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On supports of induced representations for p-adic special orthogonal and general spin groups

Chris Jantzen [a], Caihua Luo [b,*]

[a] Department of Mathematics, East Carolina University, Greenville, NV 27858, USA
[b] Department of Mathematics, Bar-Ilan University, Ramat-Gan, 52900, Israel

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ABSTRACT

To an irreducible representation of p-adic even special orthogonal groups, unitary groups, and general spin groups, we associate a family of irreducible representations of smaller rank groups of the same type based on the supercuspidal support. We further show that this correspondence respects temperedness, square-integrability, duality, and parabolic induction.

1. Introduction

Roughly speaking, the paper [15] gives a decomposition of an irreducible representation of $SO_{2n+1}(F)$ or $Sp_{2n}(F)$, F p-adic, into a product of irreducible representations of lower-rank classical groups of the same type based on the different supercuspidal repre-

* Corresponding author.
E-mail addresses: jantzenc@ecu.edu (C. Jantzen), chluo@amss.ac.cn (C. Luo).

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sentations of general linear groups which appear in the supercuspidal support, and in a manner which respects many key properties (see [15, Theorem 9.3]). This was extended to cover $O_{2n}(F)$ in [16]. The purpose of this paper is to give the corresponding result for $SO_{2n}(F)$, $U_{2n+1}(F)$, $U_{2n}(F)$, $GSpin_{2n+1}(F)$ and $GSpin_{2n}(F)$.

To be a little more precise, and thereby facilitate the discussion below, we briefly recall the correspondence of [15] focusing on the case of symplectic groups. Let $\rho_1, \cdots, \rho_k$ be irreducible unitary supercuspidal representations of general linear groups, $\tilde{\rho}_i$ be the contragredient of ρ_i and σ be an irreducible supercuspidal representation of a symplectic group. Suppose π is an irreducible representation of a symplectic group having supercuspidal support in

$$\{\nu^x\rho_1, \ \nu^{-x}\tilde{\rho}_1\}_{x\in\mathbb{R}} \cup \cdots \cup \{\nu^x\rho_k, \ \nu^{-x}\tilde{\rho}_k\}_{x\in\mathbb{R}} \cup \{\sigma\}.$$

To π, the results of [15] associate a family of irreducible representations $\pi_1, \cdots, \pi_k$ with π_i having supercuspidal support in $\{\nu^x\rho_i, \ \nu^{-x}\tilde{\rho}_i\}_{x\in\mathbb{R}} \cup \{\sigma\}$.

The decomposition thus obtained is similar to what one might get using Hecke algebra isomorphisms, but requires less overhead (though does not preserve as many properties as Hecke algebra isomorphisms). As the decomposition behaves well with respect to parabolic induction, square-integrability, duality, etc., it can serve as a useful tool for many problems–reducing the problem to that where the supercuspidal support lies in a single family $\{\nu^x\rho, \ \nu^{-x}\tilde{\rho}\}_{x\in\mathbb{R}} \cup \{\sigma\}$.

The situation for unitary groups is very similar to that of $Sp_{2n}(F)$ and $SO_{2n+1}(F)$ addressed in [15]. However, there are several issues in extending this correspondence to $SO_{2n}(F)$, $GSpin_{2n+1}(F)$, and $GSpin_{2n}(F)$. The first, which applies only to $SO_{2n}(F)$ and $GSpin_{2n}(F)$, is the lack of what [22] refer to as partial cuspidal support. In particular, if c denotes the outer automorphisms corresponding to the interchange of the last two simple roots in the Dynkin diagram, one can have the supercuspidal support of π containing both σ and $c\cdot\sigma$. As a consequence, one may have more than one candidate for a particular π_i, and some of those π_is may contain both σ and $c \cdot \sigma$ in their supercuspidal support.

The second issue, which applies only to the general spin groups, is the effect of the similitude character under Weyl conjugation. In the symplectic case, the action of the long coset representative w_0 has $w_0(\rho \otimes \sigma) = \tilde{\rho} \otimes \sigma$, so the set $\{\nu^x\rho, \ \nu^{-x}\tilde{\rho}\}_{x\in\mathbb{R}}$ is closed under Weyl conjugation. However, for $GSpin_{2n+1}(F)$, it involves the central character ω_σ of σ, e.g., $w_0(\rho \otimes \sigma) = \omega_\sigma\tilde{\rho} \otimes \sigma$. To ensure closure under Weyl conjugation, we must replace $\{\nu^x\rho, \ \nu^{-x}\tilde{\rho}\}_{x\in\mathbb{R}}$ with $\{\nu^x\rho, \ \omega_\sigma\nu^{-x}\tilde{\rho}\}_{x\in\mathbb{R}}$.

The last issue applies only to $SO_{2n}(F)$ and $GSpin_{2n}(F)$. For symplectic (or odd special orthogonal) groups, one property of the correspondence above is that if λ_1, λ_2 are representations supported on $\{\nu^x\rho_1, \ \nu^{-x}\tilde{\rho}_1\}_{x\in\mathbb{R}}$, $\{\nu^x\rho_2, \ \nu^{-x}\tilde{\rho}_2\}_{x\in\mathbb{R}}$ respectively, then $\mathrm{Ind}(\lambda_1 \otimes \lambda_2 \otimes \sigma)$ is determined by $\mathrm{Ind}(\lambda_1 \otimes \sigma)$ and $\mathrm{Ind}(\lambda_2 \otimes \sigma)$. When λ_1 and λ_2 are square-integrable, this is a consequence of the R-group results of [10], [3], and these results play a key role in the proof. In the case of $SO_{2n}(F)$ or $GSpin_{2n}(F)$, the R-group results are not so simple (cf. [10], [3]). In particular, one can have $\rho_1 \rtimes \sigma$ and $\rho_2 \rtimes \sigma$ both irreducible but $\rho_1 \times \rho_2 \rtimes \sigma$ reducible. To deal with this issue, we bundle those ρ_is where this can happen together.

We close the introduction by briefly outlining the structure of the paper. In the next section, we introduce notation and recall some basic results needed later. Section 3 gives the definition of the correspondence, with the first set of main results–the bijective nature of the correspondence (Propositions 4.4 and 4.5) and the fact that it respects duality (Lemma 4.3), essential temperedness and square-integrability (Lemma 4.1), and the Langlands classification (Lemma 4.2)–given in Section 4. In Section 5, we show that the correspondence respects parabolic induction (Corollary 5.4). The paper closes with some additional comments and observations in Section 6.

2. Notation and preliminaries

We assume $\mathrm{char}(F) = 0$ to use the results of [10], [11], [3].

Let $G = G_m(F) = SO_{2m}(F),\ GSpin_{2m+1}(F),\ GSpin_{2m}(F),\ U_{2n+1}(F)$, or $U_{2m}(F)$, where the unitary groups are determined by a fixed quadratic extension E/F. A parabolic subgroup of $G_m(F)$ has the form $P = MU$ with

$$M = \begin{cases} GL_{m_1}(F) \times \cdots GL_{m_r}(F) \times G_{m_0}(F) & \text{if } G_m = SO_{2m},\ GSpin_{2m+1},\ GSpin_{2m}, \\ GL_{m_1}(E) \times \cdots GL_{m_r}(E) \times G_{m_0}(F) & \text{if } G_m = U_{2m+1},\ U_{2m}. \end{cases}$$

Following [27], [24] and [25], if π_i is a representation of $GL_{m_i}(F)$, $i = 1, \ \cdots, \ r$, and π_0 is a representation of $G_{m_0}(F)$, we set the associated normalized parabolic induction representation of G as

$$\pi_1 \times \cdots \times \pi_r \rtimes \pi_0 = i_{G,M}(\pi_1 \otimes \cdots \otimes \pi_r \otimes \pi_0).$$

This also applies in the Grothendieck group setting, i.e., to its semi-simplification. If τ is a representation of $GL_m(F)$ (resp. $G_m(F)$), we set $d(\tau) = m$.

For χ a character of $F^\times$, we may identify χ with a character of $GL_m(F)$ $\big($resp. $GSpin_{2m+1}(F)$ or $GSpin_{2m}(F)\big)$ by $\chi \circ \det$ $\big($resp. $\chi \circ \lambda$, where λ denotes the similitude character–see [1] and [18]$\big)$. As in [27], we let $\nu = |\cdot|$. A representation π of one of these groups is essentially tempered (resp. essentially square-integrable) if there is an $\varepsilon(\pi) \in \mathbb{R}$ such that $\nu^{-\varepsilon(\pi)}\pi$ is tempered (resp. square-integrable).

For $SO_{2m}(F)$ and $GSpin_{2m}(F)$, we have an outer automorphism, denoted by c, corresponding to the interchange of the last two simple roots in the Dynkin diagram. When $m_0 = 0$ and $m_r > 1$, both M and $c(M)$ are standard Levi factors and are not conjugate in $G_m(F)$. Note that this corresponds to the situation where exactly one of $\alpha_{m-1},\ \alpha_m$–the last two simple roots–is among the simple roots for M. To distinguish between them, we use the artifice in [17] (for $SO_{2m}(F)$) and [18] (for $GSpin_{2m}(F)$): for $SO_{2m}(F)$, we let $1 \otimes e$ denote the trivial representation of $G_0(F) = 1$ if it is α_{m-1} which is a simple root for M and $1 \otimes c$ if it is α_m. The convention for $GSpin_{2m}(F)$ is similar, except that since

$G_0(F) \cong F^\times$, we use $\chi \otimes e$ (resp. $\chi \otimes c$) to denote a character of $G_0(F)$ if α_{m-1} (resp. α_m) is the simple root for M. In addition to allowing the $\rtimes$ notation above to be used unambiguously, it also means that some standard results, such as the Langlands classification and the Casselman criterion, have the same form for $SO_{2m}(F)$ (resp. $GSpin_{2m}(F)$) as for $SO_{2m+1}(F)$ (resp. $GSpin_{2m+1}(F)$). The interested reader is referred to [17], [18] for more details.

We now recall some structures that we use for Jacquet module calculations. To this end, let

$$R = \bigoplus_{m=0}^{\infty} \mathcal{R}(GL_m(F)) \text{ and } R[G] = \bigoplus_{m=0}^{\infty} \mathcal{R}(G_m(F)),$$

where $\mathcal{R}$ denotes the Grothendieck group for the category of admissible representations. Now, for a general linear group $GL_m(F)$, let $M_{(k)}$ denote the Levi factor of a standard parabolic subgroup having the form $M_{(k)} = GL_k(F) \times GL_{m-k}(F)$, and $r_{M_{(k)},GL_m}$ be the normalized Jacquet module with respect to $M_{(k)}$. Then set $m^* : R \longrightarrow R \otimes R$ by (see [27]),

$$m^* = \sum_{k=0}^{m} r_{M_{(k)},GL_m}.$$

Similarly, for $G_m(F)$, let $M_{(k)}$ denote the standard Levi subgroup having $M_{(k)} = GL_k(F) \times G_{m-k}(F)$, and $r_{M_{(k)},G_m}$ be the normalized Jacquet module with respect to $M_{(k)}$, noting that for $G_m = SO_{2m}, GSpin_{2m}$, there are two such standard Levi factors which appear symmetrically in the formula below (so need to distinguish them at this time). Following [25], let $\mu^* : R[G] \longrightarrow R \otimes R[G]$ be

$$\mu^* = \begin{cases} \displaystyle\sum_{k=0}^{m} r_{M_{(k)},G_m} & \text{for } G_m = GSpin_{2m+1},\ U_{2m+1},\ U_{2m}, \\ \displaystyle\sum_{k=0}^{m} r_{M_{(k)},G_m} + r_{c(M_{(m)}),G_m} & \text{for } G_m = SO_{2m},\ GSpin_{2m}. \end{cases}$$

In order to set up a structure formula similar to that of [25], let $\check{\pi}$ be defined by

$$\check{\pi} = \begin{cases} \tilde{\pi} \circ \kappa & \text{if } G_m = U_{2m+1},\ U_{2m}, \\ \tilde{\pi} & \text{otherwise,} \end{cases}$$

where κ denotes the non-trivial Galois conjugation associated to E/F. Let $N^* : R \longrightarrow R \otimes R \otimes R$ be

$$N^* = (\check{} \otimes m^*) \circ s \circ m^*$$

for $G_m = GSpin_{2m+1},\ U_{2m+1},\ U_{2m}$, and for $G_m = SO_{2m},\ GSpin_{2m}$, $N^* : R \longrightarrow R \otimes R \otimes R \otimes \mathbb{Z}(C)$ be

$$N^* = (\tilde{\ } \otimes m^*)_D \circ s \circ m^*,$$

where $C = \{e, c\}$, $s(\pi_1 \otimes \pi_2) := \pi_2 \otimes \pi_1$ for any two representations π_1 and π_2 of GL, and

$$(\tilde{\ } \otimes m^*)_D(\pi_1 \otimes \pi_2) = \begin{cases} \tilde{\pi}_1 \otimes m^*(\pi_2) \otimes e & \text{if } d(\pi_1) \text{ is even,} \\ \tilde{\pi}_1 \otimes m^*(\pi_2) \otimes c & \text{if } d(\pi_1) \text{ is odd.} \end{cases}$$

The following theorem is done in or directly follows from [17] for $G_m = SO_{2m}$, [22] for $G_m = U_{2m+1},\ U_{2m}$, [20] for $G_m = GSpin_{2m+1}$, and [18] (based on [21]) for $G_m = GSpin_{2m}$.

Theorem 2.1. *With notation as above,*

$$\mu^*(\tau \rtimes \pi) = N^*(\tau) \tilde{\rtimes} \mu^*(\pi),$$

where $\tilde{\rtimes}$ *is defined by*

$$(\rho_1 \otimes \rho_2 \otimes \rho_3) \tilde{\rtimes} (\rho \otimes \sigma) = \begin{cases} (\omega_\sigma \rho_1 \times \rho_2 \times \rho) \otimes (\rho_3 \rtimes \sigma) & \text{for } G_m = GSpin_{2m+1}, \\ (\rho_1 \times \rho_2 \times \rho) \otimes (\rho_3 \rtimes \sigma) & \text{for } G_m = U_{2m+1},\ U_{2m}, \end{cases}$$

and, $\iota = c^0$ *or* c,

$$(\rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \iota) \tilde{\rtimes} (\rho \otimes \sigma) = \begin{cases} (\rho_1 \times \rho_2 \times \rho) \otimes \iota(\rho_3 \rtimes \sigma) & \text{for } G_m = SO_{2m}, \\ (\omega_\sigma \rho_1 \times \rho_2 \times \rho) \otimes \iota(\rho_3 \rtimes \sigma) & \text{for } G_m = GSpin_{2m}. \end{cases}$$

We remark that in order to have a more uniform presentation that covers general spin groups, we have adopted a slightly less elegant formulation than those that are available for the other families.

The duality operator of [2] is used in a key way in this paper. Notationally, if D denotes the duality operator of [2] and π is an irreducible representation, we define $\hat{\pi} = \pm D(\pi)$, using whichever sign produces a genuine representation. Note that this convention was needed in [15] and should have been made there.

Remark 2.2. As in [18], we use $e_0^*(a_0)$, $a_0 \in F^\times$, as the center of $GSpin_{2m+1}(F)$, $GSpin_{2m}(F)$ and define the central character accordingly. With this convention, we have $c \cdot \sigma(e_0^*(a_0)) = \sigma(e_0^*(a_0))$ (by [14, Lemma 4.5]), so $\omega_\sigma = \omega_{c\cdot\sigma}$.

To close this section, we note that if $\tau \otimes \pi$ is an irreducible representation of the standard Levi subgroup $GL_{m_1}(F) \times G_{m_0}(F)$ and w_0 is the corresponding long double-coset representative from the Weyl group, then

$$w_0(\tau\otimes\pi)=\begin{cases}\check{\tau}\otimes c^{m_1}\cdot\pi & \text{for } G_m=SO_{2m},\\ \omega_\pi\check{\tau}\otimes\pi & \text{for } G_m=GSpin_{2m+1},\\ \omega_\pi\check{\tau}\otimes c^{m_1}\cdot\pi & \text{for } G_m=GSpin_{2m},\\ \check{\tau}\otimes\pi & \text{for } G_m=U_{2m+1},\ U_{2m}.\end{cases} \tag{2.1}$$

3. Definition of the correspondence

In this section, we define the basic correspondence studied in this paper.

Lemma 3.1. *Let $G_m = SO_{2m}$ or $GSpin_{2m}$. Suppose π is irreducible with $\pi \hookrightarrow \phi_1 \times \cdots \times \phi_k \rtimes \sigma$, where ϕ_i is an irreducible supercuspidal representation of $GL_{d(\phi_i)}(F)$ and σ is an irreducible supercuspidal representation of $G_{d(\sigma)}(F)$. Further, suppose that $\sigma \not\cong c\cdot\sigma$.*

(1) *If $d(\phi_i)$ is odd for some i, then both σ and $c\cdot\sigma$ occur in the supercuspidal support of π.*
(2) *If all $d(\phi_i)$ are even, then the supercuspidal support of π contains only σ.*

Proof. For (1), choose ℓ maximal such that $d(\phi_\ell)$ is odd. A commuting argument then gives

$$\begin{aligned}\pi\ &\hookrightarrow \phi_1\times\cdots\times\phi_{\ell-1}\times\phi_{\ell+1}\times\cdots\times\phi_k\times\phi_\ell\rtimes\sigma\\ &\cong \phi_1\times\cdots\times\phi_{\ell-1}\times\phi_{\ell+1}\times\cdots\times\phi_k\times\omega_\sigma\tilde{\phi}_\ell\rtimes c\cdot\sigma,\end{aligned}$$

noting the irreducibility of $\phi_\ell \rtimes \sigma$. The claim follows.

For (2), observe that $N^*(\phi_1\times\cdots\times\phi_k)$ produces only even powers of c, from which the result follows. □

To set up the discussion in what follows, we recall a bit of notation from [18], using it to more conveniently formulate the Langlands classification and Casselman criterion. For an irreducible, essentially tempered representation T of $G_{m_0}(F)$, let

$$\beta(T)=\begin{cases}\varepsilon(T) & \text{if } G_{m_0}=GSpin_{2m_0+1},\ GSpin_{2m_0} \text{ and } d(T)>0,\\ \frac{1}{2}\varepsilon(T) & \text{if } G_{m_0}=GSpin_{2m_0+1},\ GSpin_{2m_0} \text{ and } d(T)=0,\\ 0 & \text{if } G_{m_0}=SO_{2m_0},\ U_{2m_0+1},\ U_{2m_0}.\end{cases}$$

With this notation, we let $L(\tau_1,\cdots,\tau_s;\,T)$ (resp. $\mathcal{L}(\tau_1,\cdots,\tau_s)$) be the unique irreducible subrepresentation of $\tau_1\times\cdots\times\tau_s\rtimes T$ (resp. $\tau_1\times\cdots\times\tau_s$), where $\tau_1,\ \cdots,\ \tau_s$ are essentially tempered representations of general linear groups, T is an essentially tempered representation of some $G_{m_0}(F)$, and their exponents satisfy

$$\varepsilon(\tau_1)<\varepsilon(\tau_2)<\cdots<\varepsilon(\tau_s)<\beta(T) \tag{3.1}$$

(resp. $\varepsilon(\tau_1)<\cdots<\varepsilon(\tau_s)$). For the Casselman criterion, if π is an irreducible essentially tempered representation of $G_m(F)$, and $\phi_1\otimes\cdots\phi_k\otimes\sigma \le r_{M,G}(\pi)$ is a supercuspidal constituent of the Jacquet module $r_{M,G}(\pi)$ with respect to a Levi subgroup M of G_m, here σ means a supercuspidal representation of some $G_{m_0}(F)$ with $m_0 \leq m$, we have

$$\begin{aligned} d(\phi_1)[\varepsilon(\phi_1) - \beta(\sigma)] &\geq 0, \\ d(\phi_1)[\varepsilon(\phi_1) - \beta(\sigma)] + d(\phi_2)[\varepsilon(\phi_2) - \beta(\sigma)] &\geq 0, \\ &\vdots \\ d(\phi_1)[\varepsilon(\phi_1) - \beta(\sigma)] + \cdots + d(\phi_k)[\varepsilon(\phi_k) - \beta(\sigma)] &\geq 0. \end{aligned} \tag{3.2}$$

Conversely, if the inequalities hold for all such Jacquet modules, then π is essentially tempered. The criterion for essential square-integrability is similar except that the inequalities are strict.

Notation 3.2. By [18], if $\pi = L(\tau_1, \cdots, \tau_s;\ T)$ defined as above, one has $\beta(T) = \beta(\sigma)$. Consequently, at times we may simply write β for $\beta(T)$ or $\beta(\sigma)$.

Lemma 3.3. *Let π be an irreducible representation of $G_m(F)$ for some $m > 1$. If both π and $\hat{\pi}$ are essentially tempered, we have*

$$\pi \hookrightarrow \phi_1 \times \cdots \times \phi_k \rtimes \sigma$$

for some irreducible supercuspidal $\phi_1, \ \cdots, \ \phi_k, \ \sigma$ having $\varepsilon(\phi_1) = \cdots = \varepsilon(\phi_k) = \beta(\sigma)$.

Proof. Suppose $\phi_1 \otimes \cdots \otimes \phi_k \otimes \sigma \leq r_{M,G}(\pi)$ is supercuspidal. By properties of duality, we then have $\omega_\sigma \check{\phi}_1 \otimes \cdots \otimes \omega_\sigma \check{\phi}_k \otimes \iota \cdot \sigma \leq r_{M,G}(\hat{\pi})$ for some $\iota \in \{1, c\}$. The inequalities required by the Casselman criterion in (3.2) then give

$$\begin{aligned} d(\phi_1)[\varepsilon(\phi_1) - \beta] &\geq 0, \\ d(\phi_1)[\varepsilon(\phi_1) - \beta] + d(\phi_2)[\varepsilon(\phi_2) - \beta] &\geq 0, \\ &\vdots \\ d(\phi_1)[\varepsilon(\phi_1) - \beta] + \cdots + d(\phi_k)[\varepsilon(\phi_k) - \beta] &\geq 0 \end{aligned}$$

for π and (since $\omega_\sigma = \omega_{\nu^{\varepsilon(\sigma)}\sigma_0} = \nu^{2\beta}\omega_{\sigma_0}$–see [18]),

$$\begin{aligned} d(\phi_1)[(2\beta - \varepsilon(\phi_1)) - \beta] &\geq 0, \\ d(\phi_1)[(2\beta - \varepsilon(\phi_1)) - \beta] + d(\phi_2)[(2\beta - \varepsilon(\phi_2)) - \beta] &\geq 0, \\ &\vdots \\ d(\phi_1)[(2\beta - \varepsilon(\phi_1)) - \beta(\sigma)] + \cdots + d(\phi_k)[(2\beta - \varepsilon(\phi_k)) - \beta] &\geq 0 \end{aligned}$$

for $\hat{\pi}$. It then follows that $\varepsilon(\phi_1) = \varepsilon(\phi_2) = \cdots = \varepsilon(\phi_k) = \beta$, from which the result is immediate. □

Corollary 3.4. *Let $G_m = SO_{2m}$ or $GSpin_{2m}$. Suppose that $c \cdot \sigma \cong \sigma$ and π is irreducible with $\pi \hookrightarrow \phi_1 \times \cdots \times \phi_k \rtimes \sigma$, where $\phi_1, \ \cdots, \ \phi_k, \ \sigma$ are supercuspidal. Then $c \cdot \pi \cong \pi$.*

Proof. The proof is by induction on k. The case $k = 0$ is trivial. For the proof below, let $\tilde{G}_m(F) = O_{2m}(F)$ or $Gpin_{2m}(F)$ (so $\tilde{G}^0_m = G_m$ with component group $C = \{e, c\}$).

Let σ^* (resp. π^*) be an irreducible representation of $\tilde{G}_{m_0}(F)$ (resp. $\tilde{G}_m(F)$) having $\sigma^* \leq \mathrm{Ind}_{G_{m_0}}^{\tilde{G}_{m_0}}(\sigma)$ (resp. $\pi^* \leq \mathrm{Ind}_{G_m}^{\tilde{G}_m}(\pi)$). As a consequence of Lemma 2.1 [4] (which is a straightforward consequence of [8] and [9]), $c \cdot \sigma \cong \sigma$ implies $\chi\sigma^* \not\cong \sigma^*$ and $\mathrm{Ind}_{G_{m_0}}^{\tilde{G}_{m_0}}(\sigma) \cong \sigma^* \oplus \chi\sigma^*$, where χ is the nontrivial character of $\tilde{G}_{m_0}(F)$ which is trivial on $G_{m_0}(F)$. Further, it also tells us that $c{\cdot}\pi \cong \pi \Leftrightarrow \chi\pi^* \not\cong \pi^*$. Thus it suffices to show that $\chi\pi^* \not\cong \pi^*$.

First, suppose π is not essentially tempered. Then, we may write $\pi = L(\tau_1, \cdots, \tau_\ell;\ T)$ in terms of the Langlands classification. By [4, Proposition 4.5], $c{\cdot}\pi = L(\tau_1, \cdots, \tau_\ell;\ c{\cdot}T)$. Since T comes from a group of lower rank (and with supercuspidal support a subset of that of π), the inductive hypothesis tells us $c \cdot T \cong T$. It then follows that $c \cdot \pi \cong \pi$, as needed.

Next, suppose that π is essentially tempered but $\hat{\pi}$ is not. By Lemma 3.5 below $c \cdot \hat{\pi} = \widehat{(c \cdot \pi)}$. Therefore, to show $c \cdot \pi \cong \pi$, it suffices to show $c \cdot \hat{\pi} \cong \hat{\pi}$. This follows from the previous case.

By Lemma 3.3, we are reduced to the case $\pi \hookrightarrow \nu^\beta\rho_1 \times \cdots \times \nu^\beta\rho_k \rtimes \sigma$. Then,

$$\pi^* \hookrightarrow \mathrm{Ind}_{G_m}^{\tilde{G}_m}(\nu^\beta\rho_1 \times \cdots \times \nu^\beta\rho_k \rtimes \sigma) \Rightarrow \pi^* \hookrightarrow \nu^\beta\rho_1 \times \cdots \times \nu^\beta\rho_k \rtimes \sigma^*$$

for some irreducible $\sigma^* \leq \mathrm{Ind}_{G_{m_0}}^{\tilde{G}_{m_0}}(\sigma)$. Since $c \cdot \sigma \cong \sigma$, we have $\chi\sigma^* \not\cong \sigma^*$. Then, by [6, Proposition 1.9],

$$\chi\pi^* \hookrightarrow \nu^\beta\rho_1 \times \cdots \times \nu^\beta\rho_k \rtimes \chi\sigma^*.$$

It now follows immediately from partial cuspidal support considerations that $\chi\pi^* \not\cong \pi^* \Rightarrow c \cdot \pi \cong \pi$, as needed. □

The following seems like it ought to be well-known, but we have not found a reference.

Lemma 3.5. *Let $G_m = SO_{2m}$ or $GSpin_{2m}$. Then $c \cdot \hat{\pi} = \widehat{(c \cdot \pi)}$*

Proof. If $\Phi \subset \Pi$ is a subset of simple roots and P_Φ is the corresponding standard parabolic subgroup, then $c \cdot \Phi$ is also a subset of simple roots and $c(P_\Phi) = P_{c\cdot\Phi}$ is the corresponding standard parabolic subgroup. Further, it is a straightforward matter to check that $c \circ i_{G,M_\Phi} \cong i_{G,M_{c\cdot\Phi}} \circ c$ and $c \circ r_{M_\Phi,G} \cong r_{M_{c\cdot\Phi},G} \circ c$. Then, by definition ([2]),

$$\begin{aligned} c \circ D_G &= \sum_{\Phi\subset\Pi} (-1)^{|\Phi|} c \circ i_{G,M_\Phi} \circ r_{M_\Phi,G} \\ &= \sum_{\Phi\subset\Pi} (-1)^{|\Phi|} i_{G,M_{c\cdot\Phi}} \circ c \circ r_{M_\Phi,G} \\ &= \sum_{\Phi\subset\Pi} (-1)^{|c\cdot\Phi|} i_{G,M_{c\cdot\Phi}} \circ r_{M_{c\cdot\Phi},G} \circ c. \end{aligned}$$

As $c \cdot \Phi$ also runs through the subsets of Π, we then get $c \circ D_G = D_G \circ c$, from which the result is immediate. $\square$

We now set up some notation for what follows. First, we allow Σ to be a single irreducible supercuspidal representation of $G_{m_0}(F)$ or, for $SO_{2m_0}(F)$ or $GSpin_{2m_0}(F)$, a set $\{\sigma,\ c \cdot \sigma\}$ if $c \cdot \sigma \ncong \sigma$. To unify the presentation, set

$$\omega_\Sigma = \begin{cases} 1 & \text{if } G_{m_0} = SO_{2m_0},\ U_{2m_0+1},\ U_{2m_0}, \\ \omega_\sigma & \text{if } G_{m_0} = GSpin_{2m_0+1},\ GSpin_{2m_0}, \end{cases}$$

which is well-defined by Remark 2.2. With notation as above, for irreducible unitary supercuspidal representations ρ_i of $GL_{d(\rho_i)}(F)$, we let $\mathcal{S}(\rho_1, \cdots, \rho_n; \Sigma)$ be a set of the form

$$\mathcal{S}(\rho_1, \cdots, \rho_n; \Sigma) = \{\nu^x \rho_1,\ \omega_\Sigma \nu^{-x} \check{\rho}_1\}_{x \in \mathbb{R}} \cup \cdots \cup \{\nu^x \rho_n,\ \omega_\Sigma \nu^{-x} \check{\rho}_n\}_{x \in \mathbb{R}} \cup \Sigma$$

and $Irr(\mathcal{S}(\rho_1, \cdots, \rho_n; \Sigma))$ be the set of all (equivalence classes of) irreducible representations of even G_m groups which have supercuspidal support in $\mathcal{S}(\rho_1, \cdots, \rho_n; \Sigma)$. In a similar vein, when the focus is on general linear groups, we let

$$\mathcal{S}_\Sigma(\rho_1, \cdots, \rho_n) = \{\nu^x \rho_1,\ \omega_\Sigma \nu^{-x} \check{\rho}_1\}_{x \in \mathbb{R}} \cup \cdots \cup \{\nu^x \rho_n,\ \omega_\Sigma \nu^{-x} \check{\rho}_n\}_{x \in \mathbb{R}}.$$

Definition 3.6. With notation as above, fix Σ and let X be a set of the form

$$X = \{\nu^x \rho_1,\ \nu^{-x} \omega_\Sigma \check{\rho}_1\}_{x \in \mathbb{R}} \cup \cdots \cup \{\nu^x \rho_k,\ \nu^{-x} \omega_\Sigma \check{\rho}_k\}_{x \in \mathbb{R}}.$$

For π as above, we define $\mu^*_X(\pi)$ to be the sum of all irreducible representations $\tau \otimes \theta$ in $\mu^*(\pi)$ satisfying the following:

(1) The supercuspidal support of τ is contained in X.
(2) The supercuspidal support of θ contains no terms in X.

By a commuting argument, $\mu^*_X(\pi)$ is nontrivial. Further, by central character considerations (on the minimal nonzero Jacquet module of a term in $\mu^*(\pi)$) and [15, Lemma 5.5], we have $\pi \hookrightarrow \tau \rtimes \theta$ for some irreducible representation $\tau \otimes \theta \leq \mu^*_X(\pi)$.

Lemma 3.7. *Let $\pi \in Irr(\mathcal{S}(\rho_1, \cdots, \rho_n; \Sigma))$ and $X = \mathcal{S}_\Sigma(\rho_{i_1}, \cdots, \rho_{i_k}) \subset \mathcal{S}_\Sigma(\rho_1, \cdots, \rho_n)$. Choose $\tau \otimes \theta \leq \mu^*_X(\pi)$ with $\pi \hookrightarrow \tau \rtimes \theta$. Let $\tau' \otimes \theta' \leq \mu^*_X(\pi)$.*

(1) *If $c \cdot \sigma \cong \sigma$, then $\theta' \cong \theta$.*
(2) *If $d(\rho_{i_1}),\ \cdots,\ d(\rho_{i_k})$ are all even, then $\theta' = \theta$.*
(3) *Assume that there exists ρ_i such that $d(\rho_i)$ is odd. If X contains all ρ_i having $d(\rho_i)$ odd and $\Sigma = \{\sigma,\ c \cdot \sigma\}$ (with $c \cdot \sigma \ncong \sigma$ and both in the supercuspidal support of π),*

then $\theta' = \theta$ or $c \cdot \theta$, with θ and $c \cdot \theta$ distinct and both occurring. Further, θ and $c \cdot \theta$ may be distinguished by their supercuspidal support–one has only σ, the other only $c \cdot \sigma$. (Note that this case requires $G_m = SO_{2m}$ or $GSpin_{2m}$.)

Proof. We start with a formal calculation: write $\mu^*(\theta) = \sum_\ell \lambda_\ell \otimes \theta_\ell$ (a sum of irreducible representations, repetition possible) and

$$\begin{aligned} N^*(\tau) &= (\tilde{\ } \otimes m^*)_D \circ s \circ m^*(\tau) \\ &= (\tilde{\ } \otimes m^*)_D \circ s(\textstyle\sum_i \tau_i \otimes \tau_i') \\ &= (\tilde{\ } \otimes m^*)_D(\textstyle\sum_i \tau_i' \otimes \tau_i) \\ &= \textstyle\sum_{i,j} \tilde{\tau}_i' \otimes \tau_{i,j}^{(1)} \otimes \tau_{i,j}^{(2)} \otimes c^{d(\tau_i')}, \end{aligned} \tag{3.3}$$

where the $c^{d(\tau_i')}$ term may be disregarded for $G_m = GSpin_{2m+1},\ U_{2m+1},\ U_{2m}$. Then, noting $\omega_{\theta_\ell} = \omega_\Sigma$,

$$\mu^*(\tau \rtimes \theta) = \sum_{i,j,\ell} \omega_\Sigma \tilde{\tau}_i' \times \tau_{i,j}^{(1)} \times \lambda_\ell \otimes \tau_{i,j}^{(2)} \rtimes c^{d(\tau_i')} \cdot \theta_\ell. \tag{3.4}$$

Observe that for the first factor to have supercuspidal support in X, we must have $\lambda_\ell = 1$ (as $\lambda_\ell \otimes \theta_\ell \leq \mu^*(\theta)$ and the supercuspidal support of θ has no terms in X). In particular, we must have $\theta_\ell = \theta$. Similarly, for $\tau_{i,j}^{(2)} \rtimes c^{d(\tau_i')} \cdot \theta_\ell$ to have no terms in X, we must have $\tau_{i,j}^{(2)} = 1$ (as the supercuspidal support of $\tau_{i,j}^{(2)}$ is contained in that of τ). Thus

$$\theta' \leq \tau_{i,j}^{(2)} \rtimes c^{d(\tau_i')} \cdot \theta_\ell = c^{d(\tau_i')} \cdot \theta.$$

For (1), we note that it follows from the hypotheses and Corollary 3.4 that $c^{d(\tau_i')} \cdot \theta = \theta$. For (2), we observe that the hypotheses imply $d(\tau_i')$ is even for all i, from which the conclusion is immediate. For (3), that the only possibilities are $\theta' = \theta$ or $c \cdot \theta$ is clear. Since odd $d(\tau_i')$ occurs (Lemma 3.1), we do pick up copies of $c \cdot \theta$. To see that $c \cdot \theta \not\cong \theta$, note that the supercuspidal support of θ contains no terms in X, hence has only representations of even general linear groups. By Lemma 3.1 (2), θ contains only σ (and not $c \cdot \sigma$) in its supercuspidal support. Then $c \cdot \theta$ contains only $c \cdot \sigma$ in its supercuspidal support, so $c \cdot \theta \not\cong \theta$, as claimed. □

We are now ready to set up one direction of the correspondence.

Definition 3.8. Let $\pi \in Irr(\mathcal{S}(\rho_1, \cdots, \rho_n; \Sigma))$; if $|\Sigma| = 2$, we assume that both σ and $c \cdot \sigma$ appear in the supercuspidal support of π. Set

$$X_i = \mathcal{S}_\Sigma(\rho_1, \cdots, \rho_{i-1}, \rho_{i+1}, \cdots, \rho_n) \tag{3.5}$$

and let $\tau \otimes \theta \leq \mu^*_{X_i}(\pi)$ be an irreducible representation with $\pi \hookrightarrow \tau \rtimes \theta$. We consider two cases, depending on whether $c \cdot \sigma \cong \sigma$ or not.

(1) $c \cdot \sigma \not\cong \sigma$ (so $G_m = SO_{2m}$ or $GSpin_{2m}$).

(a) $d(\rho_i)$ is even. If we have some ρ_j with $d(\rho_j)$ odd, all such ρ_j appear in X_i. By Lemma 3.7 (3), $\mu^*_{X_i}(\pi)$ contains both θ and $c \cdot \theta$. Further, they may be distinguished by their supercuspidal supports–one has only σ; the other, only $c \cdot \sigma$. We then define

$$\psi^{(\sigma)}_{\rho_i}(\pi) = \theta \text{ or } c \cdot \theta,$$

whichever contains σ in its supercuspidal support. If there are no ρ_j with $d(\rho_j)$ odd, only θ occurs (by Lemma 3.7 (2)); the superscript (σ) is not needed but is still included to simplify notation in what follows.

(b) $d(\rho_i)$ odd. In this case, as one cannot always separate the effects of individual ρ_j having $d(\rho_j)$ odd (as may be seen in the more complicated R-group structure in [10], [3]), we bundle them together. Set

$$X_{odd} = \mathcal{S}_\Sigma(\rho_{i_1}, \cdots, \rho_{i_\ell}),$$

with $\rho_{i_1}, \cdots, \rho_{i_\ell}$ consisting of all ρ_i having $d(\rho_i)$ even. We then set

$$\psi_{odd}(\pi) = \theta,$$

where $\tau \otimes \theta$ is an irreducible representation in $\mu^*(\pi)$ with $\pi \hookrightarrow \tau \rtimes \sigma$, noting that this is well-defined by Lemma 3.7 (3).

(2) $c \cdot \sigma \cong \sigma$.
In this case, we set

$$\psi^{(\sigma)}_{\rho_i}(\pi) = \theta$$

as above, noting that this is well-defined by Lemma 3.7 (1). We remark that the superscript (σ) is also not needed in this case, but again simplifies the notation in what follows.

In cases where $\psi_{odd}(\pi)$ is not defined, we set $\psi_{odd}(\pi) = 1$ to allow for a more uniform presentation.

Note that for $SO_{2m}(F)$ (resp. $GSpin_{2m}(F)$), we have $\sigma = 1 \otimes e$ (resp. $\sigma = \chi \otimes e$) satisfying $c \cdot \sigma = 1 \otimes c$ (resp. $c \cdot (\chi \otimes e) = \chi \otimes c$). In particular, we have $c \cdot \sigma \not\cong \sigma$, so use (1) in the definition.

To set up one direction of the correspondence, we assume without loss of generality

$$\rho_1, \ \cdots, \ \rho_k \text{ have } d(\rho_i) \text{ odd and } \rho_{k+1}, \ \cdots, \ \rho_n \text{ have } d(\rho_i) \text{ even.} \tag{3.6}$$

Then

$$\begin{aligned} Irr(\mathcal{S}(\rho_1,\cdots,\rho_n;\Sigma)) &\longrightarrow Irr(\mathcal{S}(\rho_1,\cdots,\rho_k;\Sigma))\times Irr(\mathcal{S}(\rho_{k+1};\Sigma))\times\cdots\times Irr(\mathcal{S}(\rho_n;\Sigma)) \\ \pi &\longmapsto \big(\psi_{odd}(\pi),\ \psi^{(\sigma)}_{\rho_{k+1}}(\pi),\ \cdots,\ \psi^{(\sigma)}_{\rho_n}(\pi)\big), \end{aligned} \tag{3.7}$$

treating k as 0 for $G_m = U_{2m+1}$, U_{2m}, and $GSpin_{2m+1}$ or when $c\cdot\sigma \cong \sigma$.

Note 3.9. For $G_m = SO_{2m},\ GSpin_{2m}$, we note that, $\tau\otimes\theta$ is an irreducible representation in $\mu^*(\pi)$,

$$\pi \hookrightarrow \tau \rtimes \theta \Rightarrow c\cdot\pi \hookrightarrow \tau \rtimes c\cdot\theta.$$

Therefore

$$c\cdot\psi_{odd}(\pi) = \psi_{odd}(c\cdot\pi) \text{ and } c\cdot\psi^{(\sigma)}_{\rho_i}(\pi) = \psi^{(\sigma)}_{\rho_i}(c\cdot\pi).$$

4. Main result

In this section, we serve you the main result of the paper. We start with a couple of basic properties.

Lemma 4.1. *With notation as in Definition 3.8 and (3.6), (3.7) above, suppose that* $\pi \in Irr(\mathcal{S}(\rho_1,\cdots,\rho_n;\Sigma))$*. Then* π *is essentially tempered (resp. essentially square-integrable) if and only if* $\psi_{odd}(\pi),\ \psi^{(\sigma)}_{\rho_{k+1}}(\pi),\ \cdots,\ \psi^{(\sigma)}_{\rho_n}(\pi)$ *are all essentially tempered (resp. all essentially square-integrable).*

Proof. We do the essentially tempered case; essential square-integrability is similar.
($\Rightarrow$): Suppose π is essentially tempered but $\psi_{odd}(\pi)$ or some $\psi^{(\sigma)}_{\rho_i}(\pi)$ is not. For concreteness, suppose it is $\psi^{(\sigma)}_{\rho_n}(\pi)$ that is not essentially tempered. Write

$$\psi^{(\sigma)}_{\rho_n}(\pi) = L(\tau_1(\rho_n),\cdots,\tau_s(\rho_n);\ T(\rho_n;\sigma))$$

with $\tau_i(\rho_n) \in Irr(\mathcal{S}_\Sigma(\rho_n))$ for $1\le i\le s$, $T(\rho_n;\sigma)$ essentially tempered and satisfying (3.1). Then, for some $\lambda(X_n)\in Irr(X_n)$,

$$\begin{aligned} \pi \hookrightarrow \lambda(X_n)\rtimes\psi^{(\sigma)}_{\rho_n}(\pi) &\hookrightarrow \lambda(X_n)\times\tau_1(\rho_n)\times\cdots\times\tau_s(\rho_n)\rtimes T(\rho_n;\sigma) \\ &\cong \tau_1(\rho_n)\times\cdots\times\tau_s(\rho_n)\times\lambda(X_n)\rtimes T(\rho_n;\sigma). \end{aligned}$$

By Frobenius reciprocity, $\mu^*(\pi)$ contains a term of the form $\tau_1(\rho_n)\otimes\cdots$. We claim that this contradicts the Casselman criterion for the essential temperedness of π.

To this end, suppose that $\phi_1\otimes\cdots\otimes\phi_r \le r_{M,GL}(\tau_1(\rho_n))$ supercuspidal (noting that ϕ_i is a twist of ρ_n or $\omega_\Sigma\check{\rho}_n$ for all i). Then, by (3.1),

$$\frac{\varepsilon(\phi_1)+\cdots+\varepsilon(\phi_r)}{r} = \varepsilon(\tau_1(\rho_n)) < \beta \Rightarrow \varepsilon(\phi_1)+\cdots+\varepsilon(\phi_r) < r\beta.$$

On the other hand, since π is assumed essentially tempered, by (3.2),

$$d(\phi_1)(\varepsilon(\phi_1)-\beta)+\cdots+d(\phi_r)(\varepsilon(\phi_r)-\beta) \geq 0 \Rightarrow \varepsilon(\phi_1)+\cdots+\varepsilon(\phi_r) \geq r\beta,$$

a contradiction. The result follows.

($\Leftarrow$): Suppose $\psi_{odd}(\pi)$ and all the $\psi^{(\sigma)}_{\rho_i}(\pi)$ are essentially tempered but π is not. We may then write $\pi = L(\tau_1, \cdots, \tau_s;\ T)$. Further, we may write each $\tau_i = \tau_i(odd) \times \tau_i(\rho_{k+1}) \times \cdots \times \tau_i(\rho_n)$, with at least one factor nontrivial, here $\tau_i(\rho_j) \in Irr(\mathcal{S}_\Sigma(\rho_j))$ for $k+1 \leq j \leq n$. Suppose for the sake of concreteness that at least one of the $\tau_i(\rho_n)$ is nontrivial. By a result of Harish-Chandra (see [26, Proposition III.4.1]), we may write, for some essentially square-integrable $\delta(odd) \in Irr(\mathcal{S}_\Sigma(\rho_1, \cdots, \rho_k))$, $\delta(\rho_{k+1}) \in Irr(\mathcal{S}_\Sigma(\rho_{k+1}))$, $\cdots$, $\delta(\rho_{n-1}) \in Irr(\mathcal{S}_\Sigma(\rho_{n-1}))$,

$$T \hookrightarrow \delta(odd) \times \delta(\rho_{k+1}) \times \cdots \times \delta(\rho_{n-1}) \rtimes \psi^{(\sigma)}_{\rho_n}(T) = \delta(X_n) \rtimes \psi^{(\sigma)}_{\rho_n}(T)$$

and $\tau_i = \tau_i(X_n) \times \tau_i(\rho_n)$, we have

$$\begin{aligned}\pi &\hookrightarrow \tau_1(X_n) \times \tau_1(\rho_n) \times \cdots \times \tau_s(X_n) \times \tau_s(\rho_n) \times \delta(X_n) \rtimes \psi^{(\sigma)}_{\rho_n}(T) \\ &\cong \tau_1(X_n) \times \cdots \times \tau_s(X_n) \times \delta(X_n) \times \tau_1(\rho_n) \times \cdots \times \tau_s(\rho_n) \rtimes \psi^{(\sigma)}_{\rho_n}(T).\end{aligned}$$

Taking Jacquet modules in stages, there is some $\lambda \otimes \theta \leq \mu^*(\pi)$ having

$$r_{L,M}(\lambda \otimes \theta) \geq \tau_1(X_n) \otimes \cdots \otimes \tau_s(X_n) \otimes \delta(X_n) \otimes \tau_1(\rho_n) \otimes \cdots \otimes \tau_s(\rho_n) \otimes \psi^{(\sigma)}_{\rho_n}(T);$$

by supercuspidal support considerations, necessarily $\lambda \otimes \theta \leq \mu^*_{X_n}(\pi)$. In particular, $\theta = \psi^{(\sigma)}_{\rho_n}(\pi)$ or $c \cdot \psi^{(\sigma)}_{\rho_n}(\pi)$ (Lemma 3.7), so $\psi^{(\sigma)}_{\rho_n}(\pi)$ contains $\tau_1(\rho_n) \otimes \cdots \otimes \tau_s(\rho_n) \otimes \psi^{(\sigma)}_{\rho_n}(T)$ or $\tau_1(\rho_n) \otimes \cdots \otimes \tau_s(\rho_n) \otimes c \cdot \psi^{(\sigma)}_{\rho_n}(T)$ in its Jacquet module. In either case, as we have some $\tau_i(\rho_n)$ nontrivial, this contradicts the Casselman criterion for the temperedness of $\psi^{(\sigma)}_{\rho_n}(\pi)$ as in ($\Rightarrow$) above, finishing the proof. □

Lemma 4.2. *With notation as in Definition 3.8 and (3.7), (3.6) above, suppose that $\pi \in Irr(\mathcal{S}(\rho_1, \cdots, \rho_n; \Sigma))$ has $\pi = L(\tau_1, \cdots, \tau_s;\ T)$. For $1 \leq j \leq s$, write $\tau_j = \tau_j(odd) \times \tau_j(\rho_{k+1}) \times \cdots \times \tau_j(\rho_n)$, with $\tau_j(odd) \in Irr(\mathcal{S}_\Sigma(\rho_1, \cdots, \rho_k))$ and $\tau_j(\rho_i) \in Irr(\mathcal{S}_\Sigma(\rho_i))$ for $k+1 \leq i \leq n$. Then*

$$\psi_{odd}(\pi) = L\big(\tau_1(odd), \cdots, \tau_s(odd);\ \psi_{odd}(T)\big)$$

and

$$\psi^{(\sigma)}_{\rho_i}(\pi) = L\big(\tau_1(\rho_i), \cdots, \tau_s(\rho_i);\ \psi^{(\sigma)}_{\rho_i}(T)\big).$$

Proof. For concreteness, we look at the case $i = n$; the other cases are similar. Write $T \hookrightarrow \delta(odd) \times \delta(\rho_{k+1}) \times \cdots \times \delta(\rho_{n-1}) \rtimes \psi^{(\sigma)}_{\rho_n}(T)$ for some essentially square-integrable $\delta(odd) \in Irr(\mathcal{S}_\Sigma(\rho_1, \cdots, \rho_k))$, $\delta(\rho_{k+1}) \in Irr(\mathcal{S}_\Sigma(\rho_{k+1}))$, $\cdots$, $\delta(\rho_{n-1}) \in Irr(\mathcal{S}_\Sigma(\rho_{n-1}))$. Then

$$\pi \hookrightarrow \tau_1(odd) \times \tau_1(\rho_{k+1}) \times \cdots \times \tau_1(\rho_n) \times \cdots \times \tau_s(odd) \times \tau_s(\rho_{k+1}) \times \cdots \times \tau_s(\rho_n) \\ \times \delta(odd) \times \delta(\rho_{k+1}) \times \cdots \times \delta(\rho_{n-1}) \rtimes \psi^{(\sigma)}_{\rho_n}(T).$$

Write $\tau_j(X_n) = \tau_j(odd) \times \tau_j(\rho_{k+1}) \times \cdots \times \tau_j(\rho_{n-1})$ and similarly for $\delta(X_n)$, a commuting argument then gives

$$\pi \hookrightarrow \tau_1(X_n) \times \cdots \times \tau_s(X_n) \times \delta(X_n) \times \tau_1(\rho_n) \times \cdots \times \tau_s(\rho_n) \rtimes \psi^{(\sigma)}_{\rho_n}(T).$$

By [15, Lemma 5.5], $\pi \hookrightarrow \lambda \rtimes \theta$ for some irreducible $\lambda \le \tau_1(X_n) \times \cdots \times \tau_s(X_n) \times \delta(X_n)$ and $\theta \le \tau_1(\rho_n) \times \cdots \times \tau_s(\rho_n) \rtimes \psi^{(\sigma)}_{\rho_n}(T)$; by Lemma 3.7 and the definition of $\psi^{(\sigma)}_{\rho_n}$, $\theta = \psi^{(\sigma)}_{\rho_n}(\pi)$. In particular, $\psi^{(\sigma)}_{\rho_n}(\pi) \le \tau_1(\rho_n) \times \cdots \times \tau_s(\rho_n) \rtimes \psi^{(\sigma)}_{\rho_n}(T)$. On the other hand, by Frobenius reciprocity and Jacquet modules in stages, we have

$$r_{M,G}(\psi^{(\sigma)}_{\rho_n}(\pi)) \ge \tau_1(\rho_n) \otimes \cdots \otimes \tau_s(\rho_n) \otimes \psi^{(\sigma)}_{\rho_n}(T).$$

It follows from properties of the Langlands classification (e.g., [5, Proposition 5.3]) that the only irreducible subquotient of $\tau_1(\rho_n) \times \cdots \times \tau_s(\rho_n) \rtimes \psi^{(\sigma)}_{\rho_n}(T)$ containing $\tau_1(\rho_n) \otimes \cdots \otimes \tau_s(\rho_n) \otimes \psi^{(\sigma)}_{\rho_n}(T)$ in its Jacquet module is $L\big(\tau_1(\rho_n), \cdots, \tau_s(\rho_n);\ \psi^{(\sigma)}_{\rho_n}(T)\big)$. Thus $\psi^{(\sigma)}_{\rho_n}(\pi) = L\big(\tau_1(\rho_n), \cdots, \tau_s(\rho_n);\ \psi^{(\sigma)}_{\rho_n}(T)\big)$, as claimed. □

Lemma 4.3. *With notation as in Definition 3.8 and (3.6), (3.7) above, suppose that* $\pi \in Irr(\mathcal{S}(\rho_1, \cdots, \rho_n; \Sigma))$. *Then* $\widehat{\psi_{odd}(\pi)} = \psi_{odd}(\hat{\pi})$ *and* $\widehat{\psi^{(\sigma)}_{\rho_i}(\pi)} = \psi^{(\sigma)}_{\rho_i}(\hat{\pi})$ *for* $k+1 \le i \le n$.

Proof. This follows directly from [2, Theorem 1.7 (2)] and the definitions of ψ_{odd}, $\psi^{(\sigma)}_{\rho_i}$. We remark that in cases where it matters, $d(\rho_i)$ is even so $\widehat{\psi^{(\sigma)}_{\rho_i}(\pi)}$ contains σ rather than $c \cdot \sigma$ in its supercuspidal support. □

Proposition 4.4. *With notation as above, suppose* $\pi_{odd} \in Irr(\mathcal{S}(\rho_1, \cdots, \rho_k; \Sigma))$ *and* $\pi_i \in Irr(\mathcal{S}(\rho_i; \sigma))$ *for* $k+1 \le i \le n$. *Then there is some* $\pi \in Irr(\mathcal{S}(\rho_1, \cdots, \rho_n; \Sigma))$ *satisfying* $\psi_{odd}(\pi) = \pi_{odd}$ *and* $\psi^{(\sigma)}_{\rho_i}(\pi) = \pi_i$ *for* $k+1 \le i \le n$.

Proof. For concreteness, suppose $\pi_{odd} \ne 1$; the case $\pi_{odd} = 1$ is similar. Write $\pi_{odd} \hookrightarrow \tau(odd) \rtimes \sigma$ and $\pi_i \hookrightarrow \tau(\rho_i) \rtimes \sigma$ for $k+1 \le i \le n$. We prove by induction on $\ell = |\{\rho_{k+1}, \cdots, \rho_n\}| = n-k$ that $\tau(odd) \times \tau(\rho_{k+1}) \times \cdots \times \tau(\rho_n) \rtimes \sigma$ has an irreducible subquotient with the required properties. The case $\ell = 0$ is trivial. (If $\pi_{odd} = 1$, the case $\ell = 1$ is trivial and starts the induction.)

Now, suppose the result holds for $\ell-1$. Let $\pi' \in Irr(\mathcal{S}(\rho_1,\cdots,\rho_{n-1};\Sigma))$ having $\psi_{odd}(\pi')=\pi_{odd}$ and $\psi^{(\sigma)}_{\rho_i}(\pi')=\pi_i$ for $k+1\leq i\leq n-1$. We first claim that any irreducible $\pi^*\leq \tau(\rho_n)\rtimes\pi'$ has $\psi_{odd}(\pi^*)=\pi_{odd}$ and $\psi^{(\sigma)}_{\rho_i}(\pi^*)=\pi_i$ for $k+1\leq i\leq n-1$.

We check that $\psi_{odd}(\pi^*)=\pi_{odd}$; the other cases are similar. Looking at (3.3) and (3.4) with $\tau=\tau(\rho_n)$ and $\theta=\pi'$, we see that to contribute to $\mu^*_{X_{odd}}(\tau(\rho_n)\rtimes\pi')$, we must have $\lambda_\ell=1\Rightarrow\theta_\ell=\pi'$. Similarly, $\tau^{(2)}_{i,j}=1$. As $d(\rho_n)$ is even, we have $c^{d(\tau'_i)}=e$, from which it follows that $\psi_{odd}(\pi^*)=\psi_{odd}(\pi')=\pi_{odd}$, as needed.

Thus it remains to show that there is an irreducible subquotient $\pi^*\leq\tau(\rho_n)\rtimes\pi'$ having $\psi^{(\sigma)}_{\rho_n}(\pi^*)=\pi_n$. For this, it is enough to show that there is a term of the form $\lambda\otimes\pi_n\leq\mu^*_{X_n}(\tau(\rho_n)\rtimes\pi')$. Take a term $\lambda\otimes\sigma\leq\mu^*(\pi')$. Since $1\otimes1\otimes\tau(\rho_n)\otimes e\leq N^*(\tau(\rho_n))$ (disregarding the part $\otimes e$ if $G_m\neq SO_{2m}$ or $GSpin_{2m}$), we have

$$\mu^*(\tau(\rho_n)\rtimes\pi')\geq(1\otimes1\otimes\tau(\rho_n)\otimes e)\tilde{\rtimes}(\lambda\otimes\sigma)=\lambda\otimes(\tau(\rho_n)\rtimes\sigma).$$

By supercuspidal support considerations, this lies in $\mu^*_{X_n}(\tau(\rho_n)\rtimes\pi')$. Since $\pi_n\leq\tau(\rho_n)\rtimes\sigma$, the result follows. □

Proposition 4.5. *Suppose $\pi,\pi'\in Irr(\rho_1,\cdots,\rho_n;\Sigma)$. If $\psi_{odd}(\pi)=\psi_{odd}(\pi')$ and $\psi^{(\sigma)}_{\rho_i}(\pi)=\psi^{(\sigma)}_{\rho_i}(\pi')$ for $i=k+1,\ \cdots,\ n$, then $\pi\cong\pi'$.*

Proof. The proof is by induction on ℓ, where ℓ is the number of factors in a standard Levi subgroup $M=GL_{m_1}\times\cdots\times GL_{m_\ell}\times G_{m_0}$ which supports a minimal nonzero Jacquet module. The case $\ell=1$ is trivial. We break the inductive step into three cases.

Case 1: π not essentially tempered.

In this case, it follows from Lemma 4.1 that π' is also nontempered. Set $\pi=L(\tau_1,\cdots,\tau_s;\ T)$ and $\pi'=L(\tau'_1,\cdots,\tau'_t;\ T')$. Observe that it follows from Lemma 4.2 that s is the number of different exponents which appear in the Langlands data for $\psi_{odd}(\pi),\ \psi^{(\sigma)}_{\rho_{k+1}}(\pi),\ \cdots,\ \psi^{(\sigma)}_{\rho_n}(\pi)$, and similarly for π'. Thus $s=t$. Further, since $\psi_{odd}(\pi)=\psi_{odd}(\pi')$, it follows from Lemma 4.2 that

$$\begin{gathered}L\big(\tau_1(odd),\cdots,\tau_s(odd);\ \psi_{odd}(T)\big)=L\big(\tau'_1(odd),\cdots,\tau'_s(odd);\ \psi_{odd}(T')\big)\\ \Downarrow\\ \psi_{odd}(T)=\psi_{odd}(T'),\ \text{and}\ \tau_1(odd)=\tau'_1(odd),\ \ \cdots,\ \ \tau_s(odd)=\tau'_s(odd).\end{gathered}$$

Similarly,

$$\psi^{(\sigma)}_{\rho_i}(T)=\psi^{(\sigma)}_{\rho_i}(T'),\ \text{and}\ \tau_1(\rho_i)=\tau'_1(\rho_i),\ \ \cdots,\ \ \tau_s(\rho_i)=\tau'_s(\rho_i)$$

for $k+1\leq i\leq n$. We then have

$$\tau_j=\tau_j(odd)\times\tau_j(\rho_{k+1})\times\cdots\times\tau_j(\rho_n)=\tau'_j$$

for $1 \leq j \leq s = t$. Further, by the inductive hypothesis, $\psi_{odd}(T) = \psi_{odd}(T')$ and $\psi_{\rho_i}^{(\sigma)}(T) = \psi_{\rho_i}^{(\sigma)}(T')$ for $k+1 \leq i \leq n$ implies $T = T'$. It now follows that $\pi = \pi'$, as needed.

Case 2: π essentially tempered but $\hat{\pi}$ not essentially tempered.

This follows directly from Lemma 4.3 and the preceding case.

Case 3: both π and $\hat{\pi}$ essentially tempered.

By Lemma 3.3, we have

$$\begin{array}{c} \pi \hookrightarrow \phi_1 \times \cdots \times \phi_t \rtimes \sigma \\ \Downarrow \quad ([18]) \\ \nu^{-\beta}\pi \hookrightarrow \nu^{-\beta}\phi_1 \times \cdots \times \nu^{-\beta}\phi_t \rtimes \nu^{-\varepsilon(\sigma)}\sigma \end{array} \tag{4.1}$$

a tempered situation, to which the results of [10], [11], [3] may be applied. Further, by commuting and inverting (i.e., replacing ρ_i by $\omega_\sigma \check{\rho}_i$ and σ by $c^{d(\rho_i)} \cdot \sigma$) as needed, we may without loss of generality write

$$\nu^{-\beta}\pi \hookrightarrow \underbrace{\rho_1 \times \cdots \times \rho_1}_{\ell_1} \times \cdots \times \underbrace{\rho_n \times \cdots \times \rho_n}_{\ell_n} \rtimes \nu^{-\varepsilon(\sigma)}\sigma \tag{4.2}$$

(see Note 3.9). From the definition and [18], one has $\nu^{\beta}\psi_{odd}(\pi) = \psi_{odd}(\nu^{\beta}\pi)$, etc., so it suffices to deal with the tempered case. Thus we assume that $\beta = 0$ and π, $\hat{\pi}$ are tempered below.

First, we claim that to have $\psi_{odd}(\pi) = \psi_{odd}(\pi')$ and $\psi_{\rho_i}^{(\sigma)}(\pi) = \psi_{\rho_i}^{(\sigma)}(\pi')$ for all i, we must have the same embedding for $\nu^{-\beta}\pi'$ as well. If this is not the case, supercuspidal support considerations and the same argument as above would tell us

$$\pi' \hookrightarrow \underbrace{\rho_1 \times \cdots \times \rho_1}_{\ell_1} \times \cdots \times \underbrace{\rho_n \times \cdots \times \rho_n}_{\ell_n} \rtimes \nu^{-\varepsilon(\sigma)} c \cdot \sigma.$$

Note that this possibility requires at least one ρ_i to have $d(\rho_i)$ odd. Further, in order to have

$$\underbrace{\rho_1 \times \cdots \times \rho_1}_{\ell_1} \times \cdots \times \underbrace{\rho_n \times \cdots \times \rho_n}_{\ell_n} \rtimes \nu^{-\varepsilon(\sigma)}\sigma \ncong \underbrace{\rho_1 \times \cdots \times \rho_1}_{\ell_1} \times \cdots \times \underbrace{\rho_n \times \cdots \times \rho_n}_{\ell_n} \rtimes \nu^{-\varepsilon(\sigma)} c \cdot \sigma,$$

all ρ_i having $d(\rho_i)$ odd must have $\omega_\Sigma \check{\rho}_i \ncong \rho_i$. However, in this case, one has

$$\underbrace{\rho_1 \times \cdots \times \rho_1}_{\ell_1} \times \cdots \times \underbrace{\rho_k \times \cdots \times \rho_k}_{\ell_k} \rtimes \nu^{-\varepsilon(\sigma)}\sigma$$

and

$$\underbrace{\rho_1 \times \cdots \times \rho_1}_{\ell_1} \times \cdots \times \underbrace{\rho_n \times \cdots \times \rho_n}_{\ell_n} \rtimes \nu^{-\varepsilon(\sigma)} c \cdot \sigma$$

both irreducible and inequivalent. Thus, $\psi_{odd}(\pi') \neq \psi_{odd}(\pi)$, a contradiction. The claim follows.

We now obtain the result from [10], [11], [3]. We do the case where $\psi_{odd}(\pi) \neq 1$, i.e., $k > 0$ and $G_m = SO_{2m}$ or $GSpin_{2m}$; the case where $\psi_{odd}(\pi) = 1$ is similar, but easier (as the R-group R_{odd} below is trivial). Recall that the components of (4.2) are parameterized by representations of its R-group. We note that in what follows, we use the representations of R as a convenient parameterization of the components; other properties of the R-group–e.g., those involving actions of normalized standard intertwining operators–are not needed nor any results about them claimed. Now, [10], [11], [3] tell us that if R is the R-group for (4.2), then $R \cong R_{odd} \times R_{\rho_{k+1}} \times \cdots \times R_{\rho_n}$, where R_{odd} is the R-group for $\rho_1 \times \cdots \times \rho_k \rtimes \sigma$ and R_{ρ_i} is the R-group for $\rho_i \rtimes \sigma$. Further, $R_{odd} \cong \mathbb{Z}_2^{k'-1}$, where $k' = |\{i \,|\, \omega_\Sigma \tilde{\rho}_i \cong \rho_i \text{ with } 1 \leq i \leq k\}|$, and

$$R_{\rho_i} \cong \begin{cases} \mathbb{Z}_2 & \text{if } \rho_i \rtimes \sigma \text{ is reducible,} \\ 1 & \text{if not.} \end{cases}$$

Note that this implies (4.2) decomposes with multiplicity one. It also follows from [3] that if $\mathcal{T}(\rho_i;\sigma)$ is a component of $\rho_i \rtimes \sigma$, then

$$\underbrace{\rho_i \times \cdots \times \rho_i}_{\ell_i - 1} \rtimes \mathcal{T}(\rho_i;\sigma)$$

is irreducible; likewise, if $\mathcal{T}(\rho_1, \cdots, \rho_k;\Sigma)$ is a component of $\rho_1 \cdots \times \rho_k \rtimes \sigma$, then

$$\underbrace{\rho_1 \times \cdots \times \rho_1}_{\ell_1 - 1} \times \cdots \times \underbrace{\rho_k \times \cdots \times \rho_k}_{\ell_k - 1} \rtimes \mathcal{T}(\rho_1, \cdots, \rho_k;\Sigma)$$

is irreducible. Now, for an irreducible $\mathcal{T}$ appearing in (4.2) and $k+1 \leq i \leq n$, there is a unique $\mathcal{T}_{\hat{r}_i}(\rho_i;\sigma)$, $\hat{r}_i \in \hat{R}_{\rho_i}$ the character group of R_{ρ_i}, such that

$$\begin{aligned} \mathcal{T} \hookrightarrow \quad & \underbrace{\rho_1 \times \cdots \times \rho_1}_{\ell_1} \times \cdots \times \underbrace{\rho_{i-1} \times \cdots \times \rho_{i-1}}_{\ell_{i-1}} \times \underbrace{\rho_{i+1} \times \cdots \times \rho_{i+1}}_{\ell_{i+1}} \\ & \times \underbrace{\rho_n \times \cdots \times \rho_n}_{\ell_n} \rtimes \big(\underbrace{\rho_i \times \cdots \times \rho_i}_{\ell_i - 1} \rtimes \mathcal{T}_{\hat{r}_i}(\rho_i;\sigma) \big). \end{aligned}$$

Similarly, there is a unique $\mathcal{T}_{\hat{r}_{odd}}(\rho_1, \cdots, \rho_k;\Sigma)$, $\hat{r}_{odd} \in \hat{R}_{odd}$ the character group of R_{odd}, such that

$$\begin{aligned} \mathcal{T} \hookrightarrow \quad & \underbrace{\rho_{k+1} \times \cdots \times \rho_{k+1}}_{\ell_{k+1}} \times \cdots \times \underbrace{\rho_n \times \cdots \times \rho_n}_{\ell_n} \\ & \times \big(\underbrace{\rho_1 \times \cdots \times \rho_1}_{\ell_1 - 1} \times \cdots \times \underbrace{\rho_k \times \cdots \times \rho_k}_{\ell_k - 1} \rtimes \mathcal{T}_{\hat{r}_{odd}}(\rho_1, \cdots, \rho_k;\Sigma) \big). \end{aligned}$$

From the isomorphism $R \cong R_{odd} \times R_{\rho_{k+1}} \times \cdots \times R_{\rho_n}$ (and the structure of R_{odd} and the R_{ρ_i}), let $\hat{R}$ be the character group of R, we have a bijection

$$\begin{aligned} \hat{R} &\longleftrightarrow \hat{R}_{odd} \times \hat{R}_{\rho_{k+1}} \times \cdots \times \hat{R}_{\rho_n} \\ \hat{r} &\longleftrightarrow (\hat{r}_{odd},\ \hat{r}_{\rho_{k+1}},\ \cdots,\ \hat{r}_{\rho_n}). \end{aligned}$$

Further, from the definitions, we have $\psi^{(\sigma)}_{\rho_i}(\mathcal{T}) = \underbrace{\rho_i \times \cdots \times \rho_i}_{\ell_i - 1} \rtimes \mathcal{T}_{\hat{r}_i}(\rho_i;\sigma)$ for all i, and $\psi_{odd}(\mathcal{T}) = \underbrace{\rho_1 \times \cdots \times \rho_1}_{\ell_1-1} \times \cdots \times \underbrace{\rho_k \times \cdots \times \rho_k}_{\ell_k-1} \rtimes \mathcal{T}_{\hat{r}_{odd}}(\rho_1,\cdots,\rho_k;\Sigma)$. Thus, if $\mathcal{T},\mathcal{T}'$ are two components with $\psi_{odd}(\mathcal{T}) = \psi_{odd}(\mathcal{T}')$ and $\psi^{(\sigma)}_{\rho_i}(\mathcal{T}) = \psi^{(\sigma)}_{\rho_i}(\mathcal{T}')$ for all i, then

$$\hat{r}_{odd} = \hat{r}'_{odd} \text{ and } \hat{r}_{\rho_i} = \hat{r}'_{\rho_i} \text{ for } k+1 \le i \le n \Rightarrow \hat{r} = \hat{r}' \Rightarrow \mathcal{T} \cong \mathcal{T}',$$

as needed. □

Remarks 4.6.

(1) The discussion in Case 3 is the underlying reason that the ρ_i having $d(\rho_i)$ odd are considered together for $G_m = SO_{2m}$ or $GSpin_{2m}$: if $k > 1$, the $\rho_i \rtimes \sigma$ are irreducible for all i, but $\rho_1 \times \cdots \times \rho_k \rtimes \sigma$ is reducible. This interdependence prevents the effects of the different ρ_i from being separated.
(2) From [7], in the case of odd residual characteristic, the only self-contragredient ρ having $d(\rho)$ odd are the order two characters. Thus, in the case of $G_m = SO_{2m}$, we have $k \le 3$.

5. Parabolic induction

In this section, we show that the correspondence behaves well with respect to parabolic induction.

Definition 5.1. For π a representation of $G_m(F)$, we define $r_{GL}(\pi)$ to be the sum of all terms in $\mu^*(\pi)$ which have the form $\tau \otimes \sigma$ or $\tau \otimes c \cdot \sigma$.

Lemma 5.2. *We follow the conventions of (3.6). Suppose* $\pi \in Irr\big(\mathcal{S}(\rho_1,\cdots,\rho_n;\Sigma)\big)$ *with both* π *and* $\hat{\pi}$ *essentially tempered. Write*

$$r_{GL}(\psi_{odd}(\pi)) = \sum_{j_{odd}} c_{j_{odd}}(odd)\kappa_{j_{odd}}(odd) \otimes c^{a_{j_{odd}}} \cdot \sigma$$

and, for $k+1 \le i \le n$,

$$r_{GL}(\psi^{(\sigma)}_{\rho_i}(\pi)) = \sum_{j_i} c_{j_i}(\rho_i)\kappa_{j_i}(\rho_i) \otimes \sigma,$$

where $\kappa_{j_i}(\rho_i)$ *(resp.* $\kappa_{j_{odd}}(odd)$*) is an irreducible* $GL(F)$ *(resp.* G*) representation supported on* $\mathcal{S}_\Sigma(\rho_i)$ *(resp.* $\mathcal{S}_\Sigma(\rho_1, \cdots, \rho_k)$*) appearing with multiplicity* $c_{j_i}(\rho_i)$ *(resp.* $c_{j_{odd}}(odd)$*). Then*

$$r_{GL}(\pi) = \sum_{j_{odd}, j_{k+1}, \cdots, j_n} c_{j_{odd}}(odd) c_{j_{k+1}}(\rho_{k+1}) \cdots c_{j_n}(\rho_n) \kappa_{j_{odd}}(odd) \times \kappa_{j_{k+1}}(\rho_{k+1}) \times \cdots \times \kappa_{j_n}(\rho_n) \otimes c^{a_{j_{odd}}} \cdot \sigma.$$

Proof. As in Case 3 in the proof of Proposition 4.5, we may write

$$\pi \hookrightarrow \underbrace{\nu^\beta \rho_1 \times \cdots \times \nu^\beta \rho_1}_{\ell_1} \times \cdots \times \underbrace{\nu^\beta \rho_n \times \cdots \times \nu^\beta \rho_n}_{\ell_n} \rtimes \sigma.$$

As in (4.2), we may twist by a character to normalize things so that $\beta = 0$ and σ is unitary.

To calculate r_{GL}, we need to distinguish certain properties of the various ρ_i. To this end, when $c \cdot \sigma \ncong \sigma$ (noting that this includes the case $m_0 = d(\sigma) = 0$ for $G_m = SO_{2m}$ or $GSpin_{2m}$), we let

$$\begin{aligned} \{\gamma_1, \ \cdots, \ \gamma_{k_1}\} &= \{\rho_i | \ \rho_i \ncong \omega_\Sigma \check{\rho}_i \text{ and } d(\rho_i) \text{ odd}\}, \\ \{\eta_1, \ \cdots, \ \eta_{k_2}\} &= \{\rho_i | \ \rho_i \cong \omega_\Sigma \check{\rho}_i \text{ and } d(\rho_i) \text{ odd}\}, \end{aligned}$$

and

$$\begin{aligned} \{\zeta_1, \ \cdots, \ \zeta_{k_3}\} &= \{\rho_i | \ \rho_i \rtimes \sigma \text{ irreducible and } d(\rho_i) \text{ even}\}, \\ \{\xi_1, \ \cdots, \ \xi_{k_4}\} &= \{\rho_i | \ \rho_i \rtimes \sigma \text{ reducible and } d(\rho_i) \text{ even}\}, \end{aligned}$$

(noting that $\rho_i \rtimes \sigma$ reducible requires $d(\rho_i)$ even and $\rho_i \cong \omega_\Sigma \check{\rho}_i$). Note that for $c \cdot \sigma \cong \sigma$ and $m_0 > 0$, in the argument below, one eliminates $\gamma_i, \ \eta_i$ and drops the parity requirement on $d(\rho_i)$ for $\zeta_i, \ \xi_i$.

Using the notation above, we write

$$T \hookrightarrow \prod_{i=1}^{k_1} \underbrace{(\gamma_i \times \cdots \times \gamma_i)}_{\ell_{\gamma_i}} \times \prod_{i=1}^{k_2} \underbrace{(\eta_i \times \cdots \times \eta_i)}_{\ell_{\eta_i}} \times \prod_{i=1}^{k_3} \underbrace{(\zeta_i \times \cdots \times \zeta_i)}_{\ell_{\zeta_i}} \times \prod_{i=1}^{k_4} \underbrace{(\xi_i \times \cdots \times \xi_i)}_{\ell_{\xi_i}} \rtimes \sigma.$$

Now, recall from [10], [11], [3] that such an irreducible T may then be written as

$$T \cong \prod_{i=1}^{k_1} \underbrace{(\gamma_i \times \cdots \times \gamma_i)}_{\ell_{\gamma_i}} \times \prod_{i=1}^{k_2} \underbrace{(\eta_i \times \cdots \times \eta_i)}_{\ell_{\eta_i}-1} \times \prod_{i=1}^{k_3} \underbrace{(\zeta_i \times \cdots \times \zeta_i)}_{\ell_{\zeta_i}} \times \prod_{i=1}^{k_4} \underbrace{(\xi_i \times \cdots \times \xi_i)}_{\ell_{\xi_i}-1} \rtimes T_{ell} \tag{5.1}$$

for some (irreducible)

$$T_{ell} \hookrightarrow \eta_1 \times \eta_2 \times \cdots \times \eta_{k_2} \times \xi_1 \times \xi_2 \times \cdots \times \xi_{k_4} \rtimes \sigma.$$

We claim that

$$r_{GL}(T_{ell}) = \begin{cases} \eta_1 \times \cdots \times \eta_{k_2} \times \xi_1 \times \cdots \times \xi_{k_4} \otimes [\sigma + c \cdot \sigma] & \text{if } c \cdot \sigma \ncong \sigma \text{ and } k_2 > 0, \\ \xi_1 \times \xi_2 \times \cdots \times \xi_{k_4} \otimes \sigma & \text{if } c \cdot \sigma \cong \sigma \text{ or } k_2 = 0. \end{cases} \tag{5.2}$$

That $r_{GL}(T_{ell})$ contains at least these terms follows from their irreducibility and, when $k_2 > 0$, Lemma 3.1 (if $c \cdot \sigma \ncong \sigma$). To see there is nothing more, note that for $k_2 > 0$ with $c \cdot \sigma \ncong \sigma$,

$$\begin{aligned} &r_{GL}(\eta_1 \times \cdots \times \eta_{k_2} \times \xi_1 \times \cdots \times \xi_{k_4} \rtimes \sigma) \\ &= 2^{k_2+k_4-1} (\eta_1 \times \cdots \times \eta_{k_2} \times \xi_1 \times \cdots \times \xi_{k_4} \otimes \sigma + \eta_1 \times \cdots \times \eta_{k_2} \times \xi_1 \times \cdots \times \xi_{k_4} \otimes c \cdot \sigma) \end{aligned}$$

Since there are $2^{k_2+k_4-1}$ components in this case ([10], [3]), the claim follows. The result when $k_2 = 0$ or $c \cdot \sigma \cong \sigma$ is similar. It then follows from (5.1), (5.2) and Theorem 2.1 that for the case $k_2 > 0$ with $c \cdot \sigma \ncong \sigma$,

$$\begin{aligned} r_{GL}(T) = &\Bigg(\prod_{i=1}^{k_1} \underbrace{(\gamma_i + \omega_\Sigma \check{\gamma}_i) \times \cdots \times (\gamma_i + \omega_\Sigma \check{\gamma}_i)}_{\ell_{\gamma_i}} \times 2^{L_\eta} \prod_{i=1}^{k_2} \underbrace{(\eta_i \times \cdots \times \eta_i)}_{\ell_{\eta_i}} \\ &\times \prod_{i=1}^{k_3} \underbrace{(\zeta_i + \omega_\Sigma \check{\zeta}_i) \times \cdots \times (\zeta_i + \omega_\Sigma \check{\zeta}_i)}_{\ell_{\zeta_i}} \times 2^{L_\zeta} \prod_{i=1}^{k_4} \underbrace{(\xi_i \times \cdots \times \xi_i)}_{\ell_{\xi_i}} \Bigg) \otimes [\sigma + c \cdot \sigma] \end{aligned}$$

where

$$L_\eta = (\ell_{\eta_1} - 1) + \cdots + (\ell_{\eta_{k_2}} - 1), \qquad L_\xi = (\ell_{\xi_1} - 1) + \cdots + (\ell_{\xi_{k_4}} - 1)$$

(and noting that one has the possibility $\zeta_i \cong \omega_\Sigma \check{\zeta}_i$). If $k_2 = 0$ or $c \cdot \sigma \cong \sigma$, we have

$$\begin{aligned} r_{GL}(T) = &\prod_{i=1}^{k_1} \left[\sum_{j_i=0}^{\ell_{\gamma_i}} \binom{\ell_{\gamma_i}}{j_i} \underbrace{(\omega_\Sigma \check{\gamma}_i \times \cdots \times \omega_\Sigma \check{\gamma}_i)}_{j_i} \times \underbrace{(\gamma_i \times \cdots \times \gamma_i)}_{\ell_{\gamma_i} - j_i} \right] \\ &\times \prod_{i=1}^{k_3} \underbrace{(\zeta_i + \omega_\Sigma \check{\zeta}_i) \times \cdots \times (\zeta_i + \omega_\Sigma \check{\zeta}_i)}_{\ell_{\zeta_i}} \times 2^{L_\zeta} \prod_{i=1}^{k_4} \underbrace{(\xi_i \times \cdots \times \xi_i)}_{\ell_{\xi_i}} \otimes c^{j_1 + \cdots + j_{k_1}} \cdot \sigma. \end{aligned}$$

Specializing the above, we see that

$$r_{GL}\big(\psi_{\xi_i}^{(\sigma)}(T)\big) = 2^{\ell_{\xi_i}-1} \underbrace{(\xi_i \times \cdots \times \xi_i)}_{\ell_{\xi_i}} \otimes \sigma,$$

$$r_{GL}\big(\psi^{(\sigma)}_{\zeta_i}(T)\big) = \underbrace{(\zeta_i + \omega_\Sigma\check{\zeta}_i) \times \cdots \times (\zeta_i + \omega_\Sigma\check{\zeta}_i)}_{\ell_{\zeta_i}} \otimes\sigma,$$

and for the odd case,

$$r_{GL}\big(\psi_{odd}(T)\big) = \left(\prod_{i=1}^{k_1} \underbrace{(\gamma_i + \omega_\Sigma\check{\gamma}_i) \times \cdots \times (\gamma_i + \omega_\Sigma\check{\gamma}_i)}_{\ell_{\gamma_i}} \times 2^{L_\eta} \prod_{i=1}^{k_2} \underbrace{(\eta_i \times \cdots \times \eta_i)}_{\ell_{\eta_i}}\right) \otimes [\sigma + c\cdot\sigma]$$

if $k_2 > 0$ and $c \cdot \sigma \ncong \sigma$, and

$$r_{GL}\big(\psi_{odd}(T)\big) = \prod_{i=1}^{k_1} \left[\sum_{j_i=0}^{\ell_{\gamma_i}} \binom{\ell_{\gamma_i}}{j_i} \underbrace{(\gamma_i \times \cdots \times \gamma_i)}_{j_i} \times \underbrace{(\omega_\Sigma\check{\gamma}_i \times \cdots \times \omega_\Sigma\check{\gamma}_i)}_{\ell_{\gamma_i} - j_i}\right] \otimes c^{j_1 + \cdots + j_{k_1}} \cdot \sigma$$

if $k_2 = 0$ or $c\cdot\sigma \cong \sigma$. Comparing coefficients (and noting that $a_{j_{odd}} \equiv j_1 + \cdots + j_{k_1} \mod 2$) finishes the lemma. □

By Propositions 4.4 and 4.5, the map in (3.7) is a bijection. We let

$$\Psi^{(\sigma)} : Irr(\mathcal{S}(\rho_1, \cdots, \rho_k; \Sigma)) \times Irr(\mathcal{S}(\rho_{k+1}; \sigma)) \times \cdots \times Irr(\mathcal{S}(\rho_n; \sigma)) \longrightarrow Irr(\mathcal{S}(\rho_1, \cdots, \rho_n; \Sigma))$$

denote its inverse.

Proposition 5.3. *Suppose that $\theta(odd; \Sigma)$, $\theta(\rho_{k+1}; \sigma)$, $\cdots$, $\theta(\rho_n; \sigma)$ are irreducible G representations supported on $\mathcal{S}(\rho_1, \cdots, \rho_k; \Sigma)$, $\mathcal{S}(\rho_{k+1}; \sigma)$, $\cdots$, $\mathcal{S}(\rho_n; \sigma)$ respectively. Suppose that*

$$r_{GL}(\theta(\rho_i; \sigma)) = \sum_{j_i} c_{j_i}(\rho_i)\kappa_{j_i}(\rho_i) \otimes \sigma,$$

where $\kappa_{j_i}(\rho_i)$ is an irreducible $GL(F)$-representation supported on $\mathcal{S}_\Sigma(\rho_i)$ appearing with multiplicity $c_{j_i}(\rho_i)$. Similarly, write

$$r_{GL}(\theta(odd; \Sigma)) = \sum_{j_{odd}} c_{j_{odd}}(odd)\kappa_{j_{odd}}(odd) \otimes c^{a_{j_{odd}}} \cdot \sigma.$$

Write $\pi = \Psi^{(\sigma)}(\theta(odd; \Sigma), \theta(\rho_{k+1}; \sigma), \cdots, \theta(\rho_n; \sigma))$. Then, for $X = \mathcal{S}_\Sigma(\rho_{k+1}, \rho_{k+2}, \cdots, \rho_\ell)$, we have

$$\mu^*_X(\pi) = \sum_{j_{k+1}, \cdots, j_\ell} (c_{j_{k+1}}(\rho_{k+1}) \cdots c_{j_\ell}(\rho_\ell))\kappa_{j_{k+1}}(\rho_{k+1}) \times \cdots \times \kappa_{j_\ell}(\rho_\ell) \otimes \Psi^{(\sigma)}(\theta(odd; \Sigma), \theta(\rho_{\ell+1}; \sigma), \cdots, \theta(\rho_n; \sigma)).$$

Similarly, if $X = \mathcal{S}_\Sigma(\rho_1, \rho_2, \cdots, \rho_\ell)$ *with* $\ell \geq k$,

$$\mu_X^*(\pi) = \sum_{j_{odd}, j_{k+1}, \cdots, j_\ell} (c_{j_{odd}}(odd) c_{j_{k+1}}(\rho_{k+1}) \cdots c_{j_\ell}(\rho_\ell)) \kappa_{j_{odd}}(odd) \times \kappa_{j_{k+1}}(\rho_{k+1}) \times \cdots \times \kappa_{j_\ell}(\rho_\ell) \otimes \Psi^{(\sigma)}(\theta(\rho_{\ell+1}; \sigma), \cdots, \theta(\rho_n; \sigma)).$$

Proof. The proof is by induction on the number of GL factors in a standard Levi subgroup supporting a minimal nonzero Jacquet module and follows the same basic lines as does in Proposition 4.5.

We first look at the case r_{GL} (which corresponds to $X = \mathcal{S}_\Sigma(\rho_1, \rho_2, \cdots, \rho_n)$), then move on to the general claim.

Case 1: π not essentially tempered.

By Lemma 4.2, write $\pi = L(\tau_1, \cdots, \tau_s;\ T)$ with

$$\theta(\rho_i; \sigma) = \psi_{\rho_i}^{(\sigma)}(\pi) = L\big(\tau_1(\rho_i), \cdots, \tau_s(\rho_i);\ T(\rho_i; \sigma)\big)$$

and

$$\theta(odd; \Sigma) = \psi_{odd}(\pi) = L\big(\tau_1(odd), \cdots, \tau_s(odd);\ T(odd; \Sigma)\big).$$

We note that at least one of $\theta(odd; \Sigma)$, $\theta(\rho_{k+1}; \sigma)$, $\cdots$, $\theta(\rho_n; \sigma)$ is not essentially tempered; for concreteness, we assume that $\theta(\rho_{k+1}; \sigma)$ is not essentially tempered. We proceed by first calculating $\mu^*_{X_{k+1}}(\pi)$, with X_{k+1} as in (3.5), then use this to verify the $r_{GL}(\pi)$ case.

First, observe that

$$\begin{aligned}
&\mathcal{L}\big(\tau_1(\rho_{k+1}), \cdots, \tau_s(\rho_{k+1})\big) \rtimes L\big(\tau_1(odd) \times \tau_1(\rho_{k+2}) \times \cdots \times \tau_1(\rho_n), \\
&\qquad \cdots, \tau_s(odd) \times \tau_s(\rho_{k+2}) \times \cdots \times \tau_s(\rho_n);\ T\big) \\
&\hookrightarrow (\tau_1(\rho_{k+1}) \times \cdots \times \tau_s(\rho_{k+1})) \times (\tau_1(odd) \times \tau_1(\rho_{k+2}) \times \cdots \times \tau_1(\rho_n)) \times \\
&\qquad \cdots \times (\tau_s(odd) \times \tau_s(\rho_{k+2}) \times \cdots \times \tau_s(\rho_n)) \rtimes T) \\
&\cong (\tau_1(odd) \times \tau_1(\rho_{k+1}) \times \cdots \times \tau_1(\rho_n)) \times \\
&\qquad \cdots \times (\tau_s(odd) \times \tau_s(\rho_{k+1}) \times \cdots \times \tau_s(\rho_n)) \rtimes T).
\end{aligned}$$

As this last induced representation has π as the unique irreducible subrepresentation (Langlands subrepresentation), we get

$$\begin{aligned}
\pi \hookrightarrow &\mathcal{L}\big(\tau_1(\rho_{k+1}), \cdots, \tau_s(\rho_{k+1})\big) \\
&\rtimes L\big(\tau_1(odd) \times \tau_1(\rho_{k+2}) \times \cdots \times \tau_1(\rho_n), \cdots, \tau_s(odd) \times \tau_s(\rho_{k+2}) \times \cdots \times \tau_s(\rho_n);\ T\big).
\end{aligned} \tag{5.3}$$

Further, π appears with multiplicity one (see [5, Proposition 5.3]).

Next, by the inductive hypothesis and an argument like the formal calculation in the proof of Lemma 3.7,

$$\begin{aligned}
&\mu^*_{X_{k+1}}\Big(L\big(\tau_1(odd)\times\tau_1(\rho_{k+2})\times\cdots\times\tau_1(\rho_n),\ \cdots,\tau_s(odd)\times\tau_s(\rho_{k+2})\times\cdots\times\tau_s(\rho_n);\ T\big)\Big)\\
&\quad=\sum_{j_{odd},j_{k+2},\cdots,j_n}\big(c_{j_{odd}}(odd)c_{j_{k+2}}(\rho_{k+2})\cdots c_{j_n}(\rho_n)\big)\kappa_{j_{odd}}(odd)\times\kappa_{j_{k+2}}(\rho_{k+2})\times\cdots\times\kappa_{j_n}(\rho_n)\\
&\qquad\qquad\otimes T(\rho_{k+1};\sigma)\\
&\qquad\qquad\Downarrow\\
&\mu^*_{X_{k+1}}\Big(\mathcal{L}\big(\tau_1(\rho_{k+1}),\ \cdots,\ \tau_s(\rho_{k+1})\big)\rtimes L\big(\tau_1(odd)\times\tau_1(\rho_{k+2})\times\cdots\times\tau_1(\rho_n),\\
&\qquad\cdots,\ \tau_s(odd)\times\tau_s(\rho_{k+2})\times\cdots\times\tau_s(\rho_n);\ T\big)\Big)\\
&\quad=\sum_{j_{odd},j_{k+2},\cdots,j_n}\big(c_{j_{odd}}(odd)c_{j_{k+2}}(\rho_{k+2})\cdots c_{j_n}(\rho_n)\big)\kappa_{j_{odd}}(odd)\times\kappa_{j_{k+2}}(\rho_{k+2})\times\cdots\times\kappa_{j_n}(\rho_n)\\
&\qquad\qquad\otimes\mathcal{L}(\tau_1(\rho_{k+1}),\ \cdots,\ \tau_s(\rho_{k+1}))\rtimes T(\rho_{k+1};\sigma).
\end{aligned}$$

Consequently, from properties of the Langlands classification, we have the multiplicity

$$\begin{aligned}
&mult\Big(\kappa_{j_{odd}}(odd)\times\kappa_{j_{k+2}}(\rho_{k+2})\times\cdots\times\kappa_{j_n}(\rho_n)\otimes L\big(\tau_1(\rho_{k+1}),\cdots,\tau_s(\rho_{k+1});\ T(\rho_{k+1};\sigma)\big),\\
&\quad\mu^*_{X_{k+1}}\big(\mathcal{L}\big(\tau_1(\rho_{k+1}),\ \cdots,\ \tau_s(\rho_{k+1})\big)\rtimes L\big(\tau_1(odd)\times\tau_1(\rho_{k+2})\times\cdots\times\tau_1(\rho_n),\\
&\qquad\cdots,\ \tau_s(odd)\times\tau_s(\rho_{k+2})\times\cdots\times\tau_s(\rho_n);\ T)\big)\Big)\\
&\qquad=c_{j_{odd}}(odd)c_{j_{k+2}}(\rho_{k+2})\cdots c_{j_n}(\rho_n).
\end{aligned}$$

Any irreducible

$$\begin{aligned}
\pi'\le\mathcal{L}\big(\tau_1(\rho_{k+1}),\ \cdots,\ \tau_s(\rho_{k+1})\big)\rtimes L\big(&\tau_1(odd)\times\tau_1(\rho_{k+2})\times\cdots\times\tau_1(\rho_n),\\
&\cdots,\ \tau_s(odd)\times\tau_s(\rho_{k+2})\times\cdots\times\tau_s(\rho_n);\ T\big)
\end{aligned}$$

has

$$\begin{aligned}
\psi^{(\sigma)}_{\rho_i}(\pi')=\psi^{(\sigma)}_{\rho_i}\Big(L\big(&\tau_1(odd)\times\tau_1(\rho_{k+2})\times\cdots\times\tau_1(\rho_n),\ \cdots,\ \tau_s(odd)\times\tau_s(\rho_{k+2})\times\\
&\cdots\times\tau_s(\rho_n);\ T\big)\Big)=\theta(\rho_i;\sigma)
\end{aligned}$$

for $i=k+2,\ \cdots,\ n$, and similarly for ψ_{odd}. Thus

$$\begin{aligned}
mult\Big(&\kappa_{j_{odd}}(odd)\times\kappa_{j_{k+2}}(\rho_{k+2})\times\cdots\times\kappa_{j_n}(\rho_n)\otimes L\big(\tau_1(\rho_{k+1}),\ \cdots,\ \tau_s(\rho_{k+1});\\
&T(\rho_{k+1};\sigma)\big),\mu^*_{X_{k+1}}(\pi')\Big)=0
\end{aligned}$$

for any irreducible

$$\begin{aligned}
\pi'\le\mathcal{L}\big(\tau_1(\rho_{k+1}),\ \cdots,\ \tau_s(\rho_{k+1})\big)\rtimes L\big(&\tau_1(odd)\times\tau_1(\rho_{k+2})\times\cdots\times\tau_1(\rho_n),\\
&\cdots,\ \tau_s(odd)\times\tau_s(\rho_{k+2})\times\cdots\times\tau_s(\rho_n);\ T\big)
\end{aligned}$$

with $\pi' \neq \pi$ (if not, $\psi_{odd}(\pi') = \psi_{odd}(\pi)$ and $\psi^{(\sigma)}_{\rho_i}(\pi') = \psi^{(\sigma)}_{\rho_i}(\pi)$ for all i, contradicting Proposition 4.5). Consequently, as π appears with multiplicity one in

$$\mathcal{L}\big(\tau_1(\rho_{k+1}), \ \cdots, \ \tau_s(\rho_{k+1})\big) \rtimes L\big(\tau_1(odd) \times \tau_1(\rho_{k+2}) \times \cdots \times \tau_1(\rho_n), \\ \cdots, \ \tau_s(odd) \times \tau_s(\rho_{k+2}) \times \cdots \times \tau_s(\rho_n); \ T\big),$$

we have

$$mult\Big(\kappa_{j_{odd}}(odd) \times \kappa_{j_{k+2}}(\rho_{k+2}) \times \cdots \times \kappa_{j_n}(\rho_n) \otimes L\big(\tau_1(\rho_{k+1}), \ \cdots, \ \tau_s(\rho_{k+1}); \\ T(\rho_{k+1};\sigma)\big), \mu^*_{X_{k+1}}(\pi)\Big) = c_{j_{odd}}(odd)c_{j_{k+2}}(\rho_{k+2}) \cdots c_{j_n}(\rho_n).$$

Further, by $\psi^{(\sigma)}_{\rho_{k+1}}(\pi)$ considerations,

$$mult\Big(\kappa_{j_{odd}}(odd) \times \kappa_{j_{k+2}}(\rho_{k+2}) \times \cdots \times \kappa_{j_n}(\rho_n) \otimes \theta'(\rho_{k+1};\sigma), \ \mu^*_{X_{k+1}}(\pi)\Big) = 0$$

if $\theta'(\rho_{k+1};\sigma)$ is irreducible and $\theta'(\rho_{k+1};\sigma) \neq L\big(\tau_1(\rho_{k+1}), \ \cdots, \ \tau_s(\rho_{k+1}); \ T(\rho_{k+1};\sigma)\big)$. Thus

$$\mu^*_{X_{k+1}}(\pi) = \sum_{j_{odd}, j_{k+2}, \cdots, j_n} \big(c_{j_{odd}}(odd)c_{j_{k+2}}(\rho_{k+2}) \cdots c_{j_n}(\rho_n)\big)\kappa_{j_{odd}}(odd) \times \kappa_{j_{k+2}}(\rho_{k+2}) \times \\ \cdots \times \kappa_{j_n}(\rho_n) \otimes L\big(\tau_1(\rho_{k+1}), \ \cdots, \ \tau_s(\rho_{k+1}); \ T(\rho_{k+1};\sigma)\big),$$

as needed.

From here, it is easy to finish Case 1. The preceding paragraph tells us

$$\mu^*_{X_{k+1}}(\pi) = \sum_{j_{odd}, j_{k+2}, \cdots, j_n} \big(c_{j_{odd}}(odd)c_{j_{k+2}}(\rho_{k+2}) \cdots c_{j_n}(\rho_n)\big)\kappa_{j_{odd}}(odd) \times \kappa_{j_{k+2}}(\rho_{k+2}) \times \\ \cdots \times \kappa_{j_n}(\rho_n) \otimes \theta(\rho_{k+1};\sigma)$$

$$\Downarrow$$

$$\kappa_{j_{odd}}(odd) \times \kappa_{j_{k+2}}(\rho_{k+2}) \times \cdots \times \kappa_{j_n}(\rho_n) \otimes \kappa_{j_{k+1}}(\rho_{k+1}) \\ \otimes c^{a_{j_{odd}}} \cdot \sigma \text{ appears with multiplicity} \\ c_{j_{odd}}(odd)c_{j_{k+2}}(\rho_{k+2}) \cdots c_{j_n}(\rho_n)c_{j_{k+1}}(\rho_{k+1}) \text{ in } r_{M_\alpha,G}(\pi).$$

$$\Downarrow$$

$$r_{GL}(\pi) = \sum_{j_{odd}, j_{k+1}, \cdots, j_n} \big(c_{j_{odd}}(odd)c_{j_{k+1}}(\rho_{k+1}) \cdots c_{j_n}(\rho_n)\big)\kappa_{j_{odd}}(odd) \times \kappa_{j_{k+1}}(\rho_{k+1}) \times \\ \cdots \times \kappa_{j_n}(\rho_n) \otimes c^{a_{j_{odd}}} \cdot \sigma,$$

where M_α is the standard Levi subgroup for $\tau_{j_{odd}}(odd) \times \tau_{j_{k+2}}(\rho_{k+2}) \times \cdots \times \tau_{j_n}(\rho_n) \otimes \tau_{j_{k+1}}(\rho_{k+1}) \otimes c^a \cdot \sigma$. This finishes Case 1.

<u>Case 2</u>: π essentially tempered, $\hat{\pi}$ not essentially tempered.

First, by [2, Théorème 1.7],

$$\begin{aligned} r_{GL}(\theta(\widehat{odd;\Sigma})) &= Ad(w_{\Phi_{odd}}) \circ \widehat{\ } \circ r_{GL}(\theta(odd;\Sigma)) \\ &= Ad(w_{\Phi_{odd}}) \circ \widehat{\ } \left(\sum_{j_{odd}} c_{j_{odd}}(odd)\kappa_{j_{odd}}(odd) \otimes c^{a_{j_{odd}}} \cdot \sigma\right) \\ &= Ad(w_{\Phi_{odd}}) \left(\sum_{j_{odd}} c_{j_{odd}}(odd)\widehat{\kappa_{j_{odd}}(odd)} \otimes c^{a_{j_{odd}}} \cdot \sigma\right) \\ &= \sum_{j_{odd}} c_{j_{odd}}(odd)\omega_{\Sigma}\left(\widehat{\kappa_{j_{odd}}(odd)}\right)^{\vee} \otimes c^{d+a_{j_{odd}}} \cdot \sigma, \end{aligned}$$

where $d = d(\theta(odd;\Sigma)) - d(\sigma)$ (number of inversions under $w_{\Phi_{odd}}$). Similarly, for $k+1 \leq i \leq n$,

$$r_{GL}(\theta(\widehat{\rho_i;\sigma})) = \sum_{j_i} c_{j_i}(\rho_i)\omega_{\Sigma}\left(\widehat{\kappa_{j_i}(\rho_i)}\right)^{\vee} \otimes \sigma.$$

Note that by Lemma 4.3 and Proposition 4.5, $\hat{\pi} = \Psi^{(\sigma)}\left(\theta(\widehat{odd;\Sigma}), \theta(\widehat{\rho_{k+1};\sigma}), \cdots, \theta(\widehat{\rho_n;\sigma})\right)$. So, by Case 1,

$$\begin{aligned} r_{GL}(\hat{\pi}) = \sum_{j_{odd},j_{k+1},\cdots,j_n} &\left(c_{j_{odd}}(odd)c_{j_{k+1}}(\rho_{k+1})\cdots c_{j_n}(\rho_n)\right)\omega_{\Sigma}\widehat{\kappa_{j_{odd}}(odd)}^{\vee} \times \omega_{\Sigma}\widehat{\kappa_{j_{k+1}}(\rho_{k+1})}^{\vee} \times \\ &\cdots \times \omega_{\Sigma}\widehat{\kappa_{j_n}(\rho_n)}^{\vee} \otimes c^{d+a_{j_{odd}}} \cdot \sigma. \end{aligned}$$

Since $\hat{}$ and $\check{}$ commute and both respect induction, we see that

$$\begin{aligned} r_{GL}(\hat{\pi}) = \sum_{j_{odd},j_{k+1},\cdots,j_n} &c_{j_{odd}}(odd)c_{j_{k+1}}(\rho_{k+1})\cdots c_{j_n}(\rho_n)\omega_{\Sigma}\Big(\check{\kappa}_{j_{odd}}(odd) \times \check{\kappa}_{j_{k+1}}(\rho_{k+1}) \times \\ &\cdots \times \check{\kappa}_{j_n}(\rho_n)\Big)^{\widehat{\ }} \otimes c^{d+a_{j_{odd}}} \cdot \sigma. \end{aligned}$$

Next, again by [2, Théorème 1.7],

$$\begin{aligned} r_{GL}(\pi) &= Ad(w_{\Phi}) \circ \widehat{\ } \circ r_{GL}(\hat{\pi}) \\ &= Ad(w_{\Phi}) \circ \widehat{\ } \Big(\sum_{j_{odd},j_{k+1},\cdots,j_n} c_{odd}(odd)c_{j_{k+1}}(\rho_{k+1})\cdots c_{j_n}(\rho_n)\omega_{\Sigma}\big(\check{\kappa}_{j_{odd}}(odd) \\ &\qquad \times \check{\kappa}_{j_{k+1}}(\rho_{k+1}) \times \cdots \times \check{\kappa}_{j_n}(\rho_n)\big)^{\widehat{\ }} \otimes c^{d+a_{j_{odd}}} \cdot \sigma\Big). \end{aligned}$$

Noting that the Aubert duality commutes with twisting by a character of the group–an immediate consequence of the definition and [6, Proposition 19.9 (f)]–we have

$$
\begin{aligned}
r_{GL}(\pi) &= Ad(w_\Phi)\Big(\sum_{j_{odd},j_{k+1},\cdots,j_n} c_{j_{odd}}(odd)c_{j_{k+1}}(\rho_{k+1})\cdots c_{j_n}(\rho_n)\omega_\Sigma(\check{\kappa}_{odd}(odd) \\
&\qquad\qquad \times\check{\kappa}_{j_{k+1}}(\rho_{k+1})\times\cdots\times\check{\kappa}_{j_n}(\rho_n))\otimes c^{d+a_{j_{odd}}}\cdot\sigma\Big) \\
&= \sum_{j_{odd},j_{k+1},\cdots,j_n} \big(c_{j_{odd}}(odd)c_{j_{k+1}}(\rho_{k+1})\cdots c_{j_n}(\rho_n)\big)\kappa_{j_{odd}}(odd)\times\kappa_{j_{k+1}}(\rho_{k+1})\times \\
&\qquad\qquad \cdots\times\kappa_{j_n}(\rho_n)\otimes c^{d'+d+a_{j_{odd}}}\cdot\sigma,
\end{aligned}
$$

where $d' = d(\pi) - d(\sigma)$ (and noting that for general spin groups, $\tilde{\omega}_\Sigma\omega_\Sigma = 1$). As $d' \equiv d$ mod 2, this finishes Case 2.

Case 3: π, $\hat{\pi}$ both essentially tempered.

In this case, the result follows from Lemma 5.2.

We now turn to $X = \mathcal{S}_\Sigma(odd, \rho_{k+1}, \rho_{k+2}, \cdots, \rho_\ell)$; the case $X = \mathcal{S}_\Sigma(\rho_{k+1}, \rho_{k+2}, \cdots, \rho_\ell)$ is similar. An immediate consequence of the $r_{GL}(\pi)$ result is that

$$
\begin{aligned}
mult\Big(\kappa_{j_{odd}}(odd)\otimes\kappa_{j_{k+1}}(\rho_{k+1})\otimes\cdots\otimes\kappa_{j_n}(\rho_n)\otimes c^{a_{j_{odd}}}\cdot\sigma,\ r_{M_\alpha,G}(\pi)\Big) \\
= c_{j_{odd}}(odd)c_{j_{k+1}}(\rho_{k+1})\cdots c_{j_n}(\rho_n),
\end{aligned}
$$

where M_α is the standard Levi subgroup for $\kappa_{j_{odd}}(odd)\otimes\kappa_{j_{k+1}}(\rho_{k+1})\otimes\cdots\otimes\kappa_{j_n}(\rho_n)\otimes c^{a_{j_{odd}}}\cdot\sigma$ (see [15, Corollary 5.6]). Observe that

$$
\begin{aligned}
& mult\Big(\kappa_{j_{odd}}(odd)\times\kappa_{j_{k+1}}(\rho_{k+1})\times\cdots\times\kappa_{j_n}(\rho_n)\otimes c^{a_{j_{odd}}}\cdot\sigma, r_{GL}(\pi)\Big) \\
&= \sum_{\Psi'} mult\Big(\kappa_{j_{odd}}(odd)\otimes\kappa_{j_{k+1}}(\rho_{k+1})\otimes\cdots\otimes\kappa_{j_\ell}(\rho_\ell)\otimes\Psi',\ r_{M_\beta,G}(\pi)\Big) \\
&\qquad \times mult\Big(\kappa_{j_{odd}}(odd)\otimes\kappa_{j_{k+1}}(\rho_{k+1})\otimes\cdots\otimes\kappa_{j_n}(\rho_n)\otimes c^{a_{j_{odd}}}\cdot\sigma, \\
&\qquad\qquad r_{M_\alpha,M_\beta}(\kappa_{j_{odd}}(odd)\otimes\kappa_{j_{k+1}}(\rho_{k+1})\otimes\cdots\otimes\kappa_{j_\ell}(\rho_\ell)\otimes\Psi')\Big),
\end{aligned}
$$

where M_β is the standard Levi subgroup for

$$
\kappa_{j_{odd}}\otimes\kappa_{j_{k+1}}(\rho_{k+1})\otimes\cdots\otimes\kappa_{j_\ell}(\rho_\ell)\otimes\Psi^{(\sigma)}\big(\theta(\rho_{\ell+1};\sigma),\ \cdots,\ \theta(\rho_n;\sigma)\big)
$$

and the sum is over all irreducible Ψ' with

$$
\kappa_{j_{odd}}(odd)\otimes\kappa_{j_{k+1}}(\rho_{k+1})\otimes\cdots\otimes\kappa_{j_\ell}(\rho_\ell)\otimes\Psi' \leq r_{M_\beta,G}(\pi).
$$

Now, by $\psi^{(\sigma)}_{\rho_{\ell+1}}(\pi),\ \cdots,\ \psi^{(\sigma)}_{\rho_n}(\pi)$ considerations, we see that

$$
mult\Big(\kappa_{j_{odd}}(odd)\otimes\kappa_{j_{k+1}}(\rho_{k+1})\otimes\cdots\otimes\kappa_{j_\ell}(\rho_\ell)\otimes\Psi',\ r_{M_\beta,G}(\pi)\Big) = 0
$$

for any irreducible $\Psi' \neq \Psi^{(\sigma)}\big(\theta(\rho_{\ell+1};\sigma),\ \cdots,\ \theta(\rho_n;\sigma)\big)$. Therefore,

$$\begin{aligned}
&mult\Big(\kappa_{j_{odd}}(odd) \otimes \kappa_{j_{k+1}}(\rho_{k+1}) \otimes \cdots \otimes \kappa_{j_n}(\rho_n) \otimes c^{a_{j_{odd}}} \cdot \sigma,\ r_{M_\alpha,G}(\pi)\Big) \\
&= mult\Big(\kappa_{j_{odd}}(odd) \otimes \kappa_{j_{k+1}}(\rho_{k+1}) \otimes \cdots \otimes \kappa_{j_\ell}(\rho_\ell) \otimes \Psi^{(\sigma)}\big(\theta(\rho_{\ell+1};\sigma), \\
&\qquad\qquad \cdots,\ \theta(\rho_n;\sigma)\big),\ r_{M_\beta,G}(\pi)\Big) \\
&\cdot mult\Big(\kappa_{j_{odd}}(odd) \otimes \kappa_{j_{k+1}}(\rho_{k+1}) \otimes \cdots \otimes \kappa_{j_n}(\rho_n) \otimes c^{a_{j_{odd}}} \cdot \sigma,\ r_{M_\alpha,M_\beta}\big(\kappa_{j_{odd}}(odd) \\
&\qquad \otimes \kappa_{j_{k+1}}(\rho_{k+1}) \otimes \cdots \otimes \kappa_{j_\ell}(\rho_\ell) \otimes \Psi^{(\sigma)}\big(\theta(\rho_{\ell+1};\sigma),\ \cdots,\ \theta(\rho_n;\sigma)\big)\big)\Big).
\end{aligned}$$

The r_{GL}-result applied to $\Psi^{(\sigma)}(\theta(\rho_{\ell+1};\sigma),\cdots,\theta(\rho_n;\sigma))$ tells us that

$$\begin{aligned}
&mult\Big(\kappa_{j_{odd}}(odd) \otimes \kappa_{j_{k+1}}(\rho_{k+1}) \otimes \cdots \otimes \kappa_{j_n}(\rho_n) \otimes c^{a_{j_{odd}}} \cdot \sigma,\ r_{M_\alpha,M_\beta}\big(\kappa_{j_{odd}}(odd) \\
&\qquad \otimes \kappa_{j_{k+1}}(\rho_{k+1}) \otimes \cdots \otimes \kappa_{j_\ell}(\rho_\ell) \otimes \Psi^{(\sigma)}\big(\theta(\rho_{\ell+1};\sigma),\ \cdots,\ \theta(\rho_n;\sigma)\big)\big)\Big) \\
&\qquad = c_{j_{\ell+1}}(\rho_{\ell+1}) \cdots c_{j_n}(\rho_n).
\end{aligned}$$

Thus

$$\begin{aligned}
&mult\Big(\kappa_{j_{odd}}(odd) \otimes \kappa_{j_{k+1}}(\rho_{k+1}) \otimes \cdots \otimes \kappa_{j_\ell}(\rho_\ell) \otimes \Psi^{(\sigma)}\big(\theta(\rho_{\ell+1};\sigma), \\
&\qquad \cdots,\ \theta(\rho_n;\sigma)\big),\ r_{M_\beta,G}(\pi)\Big) \\
&\qquad = c_{j_{odd}}(odd) c_{j_{k+1}}(\rho_{k+1}) \cdots c_{j_\ell}(\rho_\ell).
\end{aligned}$$

As a consequence, the multiplicity of

$$\kappa_{j_{odd}}(odd) \times \kappa_{j_{k+1}}(\rho_{k+1}) \times \cdots \times \kappa_{j_\ell}(\rho_\ell) \otimes \Psi^{(\sigma)}\big(\theta(\rho_{\ell+1};\sigma),\ \cdots,\ \theta(\rho_n;\sigma)\big)$$

in $\mu^*_X(\pi)$ is $c_{j_{odd}}(odd) \cdot c_{j_{k+1}}(\rho_{k+1}) \cdots c_{j_\ell}(\rho_\ell)$. This finishes the proof. □

Corollary 5.4. *Let $\tau(odd)$, $\tau(\rho_{k+1}),\ \cdots,\ \tau(\rho_n)$ be irreducible representations of general linear groups with supports contained in $\mathcal{S}_\Sigma(\rho_1,\cdots,\rho_k)$, $\mathcal{S}_\Sigma(\rho_{k+1}),\ \cdots,\ \mathcal{S}_\Sigma(\rho_n)$ respectively, and $\theta(odd;\Sigma)$, $\theta(\rho_{k+1};\sigma),\ \cdots,\ \theta(\rho_n;\sigma)$ be irreducible $G(F)$ representations with supports contained in $\mathcal{S}(\rho_1,\cdots,\rho_k;\Sigma)$, $\mathcal{S}(\rho_{k+1};\sigma),\ \cdots,\ \mathcal{S}(\rho_n;\sigma)$ respectively. (We allow the possibility that $\tau(odd)$, $\tau(\rho_i)=1$ or $\theta(odd;\Sigma)$, $\theta(\rho_i;\sigma)=\sigma$.) Suppose that*

$$\tau(\rho_i) \rtimes \theta(\rho_i;\sigma) = \sum_{\ell_i} m_{\ell_i}(\rho_i)\theta_{\ell_i}(\rho_i;\sigma)$$

and

$$\tau(odd) \rtimes \theta(odd;\Sigma) = \sum_{\ell_{odd}} m_{\ell_{odd}}(odd)\theta_{\ell_{odd}}(odd;\Sigma),$$

with $\theta_{\ell_i}(\rho_i;\sigma)$, $\theta_{\ell_{odd}}(odd;\Sigma)$ irreducible and $m_{\ell_i}(\rho_i)$, $m_{\ell_{odd}}(odd)$ their respective multiplicities. Then

$$\begin{aligned}&\big(\tau(odd)\times\tau(\rho_{k+1})\times\cdots\times\tau(\rho_n)\big)\rtimes\Psi^{(\sigma)}\big(\theta(odd;\Sigma),\theta(\rho_{k+1};\sigma),\cdots,\theta(\rho_n;\sigma)\big)\\&=\sum_{\ell_{odd},\ell_{k+1},\cdots,\ell_n}\big(m_{\ell_{odd}}(odd)m_{\ell_{k+1}}(\rho_{k+1})\cdots m_{\ell_n}(\rho_n)\big)\Psi^{(\sigma)}\big(\theta_{\ell_{odd}}(odd;\Sigma),\theta_{\ell_{k+1}}(\rho_{k+1};\sigma),\\&\qquad\cdots,\theta_{\ell_n}(\rho_n;\sigma)\big).\end{aligned}$$

Proof. Write Ψ for $\Psi^{(\sigma)}\big(\theta(odd;\Sigma),\theta(\rho_{k+1};\sigma),\cdots,\theta(\rho_n;\sigma)\big)$. First, we consider $\tau(\rho_n)\rtimes\Psi$. By Theorem 2.1, Proposition 5.3 and with notation as in Definition 3.8, we have

$$\begin{aligned}&\mu^*_{X_n}\big(\tau(\rho_n)\rtimes\Psi\big)\\&=\sum_{j_{odd},j_{k+1},\cdots,j_{n-1}}\big(c_{j_{odd}}(odd)c_{j_{k+1}}(\rho_{k+1})\cdots c_{j_{n-1}}(\rho_{n-1})\big)\kappa_{j_{odd}}(odd)\times\kappa_{j_{k+1}}(\rho_{k+1})\times\\&\qquad\cdots\times\kappa_{j_{n-1}}(\rho_{n-1})\otimes\tau(\rho_n)\rtimes\theta(\rho_n;\sigma)\\&=\sum_{\ell_n,j_{odd},j_{k+1},\cdots,j_{n-1}}\big(m_{\ell_n}(\rho_n)c_{j_{odd}}(odd)c_{j_{k+1}}(\rho_{k+1})\cdots c_{j_{n-1}}(\rho_{n-1})\big)\kappa_{j_{odd}}(odd)\times\kappa_{j_{k+1}}(\rho_{k+1})\times\\&\qquad\cdots\times\kappa_{j_{n-1}}(\rho_{n-1})\otimes\theta_{\ell_n}(\rho_n;\Sigma).\end{aligned}$$

As in the proof of Proposition 4.4, for any irreducible $\pi\le\tau(\rho_n)\rtimes\Psi$, we have $\psi_{odd}(\pi)=\theta(odd;\Sigma)$ and $\psi^{(\sigma)}_{\rho_i}(\pi)=\theta(\rho_i;\sigma)$ for $i=k+1,\ \cdots,\ n-1$. Thus

$$\tau(\rho_n)\rtimes\Psi=\sum_{\ell_n}m_{\ell_n}\Psi^{(\sigma)}\big(\theta(odd;\Sigma),\theta(\rho_{k+1};\sigma),\cdots,\theta(\rho_{n-1};\sigma),\theta_{\ell_n}(\rho_n;\sigma)\big)$$

for multiplicities m_{ℓ_n}. We claim that $m_{\ell_n}=m_{\ell_n}(\rho_n)$. This is straightforward. Write

$$\Psi_\ell=\Psi^{(\sigma)}\big(\theta(odd),\theta(\rho_{k+1};\sigma),\cdots,\theta(\rho_{n-1};\sigma),\theta_\ell(\rho_n;\sigma)\big).$$

By Proposition 5.3,

$$\begin{aligned}\mu^*_{X_n}(\Psi_\ell)=&\sum_{j_{odd},j_{k+1},\cdots,j_{n-1}}\big(c_{j_{odd}}(odd)c_{j_1}(\rho_{k-1})\cdots c_{j_{n-1}}(\rho_{n-1})\big)\kappa_{j_{odd}}(odd)\times\kappa_{j_{k+1}}(\rho_{k+1})\times\\&\qquad\cdots\times\kappa_{j_{n-1}}(\rho_{n-1})\otimes\theta_\ell(\rho_n;\sigma).\end{aligned}$$

Now, if M_α is the standard Levi subgroup for $\kappa_{j_{odd}}(odd)\times\kappa_{j_{k+1}}(\rho_{k+1})\times\cdots\times\kappa_{j_{n-1}}(\rho_{n-1})\otimes\theta_\ell(\rho_n;\sigma)$,

$$mult\Big(\kappa_{j_{odd}}(odd)\times\kappa_{j_{k+1}}(\rho_{k+1})\times\cdots\times\kappa_{j_{n-1}}(\rho_{n-1})\otimes\theta_\ell(\rho_n;\sigma),\ r_{M_\alpha,G}(\Psi')\Big)=0$$

for any irreducible $\Psi'\le\tau(\rho_n)\rtimes\Psi$ with $\Psi'\ne\Psi_\ell$ (or else $\psi_{odd}(\Psi')=\psi_{odd}(\Psi_\ell)$ and $\psi^{(\sigma)}_{\rho_j}(\Psi')=\psi^{(\sigma)}_{\rho_j}(\Psi_\ell)$ for $j=k+1,\ \cdots,\ n$, contradicting Proposition 4.5). Thus we get $m_{\ell_n}=m_{\ell_n}(\rho_n)$, as needed.

To get the general result, we iterate. The same argument tells us

$$\tau(\rho_{n-1}) \rtimes \Psi_{\ell_n} = \sum_{\ell_{n-1}} m_{\ell_{n-1}}(\rho_{n-1})\Psi^{(\sigma)}\big(\theta(odd;\Sigma), \theta(\rho_{k+1};\sigma), \\ \cdots, \theta(\rho_{n-2};\Sigma), \theta_{\ell_{n-1}}(\rho_{n-1};\sigma), \theta_{\ell_n}(\rho_n;\sigma)\big).$$

Summing over ℓ_n gives

$$(\tau(\rho_{n-1}) \times \tau(\rho_n)) \rtimes \Psi = \sum_{\ell_{n-1},\ell_n} m_{\ell_{n-1}}(\rho_{n-1}) m_{\ell_n}(\rho_n)\Psi^{(\sigma)}\big(\theta(odd;\Sigma), \theta(\rho_{k+1};\sigma), \\ \cdots, \theta(\rho_{n-2};\sigma), \theta_{\ell_{n-1}}(\rho_{n-1};\sigma), \theta_{\ell_n}(\rho_n;\sigma)\big).$$

Repeating the argument for $n-2,\ n-3,\ \cdots,\ k+1$ and the odd contribution gives the corollary. □

6. Additional remarks

Although a number of properties and results are needed to prove the correspondence of this paper (and in [15], [16]) there are two which seem to be most important. The first is that standard Levi factors have the form $M = GL_{m_1}(F) \times \cdots \times GL_{m_k}(F) \times G_{m_0}(F)$, needed to produce something like the right-hand side of (3.7). Other groups which are known to have these Levi factors include nonsplit special orthogonal groups, corresponding similitude groups ($GSp_{2n}(F)$, $GSO_{2n}(F)$, $GU_N(F)$, etc.), and the metaplectic groups, at least for odd residual characteristic ([23]). For simplicity, we refer to those groups as "good" reductive groups.

For the other property needed, note the main result of this paper can be interpreted as saying that the different $\mathcal{S}(\rho_i;\sigma)$ do not "interact" with each other in parabolic induction. For simplicity, we refer to such a nice structure as Jantzen "product formula". In order to be true in general, this needs to be true in the tempered case. This requires R-group results like those in [10], [11], [3], where one can decompose the R-group based on the different ρ_i's which occur (possibly bundling, as was done here for $SO_{2m}(F)$ and $GSpin_{2m}(F)$). Those "good" reductive groups where the R-group is known and has a suitable form include the similitude groups $GSp_{2n}(F)$, $GSO_{2n}(F)$, $GU_N(F)$ ([12]) and metaplectic groups (cf. [13], [19]). For them, one also expects a "non-interaction" between different families, allowing the possibility of a decomposition along the lines of (3.7), but complicated by different actions of similitude factors under Weyl conjugation, etc.

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