REMERGING FEIGENBAUM TREES IN DYNAMICAL SYSTEMS

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Finite sequences of remerging period-doubling bifurcations have been recently observed in a variety of physically interesting dynamical systems. We show here that such remerging Feigenbaum trees are quite common in models with more than one parameter and discuss a number of criteria under which they are generally observed. These criteria are applied to simple mappings as well as the conservative Duffing's equation where the formation of a primary "bubble" is seen to lead to higher-order bubbles and hence to remerging Feigenbaum sequences. In the case of Duffing's equation, we follow the development of one such sequence, with the aid of the variation of the winding number along a symmetry axis of the problem.

Period-doubling bifurcations, as a universal route to chaos, is one of the most exciting discoveries of the last few years in the field of nonlinear dynamical systems [1–4]. The remarkable number of papers it has inspired, in so many areas of pure and applied science, is but one indication of its importance in improving our understanding of the qualitative and quantitative behavior of dynamical systems [5]. Almost everyone, nowadays, who is studying numerically or experimentally a dynamical system is looking out for infinite sequences of period-doubling bifurcations leading to widespread chaotic behavior.

Recently, however, a number of researchers have discovered, in different dynamical systems, *finite* period-doubling sequences [6–9] [‡1] "merging" as it were with inversely advancing ones, to form a finite number of "bubbles' on some cross sections of the full parameter space. An important consequence of such a remerging is that low-order periodic orbits become again stable and relatively large regions reappear around them, where the motion is regular and predictable.

The purpose of this letter is twofold: to demonstrate firstly that such remerging period-doubling sequences (or Feigenbaum trees) commonly arise in some of the simplest nonlinear dynamical systems involving the variation of more than one parameter. Secondly, we aim to isolate, with the aid of such simple systems, a minimum number of conditions, under which such finite remerging trees – or, more affectionately, period-bubbling! – can be expected to occur.

Our main result is that remergence is likely to occur when there is a symmetry transformation, under which the equations of motion remain invariant. Moreover, the first pitchfork bifurcations from period 1 to 2, say, must occur in the proper direction to form the primary bubble, from which all higher-order bubbles will follow. Such conditions are satisfied by most dynamical systems, where remerging Feigenbaum trees have been observed in the literature [7–9]. In Contopoulos' galactic models [6], it appears that rotation plays the role of the symmetry transformation mentioned above.

Before studying remerging Feigenbaum trees in Duffing's equation, let us examine first how they can arise in simple one- and two-dimensional mappings. As observed already by Schmidt and co-workers [9] it is not difficult to find examples of such mappings, exhibiting finite remerging period-doubling sequences. We would like here, however, to condense our results to a number of conditions under which this phenomenon is to be expected.

The first condition is that the one-dimensional map depends on two parameters, say A and Q:

$$x_{t+1} = f(x_t; A, Q), \quad t = 0, 1, 2, \ldots \quad (1)$$

‡1 In the preprint of Bialek et al. [9] similar results to ours are reported.

In this way x, A, Q form a three-dimensional space in which, by varying Q, we could find, at different A values, different "projections" of the Feigenbaum trees.

The second condition is the existence of a primary bubble of period 2, emerging from and remerging with the same period 1 orbit at two different values of Q. As period 1 and 2 solutions of eq. (1) are extremely easy to check for existence, uniqueness, stability etc. this condition can be readily verified, for any given $f(x_t; A, Q)$ even analytically.

Consider, for example, the mapping

$$x_{t+1} = Q + Ax_t/(x_t^2 + 1)\,, \quad t = 0, 1, 2, \dots\,, \tag{2}$$

for $A > 0$. Note that due to symmetry, we need only consider $Q > 0$, since every orbit of (2) has an equivalent one with $x_t \to -x_t$ and $Q \to -Q$. We easily find that the fixed-point equation $x = Q + Ax/(1 + x^2)$ has three solutions, which we plot in fig. 1a as a function of Q. In particular, there is one unstable period 1 orbit passing through the origin of the x, Q plane and one stable one, which are "born" together at $(x, Q) = (0.77, 2.13)$, for $A = 6.0$, see fig. 1a. By symmetry, the continuation of this stable period 1, at $Q < 0$, is identical to the third fixed point (with $x > 0$) and is easily determined by the condition $|f'(x)| < 1$ to be stable for all Q at this value of A.

However, as A increases, a bifurcation occurs and the stable $x > 0$, $Q < 0$ branch develops a stable period-2 bubble, see fig. 1b. At even higher values of A, this primary bubble bifurcates into secondary period 4 bubbles, in fig. 1c, which develop further bubbles as A gets larger, until an infinitely branched tree finally appears at $A \approx 11.8$. We have checked that the rate at which these bifurcations occur as A grows is Feigenbaum's universal $\delta = 4.669$ [1–5] – the same as the one characterizing the bifurcations with respect to Q.

Another example of remerging trees is provided by the mapping

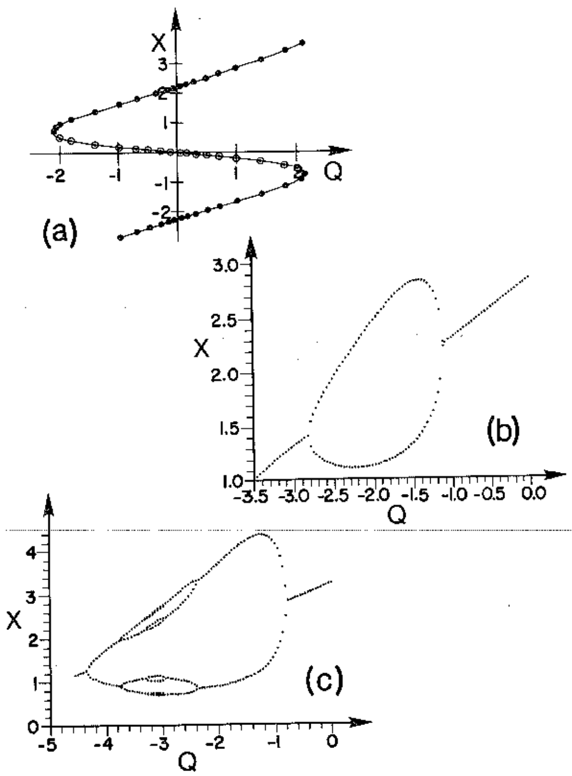

Fig. 1. (a) Fixed points of the mapping (2) at $A = 6.0$; • denotes the stable ones and o the unstable one. (b) The primary bubble of (2) at $A = 9.0$. (c) Period 2, 4 and 8 bubbles of (2) at $A = 11.5$.

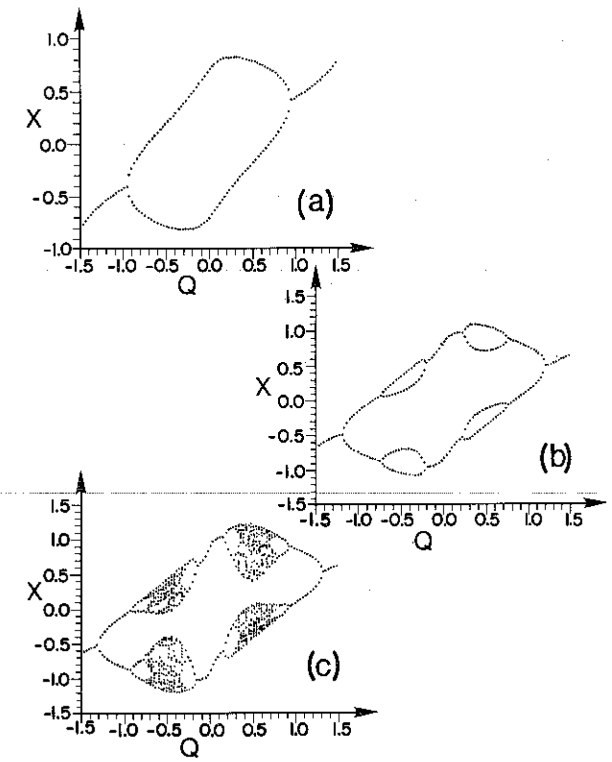

Fig. 2. (a) The primary bubble of the mapping (3) at $A = 1.5$. (b) Period 2 and 4 bubbles of (3) at $A = 1.7$. (c) Infinite period-bubbling of (3) at $A = 1.8$.

$$x_{t+1} = Q - Ax_t + x_t^3 , \qquad (3)$$

where a primary period 2 bubble already exists at $A = 1.5$, see fig. 2a, due to the symmetry $x_t \to -x_t$ and $Q \to -Q$. Again, increasing A we observe, as before, that period 4 bubbles appear on the period-2 branches (see fig. 2b), and eventually bloom into fully grown trees in fig. 2c.

It appears from the above that mappings with certain reflection symmetries in the x, A, Q space are likely to have remerging trees. However, symmetry is not sufficient; a low-order (period 1, say) orbit must undergo a pitchfork bifuraction "facing" a symmetry line [like $Q = 0$ for (2) and (3) above] in order to allow for the formation of a primary bubble. For example, the mapping

$$x_{t+1} = Q + Ax_t - x_t^3 , \qquad (4)$$

although symmetric under $x_t \to -x_t$, $Q \to -Q$ exhibits only period-doubling (pitchfork) bifurcations "facing" away from $Q = 0$, and hence never gives rise to bubbles and remerging trees, see fig. 3.

Now, as pointed out by Schmidt and Wang [9] one can discover similar phenomena in area-preserving, two-dimensional maps of the deVogelaere type

$$x_{t+1} = -y_t + f(x_t; A, Q) ,$$

$$y_{t+1} = x_t - f(x_{t+1}; A, Q) , \quad t = 0, 1, 2, 3 . \qquad (5)$$

To see this, write eq. (5) into a single second-differ-

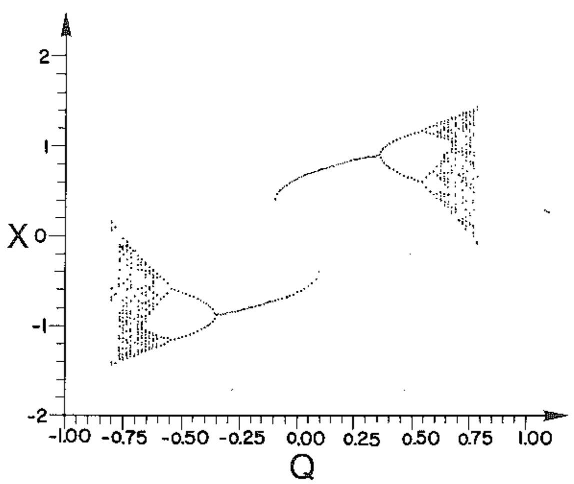

Fig. 3. The full Feigenbaum tree for mapping (4) at $A = 1.4$. No remerging bifurcations are observed in this case.

ence equation eliminating y_t:

$$x_{t+1} + x_{t-1} = 2f(x_t; A, Q) . \qquad (6)$$

It is now an easy exercise to show that the fixed-points, their stability properties, and the period 2 orbits of (6) are identical to those of the associated one-dimensional mapping (1). Hence, since the symmetries and the existence of a primary bubble of (1) is inherited by (6) it is not surprising that as A and Q are varied area-preserving mappings (5) also exhibit bubbles and remerging trees of the type shown in figs. 1–3 [9].

Remerging bifurcation trees do not arise only in simple mappings; they are also quite common in dynamical systems described by ordinary differential equations [6–8]. A well-known example of such a system is Duffing's equation

$$\ddot{x} + Ax + x^3 + Q \cos 2t = 0, \qquad (7)$$

describing the motion of a periodically forced, anharmonic oscillator in the absence of dissipation. This equation, with and without damping, has been well studied in the literature [10] and is known to exhibit most of the fascinating phenomena typically expected of non-integrable dynamical systems.

Here, we demonstrate, using Poincaré's surfaces of section, that (7) has remerging period-doubling trees starting with a primary bubble formed by its lowest-order periodic solutions. We give approximate formulas for the location and stability properties of these solutions, and examine further the development of remerging trees with respect to the variations of the rotation number along the $\dot{x} = 0$ symmetry axis of the $x, \dot{x}$ surface of section of (7).

By lowest-order periodic solutions (or orbits) we mean the ones with period π and 2π, intersecting the surface of section at one and two points respectively, see fig. 4. At $Q = 0$, the only period 1 orbit is the origin of the $x, \dot{x}$ plane (fig. 4a) while period 2 corresponds to a full invariant curve, every point of which comes back upon itself after $t = 2\pi$. As Q becomes non-zero, this curve "breaks up" into two pairs of period 2 orbits with one stable and one unstable in each pair, see fig. 4b. As Q increases further these period 2 orbits are absorbed by period 1 at $Q \approx 2.28$ for $A = 0.3$, see fig. 4c.

Since eq. (7) remains invariant under the transformation $x \to -x$, $Q \to -Q$, a "remerging" identical

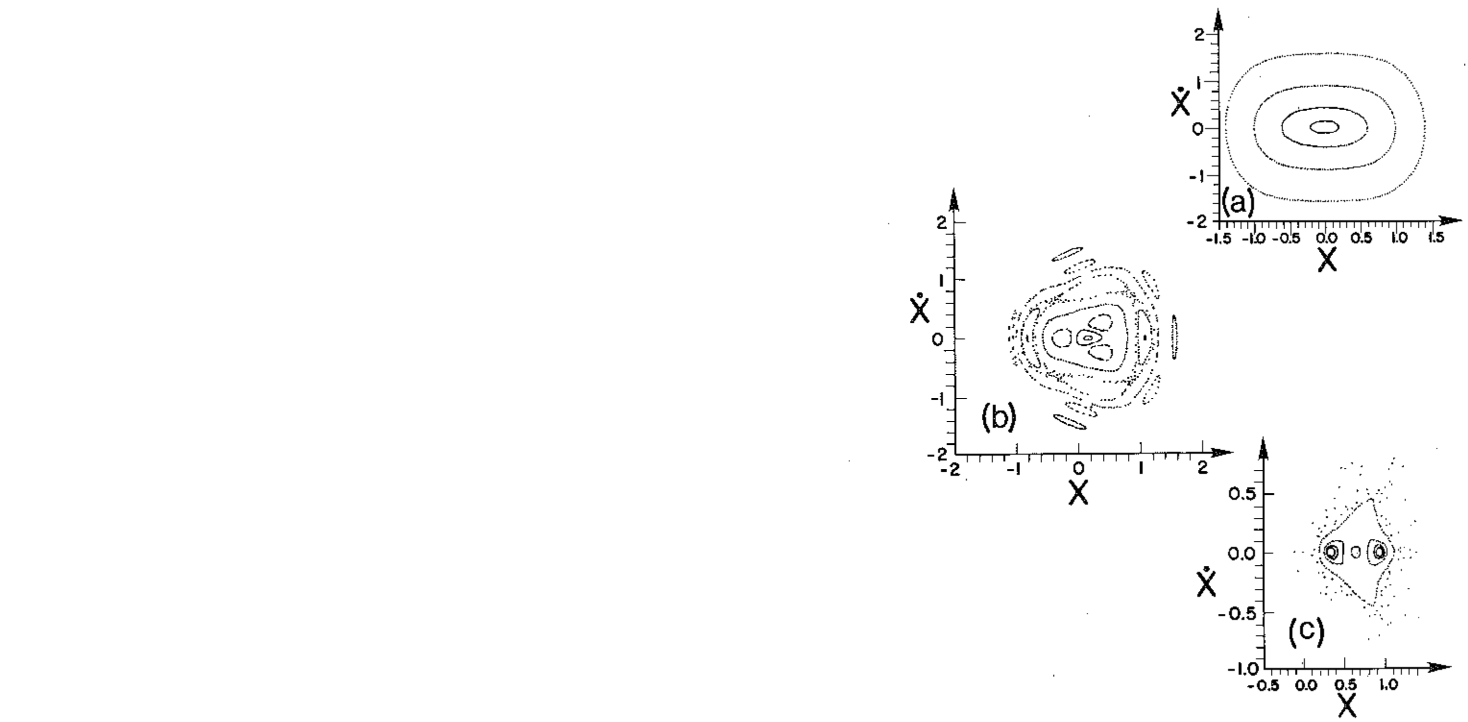

Fig. 4. Surfaces of section of Duffing's equation (7) at $A = 0.3$ and (a) $Q = 0$, (b) $Q = 0.6$ and (c) $Q = 2.2$.

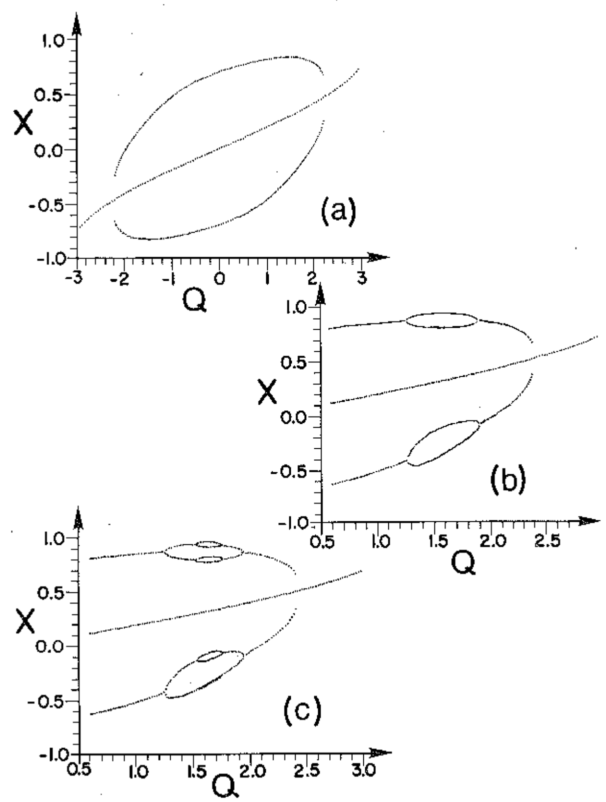

Fig. 5. (a) The primary bubble of Duffing's equation (7) formed by the period 1 and 2 orbits at $A = 0.3$. (b) Period 4 secondary bubbles of (7) at $A = 0.215$. (c) Period 2, 4 and 8 bubbles of (7) at $A = 0.2089$.

to the one described above occurs for $Q < 0$, leading to the formation of the primary bubble shown in fig. 5a. By varying A now, as in the case of the cubic mapping (3), we discover bubbles of period 4 and period 8, in figs. 5b and 5c. However, no remerging trees were found in Duffing's equation (7) with the $+x^3$ term replaced by $-x^3$. Just as in the cubic mapping (4), no primary bubble was formed by the lowest-order periodic orbits, and period-doubling bifurcations never seemed to be followed by an inverse cascade as A and Q were varied.

Thus, it becomes clear from the above that the bifurcation properties of period 1 and 2 orbits play an important role in determining whether (and where) period-bubbling will occur. It would, therefore, be useful to have approximate expressions for the location and stability of these orbits. Such formulas can be obtained by using the rapidly converging Fourier series expansions [11,12], $x(t) = \Sigma A_n \exp(in\nu t)$, $A_n = A_{-n}$, where $\nu = 2$ for the period 1 and $\nu = 1$ for the period 2 case.

Substituting these Fourier series in (7) one arrives at recursion relations for the A_n, which can be solved iteratively on the computer to obtain periodic solutions to any desired accuracy [11]. From the first 2–3 of these equations one can easily derive approximate expressions for A_0, A_1, A_2 as functions of the parameters of the problem. For example, in the case of period 1, this procedure yields $A_0 = 0$, $3A_1^3 + (A - 4)A_1 + \frac{1}{2}Q = 0$, where the three real roots of the latter equation correspond to three actual period 1 (π) solutions of eq. (7). Similarly, period 2 orbits can be constructed by iterating the recursion relations for A_0, A_1, A_2 and A_3. We have compared such estimates with numerical computation and found very good agreement, for the full range of parameter values of interest.

The stability properties of these low-order periodic orbits can also be accurately determined from the above analysis. For example in the case of the period 1 orbits $\hat{x} = 2A_1 \cos 2t$, we perturb eq. (7) about it by setting $x = \hat{x} + z(t)$ and analyzing the resulting Mathieu equation $\ddot{z} + (a - 2b \cos 2t)z = 0$, where $a \equiv (A +$

$6A_1^2)/4$, and $b \equiv -3A_1^2/4$. The resulting stability predictions [13] were in full agreement with surface of section results for all three period 1 orbits. A similar analysis of the period 2 orbit, using a more general Hill's equation [14], described well the primary bubble of fig. 5a. It was not accurate enough, however, to also predict the destabilization of period 2 and the formation of the secondary bubble of fig. 5b.

Finally, we present here an alternate description of the phenomena occurring along the $\dot{x} = 0$ symmetry axis of the Poincaré map of (7), using the notion of the winding (or rotation) number W, defined by

$$W \equiv (1/2\pi) \lim_{n\to\infty} n^{-1}\left(\sum_{i=1}^{n} \phi_i\right), \qquad (8)$$

where ϕ_i denotes the angle between the $(i-1)$st and the ith intersection of an orbit around the fixed point of figs. 4–7.

At $Q = 0$, W is a smooth function of $x\,[\equiv x(0)$ with $\dot{x}(0) = 0]$ increasing and decreasing monotonically about its maximum at $x = 0$. The value of this maximum is easily determined from $\ddot{x} = -Ax$ to be $W_{max} = 1 - \frac{1}{2}A^{1/2} = 0.726$ for $A = 0.3$ (the ϕ_i are measured in the counterclockwise direction).

As Q becomes non-zero, all invariant curves in fig. 4a with rational winding number $W = p/q$ "break-up" into strings of q islands according to Birkhoff's theorem [4], over which W remains nearly constant. Thus the graph of W versus x develops an infinite number of "plateaus" corresponding to all the rationals which have islands intersecting the x axis, and a kind of "devil's staircase" is formed [15,16] on both sides of W_{max}, in fig. 6b.

Note in fig. 7 the comparison between the extent of these plateaus and the width of the corresponding islands on the x axis. For larger values of Q, the top of the staircase comes down and periodic orbits remerge with the central fixed point as W_{max} crosses their winding number W. This happens for example with period 3, at $Q \gtrsim 1.0$, where $W_{max} = 2/3$, see fig. 7.

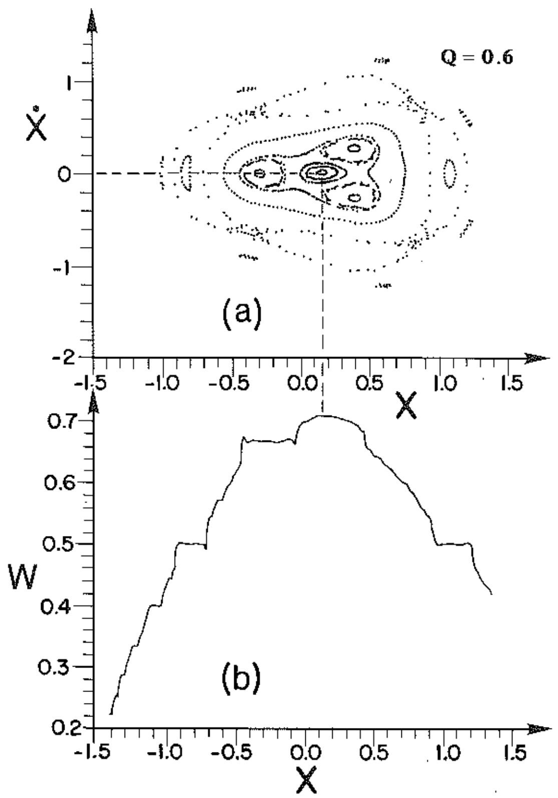

Fig. 6. (a) Surfacesof section, and (b) winding number variation of eq. (7) at $Q = 0.6$ and $A = 0.3$. Note the plateaus at rational values of W.

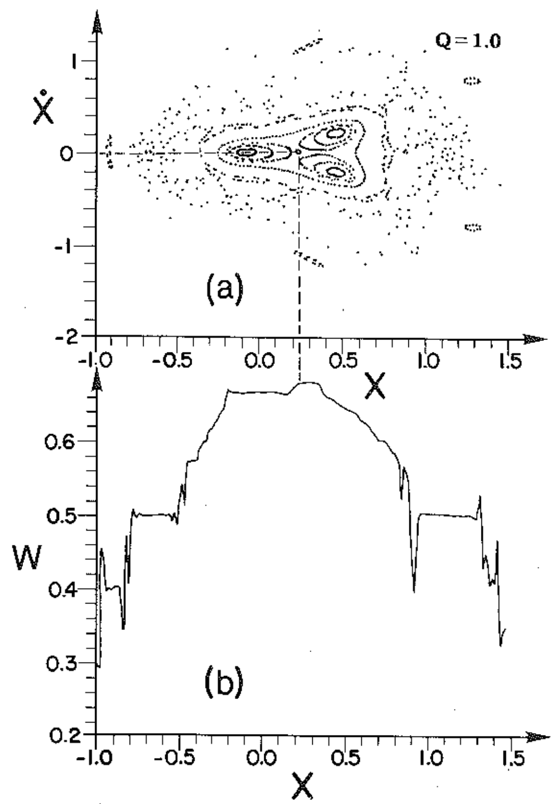

Fig. 7. Same as fig. 6 at $Q = 1.0$. Period 3 is on the verge of being absorbed by the central fixed point.

The onset of large chaotic regions between the islands in fig. 7a is evidenced in the x, W graph by the "spikes" in between plateaus, where the limit (8) does not even appear to exist. On the other hand, the width of the plateaus provides, at any given value of Q, an estimate of the "size" of the stable regions about the elliptic periodic orbits – currently, a problem of great interest in applications of nonlinear dynamics.

As far as the remerging of the primary (period 2) bubble is concerned this occurs at $Q \approx 2.28$, where $W_{max} = 1/2$, with W_{max} continuing to decrease at higher values of Q. Similarly, treating the right-hand-site point of period 2 as a period 1 point of period 2π, one can study the formation of the secondary bubble using appropriately scaled pictures of the variation of W in a neighborhood of that point, and so on for higher-order bubbles of this period-doubling (or bubbling) sequence.

Period-doubling bifurcations in dynamical systems with two (or more) parameters often form finite sequences which "merge" in some cross sections of the parameter space, inhibiting the growth of infinitely branched Feigenbaum trees and the development of the associated universal route to chaos. Such remerging Feigenbaum trees, however, having a finite number of "bubbles" occur only within some range of parameter values. Infinite period-doubling sequences and universal exponents are generally observed when the full range of parameters is explored.

We have demonstrated on simple mappings as well as Duffing's differential equation that remerging trees require a certain symmetry as well as a first bifurcation property that will ensure the formation of what we called the primary bubble. From then on, the appearance of higher-order bubbles is essentially guaranteed by the well-known property of self similarity under scaling of period-doubling bifurcations. Moreover, in all the cases we studied, period-doubling (and bubbling!) proceeded in all parameter directions in the same way as usual, and with, of course, the same universal exponents.

One of the reasons why we undertook this study was the fact that remerging bifurcation phenomena appear to occur quite frequently in physically interesting dynamical systems. In this regard, one of their most appealing characteristics is that they are connected with the restabilization of low-order periodic orbits and hence lead to the reappearance of relatively large regions in which the motion becomes once again stable or attracting. We hope, therefore, that the results presented here, will help guide researchers to such regions of stability as they follow in more than one parameter space the fascinating world of bifurcations and chaotic behavior in dynamical systems.

We wish to thank G. Schmidt for stimulating our interest in remerging trees and for many interesting discussions. Several conversations with G. Contopoulos and I. Moroz considerably improved our understanding of the occurrence of remerging trees in physically relevant models. This work was supported in part by the D.O.E. grants DEFGO2-84ER40144 and DEAC03-77ER01538.

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