RESEARCH STATEMENT

GUGLIELMO FUCCI

My main research interests lie in mathematical physics; more precisely in the analysis of spectral functions and their application to various areas of geometric analysis on manifolds, spectral geometry, quantum field theory, and quantum gravity. Due to the intrinsic interdisciplinary nature of mathematical physics my interests often overlap with other areas of mathematics such as partial differential equations, complex analysis, special functions, and asymptotic analysis.

In many areas of mathematics and physics one is often confronted with the problem of extracting relevant information from the spectrum of, usually, Laplace-type differential operators. For this purpose, it is convenient to organize the spectrum in the form of so called spectral functions. The two most widely used spectral functions in mathematics and physics are the heat kernel and the spectral zeta function [9, 10, 11, 23, 25] which are the main focus of my research. The heat kernel is the fundamental solution of the heat equation endowed with a point-like source and appropriate boundary conditions. The study of its small time asymptotic behavior is used in order to extract geometric information about the underlying manifold [24, 25, 29] and in spectral analysis thanks to a relation that exists between the heat equation and the Atiyah-Singer index theorem [1, 2]. The spectral zeta function represents a generalization of the more familiar Riemann zeta function in which the integers are replaced by the non-vanishing eigenvalues of a differential operator. This function is of fundamental importance for the analysis of complex powers of elliptic operators and the study of functional determinants [26, 27]. The widespread use of these functions in physics can be found especially in the area of quantum field theory. Many essential characteristics of quantum fields are encoded in the effective action [8] which is a functional that describes how the classical equations of motion are modified by quantum effects. The effective action can be expressed in terms of the functional determinant of an elliptic operator and, therefore, spectral zeta function techniques are particularly suitable for its analysis [7]. The process of quantization of systems with infinitely many degrees of freedom inevitably leads to the appearance of divergent quantities and anomalies [8, 9]. These objects are best studied by exploiting the small time asymptotic expansion of the heat kernel [29].

My research interests are currently focused on the development of analytic methods for the study of the spectral zeta function and the heat kernel expansion including their application to various areas of mathematics and physics. I exploit these methods principally in the following broad areas:

- The study of functional determinants and effective action of quantum fields on both smooth and singular Riemannian manifolds with or without boundary [3, 12, 13, 16].
- The analysis of vacuum energy and quantum fields under the influence of boundaries and external fields [4, 15, 17, 18, 19, 20].
- The development of improved techniques for the analysis of the spectral zeta function and heat kernel expansion in the setting of more general geometries [22].

These topics belong to extremely active fields of research dealing with problems of very current interest. The importance of these investigations lies in their potential for providing a deeper understanding of the relation between analysis and geometry and in the development of novel methods, or improvement of old ones, to deal with new challenges that often arise during the process of research. In addition, spectral functions are powerful tools for the description of physical phenomena at both the microscopic and macroscopic scale. The understanding of these phenomena will provide us with a better knowledge of the natural world and could potentially lead to technological applications.

The methods and techniques used in the analysis of spectral functions and their application to physical problems are accessible to advanced undergraduate and graduate students with a background preparation in complex analysis, partial differential equations and asymptotic methods. Students concentrating in either mathematics or physics would be able to participate, under my guidance, in research projects that are of current interest. Students would play an active role during the entire research process, from the formulation of the problem to the submission of the finished manuscript. In order to increase student participation in research projects, I would be very interested in working towards the development of a program in mathematical physics offered in conjunction with interested faculty members both in the mathematics and physics department. Such program would attract students who wish to pursue a career in mathematical physics and allow them to approach, already at an early stage, certain aspects of current research topics.

Research projects regarding the application of spectral functions to problems related to physics are very likely to be funded by national agencies. For instance, one of my interests is in the area of the Casimir effect which is a macroscopic manifestation of zero point energy of quantum fields. The understanding of this phenomenon is of extreme importance in nanotechnological applications and micromechanical devices which are of topical technological interest [6]. This represents only one possible direction and I am committed to explore also other avenues that would lead to opportunities for external funding.

Additional Details

Spectral Functions on Riemannian Manifolds. String theory and brane-world scenarios have attracted an enormous amount of interest in recent years because they provide a framework for the development of a theory of grand unification of all fundamental interactions. Moreover, brane-world models might be able to solve, amongst other issues, the hierarchy problem (i.e. why is gravity so much weaker than the other fundamental forces?) and cosmological constant problem (namely why is the cosmological constant so small?). Orbifolds, which are defined locally as the quotient space of a smooth manifold X and a discrete isometry group G acting linearly on X, play an important role in such theories. In general the action of the group G on X will have fixed points, these points are then mapped to conical singularities in the quotient space. We can thus say that orbifolds are locally represented as generalized cones (which possess line element $ds^2 = dr^2 + r^2 d\Sigma^2$ with $r \in I \subset \mathbb{R}$ and $d\Sigma^2$ being the line element on the base manifold). In [16] we computed the small mass expansion for the functional determinant of a scalar Laplacian, with both Dirichlet and Robin boundary conditions, defined on the generalized cone. This result is particularly important for the study of the effective action of massive scalars in field theoretical models containing orbifold compactifications. The obtained analytic expressions for the small mass expansion, combined with large mass expansions coming from the heat kernel, has helped reduce drastically the numerical work necessary for the analysis of intermediate values of the mass.

A singular Riemannian manifold which has occupied part of my research is the Riemann cap (or spherical suspension). Locally, such manifold is described by the line element $ds^2 = d\theta^2 + \sin^2\theta d\Sigma^2$ where $\theta \in (0, \theta_0)$, with $\theta_0 \in (0, \pi)$, and $d\Sigma^2$ represents the metric of a smooth Riemannian manifold. These types of geometries provide a generalization of the D-dimensional spherical cap and occur frequently in the ambit of cosmology, since sectors of de Sitter space belong to this class. In [12] we have obtained an expression for the functional determinant of the Laplacian acting on scalar functions, defined on the Riemann cap, endowed with Dirichlet boundary conditions. Apart from generalizing previous results available in the literature, which were limited to spherical caps, this work provides a way to analyze singular Riemannian manifolds that are not globally generalized cones. For this type of manifold we have also computed the coefficients of the heat kernel expansion for Laplace operators acting on scalar functions subjected to Dirichlet boundary conditions [21]. By exploiting a technique relying on a contour integral representation of the spectral zeta function we found its analytic continuation valid in the entire complex plane and its associated meromorphic structure. Thanks to a relation that exists between the zeta function and the heat kernel obtainable via Mellin transform we computed the coefficients of the heat kernel asymptotic expansion in arbitrary dimensions. In the literature the contour integral technique was used almost exclusively in cases when the eigenfunctions are Bessel functions; the novelty of this paper resides in the fact that we have explicitly shown how to apply this technique in the case when the eigenfunctions are associated Legendre functions.

Quantum Field Theory under the influence of external conditions. The Casimir effect is, perhaps, one of the most important macroscopic manifestations of zero point energy of quantized fields under the influence of external conditions or in spaces with non-trivial topology. This phenomenon is highly interesting not only because it provides a theoretical understanding of the lowest energy state of a quantized field but also for its relevance in nanoscale physics and nanotechnology. Casimir energy calculations, however, are plagued with divergencies. The divergent part is proportional to a particular heat kernel coefficient and is, hence, related to the geometry of the object for which the energy is being calculated. An answer that is free of divergences can be obtained if one considers the Casimir force between two separate objects (the most relevant physical case). Piston configurations (two chambers separated by a common boundary) belong to the class of systems for which the Casimir force is, in the majority of the cases, free from divergences. In [18], by using spectral zeta function methods, we studied the Casimir energy and force for massless scalar fields in the setting of a conical piston geometry when Dirichlet and Neumann boundary conditions are imposed. This work provides, for the first time, the analysis of a piston geometry that is more general than the ones considered in the literature where the two chambers have fundamentally the same type of geometry. In the conical piston the two separate regions (or chambers) have different geometries since one of them contains a geometric (conical) singularity while the other does not. This is a feature that makes the conical piston particularly interesting because, unlike usual piston configurations, we found that the Casimir force is not symmetric with respect to the center of the piston. This work represents a step forward in understanding how the geometry of the object and the presence of singularities influence, in general, the Casimir energy and the associated force. In a subsequent work [19] we have analyzed the case of a conical piston endowed with hybrid boundary conditions in a further effort to better understand how boundary conditions influence the Casimir force.

The idealized boundary conditions often studied in Casimir pistons usually do not closely resemble characteristics of real physical objects. For this reason, more general and somewhat non-standard boundary conditions need to be considered especially for real-world applications. I am currently interested in the analysis of Casimir energy in pistons with an arbitrary cross-section modelled by potentials of compact support [5, 17]. The potential mimics physical properties of a piston that cannot be studied with the use of standard boundary conditions. These investigations are of crucial importance in order to obtain results that more closely represent physical situations of interest.

My interest in investigating how geometry influences physical phenomena has led me to work with theories predicting the existence of additional dimensions. Field theoretical models in string theory describe our world as a four dimensional manifold immersed in a bulk manifold of higher dimensions. The additional dimensions are supposed to be compactified, however they have remained unobservable in current experimental settings. It is therefore of utmost importance to understand whether or not particular physical systems or phenomena can be useful in the detection of these extra dimensions within the current experimental limitations. The spectral zeta function is a powerful tool in the analysis of partition functions of systems described by quantum statistical mechanics. An important and physical relevant case is, for instance, the Bose-Einstein condensation where a gas of spin-0 particles (bosons) undergoes a process of phase transition where the particles tend to reside in the lowest energy state. In [20] we investigated the phenomenon of Bose-Einstein condensation on manifolds constructed as a product of a three-dimensional Euclidian space and a general smooth, compact d-dimensional manifold representing the extra dimensions. We found that the critical temperature, at which the condensate appears when the gas is confined in an anisotropic oscillator potential, depends on the geometry and topology of the extra dimensions and can be used to set a bound on their compactification size.

Future Directions of Research

The work that I have conducted represents a first step towards a wide range of subsequent explorations. I am attracted to any problem that lies on the boundary of analysis and geometry, and to the development of mathematical methods and techniques applied to problems related to physics. For the next few years I plan on dedicating my research to topics that will help to gain a better understanding of how geometry influences physical systems. For this purpose, I intend on developing new techniques for the study of spectral functions and generalize already known methods to new and relevant situations. In addition, I would be interested in broadening the use of spectral functions in areas of applied mathematics and theoretical physics where they either have not been exploited or underexploited but where they could potentially be very fruitful. One of such areas, for instance, concerns the description of physical phenomena involving vacuum instability and particle creation.

The research I will be focused on is essentially interdisciplinary and collaborations with individuals from both the mathematics and physics department could certainly be formed. For example, some projects will require a certain amount of numerical work which would involve the participation of faculty members and students with a background in numerical methods. This is not, however, the only field with which collaborative ties could be formed. In fact my research interests allow me to work in collaboration with interested individuals in the areas of analysis, differential equations, differential geometry as well as in areas such as quantum field theory, quantum gravity, and string theory.

Some of the research projects I would be interested in working on, suitable also for students involvement, are described below.

In the setting of the spherical suspension, I would be interested to use techniques stemming from the integral representation of the spectral zeta function for the study of the effective action for scalar fields endowed with more general Robin boundary conditions. This research would be of particular importance in order to describe how the electromagnetic field propagates in this geometry. In addition, similar types of investigation can be conducted for massive spinor fields; this would be relevant in the study of the behavior of fermions (like electrons) propagating on the spherical suspension. I am currently working on applying spectral zeta function methods in order to analyze the functional determinant and the heat kernel asymptotic expansion for Laplace-type operators on warped product manifolds of the type $I \times_f N$ [22] (these manifolds have line element $ds^2 = dr^2 + f^2(r)d\Sigma^2$ with some function $f(r) > 0$ where $r \in I = [a, b] \subset \mathbb{R}$). These geometries are of extreme importance in the generalization of Randall-Sundrum models and Kaluza-Klein theories which are believed to be the best candidates for a theory of grand unification. A project on which I plan on working is related to warped geometries but with the difference of allowing the function $f(r)$ to be zero either at one or both the end-points of the interval I. In this way the manifold becomes singular and the type of singularity (conical or cuspidal) depends on how fast $f(r)$ tends to zero in the neighborhood of that point. The study of spectral functions on cuspidal manifolds would generalize previous investigations which are limited to singular Riemannian manifolds with conical singularities.

An additional phenomenon which I intend to describe with the use of spectral functions is the so called Schwinger mechanism in which pairs of massive charged particles are created under the influence of a strong electric field [28]. I plan on investigating this phenomenon in the setting of geometries with extra compactified dimensions. In particular, I would consider a homogeneous electric field and massive scalar and spinor fields propagating on a manifold of the type $M_4 \times \mathscr{N}$ where M_4 is a four dimensional Minkowski space and $\mathscr{N}$ is a smooth compact Riemannian manifold representing the additional Klauza-Klein dimensions. I expect that the pair production rate (namely the number of created particles per unit time and volume) will involve corrections due to the geometry and topology of the manifold $\mathscr{N}$. A comparison between the production rate in this setting and the known result that is obtained by considering only M_4 will give an estimate on how the extra dimensions influence this phenomenon. Part of this project requires the knowledge of the real and imaginary part of the Hurwitz zeta function of imaginary second argument and its derivative at the points $s \in \mathbb{Z}\backslash\{1\}$, a study that I have performed in [14] and that will allow me to continue these investigations.

These are just some of the directions of research I would be interested in actively pursuing. The field of spectral functions in mathematical physics is a very fertile one and I will focus my efforts on any problem that is deemed relevant and that attracts my interest.

References

[1] Atiyah M. F. and Singer I. M., The index of elliptic operators on compact manifolds, *Bull. Am. Math. Soc.* **69**, 422 (1963)

[2] Atiyah M. F., Bott R., and Patodi V. K., On the heat equation and the index theorem, *Invent. Math.* **19**, 279 (1973); errata, ibid. **28**, 277 (1975)

[3] Avramidi I. G. and Fucci G., Low-Energy Effective Action in Non-Perturbative Electrodynamics in Curved Spacetime, *J. Math. Phys.* **50**, 102302 (2009)

[4] Avramidi I. G. and Fucci G., Non-perturbative Heat Kernel Asymptotics on Homogeneous Abelian Bundles, *Comm. Math. Phys.* **291**, 543 (2009)

[5] Beauregard M., Fucci G., Kirsten K., and Morales P., Casimir effect in the presence of external fields, (in preparation)

[6] Bordag M., Klimchitskaya G. L., Mohideen U., and Mostepanenko V. M., Advances in the Casimir Effect, (Oxford University Press, Oxford) (2009)

[7] Bytsenko A. A., Cognola G., Elizalde E., Moretti V., and Zerbini S., Analytic Aspects of Quantum Fields, (World Scientific Publishing, Singapore) (2003)

[8] DeWitt B. S., Quantum field theory in curved spacetime, *Phys. Rep.* C **19**, 295 (1975)

[9] DeWitt B. S., The Global Approach to Quantum Field Theory, (Oxford University Press, Oxford) (2003)

[10] Elizalde E., Odintsov S. D., Romeo A., Bytsenko A., and Zerbini S., Zeta Regularization Techniques with Applications, (World Scientific, Singapore) (1994)

[11] Elizalde E., Ten Physical Applications of the Spectral Zeta Function, (Springer-Verlag, Berlin) (1995)

[12] Flachi A. and Fucci G., Zeta determinant for Laplace operators on Riemann caps, *J. Math. Phys.* **52**, 023503 (2011)

[13] Fucci G., Nonperturbative one-loop effective action for electrodynamics in curved space-time, *J. Math. Phys.* **50**, 102301 (2009)

[14] Fucci G., On the Hurwitz zeta function of imaginary second argument, (under review)

[15] Fucci G. and Avramidi I. G., On the Gravitationally Induced Schwinger Mechanism In: "Quantum Field Theory under the Influence of External Conditions" (QFEXT09). Eds. Kimball A. Milton and Michael Bordag (Singapore: World Scientific, 2010), pp. 485-491

[16] Fucci G. and Kirsten K., Small mass expansion of functional determinants on the generalized cone, *J. Phys. A* **43**, 365204 (2010)

[17] Fucci G., Kirsten K., and Morales P., Pistons modelled by Potentials, In: "Cosmology, Quantum Vacuum, and Zeta Functions", Eds. S. Odintsov, D. Sáez-Gómez, S. Xambó, (Springer-Verlag, Berlin, 2011) pp 313

[18] Fucci G. and Kirsten K., The Casimir effect for conical pistons, *J. High Energy Phys.* **03**, 016 (2011)

[19] Fucci G. and Kirsten K., Conical Casimir Pistons with hybrid boundary conditions, *J. Phys. A: Math. Theor.* **44**, 295403 (2011)

[20] Fucci G. and Kirsten K., Bose-Einstein Condensation on Product Manifolds, *J. Phys. A: Math. Theor.* **44**, 332002 (2011)

[21] Fucci G. and Kirsten K., Heat Kernel Coefficients for Laplace Operators on the Spherical Suspension, arXiv, math-ph: 1107.3752 (under review)

[22] Fucci G. and Kirsten K., The spectral zeta function for Laplace operators on warped product manifolds of the type $I \times_f N$, (in preparation)

[23] Gilkey P. B., Invariance Theory the Heat Equation and the Atiyah-Singer Index Theorem, (Boca Raton: CRC Press) (1995)

[24] Kac M., Can one hear the shape of a drum?, *Am. Math. Mon.* **73**, 1 (1966)

[25] Kirsten K., Spectral Functions in Mathematics and Physics, (Boca Raton: CRC Press) (2001)

[26] Hawking S.W., Zeta Function Regularization of Path Integrals in Curved Space-Time, *Comm. Math. Phys.* **55**, 133 (1977)

[27] Ray D.B. and Singer I.M., R-torsion and the Laplacian on Riemannian manifolds, *Adv. Math.* **7**, 145 (1971)

[28] Schwinger J. S., On gauge invariance and vacuum polarization, *Phys. Rev.* **82**, 664 (1951)

[29] Vassilevich D. V., Heat kernel expansion: user's manual, *Phys. Rep.* **388**, 279 (2003)