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A reaction–diffusion model for market fluctuations – A relation between price change and traded volumes

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A B S T R A C T

Two decades ago Bak et al. (1997) [3] proposed a reaction–diffusion model to describe market fluctuations. In the model buyers and sellers diffuse from opposite ends of a 1D interval that represents a price range. Trades occur when buyers and sellers meet. We show analytically and numerically that the model well reproduces the square-root relation between traded volumes and price changes that is observed in real-life markets. The result is remarkable as this relation has commonly been explained in terms of more elaborate trader strategies. We furthermore explain why the square-root relation is robust under model modifications and we show how real-life bond market data exhibit the square-root relation.

1. Introduction

The markets for stocks, bonds, and other financial instruments are quintessential complex systems that are ready-made for study. What goes into the market and comes out of the market are publicly accessible numbers. Beneath the mere transactions there is a plethora of interacting parts.

Much of the search for patterns in markets has been done with a straightforward focus on price changes. But considerable research effort has also been directed toward understanding traded volumes and a relation between price changes and traded volumes. Many databases list traded volumes along with the price.

In 2002 Plerou et al. found an interesting pattern involving traded volumes after discriminating between buyer initiated traded volume, V_b, and seller initiated volume, V_s [1]. As buyer initiated traded volume derives from demand it drives the price up. Likewise, seller initiated traded volume derives from supply and drives the price down. Plerou et al. took the difference $\Delta V = V_b - V_s$ and next observed how in real market data the price change, Δx, increased monotonically with ΔV. What was perhaps not expected was that the curve was a sigmoid through the origin. A year later Gabaix et al. proposed a theory to lead to the apparent sigmoidal shape [2]. The theory involves traders making assessments and next following sophisticated buying or selling protocols to maximize their acquired value. It eventually leads to $\Delta x \propto \sqrt{\Delta V}$ for $\Delta V > 0$ and $\Delta x \propto -\sqrt{-\Delta V}$ for $\Delta V < 0$. In the Discussion section of this Letter we will reiterate the theory.

What we show below is a much simpler explanation for the sigmoid. In 1997 Bak, Paczuski, and Shubik proposed a model in which buyers and sellers are merely making random adjustments to the price at which they are willing to trade [3]. In that model buyers and sellers ultimately diffuse from opposite directions to a "price point." Studies of the model have generally focused on fluctuations of the price. Below it will be explained and numerically illustrated how the Bak model leads to the sigmoidal relation between ΔV and the price change Δx.

Real-life stock markets appear to bear out the sigmoidal relation between price and volume [1,2]. We collect data from the bond market and we find that there too, the square root is a very good fit.

2. The model of Bak et al.

In 1997 Bak, Paczuski, and Shubik proposed the setup depicted in Fig. 1 to model fluctuations in the stock market [3]. Similar models had at that point already been considered in the study of reaction–diffusion systems [4], but Bak et al. were the first ones to propose such a setup as an agent-based, "zero-intelligence" model [5] for market dynamics.

Two species of particles, A and B, are diffusing. Time is discrete. At each timestep each particle moves by one unit to either the left or the right, both with a probability of 1/2. The interval is bounded

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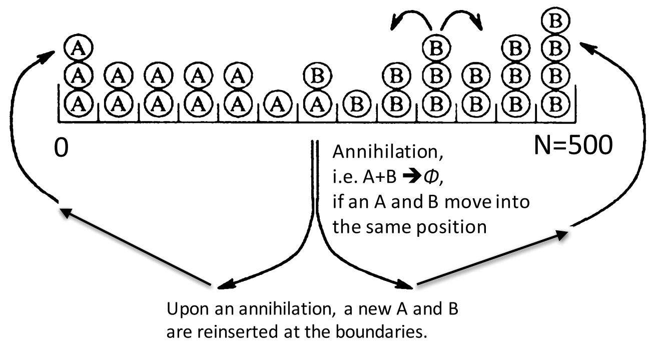

Fig. 1. The model of Bak et al. that is central in this letter. Particles (agents) A and B are diffusing on an interval with a reflecting barrier on each side. Whenever an A and a B "meet," they annihilate, i.e. $A+B \rightarrow \emptyset$. Immediately upon such annihilation, a new A and B are inserted at the left and right barrier, respectively.

by reflecting barriers. Whenever an A and B find themselves at the same position or when they cross (i.e. an A goes from $i-1$ to i and at the same timestep a B goes from i to $i-1$), they annihilate each other. Immediately upon an annihilation a new A is inserted at the reflecting barrier on the left and a new B particle is inserted at the reflecting barrier on the right.

In the chemical context, A and B are two reactants that follow Fick's Law and diffuse towards a front from opposite directions. Whenever an A and a B meet, an irreversible reaction, i.e. $A+B \rightarrow \emptyset$, occurs immediately. In the economic context, the setup represents a market with buyers and sellers that, at each timestep, are varying the price at which they are willing to trade. The A particles represent buyers and the B particles represent sellers. For each agent the position on the horizontal axis represents the price at which the agent is prepared to trade. Obviously, a trade occurs when the random motions put a buyer and a seller at the same position.

The system depicted in Fig. 1 is simple and involves no sophisticated trader tactics or strategies. Nevertheless, it realistically models a market in which trade is algorithmically driven by so-called "limit orders." A limit order is an instruction to a bank or broker to buy (sell) a set amount of a financial instrument if a price is below (above) a certain level. The random inputs in our model represent changes of this level. Models based on limit order trading have been analyzed with econophysics methods [6].

There are N possible positions between the reflecting barriers. As in Ref. [3], we take $N=500$ in our numerical realizations. Of both A and B there are M particles and we take $M=1000$.

The model of Fig. 1 is readily simulated. The behavior of the model can also in part be understood through analytic approximations.

Studies of models like in Fig. 1 have mostly focused on fluctuations of the position of the front [7]. In the chemical context, the location of the front is commonly a visible feature of the reaction–diffusion system. In the economic context the position of the front represents the all-important price of a financial instrument. In this work we focus on the economic context. In particular, we study the relation between the price change in a time interval T and the volume traded in that same time interval. In the chemical context, the volume represents a number of annihilations.

Traded stock volumes have been subject to considerable research effort [8,9]. What will be shown in Section 3 of this article is that the simple mechanistic model of Fig. 1 reproduces a relation between price and volume that has been empirically established and previously explained in terms of more elaborate judgments, strategies, and tactics on the part of traders [2].

In the analysis that follows, we will use the terms "annihilation" and "trade" interchangeably and pick the term that is more intuitive within the particular context.

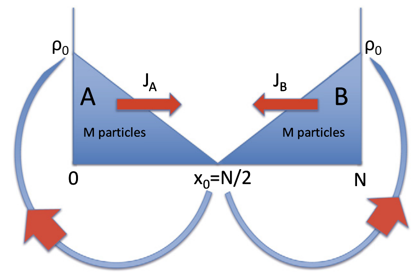

Fig. 2. The steady state for the setup shown in Fig. 1. The rate at which particles are annihilated and reinserted is also the constant uniform flow towards the point of annihilation. Such constant uniform flow requires linear density profiles with the point of annihilation being the midpoint between the two reflecting barriers (cf. Eq. (1)).

3. The continuum limit

Section 2 describes a Langevin setup, i.e. rules are given to generate particle trajectories and there is stochastic input into these trajectories. In this section we provide an almost equivalent description in which we take a probability distribution of particles and use a diffusion equation to describe how this distribution evolves deterministically in time.

We let x be the continuous position coordinate and let $\rho(x)$ be the probability density function. Fig. 2 depicts the stationary state of the continuum version of the Model in Fig. 1. There is a constant flow J from the reflecting barriers to the annihilation point at $x_0 = N/2$. On both sides we have Fick's Law, i.e.

$$J = -D\partial_x \rho. \tag{1}$$

The value of the diffusion coefficient D is readily obtained. Every unit of time, there is a "hop" to the position one unit to the left or right, i.e. $\Delta x = \pm 1$ as $\Delta t = 1$. We thus have $\langle \Delta x^2 \rangle = \Delta t$. With the well-known $\langle x^2 \rangle = 2Dt$ for 1D diffusion we then find $D = 1/2$ for the diffusion coefficient. With Eq. (1) it is next easily verified that, in terms of N and M, we have

$$\rho_0 = 4M/N \qquad |J| = 4M/N^2, \tag{2}$$

where ρ_0 is the density at both of the reflecting barriers (cf. Fig. 2). It is important to realize that $|J|$ is also the annihilation rate at $N/2$. In the econophysics context it represents the number of trades per unit of time.

The situation depicted in Fig. 2 is an attractor. Consider a fluctuation such as depicted in Fig. 3a, i.e. a "bump" in the linear profile. The number of particles between x_1 and x_2 ($\int_{x1}^{x2} \rho(x)\,dx$) is the same as would be the case without the bump. Ultimately this bump will straighten out and a relaxation to a linear profile will occur: following Eq. (1) the steeper part between the red dots will speed up and thus fill the gap in front of x_2 where there is a shortage of particles. In the fluctuation depicted in Fig. 3b, the "price," x_0, is shifted away from the middle. As $M_A = M_B$, the slope on the right is steeper than the slope on the left. This leads to $|J_B| > |J_A|$. As the ultimate annihilation rate will be the same for both species, the B particles will "eat into" the A particles and the price will be driven back towards the middle. However, this relaxation is very slow.

On the intermediate timescale the price is seen to move subdiffusively with a Hurst exponent of $H = 1/4$, i.e. $\sqrt{\langle (\Delta x)^2 \rangle} \propto (\Delta t)^{1/4}$. Because of the reflecting barriers and the small force driving the price to the middle of the interval, the diffusion terminates on the very long timescale. For an ordinary random walk where the increments are drawn from a zero average Gaussian distribution, we have "normal" diffusion with $H = 1/2$. At first sight it appears remarkable that we find anomalous subdiffusive behavior here. In

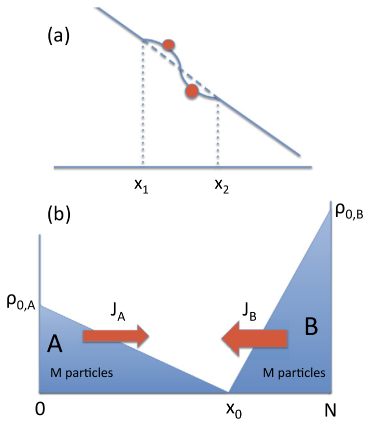

Fig. 3. The symmetric situation with linear probability distributions as shown in Fig. 2 is a stable attractor. In the text it is explained how an irregularity as shown in the (a) panel eventually "straightens out" and how there is a "force" pushing x_0 back to the middle if an asymmetry as in the (b) panel arises.

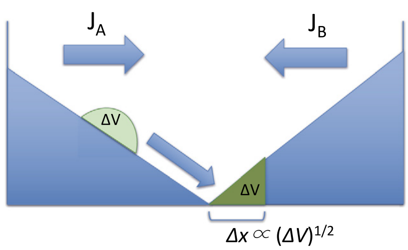

Fig. 4. Stochastic fluctuations in the flows J_A and J_B will cause diffusive movement of the annihilation point x_0, i.e. flow fluctuations translate into price fluctuations.

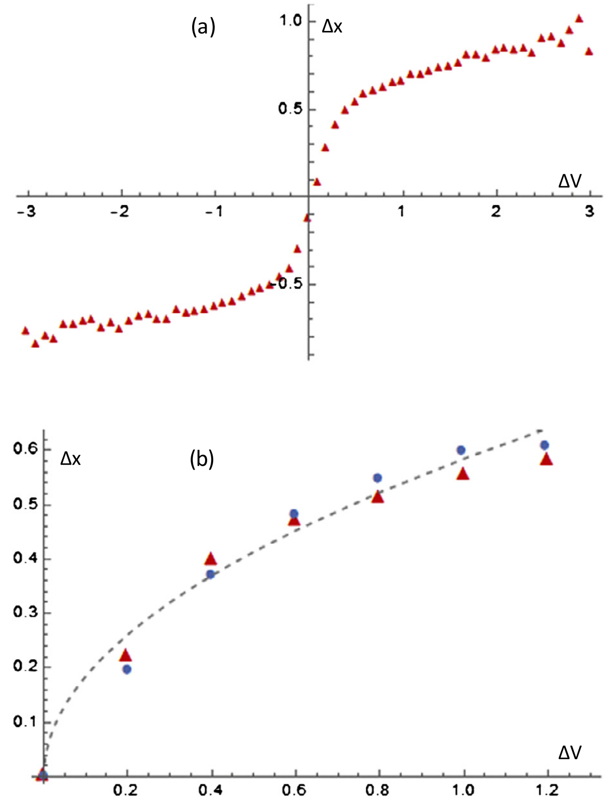

Fig. 5. (a) The relation between the price change (Δx) and the difference between buyer-initiated and seller-initiated traded volume (ΔV) for data from the bond market. The same pattern was found for data from the stock market [2]. (b) Bond market data (red triangles from Fig. 5a), results of a stochastic simulation of the model by Bak et al. (blue dots), and the theoretically predicted square root relation (dashed line, cf. Eq. (3)) appear in very good agreement. See the main text for how the data points were obtained. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Ref. [7] there is a heuristic derivation of the $H = 1/4$ of our system that we briefly reiterate below. In the remainder of this letter, we will build further on this explanation.

To derive $H = 1/4$ we refer to Fig. 4. Take the price to be close to the middle ($x_0 = N/2$) and take buyer and seller distributions to be close to the steady state triangular ones as in Fig. 2. The "force" driving the system towards the steady state will then be minimal and we have $|J_A| \approx |J_B|$. In a time $\Delta\tau$ we find for the volume that reaches the annihilation point: $\langle V_A \rangle = |J_A|\Delta\tau$. The setup in Fig. 1 behaves like a Poisson Process in the sense that the probability for a trade to occur in some small time interval remains roughly constant in time. For a Poisson Process the variance equals the average. So for the volume arriving at the annihilation point in time $\Delta\tau$, the average of $\langle V_A \rangle = |J_A|\Delta\tau$ comes with a standard deviation of $\Delta V_A \approx \sqrt{\langle V_A \rangle} \propto \sqrt{\Delta\tau}$. If excess volume of such magnitude arrives at the annihilation point, it will "eat into" the B distribution (cf. Fig. 4). For a triangle with an area that is proportional to $\Delta V \propto (\Delta\tau)^{1/2}$, the sides have lengths that are proportional to $|\Delta x| \propto (\Delta V)^{1/2} \propto (\Delta\tau)^{1/4}$. It is thus that volume fluctuations with standard-deviation-magnitude lead to subdiffusive behavior with a Hurst exponent of 1/4.

4. Relation between price change and traded volumes

In 2003 Gabaix et al. published a study in which, for traded volume, they make a distinction between buyer-initiated volume (V_b) and seller-initiated volume (V_s) [2]. It is intuitively obvious that seller-initiated traded volume drives the price down, whereas buyer-initiated traded volume drives the price up. In their much-cited article [2], the authors find that the price change, Δx, of a stock is proportional to the square root of the difference $\Delta V = V_b - V_s$:

$$\Delta x \propto \operatorname{sgn}[\Delta V]\sqrt{|\Delta V|}. \tag{3}$$

In Ref. [2] there is a "microfoundation" of this relation, i.e., a heuristic derivation involving tactical timing on the part of buyers and sellers as they are trading. The relation is also observed in actual data. A square-root relation as Eq. (3) was already identified in the 1990s for the impact of a single trade on a stocks' price (see Ref. [10] and references therein). With Fig. 4 we show that Eq. (3) also ensues from the model by Bak et al.

The model presented in Fig. 1 readily leads to Eq. (3). The relation $|\Delta x| \propto (\Delta V)^{1/2}$ is implicit in the subdiffusive behavior that we derived at the end of the last section. Fig. 5 shows the square root curve of Eq. (3) together with the results from a stochastic simulation of the system depicted in Fig. 1. Furthermore shown in Fig. 5 are data taken from the bond market.

For Fig. 5a we took data (years 2003–2016) for the 3481 most actively traded bonds during 2016. These data came from the Trade Reporting and Compliance Engine (TRACE) and are made available through Wharton Research Data Services (WRDS). To generate Fig. 5a, trade direction (buy or sell) was inferred by comparison to the price immediately before each time interval. We took time intervals of one week. Trades of more than one million or five million dollars (depending on the type of bond) are reported at the threshold value of one or five million dollars. For these trades we took the reported threshold value. A week is discarded if more than 5% of the trades during that week are reported as such "more than one/five million dollar" estimates. Such weeks, however, are very rare. The data points were binned in ΔV intervals. The Δx values represent the ultimate expectation value in each bin. For the graph, Δx and ΔV were normalized to zero mean and unit variance. All points were included in the normalizations with all bonds contributing equally. Our procedure for organizing these data is very similar to that used in Ref. [2] with stock market data.

Fig. 5b shows data from Fig. 5a, data from a simulation of the stochastic model by Bak et al., and the theoretical square-root curve (cf. Eq. (3)). For the simulation, the parameters used were $N = 500$ for the price range and $M = 1000$ for both the number of buyers and the number of sellers. We took time intervals of 50 steps. We let each time interval start with two new random triangular distributions, one for the buyers and one for the sellers. The initial price is taken as $x_0 = N/2$ (cf. Fig. 2). Initial and final prices are determined as the midpoint between the highest buyer or lowest seller. Additionally, any excess particles which diffused across the starting price, but had not yet annihilated at the end of the time interval, were counted as traded volume of their respective type. Data shown represent the average over four million 50-step intervals.

All in all, the stochastic simulations as well as the real-life bond-market data appear to closely follow the simple square-root relation that ensues from the continuum model.

5. Results and discussion

The original setup in Ref. [3] is slightly different from ours. Following a trade, Ref. [3] has the new seller and new buyer re-enter the interval not at the reflecting barriers, but at arbitrary points in the interval. For buyers the entry points are drawn from a flat distribution between the left barrier and x_0 and for sellers the entry points are drawn from a flat distribution between x_0 and the right barrier. It is easy to derive how this changes the situation. Consider Fig. 2. Let J_A^0 be the annihilation rate at x_0 and let $0 < x < x_0$. With the distributed re-entry, only particles that enter to the left of x will pass through x on their way to x_0. So with the flat re-entry distribution we have for the net flow of particles at any x: $J_A(x) = (x/x_0) J_A^0$. With Fick's Law, i.e. $J(x) = -D\partial_x \rho(x)$, it is next readily derived that $\rho(x) = C(x_0^2 - x^2)$, where C is a positive constant. So for the steady state probability densities (cf. Figs. 2, 3, and 4), instead of a linear profiles, we get parabolic ones. However, even if $\rho(x) = C(x_0^2 - x^2)$ we can still approximate $\rho(x)$ as being linear between $x_0 - \Delta x$ and x_0 as long as Δx is sufficiently small. So Eq. (3), a Hurst exponent of $H = 1/4$, and the mechanism depicted in Fig. 4 will still be valid for the parabolic profile.

Equation (3) describes the actual data well. But the subdiffusion implied by $H = 1/4$ is not in agreement with what real-life markets exhibit. For real markets we see superdiffusion, i.e. $H > 1/2$, on an intermediate timescale and $H \approx 0.5$ on a very long timescale [8]. In Ref. [3] features are added to the basic model and eventually the authors arrive at the superdiffusive behavior.

Reference [3] implements crowd behavior in the following way. After a trade the new buyer and the new seller randomly pick a buyer and a seller and copy the prices of their respective picks. With such imitation behavior, traders will naturally cluster and confinement by means of reflecting barriers is then actually no longer necessary. In the context of Fig. 4, this imitation behavior means that a $|\Delta V|$ fluctuation will grow on its way down. We still have $(\Delta x)^2 \propto |\Delta V|$ when the fluctuations arrive at the annihilation point, but the augmentation of the Poisson-Process-fluctuations on the way down means that the relation $\Delta V \propto (\Delta\tau)^{1/2}$ will no longer be valid. As ΔV will increase faster than $(\Delta\tau)^{1/2}$, we ultimately get a Hurst exponent that is larger than 1/4. In Ref. [3] Bak et al. observe $H = 1/2$ in their numerical simulations.

A so-called "volatility feedback" is the next enhancement that is researched in Ref. [3]. The authors write: "... if the price change during the last period of 50 time units is ΔP, an agent updating her price will increase or decrease her price randomly by an amount ΔP." This added feature models the well-known fact that a large price fluctuation today increases the likelihood for a large price fluctuation tomorrow, where tomorrow's fluctuation can be in either direction. It is obvious that we have a positive feedback mechanism here – a positive feedback mechanism where large fluctuations can cause large fluctuations in the near future. From the point of view of the continuum model that was discussed in Section 2, the variable stepsize that this enhancement introduces leads to a variable diffusion coefficient D. Varying D in this manner adds another source of variance on top of that of the augmented-Poisson variance that was described in the previous paragraph. With this enhancement the authors of Ref. [3] arrive at the superdiffusive behavior that is observed in real-life markets.

The enhancements described in the last two paragraphs *do not change the mechanism that leads to Eq.* (3). For sufficiently small $\Delta\tau$'s and Δx's, the densities $\rho(x)$ of buyers and sellers will still be well approximated as linear near x_0. The enhancements can lead to larger values for $|\Delta V|$, but the relation between ΔV and the price change, Eq. (3), will be very robust.

The article by Gabaix et al. [2] gives the following "microfoundation" for Eq. (3). A trader is going to buy if he or she perceives a "real value" that is M above the market price. His or her working assumption is next that the market will correct the discrepancy at a linear rate μ. To acquire a desired volume ΔV, the trader needs, of course, to find sellers. To that end the trader offers a price that is Δx above the market value, but still below the perceived "real" value. The time τ that it takes to get the necessary sellers is proportional to ΔV and inversely proportional to Δx, i.e. $\tau \propto \Delta V/\Delta x$. The trader minimizes his or her cost if $M - \mu\tau - \Delta x$ is maximal. This means that he or she has to pick the Δx that minimizes $\mu\tau + a\Delta V/\tau$, where a is a positive constant. This leads to $\tau \propto \sqrt{\Delta V}$. Combining this with $\tau \propto \Delta V/\Delta x$, we obtain $\Delta x \propto \sqrt{\Delta V}$, i.e. Eq. (3). More sophisticated microfoundations involving, for instance, risk assessment were formulated in the wake of the article by Gabaix et al. [11,12]. All of these microfoundations start with a number of very specific model assumptions about trader assessment and trader behavior.

It is worth pointing out that the relation $\tau \propto \Delta V/\Delta x$ that is assumed in the derivation in the previous paragraph emerges from the model of Bak et al. in a straightforward fashion. In Fig. 4 we see that $\Delta V \propto (\Delta x)^2$. At constant flow speeds, J_A and J_B, the time τ that it takes to cover an interval Δx (and "swallow" an excess ΔV on that interval) is proportional to Δx. We thus obtain $\Delta V \propto \tau\Delta x$, i.e. the $\tau \propto \Delta V/\Delta x$ that Ref. [2] postulates.

A "zero-intelligence" model for limit-order trading that is mindful of the model by Bak et al. is presented in Ref. [13]. With that model the square-root relation for the price impact of a single trade can be accounted for. There has been increasing research interest in zero-intelligence models and such models have commonly been successful [5]. The model by Bak et al. that we studied in this Letter can be considered to be the original zero-intelligence model. It is remarkable that already this simplest of models leads to the observed price-volume relationship. This price-volume relation, Eq. (3), moreover appears to be very robust under modifications to the model. As the stochastic model of Bak et al. is much simpler than any proposed microfoundation, it is, by Occam's razor, the superior model. The implication is that traders do not strategize and optimize. They merely move randomly and "diffuse."

Acknowledgements

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