MOMENT SETS AND THE UNITARY DUAL OF A NILPOTENT LIE GROUP

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ABSTRACT. Let G be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and unitary dual $\widehat{G}$. The moment map for $\pi \in \widehat{G}$ sends smooth vectors in the representation space of π to $\mathfrak{g}^*$. The closure of the image of the moment map for π is called its *moment set*. N. Wildberger has proved that the moment set for π coincides with the closure of the convex hull of the corresponding coadjoint orbit. We say that $\widehat{G}$ is *moment separable* when the moment sets differ for any pair of distinct irreducible unitary representations. Our main results provide sufficient and necessary conditions for moment separability in a restricted class of nilpotent groups.

1. INTRODUCTION

Let G be a real Lie group with Lie algebra $\mathfrak{g}$ and π be a (strongly continuous) unitary representation of G in some Hilbert space $\mathcal{H}$. The *moment map* for π is defined as

$$\psi_\pi : \mathcal{H}^\infty \setminus \{0\} \to \mathfrak{g}^*, \quad \psi_\pi(v)(X) = \frac{1}{i}\frac{\langle d\pi(X)v, v\rangle}{\langle v, v\rangle}, \tag{1.1}$$

where $d\pi$ denotes the derived representation of $\mathfrak{g}$ in the space $\mathcal{H}^\infty$ of smooth vectors. When π is finite dimensional, this notion, which is due to N. Wildberger, reduces to that of the usual moment map for the Hamiltonian action of G via π on the projective space $P(\mathcal{H})$ [Wil92]. The *moment set* for π is defined as

$$I_\pi = \overline{\{\psi_\pi(v) \ : \ v \in \mathcal{H}^\infty \setminus \{0\}\}},$$

the closure of the image of ψ_π in $\mathfrak{g}^*$. We let $\widehat{G}$ denote the set of irreducible unitary representations of G (up to unitary equivalence) and consider the moment sets I_π for $\pi \in \widehat{G}$. One says that $\widehat{G}$ is *moment separable* if $I_\pi \neq I_{\pi'}$ for all $\pi, \pi' \in \widehat{G}$ with $\pi \not\simeq \pi'$.

In the case of a compact group G, the moment set I_π of an irreducible representation π need not be convex. Wildberger has shown, however, that the set of extremal points of the convex hull of I_π is a single coadjoint orbit, namely the orbit through the highest weight of the representation π. Thus the moment set completely determines the representation.

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In this paper we consider the situation where G is a connected and simply connected nilpotent Lie group. In this setting, a fundamental result of Wildberger relates the moment set I_π for $\pi \in \widehat{G}$ to the coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ associated to π via the Kirillov method [Kir62]. Namely

$$I_\pi = \overline{Conv(\mathcal{O})}, \tag{1.2}$$

the closure of the convex hull of $\mathcal{O}$ in $\mathfrak{g}^*$ [Wil89]. We remark that this result has been generalized to encompass connected solvable Lie groups by D. Arnal and J. Ludwig in [AL92]. In view of Equation 1.2, $\widehat{G}$ is moment separable if and only if

$$\overline{Conv(\mathcal{O})} = \overline{Conv(\mathcal{O}')} \Rightarrow \mathcal{O} = \mathcal{O}'$$

for all coadjoint orbits $\mathcal{O}, \mathcal{O}' \subset \mathfrak{g}^*$.

In [Wil89], Wildberger presents an example which shows that $\widehat{G}$ need not be moment separable in the nilpotent case. It is thus natural to seek a characterization of the class of connected and simply connected nilpotent Lie groups for which the moment sets do separate $\widehat{G}$. Our main results in this direction are given below in Theorems 3.4 and 3.5. These provide sufficient and necessary conditions, respectively, for moment separability of $\widehat{G}$. In fact, our results apply only to a restricted class of nilpotent groups, namely those which satisfy Condition (C) formulated below in Section 3.

The sufficient and necessary conditions in Theorems 3.4 and 3.5 involve properties of the Pukanszky polynomials which parameterize the coadjoint orbits. We review this orbit parameterization in Section 2 in order to introduce notation needed to formulate our results. The orbits are grouped in layers and the polynomial functions which determine the orbits within a layer have a common domain. It is, however, possible for the convex hull of a coadjoint orbit to pass though other orbit layers. This fact greatly complicates the use of Pukanszky polynomials in this problem. We are led to formulate Condition (C) for use as a hypothesis in Theorems 3.4 and 3.5. This condition implies that the moment sets for representations whose coadjoint orbits lie in different layers are necessarily distinct. We show that all three step groups satisfy Condition (C), so these are among the groups to which our results apply.

We present several examples in Section 4. These show that the sufficient condition for separability in Theorem 3.4 is not necessary and that the necessary condition in Theorem 3.5 is not sufficient. In addition, we present an example to show that the convex hull of a coadjoint orbit need not lie within a layer. In Section 5 we discuss some questions that remain open in this area.

We conclude this introduction by noting that the definition given in Equation 1.1 extends to yield a *generalized moment map*,

$$\Psi_\pi : \mathcal{H}^\infty \setminus \{0\} \to \mathcal{U}(\mathfrak{g})^*,$$

where $\mathcal{U}(\mathfrak{g})$ denotes the complexified universal enveloping algebra. One can define the *generalized moment set* $J_\pi \subset \mathcal{U}(\mathfrak{g})^*$ as

$$J_\pi = \overline{Conv(\Psi_\pi(\mathcal{H}^\infty \setminus \{0\}))}.$$

It is shown in [BLS98] that such generalized moment sets always separate the unitary dual $\widehat{G}$ of any connected and simply connected nilpotent Lie group.

2. Preliminaries on Orbit Parameterization

We begin this section by reviewing Pukanszky's parameterization of the coadjoint orbits for a nilpotent Lie group. This material is quite standard. We refer the reader to [Puk67], [Ped84] or [CG90] for details. Throughout, G will always denote a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Let

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$$

be any fixed Jordan-Hölder sequence in $\mathfrak{g}$ and

$$X_1, X_2, \ldots, X_n$$

be an associated strong Malcev basis with $X_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j-1}$. We denote by

$$X_1^*, X_2^*, \ldots, X_n^*$$

the dual basis for $\mathfrak{g}^*$.

For $\ell \in \mathfrak{g}^*$ we let $\mathcal{O}_\ell = Ad^*(G)\ell$ denote the coadjoint orbit through ℓ and

$$\mathfrak{g}(\ell) = \{X \in \mathfrak{g} \ : \ \ell[X, \mathfrak{g}] = \{0\}\},$$

the Lie algebra of the stabilizer of ℓ. A value $j \in \{1, \ldots, n\}$ is a *jump index* for ℓ if

$$X_j \notin \mathfrak{g}(\ell) + \mathfrak{g}_{j-1}.$$

We let

$$e(\ell) = \{j \ : \ j \text{ is a jump index for } \ell\}, \quad \widetilde{e}(\ell) = \{1, \ldots, n\} \setminus e(\ell)$$

and

$$\mathcal{E} = \{e(\ell) \ : \ \ell \in \mathfrak{g}^*\}.$$

The set $e(\ell)$ contains exactly $\dim(\mathcal{O}_\ell)$ indices, which is necessarily an even number. The sets

$$\Omega_e = \{\ell \in \mathfrak{g}^* \ : \ e(\ell) = e\}$$

for each $e \in \mathcal{E}$ are the *layers* in $\mathfrak{g}^*$. One has $\mathcal{O}_\ell \subset \Omega_{e(\ell)}$ for $\ell \in \mathfrak{g}^*$.

Each layer Ω_e is a semi-algebraic set in $\mathfrak{g}^*$. To understand this fact, one first introduces the strict total ordering $\prec$ on $\mathcal{E}$ defined as follows. For $e, e' \in \mathcal{E}$ we have $e \prec e'$ if either

1. $e = \{j_1 < j_2 < \cdots < j_d\}$, $e' = \{j'_1 < j'_2 < \cdots < j'_{d'}\}$ where $j_1 = j'_1, \ldots, j_{k-1} = j'_{k-1}$ and $j_k < j'_k$ for some $k \le \min(d, d')$, or
2. $e' \subsetneq e$.

Note that, in view of the second condition, the empty set $e = \emptyset$ is the maximal element in $\mathcal{E}$. The layer $\Omega_\emptyset$ corresponds to the one-dimensional representations in $\widehat{G}$. The layer Ω_e given by the minimal element e in $\mathcal{E}$ contains the *generic orbits* and forms a Zariski open set in $\mathfrak{g}^*$. More generally, one has that for $e \in \mathcal{E}$, the layer Ω_e is the intersection of a Zariski open set with $\bigcup_{e' \succeq e} \Omega_{e'}$, which is Zariski closed. In fact, there are $Ad^*(G)$-invariant polynomial functions

$$P_e : \mathfrak{g}^* \to \mathbb{R}$$

for each $e \in \mathcal{E}$ with the property that

$$\Omega_e = \{\ell \in \mathfrak{g}^* \ : \ P_e(\ell) \neq 0 \text{ and } P_{e'}(\ell) = 0 \text{ for } e' \prec e\}.$$

These are defined explicitly as $P_\emptyset = 1$ and

$$P_e(\ell) = Pf(M_e(\ell)), \quad \text{where} \quad M_e(\ell) = \Big(\ell[X_i, X_j]\Big)_{i,j \in e} \tag{2.1}$$

for $e \prec \emptyset$. That is, $P_e(\ell)$ is the Pfaffian of the even skew-symmetric matrix $M_e(\ell)$.

Given any subset S of $\{1, 2, \ldots, m\}$, we let

$$V_S = Span\{X_j^* \ : \ j \in S\} \subset \mathfrak{g}^*.$$

For $e \in \mathcal{E}$ and $\ell \in \Omega_e$, there is a polynomial function

$$r^e(\ell, \cdot) : V_e \to V_{\widetilde{e}}$$

for which $r^e(\ell, 0) = 0$ and

$$\mathcal{O}_\ell = \ell + Graph(r^e(\ell, \cdot)) = \ell + \{x + r^e(\ell, x) \ : \ x \in V_e\}. \tag{2.2}$$

Writing

$$r^e(\ell, x) = \sum_{j \in \widetilde{e}} r_j^e(\ell, x) X_j^*,$$

the polynomial $r_j^e(\ell, \cdot) : V_e \to \mathbb{R}$ depends only on $\ell|_{\mathfrak{g}_{j-1}}$ and on the value of x on $\mathfrak{g}_{j-1}$.

For $\ell \in \mathfrak{g}^*$ we define

$$\widetilde{e}_1(\ell) = \{j \in \widetilde{e}(\ell) \ : \ r_j^{e(\ell)}(\ell, x) = 0 \ \forall x \in V_{e(\ell)}\}, \quad \widetilde{e}_2(\ell) = \widetilde{e}(\ell) \setminus \widetilde{e}_1(\ell).$$

The orbit $\mathcal{O}_\ell$ is constant in the directions X_j^* for each $j \in \widetilde{e}_1(\ell)$. For $e = e(\ell)$, the polynomial $r^e(\ell, x) = \sum_{j \in \widetilde{e}(\ell)} r_j^e(\ell, x) X_j^* = \sum_{j \in \widetilde{e}_2(\ell)} r_j^e(\ell, x) X_j^*$ takes values in the subspace $V_{\widetilde{e}_2(\ell)}$ of $V_{\widetilde{e}(\ell)}$. We will also write this polynomial as

$$r_\ell : V_{e(\ell)} \to V_{\widetilde{e}_2(\ell)}.$$

For $e \in \mathcal{E}$, the set

$$\mathcal{U}_e = \Omega_e \cap V_{\widetilde{e}}$$

is a cross section to the coadjoint orbits in Ω_e. That is, $\mathcal{O}_\ell$ meets $\mathcal{U}_{e(\ell)}$ in exactly one point. For $\ell = \sum_{j \in \widetilde{e}} \ell_j X_j^* \in \mathcal{U}_e$ we write

$$\ell^{(1)} = \sum_{j \in \widetilde{e}_1(\ell)} \ell_j X_j^* \in V_{\widetilde{e}_1(\ell)}, \quad \ell^{(2)} = \sum_{j \in \widetilde{e}_2(\ell)} \ell_j X_j^* \in V_{\widetilde{e}_2(\ell)},$$

so that $\ell = \ell^{(1)} + \ell^{(2)}$. With this notation, Equation 2.2 now yields

$$\mathcal{O}_\ell = \ell^{(1)} + \Big(\ell^{(2)} + Graph(r_\ell)\Big) \subset V_{\widetilde{e}_1(\ell)} \oplus \Big(V_e \oplus V_{\widetilde{e}_2(\ell)}\Big), \tag{2.3}$$

$$Conv(\mathcal{O}_\ell) = \ell^{(1)} + \Big(\ell^{(2)} + Conv(Graph(r_\ell))\Big), \quad \text{and} \tag{2.4}$$

$$\overline{Conv(\mathcal{O}_\ell)} = \ell^{(1)} + \Big(\ell^{(2)} + \overline{Conv(Graph(r_\ell))}\Big). \tag{2.5}$$

The indices $\widetilde{e}_1(\ell)$ can be characterized algebraically as follows.

Lemma 2.1. $\widetilde{e}_1(\ell) = \{j \in \widetilde{e}(\ell) \ : \ X_j \in \mathfrak{a}(\ell)\}$ *where* $\mathfrak{a}(\ell)$ *denotes the largest ideal of* $\mathfrak{g}$ *that is contained in* $\mathfrak{g}(\ell)$.

Proof. Note that the largest ideal in $\mathfrak{g}$ that is contained in $\mathfrak{g}(\ell)$ is

$$\mathfrak{a}(\ell) = \bigcap_{g\in G} Ad(g)(\mathfrak{g}(\ell)) = \{X \in \mathfrak{g} \ : \ Ad(g)(X) \in \mathfrak{g}(\ell) \ \forall g \in G\}.$$

Let $j \in \widetilde{e}(\ell)$. Now $j \in \widetilde{e}_1(\ell)$ if and only if the map $G \to \mathbb{R}$, $g \mapsto (Ad^*(g)\ell)(X_j)$ is constant. Equivalently, we must have

$$\frac{d}{dt}\bigg|_{t=0} \Big(Ad^*(exp(tY)g)\ell\Big)(X_j) = 0$$

for all $g \in G$ and all $Y \in \mathfrak{g}$. That is, $j \in \widetilde{e}_1(\ell)$ if and only if $(Ad^*(g)\ell)[X_j, Y] = 0$ for all g, Y. Equivalently, $X_j \in \mathfrak{g}(Ad^*(g)\ell) = Ad(g)(\mathfrak{g}(\ell))$ for all $g \in G$. □

Lemma 2.1 shows that we can have $e(\ell) = e(\ell')$ but $\widetilde{e}_1(\ell) \neq \widetilde{e}_1(\ell')$. The ideal $\mathfrak{a}(\ell)$ can be characterized in terms of the representation π asociated to the coadjoint orbit $\mathcal{O}_\ell$ through ℓ:

$$\mathfrak{a}(\ell) = \{X \in \mathfrak{g} \ : \ d\pi(X) \text{ is a scalar operator}\} = \{X \in \mathfrak{g} \ : \ d\pi(X) = i\ell(X)I\}. \tag{2.6}$$

We refer the reader to [Lud78] for this fact, the proof of which requires details concerning the Kirillov correspondence. Lemma 2.3 below recasts Equation 2.6 in terms of the moment map for π. The proof uses a preliminary result.

Lemma 2.2. *Let* $X \in \mathfrak{g}$, $\pi \in \widehat{G}$ *and* $\lambda \in \mathbb{R}$ *be given. We have* $\psi_\pi(v)(X) = \lambda$ *for all* $v \in \mathcal{H}_\pi^\infty \setminus \{0\}$ *if and only if* $d\pi(X) = i\lambda I$.

Proof. It is clear that if $d\pi(X) = i\lambda Id$, then $\psi_\pi(v)(X) = \lambda$ for all $v \in \mathcal{H}_\pi^\infty \setminus \{0\}$. Conversely, suppose that $\psi_\pi(v)(X) = \lambda$ for all $v \in \mathcal{H}_\pi^\infty \setminus \{0\}$. Thus

$$\langle d\pi(X)v, v\rangle = i\lambda\|v\|^2. \tag{2.7}$$

Replacing v by $u + v$ in this equation ($u, v \in \mathcal{H}_\pi^\infty \setminus \{0\}$ with $v \neq -u$) gives

$$\mathrm{Im}(\langle d\pi(X)v, u\rangle) = \lambda \mathrm{Re}(\langle v, u\rangle).$$

On the other hand, replacing v in Equation 2.7 by $u + iv$ yields

$$\mathrm{Re}(\langle d\pi(X)v, u\rangle) = -\lambda \mathrm{Im}(\langle v, u\rangle).$$

Thus

$$\langle d\pi(X)v, u\rangle = i\lambda\langle v, u\rangle,$$

and hence $d\pi(X)v = i\lambda v$ for all $v \in \mathcal{H}_\pi^\infty \setminus \{0\}$. □

Lemma 2.3. *Let $\ell \in \mathfrak{g}^*$ and $\pi \in \widehat{G}$ be the representation corresponding to $\mathcal{O}_\ell$. Then the largest ideal $\mathfrak{a}(\ell)$ in $\mathfrak{g}$ contained in $\mathfrak{g}(\ell)$ is*

$$\begin{aligned}\mathfrak{a}(\ell) &= \{X \in \mathfrak{g} \ : \ v \mapsto \psi_\pi(v)(X) \textit{ is constant on } \mathcal{H}_\pi^\infty\} \\ &= \{X \in \mathfrak{g} \mid \psi_\pi(v)(X) = \ell(X) \ \forall \ v \in \mathcal{H}_\pi^\infty\}.\end{aligned}$$

Proof. Lemma 2.2 shows that $v \mapsto \psi_\pi(v)(X)$ is constant on $\mathcal{H}_\pi^\infty$ if and only if $d\pi(X)$ is a scalar operator. The result now follows immediately from Equation 2.6. □

Lemma 2.4. *Let $\pi, \pi' \in \widehat{G}$ correspond to coadjoint orbits $\mathcal{O}_\ell$ and $\mathcal{O}_{\ell'}$. If $I_\pi = I_{\pi'}$ then $\mathfrak{a}(\ell) = \mathfrak{a}(\ell')$ and $\ell = \ell'$ on $\mathfrak{a}(\ell) = \mathfrak{a}(\ell')$.*

Proof. Let $X \in \mathfrak{a}(\ell)$. We will show that $X \in \mathfrak{a}(\ell')$ and that $\ell'(X) = \ell(X)$. Let $v' \in \mathcal{H}_{\pi'}^\infty \setminus \{0\}$. Since $\psi_{\pi'}(v') \in I_{\pi'}$ and $I_{\pi'} = I_\pi$ we have that $\psi_\pi(u_n) \to \psi_{\pi'}(v')$ for some sequence $(u_n)_{n=1}^\infty$ in $\mathcal{H}_\pi^\infty \setminus \{0\}$. Using Lemma 2.3 and the fact that $X \in \mathfrak{a}(\ell)$ we obtain

$$\psi_{\pi'}(v')(X) = \lim_{n\to\infty} \psi_\pi(u_n)(X) = \lim_{n\to\infty} \ell(X) = \ell(X).$$

Thus $v' \mapsto \psi_{\pi'}(v')$ takes the constant value $\ell(X)$ on $H_{\pi'}^\infty \setminus \{0\}$. Lemma 2.3 now shows that $X \in \mathfrak{a}(\ell')$ and that $\ell'(X) = \ell(X)$. □

Corollary 2.5. *If $\ell, \ell' \in \mathfrak{g}^*$ satisfy $e(\ell) = e(\ell')$ and $\overline{Conv(\mathcal{O}_\ell)} = \overline{Conv(\mathcal{O}_{\ell'})}$ then $\widetilde{e}_1(\ell) = \widetilde{e}_1(\ell')$ and $\widetilde{e}_2(\ell) = \widetilde{e}_2(\ell')$.*

Proof. In view of Equation 1.2, this result is an immediate consequence of Lemmas 2.1 and 2.4. □

3. Moment Separability

In this section we will consider connected and simply connected nilpotent Lie groups G that satisfy the following condition:

(C) For every $\ell \in \mathfrak{g}^*$, $Conv(\mathcal{O}_\ell) \subset \bigcup_{e' \succeq e(\ell)} \Omega_{e'}$.

Example 4.5 in Section 4 shows that Condition (C) need not hold in general. Since $\bigcup_{e' \succeq e(\ell)} \Omega_{e'}$ is closed, we also will have $\overline{Conv(\mathcal{O}_\ell)} \subset \bigcup_{e' \succeq e(\ell)} \Omega_{e'}$ when Condition (C) holds. Note that each of the following conditions imply (C):

- $Conv(\mathcal{O}_\ell) \subset \Omega_{e(\ell)}$ for all $\ell \in \mathfrak{g}^*$.
- The connected components of each layer Ω_e ($e \in \mathcal{E}$) are convex.

In many examples it is easy to verify these latter properties. For example, the coadjoint orbits, layers and Pukanszky polynomials for all groups G of dimension at most 6 are computed explicitly by O. Nielsen in [Nie83]. One sees very easily from this that the connected components of the layers are convex in all but two of these cases. The two exceptions are the generic layers for the six dimensional groups denoted $G_{6,17}$ and $G_{6,23}$ in [Nie83]. The orbits in these layers are, however, flat (of dimension 4) and hence we have $Conv(\mathcal{O}_\ell) = \mathcal{O}_\ell \subset \Omega_{e(\ell)}$ for such orbits. Thus we see that Condition (C) holds for all cases where $\dim(G) \leq 6$. The following proposition gives two further classes of examples which satisfy Condition (C).

Proposition 3.1. *a) If G is at most step three then G satisfies Condition* (C)*.*
b) If all coadjoint orbits for G have dimension at most two then G satisfies (C)*.*

Proof. Suppose that G is nilpotent of step at most three and let $\ell \in \mathfrak{g}^*$. We will show that $Conv(\mathcal{O}_\ell) \subset \Omega_{e(\ell)}$. For $\ell' \in Conv(\mathcal{O}_\ell)$ we have

$$\ell' = \lambda_1 \ell \circ Ad(exp(Y_1)) + \cdots + \lambda_N \ell \circ Ad(exp(Y_N))$$

for some $Y_1, \ldots, Y_N \in \mathfrak{g}$ and values $\lambda_1, \ldots, \lambda_N \in (0,1)$ with $\lambda_1 + \cdots + \lambda_N = 1$. Since the layers are determined by the polynomials $\{P_e \ : \ e \in \mathcal{E}\}$, we need only show that $P_e(\ell') = P_e(\ell)$ for all $e \in \mathcal{E}$. This is immediate for $e = \emptyset$. For $e \neq \emptyset$ we have $P_e(\ell') = Pf(M_e(\ell'))$ where $M_e(\ell') = \Big(\ell'[X_i, X_j]\Big)_{i,j \in e}$. As G is at most 3-step, we have

$$\begin{aligned}
\ell'[X_i, X_j] &= \sum_{k=1}^{N} \lambda_k \ell(Ad(exp(Y_k))[X_i, X_j]) \\
&= \sum_{k=1}^{N} \lambda_k \ell([X_i, X_j] + [Y_k, [X_i, X_j]]) \\
&= \ell[X_i, X_j] + \ell\left[\sum_{k=1}^{N} \lambda_k Y_k, [X_i, X_j]\right] \\
&= \ell \circ Ad\left(exp\left(\sum_{k=1}^{N} \lambda_k Y_k\right)\right)[X_i, X_j].
\end{aligned}$$

Letting $g = exp(-\sum \lambda_k Y_k)$ we have show that

$$M_e(\ell') = M_e(Ad^*(g)\ell).$$

Thus also $P_e(\ell') = P_e(Ad^*(g)\ell) = P_e(\ell)$ by $Ad^*(G)$-invariance of P_e. This proves the first assertion in Proposition 3.1

Next suppose that all coadjoint orbits for G are at most two dimensional. For $e \in \mathcal{E}$ let $H_e = \{\ell \in \mathfrak{g}^* \ : \ P_e(\ell) = 0\}$. If $e \prec \emptyset$ then $\#(e) = 2$ since the orbits in Ω_e are two dimensional. If $e = \{j_1 < j_2\}$ then we see that $P_e(\ell) = \ell[X_{j_1}, X_{j_2}]$. Thus H_e is a codimension 1 subspace of $\mathfrak{g}^*$ for each $e \prec \emptyset$. We have $\Omega_e = \Big(\bigcap_{e' \prec e} H_{e'}\Big) \cap \Big(\mathfrak{g}^* \setminus H_e\Big)$ for $e \prec \emptyset$ and $\Omega_\emptyset = \bigcap_{e' \prec \emptyset} H_{e'}$. We see that the connected components for each layer Ω_e are convex sets. $\square$

Remark 3.1. In [Wil89] it is shown that any pair of two-dimensional coadjoint orbits $\mathcal{O}_1$, $\mathcal{O}_2$ is "*convex distinguishable*". That is, $Conv(\mathcal{O}_1) = Conv(\mathcal{O}_2)$ only when $\mathcal{O}_1 = \mathcal{O}_2$. It is thus probably true that $\widehat{G}$ is moment separable when all coadjoint orbits have dimension at most 2. In particular, this is the case if, in addition, the convex hull of each coadjoint orbit is closed. On the other hand, examples presented below in Section 4 show that $\widehat{G}$ may or may not be moment separable when G is 3-step.

Proposition 3.2. *If G satisfies Condition (C), then $\widehat{G}$ is moment separable if and only if the following condition holds: For each $e \in \mathcal{E}$, given $f, g \in \mathcal{U}_e$, we have $\overline{Conv(\mathcal{O}_f)} = \overline{Conv(\mathcal{O}_g)}$ implies $f = g$.*

Proof. The condition is clearly necessary for moment separability. Now suppose that the condition holds, and we are given $f, g \in \mathfrak{g}^*$ with $\overline{Conv(\mathcal{O}_f)} = \overline{Conv(\mathcal{O}_g)}$. In view of Condition (C) one has

$$g \in \overline{Conv(\mathcal{O}_g)} = \overline{Conv(\mathcal{O}_f)} \subset \bigcup_{e' \succeq e(f)} \Omega_{e'}$$

and hence $e(g) \succeq e(f)$. Likewise $e(f) \succeq e(g)$ and hence $e(g) = e(f)$.

Now we can take f' in $\mathcal{U}_e \cap \mathcal{O}_f$ and g' in $U_e \cap \mathcal{O}_g$. Then our condition gives $f' = g'$, and hence $\mathcal{O}_f = \mathcal{O}_g$. Thus G is moment separable. □

Lemma 3.3. *Let ℓ, ℓ' be points in a coadjoint orbit $\mathcal{O} \subset \Omega_e$. Then*

$$r^e(\ell', x) = r^e(\ell, x + y) - r^e(\ell, y)$$

for some y in V_e.

Proof. By our description of the coadjoint orbit $\mathcal{O} = \mathcal{O}_\ell$, we have $\ell' = \ell + y + r^e(\ell, y)$ for some y in V_e. On the other hand, an arbitrary point in $\mathcal{O} = \mathcal{O}_{\ell'}$ is of the form $\ell' + x' + r^e(\ell', x')$ or

$$\begin{aligned} \ell + x + r^e(\ell, x) &= \ell' - y - r^e(\ell, y) + x + r^e(\ell, x) \\ &= \ell' + x - y + r^e(\ell, x) - r^e(\ell, y). \end{aligned}$$

Thus $x' = x - y$, and $r^e(\ell', x') = r^e(\ell, x) - r^e(\ell, y) = r^e(\ell', x' + y) - r^e(\ell, y)$. □

Our main results concerning groups G subject to Condition (C) are Theorems 3.4 and 3.5 below. These provide a sufficient and a necessary condition, respectively, for moment separability of $\widehat{G}$. We will see via examples in Section 4 that the sufficient condition provided in the first of these theorems is not necessary and that the necessary condition in the second theorem is not sufficient.

Theorem 3.4. *Suppose that G satisfies Condition* (C) *and, for all $\ell \in \mathfrak{g}^*$, $e = e(\ell)$, we have $r^e_j(\ell, V_e) \neq \mathbb{R}$ for all $j \in \widetilde{e}$. Then $\widehat{G}$ is moment separable.*

Proof. In light of Lemma 3.3, we see that the condition on $r_j^e(\ell,\cdot)$ holds for all ℓ in a layer Ω_e iff it holds for all ℓ in the cross-section $\mathcal{U}_e$. Thus we consider only points in the cross-sections.

Given $e \in \mathcal{E}$, take $f, g \in \mathcal{U}_e$ with $\overline{Conv(\mathcal{O}_g)} = \overline{Conv(\mathcal{O}_f)}$. We will show that $f = g$. Corollary 2.5 yields $\widetilde{e}_1(f) = \widetilde{e}_1(g)$ and $\widetilde{e}_2(f) = \widetilde{e}_2(g)$. Equation 2.5 implies, moreover, that $f^{(1)} = g^{(1)}$ must hold. It remains to show that $f^{(2)} = g^{(2)}$.

We write $\widetilde{e}_1$ and $\widetilde{e}_2$ for the common sets $\widetilde{e}_1(f) = \widetilde{e}_1(g)$ and $\widetilde{e}_2(f) = \widetilde{e}_2(g)$ respectively. We suppose below that $\widetilde{e}_2 \neq \emptyset$ as otherwise we are done. Let

$$\mathcal{C}_f = Conv(Graph(r_f)), \ \ \mathcal{C}_g = Conv(Graph(r_g)) \ \ \subset \ V_e \oplus V_{\widetilde{e}_2}.$$

From above we have that

$$f^{(2)} + \overline{\mathcal{C}}_f = g^{(2)} + \overline{\mathcal{C}}_g.$$

For $j \in \widetilde{e}_2$, let $p_j : V_e \oplus V_{\widetilde{e}_2} \to \mathbb{R}$ be the X_j^*-coordinate map. That is, $p_j(\ell) = \ell(X_j^*)$. As p_j is linear, we have

$$\begin{aligned} p_j(f^{(2)} + \mathcal{C}_f) &= f_j + p_j(\mathcal{C}_f) \\ &= f_j + Conv(p_j(Graph(r_f))) \\ &= f_j + Conv(r_j^e(f, V_e)) \\ &= f_j + r_j^e(f, V_e). \end{aligned}$$

Since $j \in \widetilde{e}_2$, $r_j^e(f,\cdot) : V_e \to \mathbb{R}$ is a non-constant polynomial, hence $r_j^e(f, V_e)$ is an unbounded interval in $\mathbb{R}$. By assumption we have $r_j^e(f, V_e) \neq \mathbb{R}$, so $r_j^e(f, V_e)$ is a proper unbounded interval, $(-\infty, a)$, $(-\infty, a]$, (a, ∞), or $[a, \infty)$ for some $a \in \mathbb{R}$.

We also have

$$f_j + r_j^e(f, V_e) \subset p_j(f^{(2)} + \overline{\mathcal{C}}_f) \subset f_j + \overline{p_j(\mathcal{C}_f)} \subset f_j + \overline{r_j^e(f, V_e)}$$

where $r_j^e(f, V_e)$ is a proper unbounded interval in $\mathbb{R}$, and a similar string of inclusions for g. Since $f^{(2)} + \overline{\mathcal{C}}_f = g^{(2)} + \overline{\mathcal{C}}_g$ we conclude that

$$f_j + \overline{r_j^e(f, V_e)} = g_j + \overline{r_j^e(g, V_e)} \tag{3.1}$$

for all $j \in \widetilde{e}_2$.

Let $j \in \widetilde{e}_2$ and assume inductively that $f_i = g_i$ for all $i \in \widetilde{e}_2$ with $i < j$. (In particular, this is automatic when $j = \min(\widetilde{e}_2)$.) We have that $f|_{\mathfrak{g}_{j-1}} = g|_{\mathfrak{g}_{j-1}}$. Indeed, for $i < j$ we have either

- $i \in e$, in which case $f_i = 0 = g_i$ since $f, g \in \mathcal{U}_e$,
- $i \in \widetilde{e}_1$, in which case $f_i = g_i$ since $f^{(1)} = g^{(1)}$, or
- $i \in \widetilde{e}_2$, in which case $f_i = g_i$ by the inductive hypothesis.

As $r_j(f,\cdot)$ and $r_j(g,\cdot)$ depend only on $f|_{\mathfrak{g}_{j-1}}$ and $g|_{\mathfrak{g}_{j-1}}$ respectively, we conclude that $r_j^e(f, V_e) = r_j^e(g, V_e)$. Letting $A = \overline{r_j^e(f, V_e)} = \overline{r_j^e(g, V_e)}$, Equation 3.1 becomes

$$(f_j - g_j) + A = A.$$

As A is of the form $(-\infty, a]$ or $[a, \infty)$ we now conclude that $f_j = g_j$ as desired. $\square$

Theorem 3.5. *Suppose that G satisfies Condition* (C) *and that $\widehat{G}$ is moment separable. Then for all $\ell \in \mathfrak{g}^*$, either $\widetilde{e}_2(\ell) = \emptyset$ or $Conv(Graph(r_\ell)) \neq V_{e(\ell)} \oplus V_{\widetilde{e}_2(\ell)}$.*

Note that when $\widetilde{e}_2(\ell) = \emptyset$ we have $\mathcal{O}_\ell = \ell + V_e$. These are *flat orbits* and the associated representations are square integrable modulo their kernels [MW73]. In such cases we have, in particular, that $\mathcal{O}_\ell$ is convex.

When $\widetilde{e}_2(\ell) \neq \emptyset$ then $Conv(Graph(r_\ell)) \neq V_{e(\ell)} \oplus V_{\widetilde{e}_2(\ell)}$ if and only if $Graph(r_\ell)$ lies to one side of a hyperplane in $V_{e(\ell)} \oplus V_{\widetilde{e}_2(\ell)}$. Thus the condition in Theorem 3.5 can also be stated as:

- For all $\ell \in \mathfrak{g}^*$ either $\widetilde{e}_2(\ell) = \emptyset$ or there exist real numbers $\{a_j \ : \ j \in e(\ell)\}$, $\{b_j \ : \ j \in \widetilde{e}_2(\ell)\}$, $c \in \mathbb{R}$ with
$$\sum_{j \in e(\ell)} a_j x_j + \sum_{j \in \widetilde{e}_2(\ell)} b_j r_j^{e(\ell)}(\ell, x) > c$$
for all $x = \sum_{j \in e(\ell)} x_j X_j^* \in V_{e(\ell)}$.

Equivalently:

- For all $\ell \in \mathfrak{g}^*$ either $\widetilde{e}_2(\ell) = \emptyset$ or there exist elements $A \in Span\{X_j \ : \ j \in e(\ell)\}$, $B \in Span\{X_j \ : \ j \in \widetilde{e}_2(\ell)\}$, $c \in \mathbb{R}$ with
$$x(A) + (r_\ell(x))(B) > c$$
for all $x \in V_{e(\ell)}$.

Proof of Theorem 3.5. As in the proof of Theorem 3.4, by Lemma 3.3 we only need to consider points in the cross-sections $\mathcal{U}_e$. Suppose that G satisfies Condition (C) and $\widetilde{e}_2(f) \neq \emptyset$ for some $f \in \mathcal{U}_e$, but that $Conv(Graph(r_f)) = V_{e(f)} \oplus V_{\widetilde{e}_2(f)}$. Let
$$g = f + \varepsilon X_k^*$$
where $k = \max(\widetilde{e}_2(f))$ and $\varepsilon > 0$ is chosen small enough so that $P_e(g) \neq 0$. The latter is possible since $P_e(f) \neq 0$ and P_e is continuous. We will show that $g \in \mathcal{U}_e$ and that $Conv(\mathcal{O}_g) = Conv(\mathcal{O}_f)$. As $g \neq f$ and $\mathcal{U}_e$ is a cross section for Ω_e, this shows that that $\widehat{G}$ is not moment separable.

From Equation 2.4 we obtain
$$Conv(\mathcal{O}_f) = f^{(1)} + \Big(f^{(2)} + Conv(r_f)\Big) = f^{(1)} + \Big(V_e \oplus V_{\widetilde{e}_2(f)}\Big).$$
As $g = f^{(1)} + (f^{(2)} + \varepsilon X_k^*) \in f^{(1)} + \Big(V_e \oplus V_{\widetilde{e}_2(f)}\Big)$, we see that $g \in Conv(\mathcal{O}_f)$. Condition (C) now implies that $g \in \bigcup_{e' \succeq e} \Omega_{e'}$ and hence $P_{e'}(g) = 0$ for all $e' \in \mathcal{E}$ with $e' \prec e$. Since $P_e(g) \neq 0$ we now conclude that $g \in \Omega_e = \{\ell \in \mathfrak{g}^* \ : \ P_e(\ell) \neq 0 \text{ and } P_{e'}(\ell) = 0 \text{ for } e' \prec e\}$. Moreover, $g = f + \varepsilon X_k^* \in V_{\widetilde{e}}$ so $g \in \mathcal{U}_e = \Omega_e \cap V_{\widetilde{e}}$.

We claim that $r^e(g, \cdot) = r^e(f, \cdot)$, or equivalently,
$$r_j^e(g, \cdot) = r_j^e(f, \cdot) \ \text{ for all } j \in \widetilde{e}.$$
For this we consider two cases:

1. If $j \le k$ then $g|_{\mathfrak{g}_{j-1}} = f|_{\mathfrak{g}_{j-1}}$ and hence $r_j^e(g,\cdot) = r_j^e(f,\cdot)$.
2. If $j > k$ then $j \in \widetilde{e}_1(f)$ since $k = \max(\widetilde{e}_2(f))$. But as noted above, $g \in Conv(\mathcal{O}_f) = f^{(1)} + \left(V_e \oplus V_{\widetilde{e}_2(f)}\right)$ and as $Conv(\mathcal{O}_f)$ is an $Ad^*(G)$-invariant set we have
$$\mathcal{O}_g \subset f^{(1)} + \left(V_e \oplus V_{\widetilde{e}_2(f)}\right).$$
Hence the polynomial $g_j + r_j^e(g,\cdot) = f_j + r_j^e(g,\cdot)$ which gives the orbit $\mathcal{O}_g$ in the X_j^*-direction must be constant ($g_j = f_j$). That is, $r_j^e(g,\cdot) = 0 = r_j^e(f,\cdot)$ in this case.

It now follows that $\widetilde{e}_1(g) = \widetilde{e}_1(f)$, $\widetilde{e}_2(g) = \widetilde{e}_2(f)$ and $r_g = r_f$. Finally we compute
$$\begin{aligned}
Conv(\mathcal{O}_g) &= g^{(1)} + \left(g^{(2)} + Conv(Graph(r_g))\right)\\
&= f^{(1)} + \left(g^{(2)} + Conv(Graph(r_f))\right)\\
&= f^{(1)} + \left(g^{(2)} + V_e \oplus V_{\widetilde{e}_2(f)}\right)\\
&= f^{(1)} + \left(V_e \oplus V_{\widetilde{e}_2(f)}\right)\\
&= Conv(\mathcal{O}_f).
\end{aligned}$$
□

Examples in Section 4 show that the converses for Theorems 3.4 and 3.5 do not hold in general. The necessary condition from Theorem 3.5 is, however, sufficient in the special case where each coadjoint orbit has at most one non-constant direction:

Proposition 3.6. *Suppose that G satisfies Condition* (C) *and that $\#(\widetilde{e}_2(\ell)) \le 1$ for all $\ell \in \mathfrak{g}^*$. Then $\widehat{G}$ is moment separable if and only if $Conv(Graph(r_\ell)) \ne V_{e(\ell)} \oplus V_{\widetilde{e}_2(\ell)}$ for all $\ell \in \mathfrak{g}^*$ with $\widetilde{e}_2(\ell) \ne \emptyset$.*

Proof. In view of Theorem 3.5, we need only prove that $\mathcal{O}_f = \mathcal{O}_g$ follows from $\overline{Conv(\mathcal{O}_f)} = \overline{Conv(\mathcal{O}_g)}$. By Proposition 3.2, we can assume that $f, g \in \mathcal{U}_e$ for some $e \in \mathcal{E}$. As in the proof of Theorem 3.4, we obtain $\widetilde{e}_1(f) = \widetilde{e}_1(g)$ and $f^{(1)} = g^{(1)}$. We suppose that $\widetilde{e}_2(f) = \widetilde{e}_2(g) = \{j\}$, and wish to show that $f_j = g_j$. As in the proof of Theorem 3.4 we have
$$f_j X_j^* + \overline{Conv(Graph(r_j^e(f,\cdot)X_j^*))} = g_j X_j^* + \overline{Conv(Graph(r_j^e(g,\cdot)X_j^*))}.$$
Moreover, as $f^{(1)} = g^{(1)}$ and $f, g \in \mathcal{U}_e$ we have $f|\mathfrak{g}_{j-1} = g|\mathfrak{g}_{j-1}$ and hence $r_j^e(f,\cdot) = r_j^e(g,\cdot)$. Thus
$$Conv(Graph(r_j^e(f,\cdot)X_j^*)) = Conv(Graph(r_j^e(g,\cdot)X_j^*)) = \mathcal{C}$$
say. We have now
$$(f_j - g_j)X_j^* + \overline{\mathcal{C}} = \overline{\mathcal{C}}$$
and, by iteration, $n(f_j - g_j)X_j^* + \overline{\mathcal{C}} = \overline{\mathcal{C}}$ for all $n \in \mathbb{Z}$.

Now suppose that $f_j \neq g_j$. Then, by convexity, we have $\mathbb{R}X_j^* + \overline{\mathcal{C}} = \overline{\mathcal{C}}$. Since $\mathcal{C}$ is the graph of a function from V_e to $\mathbb{R}X_j^*$, we conclude that $\overline{\mathcal{C}} = V_e \oplus \mathbb{R}X_j^*$. This contradicts the hypothesis that $\mathcal{C} \neq V_e \oplus \mathbb{R}X_j^*$ and hence $f_j = g_j$. □

4. Examples

As remarked in [Wil89], $\widehat{G}$ is moment separable for all connected and simply connected nilpotent groups G with $\dim(G) \leq 5$. Examples 4.1 and 4.2 below illustrate the application of Theorems 3.4 and 3.5 to groups of dimension 6. When $\dim(G) = 6$, as is noted in [Wil89], $\widehat{G}$ may or may not be moment separable.

Examples 4.3, 4.4 and 4.5 demonstrate some limitations to the results obtained in this paper. Example 4.3 shows that $\widehat{G}$ may be moment separable in cases where Condition (C) holds but G does not satisfy the remaining hypothesis in Theorem 3.4. Example 4.4 shows that G may satisfy Condition (C) and the conclusion of Theorem 3.5 but $\widehat{G}$ may fail to be moment separable. Example 4.5 shows that not all nilpotent groups satisfy Condition (C).

Example 4.6 suggests that some of our results should carry over to the setting of exponential solvable Lie groups.

Example 4.1. Let G be the 6-dimensional group with Lie algebra $\mathfrak{g}$ having strong Malcev basis $X_1, \ldots, X_6$ where

$$[X_6, X_5] = X_4, \quad [X_6, X_4] = X_1, \quad [X_3, X_2] = X_1,$$

and other brackets of basis elements vanish. This is the group denoted $G_{6,1}$ in [Nie83]. G is a 3-step group with center $\mathbb{R}X_1$. Condition (C) holds in view of Proposition 3.1. In fact, the jump sets and layers are, from [Nie83]:

$$\mathcal{E} = \Big\{ \{2,3,4,6\} \prec \{5,6\} \prec \emptyset \Big\}, \quad \text{where}$$

$$\Omega_{\{2,3,4,6\}} = \{\ell \ : \ \ell_1 \neq 0\}, \quad \Omega_{\{5,6\}} = \{\ell \ : \ \ell_1 = 0, \ell_4 \neq 0\}, \quad \Omega_\emptyset = \{\ell \ : \ \ell_1 = \ell_4 = 0\}.$$

Here we write elements $\ell \in \mathfrak{g}^*$ as $\ell = \ell_1 X_1^* + \cdots + \ell_6 X_6^*$. For $e = \{2,3,4,6\}$ and $\ell \in \mathcal{U}_e = \{\ell_1 X_1^* + \ell_5 X_5^* \ : \ \ell_1 \neq 0\}$, one has $\widetilde{e}_1(\ell) = \{1\}$, $\widetilde{e}_2(\ell) = \{5\}$ and $r_5^e(\ell, \cdot) : V_e \to \mathbb{R}$ is

$$r_5^e(\ell, x_2, x_3, x_4, x_6) = \frac{1}{2\ell_1} x_4^2$$

Thus $r_5^e(\ell, V_e) \neq \mathbb{R}$. For $\ell \in \Omega_{\{5,6\}} \cup \Omega_\emptyset$ we have $\widetilde{e}_2(\ell) = \emptyset$. (The orbit $\mathcal{O}_\ell$ is flat of dimension two or is a single point.) Thus Theorem 3.4 shows that $\widehat{G}$ is moment separable.

Example 4.2. Next consider the 6-dimensional group G whose Lie algebra $\mathfrak{g}$ is given by the strong Malcev basis $X_1, \ldots, X_6$ with non-zero brackets

$$[X_6, X_5] = X_3, \quad [X_6, X_4] = X_2, \quad [X_5, X_2] = X_1, \quad [X_4, X_3] = X_1.$$

This is the group $G_{6,4}$ in [Nie83]. As in Example 4.1, $\mathfrak{g}$ is 3-step with one dimensional center $\mathbb{R}X_1$. Condition (C) holds and we have

$$\mathcal{E} = \Big\{\{2,3,4,5\} \prec \{4,6\} \prec \{5,6\} \prec \emptyset\Big\}, \quad \text{where}$$

$$\begin{aligned} \Omega_{\{2,3,4,5\}} &= \{\ell \ : \ \ell_1 \neq 0\}, & \Omega_{\{4,6\}} &= \{\ell \ : \ \ell_1 = 0, \ell_2 \neq 0\}, \\ \Omega_{\{5,6\}} &= \{\ell \ : \ \ell_1 = \ell_2 = 0, \ell_3 \neq 0\}, & \Omega_{\emptyset} &= \{\ell \ : \ \ell_1 = \ell_2 = \ell_3 = 0\}. \end{aligned}$$

For $e = \{2,3,4,5\}$ and $\ell \in \mathcal{U}_e = \{\ell_1 X_1^* + \ell_6 X_6^* \ : \ \ell_1 \neq 0\}$ one has $\widetilde{e}_1(\ell) = \{1\}$, $\widetilde{e}_2(\ell) = \{6\}$ and $r_\ell : V_e \to V_{\widetilde{e}_2(\ell)}$ is the map

$$r_\ell(x_2, x_3, x_4, x_5) = -\frac{1}{\ell_1} x_2 x_3 X_6^*.$$

This gives a saddle in the the space spanned by $\{X_2^*, X_3^*, X_6^*\}$. It follows that $Conv(Graph(r_\ell)) = V_e \oplus V_{\widetilde{e}_2(\ell)}$ for $\ell \in \Omega_{\{2,3,4,5\}}$. Theorem 3.5 now implies that $\widehat{G}$ is not moment separable.

We remark that Examples 4.1 and 4.2 taken together show that the unitary dual of a 3-step group may or may not be moment separable.

Example 4.3. The *n-step ladder group* G_n ($n \geq 2$) has Lie algebra $\mathfrak{g}_n$ of dimension $n+1$ with strong Malcev basis $X_1, \ldots, X_{n+1}$ where

$$[X_{n+1}, X_j] = X_{j-1}$$

for $j = 2, \ldots, n$. In this example one has

$$\mathcal{E} = \{e_1 \prec e_2 \prec \cdots \prec e_n\} \ \text{ where } \ e_j = \{j+1, n+1\}$$

for $j = 1, \ldots, n-1$ and $e_n = \emptyset$. The layers Ω_{e_j} are

$$\Omega_{e_j} = \{\ell \in \mathfrak{g}^* \ : \ \ell_1 = \cdots = \ell_{j-1} = 0, \ell_j \neq 0\}$$

for $j = 1, \ldots, n-1$, and

$$\Omega_{e_n} = \Omega_\emptyset = \{\ell \in \mathfrak{g}^* \ : \ \ell_1 = \cdots = \ell_{n-1} = 0\} = \{\ell_n X_n^* + \ell_{n+1} X_{n+1}^* \ : \ \ell_n, \ell_{n+1} \in \mathbb{R}\}.$$

We see that the connected components of each layer are convex, so Condition (C) holds. Alternatively, this follows from Proposition 3.1 since each coadjoint orbit has dimension at most two. We'll show that $\widehat{G}_n$ is moment separable but that for $n \geq 4$ the condition in Theorem 3.4 does not hold. The case $n = 2$, the three dimensional Heisenberg group, is transparent. So we suppose below that $n \geq 3$.

First note that there is an obvious algebra isomorphism $\mathfrak{g}_n/\mathbb{R}X_1 \cong \mathfrak{g}_{n-1}$. Coadjoint orbits in layers Ω_{e_j} with $j > 1$ are diffeomorphic to coadjoint orbits in $\mathfrak{g}_{n-1}^*$ via the associated linear map $\mathfrak{g}_{n-1}^* \to \mathfrak{g}_n^*$. Thus an inductive argument shows that $\widehat{G}_n$ is moment separable if and only if $\overline{Conv(\mathcal{O})} = \overline{Conv(\mathcal{O}')} \Rightarrow \mathcal{O} = \mathcal{O}'$ for coadjoint orbits $\mathcal{O}, \mathcal{O}' \subset \Omega_{e_1}$.

Writing $e = e_1$ and functionals $\ell = \ell_1 X_1^* + \cdots + \ell_{n+1} X_{n+1}^*$ as $\ell = (\ell_1, \ldots, \ell_{n+1})$, we have

$$\mathcal{U}_e = \{(\ell_1, 0, \ell_3, \ldots, \ell_n, 0) \ : \ \ell_1 \neq 0\}.$$

A brute-force computation shows that for $\ell \in \mathcal{U}_e$, $\ell = (\ell_1, 0, \ell_3, \ldots, \ell_n, 0)$, one has

$$\begin{aligned}\mathcal{O}_\ell = \Big\{ \Big(& \ell_1, x_2, \ell_3 + \frac{1}{2\ell_1} x_2^2, \ell_4 + \frac{\ell_3}{\ell_1} x_2 + \frac{1}{6\ell_1^2} x_2^3, \ell_5 + \frac{\ell_4}{\ell_1} x_2 + \frac{\ell_3}{2\ell_1^2} x_2^2 + \frac{1}{24\ell_1^3} x_2^4, \ldots, \\ & \ell_n + \frac{\ell_{n-1}}{\ell_1} x_2 + \frac{\ell_{n-2}}{2\ell_1^2} x_2^2 + \cdots + \frac{\ell_3}{(n-3)!\ell_1^{n-3}} x_2^{n-3} + \frac{1}{(n-1)!\ell_1^{n-2}} x_2^{n-1}, x_{n+1} \Big) \\ & : \ x_2, x_{n+1} \in \mathbb{R} \Big\}.\end{aligned}$$

The Pukanszky polynomials $r_j^e(\ell, \cdot) : V_e \to \mathbb{R}$ are here given by

$$r_j^e(\ell, x_2, x_{n+1}) = \sum_{i=1}^{j-1} \frac{\ell_{j-i}}{i!\ell_1^i} x_2^i = \sum_{i=1}^{j-3} \frac{\ell_{j-i}}{i!\ell_1^i} x_2^i + \frac{1}{(j-1)!\ell_1^{j-2}} x_2^{j-1}.$$

We have $\widetilde{e}_1(\ell) = \{1\}$ and $\widetilde{e}_2(\ell) = \{3, \ldots, n\}$.

Suppose now that $\ell = (\ell_1, 0, \ell_3, \ldots, \ell_n, 0)$ and $\ell' = (\ell_1', 0, \ell_3', \ldots, \ell_n', 0)$ are two points in $\mathcal{U}_e$ with $\overline{Conv(\mathcal{O}_{\ell'})} = \overline{Conv(\mathcal{O}_\ell)}$. It follows that $\ell_1' = \ell_1$. We can suppose here that $\ell_1 > 0$. All points ℓ'' in $\mathcal{O}_\ell$ have $\ell_3'' \geq \ell_3$. Hence the same is true for all points ℓ'' in $\overline{Conv(\mathcal{O}_\ell)}$. Thus we have $\ell_3' \geq \ell_3$. Interchanging the roles of ℓ and ℓ' we obtain also $\ell_3 \geq \ell_3'$ and hence $\ell_3' = \ell_3$. We note, moreover, that all points $\ell'' \in \overline{Conv(\mathcal{O}_\ell)}$ with $\ell'' \neq \ell$ have $\ell_3'' > \ell_3$. Since $\ell' \in \overline{Conv(\mathcal{O}_\ell)}$ and $\ell_3' = \ell_3$, we now conclude that $\ell' = \ell$ must hold. This shows that $\widehat{G}_n$ is moment separable.

On the other hand, for $n \geq 4$ we see that $r_4^e(\ell, x) = (\ell_3/\ell_1)x_2 + (1/6\ell_1^2)x_2^3$ has $r_4^e(\ell, V_e) = \mathbb{R}$ for all $\ell \in \Omega_e$. Thus the condition in Theorem 3.4 does not hold here.

Example 4.4. Consider the 7-dimensional group G with Lie algebra $\mathfrak{g}$ given by the strong Malcev basis $X_1, \ldots, X_7$ where:

$$\begin{array}{lll} [X_5, X_4] = X_3, & [X_5, X_3] = X_2, & [X_5, X_2] = X_1, \\ [X_6, X_5] = X_3, & [X_6, X_4] = X_2, & [X_6, X_3] = X_1, \\ [X_7, X_5] = X_1, & & \end{array}$$

and other brackets of basis elements are zero. $\mathfrak{g}$ is 4-step with center $\mathbb{R}X_1$. The jump sets for this example are

$$\mathcal{E} = \Big\{ \{2,3,5,6\} \prec \{3,4,5,6\} \prec \{4,5\} \prec \emptyset \Big\}$$

with layers given by:

$$\begin{array}{ll} \Omega_{\{2,3,5,6\}} = \{\ell \in \mathfrak{g}^* \ : \ \ell_1 \neq 0\}, & \Omega_{\{3,4,5,6\}} = \{\ell \in \mathfrak{g}^* \ : \ \ell_1 = 0, \ell_2 \neq 0\}, \\ \Omega_{\{4,5\}} = \{\ell \in \mathfrak{g}^* \ : \ \ell_1 = \ell_2 = 0, \ell_3 \neq 0\}, & \Omega_\emptyset = \{\ell \in \mathfrak{g}^* \ : \ \ell_1 = \ell_2 = \ell_3 = 0\}, \end{array}$$

where $\ell = \ell_1 X_1^* + \cdots + \ell_7 X_7^*$. Condition (C) holds because the connected components of each layer are convex. We will show that the condition in Theorem 3.5 holds but that $\widehat{G}$ is not moment separable.

The cross sections $\mathcal{U}_e$ are given by

$$\mathcal{U}_{\{2,3,5,6\}} = \{\ell_1 X_1^* + \ell_4 X_4^* + \ell_7 X_7^* \ : \ \ell_1 \neq 0\}, \quad \mathcal{U}_{\{3,4,5,6\}} = \{\ell_2 X_2^* + \ell_7 X_7^* \ : \ \ell_2 \neq 0\},$$
$$\mathcal{U}_{\{4,5\}} = \{\ell_3 X_3^* + \ell_6 X_6^* + \ell_7 X_7^* \ : \ \ell_3 \neq 0\}, \qquad \mathcal{U}_{\emptyset} = \{\ell_4 X_4^* + \cdots \ell_7 X_7^*\}.$$

Writing elements $\ell \in \mathfrak{g}^*$ as "$(\ell_1, \ldots, \ell_7)$", the coadjoint orbit through a point ℓ in $\mathcal{U}_{\{2,3,5,6\}}, \mathcal{U}_{\{3,4,5,6\}}, \mathcal{U}_{\{4,5\}}$, or $\mathcal{U}_{\emptyset}$ is:

$$\begin{aligned}
&\mathcal{O}_\ell = \{(\ell_1, x_2, x_3, \ell_4 + x_2 x_3/\ell_1 - x_2^3/(3\ell_1^2), x_5, x_6, \ell_7 - x_2) \ : \ x_2, x_3, x_5, x_6 \in \mathbb{R}\},\\
&\mathcal{O}_\ell = \{(0, \ell_2, x_3, x_4, x_5, x_6, \ell_7) \ : \ x_3, x_4, x_5, x_6 \in \mathbb{R}\},\\
&\mathcal{O}_\ell = \{(0, 0, \ell_3, x_4, x_5, \ell_6 - x_4, \ell_7) \ : \ x_4, x_5 \in \mathbb{R}\}, \text{ or}\\
&\mathcal{O}_\ell = \{\ell\}
\end{aligned}$$

respectively.

One can read off the Pukanszky polynomials from these orbit descriptions. For $e = \{2,3,5,6\}$ and $\ell \in \mathcal{U}_e$, we see that $\widetilde{e}_2(\ell) = \{4,7\}$ and that the map $r_\ell : V_e \to V_{\widetilde{e}_2(\ell)}$ is

$$r_\ell(x_2, x_3, x_5, x_6) = \left(\frac{x_2 x_3}{\ell_1} - \frac{x_2^3}{3\ell_1^2} \right) X_4^* - x_2 X_7^*.$$

Thus $Graph(r_\ell)$ is contained in the linear subspace $\{\ell' \ : \ \ell_2' + \ell_7' = 0\}$ of $V_e \oplus V_{\widetilde{e}_2(\ell)}$. Hence the same is true for $Conv(Graph(r_\ell))$. This shows that $Conv(Graph(r_\ell)) \neq V_e \oplus V_{\widetilde{e}_2(\ell)}$ for all $\ell \in \mathcal{U}_{\{2,3,5,6\}}$. For $e = \{4,5\}$ and $\ell \in \mathcal{U}_e$, we have $\widetilde{e}_2(\ell) = \{6\}$ and $r_\ell : V_e \to V_{\widetilde{e}_2(\ell)}$ is

$$r_\ell(x_4, x_5) = -x_4 X_6^*.$$

We see that $Conv(Graph(r_\ell)) = Graph(r_\ell) \neq V_e \oplus V_{\widetilde{e}_2(\ell)}$ for all $\ell \in \Omega_{\{4,5\}}$. For $e = \{3,4,5,6\}$ or $e = \emptyset$ and $\ell \in \mathcal{U}_e$ we have $\widetilde{e}_2(\ell) = \emptyset$. Hence the condition in the conclusion of Theorem 3.5 holds for this example.

On the other hand, $\widehat{G}$ is *not* moment separable. Indeed, let $f = X_1^*$ and $g = X_1^* + X_4^*$. Since $f, g \in \mathcal{U}_{\{2,3,5,6\}}$ and $f \neq g$, we have that $\mathcal{O}_f \neq \mathcal{O}_g$. In fact, these orbits are:

$$\begin{aligned}
&\mathcal{O}_f = \{(1, x_2, x_3, x_2 x_3 - x_2^3/3, x_5, x_6, -x_2) \ : \ x_2, x_3, x_5, x_6 \in \mathbb{R}\},\\
&\mathcal{O}_g = \{(1, x_2, x_3, 1 + x_2 x_3 - x_2^3/3, x_5, x_6, -x_2) \ : \ x_2, x_3, x_5, x_6 \in \mathbb{R}\}.
\end{aligned}$$

Since $(1, 1, 1, 2/3, 0, 0, -1) \in \mathcal{O}_f$ and $(1, -1, -1, 4/3, 0, 0, 1) \in \mathcal{O}_f$, we have

$$g = (1, 0, 0, 1, 0, 0, 0) = \frac{1}{2}(1, 1, 1, 2/3, 0, 0, -1) + \frac{1}{2}(1, -1, -1, 4/3, 0, 0, 1) \in Conv(\mathcal{O}_f).$$

Thus $Conv(\mathcal{O}_g) \subset Conv(\mathcal{O}_f)$. But we also have $(1,1,-1,-1/3,0,0,-1) \in \mathcal{O}_g$ and $(1,-1,1,1/3,0,0,1) \in \mathcal{O}_g$. So

$$f = (1,0,0,0,0,0,0) = \frac{1}{2}(1,1,-1,-1/3,0,0,-1)+\frac{1}{2}(1,-1,1,1/3,0,0,1) \in Conv(\mathcal{O}_g)$$

and hence $Conv(\mathcal{O}_f) \subset Conv(\mathcal{O}_g)$. As $\mathcal{O}_f \neq \mathcal{O}_g$ but $Conv(\mathcal{O}_f) = Conv(\mathcal{O}_g)$, $\widehat{G}$ is not moment separable.

This example shows that the necessary condition for moment separability furnished by Theorem 3.5 is not sufficient.

Example 4.5. Let G be the group of 5×5 unipotent upper-triangular matrices. This is a 4-step group of dimension 10 with one dimensional center. We'll show that Condition (C) does not hold for this example. The Lie algebra $\mathfrak{g}$ for G is the set of 5×5 upper triangular matrices with 0's on the diagonal. Let $E_{i,j}$ denote the matrix with a 1 in position (i,j) and other entries zero. Then $\{E_{i,j} \ : \ i < j\}$ is a strong Malcev basis for $\mathfrak{g}$ when ordered as:

$$E_{1,5},\ E_{1,4},\ E_{2,5},\ E_{1,3},\ E_{2,4},\ E_{3,5},\ E_{1,2},\ E_{2,3},\ E_{3,4},\ E_{4,5}.$$

The structure equations are

$$[E_{i,j}, E_{k,\ell}] = \delta_{j,k} E_{i,\ell}.$$

There are nine layers Ω_e in $\mathfrak{g}^*$. Rather than writing "$X_1 = E_{1,5}, \ldots, X_{10} = E_{4,5}$" and listing jump sets e as subsets of $\{1, \ldots, 10\}$, we will specify the basis vectors $E_{i,j}$ that give the jump indices in each case. One has

$$\mathcal{E} = \{e_1 \prec e_2 \prec \cdots \prec e_9\}$$

where:

$$\begin{aligned}
e_1 &= \{E_{1,4},\ E_{2,5},\ E_{1,3},\ E_{3,5},\ E_{1,2},\ E_{2,3},\ E_{3,4},\ E_{4,5}\},\\
e_2 &= \{E_{1,4},\ E_{2,5},\ E_{1,3},\ E_{3,5},\ E_{1,2},\ E_{4,5}\},\\
e_3 &= \{E_{1,3},\ E_{2,4},\ E_{3,5},\ E_{1,2},\ E_{2,3},\ E_{3,4}\},\\
e_4 &= \{E_{1,3},\ E_{2,4},\ E_{1,2},\ E_{3,4}\},\\
e_5 &= \{E_{2,4},\ E_{3,5},\ E_{2,3},\ E_{4,5}\},\\
e_6 &= \{E_{1,2},\ E_{2,3},\ E_{3,4},\ E_{4,5}\},\\
e_7 &= \{E_{2,3},\ E_{3,4}\},\\
e_8 &= \{E_{3,4},\ E_{4,5}\},\\
e_9 &= \emptyset.
\end{aligned}$$

Writing elements $\ell \in \mathfrak{g}^*$ as $\ell = \sum_{i<j} \ell_{i,j} E^*_{i,j}$, the corresponding layers Ω_{e_j} are:

$$\begin{aligned}
\Omega_{e_1} &= \{\ell \in \mathfrak{g}^* \ : \ \ell_{1,5} \neq 0,\ \ell_{1,4}\ell_{2,5} - \ell_{2,4}\ell_{1,5} \neq 0\},\\
\Omega_{e_2} &= \{\ell \in \mathfrak{g}^* \ : \ \ell_{1,5} \neq 0,\ \ell_{1,4}\ell_{2,5} - \ell_{2,4}\ell_{1,5} = 0\},\\
\Omega_{e_3} &= \{\ell \in \mathfrak{g}^* \ : \ \ell_{1,5} = 0,\ \ell_{1,4}\ell_{2,5} \neq 0\},\\
\Omega_{e_4} &= \{\ell \in \mathfrak{g}^* \ : \ \ell_{1,5} = \ell_{2,5} = 0,\ \ell_{1,4} \neq 0\},\\
\Omega_{e_5} &= \{\ell \in \mathfrak{g}^* \ : \ \ell_{1,5} = \ell_{1,4} = 0,\ \ell_{2,5} \neq 0\},\\
\Omega_{e_6} &= \{\ell \in \mathfrak{g}^* \ : \ \ell_{1,5} = \ell_{1,4} = \ell_{2,5} = 0,\ \ell_{1,3}\ell_{3,5} \neq 0\},\\
\Omega_{e_7} &= \{\ell \in \mathfrak{g}^* \ : \ \ell_{1,5} = \ell_{1,4} = \ell_{2,5} = \ell_{1,3} = 0,\ \ell_{2,4} \neq 0\},\\
\Omega_{e_8} &= \{\ell \in \mathfrak{g}^* \ : \ \ell_{1,5} = \ell_{1,4} = \ell_{2,5} = \ell_{1,3} = \ell_{2,4} = 0,\ \ell_{3,5} \neq 0\},\\
\Omega_{e_9} &= \{\ell \in \mathfrak{g}^* \ : \ \ell_{1,5} = \ell_{1,4} = \ell_{2,5} = \ell_{1,3} = \ell_{2,4} = \ell_{3,5} = 0\}.
\end{aligned}$$

Orbits in Ω_{e_1} are 8-dimensional with one non-constant direction ($E^*_{2,4}$). Orbits in Ω_{e_2} are 6-dimensional with three non-constant directions ($E^*_{2,4}, E^*_{2,3}, E^*_{3,4}$). Orbits in Ω_{e_3} are 6-dimensional with one non-constant direction ($E^*_{4,5}$). Orbits in Ω_{e_4} are 4-dimensional with one non-constant direction ($E^*_{2,3}$). Orbits in Ω_{e_5} are 4-dimensional with one non-constant direction ($E^*_{3,4}$). Orbits in Ω_{e_6} are flat and 4-dimensional. Orbits in Ω_{e_7} are 2-dimensional with one non-constant direction ($E^*_{4,5}$). Orbits in Ω_{e_8} are flat and 2-dimensional. Orbits in $\Omega_{e_9} = \Omega_\emptyset$ are single points.

From above we see that the cross sections for the layers Ω_{e_1} and Ω_{e_2} are:

$$\mathcal{U}_{e_1} = \{aE^*_{1,5} + bE^*_{2,4} \ : \ a \neq 0, b \neq 0\}, \quad \mathcal{U}_{e_2} = \{aE^*_{1,5} + bE^*_{2,3} + cE^*_{3,4} \ : \ a \neq 0\}.$$

One computes that the coadjoint orbit through a point $f = aE^*_{1,5} + bE^*_{2,3} + cE^*_{3,4} \in \mathcal{U}_{e_2}$ ($a \neq 0$) is

$$\mathcal{O}_f = \left\{ \begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} & a \\ 0 & 0 & b + \frac{x_{25}x_{13}}{a} & \frac{x_{14}x_{25}}{a} & x_{25} \\ 0 & 0 & 0 & c + \frac{x_{14}x_{35}}{a} & x_{35} \\ 0 & 0 & 0 & 0 & x_{45} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \ : \ x_{14}, x_{25}, x_{13}, x_{35}, x_{12}, x_{45} \in \mathbb{R} \right\}.$$

Note that both $aE^*_{1,5} + 2E^*_{1,4} + bE^*_{2,3} + cE^*_{3,4}$ and $aE^*_{1,5} + 2E^*_{2,5} + bE^*_{2,3} + cE^*_{3,4}$ belong to $\mathcal{O}_f$. Hence

$$\begin{aligned}
g &= aE^*_{1,5} + E^*_{1,4} + E^*_{2,5} + bE^*_{2,3} + cE^*_{3,4}\\
&= \frac{1}{2}(aE^*_{1,5} + 2E^*_{1,4} + bE^*_{2,3} + cE^*_{3,4}) + \frac{1}{2}(aE^*_{1,5} + 2E^*_{2,5} + bE^*_{2,3} + cE^*_{3,4})\\
&\in Conv(\mathcal{O}_f).
\end{aligned}$$

Since $g \in \Omega_{e_1}$ we see that

$$Conv(\mathcal{O}_f) \not\subset \left[\bigcup_{e' \succeq e(f)} \Omega_{e'} = \bigcup_{j=2}^{9} \Omega_{e_j} \right].$$

Thus, Condition (C) fails for this example.

We note, moreover, that $\widehat{G}$ fails to be moment separable in a rather spectacular fashion. The coadjoint orbit through a point $g = aE^*_{1,5} + bE^*_{2,4} \in \mathcal{U}_{e_1}$ $(a \neq 0, b \neq 0)$ is

$$\mathcal{O}_g = \left\{ \begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} & a \\ 0 & 0 & x_{23} & b + \frac{x_{14}x_{25}}{a} & x_{25} \\ 0 & 0 & 0 & x_{34} & x_{35} \\ 0 & 0 & 0 & 0 & x_{45} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \ : \ x_{14}, x_{25}, x_{13}, x_{35}, x_{12}, x_{23}, x_{34}, x_{45} \in \mathbb{R} \right\}.$$

From this we see that $Conv(\mathcal{O}_g) = \{\ell \in \mathfrak{g}^* \ : \ \ell_{1,5} = a\}$ for all such g. (The key observation here is that $\mathcal{O}_g$ forms a saddle in the subspace spanned by $\{E^*_{1,4}, E^*_{2,5}, E^*_{2,4}\}$.) But we saw above that for $f = aE^*_{1,5} + bE^*_{2,3} + cE^*_{3,4} \in \mathcal{U}_{e_2}$, $Conv(\mathcal{O}_f)$ meets Ω_{e_1}. Letting $g \in Conv(\mathcal{O}_f) \cap \Omega_{e_1}$ we have

$$\{\ell \in \mathfrak{g}^* \ : \ \ell_{1,5} = a\} = Conv(\mathcal{O}_g) \subset Conv(\mathcal{O}_f) \subset \{\ell \in \mathfrak{g}^* \ : \ \ell_{1,5} = a\}.$$

This shows that for any point $\ell \in \Omega_{e_1} \cup \Omega_{e_2}$,

$$Conv(\mathcal{O}_\ell) = \{\ell' \in \mathfrak{g}^* \ : \ \ell'_{1,5} = \ell_{1,5}\}.$$

We remark that one can also find pairs of coadjoint orbits in each of the layers Ω_{e_3}, Ω_{e_4} and Ω_{e_5} whose convex hulls coincide.

Example 4.6. The aim here is to examine our results from the nilpotent situation in an exponential solvable case. Another aim is the following: In Remark 3.1 we mention that in [Wil89], it is shown that any pair of two dimensional coadjoint orbits $\mathcal{O}_1$, $\mathcal{O}_2$ is "convex distinguishable". We are going to show that this does not hold in the more general context of exponential groups.

Let G be the 4-dimensional group with the Lie algebra $\mathfrak{g}$ having strong Malcev basis $\{Z, X, Y, A\}$ where

$$[A, X] = -X, \quad [A, Y] = Y, \quad [X, Y] = Z,$$

and other brackets of basis elements vanish. This group is completely solvable and hence exponential solvable. Let $\{Z^*, X^*, Y^*, A^*\}$ be the basis of $\mathfrak{g}^*$ dual to $\{Z, X, Y, A\}$. Let $\ell_{\lambda,\nu,\alpha,\eta} = \lambda Z^* + \nu X^* + \alpha Y^* + \eta A^*$ and

$$\theta = \exp(aA) \cdot \exp(xX) \cdot \exp(yY) \cdot \exp(zZ), \quad a, x, y, z \in \mathbb{R}.$$

Then a routine calculation shows that:

$$Ad^*(\theta)\ell_{\lambda,\nu,\alpha,\eta} = \lambda Z^* + e^a(\nu + y\lambda)X^* + e^{-a}(\alpha - x\lambda)Y^* + (\eta - x\nu + y\alpha - xy\lambda)A^*.$$

From this it follows that for $\lambda \neq 0$, the orbit $\mathcal{O}_{(\lambda,\nu,\alpha,\eta)}$ through $\ell_{\lambda,\nu,\alpha,\eta}$ is given by:

$$\mathcal{O}_{(\lambda,\nu,\alpha,\eta)} = \left\{ \lambda Z^* + uX^* + vY^* + \left(\frac{uv - \alpha\nu}{\lambda} + \eta \right) A^* \ : \ u, v \in \mathbb{R} \right\}.$$

We see that for $\ell = \lambda Z^* + \eta A^*$ with $\lambda \neq 0$ one has $e(\ell) = \{2, 3\}$ and $\widetilde{e}_2(\ell) = \{4\}$. The map $r_\ell : V_{e(\ell)} \to V_{\widetilde{e}_2(\ell)}$ is here

$$r_\ell(u, v) = \frac{uv}{\lambda} A^*.$$

We see that $Conv(Graph(r_\ell)) = V_{e(\ell)} \oplus V_{\widetilde{e}_2(\ell)}$. In view of Theorem 3.5, we expect that $\widehat{G}$ is not moment separable. This is indeed the case. In fact, the orbits $\mathcal{O}_{(\lambda,0,0,\eta)}$ for fixed $\lambda \neq 0$ are not convex distinguishable.

Consider for example the following pair of coadjoint orbits:

$$\mathcal{O} = \mathcal{O}_{(1,0,0,0)} = \{\ell_{(1,u,v,uv)} \ : \ u, v \in \mathbb{R}\},$$
$$\mathcal{O}' = \mathcal{O}_{(1,0,0,1)} = \{\ell_{(1,u,v,uv+1)} \ : \ u, v \in \mathbb{R}\}.$$

We see that $\mathcal{O}' \neq \mathcal{O}$ since $\ell_{(1,0,0,1)} \notin \mathcal{O}$. As $\ell_{(1,1,1,1)} \in \mathcal{O}$ and $\ell_{(1,-1,-1,1)} \in \mathcal{O}$ satisfy

$$\frac{1}{2}\ell_{(1,1,1,1)} + \frac{1}{2}\ell_{(1,-1,-1,1)} = \ell_{(1,0,0,1)}$$

we have $Conv(\mathcal{O}') \subset Conv(\mathcal{O})$. On the other hand

$$\frac{1}{2}\ell_{(1,1,-1,0)} + \frac{1}{2}\ell_{(1,-1,1,0)} = \ell_{(1,0,0,0)}$$

with $\ell_{(1,1,-1,0)}, \ell_{(1,-1,1,0)} \in \mathcal{O}'$ and thus $Conv(\mathcal{O}) \subset Conv(\mathcal{O}')$. This shows that $Conv(\mathcal{O}) = Conv(\mathcal{O}')$, so that $\mathcal{O}$, $\mathcal{O}'$ are a pair of (two dimensional) coadjoint orbits which are not convex distinguishable.

5. Concluding Remarks and Questions

For a given group G, the conditions in Theorems 3.4 and 3.5 are usually easy to check. This is illustrated by the examples described above in Section 4. It would, however, be desirable to have a single condition which is both necessary and sufficient for moment separability. One might also hope to eliminate the use of Condition (C) as a hypothesis, although this seems difficult provided one sticks to the framework of Pukanszky polynomials. In this regard, we ask whether $\widehat{G}$ necessarily fails to be moment separable whenever Condition (C) fails.

Finally, we remark that the closure in Equation 1.2 plays a mysterious role. In fact, we know of no example where the convex hull $Conv(\mathcal{O})$ of some coadjoint orbit is not itself closed. This is a subtle issue because the convex hull for the graph of a polynomial function $\mathbb{R}^n \to \mathbb{R}^m$ *can* fail to be closed. (The polynomial $p : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $p(x, y) = (x^2, x^2y^2)$ is one example.) In any case, we ask whether convex distinguishability implies moment separability. That is, if $Conv(\mathcal{O}) = Conv(\mathcal{O}') \Rightarrow \mathcal{O} = \mathcal{O}'$ for all coadjoint orbits $\mathcal{O}, \mathcal{O}' \subset \mathfrak{g}^*$ then is $\widehat{G}$ moment separable?

References

[AL92] D. Arnal and J. Ludwig. La convexité de l'application moment d'un groupe de Lie. *J. Funct. Anal.*, 105:256–300, 1992.

[BLS98] A. Baklouti, J. Ludwig, and M. Selmi. Séparation des représentations unitaires irréductibles des groupes de Lie nilpotents. In H-D Doebner, J. Hilgert, and V. Dobrev, editors, *Lie Theory and its Applications in Physics II.* 1998.

[CG90] L. Corwin and F. Greenleaf. *Representations of nilpotent Lie groups and their applications, Part 1: Basic theory and examples*, volume 18 of *Studies in advanced math.* Cambridge Univ. Press, Cambridge, 1990.

[Kir62] A. A. Kirillov. Unitary representations of nilpotent Lie groups. *Uspeki Mat. Nauk*, 17:57–110, 1962. Russian Math. Surveys 17 (1962), 57-110.

[Lud78] J. Ludwig. Good ideals in the group algebra of a nilpotent Lie group. *Math. Z.*, 161:195–210, 1978.

[MW73] C. Moore and J. Wolf. Square integrable representations of nilpotent groups. *Trans. Amer. Math. Soc.*, 185:445–462, 1973.

[Nie83] O. Nielsen. *Unitary Representations and Coadjoint Orbits of Low-Dimensional Nilpotent Lie Groups.* Queen's Papers in Pure and Applied Mathematics, Queen's University, Kingston, Ont. Canada, 1983.

[Ped84] N. Pedersen. On the infinitesimal kernel of irreducible representations of nilpotent Lie groups. *Bull. Soc. Math. France*, 112:423–467, 1984.

[Puk67] L. Pukanszky. *Leçons sur les représentations des groupes.* Dunod, Paris, 1967.

[Wil89] N. Wildberger. Convexity and unitary representations of nilpotent Lie groups. *Invent. Math.*, 98:281–292, 1989.

[Wil92] N. Wildberger. The moment map for a Lie group representation. *Trans. Amer. Math. Soc.*, 330:257–268, 1992.

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