A Singularity Analysis Approach to the Solutions of Duffing's Equation

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The singularity structure of Duffing's equation in the complex t-plane is investigated analytically and numerically. A series expansion for the general solution around each singularity $t_* = t_R + it_I$ is given, and is subsequently used to approximate the locations of singularity "spirals" $t_*^{(n)}$, $n = 1,2,\ldots$, around every t_*. The main "chimney" patterns--on which singularities are observed to accumulate--are explained by deriving a simple expression for the distances between singularities on the "walls" of these chimneys $|t_* - t_*^{(1)}| \sim Q^{-1/4}\exp(-t_I/2)$, Q being the amplitude of the (periodic) driving force. Thus, singularity patterns are seen to further "condense", as Q increases and the motion becomes **globally** *more chaotic in real t. These results suggest that series expansions near singularities in the complex t-plane can provide useful representations of the general solution of Duffing's equation.*

In recent years, there has been considerable progress in the understanding of the global behaviour of the solutions of nonlinear dynamical systems, from the viewpoint of their singularity analysis in the complex time plane [1-4]. Most of the activity so far has focused on completely integrable systems [5-7], many new examples of which have been discovered by the requirement that their solutions possess the Painlevé property, i.e., that their only movable singularities in the complex t-plane are poles.

More recently, however, with the help of some important theorems due to Ziglin [8,9], it has been realized that the singularity analysis of **non-integrable** systems can be put on a more rigorous footing, as well. In particular, the work of Yoshida [10] and others [11,12] has shown that it is possible to prove **non-integrability** of 2-degree of freedom Hamiltonian systems by examining their singularity structure. On the other hand, the splitting of separatrices has been rigorously connected with multi-valued, infinitely sheeted solutions, even in the case of non-Hamiltonian, near-integrable systems [13]. For a more recent review of the results of singularity analysis to date, see [14].

Research Reports in Physics Symmetries and Singularity Structures
Editors: M. Lakshmanan · M. Daniel

In this paper, we develop the singularity analysis of non-integrable systems further, by demonstrating the usefulness of series expansions of the solutions near their singularities in the complex t-plane. As an example, we have chosen to study here the perennial paradigm of non-integrable dynamical systems, Duffing's equation,

$$\ddot{x} + \delta\dot{x} + Ax + x^3 = Q\cos 2t \tag{1}$$

whose solutions, near a singularity t_*, are of the form [13]

$$x(t) = \frac{1}{\tau}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\alpha_{rj}\tau^r(\tau^4\ln\tau)^j, \quad \tau \equiv t - t_*, \quad \tau\to 0. \tag{2}$$

Here, we will set $\delta = 0$ in (1) but entirely similar results hold in the dissipative case ($\delta > 0$) as well [15].

It has already been demonstrated numerically that the movable singularities t_* of the solutions of (1) lie on chimney patterns, which have smaller subchimneys of singularities extending outward from the "walls" of the primary chimneys [14,15]. Using asymptotic reductions of the solution (2), and concentrating mainly on its leading $r = 0$ part, we shall demonstrate here analytically that the distance ρ_1, between two neighbouring singularities t_* and t_*', on the walls of the main chimneys, behaves (far enough from the Ret axis) as

$$\rho_1 \sim Q^{-1/4}\exp(-|\mathrm{Im}t_*|/2) .$$

This explains both the exponential accumulation of t_* on these chimneys as well as their increased "condensation", observed numerically as Q increases (see Fig. 1). Moreover, our results also explain

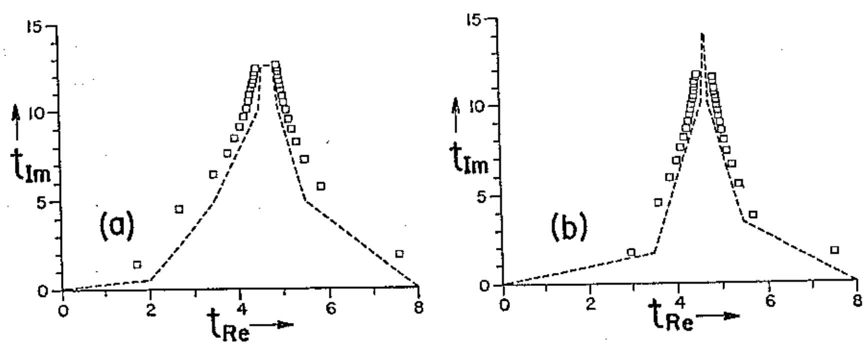

Fig. 1 Chimney patterns of singularities of Duffing's equation (1) for A = −1, $\delta = 0.25$ and (a) Q = 0.2, (b) Q = 0.4 (taken from ref. 13). Note the relative "condensation" of the pattern from (a) to (b). The picture is very similar for $\delta = 0$.

the presence of subchimneys, by demonstrating that around each t_* there is a **spiral** of singularities at distances $\rho_1 > \rho_2 > \rho_3 > \ldots$ lying on different Riemann sheets.

Consider, for simplicity, the conservative Duffing's equation

$$\ddot{x} + Ax + x^3 + Q\cos 2t = 0 . \qquad (3)$$

When $Q = 0$, of course, its solutions are well known: They are Jacobi elliptic functions [16], whose singularity patterns consist of a doubly periodic lattice of poles near which the solutions can be expressed as Laurent series

$$x(t) = \frac{1}{\tau}\sum_{\tau=0}^{\infty} \alpha_r \tau^r , \quad \tau \equiv t - t_* , \qquad (4)$$

t_* being the (arbitrary) location of these poles determined by the initial conditions for each orbit.

Now, while orbits of (3), for $Q = 0$, are periodic, and the motion in the x, $\dot{x}$ plane is regular and well understood, the situation for $Q \neq 0$ is completely different: The orbits are free to explore a 3-dimensional phase space, the Hamiltonian of the system is no longer conserved, and no general solution of the equation of motion is analytically available. The x, $\dot{x}$ plane has become a Poincaré surface of section [17], and new, **chaotic** orbits exist, which are seen to intersect the $x,\dot{x}$ surface of section in an apparently irregular and unpredictable manner.

The question we ask here is: Is this interesting behaviour of the solutions reflected to some extent in their singularity patterns in the complex t-plane? And if so, what can we learn from these patterns that will improve our understanding of the motion in **real** t?

To answer this question, note first that these singularities are no longer poles, as in the $Q = 0$ case (cf. (4)). In fact, near them, the asymptotic expansions of the solutions are known to contain logarithmic terms, as shown in (2), and thus be multivalued functions of t, with generally infinitely many Riemann sheets [9]. All α_{rj} coefficients in (2), are solved, as usual, from recursion relations, obtained by substituting (2) in (3), except for α_{40}, which is the **second** free constant of the problem specified by the initial conditions (the first one is t_*) [13].

To analyze the structure of these chimney patterns, we shall apply the method of **asymptotic reductions**, which has been used successfully in the singularity analysis of other non-integrable dynamical systems [2,3,15]. In the case of Duffing's equation one starts by writing (2) in the form

$$x(t) = \frac{1}{\tau} \sum_{r=0}^{\infty} \tau^r f_r(u), \quad u \equiv \tau^4 \ln\tau \tag{5}$$

where the functions $f_r(u)$ are assumed to be analytic near $u = 0$. They are, of course, expected to contain--as r increases--terms of smaller and smaller magnitude near $u = 0$. By this reasoning, we can consider the first term in (5), i.e., $x(t) \simeq f_0/\tau$, as the **lowest order** approximation to the solution (2), containing the most divergent terms as $\tau \to 0$. Substituting (5) in (3), and equating terms of the same order in τ, yields a nonlinear ODE for $f_0(u)$,

$$16u^2 f_0''(u) + 4uf_0'(u) + 2f_0(u) + f_0^3(u) = 0 , \tag{6}$$

and a hierarchy of **linear** ODEs for the higher order approximants $f_r(u)$:

$$16u^2 f_r'' + 4(2r + 1)uf_r' + (r^2 - 3r + 2 + 3f_0^2)f_r = F_r(f_0,\ldots,f_{r-1}), \tag{7}$$

where

$$F_1 = 0, \quad F_2 = -Af_0, \quad F_3 = -Q\cos 2t_* , \text{ etc.}$$

Now, (6) turns out to be analytically solvable! Making the transformation

$$f_0 = \nu\theta_0(\nu), \quad \nu \equiv u^{1/4} \tag{8}$$

one obtains from (6):

$$\Theta_0''(\nu) + \Theta_0^3(\nu) = 0$$

which is easily solved in terms of Jacobi elliptic functions to furnish

$$f_0(u) = -\lambda u^{1/4} \operatorname{cn}(\lambda u^{\frac{1}{4}} + iK, 1/\sqrt{2}) \tag{9}$$

where $K = K(1/\sqrt{2})$ is the elliptic integral of the first kind. In fact, the more general transformation

$$f_r(u) = \nu^{1-r}\theta_r(\nu), \quad u \equiv u^{1/4}, \quad r \geq 0 ,$$

simplifies (7) to an (inhomogeneous) equation of the Lamé-type [18]:

$$\Theta_r''(\nu) + 3\Theta_0^2(\nu)\Theta_r(\nu) = \nu^{3-r}F_r , \tag{10}$$

whose homogeneous part can be exactly solved in terms of standard hypergeometric functions in the variable $w = cn^4(\lambda\nu+iK,\ 1/\sqrt{2})$. Thus, to the extent that the solutions of these linear equations (10) are tractable, one can argue that (2) represents the **general solution** of Duffing's equation near its singularities in the complex t-plane.

As a first attempt to make a quantitative use of these results, let us substitute (2), in (3) and equate like powers of τ, to find

$$\alpha_{00} = f_0(0) = i\sqrt{2}, \quad \alpha_{01} = f_0'(0) = (2Q/5)\sin 2t_*. \tag{11}$$

Combining (11) with (9) gives an equation for λ

$$\lambda^4 = -4\sqrt{2}Q \exp(-2it_R + 2t_I), \tag{12}$$

where we have set $t_* = t_R + it_I$ and taken t_I (> 0) large enough, so that the $\exp(-2t_I)$ term in $\sin 2t_*$ can be neglected.

Since $u = 0$, about which $f_0(u)$ is **analytic**, corresponds to $t = t_*$, this means that the singularity u_* of $f_0(u)$ closest to $u = 0$, will correspond to t_*', i.e., the closest singularity to t_* of $x(t)$ in the complex t-plane. And since f_0 is known, from the theory of elliptic functions, the location of u_* is also known:

$$u_* \equiv 16K^4/\lambda^4 = (2\sqrt{2}K^4/Q)\exp(2it_R - 2t_I), \text{(cf. (12))}. \tag{13}$$

Defining then in (13)

$$u_* \equiv \tau_*^4 \ln\tau_*, \quad \tau_* \equiv t_*' - t_* = \rho e^{i\varphi}, \tag{14}$$

we finally arrive at explicit expressions for ρ and φ, for the approximate location of t_*' closest to t_*:

$$\rho^4(\ln^2\rho + \varphi^2)^{1/2} = 2(\sqrt{2}\,K^4/Q)\exp(-2t_I), \tag{14a}$$

$$4\varphi + \tan^{-1}(\varphi/\ln\rho) = 2t_R. \tag{14b}$$

These results can be used to study the chimney structure of the singularities of Duffing's equation in more detail. However, we can already deduce some useful information from the above analysis: If we denote by ρ_1, φ_1 the solution of (14) with the smallest φ_1 angle, it is not difficult to see that

$$\rho_1 \sim Q^{-1/4}\exp(-t_I/2), \tag{15}$$

which implies that ρ_1 decreases exponentially in the $\mathrm{Im}\,t > 0$ direction, as the numerically observed chimney patterns of Fig. 1 clearly show.

Perhaps a more important observation about (15) is the dependence of ρ_1 on Q, the amplitude of the driving force. It is well known that, as Q increases, the "size" of the chaotic regions on the surface of section of (3) also increases [15]. What (15) is telling us is that, as one forces the system harder, and the motion becomes **globally** more chaotic, ρ_1 should decrease approximately as $Q^{-1/4}$, and the chimney patterns become more "condensed" as we have indeed observed in earlier publications (see Fig. 1).[1]

In deriving higher order terms in the expansion (5) we have obtained $f_r(u)$, $r = 1,2,3$, in closed form, as particular solutions $g_r(u)$ of (7) in terms of Jacobi elliptic functions [15]. Moreover, we have found $f_4(u) = c_1 f_0'(u) + g_4(u)$, where c_1 is the second free constant of the solution.

Computing, however, from all of the above, higher order corrections to the locations of the nearest singularities t_*', we found that they were **orders of magnitude** smaller than what one gets from (14). In particular, at $r = 4$, τ_* is corrected by an additive term $\sim c_1/(4\ln\tau_*+1)$ [15]. Since $|\ln\tau_*| \to\infty$, as $\tau_* \to 0$, this term also turns out to be too small to significantly affect the ρ, φ values obtained from (14), thus demonstrating again the insensitivity of the "upper" part of the chimneys to changes in the initial conditions [13].

We may now use Eq. (14) to determine the location of singularities higher up on the chimneys of the solutions of (3). To this end, we start by assuming that we know the location of one of them, $t_* = t_R + it_I$, say, and solve (14) for ρ, φ, by a simple Newton algorithm, solving first (14a) for φ in terms of ρ and substituting in (14b).

We thus obtain an infinite number of distinct roots $\rho_1 > \rho_2 > \ldots$ to which there corresponds a monotonically increasing set of angles $\varphi_1 < \varphi_2 < \ldots$. Denoting now by ρ_{th} and φ_{th} the first solution pair of (14), ρ_1 and φ_1, corresponding to the t_*' just above t_* on the main chimney "wall"--we list them in Tables 1 and 2 for several t_*, against their experimentally computed values, ρ_{exp} and φ_{exp}.

Note first that, for both sets of parameter values and initial conditions in Tables 1,2, the theoretical results clearly show, qualitatively, the presence of chimney patterns in the complex t-plane. Quantitatively also, there is satisfactory agreement between theory and experiment, since ρ_{th} and φ_{th} are generally within 10-20% of their experimental values.

[1]Note that in [13] this dependence on Q was erroneously printed as Q^{-1}, instead of the correct $Q^{-1/4}$.

Table 1 Q = 0.1; A = 0.2089 : A periodic orbit of (5) of period π

Sing # n above Ret	$\rho_{th}^{(n)}$	$\varphi_{th}^{(n)}$	$\rho_{exp}^{(n)}$	$\varphi_{exp}^{(n)}$
10	0.1490	1.2411	0.141	1.489
11	0.1380	1.2425	0.128	1.493
12	0.1300	1.2437	0.119	1.498
13	0.1207	1.2446	0.110	1.502
14	0.1136	1.2454	0.102	1.505

Table 2 Q = 2.8; A = 0.3 : A chaotic orbit of (5)

Sing. # n above Ret	$\rho_{th}^{(n)}$	$\varphi_{th}^{(n)}$	$\rho_{exp}^{(n)}$	$\varphi_{exp}^{(n)}$
10	0.1480	1.2378	0.140	1.4847
11	0.1370	1.2390	0.127	1.4899
12	0.1279	1.2410	0.118	1.4955
13	0.1198	1.2420	0.109	1.5000
14	0.1129	1.2430	0.101	1.5026

Finally, it is interesting to note that, similar to what has been found for other nonintegrable systems, the singularity patterns of Duffing's equation also have a lot more structure, than what is shown in Fig. 1. Integrating (3) along a path leading outward from one of the main chimenys, secondary chimneys are discovered, which appear to have yet smaller chimneys on them, etc., as seen e.g. in Fig. 2. For

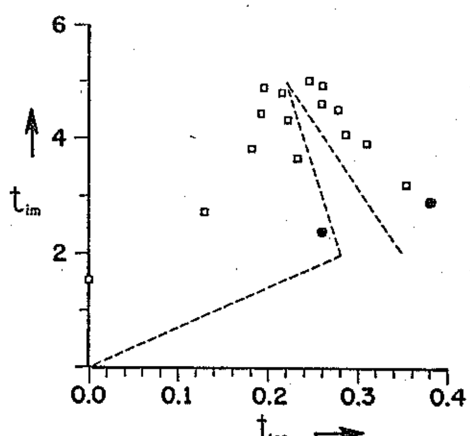

Fig. 2 Secondary chimney singularities marked by □. The dashed line indicates the integration path passing between the second and third singularity (from the left) of Fig. 2(a), marked here by solid dots (●).

a dtailed analysis of how this "subchimney" structure is described by the roots of (14), ρ_n, φ_n, $n \geq 2$ see ref. [19].

We have presented in this paper, analytical and numerical results on the singularity structure of the solutions of Duffing's equation in the complex t-plane. Our main goal was to use singularity analysis to obtain some useful information about the general solutions of this non-integrable system.

We have shown that the singularities of the solutions of equation (3) lie on "chimney" patterns, on which singularities accumulate densely on the same Riemann sheet. We have studied these patterns by asymptotic reductions of the solutions near their singularities t_* and obtained simple equations for the approximate location of singularities t_*' closest to t_*.

Judging from the accuracy of the above results, it would be reasonable to expect that the series (2) converges over a large region around each t_*, perhaps all the way to the nearest singularities. Thus, one could "patch up" the complex plane with such regions and propose this series as a useful representation of the solution of Duffing's equation, valid especially in places where Taylor series become inaccurate.

Finally, from a more "practical" point of view, it would be very interesting, if relationships such as (15), connecting singularity patterns to global chaotic properties, were to be found in higher dimensional systems as well. One could then look for evidence of global chaos in the accumulation properties of singularity patterns near $t = 0$, rather than integrating many orbits for long times in the Ret direction.

ACKNOWLEDGEMENTS AND A NOTE

Many useful discussions are acknowledged here with C. Cosgrove and M. Tabor. In fact, this paper is a brief review of results obtained already in [15] and reported in [14]. Results very similar to our own on Duffing's equation have also been obtained independently by Fournier, Levine and Tabor (see ref. [19]).

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