Physica 24D (1987) 292–304
North-Holland, Amsterdam

ON THE SINGULARITY ANALYSIS OF INTERSECTING SEPARATRICES IN NEAR-INTEGRABLE DYNAMICAL SYSTEMS

Tassos BOUNTIS*
Dept. of Theoretical Mechanics, University of Thessaloniki, Thessaloniki, Greece

V. PAPAGEORGIOU and M. BIER
Dept. of Mathematics and Computer Science, Clarkson University, Potsdam, NY 13676, USA

Received 22 January 1986
Revised manuscript received 28 May 1986

It has been recently proved by S. L. Ziglin that transversal intersections of separatrices (invariant manifolds) in near-integrable Hamiltonian systems of two degrees of freedom imply the existence of multi-valued solutions with infinitely many Riemann sheets. Ziglin's theorem is illustrated here on a simple example and then extended and applied to *non-Hamiltonian*, analytic flows $\dot{x} = f(x) + \varepsilon g(x, t)$, with $x \equiv (x, y)$ and $g(x, t) = g(x, t + 2\pi)$, which for $\varepsilon = 0$ possess the Painlevé property. On the other hand, the theoretically expected logarithmic singularities for $\varepsilon \neq 0$ are obtained explicitly in solutions near the intersecting separatrices. Thus, we conjecture that dynamical systems with the Painlevé property, can have no separatrix intersections and hence no strange attractors, etc. These singularities are then numerically located and found to form certain very interesting "chimney" patterns in the complex t-plane, on which they accumulate densely. The upper part of the chimneys (away from the Re t axis) is asymptotically quite *insensitive* to changes in parameter values or initial conditions. The singularity pattern itself, however, becomes "*denser*" and each chimney is seen to gradually "collapse" towards the Re t axis, as the amplitude of the driving term increases and the motion becomes more chaotic.

1. Introduction

In recent years, a novel approach has been developed in the study of nonlinear dynamical systems, based on the analysis of their singularities in the complex time (t-) plane [1–7]. These singularities of the (generally) *unknown* solutions of the equations of motion of the system,

$$(\mathrm{d}x/\mathrm{d}t \equiv)\, \dot{x} = f(x; t), \qquad x(t) \equiv (x_1(t), \ldots, x_n(t)), \tag{1.1}$$

are called movable [8], since their location in the complex t-plane depends on the initial conditions $x(t_0)$, required to specify uniquely each solution (or orbit) of eq. (1.1) for all t. In particular, one starts by looking for all possible leading behaviors of the form

$$x_k(t) \sim \tau^{p_k} (\log \tau)^{q_k}, \qquad \tau \equiv t - t_* \quad (k = 1, \ldots, n) \tag{1.2}$$

near such a singularity at $t = t_*$, and expands $x_k(t)$ in powers of τ (and $\log \tau$) so as to satisfy eq. (1.1) to all orders in $\tau^r (\log \tau)^s$ $(r \geq p_k)$.

If, in these series expansions, no logarithms are found to enter, and all powers of τ are integers [with at least one $p_k < 0$ in (1.2)], (1.1) is said to obey necessary conditions for possessing the *Painlevé property* of having *only poles* in $t \in \mathbb{C}$ [7, 8]. One might naturally expect that systems, which could be proved to possess this property, would be easier to integrate, or perhaps even analytically solve, describing thus predictable and globally "regular" motions. This expectation indeed turned out to be true: many new, such "completely integrable" dynamical systems – Hamiltonian or

*On leave from the Dept. of Mathematics & Computer Science, Clarkson University, Potsdam, NY 13676, USA.

not – were discovered by the requirement that their solutions possess only poles [12], and were subsequently integrated explicitly (sometimes even solved completely), establishing thus sufficient conditions for having the Painlevé property [9–13].

When the Painlevé property is violated in a systematic way – by varying e.g. the values of certain parameters of the system, or introducing additional terms in the equations of motion – *logarithmic terms* are generally seen to arise in the expansions of $x_k(t)$ near a singularity at $t = t_*$ [6, 7]. Moreover, it was observed, on numerous examples, that *the higher* the order $t^r (r \geq p_k)$ at which the $\log \tau$ terms entered, *the "weaker"* the chaotic properties of the system as a whole, as measured e.g. by the "size" of large scale chaotic regions at comparable values of the total energy, etc. [6, 7]. These results, however interesting by themselves, are for the most part non-rigorous and empirical, and hence have remained to date outside a more comprehensive treatment of the singularities of chaos in "non-integrable" dynamical systems.

In this paper, we obtain rigorous results connecting the transversal intersections of separatrices (or, invariant manifolds) in *near-integrable* dynamical systems to *multi-valued* solutions with infinitely many Riemann sheets in the complex t-plane. Our point of departure is a recent theorem due to S. L. Ziglin [14], establishing such a connection in the Hamiltonian case, and our main purpose is to extend the applicability of Ziglin's theorem to *non-Hamiltonian* systems. In the process, we shall demonstrate how these results, remarkably enough, bring together the celebrated work of Melnikov [15, 16] and the singularity analysis of dynamical systems in the complex t-plane.

Thus, this important phenomenon of transversal separatrix intersections – intimately connected with chaotic behavior in non-integrable systems [16–18] – is shown to imply the existence of infinitely branched, multi-valued solutions, in contrast with the Painlevé property, which allows only poles, in completely integrable systems. Based on these results we may assert that dynamical systems possessing the Painlevé property can have none of the chaotic phenomena associated with intersecting invariant manifolds, and more generally, we conjecture that they can have no strange attractors [16, 18].

Multi-valued solutions with infinitely many Riemann sheets have already been observed in several non-integrable two-degree-of-freedom Hamiltonian systems [3–5]. Their series expansions, near one of their singularities at $t = t_*$, involve *irrational* or *complex* powers τ^α of $\tau \equiv t - t_*$, while these singularities appear to form self-similar *natural boundaries* in the complex t-plane [3–5]. By comparison, we find in this paper that an equally – if not more – typical situation in non-integrable oscillating systems is the occurrence of *logarithmic* singularities, forming rows of "*chimneys*" in the complex t-plane, on which these singularities accumulate infinitely densely.

In section 2, we state Ziglin's theorem on the transversal intersection of invariant manifolds in the perturbed two-degree-of-freedom Hamiltonian

$$\mathscr{H} = \mathscr{H}_0(x, y, I) + \varepsilon \mathscr{H}_1(x, y, I, \phi) \qquad (2\pi\text{-periodic in } \phi), \quad (1.3)$$

and illustrate it on a simple example where $\phi = t$. For $\varepsilon = 0$, these invariant manifolds join "smoothly" in a single separatrix, or homo(hetero)-clinic orbit of the completely integrable unperturbed problem, whose solutions have only poles in complex t. For $\varepsilon \neq 0$, however, these manifolds intersect at infinitely many points and infinitely branched multi-valued solutions appear, as predicted by Ziglin, with logarithmic singularities.

Following Ziglin's approach, in section 3, we extend and apply his theorem to periodically driven, generally *non-Hamiltonian* systems of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \varepsilon \boldsymbol{g}(\boldsymbol{x}, t), \quad \boldsymbol{g}(\boldsymbol{x}, t) = \boldsymbol{g}(\boldsymbol{x}, t + 2\pi),$$
$$\boldsymbol{x} \equiv (x, y), \qquad (1.4)$$

where $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ has the Painlevé property and contains a "smooth" separatrix. We prove for all such functions $\boldsymbol{f}(\boldsymbol{x})$, analytic in x, y and a general, analytic perturbation $\boldsymbol{g}(\boldsymbol{x}, t)$, that the "splitting" of this separatrix into transversally intersecting invariant manifolds (for $\varepsilon \neq 0$), necessarily implies the existence of infinitely branched, multi-valued solutions $\boldsymbol{x}(t)$ of (1.4). Their singularities are written down explicitly and are found again to be of logarithmic type, as in the Hamiltonian example of section 2.

In section 4, we investigate the *patterns* formed by the singularities of (1.4) in the complex t-plane, using the ATOMCC computer program of Chang and Corliss [19] to solve numerically (in the complex domain) typical examples of systems (1.4), like Duffing's equation.

$$\dot{x} = y, \quad \dot{y} = -Ax - x^3 + \varepsilon(\gamma \cos t - \delta y). \tag{1.5}$$

In the Painlevé integrable case, $\varepsilon = 0$, the (Jacobi elliptic function) solutions of (1.5) have a doubly periodic, infinite lattice of poles in the complex t-plane [20]. With $\varepsilon \neq 0$, however, these poles become logarithmic singularities; and the lattice "collapses" to two rows of "*chimneys*" (one above and one below the $\operatorname{Re} t$ axis) each chimney extending, in general, to a finite distance and becoming infinitely thin in the $\operatorname{Im} t$ direction. On these chimneys, singularities appear to accumulate densely, in a way which is asymptotically *insensitive* to changes in the parameter values or initial conditions.

Such chimney patterns have been observed by other researchers in non-integrable, two-degree of freedom Hamiltonian systems, with *complex* power behavior $(t - t_*)^\alpha$ near a singularity [3]. Using some of their ideas, we have been able to explain some interesting features of the chimneys of eq. (1.5): for example, we have derived analytical formulas which describe the numerically observed "condensation" of the singularity pattern, and the gradual "collapse" of the chimneys towards the $\operatorname{Re} t$ axis, as $\gamma > 0$ is increased in (1.3) and the motion becomes more chaotic.

Finally, in section 5, we offer some concluding remarks and discuss the importance of our results in the broader context of the singularity analysis of nonlinear dynamical systems.

2. Ziglin's theorem on a simple Hamiltonian example

In this section, we shall state Ziglin's theorem on the splitting of separatrices in 2-degree of freedom Hamiltonian systems and illustrate it explicitly on a simple example. Consider the Hamiltonian

$$\mathscr{H} = \mathscr{H}_0(x, y, I) + \varepsilon \mathscr{H}_1(x, y, I, \phi), \tag{2.1}$$

where (y, x) and (I, ϕ) are canonically conjugate pairs of momentum-position and action-angle variables respectively. With Ziglin [14], we now make the following assumptions about (2.1):

(A1) $\mathscr{H}$ is real and analytic in some domain of $\boldsymbol{x} \equiv (x, y)$, $|I - I_0| < \mu$, $|\varepsilon| < \mu$ and 2π-periodic in ϕ.

(A2) For $\varepsilon = 0$, $I = I_0$, (2.1) has two hyperbolic fixed points $\boldsymbol{x}_+$, $\boldsymbol{x}_-$ (not necessarily distinct), joined by a doubly asymptotic solution $\hat{\boldsymbol{x}}(t)$ with $\hat{\boldsymbol{x}}(t) \to \boldsymbol{x}_\pm$ as $t \to \pm\infty$.

(A3) $(\partial \mathscr{H}_0 / \partial I)(\hat{\boldsymbol{x}}(t), I_0) > c > 0$ for all t, and the solution

$$\hat{z}(t) = \left(\hat{\boldsymbol{x}}(t), I_0, \hat{\phi}(t)\right),$$

$$\hat{\phi}(t) = \int^t \frac{\partial \mathscr{H}_0}{\partial I} \, \mathrm{d}t', \tag{2.2}$$

can be analytically continued to the strip $\Pi: 0 \leq \operatorname{Im} t \leq 2\pi/\lambda_+$ (where λ_+ is the positive eigenvalue of the linearized system about $\boldsymbol{x}_+$) and has no more than a finite number of singular points in Π.

(A4) $\mathscr{H}(z, \varepsilon)$ can be analytically continued for complex z and $(\partial \mathscr{H}_0 / \partial I)(\hat{z}(t))$ and $(\partial \mathscr{H}_1 / \partial \phi)(\hat{z}(t, \phi_0))$ are single valued in Π, for all $\phi_0 \in \mathbb{R}$, where $\hat{z}(t, \phi_0)$ denotes the solution $\hat{z}(t)$ of (2.2) with $\hat{\phi}$ replaced by $\hat{\phi} + \phi_0$.

Theorem (Ziglin [14]). Under the above assumptions and if $(\partial \mathscr{H}_1 / \partial \phi)(\hat{z}(t, \phi_0))$ has nonzero sum of residues in Π (for at least one ϕ_0), the system (2.1) possesses *multiple-valued* solutions $I(t) = I_0$

$+\varepsilon I_1(t)+\ldots$, since

$$\Delta I_1=\oint_\Gamma \frac{\mathrm{d}I_1}{\mathrm{d}t}\,\mathrm{d}t=-\oint_\Gamma \frac{\partial \mathscr{H}_1}{\partial \phi}(\hat{z}(t,\phi_0))\,\mathrm{d}t \neq 0 \tag{2.3}$$

for some contour $\Gamma \subset \Pi$. In fact, since for any given ϕ_0, going around Γ changes the value of I_1 by the same amount ΔI_1, we conclude that $I(t)$ is *infinitely* branched in the complex t-plane much like a $\log t$ function.

The connection between (2.3) above and the splitting of separatrices comes from Ziglin's proof [14] that (2.3) implies that the following integral does not vanish identically:

$$J(\phi_0)=\int_{-\infty}^{\infty}\{H_0,H_1\}(\hat{z}(t),\phi+\phi_0)\,\mathrm{d}t \not\equiv 0, \tag{2.4}$$

where $\{.,.\}$ denotes the Poisson bracket, and H_0, H_1 are related to the original Hamiltonians by solving (2.1) for a (single-valued) I on a constant energy surface

$$-I=H_0(x,y)+\varepsilon H_1(x,y,\phi)+\ldots. \tag{2.5}$$

It is well-known from the work of Poincaré [21], Melnikov [15] and others [16] that as a direct consequence of (2.4) the stable manifold of $\boldsymbol{x}_+$ and the unstable one of $\boldsymbol{x}_-$ "*split*", and *intersect transversally* if $J(\phi_0)$ has simple zeros in ϕ_0.

More than that, Ziglin also proves that (2.3) implies that the Hamiltonian system (2.1) does *not* possess a second analytic integral independent of $\mathscr{H}$ for any sufficiently small $|\varepsilon| \neq 0$ [14]. We concentrate here, however, on the connection between transversal intersections of separatrices and infinitely branched multi-valued solutions since it is this aspect of Ziglin's result that we wish to extend to non-Hamiltonian systems in the next section.

Let us first attempt to elucidate this connection here on a simple example described by the Hamiltonian

$$H=y(x-x^2)-\varepsilon x\cos t=H_0+\varepsilon H_1, \tag{2.6}$$

whose associated equations of motion

$$\dot{x}=x-x^2, \quad \dot{y}=-y+2xy+\varepsilon\cos t, \tag{2.7}$$

are easily seen to possess, for $\varepsilon=0$, two hyperbolic fixed points $\boldsymbol{x}_-=(0,0)$ and $\boldsymbol{x}_+=(1,0)$ in the x, y plane. In the unperturbed case, $\boldsymbol{x}_\pm$ are joined by a single, "smooth" separatrix, along the $(0,1)$ interval of the x-axis, as in fig 1a, on which the solution $\hat{\boldsymbol{x}}(t)=(\hat{x}(t),\hat{y}(t))$ is

$$\hat{x}(t)=[1+\exp(t_0-t)]^{-1}, \quad \hat{y}(t)=0. \tag{2.8}$$

The question is what happens to this separatrix for $\varepsilon \neq 0$ and what does this imply about the singularities of the general solution of (2.7) (see fig. 1b).

Following Ziglin's approach, we first rewrite (2.6) as a 2-degree-of-freedom Hamiltonian

$$\mathscr{H}=y(x-x^2)+I-\varepsilon x\cos\phi=\mathscr{H}_0+\varepsilon\mathscr{H}_1, \tag{2.9}$$

cf. (2.1), for which all the assumptions of Ziglin's theorem are seen to be satisfied. We now calculate the residues of $(\partial\mathscr{H}_1/\partial\phi)(\hat{z}(t,\phi_0))$, using (2.8), and find that

$$\operatorname{Res}\frac{\partial\mathscr{H}_1}{\partial\phi}(\hat{z},\phi_0)=\operatorname{Res}\frac{\sin(t+t_0)}{1+\exp(-t)}$$
$$=\sin(\mathrm{i}\pi+t_0)\not\equiv 0,$$

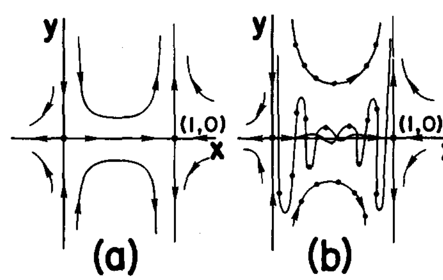

Fig. 1. Schematic representation of the time evolution in the x, y-plane of eq. (2.7) at (a) $\varepsilon=0$; and (b) on its surface of section at $\varepsilon \neq 0$.

over a contour γ enclosing the pole at $t = \mathrm{i}\pi$. We conclude, therefore, that the function $I(t)$ changes by the same amount $\Delta I \neq 0$, when going around γ, and hence is multi-valued with infinitely many Riemann sheets.

This leads, by the residue theorem, to the result [14]

$$J(\phi_0) = \int_{-\infty}^{\infty} \{H_0, H_1\}(x, \phi + \phi_0)\,\mathrm{d}t$$
$$= \int_{-\infty}^{\infty} \left[-\frac{\mathrm{d}\mathscr{H}_1}{\mathrm{d}t} + \frac{\partial \mathscr{H}_1}{\partial \phi} \right] (\hat{z}(t, \phi_0))\,\mathrm{d}t$$
$$\neq 0, \tag{2.10}$$

which, in our simple example is established directly, using $\mathscr{H}_1 = H_1 = x \cos \phi$.

The integration in (2.10) is performed over a rectangular contour Γ, whose upper side crosses the Im t axis at $\mathrm{i}\,2\pi$. But the first integral in (2.10) with ϕ replaced by t, is none other than the Poincaré–Melnikov integral [15, 16], which can also be evaluated directly by the residue theorem (over a similar contour Γ) to yield

$$J(t_0) = \int_{-\infty}^{\infty} \{H_0, H_1\}(\hat{x}, t + t_0)\,\mathrm{d}t$$
$$= \int_{-\infty}^{\infty} \frac{\cos(t + t_0)}{4\cosh^2(t/2)}\,\mathrm{d}t$$
$$= 4\pi \,\mathrm{csch}\, \pi \cos t_0, \tag{2.11}$$

where we have used (2.8). The fact that $J(t_0)$ has simple zeros in t_0 – as (2.11) clearly shows – implies that the stable and unstable manifolds of (1,0) and (0,0), along the x-interval [0,1] intersect transversally, as indicated in fig. 1b.

The infinitely branched multi-valuedness of $I(t)$ must, of course, be shared also by $y(t)$ and/or $x(t)$, since, by (2.9), I is an analytic function of x and y on every constant energy surface $\mathscr{H}$= const. But then, this could have been independently verified by analyzing the solutions $y(t)$ and $x(t)$ of eq. (2.7) directly! These can be explicitly written down for this simple example:

$$y(t) = \cosh^2\left(\frac{t - t_0}{2}\right)\left\{ y(t_0) + \varepsilon \int_0^{t - t_0} \frac{\cos(t' + t_0)}{\cosh^2(t'/2)}\,\mathrm{d}t' \right\}, \tag{2.12a}$$

$$x(t) = [1 + \exp(t_0 - t)]^{-1}, \tag{2.12b}$$

t_0, $y(t_0)$ arbitrary constants.

Now, while $x(t)$ is certainly meromorphic in t, $y(t)$ has a logarithmic singularity, as can be seen by expanding the integrand in (2.12a) and integrating term by term

$$y(t) = \varepsilon(-\tau^2/4 + \dots) \times \left(\frac{4\cos t_*}{\tau} + 4\sin t_* \log \tau + \frac{4\cos t_0}{(2k+1)\mathrm{i}\pi} - 4\sin t_0 \log[-\pi\,\mathrm{i}(2k+1)] + \dots \right) + y(t_0)(-\tau^2/4 + \dots) \times (+ \dots : \text{higher integer powers of } \tau), \tag{2.13}$$

where $\tau \equiv t - t_*$, $t_* \equiv t_0 + \mathrm{i}(2k+1)\pi$ (k any integer).

In some sense, of course, the example of this section can be considered explicitly solvable. It is certainly *separable*, and its most general solution can be reduced to a final quadrature, cf. (2.12a), which converges for all t. On the other hand, it does possess transversal separatrix intersections and heteroclinic orbits – see e.g. the surface of section of fig. 1b – usually associated with chaotic behavior in non-integrable (non-separable) Hamiltonian systems [16, 18]. Note, however, that the Poincaré map of (2.6) has *no horseshoes* [17], for $\varepsilon \neq 0$, since the stable and unstable manifolds of (0,0) and (1,0) respectively extend to infinity (see fig. 1b), and do not intersect each other transversally in the finite domain. Thus, we have here one more example of the fact that intersecting invariant manifolds do not always imply chaos (see also [22]).

3. Separatrix splitting in a non-Hamiltonian system

We now turn to the analysis of separatrix splitting in *non-Hamiltonian* systems by considering the following example of two periodically forced, first order o.d.e.'s

$$\begin{aligned} \dot{x} &= x - x^2 + 3xy \equiv f_1(x, y) + \varepsilon g_1(x, y, t), \\ \dot{y} &= -y - y^2 + 3xy + \varepsilon(\gamma \cos \omega t + \delta x) \\ &\equiv f_2(x, y) + \varepsilon g_2(x, y, t), \end{aligned} \tag{3.1}$$

where the perturbations $g_i(x, y, t) = g_i(x, y, t + 2\pi/\omega)$, $i = 1, 2$ will later be allowed to be more general functions of x and y. We shall demonstrate on this model, and for a large class of perturbations $g_i(x, y, t)$, that transversal separatrix intersections are intimately related to infinitely branched multi-valued solutions, in a similar way as in the Hamiltonian case of section 2.

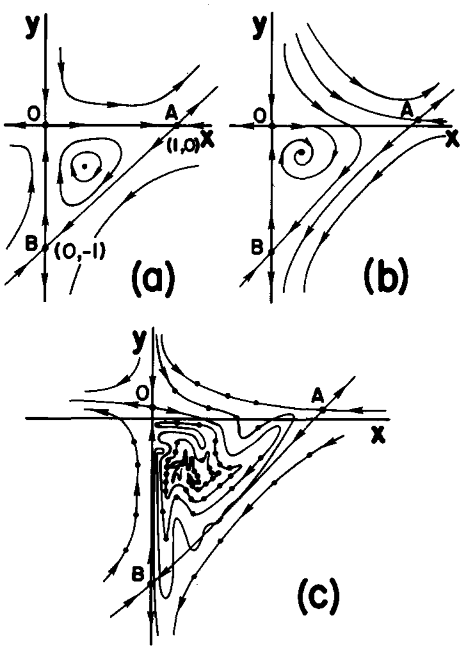

Fig. 2. Schematic representation of the time evolution in the x, y-plane of eq. (3.1) at (a) $\varepsilon = 0$; (b) $\varepsilon \neq 0$, $\delta \neq 0$, $\gamma = 0$; and (c) $\varepsilon \neq 0$, $\delta \neq 0$, and $\gamma \neq 0$ on the surface of section.

For $\varepsilon = 0$, the non-Hamiltonian system (3.1), $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, is known to possess the Painlevé property [23]. Its general solution can, in fact, be explicitly obtained in terms of elliptic functions [23], but this will not be important for the discussion here. Instead, we shall concentrate on the motion of the system in the x, y plane (or, surface of section $\varepsilon\gamma \neq 0$), depicted in fig. 2.

There are 3 hyperbolic fixed points for $\varepsilon = 0$: (0,0), (1,0) and (0, −1) joined smoothly by separatrices in the shape of a right triangle OAB (see fig. 2a). For $\varepsilon \neq 0$, and $\delta \neq 0$, but $\gamma = 0$, these separatrices split as indicated in fig. 2b, with the *stable* manifolds of (1,0) and (0, −1) coming in from (0, +∞), while the *unstable* manifold of (0,0) is attracted by the stable fixed point. We wish to examine what happens to these manifolds for $\varepsilon\gamma\delta \neq 0$:

We begin our analysis by evaluating Melnikov's integral [16] for the splitting of separatrix AB,

$$J(t_0) = \int_{-\infty}^{\infty} [f_1 g_2 - f_2 g_1](\hat{\boldsymbol{x}}, t) \times \exp\left[-\int_0^{t-t_0} \mathrm{Tr}\, \mathbf{D}\boldsymbol{f}(\hat{\boldsymbol{x}})\, \mathrm{d}t'\right] \mathrm{d}t, \tag{3.2}$$

on which the solution of the *unperturbed* system is

$$\hat{\boldsymbol{x}} \equiv (\hat{x}(t), \hat{y}(t)) = \left(\frac{1}{1 + \exp[2(t - t_0)]}, \frac{-\exp[2(t - t_0)]}{1 + \exp[2(t - t_0)]}\right), \tag{3.3}$$

and $\mathbf{D}\boldsymbol{f}$ denotes the Jacobian matrix of the unperturbed vector field $\boldsymbol{f} = (f_1, f_2)$ in (3.1). Substituting (3.3) in (3.2) and using (3.1) yields the following integral:

$$J(t_0) = \int_{-\infty}^{\infty} \frac{-\exp(t - t_0)}{1 + \exp[2(t - t_0)]} \times \left(\gamma \cos \omega t + \frac{\delta}{1 + \exp[2(t - t_0)]}\right) \mathrm{d}t. \tag{3.4}$$

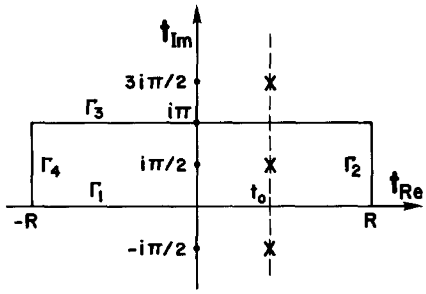

Fig. 3. Contour Γ in the complex time plane used to solve eq. (3.8).

By contour integration around a rectangular contour Γ, whose upper side crosses the $\operatorname{Im} t$ axis at $t = i\pi$ (enclosing thus only one pole of the integrand at $t_* = t_0 + i\pi/2$, see fig. 3), (3.4) is evaluated by the residue theorem:

$$J(t_0) = \frac{-\pi\gamma}{e^{\pi\omega/2} + e^{-\pi\omega/2}} \cos\omega t_0 - \frac{\pi\delta}{4} \not\equiv 0. \tag{3.5}$$

Note, first, that $J(t_0)$ has *simple zeroes*, and hence, according to Melnikov, there are transversal intersections of separatrices between A and B, if

$$\left|\frac{\gamma}{\delta}\right| > \frac{\cosh(\pi\omega/2)}{2} \equiv R_\omega^{AB}. \tag{3.6}$$

As a check for the accuracy of these formulas we calculated for $\omega = 2$, $R_2^{AB} = 5.796$ from (3.6) and found that for $\gamma = 0.2$ the critical δ value for separatrix intersections is $|\delta_{th}| = 0.0345$. Taking initial conditions along vertical directions above $x > 0$ and near the y-axis of the Poincaré surface of section (see fig. 2c), we observed numerically that separatrices intersect for $|\delta_{num}| \lesssim 0.038$, a nearly 10% discrepancy from the theoretical value.

As fig. 2c shows, at $|\delta_{num}| \gtrsim 0.038$ the stable and unstable manifolds of the fixed points A and O respectively, have not intersected as yet. This is also predicted theoretically by calculating R_ω^{OA}, cf. (3.6), using the corresponding Melnikov integral, exactly as was done above for separatrix AB. For $\omega = 2$, $R_2^{OA} = 6.352$ and $|\delta_{th}^{OA}| = 0.0315$, for $\gamma = 0.2$, suggesting that there are no transversal intersections between O and A at $|\delta| \approx 0.038$, in agreement with fig. 2c.

Returning to the solutions between A and B for $\varepsilon \neq 0$, direct singularity analysis yields the following asymptotic expansions [6,7]:

$$\begin{aligned} x(t) &= \frac{-1}{2\tau} + c_0 + c_1\tau + \frac{\varepsilon}{4}\Big(2\gamma\cos\omega t_* \\ &\quad - \delta - \frac{19}{64}\delta^2\varepsilon\Big)\tau\ln\tau + \mathcal{O}(\tau^2\ln\tau), \\ y(t) &= \frac{-1}{2\tau} + d_0 + d_1\tau - \frac{\varepsilon}{4}\Big(2\gamma\cos\omega t_* \\ &\quad - \delta - \frac{19}{64}\delta^2\varepsilon\Big)\tau\ln\tau + \mathcal{O}(\tau^2\ln\tau), \end{aligned} \tag{3.7}$$

$\tau \equiv t - t_*$, t_* being the location of the singularity, c_0, d_0, d_1 known constants and t_*, c_1 *free* to be determined by the initial conditions. It is easy to verify that for $\varepsilon = 0$ (3.7) reduces to the series expansions of $\hat{x}(t)$, $\hat{y}(t)$ of (3.3) with $t_* = t_0 + (2k + 1)\pi i/2$. On the other hand, between the fixed points O and A in fig. 2b, c the solutions of (3.1) possess a singularity type different than (3.7): At leading order, as $\tau = (t - t_*) \to 0$, $x \sim 1/\tau$ and $y \sim -\varepsilon\delta/3$, with logarithms entering at higher orders, which reduces to the separatrix OA at $\varepsilon = 0$.

In seeking to establish a direct connection between (infinitely branched) multi-valuedness of solutions and separatrix intersections, we isolate first the part of the Melnikov integral (3.2) that depends on t_0. For the specific example of (3.1) this is

$$\begin{aligned} J_1(t_0) &= \int_{-\infty}^{\infty} \hat{x}(1 - \hat{x} + 3\hat{y})\cos\omega(t + t_0) \\ &\quad \times \exp\left[-\int_0^t (\hat{x} + \hat{y})\,dt'\right] dt, \end{aligned} \tag{3.8}$$

which has simple zeros in $t_0(dJ_1/dt_0 \not\equiv 0)$, as was already discussed above, and its value is given by the $\cos\omega t_0$ dependent term in (3.5). The evaluation of (3.8), over the contour Γ of fig. 3, proceeds by observing that the integral over Γ_3 is proportional to $J_1(t_0)$ (i.e. that over Γ_1), while the ones over Γ_2 and Γ_4 vanish in the limit ($R \to \infty$). Thus if we could find a function $h(x, y, t) = \varepsilon h_1(x, y, t) + \mathcal{O}(\varepsilon^2)$ such that

$$\Delta h = \oint \frac{\mathrm{d}h}{\mathrm{d}t}\bigg|_{x=\hat{x}} \mathrm{d}t \propto \varepsilon \int_{-\infty}^{\infty} \hat{x}(1 - \hat{x} + 3\hat{y}) \times \cosh(t - t_0) \cos\omega(t + t_0)\, \mathrm{d}t \not\equiv 0, \tag{3.9}$$

[where (3.3) has been used to evaluate the exponential in (3.8)] h_1, and hence $h(x, y, t)$, would be a multi-valued, infinitely branched function of t, when evaluated along a solution $\boldsymbol{x}(t) = \hat{\boldsymbol{x}}(t) + \mathcal{O}(\varepsilon)$ of the *perturbed* problem. This $h(x, y, t)$ would then play a role analogous to that of the action I in Ziglin's theorem for Hamiltonian systems, cf. (2.3), (2.4) and section 2.

The important question now is: Does there exist such an $h(x, y, t)$ and if so, what kind of function of x, y, t is it? To answer this question, we first rewrite (3.9) as a linear, first order partial differential equation, valid for $\boldsymbol{x} = \hat{\boldsymbol{x}} + \mathcal{O}(\varepsilon)$, with $h = \varepsilon h_1 + \mathcal{O}(\varepsilon^2)$, at order ε:

$$f_1(x, y)\frac{\partial h_1}{\partial x} + f_2(x, y)\frac{\partial h_1}{\partial y} + \frac{\partial h_1}{\partial t} = x(1 - x + 3y)\cosh(t - t_0)\cos\omega(t + t_0), \tag{3.10}$$

where we have used (3.1) and omitted a proportionality constant in front of the rhs. It is now simple to ascertain that there exists a domain D, near the origin, in complex $x, y, t - t_*$ space, over which the rhs of (3.10) and, of course, the coefficients of the first derivatives on the lhs are analytic, single-valued functions of x, y and t, cf. (3.1), (3.3).

Under the above conditions (and after specifying equally "smooth" data at $t = t_*$ and on ∂D), the celebrated Cauchy–Kowalevskaya theorem can be invoked [24] to guarantee that (3.10) has a *unique* solution $h_1(x, y, t)$, which is analytic and single-valued in x, y, t over D. But since according to (3.9), $h(t)$ is *multi-valued* as a function of t *alone*, it clearly follows, from its "smooth" dependence on x, y, that this multivaluedness must be a property of the solutions $x(t)$, $y(t)$ of the perturbed problem.

Thus, we have established, on the non-Hamiltonian system (3.1), a direct connection between transversal intersection of invariant manifolds and infinitely branched, multi-valued solutions, at least for small values of the perturbation parameter ε. Our arguments parallel those of Ziglin, on near-integrable two-degree of freedom Hamiltonian systems [14]. Our result, although obtained with the aid of a specific example of $g_i(\boldsymbol{x}, t)$ actually holds for much more general perturbations in (3.1) provided they are analytic, single-valued functions of x, y and periodic in t: One simply has to concentrate on their periodic part and form integrals like $J_1(t_0)$, cf. (3.8) leading to the existence of multivalued $h(t)$, analytic in x, y, and hence to multivalued $x(t)$ and/or $y(t)$. One may have to expand the g_i's and $h(t)$ in Fourier series but the approach will be entirely analogous to the above.

Note also that the final form of the rhs of the partial differential equation for $h_1(x, y, t)$, cf. (3.10), will depend on the exponential term in (3.2), (3.8), involving the trace of the Jacobian $\mathbf{D}\boldsymbol{f}$ evaluated on the unperturbed separatrix $\boldsymbol{x} = \hat{\boldsymbol{x}}$. If this term is analytic in t, as e.g. in the case of the separatrix AB here, we can apply the Cauchy–Kovalevskaya theorem directly.

In fact, one can always find a function $E(x, y)$ analytic in a neighborhood of the separatrix, which, when evaluated on the unperturbed separatrix, equals the exponential factor in (3.2) *exactly*. This $E(x, y)$ is a solution of the linear partial differential equation

$$f_1(\partial E/\partial x) + f_2(\partial E/\partial y) = -E(\mathrm{Tr}\,\mathbf{D}\boldsymbol{f}), \tag{3.11}$$

as can be easily verified by differentiating $\hat{E}(t) = E(x, y)|_{\boldsymbol{x}=\hat{\boldsymbol{x}}}$ w.r. to t, using $\mathrm{d}\hat{E}/\mathrm{d}t = -\hat{E}(\mathrm{Tr}\,\mathbf{D}\boldsymbol{f}|_{\boldsymbol{x}=\hat{\boldsymbol{x}}})$. The analyticity of $E(x, y)$ then follows by applying the Cauchy–Kovalevskaya theorem to (3.11), and then the extension of Ziglin's theorem to general non-Hamiltonian systems (3.1), with analytic $\boldsymbol{f}$ and $\boldsymbol{g}$, is complete.

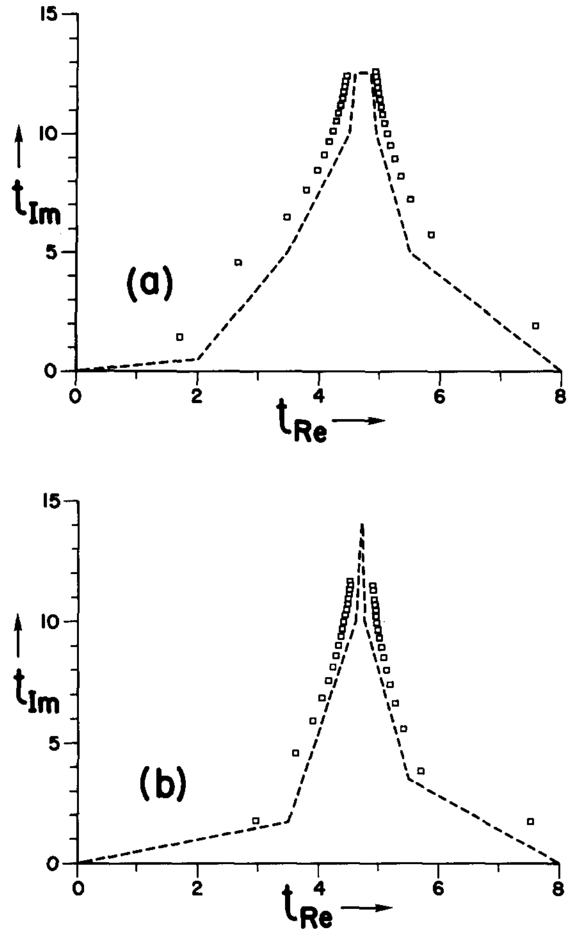

Fig. 4. Accumulation of singularities into "chimney-patterns" for eq. (4.1) at $A = -1$ and $\delta = 0.25$ for (a) $\gamma = 0.2$; and (b) $\gamma = 0.4$.

4. Accumulation of singularities in the complex t-plane

Let us now turn from the analysis of the *type* of singularities present in the solutions of a dynamical system to their actual *location* in the complex t-plane. We will discover that they form there certain characteristic patterns from which we can infer a number of fundamental properties of the system, e.g. its non-integrability and tendency to become more chaotic "globally", as some of its parameters are varied. Concerning the behavior of individual orbits, on the other hand, it appears that it is mainly the singularities closest to the Re t axis, which indicate whether the orbit is in a chaotic region, approaches asymptotically a simple attractor, etc.

In order to systematically study singularity patterns in dynamical systems, we have decided to concentrate first on their significance in periodically driven, simple anharmonic oscillators, with and without damping. To this end, we have studied, in some detail, Duffing's equation

$$\ddot{x} + \delta\dot{x} + Ax + x^3 = \gamma\cos\omega t, \tag{4.1}$$

using the very efficient algorithm ATOMCC of Chang and Corliss [2, 3, 19]. This program calculates the first thirty terms in a Taylor expansion from which a radius of convergence is obtained. It then goes on a distance of one radius via a prescribed path and repeats the procedure. The intersection of the circles indicates the presence of a singularity. We located singularities of the solution of (4.1) for a wide variety of parameter values and initial conditions. We have found that *non-integrability* – in the sense of the presence of a second frequency ω in (4.1), at $\gamma \neq 0$ – was always associated with a "*chimney pattern*" in the com-

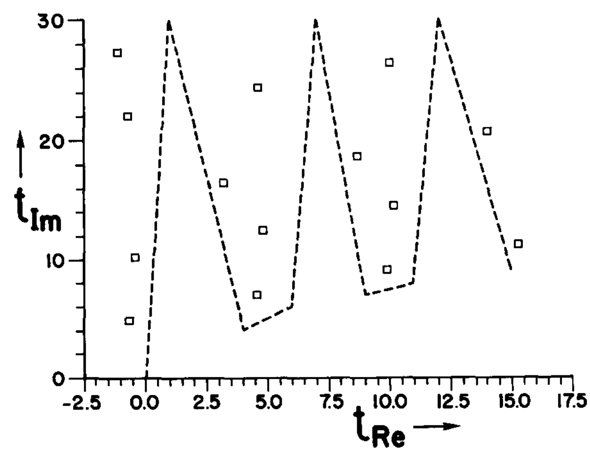

Fig. 5. Typical singularity pattern for eq. (4.1) in the absence of a driving force ($\gamma = 0$) at $A = 0.3$ and $\delta = 0.4$. Note the "sparsity" of the pattern compared with fig. 4.

plex t-plane, on which the (logarithmic) singularities accumulated densely near the top of every chimney (see fig. 4). Moreover, each of these chimneys seemed to close completely, extending to finite or – for $\delta = 0$ – infinite distances in the Im t direction.

A second result of our investigations was the fact that these chimney patterns are observed to get "denser" and gradually "collapse" towards the Re t axis as the value of γ in (4.1) is increased, cf. fig. 4a, b. In other words, the accumulation of singularities appears to increase as the system is driven harder and then global motion becomes more chaotic. On the other hand, when the forcing is completely removed [setting $\gamma = 0$ in (4.1)] the singularity patterns become, by comparison with fig. 4, distinctly more sparse (see fig. 5), showing no accumulation whatsoever in the complex t-plane.

The formation of these chimneys in the $\gamma \neq 0$ case, may be explained by the following argument: Expanding the solution of (4.1) near one of its (movable) singularities at $t = t_*$, say, we obtain a series of the form

$$x(t) = \frac{\mathrm{i}\sqrt{2}}{\tau} + a_0 + a_1\tau + a_2\tau^2 + b_1\tau^3 \ln\tau + a_3\tau^3 + \mathcal{O}\left(\tau^3(\ln\tau)^2\right), \qquad (4.2)$$

where $\tau = t - t_*$, a_3 is the second free constant (the other one is t_*) and the coefficient of the first logarithmic term is

$$b_1 = \frac{\mathrm{i}\sqrt{2}}{15}\delta^2\left(\frac{2\delta^2}{9} - A\right) + \frac{\gamma\delta}{5}\cos\omega t_* - \frac{\gamma\omega}{5}\sin\omega t_*. \qquad (4.3)$$

Clearly, as the $|\mathrm{Im}\, t_*|$ increases, the *magnitude* of b_1 grows exponentially, by virtue of the $\exp(\omega|\mathrm{Im}\, t_*|)$ contribution of the trigonometric terms in (4.3). This would progressively decrease the radii of convergence of (4.2) in that direction and thus account for the observed accumulation of singularities and the formation of chimneys, as in fig. 4. By the same token, the absence of the driving force ($\gamma = 0$) would not have such an effect on the radii of convergence, and no accumulation should be observed in that case, as indeed none was seen in fig. 5.

The above argument can be made more rigorous by a careful analysis of the so-called ψ-series expansion of the solution of (4.1) containing the most divergent terms, one from each power of τ in (4.2) [2, 3]. This analysis will be presented in detail elsewhere [25]. The main result is that the distance between two nearest singularities on the chimneys of fig. 4 is directly related to the radius of convergence of a Taylor series about the singularity, given by [25].

$$\rho \propto |\gamma\omega \sin\omega t_*|^{-1}, \qquad (4.4)$$

which supports the more heuristic argument given above in connection with (4.2) and (4.3). Moreover, besides explaining the chimney structure, (4.4) shows also why – with increasing γ – the singularity accumulation on them gets "denser" and the chimneys are gradually seen to "collapse" towards the Re t axis, as was numerically observed in fig. 4.

Singularity chimneys in the complex t-plane are not new. Other researchers have also observed them in the study of non-integrable two-degree-of-freedom Hamiltonian systems [3]. What we wish to stress here is that they are quite ubiquitous in non-integrable, nonlinear oscillator systems and need to be studied further, as they seem to withhold information about the solutions, which is to a large extent *independent* of particular parameter values and initial conditions: For example, if we number by n the "rows" of singularity pairs of each chimney, and compute the distance d_n, between the two singularities of each row, and the separation s_n, between the nth and $(n+1)$st row, we find

$$s_n \sim n^{-\alpha}, \quad d_n \sim n^{-\beta}, \quad n \text{ "large"}. \qquad (4.5)$$

Our computations show that the exponents α, β appear to converge, as n increases, to *positive* values, which are surprisingly insensitive to specific

Table I
Values of α, β in (4.5) for chimneys of eq. (4.1)

Eq. parameters/In. cond.	α	β
$\delta=0$, $A=0.3$;		
$\delta=0.5$: quasiperiodic orbits	0.93 ± 0.01	0.77 ± 0.01
$\delta=2.8$: chaotic orbits		
$\delta=0.25$, $A=-1.0$;		
$\gamma=0.05$: regular orbits, no intersecting separatrices	1.08 ± 0.01	0.83 ± 0.01
$\gamma=0.2$: regular orbit attracted by fixed point	1.07 ± 0.02	0.81 ± 0.02
$\gamma=0.2$: irregular orbit trapped temporarily by intersecting separatrices	1.05 ± 0.01	0.80 ± 0.01
$\gamma=0.4$: irregular orbit on the strange attractor	1.04 ± 0.01	0.81 ± 0.01
$\gamma=1.0$: regular orbit attracted by fixed point	1.06 ± 0.02	0.78 ± 0.02

choices of parameters and initial conditions (see table I). The fact that $\alpha > 0$ and $\beta > 0$, on the other hand, indicates that, as $n \to \infty$, the singularities accumulate infinitely densely, and each chimney, becomes infinitely thin, extending to finite (or, if $\alpha < 1$, infinite) distances in the complex t-plane.

Returning, briefly, to the examples of sections 2 and 3 of this paper we make the following remarks: First, the Hamiltonian system of section 2, eg. (2.6), (2.7), being separable and linearizable, has a very simple singularity structure: $x(t)$ and $y(t)$ have a single vertical column of evenly spaced singularities at $t_* = t_0 + (2k+1)\mathrm{i}\pi$. In the case of $x(t)$, these are movable poles, cf. (2.8), while for $y(t)$, they are *fixed* logarithmic singularities imposed by substituting the solution $x(t)$ of the first equation of (2.7) in the second, linear equation for $y(t)$.

The non-Hamiltonian system of section 3, on the other hand, being apparently non-separable (and non-linearizable) for $\varepsilon \neq 0$, possesses a much more complicated singularity structure: There are *three* distinct types of singularities t_* here, having leading behaviors (as $t \to t_*$)

$$\text{(i) } x \sim \frac{1}{\tau}, \quad y \sim \frac{-\varepsilon\delta}{3}, \quad \text{(ii) } x, y \sim \frac{-1}{2\tau}, \quad \text{(iii) } x \sim \tau^3, \quad y \sim \frac{1}{\tau}, \tag{4.6}$$

$\tau \equiv t - t_*$, corresponding to motions on the complex time extensions of separatrices OA, AB and OB respectively [see fig. 2a, (3.7) and the discussion below it]. It is interesting to note that logarithmic terms enter only in the higher orders of singularities (i) and (ii) in (4.6). Singularity type (iii) has no logarithms, for all ε, which may be related to the fact that separatrix OB has *no* transversal intersections for $\varepsilon \neq 0$, cf. fig. 2c. Searching numerically in the complex t-plane, we also saw "chimney"-looking structures here – only not as clearly as with Duffing's equation – mainly because of the entangled presence of more than one type of singularities.

Finally, we also observed, in agreement with other studies [1–3], that the oscillation amplitudes and quasi-periods of individual orbits, in real time, are directly related to the locations of the complex t-plane singularities closes to the Re t axis. Analytical formulas describing these connections precisely – at least in the case of certain periodic orbits of (4.1) – will be given in future publications [25].

5. Concluding remarks

Considerable progress has been made recently in the identification and study of new completely integrable dynamical systems, starting from the requirement that their solutions have the Painlevé property, i.e. that their only movable singularities in the complex t-plane are poles [9–13]. In particular, in Hamiltonian systems, the Painlevé property has been associated with a whole class of so-called *algebraically completely integrable* systems, which possess a full set of *rational* integrals, and whose motion is linearizable on complex tori [21, 26].

But if poles are related to complete integrability, what then is the singularity structure of chaos, in non-integrable systems? This is the question we have asked, and made a first attempt to answer, analytically as well as numerically, in the present paper. More specifically, we have concentrated on one characteristic phenomenon, occurring in non-integrable dynamical systems: the transversal intersections of invariant manifolds (separatrices), near which "most" orbits are known to exhibit chaotic behavior, at least in problems of low dimensionality [16–18].

What we have tried to show in this work is that there exists now the theoretical foundation for a comprehensive singularity analysis of intersecting separatrices in non-integrable dynamical systems. This foundation is provided by a theorem of Ziglin on two degrees of freedom Hamiltonian systems, which we have extended and applied here to periodically forced, generally *non-Hamiltonian* flows in the plane. According to this theorem, transversal intersections of separatrices necessarily imply the existence of solutions which are multi-valued, with infinitely many branches in complex t.

So far, these results apply to low-dimensional systems and can be rigorously proved only in the near-integrable case, with the aid of first order perturbation theory. We have found, however, that the singularities they predict, can be explicitly constructed and shown to retain their form, even for larger values of the perturbation parameter. Thus we may formulate, for example, the following conjecture: Dynamical systems, Hamiltonian or not , possessing the Painlevé property, can have *no* intersecting separatrices and hence *no* regions of chaotic behavior or strange attractors, both known to be intimately related with that phenomenon [16–18].

Moreover, we have discovered that complementary to this analysis, a numerical investigation of the "pattern formations" of these singularities in the complex t-plane, can be quite useful in the study of chaotic motions in dynamical systems. In particular, we have observed that, as the periodic driving force on a non-linear oscillator increases, and the system becomes more chaotic, its associated "chimneys" of singularities in the complex t-plane get denser, and progressively "collapse" further towards the $\mathrm{Re}\, t$ axis. The singularities closest to $\mathrm{Re}\, t$ follow quite accurately the real time oscillations of each orbit. On the other hand, those furthest away from $\mathrm{Re}\, t$ accumulate infinitely densely, in a manner apparently independent of specific parameter values or initial conditions, suggesting that they may perhaps be related to a mathematical property of the more general solutions of the problem.

Of course, not all infinitely branched multi-valued solutions have logarithmic singularities. They may involve instead irrational or complex powers of $\tau(\equiv t - t_*)$, as is the case with certain Hamiltonian systems, where these singularities are sometimes seen to accumulate on self-similar natural boundaries [3–5]. Thus the occurrence of such structures, compared with that of chimney patterns, requires further investigation.

Another interesting question is the quantification of the growth of chaotic regions according to the progressively more dense accumulation of singularities, observed already in the first few chimneys in the complex t-plane. The need and potential usefulness of such results, of course, becomes greater the higher the dimensionality of the system. And since singularity analysis relies on relatively short integration paths in the complex t-plane it is there that its importance will be ultimately judged, i.e. in the qualitative and quantitative study of chaos in higher-dimensional dynamical systems.

Acknowledgements

This work was supported in part by the NSF grant no. DMS-8314172 and the D.O.E. contract DE-FG02-84ER40144. Two of the authors (T.B. and V.P.) also wish to acknowledge discussions with Prof. Drs. H. Capel, R. Helleman and J. Hijmans and the hospitality of the Institute of

Theoretical Physics, University of Amsterdam, the Netherlands, where part of this paper was written.

References

[1] U. Frisch and R. Morf, Phys. Rev. A 23 (1981) 2673.
[2] M. Tabor and J. Weiss, Phys. Rev. A 24 (1981) 2157.
[3] Y.F. Chang, M. Tabor and J. Weiss, J. Math. Phys. 23 (1982) 531; see also Physica 8D (1983) 183.
[4] D. Bessis and N. Chafee, On the existence and non-existence of natural boundaries for non-integrable dynamical systems, Preprint, School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia.
[5] H. Yoshida, Self-similar natural boundaries of non-integrable dynamical systems in the complex t-plane, Preprint, Dept. of Astronomy, Univ. of Tokyo, Tokyo, Japan.
[6] T. Bountis and H. Segur, in: Mathematical Methods in Hydrodynamics, M. Tabor and Y. Treve, eds., A.I.P. Conf. Proc. vol. 88 (A.I.P., New York, 1982).
[7] T. Bountis, in: Singularities and Dynamical Systems, S. Pnevmatikos, ed. (North-Holland, Amsterdam, 1984); see also Lect. Notes in Phys. 179 (1983) 227.
[8] S. Kowalevskaya, Acta Math. 14 (1890) 81. The work of Painlevé and co-workers is well described in E.L. Ince, Ordinary Differential Equations (Dover, London, 1956).
[9] T. Bountis, H. Segur and F. Vivaldi, Phys. Rev. A 25 (1982) 1257.
[10] A. Ramani, B. Grammaticos and B. Dorizzi, J. Math. Phys. 24 (1983) 2282; see also J. Math. Phys. 25 (1984) 481.
[11] H. Yoshida, Cel. Mech. 31 (1983) 363; 381.
[12] T. Bountis, A. Ramani, B. Grammaticos and B. Dorizzi, Physica 128A (1984) 268.
[13] T. Bountis, M. Bier and J. Hijmans, Phys. Lett. A 97 (1983) 11.
[14] S.L. Ziglin, Splitting of separatrices, branching of solutions and nonexistence of an integral in the dynamics of a solid body, Trans. Moscow Math. Soc. 1 (1982) 283–298.
[15] V.K. Melnikov, Trans. Moscow Math. Soc. 12 (1963) 1–57.
[16] J. Guckenheimer and P. Holmes, Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields (Springer, Berlin, 1983).
[17] R.H.G. Helleman, in: Fundamental Problems in Statistical Mechanics, vol. 5, E.G.D. Cohen, ed. (North-Holland, Amsterdam, 1981).
[18] M. Lieberman and A. Lichtenberg, Regular and Stochastic Motion (Springer, Berlin, 1983).
[19] Y.F. Chang and G. Corliss, J. Inst. Math. Applics. 25 (1980) 349.
[20] H.T. Davis, Introduction to Non-Linear Differential and Integral Equations (Dover, London, 1962).
[21] V.V. Kozlov, Integrability and non-integrability in hamiltonian mechanics, Russian Math. Surveys 38 (1983) 1–76.
[22] R. Devaney, Am. J. Math. 100 (1978) 631.
[23] A. Ramani, B. Dorizzi, B. Grammaticos and T. Bountis, J. Math. Phys. 25 (1984) 878.
[24] Francois Treves, Basic Partial Differential Equations (Academic Press, New York, 1975).
[25] T. Bountis, M. Bier and V. Papageorgiou, Singularity patterns in complex time and chaotic behavior in Duffing's equation, in preparation.
[26] M. Adler and P. van Moerbeke, Commun. Math. Physics 83 (1982) 83; Invent. Math. 67 (1982) 297.