitutes a basis for $V_{\sigma^{B\cap M}} \subset \mathcal{H}_M \otimes_\Theta \mathbb{C}_{w_0''\chi}$. If $\tilde{\mathcal{E}}$ denotes the map giving the equivalence $\operatorname{Ind}_{\Theta}^{\mathcal{H}_M} w_0''\chi \cong (i_{MA}\chi)^B$, we see that $\{\tilde{\mathcal{E}}[\tilde{T}_i + \mathcal{K}_M(w_0''\chi)]\}_{i=1,\ldots,k} = \{\tilde{\pi}_1(\tilde{T}_i)\tilde{f}_{w_0''}\}_{i=1,\ldots,k}$ gives a basis for $V_\sigma^{B\cap M} \subset V_{i_{MA}\chi}^{B\cap M}$. By Lemma 2.1.4, $V_{i_{GM}\sigma}^{B} \subset V_{i_{GA}\chi}^{B}$ has basis

$$\{\pi_1(T_w)\pi_1(T_i)f_{w_0''}\}_{\substack{i=1,\ldots,k \\ w\in W^{MA}}}$$

On the other hand, by Lemmas 2.1.5 and 2.1.6, $\mathcal{H} \otimes_{\mathcal{H}_{M'}} V_{w_0'\sigma^{B\cap M}} \subset \mathcal{H} \otimes_\Theta \mathbb{C}_{w_0\chi}$ has basis

$$\{\pi_2(T_w)[C_{w_0'}(T_i) + \mathcal{K}(w_0\chi)]\}_{\substack{i=1,\ldots,k \\ w\in W^{MA}}}$$

(where $C_{w_0'}$ acts on $T_i \in \mathcal{H}$ the same way it acts on $\tilde{T}_i \in \mathcal{H}_M$). Both these have dimension $|W^{MA}| \cdot k$. Therefore, if

$$\mathcal{E} : sp\{\pi_2(T_w)[C_{w_0'}(T_i) + \mathcal{K}(w_0\chi)]\} \hookrightarrow sp\{\pi_1(T_w)\pi_1(T_i)f_{w_0''}\}$$

we will have that the subspaces corresponding to $(i_{GM}\sigma)^B \subset (i_{GA}\chi)^B$ and $(\mathrm{Ind}_{\mathcal{H}_{M'}}^{\mathcal{H}} w_0' \cdot \sigma^{B\cap M}) \subset (\mathrm{Ind}_{\Theta}^{\mathcal{H}} w_0\chi)$ match up, verifying the proposition.

We compute, letting $T_i' = C_{w_0'}(T_i)$:

$$\mathcal{E}\pi_2(T_w)[C_{w_0'}(T_i) + \mathcal{K}(w_0\chi)] = \pi_1(T_w T_i' T_{w_0})f_1$$

If $T_i = \sum_{x\in W_M} a_x T_x$, we get $T_i' = \sum_{x\in W_M} a_x T_{w_0' x w_0'^{-1}}$ and

$$\begin{aligned} T_w T_i' T_{w_0} &= \sum_{x\in W_M} a_x T_w T_{w_0' x w_0'^{-1}} (T_{w_0'} T_{w_0''}) \\ &= \sum_{x\in W_M} a_x T_w T_{w_0' x} T_{w_0''} \\ &= T_w T_{w_0'} T_i T_{w_0''} \end{aligned}$$

since $\ell(w_0'x) = \ell(w_0'xw_0'^{-1}) + \ell(w_0') = \ell(w_0') + \ell(x)$.

Next, define $a_w(x,y)$ by

$$T_w T_{w_0'} = \sum_{y\in W^{MA}} \sum_{x\in W_M} a_w(x,y) T_{yx}$$

so that

$$T_w T_{w_0'} T_i = \sum_{y\in W^{MA}} \sum_{x\in W_M} a_w(x,y) T_y T_x T_i$$

Now, $sp\{T_i\}$ is invariant under left multiplication by $T_x, x \in W_M$ (from the invariance of $V_{\sigma^{B\cap M}}$ under $\mathcal{H}_M$). So, we write

$$T_x T_i = \sum_{j=1}^{k} c_{x,i}(j) T_j$$

Then, we have (from above)

$$\begin{aligned} T_w T_i' T_{w_0} &= T_w T_{w_0'} T_i T_{w_0''} \\ &= \sum_{y\in W^{MA}} \sum_{x\in W_M} \sum_{j=1}^{k} a_w(x,y) c_{x,i}(j) T_y T_j T_{w_0''} \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{E}\pi_2(T_w)[C_{w_0'}(T_i) + \mathcal{K}(w_0\chi)] &= \sum_{y\in W^{MA}} \sum_{x\in W_M} \sum_{j=1}^{k} a_w(x,y) c_{x,i}(j) \pi_1(T_y T_j T_{w_0''}) f_1 \\ &= \sum_{y\in W^{MA}} \sum_{x\in W_M} \sum_{j=1}^{k} a_w(x,y) c_{x,i}(j) \pi_1(T_y)\pi_1(T_j) f_{w_0''} \end{aligned}$$

which is in $sp\{\pi_1(T_w)\pi_1(T_i)f_{w_0''}\}_{\substack{i=1,\ldots,k \\ w\in W^{MA}}}$, as needed. $\square$

2.2. The Iwahori-Matsumoto involution

We now take a look at the Iwahori-Matsumoto involution. Recall that the map j defined by

$$j: \quad T_s \longmapsto -qT_s^{-1}$$
$$j: \quad \theta \longmapsto \theta^{-1}$$

extends uniquely to an involution (automorphism) on $\mathcal{H}$. If $h \in \mathcal{H}$, we will also write $\hat{h}$ for $j(h)$.

Next, the Iwahori-Matsumoto involution induces an involution on representations of $\mathcal{H}$, hence on B-unramified representations of G, as follows: $\hat{\pi}$ has the same space as π with action defined by

$$\hat{\pi}(h)v = \pi(\hat{h})v.$$

In the remainder of this section, we study this involution and establish some properties that will be needed later. We note similar results in section 1.7 [M-W] (with a different Iwahori-Matsumoto involution).

PROPOSITION 2.2.1. – *Let M be a standard Levi of G, σ a representation of $\mathcal{H}_M$. Suppose $\pi = Ind_{\mathcal{H}_M}^{\mathcal{H}} \sigma$. Let $\pi_0 = Ind_{\mathcal{H}_M}^{\mathcal{H}} \hat{\sigma}$. Then,*

$$\hat{\pi} \cong \pi_0.$$

Further, the equivalence is given by the map $\mathcal{J} : V_\pi \longrightarrow V_{\pi_0}$, where

$$\mathcal{J} : T \otimes v + \mathcal{K}(\sigma) \longmapsto \hat{T} \otimes v + \mathcal{K}(\hat{\sigma})$$

Proof. – First, it is easy to check that the map $h \otimes v \longmapsto \hat{h} \otimes v$ sends $\mathcal{K}(\sigma)$ to $\mathcal{K}(\hat{\sigma})$. Thus, $\mathcal{J}$ is well-defined. It is easy to check that $\mathcal{J}\hat{\pi} = \pi_0\mathcal{J}$, verifying the proposition. □

We now look at the Iwahori-Matsumoto involution in the context of Iwahori-fixed vectors of group representations.

COROLLARY 2.2.2. – *Let M be a standard Levi of G, σ a B-unramified representation of M. Let $\pi = i_{GM}\sigma$. Then,*

$$\hat{\pi} \cong i_{GM}\hat{\sigma}.$$

Proof. – By Propositions 2.1.2 and 2.2.1, as $\mathcal{H}$-modules $(\hat{\pi})^B \cong \mathrm{Ind}_{\mathcal{H}_{M'}}^{\mathcal{H}} \widehat{w_0' \cdot \sigma^{B\cap M}}$ and $(i_{GM}\hat{\sigma})^B \cong \mathrm{Ind}_{\mathcal{H}_{M'}}^{\mathcal{H}} w_0' \cdot \hat{\sigma}^{B\cap M}$. An easy check shows that for $h \in \mathcal{H}_M$, $\widehat{C_{w_0'}(h)} = C_{w_0'}(\hat{h})$ (Iwahori-Matsumoto involution for $\mathcal{H}_{M'}$ on the left; $\mathcal{H}_M$ on the right). This gives $(\hat{\pi})^B \cong (i_{GM}\hat{\sigma})^B$ as $\mathcal{H}$-modules; the corollary follows from Proposition 2.1.1. □

Next, let $\mathcal{M} = \mathcal{E}\mathcal{J}\mathcal{E}^{-1}$be the map giving the equivalence of $(\widehat{i_{GA}\chi})^B$ and $(i_{GA}\chi^{-1})^B$. Then, for $w \in W$, we can write

$$\mathcal{M}f_w = \sum_{y \in W} m_w(y) f_y$$

for suitable $m_w(y)$. This is to be interpreted in the obvious way: on the left-hand side, $f_y \in V^B_{i_{GA}\chi}$; on the right-hand side, $f_y \in V^B_{i_{GA}\chi^{-1}}$. Lemma 2.2.4 gives an expression for the coefficients $m_w(y)$.

DEFINITION 2.2.3. – *Let $w \in W$. Define $a_w(y)$ by*

$$T_{w^{-1}w_0}T_{w_0} = \sum_{y\in W} a_w(y)T_y$$

LEMMA 2.2.4. – *Suppose $w \in W$. Then*

$$\mathcal{M}f_w = (-q)^{-\ell(w_0w)} \sum_{y\in W} a_w(y^{-1})f_y$$

Proof. – Suppressing the $\mathcal{K}$'s, we have

$$\begin{aligned}\mathcal{M}f_w &= \mathcal{E}\mathcal{J}\mathcal{E}^{-1}f_w\\ &= \mathcal{E}\mathcal{J}T^{-1}_{w_0w}\\ &= \mathcal{E}\widehat{T^{-1}_{w_0w}}\\ &= \mathcal{E}(\widehat{T_{w_0w}})^{-1}\\ &= \mathcal{E}(-q)^{-\ell(w_0w)}T_{w^{-1}w_0}\\ &= (-q)^{-\ell(w_0w)}\pi_1(T_{w^{-1}w_0})f_{w_0}\end{aligned}$$

Since $f_{w_0} = \pi_1(T_{w_0})f_1$,

$$\begin{aligned}\mathcal{M}f_w &= (-q)^{-\ell(w_0w)}\pi_1(T_{w^{-1}w_0})\pi_1(T_{w_0})f_1\\ &= (-q)^{-\ell(w_0w)}\sum_{y\in W} a_w(y)\pi_1(T_y)f_1\\ &= (-q)^{-\ell(w_0w)}\sum_{y\in W} a_w(y)f_{y^{-1}}\\ &= (-q)^{-\ell(w_0w)}\sum_{y\in W} a_w(y^{-1})f_y\end{aligned}$$

as claimed. □

REMARK 2.2.5. – *There is another Iwahori-Matsumoto involution which we now describe. For $x \in G$, let $q_x = |BxB|$ and $sgn(q_x) = 1$ if $q_x \in q^{2\mathbb{Z}}$;-1 if not. Then*

$$T_x \longmapsto q_x sgn(q_x)(T_{x^{-1}})^{-1}$$

gives an involution on $\mathcal{H}$ (cf. [Rod]). Suppose σ is a B-unramified representation of a standard Levi M. Let $w_0' \in W^{MA}$ of maximal length, $M' = w_0'Mw_0'^{-1}$ (a standard Levi) and $\psi = sgn \circ \delta_{P'}$, where $\delta_{P'}$ denotes the modular function of the standard parabolic with Levi M'. Set $\sigma_0 = \psi w_0' \cdot \hat{\sigma}$. Then,

$$\widehat{i_{GM}\sigma} \cong i_{GM'}\sigma_0.$$

Note that if $\pi = i_{GA}\chi$ and $\pi_0 = i_{GA}\chi_0$, the equivalence $\hat{\pi} \cong \pi_0$ is given explicitly by $\mathcal{M}' : V^B_\pi \longrightarrow V^B_{\pi_0}$ with

$$\mathcal{M}'f_w = (-q)^{\ell(w)}f_{w_0w}$$

3. Applications to R-groups

3.1. Intertwining operators and R-groups

In this section, we review some facts regarding standard intertwining operators and R-groups. First, we recall the definition and some basic properties of unnormalized standard intertwining operators. Then, we discuss normalizing factors and normalized standard intertwining operators. We close by reviewing some properties of R-groups. Note that it is not the goal of this section to give a detailed account of these subjects; just to briefly review the facts that will be needed. As we go, we will indicate where more detailed accounts may be found.

We start by discussing unnormalized standard intertwining operators. Suppose M is the Levi of a standard parabolic $P = MU$ of G corresponding to $\Phi \subset \Pi$. Let $W(\Phi) = \{w \in W | w \cdot \Phi = \Phi\}$. Suppose σ is a representation of M, $\pi = i_{GM}\sigma$. Then, formally, set

$$\mathcal{A}_w(\sigma)f(g) = \int_{U_w} f(w^{-1}ug)du$$

for $f \in V_\pi$, $w \in W(\Phi)$ and $U_w = U_{min} \cap w^{-1}\overline{U}w$. The intertwining operator $\mathcal{A}_w(\sigma)$ converges under suitable conditions on the exponent associated to σ and has meromorphic analytic continuation (*cf.* [Art2]). It intertwines the representations $\pi = i_{GM}\sigma$ and $i_{GM}w\sigma$, *i.e.*, $\mathcal{A}_w(\sigma)i_{GM}\sigma = i_{GM}w\sigma\mathcal{A}_w(\sigma)$. We note the following properties:

1. If $w_1, w_2 \in W(\Phi)$ with $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$,

$$\mathcal{A}_{w_1w_2}(\sigma) = \mathcal{A}_{w_1}(w_2\sigma)\mathcal{A}_{w_2}(\sigma).$$

2. If $\sigma \hookrightarrow i_{MA}\chi$, then $i_{GM}\sigma \hookrightarrow i_{GA}\chi$ and

$$\mathcal{A}_w(\sigma) = \mathcal{A}_w(\chi)|_{V_{i_{GM}\sigma}}$$

for $w \in W(\Phi)$ (since the defining integrals are the same).

We now turn to normalized intertwining operators. For the intertwining operator $\mathcal{A}_w(\sigma)$, let $n(w, \sigma)$ denote the normalizing factor and write

$$A_w(\sigma) = n(w, \sigma)\mathcal{A}_w(\sigma).$$

Suppose that σ has Langlands data χ, a character of A, in the subrepresentation setting (*i.e.*, σ is the unique irreducible subrepresentation of $i_{MA}\chi$). Then, we can use the normalizing factor for $\mathcal{A}_w(\chi)$ for $\mathcal{A}_w(\sigma)$, *i.e.*,

$$A_w(\chi)|_{V_{i_{GM}\sigma}} = A_w(\sigma)$$

for $w \in W(\Phi)$. This is in section 2 of [Art2] in the quotient setting of Langlands classification; it is not difficult to pass from one version to the other. To make use of this, we also give the normalizing factor explicitly in the case where χ is an unramified character of A. In this case, for s_α a simple reflection,

$$n(s_\alpha, \chi) = \frac{1 - \chi(h_\alpha(\varpi))}{1 - q^{-1}\chi(h_\alpha(\varpi))}.$$

We refer the reader to [Art3], [Sha] for more on normalizing factors.

We need the following properties of normalized intertwining operators:

1. If $w_1, w_2 \in W(\Phi)$ with $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$,

$$A_{w_1 w_2}(\sigma) = A_{w_1}(w_2\sigma) A_{w_2}(\sigma).$$

2. Suppose χ is an unramified character of A. Let $\pi = i_{GA}\chi$, $\pi' = i_{GA}s\chi$ for $s = s_\alpha$. Let $v_K = \sum_{w \in W} f_w$, and suppose $\chi(h_\alpha(\varpi)) \neq 1, q$. Then,

$$A_s(\chi) v_K = v_K$$

with $v_K \in V_\pi$ on the left and $v_K \in V_{\pi'}$ on the right. (This is immediate from $n(s_\alpha, \chi)$ above and Proposition 3.2.1.)

Finally, we turn to R-groups. Suppose that σ is unitary. Then, $\pi = i_{GM}\sigma$ is unitary as well, so that the reducibility of π is governed by the intertwining algebra $\mathrm{Hom}_G(\pi, \pi)$. Furthermore, if σ is discrete series, a basis for $\mathrm{Hom}_G(\pi, \pi)$ consists of $\{A_r(\sigma)\}_{r \in R}$, where $R \subset W(\sigma) = \{w \in W(\Phi) | w \cdot \sigma \cong \sigma\}$ denotes the R-group. For the situations we are interested in, the R-group has the following properties:

1. $\mathrm{Hom}(\pi, \pi) \cong \mathbb{C}[R]$ (with the isomorphism given by $r \mapsto A_r(\sigma)$)

2. The inequivalent components of π are parameterized by the irreducible representations of R. In addition, if π_ρ is a component of π corresponding to $\rho \in \hat{R}$, then the mutiplicity of π_ρ in π equals $\dim\rho$. We write

$$\pi = \bigoplus_{\rho \in \hat{R}} (dim\rho)\pi_\rho$$

3. The operators $A_r(\sigma)$ act on and permute the $\dim\rho$ irreducible subspaces of the π_ρ-isotypic component as the representation ρ. That is, we can write

$$A_r(\sigma)|_{V_\rho} = \rho(r),$$

where V_ρ denotes the π_ρ-isotypic subspace. For the right-hand side, we are viewing $\rho(r)$ as an element of $\mathrm{Hom}_G(V_\rho, V_\rho) \hookrightarrow \mathrm{Hom}(\pi, \pi)$.

We can ensure that the R-group has these properties by making the assumption that σ is generic. This forces the cocycle η to be trivial (*cf.* [Keys], e.g.). In this case, the properties above are given in section 1 of [Gol] and section 2 of [Art3].

For the record, we mention that there is an alternate description of the R-group in terms of Langlands parameters. We refer the reader to [Keys] or [Art1] for more details.

3.2. Computation of standard intertwining operators

For $\pi = i_{GA}\chi$, the proposition below gives the action of the unnormalized standard intertwining operator $\mathcal{A}_s(\chi)$, which intertwines π with $\pi' = i_{GA}s\chi$, on the B-fixed vectors for π. Note that $\mathcal{A}_s(\chi) : V_\pi^B \longrightarrow V_{\pi'}^B$ and we will work in both of these vector spaces with respect to the corresponding bases $\{f_w\}_{w\in W}$ (*cf.* section 2.1).

PROPOSITION 3.2.1. *Let L be the Levi of $\langle P_{min}, s\rangle$ for $s = s_\alpha$. Then,*

$$\mathcal{A}_s(\chi)f_w = \begin{cases} q^{-1}f_{sw} + (c_\alpha(\chi) - 1)f_w & \text{if } w \in W^{AL} \\ f_{sw} + (c_\alpha(\chi) - q^{-1})f_w & \text{if } w \notin W^{AL} \end{cases}$$

where

$$c_\alpha(\chi) = \frac{1 - q^{-1}\chi(h_\alpha(\varpi))}{1 - \chi(h_\alpha(\varpi))}$$

(Interpret this in the obvious way: $f_w \in V^B_{i_{GA}\chi}$ on the left-hand side, $f_w \in V^B_{i_{GA}s\chi}$ on the right-hand side).

Proof. – For regular χ, this is Theorem 3.4 in [Cas3].
First, we show that for $f \in V_\pi^B$,

$$\mathcal{A}_s(\chi)f(w) = \begin{cases} f(sw) + (c_\alpha(\chi) - 1)f(w) & \text{if } w \in W^{AL} \\ q^{-1}f(sw) + (c_\alpha(\chi) - q^{-1})f(w) & \text{if } w \notin W^{AL} \end{cases}$$

The proposition is an easy corollary of this.
For $\alpha \in \Delta$, there is a corresponding homomorphism

$$\phi_\alpha : SL_2(F) \longrightarrow G.$$

For $x \in F$, and for $t \in F^\times$, we write

$$u_\alpha(x) = \phi_\alpha\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) \quad u_\alpha^-(x) = \phi_\alpha\left(\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}\right)$$

and

$$h_\alpha(t) = \phi_\alpha\left(\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}\right).$$

Let $q^{-\delta_\alpha} = \delta^{\frac{1}{2}}\chi(h_\alpha(\varpi))$.
First, we give the following decomposition, which will play a key role later:

$$\begin{pmatrix} 1 & \\ t & 1 \end{pmatrix} = \begin{pmatrix} t^{-1} & \\ & t \end{pmatrix}\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & t^{-1} \\ & 1 \end{pmatrix}.$$

Taking ϕ_α of everything,

$$u_\alpha^-(t) = h_\alpha(t^{-1})u_\alpha(t)s_\alpha u_\alpha(t^{-1}).$$

Now,

$$\begin{aligned}\mathcal{A}_s(\chi)f(w) &= \int_{u\in U_\alpha} f(suw)du \\ &= \int_{x\in F} f(u_\alpha^-(x)sw)dx.\end{aligned}$$

<u>Case 1 :</u> $w \in W^{AL}$ (so $w^{-1} \in W^{LA}$)

$$\begin{aligned}\mathcal{A}_s(\chi)f(w) &= \int_{x\in\mathcal{O}} f(u_\alpha^-(x)sw)dx + \sum_{n=1}^{\infty}\int_{x\in\varpi^{-n}\mathcal{O}^\times} f(u_\alpha^-(x)sw)dx \\ &= f(sw) + \sum_{n=1}^{\infty} q^n \int_{\tilde{x}\in\mathcal{O}^\times} f(u_\alpha^-(\varpi^{-n}\tilde{x})sw)d\tilde{x}\end{aligned}$$

where equality follows since $u_\alpha^-(x)sw = su_\alpha(x)w = sww^{-1}u_\alpha(x)w$ and $w^{-1}u_\alpha(x)w \in B$ since $w^{-1} \in W^{LA}$. Continuing,

$$\begin{aligned}\mathcal{A}_s(\chi)f(w) &= f(sw) + \sum_{n=1}^{\infty} q^n \int_{\tilde{x}\in\mathcal{O}^\times} f(h_\alpha(\varpi^n\tilde{x}^{-1})u_\alpha(\varpi^{-n}\tilde{x})su_\alpha(\varpi^n\tilde{x}^{-1})sw)d\tilde{x} \\ &= f(sw) + \sum_{n=1}^{\infty} q^n \int_{\tilde{x}\in\mathcal{O}^\times} \delta^{\frac{1}{2}}\chi(h_\alpha(\varpi^n\tilde{x}^{-1}))f(su_\alpha(\varpi^n\tilde{x}^{-1})sw)d\tilde{x} \\ &= f(sw) + \sum_{n=1}^{\infty} q^n (q^{-n})^{\delta_\alpha} \int_{\tilde{x}\in\mathcal{O}^\times} f(w)d\tilde{x}\end{aligned}$$

since $u_\alpha(\varpi^n\tilde{x}^{-1}) \in K_1$ (so that conjugating by $(sw)^{-1}$ leaves it in $K_1 \subset B$). Continuing,

$$\begin{aligned}\mathcal{A}_s(\chi)f(w) &= f(sw) + \sum_{n=1}^{\infty} q^n (q^{-n})^{\delta_\alpha}(1-q^{-1})f(w) \\ &= f(sw) + (1-q^{-1})\frac{q^{-\delta_\alpha+1}}{1-q^{-\delta_\alpha+1}}f(w).\end{aligned}$$

The sum converges for suitable δ_α (*i.e.*, suitable χ), and analytic continuation gives it elsewhere.

<u>Case 2 :</u> $w \notin W^{AL}$

Write $w = sy$ with $y \in W^{AL}$. Then,

$$\begin{aligned}\mathcal{A}_s(\chi)f(w) &= \int_{x\in\mathcal{P}} f(u_\alpha^-(x)y)dx + \sum_{n=0}^{\infty}\int_{x\in\varpi^{-n}\mathcal{O}^\times} f(u_\alpha^-(x)y)dx \\ &= q^{-1}f(y) + \sum_{n=0}^{\infty} q^n \int_{\tilde{x}\in\mathcal{O}^\times} f(h_\alpha(\varpi^n\tilde{x}^{-1})u_\alpha(\varpi^{-n}\tilde{x})su_\alpha(\varpi^n\tilde{x}^{-1})y)dx \\ &= q^{-1}f(y) + \sum_{n=0}^{\infty} q^n (q^{-n})^{\delta_\alpha}(1-q^{-1})f(sy) \\ &= q^{-1}f(y) + (1-q^{-1})\frac{1}{1-q^{-\delta_\alpha+1}}f(sy) \\ &= q^{-1}f(sw) + (1-q^{-1})\frac{1}{1-q^{-\delta_\alpha+1}}f(w)\end{aligned}$$

by the same sort of arguments as in the first case. This finishes the proof. □

LEMMA 3.2.2. – *With $s = s_\alpha$, L as above, suppose $w \in W^{AL}$. Then,*

$$T_{w^{-1}sw_0} \cdot T_{w_0} = T_{w^{-1}w_0} \cdot T_{w_0} \cdot (q^{-1}T_s - (1-q^{-1})).$$

Proof. – Let $s' = w_0 s w_0$, also a simple reflection. Then $w^{-1}sw_0 = w^{-1}w_0 s'$. Note that $w \in W^{AL}$ implies $w^{-1}w_0 s' \in W^{L'A}$, where L' is the Levi factor of $\langle P_{min}, s'\rangle$ (this is easy if one uses the characterization $W^{AL} = \{w \in W | w^{-1}\alpha > 0\}$). Thus

$$\begin{aligned} T_{w^{-1}sw_0} \cdot T_{w_0} &= T_{w^{-1}w_0 s'} \cdot T_{w_0} \\ &= (T_{w^{-1}w_0} \cdot T_{s'}^{-1}) \cdot T_{w_0} \\ &= T_{w^{-1}w_0} \cdot T_{s'}^{-1} \cdot (T_{s'} \cdot T_{s'w_0}) \\ &= T_{w^{-1}w_0} \cdot T_{s'w_0} \\ &= T_{w^{-1}w_0} \cdot T_{w_0 s} \\ &= T_{w^{-1}w_0} \cdot T_{w_0} T_s^{-1}. \end{aligned}$$

Since $T_s^{-1} = q^{-1}T_s - (1-q^{-1})$, the lemma follows. □

LEMMA 3.2.3. – *Suppose $w \in W^{AL}$. Then,*

$$\begin{aligned} \mathcal{M} f_{sw} = (-q)^{-\ell(w_0 w)} &\sum_{y \in W^{AL}} [(q-1)a_w(y^{-1}) - q a_w(y^{-1}s)] f_y \\ + (-q)^{-\ell(w_0 w)} &\sum_{y \notin W^{AL}} [-a_w(y^{-1}s)] f_y \end{aligned}$$

Proof. – Recall, from the proof of Lemma 2.2.4, that

$$\mathcal{M} f_{sw} = (-q)^{-\ell(w_0 sw)} \pi(T_{w^{-1}sw_0} \cdot T_{w_0}) f_1$$

By Lemma 3.2.2,

$$\begin{aligned} \mathcal{M} f_{sw} &= (-q)^{-\ell(w_0 sw)} \pi(T_{w^{-1}w_0} \cdot T_{w_0}) \pi(q^{-1}T_s - (1-q^{-1})) f_1 \\ &= (-q)^{-\ell(w_0 sw)} \pi(T_{w^{-1}w_0} \cdot T_{w_0}) [q^{-1} f_s - (1-q^{-1}) f_1] \\ &= (-q)^{-\ell(w_0 sw)} \sum_{y \in W} a_w(y) \pi(T_y) [q^{-1} f_s - (1-q^{-1}) f_1] \end{aligned}$$

with $a_w(y)$ as in Definition 2.2.3. Now, $\ell(w_0sw) = \ell(w^{-1}sw_0)$ and $w^{-1}sw_0 = w^{-1}w_0s' \in W^{L'A}$. Therefore, $\ell(w_0sw) = \ell(w_0w) - 1$. Thus,

$$\begin{aligned}\mathcal{M}f_{sw} &= (-q)^{-\ell(w_0w)}\sum_{y\in W} a_w(y)\pi(T_y)[(q-1)f_1 - f_s]\\ &= (-q)^{-\ell(w_0w)}\sum_{y\in W} a_w(y)(q-1)f_{y^{-1}} - (-q)^{-\ell(w_0w)}\sum_{y\in W} a_w(y)\pi(T_yT_s)f_1\\ &= (-q)^{-\ell(w_0w)}\sum_{y\in W} a_w(y)(q-1)f_{y^{-1}}\\ &\quad - (-q)^{-\ell(w_0w)}\sum_{y\in W^{LA}} a_w(y)\pi(T_{ys})f_1\\ &\quad - (-q)^{-\ell(w_0w)}\sum_{y\notin W^{LA}} a_w(y)\pi((q-1)T_y + qT_{ys})f_1\\ &= (-q)^{-\ell(w_0w)}\sum_{y\in W} a_w(y)(q-1)f_{y^{-1}}\\ &\quad - (-q)^{-\ell(w_0w)}\sum_{y\in W^{LA}} a_w(y)f_{sy^{-1}}\\ &\quad - (-q)^{-\ell(w_0w)}\sum_{y\notin W^{LA}} a_w(y)[(q-1)f_{y^{-1}} + qf_{sy^{-1}}]\end{aligned}$$

If we substitute $x = y^{-1}$ in the first sum, $x = sy^{-1}$ in the second, and use both in the third, we get

$$\begin{aligned}\mathcal{M}f_{sw} = (-q)^{-\ell(w_0w)}\sum_{x\in W} a_w(x^{-1})(q-1)f_x + (-q)^{-\ell(w_0w)}\sum_{x\notin W^{AL}} - a_w(x^{-1}s)f_x\\ + (-q)^{-\ell(w_0w)}\sum_{x\notin W^{AL}} - a_w(x^{-1})(q-1)f_x + (-q)^{-\ell(w_0w)}\sum_{x\in W^{AL}} - qa_w(x^{-1}s)f_x\end{aligned}$$

Rearranging this sum gives the lemma. □

PROPOSITION 3.2.4.

$$\mathcal{M}\mathcal{A}_w(\chi)\mathcal{M}^{-1} = (-1)^{\ell(w)}\mathcal{A}_w(\chi^{-1})$$

Proof. – It is sufficient to show

$$\mathcal{M}\mathcal{A}_s(\chi)\mathcal{M}^{-1} = (-1)\mathcal{A}_s(\chi^{-1})$$

for all simple reflections s. Fix such a simple reflection $s = s_\alpha$. Then, for $w \in W$, by Proposition 3.2.1,

$$\begin{aligned}\mathcal{M}\mathcal{A}_s(\chi)f_w &= \mathcal{M}\begin{cases} q^{-1}f_{sw} + (c_\alpha(\chi) - 1)f_w & \text{if } w \in W^{AL}\\ f_{sw} + (c_\alpha(\chi) - q^{-1})f_w & \text{if } w \notin W^{AL}\end{cases}\\ &= \begin{cases}\sum_{y\in W}[q^{-1}m_{sw}(y) + (c_\alpha(\chi) - 1)m_w(y)]f_y & \text{if } w \in W^{AL}\\ \sum_{y\in W}[m_{sw}(y) + (c_\alpha(\chi) - q^{-1})m_w(y)]f_y & \text{if } w \notin W^{AL}\end{cases}\end{aligned}$$

where $m_w(y)$ is defined by $\mathcal{M}f_w = \sum_{y\in W} m_w(y)f_y$.

On the other hand,

$$-\mathcal{A}_s(\chi^{-1})\mathcal{M}f_w = -\mathcal{A}_s(\chi^{-1})\sum_{y\in W} m_w(y)f_y$$

Now, we note that $c_\alpha(\chi^{-1})-1 = -(c_\alpha(\chi)-q^{-1})$ and $c_\alpha(\chi^{-1})-q^{-1} = -(c_\alpha(\chi)-1)$. Thus,

$$\begin{aligned}
-\mathcal{A}_s(\chi^{-1})\mathcal{M}f_w &= \sum_{y\in W^{AL}} -\mathcal{A}_s(\chi^{-1})m_w(y)f_y + \sum_{y\notin W^{AL}} -\mathcal{A}_s(\chi^{-1})m_w(y)f_y \\
&= \sum_{y\in W^{AL}} m_w(y)[-q^{-1}f_{sy} + (c_\alpha(\chi)-q^{-1})f_y] \\
&+ \sum_{y\notin W^{AL}} m_w(y)[-f_{sy} + (c_\alpha(\chi)-1)f_y] \\
&= \sum_{y\in W^{AL}} [-m_w(sy) + (c_\alpha(\chi)-q^{-1})m_w(y)]f_y \\
&+ \sum_{y\notin W^{AL}} [-q^{-1}m_w(sy) + (c_\alpha(\chi)-1)m_w(y)]f_y
\end{aligned}$$

We now verify that these are equal. This is just a matter of showing that the coefficients of the f_y's are equal.

Case 1 : $w \in W^{AL}$, $y \in W^{AL}$
In this case, we need to show

$$q^{-1}m_{sw}(y) + (c_\alpha(\chi)-1)m_w(y) = -m_w(sy) + (c_\alpha(\chi)-q^{-1})m_w(y)$$

or

$$q^{-1}m_{sw}(y) + m_w(sy) = (1-q^{-1})m_w(y)$$

From Lemma 2.2.4, $m_w(y) = (-q)^{-\ell(w_0w)}a_w(y^{-1})$ and $m_w(sy) = (-q)^{-\ell(w_0w)}a_w(y^{-1}s)$. From Lemma 3.2.3, $m_{sw}(y) = (-q)^{-\ell(w_0w)}[(q-1)a_w(y^{-1}) - qa_w(y^{-1}s)]$. It is now immediate that $q^{-1}m_{sw}(y) + m_w(sy) = (1-q^{-1})m_w(y)$.

Case 2 : $w \in W^{AL}$, $y \notin W^{AL}$
This is similar to case 1.

Case 3 : $w \notin W^{AL}$, $y \in W^{AL}$
In this case, we need to show that

$$m_{sw}(y) + (c_\alpha(\chi)-q^{-1})m_w(y) = -m_w(sy) + (c_\alpha(\chi)-q^{-1})m_w(y)$$

or

$$m_{sw}(y) = -m_w(sy)$$

If we apply Lemma 3.2.3 to $sw \in W^{AL}$, we get

$$\mathcal{M}f_w = (-q)^{-\ell(w_0sw)} \sum_{x\in W^{AL}} [(q-1)a_{sw}(x^{-1}) - qa_{sw}(x^{-1}s)]f_x + (-q)^{-\ell(w_0sw)} \sum_{x\notin W^{AL}} [-a_{sw}(x^{-1}s)]f_x$$

From this and Lemma 2.2.4, we get $m_{sw}(y) = (-q)^{-\ell(w_0sw)}a_{sw}(y^{-1})$and $m_w(sy) = -(-q)^{-\ell(w_0sw)}a_{sw}(y^{-1})$. The desired equality follows.

Case 4 : $w \notin W^{AL}$, $y \notin W^{AL}$
This is similar to case 3.
This verifies that $\mathcal{M}\mathcal{A}_s(\chi)f_w = -\mathcal{A}_s(\chi^{-1})\mathcal{M}f_w$, finishing the proposition. □

3.3. Results on R-groups

The goal of this section is to obtain an R-group in cases where the definition of the R-group is not applicable (*cf.* Theorem 3.3.2). If $\pi = i_{GM}\sigma$ with σ square-integrable, then π has an R-group. We use the R-group for π to construct an R-group for $\hat{\pi} \cong i_{GM}\hat{\sigma}$ as follows: from the preceding section, we know the relationship between unnormalized standard intertwining operators for π and $\hat{\pi}$. By comparing normalizing factors, we obtain a relationship between normalized standard intertwining operators. Then, if we look at the normalized standard intertwining operators for $\hat{\pi}$ which correspond (under the Iwahori-Matsumoto involution) to the R-group for π, it is not difficult to check that they have the properties that we would want of an R-group for $\hat{\pi}$.

We start with the following definition:

DEFINITION 3.3.1. – *Let* $v_K = \sum_{w\in W} f_w$ *and set* $v_{St} = \mathcal{M}^{-1}v_K$.

The labeling is explained by the fact that if π is an unramified principal series, $v_K \in V_\pi$ is the K-fixed vector (unique up to scalars). Further, if $\pi = i_{GA}\delta^{\frac{1}{2}}$, then v_{St} gives a basis for the subspace of B-fixed vectors corresponding to the Steinberg representation.

Suppose that M is a standard Levi of G and σ is a B-unramified essentially square-integrable representation of M, that is, a square-integrable representation of M up to a character of M. In order to ensure that the R-group has the properties listed in section 3.1, we also require that σ be generic. In addition, we assume that $\hat{\sigma}$ has Langlands (subrepresentation) data χ (a character of A). We work mainly by embedding everything in the appropriate unramified principal series. So, let us be careful and write $\sigma_0 \cong \hat{\sigma}$, viewing σ_0 as acting on a subspace of $V_{i_{MA}\chi}^{B\cap M}$ and $\hat{\sigma}$ as acting on a subspace of $V_{i_{MA}\chi^{-1}}^{B\cap M}$. Now, if $w_0'' \in W_M$ of maximal length, σ_0 may be identified as the image of $\mathcal{A}_{w_0''}(w_0''\chi) : V_{i_{MA}w_0''\chi} \longrightarrow V_{i_{MA}\chi}$. Since χ satisfies the requirements for Langlands (subrepresentation) data, Proposition 3.2.1 shows that none of the terms in the decomposition of $\mathcal{A}_{w_0''}(w_0''\chi)$ into operators associated to simple reflections annihilates the $K\cap M$-fixed vector, *i.e.*, σ_0 has a $K\cap M$-fixed vector. Also, by Corollary 2.2.2, $\mathcal{M}$ maps the subspace $V_{i_{GM}\sigma}^B \subset V_{i_{GA}\chi^{-1}}^B$ to the subspace $V_{i_{GM}\sigma_0}^B \subset V_{i_{GA}\chi}^B$. So, by the comments in section 3.1, we can restrict the identity in Proposition 3.2.4 to the subspace $V^B_{i_{GM}\sigma_0}$ to get

$$\mathcal{M}\mathcal{A}_w(\sigma)\mathcal{M}^{-1} = (-1)^{\ell(w)}\mathcal{A}_w(\sigma_0).$$

Also,

$$n(w,\sigma_0)\mathcal{A}_w(\sigma_0)v_K = v_K \qquad n(w,\sigma)\mathcal{A}_w(\sigma)v_{St} = \lambda(w,\sigma)v_{St}$$

(this defines $\lambda(w,\sigma)$). The first of these is in section 3.1; the second results from the fact that v_{St} must be sent to a multiple of v_{St} (apply the identity above to v_K). Observe that

$$\begin{aligned} n(w,\sigma)n(w,\sigma_0)\mathcal{M}\mathcal{A}_w(\sigma)\mathcal{M}^{-1}v_K &= n(w,\sigma_0)\mathcal{M}n(w,\sigma)\mathcal{A}_w(\sigma)v_{St}\\ &= n(w,\sigma_0)\mathcal{M}\lambda(w,\sigma)v_{St}\\ &= n(w,\sigma_0)\lambda(w,\sigma)v_K. \end{aligned}$$

On the other hand,

$$n(w,\sigma)n(w,\sigma_0)(-1)^{\ell(w)}\mathcal{A}_w(\sigma_0)v_K = (-1)^{\ell(w)}n(w,\sigma)v_K.$$

Thus,

$$(-1)^{\ell(w)}n(w,\sigma) = n(w,\sigma_0)\lambda(w,\sigma).$$

This gives, for normalized intertwining operators,

$$(*) \qquad \mathcal{M}A_w(\sigma)\mathcal{M}^{-1} = \lambda(w,\sigma)A_w(\sigma_0).$$

Before proceeding to the main theorem in this section, we need one more item. Suppose σ is actually square-integrable and R is the R-group for $i_{GM}\sigma$. Since a component of an unramified principal series representation which has a K-fixed vector appears with multiplicity one, by the Iwahori-Matsumoto involution, the same is true for a component containing v_{St}. In particular, the corresponding representation of the R-group is 1-dimensional. We denote it by ρ_{St}. In this case, if $r \in R$, then $\lambda(r,\sigma) = \rho_{St}(r)$ (to see this, just apply $(*)$ to v_K).

THEOREM 3.3.2. – *Suppose σ is a generic square-integrable B-unramified representation of M. Also, suppose σ_0 has Langlands subrepresentation data χ. Let $\pi = i_{GM}\sigma$ and $\pi_0 = i_{GM}\sigma_0$ (so $\hat{\pi} \cong \pi_0$). If $\rho \in \hat{R}$, we can view ρ as an element of $Hom(V_\rho, V_\rho) \subset Hom(V_\pi, V_\pi)$. Therefore, we can also let ρ act on $Hom(V_{\pi_0}, V_{\pi_0})$, the action being given by $\mathcal{M}\rho\mathcal{M}^{-1}$. If π_ρ denotes the irreducible subrepresentation of π associated to $\rho \in \hat{R}$, set*

$$(\pi_0)_\rho = \widehat{\pi_{\rho_{St}\rho}}$$

(Note that as ρ runs through $\hat{R}$, $\rho_{St}\rho$ does as well.) Then, R is the R-group for π_0 in the sense that

1.

$$Hom(\pi_0,\pi_0) \cong \mathbb{C}[R]$$

2. *We have a correspondence between the irreducible representations of R and the equivalence classes of components of π_0. In particular,*

$$\pi_0 \cong \bigoplus_{\rho\in\hat{R}} (dim\rho)(\pi_0)_\rho$$

3. *The operators $A_r(\sigma_0)$ act on and permute the $dim\rho$ irreducible subspaces of the $(\pi_0)_\rho$-isotypic component as the representation ρ. That is, we can write*

$$A_r(\sigma_0)|_{V_\rho} = \rho(r).$$

Proof. – For the first claim, the map $A_r(\sigma) \mapsto \mathcal{M}A_r(\sigma)\mathcal{M}^{-1}$ gives an isomorphism $\mathrm{Hom}(\pi,\pi) \cong \mathrm{Hom}(\pi_0,\pi_0)$. The claim follows.

For the second claim, we start with

$$\pi \cong \bigoplus_{\rho\in\hat{R}} (dim\rho)\pi_\rho.$$

Applying the Iwahori-Matsumoto involution to this, we get

$$\begin{aligned}\hat{\pi} &\cong \bigoplus_{\rho\in\hat{R}} (dim\rho)(\widehat{\pi_\rho})\\ &\cong \bigoplus_{\rho\in\hat{R}} (dim\rho_{St}\rho)(\widehat{\pi_{\rho_{St}\rho}})\\ &\cong \bigoplus_{\rho\in\hat{R}} (dim\rho)(\pi_0)_\rho\end{aligned}$$

as needed.

For the last claim, let $v \in (V_\pi)_{\rho_{St}\rho}$ and write $\hat{v} = \mathcal{M}v \in (V_{\pi_0})_\rho$. Since $\lambda(r,\sigma) = \rho_{St}(r)$, $(*)$ implies

$$\begin{aligned}A_r(\sigma_0)\hat{v} &= \rho_{St}^{-1}(r)\mathcal{M}A_r(\sigma)\mathcal{M}^{-1}\hat{v}\\ &= \rho_{St}^{-1}(r)\mathcal{M}A_r(\sigma)v\\ &= \rho_{St}^{-1}(r)\mathcal{M}\rho_{St}(r)\rho(r)v\\ &= \mathcal{M}\rho(r)\mathcal{M}^{-1}\cdot\mathcal{M}v\\ &= \rho(r)\hat{v}\end{aligned}$$

which is the needed transformation. □

REFERENCES

[Art1] J. ARTHUR, Unipotent automorphic representations: conjectures, *Astérisque,* Vol. 171-172, 1989, pp. 13-71.

[Art2] J. ARTHUR, Intertwining operators and residues 1. weighted characters, *J. Func. Anal.*, Vol. 84, pp. 19-84.

[Art3] J. ARTHUR, On elliptic tempered characters (preprint).

[B-Z] I. BERNSTEIN and A. ZELEVINSKY, Induced representations of reductive p-adic groups *I, Ann. Sci. École Norm. Sup.*, Vol. 10, 1977, pp. 441-472.

[Bor] A. BOREL, Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup, *Invent. Math*, Vol. 35, 1976, pp. 233-259.

[Cas1] W. CASSELMAN, Introduction to the theory of admissible representation of p-adic reductive groups (preprint).

[Cas2] W. CASSELMAN, The Steinberg character as a true character, *Proceedings of Symposia in Pure Mathematics*, Vol. 26, 1973, pp. 413-417.

[Cas3] W. CASSELMAN, The unramified principal series of p-adic groups I, *Comp. Math.*, Vol. 40, 1980, pp. 387-406.

[Gol] D. GOLDBERG, Reducibility of induced representations for $Sp(2n)$ and $SO(n)$ (preprint).

[I-M] N. IWAHORI and H. MATSUMOTO, On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups, *Publ. Math. IHES*, Vol. 25, 1965, pp. 5-48.

[Keys] D. KEYS, L-indistinguishibility and R-groups for quasi-split groups: Unitary groups of even dimension, *Ann. Sci. École Norm. Sup.*, Vol. 20, 1987, pp. 31-64.

[Lus] G. LUSZTIG, Representations of affine Hecke algebras, *Astérisque,* Vol. 171-172, 1989, pp. 73-84.

[M-W] C. MOEGLIN and J.-L. WALDSPURGER, Sur l'involution de Zelevinski, *J. reine angew. Math.*, Vol. 372, 1986, pp. 136-177.

[Re1] M. REEDER, On certain Iwahori invariants in the unramified principal series, *Pac. J. Math.*, Vol. 153, 1992, pp. 313-342.

[Re2] M. REEDER, Nonstandard intertwining operators and the structure of unramified principal series representations (preprint).

[Rod] F. RODIER, Sur les représentations non ramifiées des groupes réductifs p-adiques; l'exemple de $GSp(4)$, *Bull. Soc. Math. France*, Vol. 116, 1988, pp. 15-42.

[Sha] F. SHAHIDI, A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups, *Ann. of Math.*, Vol. 132, 1990, pp. 273-330.

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