ON THE INTEGRABILITY OF SOME GENERALIZED LOTKA–VOLTERRA SYSTEMS

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Several integrable systems of nonlinear ordinary differential equations of the Lotka–Volterra type are identified by the Painlevé property and completely integrated. One such integrable case of N first order ode's is found, with $N-2$ free parameters and N arbitrary. The concept of integrability of a general dynamical system, not necessarily derived from a hamiltonian, is also discussed.

Ever since the early days of classical mechanics, the question of integrability of dynamical systems presented physicists and mathematicians with a big challenge. The main difficulty, of course, with integrating the ordinary, differential equations (ode's) of any dynamical system, is that these equations are generally nonlinear and involve several "degrees of freedom", which are coupled to each other in a non-trivial way [1,2]. Most of the progress, so far, has been in the area of integrable hamiltonian systems, where a number of rigorous results are known mainly due to ingenious as well as fortuitous applications of Lie algebraic methods [3]. However, despite the long history of the problem no general method is available to date even for deciding whether a given dynamical system is integrable, let alone integrating its equations of motion explicitly.

More recently, a direct method has been proposed for identifying integrable dynamical systems by requiring that their solutions possess no movable (i.e. initial condition dependent) singularities other than poles in the complex time plane [4]. This so called Painlevé property was originally adopted by Kowalevskaya [5], in her celebrated integration of a special case of rigid body motion, and was employed by Painlevé and co-workers [6] in their exhaustive studies of integrable (i.e. solvable) ode's of second order.

The Painlevé property has already been used successfully to identify new, integrable hamiltonian systems [7,8], as well as some integrable cases of the Lorenz equations [9]. It is the purpose of this letter to demonstrate the usefulness of the Painlevé property on non-hamiltonian systems by completely integrating certain generalized Lotke–Volterra systems of N first order ode's. We also arrive, as a result of our analysis, at a definition of integrability of dynamical systems described by ode's, which are not necessarily derived from a hamiltonian.

Before presenting our results for arbitrary N, we illustrate our approach on a system of 3 first order ode's

$$\dot{x}_k = \lambda_k x_k + x_k \sum_{j=1}^{3} c_{kj} x_j , \quad k = 1, 2, 3 , \tag{1}$$

$(\dot{\ }) \equiv \mathrm{d}(\ )/\mathrm{d}t$, where the coefficient matrix $\mathbf{C}^{(3)} \equiv (c_{kj})$ in (1) is taken, as in several models of physical interest [10], to be antisymmetric, i.e.

$$c_{kj} = -c_{jk} , \quad j, k = 1, 2, 3 . \tag{2}$$

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The general leading order behavior of the solutions of (1) near a (movable) singularity $t = t_0$ is of the form

$$x_k \sim a_k\tau^{-1}\,, \quad \tau \to 0 \quad (\tau \equiv t - t_0)\,, \tag{3}$$

t_0 being the first free constant of the asymptotic expansions. Substituting (3) in (1), with (2), we find that one of the a_k, say a_1, is the second free constant provided

$$c_{23} - c_{13} + c_{12} = 0\,. \tag{4}$$

Expressing now c_{13} in terms of c_{23}, c_{12} from (4) and scaling $x_k \to x_k/c_{23}$ the coefficient matrix $\mathbf{C}^{(3)}$ in (1) becomes

$$(c_{kj}) \equiv \mathbf{C}^{(3)} = \begin{pmatrix} 0 & 1 & 1+c_1 \\ -1 & 0 & c_1 \\ -1-c_1 & -c_1 & 0 \end{pmatrix}, \tag{5}$$

where $c_1 \equiv c_{12}/c_{23}$ is still an arbitrary parameter of the problem. We also remark at this point that the only "type" of singularity (3) of system (1) is the one for which all the a_k are non zero. Other singularity "types" (where one of the solutions $x_k \sim a_k\tau^p$, with $p > -1$) do occur in system (1) if $\mathbf{C}^{(3)}$ is not antisymmetric and lead to other new integrable cases discussed at the end of this letter.

Looking for the third free constant in our asymptotic expansions we insert $x_k = a_k\tau^{-1} + b_k\tau^{-1+r}$ in (1) and solve for the b_k from linear equations at order τ^{-2+r}. A free constant arises if the determinant of the coefficients of the b_k in these equations vanishes [4,7–9], which in the present case finally reduces to

$$(r+1)r(r-1) = 0\,. \tag{6}$$

The two free constants, at $r = -1, 0$, are already known: they are t_0 and a_1 respectively. Hence, the third (and last) one enters at $r = 1$, provided a compatibility condition is satisfied there by the nonhomogeneous part of the b_k equations:

$$\lambda_1 = \lambda_2 = \lambda_3 \equiv \lambda\,. \tag{7}$$

If (7) is not satisfied, logarithmic terms will enter in the asymptotic expansions of the solutions [11] and the Painlevé property will be violated.

Thus, system (1) with (5) and (7) has been shown to have the Painlevé property and is, therefore, expected to be integrable [12]. Instead of verifying, however, that (1) is integrable, we prefer to integrate below the Nth order anlogue of (1) [with (5) and (7)]

$$\dot{x}_k = \lambda x_k + x_k \sum_{j=1}^{N} c_{kj}x_j\,, \quad k = 1, 2, ..., N\,. \tag{8}$$

with the elements of $\mathbf{C}^{(N)} = (c_{kj})$ written as

$$c_{kj} \equiv c_{j-2} - c_{k-2}\,, \tag{8a}$$

where $c_{-1} = -1$, $c_0 = 0$, and $c_1, c_2, ..., c_{N-2}$ are arbitrary parameters of the problem. Eqs. (8), with (8a), reduce to (5) for $N = 3$, and can also be shown to possess the Painlevé property for $N = 4$ and $N = 5$ by a similar analysis [13]. Here we integrate eqs. (8), (8a) explicity for arbitrary N.

First note that by a simple scaling of the time we can set $\lambda = 1$. Note also that the linear terms on the right-hand side of (8) can be removed by changing to new variables

$$X_k \equiv x_k/v\,, \quad v \equiv e^t\,, \tag{9}$$

whence eqs. (8) become

$$X_k' = X_k \sum_{j=1}^{N} c_{kj}X_j\,, \quad k = 1, 2, ..., N\,, \tag{10}$$

where prime denotes differentiation with respect to v. By virtue of the antisymmetry of the c_{kj}, cf. (8a), eqs. (10) have the simple integral

$$\sum_{k=1}^{N} X_k = D = \text{const.} \tag{11}$$

Using (11) to express X_N in terms of the other X_k and substitute it in the first $N-1$ of eqs. (10) with (8a) yields

$$X_k' = X_k D(c_{N-2} - c_{k-2}) + X_k \sum_{j=1}^{N-1} (c_{j-2} - c_{N-2})X_j\,, \quad k = 1, ..., N-1\,. \tag{12}$$

Eqs. (12) are a special case of a system of Riccati equations of the projective type, for which there exist nonlinear superposition principles, and which can be linearized, provided one particular solution of (12) is known [14]. We can, however, solve eqs. (12)

explicitly in the following way:

Since the sum on the rhs of (12) is independent of k, we combine consecutive pairs of eqs. (12) and easily obtain the integrals

$$X_{k+1}/X_k = A_k \exp(B_k v)\,, \quad k = 1, 2, \ldots, N-2\,, \tag{13}$$

where

$$B_k \equiv D(c_{k-2} - c_{k-1})\,, \tag{13a}$$

and the A_k are $N-2$ arbitrary constants of integration. With (13) it is now easy to show that eqs. (12) all reduce to the same equation for X_1:

$$X_1' = X_1 D(c_{N-2} + 1) + f(v) X_1^2\,, \tag{14}$$

where

$$f(v) \equiv \sum_{j=1}^{N-1} (c_{j-2} - c_{N-2}) \times \left(\prod_{k=1}^{j-1} A_k\right) \exp[-vD(1 + c_{j-2})]\,. \tag{14a}$$

This is a simple Riccati equation which can be integrated by elementary methods [7] to yield

$$X_1 = -e^{vK}\left(f \int f e^{vK}\, dv + A_0 f\right)^{-1}\,, \tag{15}$$

where $K \equiv D(1 + c_{N-2})$ and A_0 is the $(N-1)$ st integration constant (the Nth one is D).

Thus, with the X_k $(k = 2, 3, \ldots, N-1)$ all related to X_1 by (13) and X_N obtained from (11) the Lotka–Volterra system (8), (8a) has been completely solved. The specific behavior of each X_k, i.e. whether it will approach a finite constant or go to infinity as $t \to \infty$, will depend on the values of the parameters of the problem. Clearly, however, no matter what these values are, all solutions of (8), (8a) will behave in a regular and predictable way; there will be no chaos, strange attractors or any other of the exotic phenomena of non-integrable systems [1,2].

We also remark that the linear term in (8) has a particularly simple form. More general linear terms can also lead to integrable systems having the Painlevé property. This point is discussed in more detail elsewhere [15,16] on some second and third order models. Finally, we point out that new integrable systems (1) can be found when one drops the antisymmetry condition (2).

Taking again for simplicity the case of equal "growth rate" (7), and scaling the x_k we may write the coefficient matrix $\mathbf{C}^{(3)}$ in the form

$$(c_{kj}) \equiv \mathbf{C}^{(3)} = \begin{pmatrix} 0 & 1 & C \\ A & 0 & 1 \\ 1 & B & 0 \end{pmatrix}, \tag{16}$$

cf. (5). Performing the singularity analysis on this system we find four cases with the Painlevé property [13,16]: in three of them A, B, C have fixed, complex values, while the fourth one has again one free parameter, C, with

$$A = -(1 + C)/C\,, \quad B = -1/(1 + C)\,. \tag{17}$$

The only difference between the analysis of (16) and that of (5) is that now more singularity "types" [where some of the a_k in (3) are zero] must be examined [13, 16].

We have confirmed the integrability of (1) with (16), (17) and (7) by removing first the linear terms via (9) and then integrating the equation explicitly in terms of the new variables

$$S \equiv X_1 + CX_2/(C+1) + CX_3\,, \quad P \equiv X_1 X_3\,. \tag{18}$$

For $PC/D > 0$ the result is [13,16]

$$S = \pm\sqrt{D}\coth u\,, \quad P = D/(4C\sinh^2 u)\,,$$
$$u \equiv \pm\tfrac{1}{2}\sqrt{D}(v - v_0)\,, \tag{19}$$

where D, v_0 are two integration constants (and similar results for the $PC/D > 0$ case). Using now (18), (19) X_3 is solved from a Riccati equation

$$X_3' = (C+1)P(v)/C - X_3 S(v)/C + X_3^2\,, \tag{20}$$

which can be transformed with $X_3 \equiv -W'/W$ to a linear second order ode in W, with variable coefficients.

We end with some comments on the concept of integrability of a generally non-hamiltonian dynamical system. Clearly, one may start by defining as integrable (or completely integrable) such a system, if it can be integrated step by step down to a final integral (or quadrature) as in the case (8), (8a), ending with (15). This definition, however, by itself would be too narrow; it must be extended to include systems which can be transformed to a (higher than first order) set of linear ode's with variable coefficients, as was the case with (1), (7) and (16) above.

Still this does not complete the list. Our singularity analysis leads us to distinguish yet a third possibility, which we may also call integrable: this is the case where a dynamical system, identified by the Painlevé property, reduces to one of the second (or possible higher) order Painlevé transcedents [7] as it happens e.g. with the Lorenz equations at some special values of their parameters [9]. These Painlevé transcedents are nonlinear equations, whose solutions are analytic (modulo poles) in the complex t-plane. Families of particular solutions of these equations are known [17], while more general solutions can be obtained by analytic continuation. Painlevé transcedents, however, are neither integrable by quadratures, nor can they be transformed to linear ode's. Still, they may be considered linearizable in the sense that they can be solved in terms of linear integral equations [18].

We do not pretend, of course, that the Painlevé property can identify all integrable dynamical systems. In fact, some examples are known [8,15] which can be integrated by quadratures and yet their solutions have (movable) singularities "worse" than poles. Still, the Painlevé property appears to be a very useful criterion for identifying integrable dynamical systems. More than that, it may also lead to a better understanding of integrability of nonlinear ode's as it did in the case of nonlinear pde's [19,20].

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