Ban, D. and C. Jantzen. (2007) "R-groups and the Action of Intertwining Operators in the Nontempered Case," International Mathematics Research Notices, Vol. 2007, Article ID rnm059, 29 pages.
doi:10.1093/imrn/rnm059

R-groups and the Action of Intertwining Operators in the Nontempered Case

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In this paper, we study the R-group $R_{\psi,\sigma}$ conjectured by Arthur, associated to the representation parabolically induced from a representation σ. We address the question of which properties of the classical Knapp-Stein R-groups carry over to Arthur's setting, with somewhat surprising results. Some of the basic properties fail: the normalized standard intertwining operators $A(r, \sigma)$, $r \in R_{\psi,\sigma}$ in general do not form a basis of the commuting algebra and the components of the induced representation are not in a bijective correspondence with the irreducible representations of $R_{\psi,\sigma}$. However, the action of standard intertwining operators on the induced space has a natural description in terms of representations of a finite group and the corresponding trace formulation holds.

1 Introductory material

1.1 Introduction

This paper focuses on closely related questions about R-groups and the action of (normalized) standard intertwining operators. To set matters up, let F be a p-adic field

Received February 16, 2007; Revised June 26, 2007; Accepted July 10, 2007
Communicated by Freydoon Shahidi

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and G the F-points of a connected quasi-split group defined over F (though we note the results in the third chapter also apply to real groups). Recall that if $P = MN$ is a standard parabolic subgroup of G and σ is a representation of M, we may consider the induced representation $I = \mathrm{Ind}_P^G \sigma$ (normalized parabolic induction). Of particular interest to us in this paper is the case where σ is an irreducible unitary representation, in which case π is also unitary and decomposes as a direct sum.

It is perhaps easiest to start the discussion by reviewing the properties of Knapp-Stein R-groups (cf. [36], [37]; [25] for the real case). Suppose σ is in the discrete series. If W is the Weyl group of G, let $W(\sigma) = \{w \in W \,|\, w\sigma \cong \sigma\}$ (implicit is $w \cdot M = M$). The R-group R_σ is a subgroup of $W(\sigma)$ which determines the intertwining algebra $\mathrm{Hom}_G(I, I)$, among other things. The R-group is defined in terms of Plancherel measures, whence the assumption σ is in the discrete series (see [36], [37] for more details).

For $w \in W(\sigma)$, let $A(\sigma, w) \in \mathrm{Hom}_G(I, I)$ denote the normalized standard intertwining operator (cf. [33],[2]). For purposes of this introduction, we assume trivial cocycle, so the normalized standard intertwining operators satisfy $A(\sigma, w_2)A(\sigma, w_1) = A(\sigma, w_2 w_1)$ for all $w_1, w_2 \in W(\sigma)$. This is known to hold in a number of important situations (e.g. if σ is generic [22] or $F = \mathbb{R}$ [24], [2]). In this case, the R-group has the following properties (cf. [22] or [6]):

Properties 1.1. With notation as above,

(1) The equivalence classes of components of I are parameterized by the irreducible representations $\hat{R}_\sigma$ of R_σ. Further, if I_ρ is a component of I corresponding to $\rho \in \hat{R}_\sigma$, then I_ρ appears with multiplicity $\dim \rho$. That is,

$$I \cong \bigoplus (\dim \rho) I_\rho. \tag{1.1}$$

(2) $\mathrm{Hom}_G(I, I) \cong \mathbb{C}[R_\sigma]$ (with the isomorphism generated by $r \longmapsto A(\sigma, r)$ in one direction).

(3) The operators $A(\sigma, r)$ act on and permute the $\dim \rho$ irreducible subspaces of the I_ρ-isotypic component as the representation ρ. This may be expressed in the following manner: for $\pi \cong I_\rho$ an equivalence class of components of I, let

$$\langle r, \pi \rangle = \mathrm{trace}\, \rho(r). \tag{1.2}$$

Then, for $r \in R_\sigma, f \in C_c^\infty(G)$,

$$\mathrm{trace}\,(A(\sigma, r)I(f)) = \sum_{\pi} \langle r, \pi \rangle\, \mathrm{trace}(\pi(f)), \tag{1.3}$$

where the sum runs over the equivalence classes of components of I. □

We note that if the cocycle is not trivial, the first statement still applies, along with modified versions of the second and third. In what follows, we also use the above numbering for the corresponding properties when the cocycle is nontrivial.

The theory of the R-group has important applications both locally and globally. Locally, the R-group governs the reducibility of induced discrete series, hence plays a key role in the classification of irreducible tempered representations for real and p-adic groups. Globally, the actions of normalized standard intertwining operators—especially as formulated in (3)—arise (as part of a corresponding global formulation) in the trace formula, and hence are of interest in automorphic forms.

On the basis of global considerations, Arthur conjectured a characterization of the R-group in terms of Arthur parameters (A-parameters), which we refer to as the Arthur R-group (cf. [1]). The Arthur R-group is conjectured to exist for more general unitary inducing representations, not just discrete series. Whereas the Knapp-Stein R-group requires the inducing representation to be in the discrete series in order to work with Plancherel measures, A-parameters do not require such a constraint. We note that when the inducing representation is in the discrete series, the Arthur R-group is known to correspond to the Knapp-Stein R-group (with properties (1)–(3) holding) in a number of situations. If $F = \mathbb{R}$, they correspond (cf. [34]). For F p-adic, the Arthur R-group is known to match the Knapp-Stein R-group when G is split and the inducing representation is a character (cf. [22]) or for classical groups when the inducing representation is generic (cf. [9]). We note, however, that the Arthur R-group is formally defined as a subquotient of $W(\sigma)$, not a subgroup, and so does not necessarily give rise to an action of intertwining operators. Thus, when dealing with (3), we work in the context of subgroups of $W(\sigma)$ where we have such actions (and in fact, Arthur's conjecture is for the subgroup $W_{\psi,\sigma}$ of $W(\sigma)$; the Arthur R-group is a quotient of $W_{\psi,\sigma}$).

A number of results on the Arthur R-group concern certain special nontempered representations (cf. [18],[5],[6],[8]). The basic strategy in these results is to use a duality operator (either the Iwahori-Matsumoto involution [16] or the duality of [4], [31]) to relate $\mathrm{Ind}_P^G\sigma$ to its dual $\widehat{\mathrm{Ind}_P^G\sigma} = \mathrm{Ind}_P^G\hat{\sigma}$ when $\mathrm{Ind}_P^G\sigma$ admits a Knapp-Stein R-group. The properties of the classical Knapp-Stein R-groups are then transferred to the nontempered representation. In fact, this approach produces more refined results—the R-groups are isomorphic and the action of normalized standard intertwining operators closely related.

This is not so surprising given that it is expected that duality admits a nice description in terms of A-parameters.

All the results mentioned above rely on relating Arthur R-groups to Knapp-Stein R-groups. The representations considered are induced from either discrete series or duals of discrete series. Arthur's definition, however, applies to a much wider class of inducing representations, and in the present paper we study the general case. We address the question of which of the properties (1)–(3) carry over to this more general setting.

The results are somewhat surprising. The short answer is that properties (1)–(2) fail to hold in general, while (3) does carry over. We note that it is actually property (3) which is singled out by Arthur.

To show that (1)–(2) do not hold in general, we construct an example where they fail. The representation we consider is

$$\pi = \mathrm{Ind}_P^G(St_{GL(2)} \otimes triv_{GL(2)}),$$

where $G = SO(9,F)$, $P = MN$ is the standard parabolic subgroup with Levi factor $M \cong GL(2,F) \times GL(2,F)$, $St_{GL(2)}$ is the Steinberg representation of $GL(2,F)$, and $triv_{GL(2)}$ is the trivial representation of $GL(2,F)$. Using Jacquet module methods (cf. [40],[19],[7],[29], etc.), we show π has three components. On the other hand, from [21], we know the A-parameter ψ of the inducing representation $\sigma = St_{GL(2)} \otimes triv_{GL(2)}$, and hence may calculate the Arthur R-group $R_{\psi,\sigma}$ for π. We find that

$$R_{\psi,\sigma} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

(not a surprising result in the light of [14]). Since π has three components but its Arthur R-group has four elements, we see that properties (1)–(2) fail to hold in general.

On the other hand, property (3) holds for any subgroup R of $W(\sigma)$. Even though properties (1)–(3) have often been discussed together in the context of R-groups, (3) is somehow more basic—just a property of normalized standard intertwining operators. In particular, (1)–(2) are not needed in proving (3).

We now discuss the results section by section. In the next section, we review some background material on A-parameters, Arthur R-groups, etc. Section 2 deals with the example of $\pi = \mathrm{Ind}_P^G(St_{GL(2)} \otimes triv_{GL(2)})$ mentioned above. Section 2.1 contains a review of the classical groups $SO(2n+1,F)$ and their Jacquet modules. In Section 2.2, we use Jacquet module methods to decompose π, showing that it has three components. In

Section 2.3, we calculate its Arthur R-group, showing $R_{\psi,\sigma} \cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. These combine to show that properties (1)–(2) fail to hold for Arthur R-groups in general. In Section Section 3.1, we give an argument for a general version of property (3) based on known R-group arguments (cf. [22],[23],[6]). In Section 3.2, we apply the results of Section 3.1 to a central extension of $R_{\psi,\sigma}$, introduced in [3] to deal with problems introduced by the nontrivial cocyle. In 3.3, we discuss conditions under which these results may be applied to the Arthur R-group, as well as revisiting the example from sections 2.1–2.3.

1.2 Notation and preliminaries

In this section, we introduce notation and recall some results that will be needed in the rest of the paper. Let F be a p-adic field with char F=0. Let $\mathbf{G}$ be a connected reductive algebraic group defined over F, G its F-points. The group $\mathbf{G}$ is said to be quasi-split if it contains a Borel subgroup which is defined over F ([38], Section 3.2). We shall assume that $\mathbf{G}$ is quasi-split over F. Fix a Borel subgroup $B \subset G$ and a maximal split torus $A_0 \subset B$. Let P be a standard parabolic subgroup, i.e. a parabolic subgroup containing B. There exists a unique Levi subgroup in P containing A_0; denote it by M. Then M is a connected reductive F-group. We denote by $i_{G,M}$ the functor of normalized parabolic induction ([10], Section 2.3). Let A be the split component of M. Denote by $X(M)_F$ and $X(A)_F$ the groups of all F-rational characters of M and A, respectively. Let

$$\mathfrak{a}^* = X(M)_F \otimes_{\mathbb{Z}} \mathbb{R} = X(A)_F \otimes_{\mathbb{Z}} \mathbb{R}$$

and $\mathfrak{a}^*_{\mathbb{C}} = \mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$.

We now give a brief discussion of Langlands parameters and Arthur parameters. The reader is referred to [11],[1] for a more detailed discussion.

Let $\hat{G}$ denote the dual of G-the complex connected reductive group whose root datum is dual to that of $\mathbf{G}$. The L-group is then

$${}^LG = \hat{G} \rtimes W_F,$$

where W_F denotes the Weil group of F. Here, the action of W_F on $\hat{G}$ is induced from the action of $Gal(\bar{F}/F)$ on $\mathbf{G}$, where $\bar{F}$ is the algebraic closure of F (cf. [11]).

Let

$$\phi : W_F \times SL(2,\mathbb{C}) \longrightarrow {}^LG$$

be a homomorphism. If ϕ commutes with projections to W_F and satisfies the conditions (1)–(5) of [28], we call ϕ a Langlands parameter (L-parameter). We let $\Phi(G)$ denote the set of all equivalence classes of L-parameters of G. If $\Pi(G)$ denotes the set of equivalence classes of irreducible admissible representations of G, the Langlands correspondence predicts that $\Pi(G)$ may be partitioned into disjoint subsets (L-packets) which are in bijective correspondence with $\Phi(G)$. For $\phi \in \Phi(G)$, we let $\Pi_\phi(G)$ denote the corresponding L-packet. This bijection is expected to have certain number-theoretic properties characterized in terms of L-functions (cf. [11]).

Let $W_F' = W_F \times SL(2,\mathbb{C})$ and let

$$\psi : W_F' \times SL(2,\mathbb{C}) \longrightarrow {}^LG$$

be a homomorphism. If $\psi|_{W_F'}$ is an L-parameter and ψ satisfies

(1) $\psi|_{W_F'}$ is tempered (i.e., the projection of $\psi(W_F)$ to $\hat{G}$ is bounded)

(2) ψ is algebraic on the second $SL(2,\mathbb{C})$,

we call ψ an Arthur parameter (A-parameter). Let $\Psi(G)$ denote the set of equivalence classes of A-parameters. It is expected that $\Pi(G)$ has subsets (A-packets) which are in bijective correspondence with $\Psi(G)$. Unlike the Langlands correspondence, A-packets need not be disjoint. If $\psi \in \Psi(G)$, we let $\Pi_\psi(G)$ denote the corresponding A-packet. This correspondence also has certain number-theoretic properties (cf. [1]).

We now turn to a discussion of Arthur R-groups. Suppose ψ is an A-parameter of G which factors through ${}^LM = \hat{M} \rtimes W_F \subset {}^LG$,

$$\psi : W_F' \times SL(2,\mathbb{C}) \longrightarrow {}^LM \subset {}^LG.$$

The group ${}^LM$ is the L-group of M. It is a Levi subgroup of ${}^LG$ (see [11], See Section 3 for definition of parabolic subgroups and Levi subgroups of ${}^LG$). Then we can regard ψ as an A-parameter of M. Suppose in addition that the image of ψ is not contained in a smaller Levi subgroup (i.e. ψ is an elliptic parameter of M).

Let S_ψ be the centralizer in $\hat{G}$ of the image of ψ and S_ψ^0 its identity component. If T_ψ is a maximal torus of S_ψ^0, define

$$W_\psi = N_{S_\psi}(T_\psi)/Z_{S_\psi}(T_\psi),$$
$$W_\psi^0 = N_{S_\psi^0}(T_\psi)/Z_{S_\psi^0}(T_\psi).$$

Lemma 2.3 of [9] and the discussion on page 326 of [9] imply that W_ψ can be identified with a subgroup of $W(G,A)$.

Let σ be an irreducible unitary representation of M. Assume σ belongs to the A-packet $\Pi_\psi(M)$. If $W(\sigma) = \{w \in W(G, A) \mid w\sigma \cong \sigma\}$, we let

$$W_{\psi,\sigma} = W_\psi \cap W(\sigma)$$
$$W^0_{\psi,\sigma} = W^0_\psi \cap W(\sigma)$$

and take

$$R_{\psi,\sigma} = W_{\psi,\sigma}/W^0_{\psi,\sigma}$$

as the Arthur R-group.

2 An example

In this section, we give an example which shows that not all properties of classical R-groups carry over to the nontempered setting. In particular, we consider the representation $St_{GL(2)} \times triv_{GL(2)} \rtimes 1$ (See Section 2.1 for notation) of $SO(9, F)$. Using Jacquet module methods, we show that this representation has 3 components (cf. Theorem 2.5). A calculation of the Arthur R-group shows that for $\sigma = St_{GL(2)} \otimes triv_{GL(2)}$, we have $R_{\psi,\sigma} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (cf. Section 2.3). In particular, $|R_{\psi,\sigma}|$ does not give the number of components; see Section 2.1; so Properties (1)–(2) from the introduction fail.

2.1 Classical groups

In this section, we review the background on special odd-orthogonal groups which will be needed in the rest of this paper.

We define $\times$ on general linear groups as in [10]: if $\rho_1, \ldots, \rho_k$ are representations of $GL(n_1, F), \ldots, GL(n_k, F)$, let $\rho_1 \times \cdots \times \rho_k$ denote the representation of $GL(n_1 + \cdots + n_k, F)$ obtained by inducing $\rho_1 \otimes \cdots \otimes \rho_k$ from the standard parabolic subgroup of $GL(n_1 + \cdots + n_k, F)$ with Levi factor $GL(n_1, F) \times \cdots \times GL(n_k, F)$.

In much of Section 2.2, we work in the Grothendieck group setting. That is, we work with the semisimplified representation. So, for any representation π and irreducible representation ρ, let $m(\rho, \pi)$ denote the multiplicity of ρ in π. We write $\pi = \pi_1 + \cdots + \pi_k$ if $m(\rho, \pi) = m(\rho, \pi_1) + \cdots + m(\rho, \pi_k)$ for every irreducible ρ. Similarly, we write $\pi \geq \pi_0$ if $m(\rho, \pi) \geq m(\rho, \pi_0)$ for every such ρ. For clarity, in Section 2 (but only

Section 2), we use = when defining something or working in the Grothendieck group; ≅ is used to denote an actual equivalence.

We now turn to odd orthogonal groups. Let

$$J_n = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}$$

denote the $n \times n$ antidiagonal matrix above. Then,

$$SO(2n+1,F) = \{X \in SL(2n+1,F) | {}^{T}X\, J_{2n+1}\, X = J_{2n+1}\}.$$

Note that the Weyl group is W ={permutations and sign changes on n letters}.

We take as minimal parabolic subgroup in $SO(2n+1,F)$ the subgroup $P_\emptyset$ consisting of upper triangular matrices. Let $\alpha = (n_1, \ldots, n_k)$ be an ordered partition of a non-negative integer $m \leq n$ into positive integers. Let $M_\alpha \subset SO(2n+1,F)$ be the subgroup

$$M_\alpha = \left\{ \begin{pmatrix} X_1 & & & & & & \\ & \ddots & & & & & \\ & & X_k & & & & \\ & & & X & & & \\ & & & & {}^{\tau}X_k & & \\ & & & & & \ddots & \\ & & & & & & {}^{\tau}X_1 \end{pmatrix} \middle| X_i \in GL(n_i,F),\ X \in SO(2(n-m)+1,F) \right\},$$

where ${}^{\tau}X = J\,{}^{T}X^{-1}J$. Then $P_\alpha = M_\alpha P_\emptyset$ is a parabolic subgroup of $SO(2n+1,F)$ and every parabolic subgroup is of this form (up to conjugation). For $\alpha = (n_1, \ldots, n_k)$, let $\rho_1, \ldots, \rho_k$ be representations of $GL(n_1,F), \ldots, GL(n_k,F)$, respectively, and σ a representation of $SO(2(n-m)+1,F)$. Let $\rho_1 \times \cdots \times \rho_k \rtimes \sigma$ denote the representation of $SO(2n+1,F)$ obtained by inducing the representation $\rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma$ of M_α (extended trivially to P_α). If $m = n$, we write $\rho_1 \times \cdots \times \rho_k \rtimes 1$, where 1 denotes the trivial representation of $SO(1,F)$ (trivial group).

We recall some structures which will be useful later (cf. Section 1 of [42] and section 4 of [39]). Let $R(GL(n,F))$ (resp., $R(SO(2n+1,F))$) denote the Grothendieck group of the category of all smooth finite-length $GL(n,F)$-modules (resp., $SO(2n+1,F)$-modules). Set $R = \oplus_{n\geq 0} R(GL_n(F))$ and $R[S] = \oplus_{n\geq 0} R(SO(2n+1,F))$. The operators $\times$ and $\rtimes$ lift naturally to

$$\times : R \otimes R \longrightarrow R \quad \text{and} \quad \rtimes : R \otimes R[S] \longrightarrow R[S].$$

With these multiplications, R becomes an algebra and $R[S]$ a module over R.

Next, we introduce some convenient shorthand for Jacquet modules (cf. [39]). If π is a representation of some $SO(2n+1,F)$ and α is a partition of $m \leq n$, let $s_\alpha(\pi)$ denote the Jacquet module with respect to M_α. Note that, by abuse of notation, we also allow s_α to be applied to representations M_β when $M_\beta > M_\alpha$ (cf. Section 2.1, [10]). We will occasionally use similar notation for representations of $GL(n,F)$: if $\alpha = (n_1, \ldots, n_k)$ is a partition of $m \leq n$, $GL(n,F)$ has a standard parabolic subgroup with Levi factor $L_\alpha \cong GL(n_1,F) \times \cdots \times GL(n_k,F) \times GL(n-m,F)$ (L_α consists of block-diagonal matrices; the corresponding parabolic subgroup of block upper triangular matrices). If π is a representation of $GL(n,F)$, we let $r_\alpha(\pi)$ denote the Jacquet module of π with respect to L_α.

We now give the Langlands classification for $GL(n,F)$ and $SO(2n+1,F)$ (cf. [12], [35],[26]; for real groups, see [27]). As in [42], let $\nu = |det|$ on $GL(n,F)$ (with the value of n clear from context). Suppose that δ is an irreducible essentially square integrable representation of $GL(n,F)$. Then, there is an $\varepsilon(\delta) \in \mathbb{R}$ such that $\nu^{-\varepsilon(\delta)}\delta$ is unitarizable. For $GL(n,F)$, let $\delta_1, \ldots, \delta_k$ be irreducible, essentially square, integrable representations satisfying $\varepsilon(\delta_1) \leq \cdots \leq \varepsilon(\delta_k)$. Then, $\delta_1 \times \cdots \times \delta_k$ has a unique irreducible subrepresentation (Langlands subrepresentation) which we denote by $\mathcal{L}(\delta_1, \ldots, \delta_k)$. For $SO(2n+1,F)$, let $\delta_1, \ldots, \delta_k$ be irreducible essentially square integrable representations satisfying $\varepsilon(\delta_1) \leq \cdots \leq \varepsilon(\delta_k) < 0$ and τ a tempered representation of $SO(2(n-m)+1,F)$ where $m = n_1 + \cdots + n_k$. Then, $\delta_1 \times \cdots \times \delta_k \rtimes \tau$ has a unique irreducible subrepresentation which we denote by $L(\delta_1, \ldots, \delta_k; \tau)$. If $m = n$, we write $L(\delta_1, \ldots, \delta_k; 1)$ simply as $L(\delta_1, \ldots, \delta_k)$. Every irreducible admissible representation of $GL(n,F)$ or $SO(2n+1,F)$ appears as such a Langlands subrepresentation; the data $\delta_1 \otimes \cdots \otimes \delta_k$ (resp., $\delta_1 \otimes \cdots \otimes \delta_k \otimes \tau$) is unique up to the order in which δ_i's having the same value of $\varepsilon(\delta_i)$ appear. More precisely, if $L(\delta_1, \ldots, \delta_k; \tau) \cong L(\delta'_1, \ldots, \delta'_\ell; \tau')$, then $k = \ell$, $\tau \cong \tau'$ and there exists a permutation p of $\{1, \ldots, k\}$ such that $\delta_{p(i)} \cong \delta'_i$ and $\varepsilon(\delta_i) = \varepsilon(\delta_{p(i)})$, $\forall i$. Note that we use the Langlands classification in the subrepresentation setting rather than the quotient setting for the following reason: in the subrepresentation setting, $\delta_1 \otimes \cdots \otimes \delta_k \otimes \tau$ will lie in the appropriate Jacquet module of $L(\delta_1, \ldots, \delta_k; \tau)$ (by Frobenius reciprocity).

We now recall some structure theory related to Jacquet modules. We discuss only the calculational aspects which are needed in what follows; the reader is referred to [42] and [39] for structure-theoretic interpretations.

Definition 2.1. (1) If τ is a representation of $GL(n,F)$, set

$$m^*\tau = \sum_{i=0}^{n} r_{(i)}\tau$$

(2) If π is a representation of $SO(2n+1,F)$, set

$$\mu^*\pi = \sum_{i=0}^{n} s_{(i)}\pi.$$

If τ_1 and τ_2 are representations of $GL(n_1,F), GL(n_2,F)$, respectively, let $s(\tau_1 \otimes \tau_2) = \tau_2 \otimes \tau_1$ and $m(\tau_1 \otimes \tau_2) = \tau_1 \times \tau_2$. If τ is a representation of $GL(n,F)$ and ϑ is a representation of $SO(2m+1,F)$, define $\rtimes$ on $(R\otimes R)\otimes(R\otimes R[S])$ by $(\tau_1\otimes\tau_2)\rtimes(\tau\otimes\vartheta) = (\tau_1 \times \tau) \otimes (\tau_2 \rtimes \vartheta)$. Set $M_S^* = (m\otimes 1)\circ(\tilde{}\otimes m^*)\circ s\circ m^*$ ($\tilde{}$ denotes contragredient).

Theorem 2.2 (Tadić)**.** If τ is a representation of $GL(n_1F)$ and ϑ a representation of $SO(2m+1_1F)$, then

$$\mu^*(\tau \rtimes \vartheta) = M_S^*(\tau) \rtimes \mu^*(\vartheta). \qquad \square$$

Proof. See [39]. ■

We mention the counterpart for general linear groups: if we define $\times$ on $R\otimes R$ by $(\tau_1\otimes\tau_2)\times(\tau_1'\otimes\tau_2') = (\tau_1\times\tau_1')\otimes(\tau_2\times\tau_2')$, then $m^*(\pi_1\times\pi_2) = m^*(\pi_1)\times m^*(\pi_2)$. See section 1.7 of [42].

2.2 Decomposition of $St_{GL(2)} \times triv_{GL(2)} \rtimes 1$

We analyze the induced representation $St_{GL(2)} \times triv_{GL(2)} \rtimes 1$ using Jacquet module methods (cf. [40],[19],[7],[29], etc., for similar arguments). Before getting into the analysis, we first introduce a couple of representations which will occur.

By Theorem 4.5 [17] or Lemma 4.2 [19], the degenerate principal series $triv_{GL(2)} \rtimes triv_{SO(3)}$ is irreducible. Therefore, by duality (cf. [4],[31]) or the Iwahori-Matusmoto involution, we see that

$$\mathcal{S} = St_{GL(2)} \rtimes St_{SO(3)}$$

is an irreducible tempered representation of $SO(7,F)$. Similarly, the degenerate principal series $triv_{GL(2)} \rtimes 1$ decomposes as $L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}}) + L(\nu^{-\frac{1}{2}};St_{SO(3)})$ (cf. Theorem 4.1 [17] or Proposition 3.6 [19]). Taking duals,

$$St_{GL(2)} \rtimes 1 = \mathcal{T}_1 + \mathcal{T}_2,$$

where $\mathcal{T}_1,\mathcal{T}_2$ are tempered representations of $SO(5,F)$ with $s_{(1)}\mathcal{T}_1 = \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}})$ and $s_{(1)}\mathcal{T}_2 = \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}) + 2\nu^{\frac{1}{2}} \otimes St_{SO(3)}$. We summarize:

Lemma 2.3. We have the following:

(1) $\nu^{-\frac{1}{2}} \rtimes triv_{SO(3)} = L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}}) + \mathcal{T}_1$

(2) $triv_{GL(2)} \rtimes 1 = L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}}) + L(\nu^{-\frac{1}{2}};St_{SO(3)})$

(3) $\nu^{-\frac{1}{2}} \rtimes St_{SO(3)} = L(\nu^{-\frac{1}{2}};St_{SO(3)}) + \mathcal{T}_2$

(4) $St_{GL(2)} \rtimes 1 = \mathcal{T}_1 + \mathcal{T}_2$.

The Jacquet modules for the irreducible representations appearing above are given in the table at the end of this section. □

Lemma 2.4. We have the following:

(1) $\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}};St_{SO(3)}) = L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}};St_{SO(3)})$

(2) $\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}}) = L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}}) + L(\nu^{-\frac{1}{2}};\mathcal{T}_2)$

(3) $St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}) = L(\nu^{-\frac{1}{2}};\mathcal{T}_1) + L(\nu^{-\frac{1}{2}};\mathcal{T}_2)$

(4) $\nu^{-\frac{1}{2}} \rtimes \mathcal{T}_1 = L(\nu^{-\frac{1}{2}};\mathcal{T}_1)$

(5) $\nu^{-\frac{1}{2}} \rtimes \mathcal{T}_2 = L(\nu^{-\frac{1}{2}};\mathcal{T}_2) + \mathcal{S}$.

The Jacquet modules for the irreducible representations appearing above are given in the table at the end of this section. □

Proof. For (1), observe that since $s_{(1)}L(\nu^{-\frac{1}{2}};St_{SO(3)}) = \nu^{-\frac{1}{2}} \otimes St_{SO(3)}$, by Theorem 2.2,

$$\begin{aligned} s_{(1)}\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}};St_{SO(3)}) &= \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};St_{SO(3)}) + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};St_{SO(3)}) \\ &\quad + \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \rtimes St_{SO(3)} \\ &= 2\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};St_{SO(3)}) + \nu^{-\frac{1}{2}} \otimes \mathcal{T}_2 + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};St_{SO(3)}) \end{aligned}$$

(cf. Lemma 2.3). Now,

$$\begin{aligned} s_{(1,1,1)}\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};St_{SO(3)}) &= \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \\ s_{(1,1,1)}\mathcal{T}_2 &= \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} + 2\nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \\ s_{(1,1,1)}\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};St_{SO(3)}) &= \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}. \end{aligned}$$

Let π_1 be an irreducible subquotient of $\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)})$ such that $s_{(1)}\pi_1 \geq \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; St_{SO(3)})$. Observe that the only irreducible representation of $GL(3,F)$ having $\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}$ in its Jacquet module is $\mathcal{L}(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}) = \nu^{-\frac{1}{2}} \times \mathcal{L}(\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}})$. Since $r_{(1,1)}$ $\mathcal{L}(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}) = 2\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} + \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}$, we have $s_{(1,1,1)}\pi_1 \geq 2\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} + \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}$. Therefore, $s_{(1)}\pi_1 \geq 2\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) + \nu^{-\frac{1}{2}} \otimes \mathcal{T}_2$. However, this then tells us $s_{(1,1,1)}\pi_1 \geq \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{\frac{1}{2}}$ (consider $s_{(1,1,1)}\nu^{-\frac{1}{2}} \otimes \mathcal{T}_2$). Now, the only irreducible representation of $GL(3,F)$ having $\nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{\frac{1}{2}}$ in its Jacquet module is $\mathcal{L}(\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}, \nu^{\frac{1}{2}}) = \mathcal{L}(\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}) \times \nu^{\frac{1}{2}}$. Since $r_{(1,1)}\mathcal{L}(\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}, \nu^{\frac{1}{2}}) = 2\nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{\frac{1}{2}} + \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}$, we have $s_{(1,1,1)}\pi_1 \geq \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}}$. Therefore, $s_{(1)}\pi_1$ also contains $\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; St_{SO(3)})$. As $s_{(1)}\pi_1$ accounts for all of $s_{(1)}\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)})$, we see that π_1 is the only component, i.e. $\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)})$ is irreducible. Further, since

$$\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) \hookrightarrow \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \rtimes St_{SO(3)}$$

has $L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; St_{SO(3)})$ as unique irreducible subrepresentation (by the Langlands classification), we must have

$$\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; St_{SO(3)}),$$

as claimed.

For (2), since $s_{(1)}L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) = 2\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}) + \nu^{-\frac{1}{2}} \otimes St_{SO(3)}$, Theorem 2.2 tells us

$$\begin{aligned} s_{(1)}\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) &= \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + 2\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}) \\ &\quad + \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \rtimes St_{SO(3)} \\ &= 3\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) + 2\nu^{-\frac{1}{2}} \otimes \mathcal{T}_1 \\ &\quad + \nu^{-\frac{1}{2}} \otimes \mathcal{T}_2 + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}), \end{aligned}$$

by Lemma 2.3. Let π_1 be an irreducible subquotient of $\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$ such that $s_{(1)}\pi_1 \geq \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$. Then, $s_{(1,1,1)}\pi_1 \geq \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}$. By Frobenius reciprocity,

$$\mathrm{Hom}_G(\pi_1, \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \rtimes 1) \cong \mathrm{Hom}_A(s_{(1,1,1)}\pi_1, \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}),$$

which is nonzero by a central character argument (cf. Lemma 8.2 [15] or section 1.3 [41]). Therefore,

$$\pi_1 \hookrightarrow \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \rtimes 1.$$

Since $\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \rtimes 1$ has unique irreducible subrepresentation $L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$ (by the Langlands classification), we see that $\pi_1 = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$. Now, $L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) = triv_{GL(2)} \rtimes triv_{SO(3)}$ and has

$$s_{(1)}L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) = 3\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) + 2\nu^{-\frac{1}{2}} \otimes \mathcal{T}_1$$

(which follows from Theorem 2.2 and Lemma 2.3). We have not yet accounted for $\nu^{-\frac{1}{2}} \otimes \mathcal{T}_2$ and $\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$. An $s_{(1,1,1)}$ argument like that used for part (1) shows that if π_2 is the irreducible subquotient with $s_{(1)}\pi_2 \geq \nu^{-\frac{1}{2}} \otimes \mathcal{T}_2$, then $s_{(1)}\pi_2$ must also contain $\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$. Further, by central character considerations like those used above,

$$0 \neq \mathrm{Hom}_{M_{(1)}}(r_{M_{(1)},G}\pi_2, \nu^{-\frac{1}{2}} \otimes \mathcal{T}_2) \cong \mathrm{Hom}_G(\pi_2, \nu^{-\frac{1}{2}} \rtimes \mathcal{T}_2),$$

so $\pi_2 \hookrightarrow \nu^{-\frac{1}{2}} \rtimes \mathcal{T}_2$. Thus, the Langlands classification tells us $\pi_2 = L(\nu^{-\frac{1}{2}}; \mathcal{T}_2)$, as claimed.

We now turn to (3). By Theorem 2.2, we have

$$\begin{aligned} s_{(1)}St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}) &= 2\nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}) + \nu^{-\frac{1}{2}} \otimes St_{GL(2)} \rtimes 1 \\ &= 2\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + 2\nu^{\frac{1}{2}} \otimes \mathcal{T}_1 + \nu^{-\frac{1}{2}} \otimes \mathcal{T}_1 + \nu^{-\frac{1}{2}} \otimes \mathcal{T}_2 \end{aligned}$$

by Lemma 2.3. Observe that $L(\nu^{-\frac{1}{2}}; \mathcal{T}_1)$ is the unique irreducible subrepresentation of $\nu^{-\frac{1}{2}} \rtimes \mathcal{T}_1$ (by the Langlands classification) and $s_{(1)}L(\nu^{-\frac{1}{2}}; \mathcal{T}_1)$ does not contain $\nu^{-\frac{1}{2}} \otimes \mathcal{T}_2$ (since $s_{(1)}\nu^{-\frac{1}{2}} \rtimes \mathcal{T}_1$ does not–an easy calculation, or cf. Lemma 3.4 [20]). The corresponding statement holds for $L(\nu^{-\frac{1}{2}}; \mathcal{T}_2)$. Let π_1 be the component of $St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}})$ such that $s_{(1)}\pi_1 \geq \nu^{-\frac{1}{2}} \otimes \mathcal{T}_1$. Then, $s_{(1)}\pi \not\geq \nu^{-\frac{1}{2}} \otimes \mathcal{T}_2$. Therefore, by the same central character considerations used in (2), we have $\pi_1 \hookrightarrow \nu^{-\frac{1}{2}} \rtimes \mathcal{T}_1$, hence $\pi_1 = L(\nu^{-\frac{1}{2}}; \mathcal{T}_1)$. Similarly, we see $\pi_2 = L(\nu^{-\frac{1}{2}}; \mathcal{T}_2)$ is also a component of $St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}})$. Further, since $s_{(1)}\pi_2 = \nu^{-\frac{1}{2}} \otimes \mathcal{T}_2 + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$ (cf. (2) above), we have

$$s_{(1)}\pi_1 \leq \nu^{-\frac{1}{2}} \otimes \mathcal{T}_1 + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + 2\nu^{\frac{1}{2}} \otimes \mathcal{T}_1.$$

Again, an $s_{(1,1,1)}$ argument like that used in part (1) tells us

$$s_{(1)}\pi_1 = \nu^{-\frac{1}{2}} \otimes \mathcal{T}_1 + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + 2\nu^{\frac{1}{2}} \otimes \mathcal{T}_1.$$

Thus, π_1 and π_2 are the only components, and so

$$St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}) = L(\nu^{-\frac{1}{2}}; \mathcal{T}_1) + L(\nu^{-\frac{1}{2}}; \mathcal{T}_2),$$

as claimed.

For (4) and (5), note that the duals (in the sense of [4],[31]) to $\nu^{-\frac{1}{2}} \rtimes \mathcal{T}_1$ and $\nu^{-\frac{1}{2}} \rtimes \mathcal{T}_2$ are $\nu^{\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)})$ and $\nu^{\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$, respectively, which have been analyzed in (1) and (2). As $\nu^{\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)})$ is irreducible, so is $\nu^{-\frac{1}{2}} \rtimes \mathcal{T}_1$. Therefore,

$$\nu^{-\frac{1}{2}} \rtimes \mathcal{T}_1 = L(\nu^{-\frac{1}{2}}; \mathcal{T}_1).$$

Similarly, as $\nu^{\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$ has two irreducible subquotients, so does $\nu^{-\frac{1}{2}} \rtimes \mathcal{T}_2$. One of them must be $L(\nu^{-\frac{1}{2}}; \mathcal{T}_2)$. Since $triv_{GL(2)} \rtimes triv_{SO(3)}$ is the other irreducible subquotient of $\nu^{\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$, its dual—i.e. $St_{GL(2)} \rtimes St_{SO(3)} = \mathcal{S}$-is the other irreducible subquotient of $\nu^{-\frac{1}{2}} \rtimes \mathcal{T}_2$. This finishes the proof of the lemma. ■

Theorem 2.5.

$$St_{GL(2)} \times triv_{GL(2)} \rtimes 1 = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_1) + L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_2) + L(\nu^{-\frac{1}{2}}; \mathcal{S})$$

The Jacquet modules of the components are given in the table at the end of this section. □

Proof. By Lemma 2.3,

$$St_{GL(2)} \times triv_{GL(2)} \rtimes 1 = St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}).$$

Thus, it suffices to analyze these two induced representations.

Let us start with $St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)})$. By Theorem 2.2,

$$\begin{aligned} s_{(1)} St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) &= 2\nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) + \nu^{-\frac{1}{2}} \otimes St_{GL(2)} \rtimes St_{SO(3)} \\ &= 2\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; St_{SO(3)}) + \nu^{-\frac{1}{2}} \otimes \mathcal{S} \end{aligned}$$

by Lemma 2.4. Now, let π_1 be a component of $St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)})$ such that $s_{(1)}\pi_1 \geq \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; St_{SO(3)})$. Then,

$$\begin{gathered} s_{(1,1,1)}\pi_1 \geq \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes St_{SO(3)} \\ \Downarrow \\ s_{(3)}\pi_1 \geq \mathcal{L}(\nu^{-\frac{1}{2}}, St_{(GL(2)}) \otimes St_{SO(3)}, \end{gathered}$$

since $\mathcal{L}(\nu^{-\frac{1}{2}}, St_{GL(2)}) = \nu^{-\frac{1}{2}} \times St_{GL(2)}$ is the only irreducible representation of $GL(3,F)$ containing $\nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}$ in its Jacquet module. Now, the Jacquet module of $\mathcal{L}(\nu^{-\frac{1}{2}}, St_{GL(2)})$

also contains $\nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}}$. Therefore,

$$\begin{gathered} s_{(1,1,1)}\pi_1 \geq \nu^{-\frac{1}{2}} \otimes \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes St_{SO(3)} \\ \Downarrow \\ s_{(1)}\pi_1 \geq \nu^{-\frac{1}{2}} \otimes \mathcal{S}. \end{gathered}$$

Thus, there is only the single component π_1, i.e. $St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)})$ is irreducible. By the now usual central character considerations, $\pi_1 \hookrightarrow \nu^{-\frac{1}{2}} \rtimes \mathcal{S}$. Therefore, by the Langlands classification,

$$St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) = \pi_1 = L(\nu^{-\frac{1}{2}}; \mathcal{S}).$$

We now turn to $\pi = St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$. By Theorem 2.2,

$$\begin{aligned} s_{(1)}St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) &= 2\nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + 2\nu^{-\frac{1}{2}} \otimes St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}) \\ &\quad + \nu^{-\frac{1}{2}} \otimes St_{GL(2)} \rtimes St_{SO(3)} \\ &= 2\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) + 2\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_2) \\ &\quad + 2\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_1) + 2\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_2) + \nu^{-\frac{1}{2}} \otimes \mathcal{S} \end{aligned}$$

by Lemma 2.4. We observe that the only terms of the form $\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \tau$ (τ irreducible) in $s_{(1,1)}\pi$ are $2\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \mathcal{T}_1$ and $2\nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \mathcal{T}_2$. Therefore, by central character considerations,

$$0 \neq \mathrm{Hom}_{\mathrm{M}_{(1,1)}}(r_{\mathrm{M}_{(1,1)},\mathrm{G}}\pi, \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \otimes \mathcal{T}_i) \cong \mathrm{Hom}_G(\pi, \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \rtimes \mathcal{T}_i)$$

for some i. That is, there is component π_1 of π such that $\pi_1 \hookrightarrow \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \rtimes \mathcal{T}_i$; by the Langlands classification, $\pi_1 = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; T_i)$. This shows $L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_i)$ is a component of $St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$ for some i. We next show that this actually holds for both $i = 1, 2$.

Observe that

$$\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_i) \hookrightarrow \nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \rtimes \mathcal{T}_i.$$

As $L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_i)$ is the unique irreducible subrepresentation of $\nu^{-\frac{1}{2}} \times \nu^{-\frac{1}{2}} \rtimes \mathcal{T}_i$, we must have

$$L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_i) \hookrightarrow \nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}}; \mathcal{T}_i).$$

Therefore, $s_{(1)}L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}};\mathcal{T}_i) \leq s_{(1)}\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}};\mathcal{T}_i)$. Now, by Theorem 2.2 and Lemma 2.4,

$$\begin{aligned} s_{(1)}\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}};\mathcal{T}_1) &= \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};\mathcal{T}_1) + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};\mathcal{T}_1) + \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \rtimes \mathcal{T}_1 \\ &\quad + 2\nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \rtimes \mathcal{T}_1 + \nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}}) \\ &= 2\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};\mathcal{T}_1) + 3\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};\mathcal{T}_1) + \nu^{\frac{1}{2}} L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}}) \\ &\quad + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};\mathcal{T}_2) \end{aligned}$$

and

$$\begin{aligned} s_{(1)}\nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}};\mathcal{T}_2) &= \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};\mathcal{T}_2) + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};\mathcal{T}_2) + \nu^{-\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \rtimes \mathcal{T}_2 \\ &\quad + \nu^{\frac{1}{2}} \otimes \nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}}) \\ &= 2\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};\mathcal{T}_2) + 2\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};\mathcal{T}_2) + \nu^{-\frac{1}{2}} \otimes \mathcal{S} \\ &\quad + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}}). \end{aligned}$$

Suppose, e.g. $\pi_1 = L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}};\mathcal{T}_1)$. The above calculations then show $\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};\mathcal{T}_2) \not\leq s_{(1)}\pi_1$. So, we let π_2 be a component such that $s_{(1)}\pi_2 \geq \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};\mathcal{T}_2)$. The same central character/Frobenius reciprocity argument used above then tells us $\pi_2 \hookrightarrow \nu^{-\frac{1}{2}} \rtimes L(\nu^{-\frac{1}{2}};\mathcal{T}_2)$, hence $\pi_2 = L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}};\mathcal{T}_2)$. We note that the same considerations would apply if we started with $\pi_1 = L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}};\mathcal{T}_2)$. Thus, both $L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}};\mathcal{T}_1)$ and $L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}};\mathcal{T}_2)$ are components of $St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}})$, as claimed.

It remains to show that $L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}};\mathcal{T}_1)$ and $L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}};\mathcal{T}_2)$ are the only components. This may be done using the same sort of $s_{(1,1,1)}$ considerations applied in showing the irreducibility of $St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}};St_{SO(3)})$. ■

Jacquet modules:

$$\begin{aligned} s_{(1)}L(\nu^{-\frac{1}{2}};\mathcal{S}) &= \nu^{-\frac{1}{2}} \otimes \mathcal{S} + 2\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}};St_{SO(3)}) \\ s_{(1)}L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}};\mathcal{T}_1) &= 2\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};\mathcal{T}_1) + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}}) \\ s_{(1)}L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}}\mathcal{T}_2) &= 2\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};\mathcal{T}_2) + \nu^{-\frac{1}{2}} \otimes \mathcal{S} + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}}) \\ &\quad + 2\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};\mathcal{T}_2) \\ s_{(1)}L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}}) &= 3\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}}) + \nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};St_{SO(3)}) + 2\nu^{-\frac{1}{2}} \otimes \mathcal{T}_1 \\ s_{(1)}L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}};St_{SO(3)}) &= 2\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};St_{SO(3)}) + \nu^{-\frac{1}{2}} \otimes \mathcal{T}_2 + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}};St_{SO(3)}) \\ s_{(1)}L(\nu^{-\frac{1}{2}};\mathcal{T}_1) &= \nu^{-\frac{1}{2}} \otimes \mathcal{T}_1 + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}}) + 2\nu^{\frac{1}{2}} \otimes \mathcal{T}_1 \\ s_{(1)}L(\nu^{-\frac{1}{2}};\mathcal{T}_2) &= \nu^{-\frac{1}{2}} \otimes \mathcal{T}_2 + \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}},\nu^{-\frac{1}{2}}) \end{aligned}$$

$$\begin{aligned}
s_{(1)}\mathcal{S} &= 2\nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) + \nu^{\frac{1}{2}} \otimes \mathcal{T}_1 + 3\nu^{\frac{1}{2}} \otimes \mathcal{T}_2 \\
s_{(1)}L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) &= 2\nu^{-\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}) + \nu^{-\frac{1}{2}} \otimes St_{SO(3)} \\
s_{(1)}L(\nu^{-\frac{1}{2}}; St_{SO(3)}) &= \nu^{-\frac{1}{2}} \otimes St_{SO(3)} \\
s_{(1)}\mathcal{T}_1 &= \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}) \\
s_{(1)}\mathcal{T}_2 &= \nu^{\frac{1}{2}} \otimes L(\nu^{-\frac{1}{2}}) + 2\nu^{\frac{1}{2}} \otimes St_{SO(3)}
\end{aligned}$$

Remark 2.6. The calculations done in this section can be generalized to (ρ, σ) having reducibility at 1/2 (cf. [40] for more details).

2.3 Calculation of the Arthur R-group

We now show that the Arthur R-group for $St_{GL(2)} \times triv_{GL(2)} \rtimes 1$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Now, $\sigma = St_{GL(2)} \otimes triv_{GL(2)}$ is a representation of the standard parabolic subgroup with Levi factor $M \cong GL(2,F) \times GL(2,F) \subset SO(9,F)$. Then, $\hat{M} \cong GL(2,\mathbb{C}) \times GL(2,\mathbb{C}) \subset Sp(8,\mathbb{C})$, where

$$Sp(2n,\mathbb{C}) = \left\{ X \in GL(2n,\mathbb{C}) \mid {}^{T}X \begin{pmatrix} & -J \\ J & \end{pmatrix} X = \begin{pmatrix} & -J \\ J & \end{pmatrix} \right\}.$$

The L-parameter of σ may be determined from [42], section 10. We apply formula (15) on page 340 of [9] to obtain the associated A-parameter

$$\psi : W_F \times SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \longrightarrow \hat{M} \subset Sp(8,\mathbb{C}).$$

In accordance with notation of [21], we write it in the form

$$\psi = (1 \otimes S_2 \otimes S_1) \oplus (1 \otimes S_1 \otimes S_2) \oplus (1 \otimes S_1 \otimes S_2) \oplus (1 \otimes S_2 \otimes S_1),$$

where 1 denotes the trivial representation of W_F and S_n the standard irreducible n-dimensional algebraic representation of $SL(2,\mathbb{C})$. In particular, S_1 is the trivial representation and S_2 may be chosen to have $S_2(x) = x$ for $x \in SL(2,\mathbb{C})$. Thus,

$$image\,\psi = \left\{ \begin{pmatrix} X_1 & & & \\ & X_2 & & \\ & & {}^{\tau}X_2^{-1} & \\ & & & {}^{\tau}X_1^{-1} \end{pmatrix} \mid X_1, X_2 \in SL(2,\mathbb{C}) \right\},$$

where ${}^{\tau}$ denotes transpose with respect to the antidiagonal.

Given the form of *image* ψ, its centralizer has the form

$$S_\psi = \begin{pmatrix} A_1 & & & B_1 \\ & A_2 & B_2 & \\ & C_2 & D_2 & \\ C_1 & & & D_1 \end{pmatrix},$$

with A_i, B_i, C_i, D_i 2×2 matrices. In particular, $W_\psi, W_\psi^0 \subset W(\sigma)$. Let

$$E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so that $EX = {}^\tau X^{-1}E$ for $X \in SL(2,\mathbb{C})$ (i.e., E gives the equivalence of representations $S_2 \cong {}^\tau S_2^{-1}$, in the obvious notation). Now,

$$\begin{pmatrix} X_1 & & & \\ & X_2 & & \\ & & {}^\tau X_2^{-1} & \\ & & & {}^\tau X_1^{-1} \end{pmatrix} \begin{pmatrix} A_1 & & & B_1 \\ & A_2 & B_2 & \\ & C_2 & D_2 & \\ C_1 & & & D_1 \end{pmatrix} = \begin{pmatrix} A_1 & & & B_1 \\ & A_2 & B_2 & \\ & C_2 & D_2 & \\ C_1 & & & D_1 \end{pmatrix} \times \begin{pmatrix} X_1 & & & \\ & X_2 & & \\ & & {}^\tau X_2^{-1} & \\ & & & {}^\tau X_1^{-1} \end{pmatrix}$$

immediately gives

$$\begin{aligned} X_iA_i &= A_iX_i & {}^\tau X_i^{-1}C_i &= C_iX_i \\ X_iB_i &= B_i{}^\tau X_i^{-1} & {}^\tau X_i^{-1}D_i &= D_i{}^\tau X_i^{-1}. \end{aligned}$$

Therefore, by Schur's lemma,

$$\begin{aligned} A_i &= \lambda_{A_i}I & C_i &= \lambda_{C_i}E \\ B_i &= \lambda_{B_i}E & D_i &= \lambda_{D_i}I \end{aligned}$$

for scalars $\lambda_{A_i}, \lambda_{B_i}, \lambda_{C_i}, \lambda_{D_i}$ (with I the 2×2 identity matrix).

Now, to have

$$\begin{pmatrix} \lambda_{A_1} I & & & \lambda_{B_1} E \\ & \lambda_{A_2} I & \lambda_{B_2} E & \\ & \lambda_{C_2} E & \lambda_{D_2} I & \\ \lambda_{C_1} E & & & \lambda_{D_1} I \end{pmatrix} \in Sp(8, \mathbb{C}),$$

we must have $\lambda_{A_i}\lambda_{C_i} = \lambda_{B_i}\lambda_{D_i} = 0$ and $\lambda_{A_i}\lambda_{D_i} + \lambda_{B_i}\lambda_{C_i} = 1$. Thus,

$$\begin{aligned} S_\psi &= \left\{ \begin{pmatrix} \lambda_{A_1} I & & & \\ & \lambda_{A_2} I & & \\ & & \lambda_{A_2}^{-1} I & \\ & & & \lambda_{A_1}^{-1} I \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} \lambda_{A_1} I & & & \\ & & \lambda_{B_2} E & \\ & \lambda_{B_2}^{-1} E & & \\ & & & \lambda_{A_1}^{-1} I \end{pmatrix} \right\} \\ &\cup \left\{ \begin{pmatrix} & & & \lambda_{B_1} E \\ & \lambda_{A_2} I & & \\ & & \lambda_{A_2}^{-1} I & \\ \lambda_{B_1}^{-1} E & & & \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} & & & \lambda_{B_1} E \\ & & \lambda_{B_2} E & \\ & \lambda_{B_2}^{-1} E & & \\ \lambda_{B_1}^{-1} E & & & \end{pmatrix} \right\}. \end{aligned}$$

Therefore, $W_{\psi,\sigma} = W_\psi \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $W^0_{\psi,\sigma} = W^0_\psi = 1$. Thus, the Arthur R-group is

$$R_{\psi,\sigma} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

as claimed.

3 Action of intertwining operators

In this section, we show that the analog of (3) of Properties 1.1 holds for intertwining operators coming from subgroups of $W(\sigma)$. More precisely, we show the appropriate generalization to accommodate nontrivial cocycle holds. Our approach is based on arguments used for Knapp-Stein R-groups (cf. [22],[23],[6]) and the adaptations to nontrivial cocycle from [3]. We remark that the results in this section can also be applied to real groups. G, M are as in Section 1.2

3.1 Actions on intertwining algebras

In this section, we do a general version of (3) in properties 1.1, which will be applied in the next section. Note that these algebraic arguments are based on known results for R-groups (cf. [22],[23],[6]), but incorporating a point-of-view more like that of [3]. We have used a notation suggestive of an R-groups, though the results will be applied more generally.

Let σ be an irreducible unitary representation of M and consider the induced representation $I = i_{G,M}(\sigma)$. Let V denote the representation space of I. Let R be a finite group, $\mathbb{C}[R]$ its group algebra. Suppose

$$r \mapsto A(r)$$

extended linearly to $\mathbb{C}[R]$ gives a homomorphism of $\mathbb{C}[R]$ into the intertwining algebra $C(\sigma) = \mathrm{Hom}_G(I, I)$. In particular, this gives rise to a representation of R on V which commutes with the action of G. It follows $\tilde{I} : R \times G \longrightarrow Aut(V)$ given by

$$\tilde{I}(r,g) = A(r)I(g)$$

is a representation of $R \times G$ on V. We have

$$\tilde{I} \cong \bigoplus_{\rho\otimes\pi} m_{\rho\otimes\pi}\, \rho \otimes \pi, \tag{3.1}$$

where ρ runs over the set of equivalence classes of irreducible representations of R and π runs over the set of equivalence classes of irreducible components of I. The integer $m_{\rho\otimes\pi} \geq 0$ is the multiplicity of $\rho \otimes \pi$ in $\tilde{I}$. Let

$$V = \bigoplus_{\rho\otimes\pi} \bigoplus_{i=1}^{m_{\rho\otimes\pi}} V_{\rho\otimes\pi}(i) \cong \bigoplus_{\rho\otimes\pi} \bigoplus_{i=1}^{m_{\rho\otimes\pi}} U^i_{\rho\otimes\pi} \otimes V^i_{\rho\otimes\pi} \tag{3.2}$$

be a decomposition of V into $\tilde{I}$-irreducible subspaces corresponding to Equation (3.1), with $U^i_{\rho\otimes\pi}$ (resp., $V^i_{\rho\otimes\pi}$) an irreducible R-invariant (resp., G-invariant) subspace of $V_{\rho\otimes\pi}(i)$. For an irreducible component π of I, define

$$\rho_\pi = \bigoplus_{\rho} m_{\rho\otimes\pi}\, \rho. \tag{3.3}$$

This representation acts on the space $U_\pi = \bigoplus_\rho \bigoplus_{i=1}^{m_{\rho\otimes\pi}} U^i_{\rho\otimes\pi} \subset V$. The spaces $V^i_{\rho\otimes\pi}$, for all ρ and $i = 1, \ldots, m_{\rho\otimes\pi}$, are mutually equivalent and we can identify each of them with a space denoted by V_π; we write $I_{\rho,i} : V_\pi \longrightarrow V^i_{\rho\otimes\pi}$ for the maps. From Equation (3.2),

$$V \cong \bigoplus_{\rho\otimes\pi} \bigoplus_{i=1}^{m_{\rho\otimes\pi}} U^i_{\rho\otimes\pi} \otimes V_\pi = \bigoplus_{\pi} \left(\bigoplus_{\rho} \bigoplus_{i=1}^{m_{\rho\otimes\pi}} U^i_{\rho\otimes\pi} \right) \otimes V_\pi = \bigoplus_{\pi} U_\pi \otimes V_\pi .$$

We may describe the isomorphism explicitly: for $u \in U_\pi$, write $u = \sum_\rho \sum_i u_{\rho,i}$ with $u_{\rho,i} \in U^i_{\rho\otimes\pi}$. For $v \in V_\pi$, we have $u \otimes v \in U_\pi \otimes V_\pi$ corresponds to

$$\mathcal{I}(u \otimes v) = \sum_\rho \sum_i u_{\rho,i} \otimes I^{-1}_{\rho,i}(v) \in V,$$

which extends to give the isomorphism. It follows that

$$\tilde{I} \cong \bigoplus_\pi \rho_\pi \otimes \pi, \qquad I \cong \bigoplus_\pi (\dim \rho_\pi)\pi. \tag{3.4}$$

For a given π, the space $U_\pi \otimes V_\pi$ is the π-isotypic subspace of V and therefore is canonically defined (although U_π and V_π are not). The representation ρ_π is also canonical. If we want to fix a decomposition of $U_\pi \otimes V_\pi$ into G-irreducible subspaces, we fix an orthonormal basis $\{u_1, \ldots, u_k\}$ of U_π. Then $U_\pi \otimes V_\pi = (u_1 \otimes V_\pi) \oplus \cdots \oplus (u_k \otimes V_\pi)$. The action of $A(r)$ on $u \otimes v \in u \otimes V_\pi$ is precisely $\rho_\pi(r)u \otimes v$.

For $f \in C_c^\infty(G)$, define

$$I(f) = \int_G f(g)I(g)dg, \qquad \pi(f) = \int_G f(g)\pi(g)dg.$$

For $v \in V$, write $\mathcal{I}^{-1}v = \sum_\pi \sum_i u^i_\pi \otimes v^i_\pi$, in accordance with the decomposition Equation (3.4) of $\tilde{I}$ ($u^i_\pi \otimes v^i_\pi \in U_\pi \otimes V_\pi$). Then, for $r \in R$ and $f \in C_c^\infty(G)$, we have

$$\begin{aligned} A(r)I(f)v = A(r)\int_G f(g)I(g)vdg &= \int_G f(g)A(r)I(g)vdg = \int_G f(g)\tilde{I}(r,g)vdg \\ &= \int_G f(g) \sum_\pi \sum_i \mathcal{I}\left(\rho_\pi(r)u^i_\pi \otimes \pi(g)v^i_\pi\right) dg \\ &= \sum_\pi \sum_i \mathcal{I}\left(\rho_\pi(r)u^i_\pi \otimes \int_G f(g)\pi(g)v^i_\pi\right) dg \\ &= \sum_\pi \sum_i \mathcal{I}\left(\rho_\pi(r)u^i_\pi \otimes \pi(f)v^i_\pi\right) = \left(\bigoplus_\pi \rho_\pi(r) \otimes \pi(f)\right) v. \end{aligned}$$

Notice that all the integrals above are essentially finite sums. It follows that

$$A(r)I(f) = \bigoplus_\pi \rho_\pi(r) \otimes \pi(f).$$

In particular,

$$\operatorname{trace}(A(r)I(f)) = \sum_\pi \operatorname{trace}(\rho_\pi(r) \otimes \pi(f)) = \sum_\pi \operatorname{trace}\, \rho_\pi(r) \operatorname{trace}\, \pi(f).$$

We have proved the following:

Lemma 3.1. Let σ be an irreducible admissible unitary representation of M and $I = i_{G,M}(\sigma)$. Let R be a finite group. Suppose $r \mapsto A(r)$ is a homomorphism of R into a multiplicative subgroup of $C(\sigma) = \mathrm{Hom}(I, I)$. To each component π of I we can attach in a canonical way a representation ρ_π of R. Then

$$I \cong \bigoplus_{\pi} (\dim \rho_\pi)\pi,$$

where π runs over equivalence classes of irreducible subrepresentations of I. If we define

$$\langle r, \pi \rangle = \text{trace } \rho_\pi(r),$$

then

$$\text{trace}(A(r)I(f)) \cong \sum_{\pi} \langle r, \pi \rangle \text{ trace } \pi(f). \tag{3.5}$$

□

3.2 Standard intertwining operators

In this section, we apply the general results of Section 3.1 to (arbitrary) subgroups of $W(\sigma)$. Since we do not assume trivial cocycle, we follow [3] and work with a central extension. In particular, the results apply to $W_{\psi,\sigma}$, consistent with the conjecture of Arthur.

Let σ be an irreducible admissible unitary representation of M. For $w \in W(\sigma)$, take a representative $\bar{w} \in K \subset G$, a good maximal compact subgroup. Let $\mathbf{A}(\nu, \sigma, \bar{w})$, $\nu \in \mathfrak{a}^*_{\mathbb{C}}$, be the standard intertwining operator defined in Section 1 of [33] (cf. [32] for a more detailed description). Then $\nu \mapsto \mathbf{A}(\nu, \sigma, \bar{w})$ is a meromorphic function of ν. If $\mathbf{A}(\nu, \sigma, \bar{w})$ is holomorphic at $\nu = 0$, then $\mathbf{A}(0, \sigma, \bar{w})$ is an intertwining operator between $i_{G,M}(\sigma)$ and $i_{G,M}(\bar{w}\sigma)$. We define a normalized intertwining operator

$$A'(\nu, \sigma, \bar{w}) = n(\nu, \sigma, \bar{w})\mathbf{A}(\nu, \sigma, \bar{w}),$$

where $n(\nu, \sigma, w)$ is a normalizing factor. We do not specify the normalizing factor used here, we just refer to [2], Theorem 2.1, for the proof of existence. Then the operators $A'(\nu, \sigma, \bar{w})$ satisfy the properties described in Theorem 2.1 of [2]. Set $A'(\sigma, \bar{w}) = A'(0, \sigma, \bar{w})$. One of the basic properties of normalized operators is

$$A'(\sigma, \bar{w}_1\bar{w}_2) = A'(\bar{w}_1\sigma, \bar{w}_2)A'(\sigma, \bar{w}_2), \tag{3.6}$$

where $\bar{w}_1, \bar{w}_2$ are representatives of $w_1, w_2 \in W(\sigma)$. Let $\langle , \rangle$ be the pairing defined in Section 2 of [32] (if $f, f' \in i_{G,M}(\sigma)$, then

$$\langle f, f' \rangle = \oint_G (f(g), f'(g)) d\mu(g),$$

where $(,)$ is the unitary pairing on the space of σ and the notation $\oint_G h(g) d\mu(g)$ is explained on page 303 of [32]). Then the property (R_4) in Theorem 2.1 of [2] implies

$$\langle A'(\sigma, \bar{w})f,\, f' \rangle = \langle f, A'(\bar{w}\sigma, \bar{w}^{-1})f' \rangle,$$

for $f \in i_{G,M}(\sigma)$ and $f' \in i_{G,M}(\bar{w}\sigma)$. Then, using Equation (3.6),

$$\langle A'(\sigma, \bar{w})f, A'(\sigma, \bar{w})f \rangle = \langle f, A'(\bar{w}\sigma, \bar{w}^{-1})A'(\sigma, \bar{w})f \rangle = \langle f, f \rangle.$$

This tells us that $A'(\nu, \sigma, \bar{w})$ is bounded at 0 and therefore holomorphic there. In particular, $A'(\sigma, \bar{w}) \in \mathrm{Hom}_G(i_{G,M}(\sigma), i_{G,M}(\bar{w}\sigma))$.

Next, we can associate to $A'(\sigma, \bar{w})$ an operator in $C(\sigma)$, as follows. Since $w\sigma \cong \sigma$, σ extends to a representation σ_w of the smallest group containing M and $\bar{w}$. Fix such an extension σ_w and define

$$A(\sigma, w) = \sigma_w(\bar{w})A'(\sigma, \bar{w}). \tag{3.7}$$

Then $A(\sigma, w) \in C(\sigma)$ and the definition is independent of the representative $\bar{w}$. Note that $\sigma_w(\bar{w}) \in \mathrm{Hom}_M(\bar{w}\sigma, \sigma)$. Let $w_1, w_2 \in W(\sigma)$, with representatives $\bar{w}_1, \bar{w}_2$. Since σ is irreducible, there exists a constant $\eta(w_1, w_2)$ such that

$$\sigma_{w_1 w_2}(\bar{w}_1 \bar{w}_2) = \eta(w_1, w_2)\sigma_{w_1}(\bar{w}_1)\sigma_{w_2}(\bar{w}_2).$$

It follows that

$$A(\sigma, w_1 w_2) = \eta(w_1, w_2)A(\sigma, w_1)A(\sigma, w_2). \tag{3.8}$$

Now, let R be a subgroup of $W(\sigma)$. Equation (3.8) implies

$$A(\sigma, r_1 r_2) = \eta(r_1, r_2)A(\sigma, r_1)A(\sigma, r_2), \quad r_1, r_2 \in R. \tag{3.9}$$

We will show that (3) of Properties 1.1 hold for R.

If $\eta = 1$, then $r \mapsto A(\sigma, r)$ is a representation of R on $C(\sigma)$ and Equation (3.5) follows directly. Assume η splits, i.e., there exists a function $\xi : R \to \mathbb{C}^\times$ such that

$$\eta(r_1, r_2) = \xi(r_1 r_2)\xi(r_1)^{-1}\xi(r_2)^{-1}.$$

Then $\xi(r_1 r_2)^{-1} A(\sigma, r_1 r_2) = \xi(r_1)^{-1} A(\sigma, r_1)\xi(r_2)^{-1} A(\sigma, r_2)$ and $r \mapsto \xi(r)^{-1} A(\sigma, r)$ is a representation of R on $C(\sigma)$. Formula (3.5) holds for $\xi(r)^{-1} A(\sigma, r)$ (noting that in this case, the cocycle is normally absorbed into the normalization so does not appear).

If η does not split, we apply Schur's theory of projective representations (cf. §53 [13], [30], [3]). Recall that $\alpha \in \{\eta\}$ means there exists a function $\xi : R \to \mathbb{C}^\times$ such that

$$\eta(r_1, r_2) = \alpha(r_1, r_2)\xi(r_1 r_2)\xi(r_1)^{-1}\xi(r_2)^{-1}.$$

Theorem 53.3 in [13] tells us the class $\{\eta\}$ has finite order n, i.e., $\eta^n \in \{1\}$. From the proof of the same theorem, the class $\{\eta\}$ has a representative α whose values $\alpha(r, s)$ are nth roots of 1 and $\alpha(1, 1) = 1$. Fix ζ, a primitive nth root of 1. For each pair $r, s \in R$, define the integer $a_{r,s}$ by

$$\alpha(r, s) = \zeta^{a_{r,s}},$$

$0 \leq a_{r,s} < n$. Let Z_α be the cyclic group generated by α. On the set $\tilde{R}$ of all ordered pairs (r, α^k), $r \in R$, $\alpha^k \in Z_\alpha$, define multiplication by

$$(r, \alpha^k)(s, \alpha^l) = (rs, \alpha^{a_{r,s}+k+l}).$$

Then $\tilde{R}$ is a group. (To see associativity, observe that Equation (3.9) for $A'(r) = \xi^{-1}(r)A(\sigma, r)$ gives $A'(r_1 r_2) = \alpha(r_1, r_2)A'(r_1)A'(r_2)$. Associativity follows from $\alpha(r_1 r_2, r_3)$ $\alpha(r_1, r_2) = \alpha(r_1, r_2 r_3)\alpha(r_2, r_3)$, which can be obtained easily from (3.9) for $A'(r_1 r_2 r_3)$.) The mapping $z \mapsto (1, z)$ is an isomorphism of Z_α into the center of $\tilde{R}$. We have

$$1 \to Z_\alpha \to \tilde{R} \to R \to 1.$$

Define a linear character $\chi : Z_\alpha \to \mathbb{C}^\times$ by $\chi(\alpha^k) = \zeta^k$. Define $\tilde{A} : \tilde{R} \to C(\sigma)$ by

$$\tilde{A}((r, z)) = \chi^{-1}(z)\xi(r)^{-1}A(\sigma, r),$$

$r \in R, z \in Z_\alpha$. Then

$$\begin{aligned}\tilde{A}((r,\alpha^k)(s,\alpha^l)) = \tilde{A}((rs,\alpha^{a_{r,s}+k+l})) &= \zeta^{-a_{r,s}-k-l}\xi(rs)^{-1}\eta(r,s)A(\sigma,r)A(\sigma,s)\\ &= \zeta^{-a_{r,s}-k-l}\xi(rs)^{-1}\alpha(r,s)\xi(rs)\xi(r)^{-1}\xi(s)^{-1}A(\sigma,r)A(\sigma,s)\\ &= \zeta^{-a_{r,s}-k-l}\zeta^{a_{r,s}}\xi(r)^{-1}\xi(s)^{-1}A(\sigma,r)A(\sigma,s) = \tilde{A}((r,\alpha^k))\tilde{A}((s,\alpha^l)).\end{aligned}$$

It follows that $\tilde{r} \mapsto \tilde{A}(\tilde{r}), \tilde{r} \in \tilde{R}$, is a homomorphism of $\tilde{R}$ into $C(\sigma)$ and formula (3.5) holds for $\tilde{A}$. In particular, attached to each component π is a representation ρ_π of the group $\tilde{R}$. For $r \in R$, define

$$\langle r,\pi\rangle = \operatorname{trace}\rho_\pi((r,1)).$$

Lemma 3.1 now implies the following:

Theorem 3.2. With notation as above,

$$\xi^{-1}(r)\operatorname{trace}(A(\sigma,r)I(f)) = \sum_\pi \langle r,\pi\rangle \operatorname{trace}\pi(f). \qquad \square$$

3.3 Arthur R-group

In this section, we discuss the application of the results of section 3.2 to Arthur R-groups. We first note that to do this, we must have a well-defined action of the Arthur R-group on the space of the induced representation. This is the case for the example from Section 2; we close by revisiting this example.

We now consider the situation described in Section 1.2. In particular, ψ is an elliptic A-parameter of M and σ belongs to the A-packet $\Pi_\psi(M)$. The group S_ψ (the centralizer in $\hat{G}$ of the image of ψ) is a reductive group. It can be shown that W_ψ/W_ψ^0 is isomorphic to a subgroup of W_ψ; denote this subgroup by R_ψ. In addition, it can be shown that the following exact sequence splits:

$$1 \longrightarrow W_\psi^0 \longrightarrow W_\psi \longrightarrow R_\psi \longrightarrow 1.$$

Then we have an embedding

$$\begin{array}{ccccccccc} 1 & \longrightarrow & W^0_{\psi,\sigma} & \longrightarrow & W_{\psi,\sigma} & \longrightarrow & R_{\psi,\sigma} & \longrightarrow & 1\\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & W^0_{\psi} & \longrightarrow & W_{\psi} & \longrightarrow & R_{\psi} & \longrightarrow & 1\end{array}$$

of short exact sequences. If $R_{\psi,\sigma} \subset W(\sigma)$ (or if $W^0_{\psi,\sigma}$ acts trivially), we have a well-defined action of $R_{\psi,\sigma}$ on the inducing space, so have (Theorem 3.2)

$$\xi^{-1}(r)\,\mathrm{trace}(A(\sigma,r)I(f)) = \sum_{\pi} \langle r,\pi\rangle\,\mathrm{trace}\,\pi(f).$$

Example 3.3. We look at what happens in the example from Section 2. In this case, $\pi = \mathrm{Ind}_P^G(\sigma) = St_{GL(2)} \times triv_{GL(2)} \rtimes 1$ and $W(\sigma) = \{1, w_1, w_2, w_3\}$, where the elements of $W(\sigma)$ are most easily described by their actions: if $m = diag(X_1, X_2, 1, {}^{\tau}X_2, {}^{\tau}X_1) \in M$ (block diagonal matrix in $SO(9,F)$), then $w_1 \cdot m = diag(X_1, {}^{\tau}X_2, 1, X_2, {}^{\tau}X_1)$, $w_2 \cdot m = diag({}^{\tau}X_1, X_2, 1, {}^{\tau}X_2, X_1)$, and $w_3 \cdot m = diag({}^{\tau}X_1, {}^{\tau}X_2, 1, X_2, X_1)$.

Recall that we have (cf. proof of Theorem 2.5)

$$St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_1) \oplus L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_2)$$
$$St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)}) = L(\nu^{-\frac{1}{2}}; \mathcal{S}).$$

We may also determine that

$$triv_{GL(2)} \rtimes \mathcal{T}_1 = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_1)$$
$$triv_{GL(2)} \rtimes \mathcal{T}_2 = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_2) \oplus L(\nu^{-\frac{1}{2}}; \mathcal{S})$$

by a similar calculation (though made much easier since the Jacquet modules of the various subquotients are already known).

We know $St_{GL(2)} \rtimes 1 = \mathcal{T}_1 \oplus \mathcal{T}_2$ and $triv_{GL(2)} \rtimes 1 = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}) \oplus L(\nu^{-\frac{1}{2}}; St_{SO(3)})$ (cf. Lemma 2.3). We use normalizations so that the nontrivial normalized standard intertwining operator for $St_{GL(2)} \rtimes 1$ acts trivially on $\mathcal{T}_2$ (generic component) and nontrivially on $\mathcal{T}_1$ (consistent with [33] by an argument from section 5 of [23] and a limit calculation); for $triv_{GL(2)} \rtimes 1$, we normalize so that the nontrivial normalized standard intertwining operator acts trivially on $L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$ (K-spherical component) and nontrivially on $L(\nu^{-\frac{1}{2}}; St_{SO(3)})$. We then use corresponding normalizations for $St_{GL(2)} \times triv_{GL(2)} \rtimes 1$ (so that $A(\sigma, w_1)$ acts trivially on $St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}})$ and nontrivially on $St_{GL(2)} \rtimes L(\nu^{-\frac{1}{2}}; St_{SO(3)})$, etc.). The action of normalized standard intertwining operators is summarized below:

	$A(\sigma,1)$	$A(\sigma,w_1)$	$A(\sigma,w_2)$	$A(\sigma,w_3)$
$\pi_1 = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_1)$	1	1	−1	−1
$\pi_2 = L(\nu^{-\frac{1}{2}}, \nu^{-\frac{1}{2}}; \mathcal{T}_2)$	1	1	1	1
$\pi_3 = L(\nu^{-\frac{1}{2}}, \mathcal{S})$	1	−1	1	−1

If we let ρ_0 denote the trivial character of $R_{\psi,\sigma} = W(\sigma)$ and ρ_i for $i = 1, 2, 3$ the nontrivial character which is trivial on w_i, then the above implies $\rho_{\pi_1} = \rho_1$, $\rho_{\pi_2} = \rho_0$ and $\rho_{\pi_3} = \rho_2$ (with $m_{\rho_3 \otimes \pi} = 0$ for all π). Thus, e.g.,

$$trace(A(\sigma, w_1)\pi(f)) = trace\ \pi_1(f) + trace\ \pi_2(f) - trace\ \pi_3(f).$$

If we twist the normalizations by a character ρ of $R_{\psi,\sigma}$, there is a corresponding twist in the ρ_{π_i}. Regarding the A-packet π_ψ corresponding to the parameter ψ, C. Moeglin explained to us that $\pi_\psi = \{\pi_1, \pi_2, \pi_3\}$, so the packet consists of three elements.

Acknowledgments

D.B. wishes to thank Colette Moeglin, Freydoon Shahidi, David Vogan and the seminar *Formes automorphes* at Jussieu for valuable comments. She also thanks Peter Schneider for his hospitality during her 1-year stay at the Universität Münster, where a part of this work was done. C.J. would also like to thank Freydoon Shahidi for valuable comments.

D.B. supported in part by NSF grant DMS-0601005 and the Research Fellowship from the Alexander von Humboldt Foundation; C.J. supported in part by NSA grant H98230-04-1-0029 and the East Carolina University College of Arts and Sciences

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