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Duality and the normalization of standard intertwining operators

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Abstract. Normalized standard intertwining operators associated to an induced representation and its dual (dual in the sense of Aubert) arise in work on a conjecture of Arthur about R-groups. The purpose of this paper is to address the question of relating the normalizing factors used.

1. Introduction

This paper is on certain questions related to R-groups for p-adic groups.

First, let us recall the classical situation. Suppose G is a split connected reductive p-adic group and $P = MU$ a parabolic subgroup. For σ an irreducible square-integrable representation of M, consider $\mathrm{Ind}_P^G(\sigma)$ (normalized induction). Associated to $\mathrm{Ind}_P^G(\sigma)$ is the R-group, a subgroup of the Weyl group which governs the reducibility of this induced representation. More precisely, the normalized standard intertwining operators $\{A(\sigma, r) \mid r \in R\}$ constitute a basis for $\mathrm{Hom}_G(\mathrm{Ind}_P^G(\sigma), \mathrm{Ind}_P^G(\sigma))$. Further, the action of these standard intertwining operators on the irreducible subspaces of $\mathrm{Ind}_P^G(\sigma)$ is by representations of R, which can be rephrased in the form

$$\mathrm{trace} A(\sigma, r)\mathrm{Ind}_P^G(\sigma) = \sum_{\rho \in \hat{R}} \mathrm{trace}\rho(r)\mathrm{trace}\pi(\rho),$$

acting on $C_c^\infty(G)$. (Here, $\pi(\rho)$ is the component of $\mathrm{Ind}_P^G(\sigma)$ associated to ρ under the identification of components of $\mathrm{Ind}_P^G(\sigma)$ with representations of R.)

Classically, the construction of the R-group relies on Plancherel measures, whence the assumption σ square-integrable. In [A1], Arthur conjectured the existence of R-groups in certain situations where the inducing representation is not square-integrable. The papers [J],[B2],[B3] deal with aspects of this conjecture, and serve as the starting point for this paper.

The basic idea behind [B2],[B3] is to use the duality operator of [Au],[ScSt] to construct one of the conjectured R-groups from a classical one ([J] relies on the Iwahori-Matsumoto involution [IM] and is more specialized). More precisely,

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the R-group for $\mathrm{Ind}_P^G(\sigma)$ is used to construct an R-group for $\mathrm{Ind}_P^G(\hat{\sigma})$ under suitable hypotheses, where $\hat{\sigma}$ denotes the dual to σ (in the sense of [Au],[ScSt]). This R-group then has the desired properties.

The intertwining operators involved are normalized standard intertwining operators. Neither [J] nor [B2],[B3] addresses the question of normalizing factors directly. Instead, they use the fact that normalizations may be chosen so that the normalized intertwining operators have certain useful properties, which are enough to obtain the results wanted. In particular, no connection is drawn between the normalizing factors for the induced representation and its dual. Such a connection is the aim of this paper.

We now give a brief description of the contents, section by section. The next section reviews some notation and background material. The third section introduces a generalization of Plancherel measure to nontempered representations, based on the behavior of standard intertwining operators. In the fourth section, we show that the duality operator behaves well with respect to restriction to the derived subgroup, using that fact to relate (generalized) Plancherel measures for a group to those for its derived subgroup. This is used in the fifth section, where we discuss normalization of standard intertwining operators. This section has the main result of the paper, which essentially says that one can use the same normalizing factors whether inducing from a representation or its dual (cf. Theorem 5.3).

2. Notation and preliminaries

In this section, we introduce notation and review some results which will be needed in the rest of this paper.

Let F be a p-adic field and G the group of F-points of a connected reductive group which splits over F. We fix a maximal split torus $A_\emptyset$ of G and a minimal parabolic subgroup $P_\emptyset$ which has $A_\emptyset$ as its split component. We let $W = W(G/A_\emptyset)$ denote the Weyl group of G with respect to $A_\emptyset$.

Let Σ denote the set of roots of G with respect to $A_\emptyset$. The choice of $P_\emptyset$ determines the set of positive roots (resp., negative roots, simple roots), which we denote by Σ^+ (resp., Σ^-, Δ).

Let $P = MU$ be the standard parabolic subgroup corresponding to $\Theta \subset \Delta$ and A the split component of M. Write $X(M)_F$ for the F-rational characters of M. Let

$$\mathfrak{a} = \mathfrak{a}_\Theta = \mathrm{Hom}(X(A)_F, \mathbb{R}) = \mathrm{Hom}(X(M)_F, \mathbb{R})$$

be the real Lie algebra of A and

$$\mathfrak{a}^* = \mathfrak{a}^*_\Theta = X(A)_F \otimes_{\mathbb{Z}} \mathbb{R} = X(M)_F \otimes_{\mathbb{Z}} \mathbb{R}$$

its dual ([H-C], §7; [S2], §0.5). Each element $\chi \in X(A)$ corresponds to a unique element of $\mathfrak{a}^*$, called the associated weight. Set $\mathfrak{a}^*_{\mathbb{C}} = \mathfrak{a}^* \otimes \mathbb{C}$. There is a homomorphism ([H-C], §7) $H_\Theta = H_M : M \to \mathfrak{a}$ such that

$$q^{\langle \chi, H_M(m) \rangle} = |\chi(m)|$$

for all $m \in M,\ \chi \in X(M)_F$. Given $\nu \in \mathfrak{a}^*$, let us write

$$\exp\ \nu = q^{\langle \nu, H_M(\cdot)\rangle}$$

for the corresponding character.

Fix a W-invariant inner product on $\mathfrak{a}_\emptyset$ and use this inner product to identify $\mathfrak{a}_\emptyset^*$ and $\mathfrak{a}_\emptyset$ ([S2], §0.5). By restriction, we obtain the inner product on $\mathfrak{a}$ and we identify $\mathfrak{a}^*$ and $\mathfrak{a}$. In this manner, the natural projection $\mathfrak{a}_\emptyset^* \to \mathfrak{a}^*$ gives rise to the inclusion

$$i : \mathfrak{a}^* \hookrightarrow \mathfrak{a}_\emptyset^*.$$

If $\nu \in \mathfrak{a}^*$, we use the same letter ν to denote $i(\nu) \in \mathfrak{a}_\emptyset^*$. Then

$$\langle \nu, H_\emptyset(a)\rangle = \langle \nu, H_\Theta(a)\rangle \quad \text{for all } a \in A_\emptyset.$$

We use $i_{G,M}$ to denote the functor of normalized parabolic induction ([BeZ]): if (σ, V) is a smooth representation of M, then $(i_{G,M}(\sigma), i_{G,M}(V))$ is the representation of G parabolically induced from (σ, V). Similarly, $r_{M,G}$ denotes the normalized Jacquet functor. For an irreducible representation π of G, we write $\hat{\pi}$ (or $D_G(\pi)$) for the dual of π in the sense of [Au],[ScSt]. This extends to an operator on the Grothendieck group of smooth, finite-length representations (for which we use the same notation).

Let $\Delta_P \subset \mathfrak{a}^*$ denote the set of simple roots corresponding to the pair (P, A). Set

$$\mathfrak{a}_-^* = \{x \in \mathfrak{a}^* \mid (x, \alpha) < 0 \text{ for all } \alpha \in \Delta_P\}.$$

We recall the Langlands classification for p-adic groups (cf. [S1], [BoW]). This version is closer to that of [S1], though we work in the subrepresentation setting rather than the quotient setting.

Theorem 2.1. *(the Langlands classification)*

(1) *Let $P = MU$ be a standard parabolic subgroup of G, τ an irreducible tempered representation of M and $\nu \in \mathfrak{a}_-^*$. Then the induced representation $i_{G,M}(\exp\ \nu \otimes \tau)$ has a unique irreducible subrepresentation, which we denote by $L(P, \nu, \tau)$.*
(2) *Let π be an irreducible admissible representation of G. Then there exists a unique triple (P, ν, τ) as in (1) such that $\pi = L(P, \nu, \tau)$.*

Suppose that σ is an irreducible admissible representation of M and $w \in W$ such that $w(\Theta) \subset \Delta$. Set

$$U_w = U_\emptyset \cap wU^- w^{-1},$$
$$U_w^- = w^{-1} U_w w = w^{-1} U_\emptyset w \cap U^-.$$

The standard intertwining operator $\mathbf{A}(\nu, \sigma, w)$ is formally defined by

$$\mathbf{A}(\nu, \sigma, w) f(g) = \int_{U_w} f(w^{-1}ug)du,$$

where $\nu \in \mathfrak{a}^*_{\mathbb{C}}$, $f \in V_{i_{G,M}(\exp\ \nu\otimes\sigma)}$ and $g \in G$ (see [B2], [Sh1], [GSh] for a more detailed discussion). It converges absolutely for the real part of ν in a certain chamber and

$$\nu \mapsto \mathbf{A}(\nu, \sigma, w)$$

has analytic continuation as meromorphic function of $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ (cf. [A2]).

We let ρ_P denote half the sum of the positive roots associated to U.

Theorem 2.2. *(Harish-Chandra, cf. page 182, [H-C]) Let σ be an irreducible tempered representation of M. Then there is a complex number $\mu(\nu, \sigma, w)$ so that*

$$\mathbf{A}(\nu, \sigma, w)\mathbf{A}(w\nu, w\sigma, w^{-1}) = \mu(\nu, \sigma, w)^{-1}\gamma_w^2(G/P),$$

where

$$\gamma_w(G/P) = \int_{U_{\bar{w}}^-} q^{\langle 2\rho_P, H_P(\bar{u})\rangle} d\bar{u}.$$

Moreover, $\nu \mapsto \mu(\nu, \sigma, w)$ is meromorphic on $\mathfrak{a}^*_{\mathbb{C}}$ and holomorphic and nonnegative on the unitary axis $i\mathfrak{a}^*$.

$\mu(\nu, \sigma, w)$ is called the Plancherel measure associated to ν, σ, w.

The following result will be used to deal with factorizations of intertwining operators in section 5. It is Lemma 2.1.2 and its corollary from [Sh1] and Corollary 6.3 from [B2]. As in those papers, if $\Theta \subset \Delta$, we let Σ_Θ be the set of roots in Σ which lie in the linear span of Θ. Letting $\Sigma(\Theta)$ denote the roots of (P_Θ, A_Θ), we have $\Sigma(\Theta) = \{\alpha|_{A_\Theta} \mid \alpha \in \Sigma - \Sigma_\Theta\}$. Given $\alpha \in \Sigma(\Theta)$, we let

$$[\alpha] = \{\beta \in \Sigma - \Sigma_\Theta \mid \beta|_{A_\Theta} = \alpha_{A_\Theta}\}.$$

We now state the lemma:

Lemma 2.3. *Suppose $\Theta, \Theta' \subset \Delta$ are associate. Take $w \in W(\Theta, \Theta') = \{w \in W \mid w \cdot \Theta = \Theta'\}$. Then, there exists a family of subsets $\Theta_1, \ldots, \Theta_{n+1} \subset \Delta$ such that the following are satisfied:*

(1) $\Theta_1 = \Theta$ *and* $\Theta_{n+1} = \Theta'$.

(2) *Fix $1 \leq i \leq n$; then there exists a root $\alpha_i \in \Delta \setminus \Theta_i$ such that Θ_{i+1} is the conjugate of Θ_i in $\Omega_i = \Theta_i \cup \alpha_i$.*

(3) *Letting $w_{l,\Theta}$ denote the longest element of the Weyl group of M_Θ, set $w_i = w_{l,\Omega_i} w_{l,\Theta_i}$ in $W(\Theta_i, \Theta_{i+1})$ for $1 \leq i \leq n$. Then $w = w_n \cdots w_1$. (We note that any such decomposition has $n = |\Sigma_r(\Theta, \Theta', w)|$–cf. (5) below.)*

(4) $\mathbf{A}(\nu, \sigma, w) = \mathbf{A}(\nu_n, \sigma_n, w_n) \cdots \mathbf{A}(\nu_1, \sigma_1, w_1)$, *where $\nu_1 = \nu$, $\sigma_1 = \sigma$, $\nu_i = w_{i-1}(\nu_{i-1})$ and $\sigma_i = w_{i-1}(\sigma_{i-1})$ for $2 \leq i \leq n$.*

(5) *Let $\Sigma_r^+(\Theta)$ be the set of all reduced roots in $\Sigma^+(\Theta) = \Sigma(\Theta) \cap \Sigma^+$. Let*

$$\Sigma_r(\Theta, \Theta', w) = \{[\beta] \in \Sigma_r^+(\Theta) \mid \beta \in \Sigma^+ - \Sigma_\Theta^+, w(\beta) \in \Sigma^-\}.$$

Fix $w = w_n \cdots w_1$ as in (3), and let $\alpha_1, \ldots, \alpha_n \in \Delta$ be the corresponding simple roots. Then

$$[\beta_i] = w_1^{-1} \cdots w_{i-1}^{-1}([\alpha_i]), \quad 1 \leq i \leq n,$$

are all distinct elements of $\Sigma_r(\Theta, \Theta', w)$. Furthermore, given $[\beta] \in \Sigma_r(\Theta, \Theta', w)$, there exists an i, $1 \leq i \leq n$, such that $[\beta] = w_1^{-1} \cdots w_{i-1}^{-1}([\alpha_i])$.

3. Plancherel measures for nontempered representations

In this section, we extend the notion of Plancherel measure and obtain a formula like that of Theorem 2.2, but valid more generally. We then relate these (generalized) Plancherel measures to those of representations having the same supercuspidal support.

We remark that Plancherel measures for nontempered representations were considered in section 3 of [Sh2] (for σ unitary and generic). While we use a different definition, in light of Theorem 3.2 below and the corresponding result in [Sh2], our Plancherel measure must be the same as his when both are defined. (While Shahidi's extension of Plancherel measure has been quite useful, we do not know whether our extension has applications beyond those considered here.)

Let σ be an irreducible representation of $M = M_\Theta$. By the Langlands classification (Theorem 2.1), there exist Langlands data $(P_0 \cap M, \nu_0, \sigma_0)$ such that

$$\sigma = L(P_0 \cap M, \nu_0, \sigma_0).$$

Let Θ_0 be the set of simple roots corresponding to P_0. Denote $\mathfrak{a}^*_{\Theta_0}$ by $\mathfrak{a}^*_0$. Then $\mathfrak{a}^* \subseteq \mathfrak{a}^*_0$. We have $\nu_0 \in (\mathfrak{a}^*_0)_-$.

Let $w \in W$ satisfy $w(\Theta) \subset \Delta$. Then $w(\Theta_0) \subset \Delta$. We have

$$U_\Theta = \prod_{\alpha \in \Sigma^+ - \Sigma^+_\Theta} U^\alpha, \qquad U^-_\Theta = \prod_{\alpha \in \Sigma^- - \Sigma^-_\Theta} U^\alpha,$$
$$U_{\Theta_0} = \prod_{\alpha \in \Sigma^+ - \Sigma^+_{\Theta_0}} U^\alpha, \qquad U^-_{\Theta_0} = \prod_{\alpha \in \Sigma^- - \Sigma^-_{\Theta_0}} U^\alpha,$$

where Σ_Θ (resp., Σ_{Θ_0}) denotes the subset of roots in the linear span of Θ (resp., Θ_0). The conditions $w(\Theta) \subset \Delta$ and $w(\Theta_0) \subset \Delta$ imply $w(\Sigma^+_\Theta) \subset \Sigma^+$ and $w(\Sigma^+_{\Theta_0}) \subset \Sigma^+$. It follows that

$$U_w = (U_0)_w = \prod_{\substack{\alpha \in \Sigma^+ \\ w(\alpha) < 0}} U^{w(-\alpha)}.$$

Lemma 3.1. *Let* $\nu \in \mathfrak{a}^*_{\mathbb{C}} \subseteq (\mathfrak{a}^*_0)_{\mathbb{C}}$. *Then*

$$\exp \nu \otimes i_{M,M_0}(\sigma_0) \cong i_{M,M_0}(\exp \nu \otimes \sigma_0).$$

Proof. This is just a rephrasing of Proposition 1.9 (f) of [BeZ]. □

It follows from Lemma 3.1 that $\exp \nu \otimes \sigma$ is a subrepresentation of i_{M,M_0} $(\exp(\nu_0 + \nu) \otimes \sigma_0)$ and therefore the standard intertwining operator $\mathbf{A}(\nu, \sigma, w)$ is the restriction of $\mathbf{A}(\nu_0 + \nu, \sigma_0, w)$, i.e.,

$$\mathbf{A}(\nu, \sigma, w) = \mathbf{A}(\nu_0 + \nu, \sigma_0, w)|_{i_{G,M}(V)},$$

where V denotes the subspace of the space of $i_{M,M_0}(\exp(\nu_0 + \nu) \otimes \sigma_0)$ corresponding to $\exp \nu \otimes \sigma$. According to Theorem 2.2, we have

$$\mathbf{A}(\nu_0 + \nu, \sigma_0, w)\mathbf{A}(w(\nu_0 + \nu), w\sigma_0, w^{-1}) = \mu(\nu_0 + \nu, \sigma_0, w)^{-1}\gamma^2_w(G/P_0).$$

We define

$$\mu(\nu, \sigma, w) = \mu(\nu_0 + \nu, \sigma_0, w)$$

and call $\mu(\nu, \sigma, w)$ the Plancherel measure associated to ν, σ, w. We remark that by the same arguments in [Sh2], $\mu(\nu, \sigma, w)$ is independent of the choice of representative for w and the choice of measures; since the Langlands data are unique, $\mu(\nu, \sigma, w)$ is well-defined.

Theorem 3.2. *Let σ be an irreducible admissible representation of M. Then*

$$\mathbf{A}(\nu, \sigma, w)\mathbf{A}(w\nu, w\sigma, w^{-1}) = \mu(\nu, \sigma, w)^{-1}\gamma_w^2(G/P)$$

and $\nu \mapsto \mu(\nu, \sigma, w)$ is meromorphic on $\mathfrak{a}_{\mathbb{C}}^*$.

Proof. Theorem 2.2 implies

$$\mathbf{A}(\nu, \sigma, w)\mathbf{A}(w\nu, w\sigma, w^{-1}) = \mu(\nu, \sigma, w)^{-1}\gamma_w^2(G/P_0).$$

It follows from $U_w = (U_0)_w$ that $\gamma_w(G/P_0) = \gamma_w(G/P)$. □

Let τ be an irreducible admissible representation of $M = M_\Theta$. If τ_0 is an irreducible supercuspidal representation of a standard Levi subgroup $M_0 \leq M$ such that τ occurs as a subquotient of $i_{M,M_0}(\tau_0)$, we say that (τ_0, M_0) is in the supercuspidal support of τ. We note that two irreducible representations of M either have the same supercuspidal support or their supercuspidal supports are disjoint.

Let Θ and Θ_0 be the sets of simple roots corresponding to M and M_0, respectively. Let $W_\Theta = W(M/A_\emptyset)$ denote the Weyl group of $M = M_\Theta$ with respect to $A_\emptyset$. Set

$$W_\Theta(\Theta_0) = \{w \in W_\Theta \mid w(\Theta_0) = \Theta_0\}.$$

Theorem 3.3. *Suppose that σ and σ' are irreducible admissible representations of M with the same supercuspidal support. Let $w \in W$ such that $w(\Theta) \subset \Delta$. Then,*

$$\mu(\nu, \sigma', w) = \mu(\nu, \sigma, w),$$

for all $\nu \in \mathfrak{a}_{\mathbb{C}}^*$.

Proof. Let σ_0 be a supercuspidal representation of M_0 such that σ is a subrepresentation of $i_{M,M_0}(\exp \nu_0 \otimes \sigma_0)$. According to [C], Corollary 7.2.2, there exists $w_0 \in W_\Theta(\Theta_0)$ such that σ' is equivalent to a subrepresentation of $i_{M,M_0}(w_0(\exp \nu_0 \otimes \sigma_0))$.

Let $w_0' = ww_0w^{-1}$. In the same way as in the proof of Lemma 7.1 [B1], we obtain

$$ww_0 = w_0'w, \tag{1}$$

$$w_0w^{-1} = w^{-1}w_0', \tag{2}$$

$$l(ww_0) = l(w) + l(w_0), \tag{3}$$

$$l(w_0'w) = l(w_0') + l(w), \tag{4}$$

$$l(w_0 w^{-1}) = l(w_0) + l(w^{-1}), \tag{5}$$

$$l(w^{-1}w_0') = l(w^{-1}) + l(w_0'). \tag{6}$$

The basic idea of the proof is similar to that of the proof of Lemma 7.1 [B1]. Let $\nu \in \mathfrak{a}_{\mathbb{C}}^*$. It follows from (1) that

$$\mathbf{A}(\nu + \nu_0, \sigma_0, w_0'w) = \mathbf{A}(\nu + \nu_0, \sigma_0, ww_0). \tag{7}$$

Using properties of standard intertwining operators, (3) and (4) then imply

$$\begin{aligned} &\mathbf{A}(w(\nu + \nu_0), w\sigma_0, w_0')\mathbf{A}(\nu + \nu_0, \sigma_0, w) \\ &\quad = \mathbf{A}(w_0(\nu + \nu_0), w_0\sigma_0, w)\mathbf{A}(\nu + \nu_0, \sigma_0, w_0). \end{aligned} \tag{8}$$

Similarly, (2), (5) and (6) imply

$$\begin{aligned} &\mathbf{A}(\nu + \nu_0, \sigma_0, w_0)\mathbf{A}(w(\nu + \nu_0), w\sigma_0, w^{-1}) \\ &\quad = \mathbf{A}(ww_0(\nu + \nu_0), ww_0\sigma_0, w^{-1})\mathbf{A}(w(\nu + \nu_0), w\sigma_0, w_0'). \end{aligned} \tag{9}$$

According to Theorem 3.2,

$$\mathbf{A}(\nu, \sigma, w)\mathbf{A}(w\nu, w\sigma, w^{-1}) = \mu(\nu, \sigma, w)^{-1}\gamma_w^2(G/P). \tag{10}$$

Since σ is a subrepresentation of $i_{M,M_0}(\exp \nu_0 \otimes \sigma_0)$, the standard intertwining operator $\mathbf{A}(\nu, \sigma, w)$ is the restriction of $\mathbf{A}(\nu + \nu_0, \sigma_0, w)$ and

$$\mathbf{A}(\nu + \nu_0, \sigma_0, w)\mathbf{A}(w(\nu + \nu_0), w\sigma_0, w^{-1}) = \mu(\nu + \nu_0, \sigma, w)^{-1}\gamma_w^2(G/P). \tag{11}$$

Now, using (11), (8) and (9), we have

$$\begin{aligned} &\mathbf{A}(w(\nu + \nu_0), w\sigma_0, w_0')\mu(\nu, \sigma, w)^{-1}\gamma_w^2(G/P) \\ &\quad = \mathbf{A}(w(\nu + \nu_0), w\sigma_0, w_0')\mathbf{A}(\nu + \nu_0, \sigma_0, w)\mathbf{A}(w(\nu + \nu_0), w\sigma_0, w^{-1}) \\ &\quad = \mathbf{A}(w_0(\nu + \nu_0), w_0\sigma_0, w)\mathbf{A}(\nu + \nu_0, \sigma_0, w_0)\mathbf{A}(w(\nu + \nu_0), w\sigma_0, w^{-1}) \\ &\quad = \mathbf{A}(w_0(\nu + \nu_0), w_0\sigma_0, w)\mathbf{A}(ww_0(\nu + \nu_0), ww_0\sigma_0, w^{-1}) \\ &\qquad \mathbf{A}(w(\nu + \nu_0), w\sigma_0, w_0') \\ &\quad = \mu(w_0(\nu + \nu_0), w_0\sigma_0, w)^{-1}\gamma_w^2(G/P)\mathbf{A}(w(\nu + \nu_0), w\sigma_0, w_0'). \end{aligned} \tag{12}$$

We conclude that

$$\mu(\nu, \sigma, w) = \mu(w_0(\nu + \nu_0), w_0\sigma_0, w) \tag{13}$$

for all ν such that $\mathbf{A}(w(\nu + \nu_0), w\sigma_0, w_0')$ is holomorphic, and by analytic continuation, for all $\nu \in \mathfrak{a}_{\mathbb{C}}^*$. Since σ' is a subrepresentation of $i_{M,M_0}(w_0(\exp \nu_0 \otimes \sigma_0))$, the standard intertwining operator $\mathbf{A}(\nu, \sigma', w)$ is the restriction of $\mathbf{A}(\nu + w_0\nu_0, w_0\sigma_0, w)$. Therefore,

$$\mu(\nu, \sigma', w) = \mu(\nu + w_0\nu_0, w_0\sigma_0, w)$$

and (13) implies

$$\mu(w_0\nu, \sigma', w) = \mu(\nu, \sigma, w).$$

Finally, since $w_0 \in W_\Theta$ and $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, we see that $w_0 \cdot \nu = \nu$ (noting that it does not matter whether ν is viewed as an element of $\mathfrak{a}_{\mathbb{C}}^*$ or $(\mathfrak{a}_0)_{\mathbb{C}}^*$ as far as this equality is concerned.) The theorem follows. □

Remark 3.4. If σ_0 is a supercuspidal representation of M_0 such that $\sigma \hookrightarrow i_{M,M_0}\sigma_0$, then the proof of the lemma tells us w_0 can be any element of $W_\Theta(\Theta_0)$ such that $\sigma' \hookrightarrow i_{M,M_0}(w\sigma_0)$.

4. Derived subgroups of reductive groups

The normalization of standard intertwining operators we use is based on the results of [KnSt], who work in the setting of semisimple groups. We would like to work in the generality of reductive groups; the results in this section allow us to do so. Our strategy is to reduce to the semisimple case via the derived subgroup.

We begin by introducing some notation. Let G' denote the derived group, which is semisimple. Without loss of generality, we may take $\Sigma, \Sigma^+, \Delta, W$ to be the same for G' as G. If $\Theta \subset \Delta$ has the associated standard parabolic subgroup $P = MU$, we let

$$P' = P \cap G', \quad M' = M \cap G', \quad \text{etc.}$$

(n.b.: P' is not the derived subgroup of P, etc.). Then, $P' = M'U'$ is (the Levi factorization of) the standard parabolic subgroup of G' associated to Θ. Further, one has $U' = U$. (We note that this seems to be well-known–cf. p. 385 [T], e.g. In any case, at the Lie algebra level, it is a fairly easy consequence of $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$ and $[H, X] = \alpha(H)X \in \mathfrak{g}'$ for $H \in \mathrm{Lie}A$ and $X \in \mathrm{Lie}U$ in the root space for α.)

Suppose σ is an irreducible admissible representation of M and $\nu \in \mathfrak{a}_{\mathbb{C}}^*$. We let $\nu' = \nu|_{\mathfrak{a}'}$ ($\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{g}'$) and choose $\sigma' \subset \mathrm{Res}^M_{M'}\sigma$ irreducible (by Lemma 2.1 [T], $\mathrm{Res}^M_{M'}\sigma$ decomposes as a finite direct sum). If $w \in W$ such that $w(\Theta) \subset \Delta$, we let

$$c(\nu, \sigma, w) = \mu^{-1}(\nu, \sigma, w)\gamma_w^2(G/P)$$

and

$$c(\nu', \sigma', w) = \mu^{-1}(\nu', \sigma', w)\gamma_w^2(G'/P').$$

We have the following:

Proposition 4.1. $c(\nu, \sigma, w) = c(\nu', \sigma', w)$.

Proof. First, we claim

$$\mathrm{Res}^G_{G'}(\mathrm{Ind}^G_P(\exp \nu \otimes \sigma)) \cong \mathrm{Ind}^{G'}_{P'}(\mathrm{Res}^M_{M'}(\exp \nu \otimes \sigma)).$$

This is a straightforward generalization of Lemma 1.1 [T]. The left-hand side acts on (the smooth vectors of)

$$V = \{f : G \to V_\sigma \mid f'(pg) = \delta^{\frac{1}{2}}(p)(\exp \nu \otimes \sigma)(p) f(g)\}$$

by right translations. The right-hand side acts on (the smooth vectors of)

$$V' = \{f' : G' \to V_\sigma \mid f'(p'g') = \delta^{\frac{1}{2}}(p')(\exp \nu \otimes \sigma)(p') f(g')\}$$

(noting that $\delta = \delta'$ on P'). The map

$$\begin{aligned} E : V &\longrightarrow V' \\ f &\longmapsto f|_{G'} \end{aligned}$$

gives the equivalence of the representations.

We now claim E behaves well with respect to standard intertwining operators. To make this precise, let $\pi = \mathrm{Ind}_P^G(\exp \nu \otimes \sigma)$ and $\pi' = \mathrm{Ind}_{P'}^{G'}(\exp \nu' \otimes \sigma')$. It follows from the preceding discussion that π' is a subrepresentation of $\mathrm{Res}^G_{G'}\pi$ which may be realized on the subspace $V_{\pi'} \subset V$, where $V_{\pi'} = \{f \in V \mid \mathrm{image}(f|_{G'}) \subset V_{\sigma'}\}$. Let $E_{\pi'} = E \circ \mathrm{proj}_{V_{\pi'}}$. Then, an easy calculation (noting $U = U'$) gives

$$\mathbf{A}(\nu', \sigma', w) E_{\pi'} = E_{\pi'_w} \mathbf{A}(\nu, \sigma, w),$$

where $\pi'_w = \mathrm{Ind}_{P'_w}^{G'}(w(\exp \nu' \otimes \sigma'))$ and P'_w is the standard parabolic subgroup with Levi factor $w(M)$ (a G'-subrepresentation of $\pi_w = \mathrm{Ind}_{P_w}^G(w(\exp \nu \otimes \sigma))$.

Finally, since

$$\mathbf{A}(\nu', \sigma', w)\mathbf{A}(w\nu', w\sigma', w^{-1}) = c(\nu', \sigma', w) I,$$

we have

$$\mathbf{A}(\nu', \sigma', w)\mathbf{A}(w\nu', w\sigma', w^{-1}) E_{\pi'_w} = c(\nu', \sigma', w) E_{\pi'_w}.$$

On the other hand,

$$\begin{aligned} \mathbf{A}(\nu', \sigma', w)\mathbf{A}(w\nu', w\sigma', w^{-1}) E_{\pi'_w} &= E_{\pi'_w} \mathbf{A}(\nu, \sigma, w)\mathbf{A}(w\nu, w\sigma, w^{-1}) \\ &= E_{\pi'_w} c(\nu, \sigma, w) I \\ &= c(\nu, \sigma, w) E_{\pi'_w}. \end{aligned}$$

The proposition follows. □

Remark 4.2. It follows from the preceding proposition that $c(\nu', \sigma', w)$ does not depend on the particular $\sigma' \subset \mathrm{Res}^M_{M'}\sigma$ used.

Lemma 4.3. *The duality operators D_G, $D_{G'}$ satisfy*

$$Res^G_{G'} \circ D_G = D_{G'} \circ Res^G_{G'}$$

(where the equalities hold in the Grothendieck group).

Proof. We begin by observing that

$$\mathrm{Res}^{M}_{M'} \circ r_{M,G} \cong r_{M',G'} \circ \mathrm{Res}^{G}_{G'}$$

(an easy verification, noting $U' = U$) and

$$\mathrm{Res}^{G}_{G'} \circ i_{G,M} \cong i_{G',M'} \circ \mathrm{Res}^{M}_{M'}$$

(cf. proof of Proposition 4.1). We now calculate:

$$\begin{aligned}\mathrm{Res}^{G}_{G'} \circ D_G &= \sum_{\Theta\subset\Delta} (-1)^{|\Theta|} \mathrm{Res}^{G}_{G'} \circ i_{G,M_\Theta} \circ r_{M_\Theta,G} \\ &= \sum_{\Theta\subset\Delta} (-1)^{|\Theta|} i_{G',M'_\Theta} \circ \mathrm{Res}^{M_\Theta}_{M'_\Theta} \circ r_{M_\Theta,G} \\ &= \sum_{\Theta\subset\Delta} (-1)^{|\Theta|} i_{G',M'_\Theta} \circ r_{M'_\Theta,G'} \circ \mathrm{Res}^{G}_{G'} \\ &= D_{G'} \circ \mathrm{Res}^{G}_{G'},\end{aligned}$$

as needed. □

The preceding lemma tells us that $\widehat{(\sigma')}$ is a suitable choice for $(\hat{\sigma})'$. Since $\hat{\sigma}$ and σ have the same supercuspidal support, Theorem 3.3 implies the following:

Corollary 4.4.

$$c(\nu, \hat{\sigma}, w) = c(\nu, \sigma, w).$$

5. Normalizing factors

In this section, we relate normalizing factors for the intertwining operators under consideration. Our approach is based on that of [KnSt], which constructs the normalizing factor directly from the Plancherel measure. This has the advantage that we can relate Plancherel measures associated to induced representations and their duals under [Au], [ScSt]. (While it would be nice to do the normalizations via L-functions as in [Sh2], this would require being able to track the data needed to calculate L-functions through the duality operator–a difficult task.) We will normalize standard intertwining operators for σ an arbitrary irreducible unitary representation. (We remark that we want σ unitary since both parts of Lemma 5.1 below are needed to apply the results of [KnSt]). Our normalized standard intertwining operators will have the following properties, the first of which is crucial to the results in [B3]:

(1) $\mathcal{A}(w_1\nu, w_1\sigma, w_2)\mathcal{A}(\nu, \sigma, w_1) = \mathcal{A}(\nu, \sigma, w_2 w_1)$
(2) $\mathcal{A}(\nu, \sigma, w)^* = \mathcal{A}(-w\bar{\nu}, w\sigma, w^{-1})$ (where * denotes adjoint and σ assumed unitary)

for suitable w, w_1, w_2 (cf. Theorem 5.3). We now give some properties of the functions $c(\nu, \sigma, w)$, which are defined as in section 4:

$$c(\nu, \sigma, w) = \mu^{-1}(\nu, \sigma, w)\gamma_w^2(G/P).$$

Lemma 5.1. $c(\nu, \sigma, w)$ *has the following properties:*

(1) $c(\nu, \sigma, w) = \overline{c(-\bar{\nu}, \sigma, w)}$
(2) *If* σ *is unitary,* $c(\nu, \sigma, w) \geq 0$ *for* $\nu \in i\mathfrak{a}^*_{\mathbf{R}}$.

Proof. Recall that (cf. Proposition 2.4.2 [Sh1])

$$\mathbf{A}(\nu, \sigma, w)^* = \mathbf{A}(-w\bar{\nu}, w\sigma, w^{-1}) \tag{14}$$

and

$$\mathbf{A}(\nu, \sigma, w)\mathbf{A}(w\nu, w\sigma, w^{-1}) = c(\nu, \sigma, w)I. \tag{15}$$

We now just follow the proof of Proposition 27, [KnSt], properties (iv) and (v), page 528. From (15) and (14), we have

$$c(\nu, \sigma, w)I = \mathbf{A}(\nu, \sigma, w)\mathbf{A}(w\nu, w\sigma, w^{-1}) = \mathbf{A}(\nu, \sigma, w)\mathbf{A}(-\bar{\nu}, \sigma, w)^*,$$

so $c(\nu, \sigma, w) \geq 0$ for ν purely imaginary. Apply * to (15) to get

$$\overline{c(\nu, \sigma, w)}I = \mathbf{A}(-\bar{\nu}, \sigma, w)\mathbf{A}(-w\bar{\nu}, w\sigma, w^{-1}) = c(-\bar{\nu}, \sigma, w)I.$$

It follows that

$$c(\nu, \sigma, w) = \overline{c(-\bar{\nu}, \sigma, w)}. \quad \square$$

We now turn to the construction of normalizing factors. Here, we restrict our attention to the case where σ is unitary (and where appropriate, $\hat{\sigma}$ unitary). First, suppose G is semisimple and $P = MU$ is a maximal parabolic subgroup (so $\dim(A_M/A_G) = 1$). Then, $\mathfrak{a}$ is one-dimensional, so $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ is effectively a complex number. By the preceding lemma, the hypotheses of Lemma 36 [KnSt] are satisfied for $c(\nu) = c(\nu, \sigma, w)$. Thus we obtain a normalizing factor $\gamma(\nu) = \gamma(\nu, \sigma, w)$ as in Lemma 36 and Proposition 37 [KnSt]. (More precisely, it is $\gamma^{-1}(\nu, \sigma, w)$ which is used for normalizations–cf. Theorem 5.3.) The normalizing factors satisfy

$$\overline{\gamma(-\bar{\nu}, \sigma, w)}\gamma(\nu, \sigma, w) = c(\nu, \sigma, w)$$

and

$$\gamma(w\nu, w\sigma, w^{-1}) = \overline{\gamma(-\bar{\nu}, \sigma, w)}.$$

We note that since $c(\nu, \hat{\sigma}, w) = c(\nu, \sigma, w)$ (an immediate consequence of Theorem 3.3), we may choose

$$\gamma(\nu, \hat{\sigma}, w) = \gamma(\nu, \sigma, w).$$

Next, suppose G is no longer assumed to be semisimple, but is allowed to be reductive (with $\dim(A_M/A_G) = 1$). In light of Proposition 4.1, we take

$$\gamma(\nu, \sigma, w) = \gamma(\nu', \sigma', w).$$

Corollary 4.4 then allows us to conclude that

$$\overline{\gamma(-\bar{\nu},\sigma,w)}\gamma(\nu,\sigma,w)=c(\nu,\sigma,w),$$
$$\gamma(w\nu,w\sigma,w^{-1})=\overline{\gamma(-\bar{\nu},\sigma,w)},$$

and

$$\gamma(\nu,\hat{\sigma},w)=\gamma(\nu,\sigma,w).$$

Finally, suppose $P=MU$ has $\dim(A_M/A_G)>1$. By Lemma 2.3, we have the decomposition

$$\mathbf{A}(\nu,\sigma,w)=\mathbf{A}(\nu_n,\sigma_n,w_n)\ldots\mathbf{A}(\nu_1,\sigma_1,w_1). \tag{16}$$

We define $\gamma(\nu,\sigma,w)$ via a corresponding product of normalizing factors:

$$\gamma(\nu,\sigma,w)=\gamma(\nu_n,\sigma_n,w_n)\ldots\gamma(\nu_1,\sigma_1,w_1).$$

To define $\gamma(\nu_i,\sigma_i,w_i)$, observe that the intertwining operators appearing in (16) have the form

$$\mathbf{A}(\nu_i,\sigma_i,w_i)=i_{G,M_i}(\mathbf{A}_{M_i}(\nu_i,\sigma_i,w_i)),$$

where $M_i=M_{\Omega_i}$ as in Lemma 2.3. Since $\dim(A_{w_{i-1}\ldots w_1(M)}/A_{M_i})=1$, we may define

$$\gamma(\nu_i,\sigma_i,w_i)=\gamma_{M_i}(\nu_i,\sigma_i,w_i) \tag{17}$$

(noting that it is not difficult to show, using Theorem 3.2, that $c(\nu_i,\sigma_i,w_i)=c_{M_i}(\nu_i,\sigma_i,w_i)$). It remains to check that $\gamma(\nu,\sigma,w)$ is well-defined. We do so in the following lemma:

Lemma 5.2. *Suppose $\Theta,\Theta'\subset\Delta$ are associate. Take $w\in W(\Theta,\Theta')$. Fix a decomposition $w=w_n\cdots w_1$ as in Lemma 2.3. Define*

$$\gamma(\nu,\sigma,w)=\gamma(\nu_n,\sigma_n,w_n)\cdots\gamma(\nu_1,\sigma_1,w_1).$$

Then $\gamma(\nu,\sigma,w)$ does not depend on the particular decomposition $w=w_n\cdots w_1$, i.e., it is well-defined.

Proof. Let $\alpha_1,\ldots,\alpha_n\in\Delta$ be the corresponding set of simple roots. Fix α_i, $1\leq i\leq n$. Let $[\beta_i]$ be as in (5) of Lemma 2.3. Suppose $w=w'_{n'}\cdots w'_1$ is another decomposition of w and $\alpha'_1,\ldots,\alpha'_{n'}\in\Delta^+$ the corresponding set of simple roots. Let α'_j be the simple root corresponding to $[\beta_i]$ under Lemma 2.3 (5). Set

$$x=w_{i-1}\cdots w_1,\quad x'=w'_{j-1}\cdots w'_1.$$

Then $\Omega'_j=x'x^{-1}(\Omega_i)$, $\sigma'_j=x'\sigma=x'x^{-1}\sigma_i$ and $\nu'_j=x'\nu=x'x^{-1}\nu_i$. It follows that

$$\gamma_{M_{\Omega_i}}(\nu_i,\sigma_i,w_i)=\gamma_{M_{\Omega'_j}}(\nu'_j,\sigma'_j,w'_j).$$

Doing this for each i, we see that $n\leq n'$; working the same way from $w'_{n'}\cdots w'_1$ gives $n\geq n'$. Thus $n=n'$, and the lemma follows. □

Observe that we also have

$$c(\nu, \sigma, w) = c(\nu_n, \sigma_n, w_n) \dots c(\nu_1, \sigma_1, w_1)$$

(an easy consequence of Lemma 2.3 and Theorem 3.2). It then follows from the corresponding properties in the case $\dim(A_M/A_G) = 1$ that

$$\overline{\gamma(-\bar{\nu}, \sigma, w)}\gamma(\nu, \sigma, w) = c(\nu, \sigma, w), \tag{18}$$

and

$$\gamma(w\nu, w\sigma, w^{-1}) = \overline{\gamma(-\bar{\nu}, \sigma, w)}. \tag{19}$$

Theorem 5.3. *Suppose $\sigma, \hat{\sigma}$ are both unitary. The normalizing factors $\gamma(\nu, \sigma, w)$ defined above satisfy*

$$\gamma(\nu, \hat{\sigma}, w) = \gamma(\nu, \sigma, w).$$

The normalized standard intertwining operators

$$\mathcal{A}(\nu, \sigma, w) = \gamma^{-1}(\nu, \sigma, w)\mathbf{A}(\nu, \sigma, w)$$

have the following properties:

(1) $\mathcal{A}(\nu, \sigma, w)^* = \mathcal{A}(-w\bar{\nu}, w\sigma, w^{-1})$ *for* $w \in W(\Theta, \Theta')$
(2) $\mathcal{A}(w_1\nu, w_1\sigma, w_2)\mathcal{A}(\nu, \sigma, w_1) = \mathcal{A}(\nu, \sigma, w_2 w_1)$ *for* $w_1 \in W(\Theta, \Theta')$ *and* $w_2 \in W(\Theta', \Theta'')$,

where $\Theta, \Theta', \Theta''$ are associate.

Proof. We first show that $\gamma(\nu, \sigma, w) = \gamma(\nu, \hat{\sigma}, w)$. We have

$$\gamma(\nu, \sigma, w) = \gamma_{M_n}(\nu_n, \sigma_n, w_n) \dots \gamma_{M_1}(\nu_1, \sigma_1, w_1).$$

The same argument tells us

$$\gamma(\nu, \hat{\sigma}, w) = \gamma_{M_n}(\nu_n, \hat{\sigma_n}, w_n) \dots \gamma_{M_1}(\nu_1, \hat{\sigma_1}, w_1).$$

Since $\dim(A_{w_{i-1}\dots w_1(M)}/A_{M_i}) = 1$, we know that

$$\gamma_{M_i}(\nu_i, \sigma_i, w_1) = \gamma_{M_i}(\nu_i, \hat{\sigma_i}, w_i).$$

From this, we see that

$$\gamma(\nu, \sigma, w) = \gamma(\nu, \hat{\sigma}, w).$$

The arguments needed for (1) and (2) are well-known (e.g., cf. [A2],[Sh2]); we include them for completeness. We begin with (1). In the case $\dim(A_M/A_G) = 1$, (1) follows from (14) and the identity $\gamma(w\nu, w\sigma, w^{-1}) = \overline{\gamma(-\bar{\nu}, \sigma, w)}$. In general,

$$\begin{aligned}\mathcal{A}(\nu, \sigma, w)^* &= [\mathcal{A}(\nu_n, \sigma_n, w_n) \dots \mathcal{A}(\nu_1, \sigma_1, w_1)]^* \\ &= \mathcal{A}(\nu_1, \sigma_1, w_1)^* \dots \mathcal{A}(\nu_n, \sigma_n, w_n)^* \\ &= \mathcal{A}(-w_1\bar{\nu}_1, w_1\sigma_1, w_1^{-1}) \dots \mathcal{A}(-w_n\bar{\nu}_n, w_n\sigma_n, w_n^{-1}).\end{aligned}$$

Now, observe that since $\sigma_n = w_{n-1} \dots w_1 \sigma$, we have $w_n \sigma_n = w\sigma$, and similarly for $-w_n \bar{\nu}_n$. Therefore, since $w^{-1} = w_1^{-1} \dots w_n^{-1}$ is a decomposition of w^{-1} satisfying Lemma 2.3, we have

$$\mathcal{A}(\nu, \sigma, w)^* = \mathcal{A}(-w\bar{\nu}, w\sigma, w^{-1}),$$

as needed.

We now turn to (2). First, suppose $w = w_1 \in W(\Theta, \Theta')$ and $u = w_2 \in W(\Theta', \Theta'')$ such that u is what might be called a generalized simple reflection–i.e., there exists an $\alpha \in \Delta - \Theta'$ with Θ'' the conjugate of Θ' in $\Omega = \Theta' \cup \{\alpha\}$, conjugate by $u = w_{l,\Omega} w_{l,\Theta'}$. If $l(uw) = l(u) + l(w)$, the results follows immediately from the multiplicativity property for unnormalized standard intertwining operators (Lemma 2.3 (3)) and that for the normalizing factors (an easy consequence of Lemma 5.2). If not, we have $l(u^{-1}) + l(uw) = l(w)$. Then,

$$\mathcal{A}(uw\nu, uw\sigma, u^{-1})\mathcal{A}(\nu, \sigma, uw) = \mathcal{A}(\nu, \sigma, w).$$

Therefore,

$$\mathcal{A}(w\nu, w\sigma, u)\mathcal{A}(uw\nu, uw\sigma, u^{-1})\mathcal{A}(\nu, \sigma, uw) = \mathcal{A}(w\nu, w\sigma, u)\mathcal{A}(\nu, \sigma, w).$$

Now, it follows from Theorem 3.2 and (18),(19) that

$$\mathcal{A}(w\nu, w\sigma, u)\mathcal{A}(uw\nu, uw\sigma, u^{-1}) = I.$$

Thus,

$$\mathcal{A}(\nu, \sigma, uw) = \mathcal{A}(w\nu, w\sigma, u)\mathcal{A}(\nu, \sigma, w),$$

as needed. The general case follows. □

Remarks 5.4. (1) The preceding theorem tells us the normalizations used are suitable for [B3]. In particular, section 6 [B3] tells us that if $\sigma, \hat{\sigma}$ unitary have normalized standard intertwining operators satisfying (2) of Theorem 5.3, then Ind(σ) having an R-group (in a sense made precise in [B3]) implies the same for Ind($\hat{\sigma}$). (E.g., if σ is assumed to be square-integrable, then Ind(σ) automatically has such an R-group, hence so does Ind($\hat{\sigma}$).) Theorem 5.3 tells us that the intertwining operators for Ind(σ) and Ind($\hat{\sigma}$) may be normalized using the same normalizing factor.

(2) The preceding theorem shows that our normlizations satisfy the properties conjectured by Langlands in [L]. These correspond to the properties in Theorem 7.9 [Sh2] and properties (R_2),(R_4) in [A2]. Our normalizations also stasfy (R_1) and (R_3) of [A2] ((R_1) is automatic; (R_3) follows from equation (17) and our construction of normalizing factors). Since we work only with standard parabolic subgroups, we do not need to worry about (R_5) of [A2]. (R_6) is for archimedean places; the status of (R_7) and (R_8) is not clear at this time.

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