ON MULTIPLICITY FREE ACTIONS

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1. Preliminaries

Much of the literature on multiplicity free actions is set in the framework of algebraic groups. We begin by summarizing the basic definitions and results we require concerning such groups and their representations.

1.1. **Algebraic groups.** The general linear group $GL(n,\mathbb{C})$ can be viewed as an algebraic group. Letting $gl(n,\mathbb{C})$ denote the space of $n \times n$ complex matrices, the group $GL(n,\mathbb{C})$ can be identified with the zero set for the polynomial function $p(A,w) = det(A)w - 1$ on $gl(n,\mathbb{C}) \times \mathbb{C}$. This determines the structure of $GL(n,\mathbb{C})$ as an affine variety. One calls G a *reductive complex (linear) algebraic group* when

- (*linear*) G is an algebraic subgroup of $GL(n,\mathbb{C})$, and
- (*reductive*) $\mathbb{C}^n$ is a direct sum of G-irreducible subspaces.

The classical examples are

$$GL(n.\mathbb{C}), SL(n,\mathbb{C}), O(n,\mathbb{C}), SO(n,\mathbb{C}), Sp(2n,\mathbb{C})$$

and direct products of these groups. The torus $(\mathbb{C}^\times)^n$ is a direct product of copies of $GL(1,\mathbb{C}) = \mathbb{C}^\times$. A reductive complex algebraic group is connected (in the Zariski topology) if and only if it is irreducible as an algebraic variety. The classical examples are all connected except for $O(n,\mathbb{C})$, which has two components. *We will assume that our algebraic groups G are connected unless noted otherwise.*

1.2. **Regular functions.** $\mathbb{C}[G]$ denotes the ring of *regular functions* on G. This is the coordinate ring of G as an affine variety. More concretely

$$\mathbb{C}[G] \text{ is the algebra generated by the matrix entries of } G \text{ and } det^{-1}.$$

A function $f : G \to \mathbb{C}$ is regular if and only if f is the restriction of a regular function on $GL(n, \mathbb{C})$. So

$$\mathbb{C}[G] \cong \mathbb{C}[GL(n, \mathbb{C})]/I(G), \text{ where } I(G) = \{f \in \mathbb{C}[GL(n, \mathbb{C})] \ : \ f(G) = 0\}.$$

Examples 1.2.1. For $g = [a_{ij}] \in GL(n, \mathbb{C})$ let $z_{ij}(g) = a_{ij}$. Then

- $\mathbb{C}[GL(n, \mathbb{C})] = \mathbb{C}[z_{ij}, det^{-1}]$,
- $\mathbb{C}[SL(n, \mathbb{C})] = \mathbb{C}[z_{ij}]$,
- $\mathbb{C}[(\mathbb{C}^\times)^n] = \mathbb{C}[z_{11}, z_{11}^{-1}, \ldots, z_{nn}, z_{nn}^{-1}]$.

1.3. **Algebraic groups as Lie groups.** As an algebraic group, G carries the Zariski topology. As a set of $n \times n$ complex matrices, G also has a subspace topology from $gl(n, \mathbb{C})$. In fact, G is a smooth submanifold of $gl(n, \mathbb{C})$, viewed as a real vector space of dimension $(2n)^2$. In this way G is seen as a (real) Lie group with Lie algebra

$$\mathfrak{g} = \{A \in gl(n, \mathbb{C}) \ : \ e^{tA} \in G \text{ for all } t \in \mathbb{R}\}.$$

Moreover $\mathfrak{g}$ is closed under multiplication by i and hence is a *complex* Lie-subalgebra of $gl(n, \mathbb{C})$. Alternatively one can define the (complex) Lie algebra $\mathfrak{g}$ for G as

$$\mathfrak{g} = \{A \in gl(n, \mathbb{C}) \ : \ f \in I(G) \implies Af \in I(G)\}$$

where

$$Af(g) = \left.\frac{d}{dt}\right|_{t=0} f(g(I + tA)).$$

When G is reductive, $\mathfrak{g}$ is a complex reductive Lie algebra. This implies that

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}'$$

where $\mathfrak{z}(\mathfrak{g})$ denotes the center of $\mathfrak{g}$ and the derived subalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is semi-simple, a direct sum of simple ideals. $\mathfrak{g}'$ is the Lie algebra of the commutator subgroup $G' = (G, G)$.

1.4. **Structure theory.** A maximal connected solvable algebraic subgroup B of G is called a *Borel subgroup*. The following facts concerning such subgroups are well known:

- Any two Borel subgroups are conjugate in G.
- Given any Borel subgroup B, there is an *opposite* Borel subgroup B^- with the property that B^-B is Zariski dense in G and contains an open neighborhood of I.
- B is the semidirect product $B = HN$ of its commutator subgroup $N = (B, B)$ with a maximal torus H in G. The group N is a maximal unipotent subgroup of G.

The Lie algebra $\mathfrak{h}$ of H is a maximal abelian subalgebra of $\mathfrak{g}$. For $\alpha \in \mathfrak{h}^*$ let

$$\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} \ : \ [X, Y] = \alpha(X)Y \text{ for all } X \in \mathfrak{h}\}.$$

Then

$$\Delta = \Delta(\mathfrak{g}, \mathfrak{h}) = \{\alpha \in \mathfrak{h}^* \ : \ \mathfrak{g}_\alpha \neq 0\}$$

is the set of *roots* for $\mathfrak{g}$ (relative to $\mathfrak{h}$). Each root space $\mathfrak{g}_\alpha$ is one dimensional and

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

There is a subset Δ^+ of Δ, called the *positive roots*, such that N has Lie algebra

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha.$$

Now

- $\Delta = \Delta^+ \cup (-\Delta^+)$,
- $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is the Lie algebra of B, and
- N^- has Lie algebra $\mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$.

For each $\alpha \in \Delta^+$ there are elements $X_\alpha \in \mathfrak{g}_\alpha$, $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$, $H_\alpha \in \mathfrak{h}$ which form an $sl(2)$-triple:

$$[H_\alpha, X_\alpha] = 2X_\alpha, \quad [H_\alpha, X_{-\alpha}] = -2X_{-\alpha}, \quad [X_\alpha, X_{-\alpha}] = H_\alpha.$$

For each $\alpha \in \Delta$ we have a *root reflection*

$$s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*, \quad s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha \rangle \alpha$$

where

$$\langle \lambda, \alpha \rangle = \lambda(H_\alpha).$$

The *Weyl group* $W = W(\mathfrak{g}, \mathfrak{h})$ is the subgroup of $GL(\mathfrak{h}^*)$ generated by $\{s_\alpha \ : \ \alpha \in \Delta\}$. It is a finite reflection group that acts by permutations on the set Δ.

Example 1.4.1. The *standard Borel subgroup* in $G = GL(n, \mathbb{C})$ is

$$B_n = \left\{ \begin{bmatrix} z_1 & & \\ \mathbf{O} & \ddots & \star \\ & & z_n \end{bmatrix} \ : \ z_j \in \mathbb{C}^\times \right\},$$

the group of invertible upper triangular matrices. We have

$$B_n = H_n N_n$$

where H_n denotes the diagonal matrices in $GL(n, \mathbb{C})$ and N_n denotes the unipotent upper triangular matrices. The opposite Borel subgroup for B_n is

$$B_n^- = H_n N_n^-$$

where B_n^- and N_n^- are the invertible and unipotent lower triangular matrices respectively.

The Lie algebra $\mathfrak{h}_n$ of H_n is the set of all diagonal matrices. Letting $\varepsilon_i \in \mathfrak{h}^*$ denote the functional

$$\varepsilon_i(diag(z_1, \ldots, z_n)) = z_i$$

one has

$$\text{roots } \Delta = \{\varepsilon_i - \varepsilon_j \ : \ i \neq j\} \text{ and positive roots } \Delta^+ = \{\varepsilon_i - \varepsilon_j \ : \ i < j\}.$$

For $\alpha = \varepsilon_i - \varepsilon_j \in \Delta^+$ we have

$$X_\alpha = E_{ij}, \quad X_{-\alpha} = E_{ji}, \quad H_\alpha = E_{ii} - E_{jj}.$$

The root reflection s_α satisfies $s_\alpha(\varepsilon_k) = \varepsilon_{\tau(k)}$ where $\tau \in S_n$ is the transposition that interchanges i with j. Thus the Weyl group is isomorphic to S_n.

1.5. **Rational representations.**

Definition 1.5.1. Let (σ, V) be a representation of G.

(1) (σ, V) is said to be *rational* (or regular) if it is finite dimensional and its matrix coefficients

$$g \mapsto \xi(\sigma(g)v), \quad \xi \in V^*, v \in V$$

all belong to $\mathbb{C}[G]$.

(2) (σ, V) is *locally rational* (or locally regular) if $dim(V) = \infty$ and for any finite dimensional subspace F of V there is a $\sigma(G)$-invariant subspace W with $F \subset W \subset V$ for which $\sigma|W$ is rational.

We let $\widehat{G}$ denote the set of equivalence classes of irreducible rational representations of G and sometimes write V_σ for the representation space of $\sigma \in \widehat{G}$.

Note that subrepresentations of rational (or locally rational) representations are rational (resp. locally rational).

Example 1.5.2. As $G = \mathbb{C}^\times$ is abelian, its irreducible representations are given by characters $\rho : \mathbb{C}^\times \to \mathbb{C}^\times$. For such a character to be rational we require $\rho \in \mathbb{C}[G] = \mathbb{C}[z, 1/z]$. So ρ is holomorphic on $\mathbb{C}^\times$ and hence determined by its restriction to the unit circle $\mathbb{T}$. One concludes that

$$(\mathbb{C}^\times)\widehat{\ } = \{\rho_n \ : \ n \in \mathbb{Z}\}$$

where $\rho_n(z) = z^n$. The character

$$\rho(z) = \frac{z}{|z|}$$

gives a representation of $\mathbb{C}^\times$ which is not rational.

This example illustrates the:

Weyl Unitarian Trick: Rational representations of G are determined by holomorphic extension from a maximal compact connected subgroup K of G (viewed as a real Lie group).

It now follows from the representation theory for compact Lie groups that rational representations are completely reducible. Moreover, the Unitarian Trick establishes a bijection between $\widehat{G}$ and the set $\widehat{K}$ of unitary equivalence classes of irreducible unitary representations of K. So one can work entirely in the compact group setting, should one so prefer. Maximal compact subgroups for the classical groups $GL(n.\mathbb{C})$, $SL(n,\mathbb{C})$, $O(n,\mathbb{C})$, $SO(n,\mathbb{C})$, $Sp(2n,\mathbb{C})$ are

$$U(n), SU(n), O(n,\mathbb{R}), SO(n,\mathbb{R}), Sp(2n) = U(2n) \cap Sp(2n,\mathbb{C}).$$

Each rational representation (σ, V) of G is smooth. That is, $t \mapsto \sigma(e^{tX})v$ is a smooth map $\mathbb{R} \to V$ for each $v \in V$, $X \in \mathfrak{g}$. Thus we obtain a derived representation

$$\sigma(X)v = \frac{d}{dt}\bigg|_{t=0} \sigma(e^{tX})v$$

of the Lie algebra $\mathfrak{g}$ on V. (Note that we are using the same notation for the representation σ on both the Lie group and the Lie algebra.)

When there is no ambiguity, we will denote the action of G (or $\mathfrak{g}$) on V by

$$\sigma(g)v = g \cdot v \text{ (or } \sigma(X)v = X \cdot v).$$

1.6. **Highest weight theory.** Let $B = HN$ be a Borel subgroup in G and (σ, V) be a rational representation. By the Lie-Kolchin Theorem, there are non-zero $\sigma(B)$-eigenvectors. That is, there are vectors $v \neq 0$ in V such that

$$\sigma(b)v = \psi(b)v \quad \text{for all } b \in B,$$

where $\psi : B \to \mathbb{C}^\times$ is a regular character. As $N = (B, B)$, we have $\psi|_N = 1$ and hence

- $\{v \in V \ : \ v \text{ is a } B\text{-eigenvector}\} = V^N$, the N-fixed vectors in V, and
- ψ is determined by $\psi|_H \in \widehat{H}$.

Highest weight theory asserts that

$$\sigma \text{ is irreducible} \iff dim(V^N) = 1.$$

For each $\sigma \in \widehat{G}$, there is a non-zero B-eigenvector $v_\sigma \in V_\sigma$, unique up to scalar multiples. This is the *highest weight vector* for σ.

The corresponding character $\psi : B \to \mathbb{C}^\times$ can be differentiated to give a functional λ on the Lie algebra $\mathfrak{b}$, with $\lambda(\mathfrak{n}) = 0$. We have $X \cdot v_\sigma = \lambda(X)v_\sigma$ for all $X \in \mathfrak{b}$. As noted above, λ is determined by its value on the Lie algebra $\mathfrak{h}$ of H.

The functional λ in $\mathfrak{h}^*$ is the *highest weight* for σ. We can extend λ to $\mathfrak{b}$ (or $\mathfrak{b}^-$) by taking $\lambda(\mathfrak{n}) = 0$ (resp. $\lambda(\mathfrak{n}^-) = 0$). Thus v_σ is, up to scalars, the unique vector in V with

$$X \cdot v_\sigma = \lambda(X)v_\sigma \text{ for all } X \in \mathfrak{b}.$$

Highest weight theory asserts, moreover, that

$$\sigma \text{ is determined up to equivalence by its highest weight.}$$

Given a representation of G with highest weight λ, we will denote the corresponding representation as V_λ. (Keep in mind that this all depends on the initial choice of a Borel subgroup.) For an element b in the subgroup B, we will denote the corresponding character by $b \mapsto b^\lambda$.

The highest weights for $GL(n,\mathbb{C})$ (with respect to the standard Borel subgroup) are

$$\{diag(h_1,\ldots,h_n) \mapsto d_1h_1 + \cdots + d_nh_n \ : \ d_1,\ldots,d_n \in \mathbb{Z} \text{ with } d_1 \geq d_2 \geq \cdots \geq d_n\}.$$

1.7. **The contragredient representation.** Given a representation of G on V, we define the *contragredient representation* of G on V^* by:

$$g \cdot \xi(v) = \xi(g^{-1} \cdot v) \text{ for } \xi \in V^*,\ v \in V.$$

If V has highest weight vector v with highest weight λ with respect to some Borel subgroup B, then V^* has a highest weight vector v^* with weight $-\lambda$ with respect to the *opposite* Borel subgroup B^-.

1.8. **Decompositions and multiplicities.** Let (ρ, W) be a rational representation of G and $\sigma \in \widehat{G}$. One has a direct sum decomposition

$$W = \bigoplus_{\sigma \in \widehat{G}} W^\sigma$$

of W into σ*-isotypic components*

$$\begin{aligned} W^\sigma &= \sum \{V \ : \ V \text{ is a } \rho(G)\text{-invariant subspace of } W \text{ with } \rho|V \simeq \sigma\} \\ &= \sum \{T(V_\sigma) \ : \ T : V_\sigma \to W \text{ intertwines } \sigma \text{ with } \rho\}. \end{aligned}$$

Then

$$W^\sigma \simeq m(\sigma,\rho)V_\sigma = \underbrace{V_\sigma \oplus \cdots \oplus V_\sigma}_{m(\sigma,\rho)}$$

as G-modules where the *multiplicity* $m(\sigma,\rho)$ of σ in ρ is given by

$$m(\sigma,\rho) = dim(W^\sigma)/dim(V_\sigma) = dim\big(Hom_G(V_\sigma, W)\big).$$

($Hom_G(V_\sigma, W)$ is the space of linear maps $V_\sigma \to W$ intertwining σ with ρ.)

The multiplicity $m(\sigma,\rho)$ can also be characterized using highest weight theory. Let B be a Borel subgroup of G and λ be the highest weight for σ. Then

$$m(\sigma,\rho) = dim\left(W^{B,\lambda}\right)$$

where

$$W^{B,\lambda} = \{w \in W \ : \ b \cdot w = b^\lambda w\}$$

is the space of weight vectors in W with weight λ.

1.9. **Group actions.** We use the notation

$$G : X$$

to indicate that there is a rational action of G on a variety X,

$$G \times X \to X, \quad (g, x) \mapsto g \cdot x.$$

This means $G \times X \to X$ is a morphism of algebraic varieties satisfying

$$(gh) \cdot x = g \cdot (h \cdot x) \quad \text{and} \quad e \cdot x = x.$$

The variety X may be affine or, more generally, quasi-projective. That is, the intersection of a closed with an open set in $\mathbb{CP}^n$.

It follows that if $f \in \mathbb{C}[X]$, the regular functions on X, then

$$x \mapsto f(g^{-1} \cdot x)$$

is a regular function. Thus we obtain a representation ρ of G on $\mathbb{C}[X]$,

$$(\rho(g)f)(x) = (g \cdot f)(x) = f(g^{-1} \cdot x).$$

Lemma 1.9.1. *The representation ρ of G on $\mathbb{C}[X]$ is locally rational.*

Remark 1.9.2. *Linear* actions $G : X$ will be our main concern. That is, X will usually be a vector space on which G acts by a rational representation. Then $\mathbb{C}[X]$ is the algebra of polynomials on the vector space X and one can give an easy proof of Lemma 1.9.1. Let $\mathcal{P}_k(X)$ denote the space of polynomials on X homogeneous of degree k. For any finite dimensional subspace F of $\mathbb{C}[X]$, we have $F \subset W$ where

$$W = \sum_{k=0}^{n} \mathcal{P}_k(X)$$

for n sufficiently large. Now W is a finite dimensional, G-invariant subspace and we have a rational representation.

Now for $\sigma \in \widehat{G}$ define the σ-isotypic component $\mathbb{C}[X]^\sigma$ as:

$$\mathbb{C}[X]^\sigma = \sum \{T(V_\sigma) \; : \; T \in Hom_G(V_\sigma, \mathbb{C}[X])\}.$$

Lemma 1.9.3. *We have an algebraic direct sum*

$$\mathbb{C}[X] = \bigoplus_{\sigma \in \widehat{G}} \mathbb{C}[X]^\sigma.$$

Thus

$$\mathbb{C}[X] \cong \bigoplus_{\sigma \in \widehat{G}} m(\sigma, \mathbb{C}[X]) V_\sigma$$

as G-modules, where the (possibly infinite) multiplicity $m(\sigma, \mathbb{C}[X])$ of σ in $\mathbb{C}[X]$ is

$$m(\sigma, \mathbb{C}[X]) = dim(Hom_G(V_\sigma, \mathbb{C}[X])) = dim\left(\mathbb{C}[X]^{B,\lambda}\right),$$

where λ is the highest weight for the representation σ.

We can now introduce our principal objects of study.

Definition 1.9.4. $G : X$ is a *multiplicity free action* if $m(\sigma, \mathbb{C}[X]) \leq 1$ for each $\sigma \in \widehat{G}$.

Given an action $G : X$ and an algebraic subgroup H of G, one has

$$H : X \text{ multiplicity free} \implies G : X \text{ multiplicity free.} \tag{1.9.1}$$

Indeed, if $G : X$ fails to be multiplicity free, then some representation $\sigma \in \widehat{G}$ occurs in $\mathbb{C}[X]$ with multiplicity greater than one. Thus all irreducible constituents of $\sigma|_H$ occur with multiplicity greater than one. So $H : X$ also fails to be multiplicity free.

1.10. **Section 1 notes.** The material in this section is standard and there are many excellent references. One is the book [17] by Goodman and Wallach. See Chapter 1 and Appendix D in [17] for further details on foundational material concerning algebraic groups and Lie groups. See Section 11.3 of [17] for proofs of the assertions concerning Borel subgroups. Our treatment of the isotypic decomposition for $\mathbb{C}[X]$ follows Section 12.1 in [17].

2. Multiplicity free actions

This work is mainly devoted to the study of linear multiplicity free actions. In this section, however, we consider actions $G : X$ in the more general context of algebraic varieties. Our main purpose is to describe some noteworthy non-linear examples.

2.1. **Borel orbits.** There is a simple criterion for multiplicity free actions.

Theorem 2.1.1. *If a Borel subgroup B in G has a (Zariski) dense orbit in X then $G : X$ is multiplicity free.*

Proof. Suppose that $B \cdot x_\circ$ is dense in X. Let $\sigma \in \widehat{G}$ occur in $\mathbb{C}[X]$ (that is $m(\sigma, \mathbb{C}[X]) > 0$) and let λ be the highest weight for σ. Let $f_1, f_2 \in \mathbb{C}[X]$ be two B-highest weight vectors with weight λ. One has

$$f_j(b^{-1} \cdot x_\circ) = b^\lambda f_j(x_\circ).$$

As f_j is regular and $B \cdot x_\circ$ dense in X, we see that f_j is completely determined by the value $f_j(x_\circ)$. In particular, $f_j(x_\circ) \neq 0$ (as $f_j \neq 0$) and we can write

$$f_2 = \frac{f_2(x_\circ)}{f_1(x_\circ)} f_1.$$

So the space of λ-highest weight vectors in $\mathbb{C}[X]$ is one dimensional and hence $m(\sigma, \mathbb{C}[X]) = 1$. $\square$

Since any two Borel subgroups are conjugate in G, the criterion in Theorem 2.1.1 does not depend on the choice of Borel subgroup B.

Suppose that $B \cdot x_\circ$ is dense in X. Since B is connected, it follows that X must be an *irreducible* variety and that $B \cdot x_\circ$ is a Zariski open set. Thus $X \backslash (B \cdot x_\circ)$ is a closed set which contains no open subsets in X. We conclude that there is *only one* dense open B-orbit in X.

Conversely, if $G : X$ is a multiplicity free action with X an *irreducible affine* variety then X contains an open (hence dense) B-orbit. We will prove this result for linear multiplicity free actions in the following section. (See Theorem 3.2.8 below.) One special case of the converse admits, however, a direct proof. This is the case where G is an algebraic torus $G = A \cong (\mathbb{C}^\times)^n$. In this case the Borel subgroup is A itself and one has the following.

Proposition 2.1.2. *Let X be an irreducible affine variety and A be a torus. If $A : X$ is multiplicity free then there is an open (hence dense) A-orbit in X.*

Proof. One can choose weight vectors $f_1, \ldots, f_r$ in $\mathbb{C}[X]$ that generate $\mathbb{C}[X]$ as an algebra. Since $A : X$ is multiplicity free the weights $\{\lambda_1, \ldots, \lambda_r\}$ for $f_1, \ldots, f_r$ must be linearly independent. Choose a point $x_\circ \in X$ for which $f_j(x_\circ) \neq 0$ for $1 \leq j \leq r$. Define a map $\rho : \mathbb{C}[X] \to \mathbb{C}[A]$ by

$$(\rho f)(a) = (a \cdot f)(x_\circ) = f(a^{-1} \cdot x_\circ).$$

We claim that ρ is injective. Indeed, for $f = f^m = f_1^{m_1} \cdots f_r^{m_r}$ one has

$$(\rho f)(a) = a^{m_1\lambda_1} \cdots a^{m_r\lambda_r} f(x_\circ) = a^{m\lambda} f(x_\circ),$$

and hence

$$(\rho f)(a) = \sum_m c_m a^{m\lambda} f^m(x_\circ)$$

for $f = \sum_m c_m f^m \in \mathbb{C}[X]$. If $(\rho f)(a) = 0$ for all $a \in A$, then linear independence of the λ_j's implies that $c_m = 0$ for all m.

Thus every regular function on X is determined by its restriction to $A \cdot x_\circ$. It follows that $A \cdot x_\circ$ is open and dense in X. $\square$

2.2. **Quasi-regular representations.** We continue to let G denote a reductive complex linear algebraic group. The left and right actions of G on $X = G$

- $g \cdot x = gx$, and
- $g \cdot x = xg^{-1}$

give rise to the *left and right regular representations*

- $L(g)f(x) = f(g^{-1}x)$, and
- $R(g)f(x) = f(xg)$ respectively

of G on $\mathbb{C}[G]$. For any algebraic subgroup H of G we let

$$\mathbb{C}[G/H] = \mathbb{C}[G]^{R(H)}, \quad \mathbb{C}[H\backslash G] = \mathbb{C}[G]^{L(H)}$$

and define the *left (resp. right) quasi-regular representation* of G as the restriction of L to $\mathbb{C}[G/H]$ (resp. R to $\mathbb{C}[H\backslash G]$). The representations L and R of G on $\mathbb{C}[G/H]$ and $\mathbb{C}[H\backslash G]$ are equivalent via the intertwining operator

$$T : \mathbb{C}[G/H] \to \mathbb{C}[H\backslash G], \quad T(f) = \check{f}, \quad (\check{f}(x) = f(x^{-1})).$$

In fact the homogeneous space G/H is a smooth quasi-projective variety with coordinate ring $\mathbb{C}[G/H]$. If H is a reductive or normal subgroup then G/H is an affine variety. We remark that one can have $\mathbb{C}[G/H] = \mathbb{C}$ even when $dim(G/H) > 0$. This situation occurs whenever G/H is a projective variety, in particular when H is a Borel subgroup of G. In any case, the left action $G : (G/H)$ is rational and gives rise to the left quasi-regular representation. Similar remarks apply for $H\backslash G$ and the right quasi-regular representation.

The isotypic decomposition for the quasi-regular representations is given by *Frobenius Reciprocity*:

Theorem 2.2.1. *As a G-module we have*

$$\mathbb{C}[G/H] \cong \bigoplus_{\sigma\in\widehat{G}} dim(V_{\sigma^*}^H)V_\sigma.$$

In particular, $\mathbb{C}[G] \cong \oplus_\sigma dim(V_\sigma)V_\sigma$.

Proof. Lemma 1.9.3 applies here since $G : (G/H)$ is a rational action. It suffices to show that $Hom_G(V_\sigma, \mathbb{C}[G/H]) \cong V_{\sigma^*}^H$. For this, one verifies that

$$\Phi : V_{\sigma^*}^H \to Hom_G(V_\sigma, \mathbb{C}[G/H]), \quad \Phi(\xi)(v)(g) = \xi(\sigma(g^{-1})v)$$

is an isomorphism with inverse

$$\Lambda : Hom_G(V_\sigma, \mathbb{C}[G/H]) \to V_{\sigma^*}^H, \quad \Lambda(T)(v) = (T(v))(e).$$

□

Corollary 2.2.2. *The action of G on G/H (or on $H\backslash G$) is multiplicity free if and only if $dim(V_\sigma^H) \leq 1$ for all $\sigma \in \widehat{G}$.*

Note that if H_1 and H_2 are algebraic subgroups of G with $H_1 \subset H_2$ then $V_\sigma^{H_2} \subset V_\sigma^{H_1}$. Thus if $G : (G/H_1)$ is a multiplicity free action then so is $G : (G/H_2)$.

The proof of Theorem 2.2.1 shows that the σ-isotypic component in $\mathbb{C}[G]$ for the left regular representation is

$$\begin{aligned}
\mathbb{C}[G]^{\sigma,L} &= \{T(v) \ : \ v \in V_\sigma, T \in Hom_G(V_\sigma, \mathbb{C}[G])\} \\
&= \{\Phi(\xi)(v) \ : \ v \in V_\sigma, \xi \in V_\sigma^*\} \\
&= \{\check{m_{\xi,v}} \ : \ v \in V_\sigma, \xi \in V_\sigma^*\},
\end{aligned}$$

where $m_{\xi,v}(g) = \xi(\sigma(g)v)$ is a matrix coefficient and $\check{m_{\xi,v}}(g) = m_{\xi,v}(g^{-1})$. The right regular representation preserves $\mathbb{C}[G]^{\sigma,L}$, since L and R commute. As a module for $G \times G$, $\mathbb{C}[G]^{\sigma,L}$ is isomorphic to $V_\sigma \otimes V_{\sigma^*} = V_\sigma \otimes V_\sigma^*$. Indeed, the map

$$\Psi : V_\sigma \otimes V_\sigma^* \to \mathbb{C}[G]^{\sigma,L}, \quad \Psi(v \otimes \xi) = \check{m_{\xi,v}}$$

intertwines $\sigma \otimes \sigma^*$ with the left-right regular representation

$$(LR)(g_1, g_2)f(x) = f(g_1^{-1}xg_2)$$

of $G \times G$ on $\mathbb{C}[G]$. This gives the *Peter-Weyl Theorem*:

Theorem 2.2.3. *The left-right action of $G \times G$ on G is multiplicity free. As a $G \times G$-module we have*
$$\mathbb{C}[G] \cong \bigoplus_{\sigma \in \widehat{G}} \sigma \otimes \sigma^*.$$

2.3. **Maximal unipotent subgroups.** Let $B = HN$ be a Borel subgroup in G. For each $\sigma \in \widehat{G}$ we have $dim(V_\sigma^N) = 1$, by highest weight theory. Theorem 2.2.1 now shows:

Theorem 2.3.1. *The action of G on G/N is multiplicity free. Moreover, each irreducible representation $\sigma \in \widehat{G}$ occurs exactly once in $\mathbb{C}[G/N]$.*

The *Borel-Weil Theorem* provides an explicit model for the irreducible representation with specified highest weight $\lambda \in \mathfrak{h}^*$ inside $\mathbb{C}[G/N^-]$.

Theorem 2.3.2. *The irreducible representation with highest weight $\lambda \in \mathfrak{h}^*$ is given by the left regular representation of G on*
$$R_\lambda = \{f \in \mathbb{C}[G] \ : \ f(gb) = b^{-\lambda} f(g) \text{ for all } b \in B^-\}.$$
Moreover, a highest weight vector in R_λ is given on the dense subset NHN^- of G by
$$f_\lambda(nhn^-) = h^{-\lambda}.$$

Proof. First observe that R_λ is a $L(G)$-invariant subspace of $\mathbb{C}[G/N^-]$. If $f \in R_\lambda$ is a B-highest weight vector then
$$f(nhn^-) = f(h) = (h^{-1} \cdot f)(e)$$
and also $f(h) = h^{-\lambda} f(e)$. Thus f has weight λ and $f = f(e) f_\lambda$. So R_λ is $L(G)$-irreducible with highest weight λ and highest weight vector f_λ. □

Corollary 2.3.3. *The span of $R_\lambda R_\mu$ is $R_{\lambda+\mu}$.*

Proof. The span of $R_\lambda R_\mu$ is a G-invariant subspace of $R_{\lambda+\mu}$. □

Remark 2.3.4. It follows from Theorem 2.1.1 that $G : (G/B)$ is also multiplicity free. But $V_\sigma^B = \{0\}$ unless $\sigma \in \widehat{G}$ is the trivial representation. So $\mathbb{C}[G/B] = \mathbb{C}$ and this is an uninteresting example. Alternatively, one can note that G/B is a flag manifold, hence projective, hence compact. So every regular function on G/B is constant.

2.4. S**-varieties.** Let $B = HN$ be a Borel subgroup in G with opposite Borel subgroup $B^- = HN^- = N^- H$. For $i = 1, \dots k$, let $\sigma_i \in \widehat{G}$ act on the space V_i, and let $v_i \in V_i$ be a B^--highest weight vector with weight $-\lambda_i$. Let $v = v_1 + \cdots + v_k \in V_1 \oplus \cdots \oplus V_k$. Then
$$X = \overline{G \cdot v},$$

the Zariski-closure of the orbit of v in $V_1 \oplus \cdots \oplus V_k$, is a G-invariant subvariety, called an *S-variety.*

Since $N^- \cdot v = \{v\}$, we see that

$$B \cdot v = NHN^- \cdot v$$

is dense in $G \cdot v$ and hence also in X. It follows from Theorem 2.1.1 that $G : X$ is a multiplicity free action. In the present context, however, we will show directly that $G : X$ is multiplicity free by exhibiting the decomposition of $\mathbb{C}[X]$.

Theorem 2.4.1. *The multiplicity free decomposition of* $\mathbb{C}[X]$ *is*

$$\mathbb{C}[X] \cong \bigoplus_{\lambda \in \Lambda} R_\lambda,$$

where $\Lambda = \{a_1\lambda_1 + \cdots + a_k\lambda_k \ : \ a_j \in \mathbb{N}\}$.

(Throughout, $\mathbb{N}$ denotes the non-negative integers, including zero.)

Proof. We can lift an element ξ of V_i^* to V, and define a function f_ξ on G by $f_\xi(g) = \xi(g \cdot v) = \xi(g \cdot v_i)$. Then for $b \in B^-$, $f_\xi(gb) = \xi(gb \cdot v_i) = b^{-\lambda_i}\xi(g \cdot v_i) = b^{-\lambda_i} f_\xi(v)$. Thus f_ξ is in the G-irreducible space R_{λ_i} defined above. Hence $\{\xi|_X : \xi \in V_i^*\}$ is equivalent to the G-irreducible R_{λ_i}.

Note that $\mathbb{C}[X]$ is generated by the restriction of elements of each V_i^* to X. So by Corollary 2.3.3, we conclude that the irreducible components are products of the subspaces R_{λ_i}. $\square$

There is another characterization of S-varieties:

Theorem 2.4.2. *Let X be an irreducible affine G-variety with a G-open orbit, such that the isotropy subgroup of any point in the open orbit contains a maximal unipotent subgroup. Then X is an S-variety.*

Proof. One can find a (rational) representation σ of G and a G-equivariant embedding of X into V_σ. (See [44].)

So assume that $X \subset V_\sigma$ and let $v \in X$ be any point in the open G-orbit. Choose a Borel subgroup B with $N \subset B$ such that $N \subset Stab_G(v)$. Then we can write $v = v_1 + \cdots + v_k$, where each v_i is a weight vector with weight λ_i, and the $\lambda_i's$ are distinct. Since v is stabilized by N, each v_i is a highest weight vector.

Thus our variety X is the closure (in V) of $G \cdot v$, where $v = v_1 + \cdots + v_k$. $\square$

2.5. **Spherical pairs.** Suppose that H is a *reductive* algebraic subgroup of G. We say that (G, H) is a *spherical pair* if $dim(V_\sigma^H) \leq 1$ for all $\sigma \in \widehat{G}$. Equivalently, the actions of G on G/H and $H\backslash G$ are multiplicity free. In this section we will summarize some results concerning spherical pairs, without proof.

Let U and K denote maximal compact connected subgroups of G and H. These are compact real Lie groups. Recall that the irreducible rational representations of G

and H correspond to irreducible unitary representations of U and K via the Unitarian Trick. So

$$(G, H) \text{ is spherical } \iff dim(V_\sigma^K) \leq 1 \text{ for all } \sigma \in \widehat{U}.$$

We say that (U, K) is a *compact Gelfand pair* in this case. It is known that (U, K) is a Gelfand pair if and only if the algebra $L^1(U//K)$ of integrable K-bi-invariant functions on U is abelian with respect to convolution.

The preceding remarks show that the spherical pairs (G, H) are precisely the complexifications of the compact Gelfand pairs (U, K). It is well known that (U, K) is a Gelfand pair whenever U/K is a Riemannian symmetric space. The classification of irreducible symmetric spaces of compact type produces ten families of examples, listed in Table 1, together with seventeen exceptional cases. In entries 8 and 9, $U(n)$

The classical compact Riemannian symmetric spaces

	U	K	G	H
1	$SO(n,\mathbb{R})$	$\{I\}$	$SO(n,\mathbb{C})$	$\{I\}$
2	$SU(n)$	$\{I\}$	$SL(n,\mathbb{C})$	$\{I\}$
3	$Sp(2n)$	$\{I\}$	$Sp(2n,\mathbb{C})$	$\{I\}$
4	$SU(n)$	$SO(n,\mathbb{R})$	$SL(n,\mathbb{C})$	$SO(n,\mathbb{C})$
5	$SU(2n)$	$Sp(2n)$	$SL(2n,\mathbb{C})$	$Sp(2n,\mathbb{C})$
6	$SU(p+q)$	$S(U(p)\times U(q))$	$SL(p+q,\mathbb{C})$	$S(GL(p,\mathbb{C})\times GL(q,\mathbb{C}))$
7	$SO(p+q,\mathbb{R})$	$SO(p,\mathbb{R})\times SO(q,\mathbb{R})$	$SO(p+q,\mathbb{C})$	$SO(p,\mathbb{C})\times SO(q,\mathbb{C})$
8	$SO(2n,\mathbb{R})$	$U(n)$	$SO(2n,\mathbb{C})$	$GL(n,\mathbb{C})$
9	$Sp(2n)$	$U(n)$	$Sp(2n,\mathbb{C})$	$GL(n,\mathbb{C})$
10	$Sp(p+q)$	$Sp(p)\times Sp(q)$	$Sp(p+q,\mathbb{C})$	$Sp(p,\mathbb{C})\times Sp(q,\mathbb{C})$

TABLE 1

is embedded in $SO(2n,\mathbb{R})$ and $Sp(2n)$ via

$$A + iB \mapsto \left[\begin{array}{r|l} A & B \\ \hline -B & A \end{array}\right].$$

In each entry of Table 1, G is a *simple* complex algebraic group (and U is a simple compact Lie group). This means that the adjoint representation for G on its Lie algebra $\mathfrak{g}$ is irreducible. Hence $dim(\mathfrak{g}^H) \leq 1$ for spherical pairs (G, H) with G simple. In particular, the center of H is at most one dimensional.

The spherical pairs (G, H) with G simple were classified by Krämer. In addition to the examples that arise from symmetric spaces, there are six further families and six exceptional cases. These are listed in Table 2. The compact form (U, K) for each entry in Table 2 is a compact Gelfand pair for which U/K is not a symmetric space.

Spherical pairs with G simple, not arising from symmetric spaces

	G	H
1	$SL(p+q,\mathbb{C})$ $(p \neq q)$	$SL(p,\mathbb{C}) \times SL(q,\mathbb{C})$
2	$SL(2n+1,\mathbb{C})$	$\mathbb{C}^\times \times Sp(2n,\mathbb{C})$
3	$SL(2n+1,\mathbb{C})$	$Sp(2n,\mathbb{C})$
4	$Sp(2n,\mathbb{C})$	$Sp(2n-2,\mathbb{C}) \times SO(2,\mathbb{C})$
5	$SO(2n+1,\mathbb{C})$	$GL(n,\mathbb{C})$
6	$SO(4n+2,\mathbb{C})$	$SL(2n+1,\mathbb{C})$
7	$SO(10,\mathbb{C})$	$Spin(7,\mathbb{C}) \times SO(2,\mathbb{C})$
8	$SO(9,\mathbb{C})$	$Spin(7,\mathbb{C})$
9	$SO(8,\mathbb{C})$	$G_2^{\mathbb{C}}$
10	$SO(7,\mathbb{C})$	$G_2^{\mathbb{C}}$
11	$E_6^{\mathbb{C}}$	$Spin(10,\mathbb{C})$
12	$G_2^{\mathbb{C}}$	$SL(3,\mathbb{C})$

TABLE 2

2.6. **Section 2 notes.** Theorem 2.1.1 is due to Servedio, [49]. The converse was proved by Vinberg in [52].

The structure of G/H as a quasi-projective algebraic variety is discussed in Section 11.2.1 of [17].

The classical Peter-Weyl Theorem concerns the decomposition of $L^2(K)$ under the action of $K \times K$ for compact Lie groups K. Let $\mathcal{E}_\sigma$ denote the space of matrix coefficients $m_{\xi,v}(k) = \xi(\sigma(k)v)$ for $\sigma \in \widehat{K}$. One has a Hilbert space direct sum

$$L^2(K) = \bigoplus_{\sigma \in \widehat{K}} \mathcal{E}_\sigma$$

with $K \times K$ acting on $\mathcal{E}_\sigma$ by a copy of $\sigma^* \otimes \sigma$. Theorem 2.2.3 yields the algebraic content of this theorem, via the Unitarian Trick. A complete proof also requires two analytic facts:

- Schur Orthogonality: $\mathcal{E}_\sigma \perp \mathcal{E}'_\sigma$ in $L^2(K)$ for $\sigma \not\simeq \sigma'$; and
- $\bigcup_{\sigma \in \widehat{K}} \mathcal{E}_\sigma$ is dense in $L^2(K)$.

See, for example, Section 5.2 in [15] for proofs.

We refer the reader to Chapter IV in [21] or Chapter 1 in [16] for information concerning Gelfand pairs (U, K). In particular, for the fact that Riemannnian symmetric spaces yield Gelfand pairs. The classification of Riemannian symmetric spaces, including the exceptional cases, can be found in Chapter X of [20]. In [17], Section 12.3, it is shown that symmetric spaces yield spherical pairs (G, H) by showing that there is an open Borel orbit in G/H. This approach involves a complex version of the Iwasawa decomposition.

Krämer's classification of spherical pairs (G, H) with G simple appeared in [36]. This classification was pushed further by Brion [8] and by Mikityuk [40] to encompass all spherical pairs (G, H) with G *semi*-simple. This results in eight additional families of examples. The classification of spherical pairs can also be found in Vinberg's recent survey article [53]. S-varieties were introduced by Vinberg and Popov [54].

3. Linear multiplicity free actions

We now restrict our attention to *linear* actions $G : V$. That is, G is a reductive algebraic group acting on a complex vector space V by some rational representation.

3.1. Connectivity of G.

We generally assume that our groups G are connected unless stated otherwise. Proposition 3.1.3 below shows that this entails no great loss of generality. For Lemma 3.1.1 and Proposition 3.1.2 we assume G is connected and B denotes a Borel subgroup.

Lemma 3.1.1. *Let $h \in \mathbb{C}[V]$ be a highest weight vector for B with prime decomposition*

$$h = p_1^{m_1} \cdots p_k^{m_k}.$$

Then each irreducible factor p_j is a B-highest weight vector.

Proof. Let $\psi : B \to \mathbb{C}^\times$ be the weight for h. Then for $b \in B$,

$$\psi(b)h = b \cdot h = (b \cdot p_1)^{m_1} \cdots (b \cdot p_k)^{m_k}$$

and each $b \cdot p_j$ is an irreducible polynomial. By uniqueness of the prime decomposition for h we conclude that, for each j, $b \cdot p_j$ is a non-zero multiple of one of the prime factors. That is, $b \cdot p_j \in \cup_{l=1}^{k} \mathbb{C}^\times p_l$. But B acts continuously on this space, and B is connected, so B must act by a scalar on each p_j. □

Proposition 3.1.2. *If $G : V$ is* not *a multiplicity free action then the multiplicities $\{m(\sigma, \mathbb{C}[V]) : \sigma \in \widehat{G}\}$ are unbounded.*

Proof. As $G : V$ is not multiplicity free we can find a pair of linearly independent B-highest weight vectors h_1, h_2 in $\mathbb{C}[V]$ with common weight $\psi : B \to \mathbb{C}^\times$. In view of Lemma 3.1.1 we can remove any common irreducible factors and assume h_1, h_2 are relatively prime. For each N and $0 \le k \le N$, $h_1^k h_2^{N-k}$ has weight ψ^N. Moreover, $\{h_1^k h_2^{N-k} \ : \ 0 \le k \le N\}$ is a linearly independent set in $\mathbb{C}[V]$. For otherwise we could express some $h_1^{k_\circ} h_2^{N-k_\circ}$ as a linear combination

$$h_1^{k_\circ} h_2^{N-k_\circ} = \sum_{k=k_\circ+1}^{N} c_k h_1^k h_2^{N-k}$$

and conclude that h_1 divides $h_2^{N-k_\circ}$. This contradicts the fact that h_1, h_2 are relatively prime. Thus the irreducible representation with highest weight ψ^N occurs in $\mathbb{C}[V]$ with multiplicity at least N. □

Proposition 3.1.3. *Let G be multiply-connected with identity component $G_\circ$. Then $G : V$ is a multiplicity free action if and only if $G_\circ : V$ is a multiplicity free action.*

Proof. If $G_\circ : V$ is multiplicity free then so is $G : V$, in view of (1.9.1). Conversely, suppose that $G : V$ is a multiplicity free action but that $G_\circ : V$ is not multiplicity free. Let

$$\mathbb{C}[V] = \bigoplus_\lambda P_\lambda$$

denote the decomposition of $\mathbb{C}[V]$ into distinct G-irreducible subspaces and $\{g_1, \ldots, g_\ell\}$ be a complete set of coset representatives for $G_\circ$ in G. Proposition 3.1.2 ensures that for any N there is some $\sigma \in \widehat{G_\circ}$ with $m(\sigma, \mathbb{C}[V]) \geq N\ell 2^\ell$. As the P_λ's are $G_\circ$-invariant, each copy of σ in $\mathbb{C}[V]$ is contained in some P_λ. Let W denote a subspace of some P_λ on which $G_\circ$ acts by a copy of σ. Since $G_\circ$ is normal in G, $g_i \cdot W \subset P_\lambda$ is $G_\circ$-invariant with $G_\circ$ acting via a copy of $\sigma_i \in \widehat{G_\circ}$, where

$$\sigma_i(g) = \sigma(g_i g g_i^{-1}).$$

As P_λ is G-irreducible, we must have

$$P_\lambda = \bigoplus_{j \in J} (g_j \cdot W) \tag{3.1.1}$$

for some subset J of $\{1, \ldots, \ell\}$. Thus P_λ is equivalent to $\sum_{j \in J} \sigma_j$ as a $G_\circ$-module.

Equation 3.1.1 contains at most ℓ factors, so at least $N2^\ell$ distinct P_λ's must contain copies of σ. As there are only 2^ℓ possibilities for J, at least N of these P_λ's must be equivalent as $G_\circ$-modules. Thus we have shown that for each N one can find N distinct irreducible G-modules that are equivalent as $G_\circ$-modules. This is impossible since $G/G_\circ$ is a finite group. □

3.2. **Borel orbits.** We will prove the converse of Theorem 2.1.1 in the context of linear actions $G : V$. We use the notation introduced in Section 1.4: Choose a Borel subgroup $B = HN$ in G and let $\Delta^+ \subset \mathfrak{h}^*$ be the associated set of positive roots.

Now suppose that $h \in \mathbb{C}[V]$ is a B-highest weight vector with weight $\lambda \in \mathfrak{h}^*$. Let

$$P = P_h = \{g \in G \ : \ g \cdot h \in \mathbb{C}^\times h\}. \tag{3.2.1}$$

This is a parabolic subgroup of G that contains B. We have $X \cdot h = \lambda(X)h$ for all $X \in \mathfrak{p}$, the Lie algebra of P.

- Let L and U be the Levi component and unipotent part of P, so $P = LU$.
- On the Lie algebra level we can write

$$\mathfrak{p} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_P} \mathfrak{g}_\alpha, \quad \text{where } \Delta_P = \{\alpha \in \Delta \mid \langle \lambda, \alpha \rangle \geq 0\}.$$

Now setting

$$\Delta_L = \{\alpha \in \Delta \mid \langle \lambda, \alpha \rangle = 0\}, \quad \Delta_U = \{\alpha \in \Delta \mid \langle \lambda, \alpha \rangle > 0\} = \{\alpha \in \Delta \mid -\alpha \notin \Delta_P\}$$

one has

$$\mathfrak{l} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_L} \mathfrak{g}_\alpha, \quad \mathfrak{u} = \sum_{\alpha \in \Delta_U} \mathfrak{g}_\alpha.$$

- Letting $\mathfrak{u}^- = \sum_{\alpha \in \Delta_U} \mathfrak{g}_{-\alpha}$ we have $\mathfrak{u} \subset \mathfrak{n}$, $\mathfrak{u}^- \subset \mathfrak{n}^-$ and

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{u}^- = \mathfrak{l} \oplus \mathfrak{u} \oplus \mathfrak{u}^-.$$

We will make extensive use of the following map (which depends on h). Let

$$V_\circ = \{z \in V \;:\; h(z) \neq 0\},$$

a principal open set in V, and define

$$\chi = \chi_h : V_\circ \to \mathfrak{g}^*$$

via

$$\chi(z)(X) = \frac{(X \cdot h)(z)}{h(z)}.$$

Lemma 3.2.1. *χ is P-equivariant.*

Proof. Note that the action of P on V preserves $V_\circ$. Also P acts on $\mathfrak{g}^*$ via the coadjoint action

$$(g \cdot f)(X) = f(Ad(g^{-1})X).$$

For $g \in P$ one has

$$\begin{aligned} \chi(g^{-1} \cdot z)(X) &= \frac{(X \cdot h)(g^{-1} \cdot z)}{h(g^{-1} \cdot z)} = \frac{(g \cdot X)(g \cdot h)(z)}{(g \cdot h)(z)} \\ &= \frac{\lambda(g)(g \cdot X)(h)(z)}{\lambda(g)h(z)} = \chi(z)(g \cdot X). \end{aligned}$$

Thus $\chi(g^{-1} \cdot z)(X) = (g^{-1} \cdot \chi(z))(X)$ so χ is P-equivariant. □

Lemma 3.2.2. *The stabilizer of λ in P is $Stab_P(\lambda) = L$. Moreover, $Stab_U(\lambda) = \{e\}$.*

Proof. Here $\lambda \in \mathfrak{h}^*$ is regarded as a functional on all of $\mathfrak{g}$ with $\lambda(\mathfrak{n}) = \{0\} = \lambda(\mathfrak{n}^-)$. For $X \in \mathfrak{p}$ one has $X \cdot \lambda = 0$ if and only if $\lambda[X, \mathfrak{g}] = \{0\}$. Clearly $\mathfrak{h}$ stabilizes λ because $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{n} + \mathfrak{n}^-$. Also for any $\alpha \in \Delta_L$ and any $Y = H + \sum_{\beta \in \Delta} c_\beta X_\beta \in \mathfrak{g}$,

$$\lambda([X_\alpha, Y]) = c_{-\alpha}\lambda(H_\alpha) = c_{-\alpha}\langle \lambda, \alpha \rangle = 0.$$

Thus X_α stabilizes λ for each $\alpha \in \Delta_L$. So $\mathfrak{l} = \mathfrak{h} + \sum_{\alpha \in \Delta_L} \mathfrak{g}_\alpha$ stabilizes λ.

We have shown that $L \subset Stab_P(\lambda)$. For the reverse inclusion, let

$$X = H + \sum_{\alpha \in \Delta_L} a_\alpha X_\alpha + \sum_{\alpha \in \Delta_U} b_\alpha X_\alpha \in \mathfrak{p}$$

stabilize λ. Then for $\beta \in \Delta_U$,

$$\lambda[X, X_{-\beta}] = b_\beta \lambda(H_\beta) = b_\beta \langle \lambda, \beta \rangle.$$

But $\langle \lambda, \beta \rangle > 0$, and hence $b_\beta = 0$. Thus $Stab_P(\lambda) \subset L$ and the stabilizer of λ in U is trivial. □

Proposition 3.2.3. *The image of $\chi : V_\circ \to \mathfrak{g}^*$ is $\lambda + \mathfrak{p}^\perp$, a single P-orbit in $\mathfrak{g}^*$.*

Proof. For all $z \in V_\circ$ and $X \in \mathfrak{p}$,

$$\chi(z)(X) = \frac{(X \cdot h)(z)}{h(z)} = \frac{\lambda(X)h(z)}{h(z)} = \lambda(X).$$

Thus $\chi(z) - \lambda$ annihilates $\mathfrak{p}$ and $\chi(V_\circ) \subset \lambda + \mathfrak{p}^\perp$. Note that $dim(\lambda + \mathfrak{p}^\perp) = dim(\mathfrak{u})$. As U is unipotent and acts without stabilizer on λ we conclude that $U \cdot \lambda$ is both open and closed in $\lambda + \mathfrak{p}^\perp$ and hence $U \cdot \lambda = \lambda + \mathfrak{p}^\perp$. As $P = LU$ and L stabilizes λ we also have $P \cdot \lambda = \lambda + \mathfrak{p}^\perp$. As χ is P-equivariant we must have $P \cdot \lambda = \chi(V_\circ)$. □

Corollary 3.2.4. *$\lambda = \chi(z_\circ)$ for some $z_\circ \in V_\circ$.*

Let

$$\Sigma = \Sigma_h = \chi^{-1}(\lambda) = \{z \in V_\circ \ : \ \chi(z) = \lambda\}. \tag{3.2.2}$$

The group L acts on Σ because χ is P-equivariant and L stabilizes λ.

Lemma 3.2.5. *$U \times \Sigma \cong V_\circ$ via $(g, z) \mapsto g \cdot z$.*

Proof. Given $z \in V_\circ$ one has $\chi(z) = g \cdot \lambda$ for some $g \in U$ in view of Proposition 3.2.3. Thus $\lambda = g^{-1} \cdot \chi(z) = \chi(g^{-1} \cdot z)$, so $g^{-1} \cdot z \in \Sigma$. Now $(g, g^{-1} \cdot z) \in U \times \Sigma$ maps to z.

To see that $(g, z) \mapsto g \cdot z$ is injective, suppose that $g \cdot z = g' \cdot z'$ for some $g, g' \in U$, $z, z' \in \Sigma$. Applying χ gives

$$g \cdot \lambda = g \cdot \chi(z) = g' \cdot \chi(z') = g' \cdot \lambda$$

and thus $g^{-1}g'$ stabilizes λ. As U acts without stabilizer on λ it follows that $g = g'$ and thus also $z = z'$. □

Lemma 3.2.6. *There is a unique parabolic subgroup $P = P_h$ of lowest possible dimension. Moreover, $P \subset P_{h'}$ for all B-highest weight vectors $h' \in \mathbb{C}[V]$.*

Proof. Suppose that $h \in \mathbb{C}[V]$ is a B-highest weight vector with prime decomposition

$$h = p_1^{m_1} \cdots p_r^{m_r}.$$

The proof of Lemma 3.1.1 shows that P_h acts by a character on each irreducible factor p_j. Thus

$$P_h = P_{p_1} \cap \cdots \cap P_{p_r}.$$

Now assume that $h, h' \in \mathbb{C}[V]$ are two highest weight vectors and that P_h has minimal dimension. Let $p_1, \ldots, p_r$ be the prime factors of h and $q_1, \ldots, q_s$ the prime factors of h'. Letting $h'' = p_1 \cdots p_r q_1 \cdots q_s$ one has

$$P_{h''} = P_{p_1} \cap \cdots \cap P_{p_r} \cap P_{q_1} \cap \cdots \cap P_{q_s} = P_h \cap P_{h'}.$$

If $P_h \not\subset P_{h'}$ then $P_{h''}$ is a proper subset of P_h and hence $dim(P_{h''}) < dim(P_h)$, a contradiction. Thus $P_h \subset P_{h'}$ as claimed. Moreover if $P_{h'}$ also has minimal dimension then $P_{h'} \subset P_h$ and now $P_h = P_{h'}$. This shows uniqueness. □

Lemma 3.2.7. *Let $P = P_h = LU$ be the unique parabolic subgroup of minimal dimension. Then (L, L) acts trivially on $\Sigma = \Sigma_h$.*

Proof. First note that as P acts on h by a character, $(L, L) \subset (P, P)$ acts trivially on h. Suppose, however, that (L, L) does not act trivially on Σ. It follows that there is some highest weight vector $h' \in \mathbb{C}[\Sigma]$ for the action $L : \Sigma$ that is not fixed by (L, L). Recall that $B = HN$ with $N \subset U$ and that $V_\circ = U \cdot \Sigma$. Thus we can extend h' to a B-semi-invariant function on $V_\circ$. For N sufficiently large, $h^N h'$ is a regular function on V. Now $h^N h' \in \mathbb{C}[V]$ is a B-highest weight vector and $P \subset P_{h^N h'}$ by Lemma 3.2.6. Thus P acts by a character on $h^N h'$ and hence (L, L) acts trivially on $h^N h'$. As (L, L) fixes both h and $h^N h'$, it must fix h', a contradiction. □

Lemma 3.2.7 implies that for the minimal parabolic $P = P_h = LU$, the action of L on $\Sigma = \Sigma_h$ is a torus action. Lemma 3.2.5 implies that the variety Σ is affine and irreducible.

Theorem 3.2.8. *If $G : V$ is multiplicity free then there is an open B-orbit in V.*

Proof. Suppose that there is no open B-orbit in V. Let $P = P_h = LU$ be the parabolic subgroup of minimal dimension and $\Sigma = \Sigma_h$.

We claim that $H : \Sigma$ is not multiplicity free. Indeed, if $H : \Sigma$ were multiplicity free then there would be an open H-orbit in Σ, $H \cdot v_\circ$ say, by Proposition 2.1.2. Now as $U \cdot \Sigma = V_\circ$, one has $UH \cdot v_\circ$ open in $V_\circ$ and hence open in V. As $U \subset N$, then $B \cdot v_\circ = HN \cdot v_\circ$ is open in V, a contradiction.

As $H : \Sigma$ is not multiplicity free we can find a pair of linearly independent highest weight vectors $h_1, h_2 \in \mathbb{C}[\Sigma]$ with a common weight for the action $L : \Sigma$. As in the proof of Lemma 3.2.7, one can extend h_1 and h_2 to B-semi-invariant functions on $V_\circ$. Now for N sufficiently large, $h^N h_1$ and $h^N h_2$ are regular on V and share a common weight. Hence the action $G : V$ fails to be multiplicity free. □

Note that $B \cdot v_\circ$ is open in V if and only if $dim(\mathfrak{b} \cdot v_\circ) = dim(B \cdot v_\circ) = dim(V)$. Thus $B \cdot v_\circ$ is open if and only if $\mathfrak{b} \cdot v_\circ = V$. We obtain an infinitesimal version of Theorems 2.1.1 and 3.2.8:

Corollary 3.2.9. *$G : V$ is multiplicity free if and only if $\mathfrak{b} \cdot v_\circ = V$ for some point $v_\circ \in V$.*

We will apply Corollary 3.2.9 to the study of examples in Section 4.

3.3. **Fundamental highest weights for a multiplicity free action.** Let $G : V$ be a linear multiplicity free action, $B = HN$ a Borel subgroup of G and $v_\circ$ a point in V with $B \cdot v_\circ$ open in V. Let

$$\Lambda \subset \mathfrak{h}^*$$

denote the set of highest weights λ for representations $\sigma_\lambda \in \widehat{G}$ that occur in $\mathbb{C}[V]$. The character $B \to \mathbb{C}^\times$ given by $\lambda \in \Lambda$ will be denoted $b \mapsto b^\lambda$.

For $\lambda \in \Lambda$, let $P_\lambda \subset \mathbb{C}[V]$ be the irreducible subspace of $\mathbb{C}[V]$ on which G acts by a copy of σ_λ. Then

$$\mathbb{C}[V] = \bigoplus_{\lambda \in \Lambda} P_\lambda \tag{3.3.1}$$

is the multiplicity free decomposition of $\mathbb{C}[V]$ under the action of G. As G preserves the subspaces $\mathcal{P}_m(V)$ of polynomials homogeneous of degree m, each P_λ is contained in some $\mathcal{P}_m(V)$.

Each P_λ contains a B-highest weight vector h_λ, which is unique up to scalar multiples. As shown in the proof of Theorem 2.1.1, we must have $h_\lambda(v_\circ) \neq 0$ and we can normalize our choice of h_λ by the condition

$$h_\lambda(v_\circ) = 1.$$

For $\lambda, \mu \in \Lambda$,

$$b \cdot (h_\lambda h_\mu) = (b \cdot h_\lambda)(b \cdot h_\mu) = (b^\lambda h_\lambda)(b^\mu h_\mu) = b^{\lambda+\mu} h_\lambda h_\mu.$$

Hence

$$\lambda + \mu \in \Lambda \quad \text{and} \quad h_{\lambda+\mu} = h_\lambda h_\mu.$$

In particular, Λ is an additive semigroup in $\mathfrak{h}^*$.

Next suppose $\lambda \in \Lambda$ and let

$$h_\lambda = p_1^{m_1} \cdots p_k^{m_k}$$

be the prime decomposition for h_λ. Lemma 3.1.1 shows that each p_j is a B-highest weight vector. Suitably normalizing the p_j's, we can now say that $p_j = h_{\lambda_j}$ for some $\lambda_j \in \Lambda$ and $\lambda = m_1\lambda_1 + \cdots + m_k\lambda_k$.

Now let

$$\Lambda' = \{\lambda \in \Lambda \;:\; h_\lambda \text{ is an irreducible polynomial}\}.$$

We know that $\Lambda' \neq \emptyset$ and that Λ' generates the semigroup Λ. We will show that Λ' is a $\mathbb{Q}$-linearly independent subset of $\mathfrak{h}^*$. Indeed, suppose that $\lambda_1, \ldots, \lambda_k \in \Lambda'$ satisfy a non-trivial linear dependence relation over $\mathbb{Q}$. Clearing denominators we obtain a non-trivial linear dependence

$$a_1\lambda_1 + \cdots + a_k\lambda_k = 0$$

with integer coefficients a_j. Let

$$L = \{j \;:\; a_j \geq 0\}$$

and set $m_j = |a_j|$ for all $1 \leq j \leq k$. Then

$$\sum_{j \in L} m_j \lambda_j = \sum_{j \notin L} m_j \lambda_j$$

and hence

$$\prod_{j \in L} h_{\lambda_j}^{m_j} = \prod_{j \notin L} h_{\lambda_j}^{m_j}$$

since these are both highest weight vectors with a common weight and take value 1 at the point $v_\circ$. But this is impossible, since the h_{λ_j}'s are distinct irreducibles and $\mathbb{C}[V]$ is a unique factorization domain.

It now follows that Λ' is a finite set with at most $dim(\mathfrak{h})$ elements.

In summary, we have proved the following.

Proposition 3.3.1. *$\Lambda' = \{\lambda \in \Lambda \ : \ h_\lambda$ is irreducible$\}$ is a $\mathbb{Q}$-linearly independent subset of $\mathfrak{h}^*$ with at most $dim(H)$ elements. The additive semigroup Λ is freely generated by Λ'. Writing*

$$\Lambda' = \{\lambda_1, \dots, \lambda_r\},$$

the decomposition for $\mathbb{C}[V]$ can be written

$$\mathbb{C}[V] = \bigoplus_\lambda P_\lambda$$

where the sum is taken over all $\mathbb{N}$-linear combinations $\lambda = m_1\lambda_1 + \cdots m_r\lambda_r$. The highest weight vector in the irreducible subspace P_λ is $h_\lambda = h_{\lambda_1}^{m_1} \cdots h_{\lambda_r}^{m_r}$.

Definition 3.3.2. The number r is the *rank* of the multiplicity free action $G : V$, $\{\lambda_1, \dots, \lambda_r\}$ are the *fundamental highest weights* and $\{h_1 = h_{\lambda_1}, \dots, h_r = h_{\lambda_r}\}$ are the *fundamental highest weight vectors.*

3.4. **Section 3 notes.** Many results in this section are due to Roger Howe. Proposition 3.3.1 is from [23]. Proposition 3.1.3 appeared in [3] but the proof was shown to the authors by Howe.

The proof given for Theorem 3.2.8 is due to Friedrich Knop [31] and based on ideas from [10]. The more standard proof (see [52]) is shorter but requires use of a hard result of Rosenlicht. For background on parabolic subgroups see Section V.7 in [29].

A linear action $G : V$ is called a *prehomogeneous vector space* when there is an open G-orbit in V. We refer the reader to [28] for a survey of this subject, including classification and applications to analysis. Theorem 3.2.8 implies that each linear multiplicity free action is, in particular, a prehomogeneous vector space.

Proposition 3.1.2 shows that linear actions $G : V$ are of three types:

(1) $G : V$ is multiplicity free.
(2) Some $\sigma \in \widehat{G}$ occurs in $\mathbb{C}[V]$ with infinite multiplicity.
(3) Each $\sigma \in \widehat{G}$ has $m(\sigma, \mathbb{C}[V]) < \infty$ but these multiplicities are unbounded.

It is known that (2) is equivalent to the existence of a non-constant G-invariant in $\mathbb{C}[V]$. (See Theorem 5.5 in [3].) In this case, *every* $\sigma \in \widehat{G}$ which occurs in $\mathbb{C}[V]$ does so with infinite multiplicity.

4. Examples of multiplicity free decompositions

4.1. $\mathbf{GL(n)} \otimes \mathbf{GL(m)}$. Here we consider the action of $G = GL(n, \mathbb{C}) \times GL(m, \mathbb{C})$ on $\mathbb{C}^n \otimes \mathbb{C}^m$ via the the outer tensor product of the defining representations for the two factors. Identifying $\mathbb{C}^n \otimes \mathbb{C}^m$ with the space $M_{n,m}(\mathbb{C})$ of complex $n \times m$-matrices (via $e_i \otimes e_j \leftrightarrow E_{i,j}$) one has

$$(g, g') \cdot v = gv(g')^t \tag{4.1.1}$$

In fact it will prove convenient to *twist* the action by composing with the automorphisms $g \mapsto (g^{-1})^t$ on both factors. This gives

$$(g, g') \cdot v = (g^{-1})^t v(g')^{-1} \tag{4.1.2}$$

and the associated representation on $\mathbb{C}[M_{n,m}(\mathbb{C})]$ becomes

$$(g, g') \cdot p(v) = p(g^t v g').$$

Of course the decompositions for $\mathbb{C}[M_{n,m}(\mathbb{C})]$ under the two actions (4.1.1) and (4.1.2) are the same. Twisting by $g \mapsto (g^{-1})^t$ has the effect of interchanging representations with their contragredients. So decomposing $\mathbb{C}[M_{n,m}(\mathbb{C})]$ with respect to (4.1.2) amounts to decomposing the symmetric algebra $S(M_{n,m}(\mathbb{C})) \cong \mathbb{C}[M_{n,m}(\mathbb{C})^*]$ with respect to (4.1.1).

Recall that the upper triangular matrices in $GL(k, \mathbb{C})$ give the standard Borel subgroup B_k with Lie algebra $\mathfrak{b}_k$. The diagonal matrices H_k in $GL(k, \mathbb{C})$ give the maximal torus with Lie algebra $\mathfrak{h}_k \cong \mathbb{C}^k$. We let

$$B = B_n \times B_m, \quad \mathfrak{b} = \mathfrak{b}_n \times \mathfrak{b}_m, \quad H = H_n \times H_m, \quad \mathfrak{h} = \mathfrak{h}_n \times \mathfrak{h}_m,$$

$\varepsilon_j \in \mathfrak{h}_n^*$ be $\varepsilon_j(diag(h_1, \ldots, h_n)) = h_j$ and likewise for $\varepsilon'_j \in \mathfrak{h}_m^*$. We sometimes use the shorthand

$$\lambda = (\mu, \nu) = (\mu_1, \ldots, \mu_n; \nu_1, \ldots, \nu_m)$$

for the weight $\lambda = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n + \nu_1\varepsilon'_1 + \cdots + \nu_m\varepsilon'_m$ in $\mathfrak{h}^*$. The dominant weights have $\mu \in \mathbb{Z}^n$, $\nu \in \mathbb{Z}^m$ with $\mu_1 \geq \cdots \geq \mu_n$ and $\nu_1 \geq \cdots \geq \nu_m$. We say that λ is *non-negative* and write $\lambda \geq 0$ when $\mu_j \geq 0$ and $\nu_j \geq 0$ for all j.

For any $(h, h') = (diag(h_1, \ldots, h_n), diag(h'_1 \ldots, h'_m)) \in H$ one has

$$(h, h') \cdot z_{i,j} = h_i h'_j z_{i,j}$$

where $z_{ij} : M_{n,m}(\mathbb{C}) \to \mathbb{C}$ is the (i, j)-th entry function. So $z_{i,j}$ is a weight vector with weight $\varepsilon_i + \varepsilon'_j$. Hence also $z^\alpha = \prod_{i,j} z_{ij}^{\alpha_{ij}}$ is a weight vector with weight

$$\sum_{i,j} \alpha_{ij}(\varepsilon_i + \varepsilon'_j) \geq 0.$$

As the z^α's form a basis for $\mathbb{C}[M_{n,m}(\mathbb{C})]$, all weight vectors in $\mathbb{C}[M_{n,m}(\mathbb{C})]$ are non-negative. Thus the highest weights $\lambda = (\mu, \nu)$ that occur in $\mathbb{C}[M_{n,m}(\mathbb{C})]$ all have

$$\mu_1 \geq \cdots \geq \mu_n \geq 0, \quad \nu_1 \geq \cdots \geq \nu_m \geq 0.$$

This explains why we are using the action (4.1.2) in place of (4.1.1).

Let us assume that $n \geq m$ and set

$$v_\circ = \left[\frac{I_m}{O}\right] \in M_{n,m}(\mathbb{C}).$$

A typical element X in the Lie algebra $\mathfrak{b}$ for B has the form

$$X = \left(\left[\begin{array}{c|c} A & B \\ \hline O & C \end{array} \right], D \right) \tag{4.1.3}$$

where A, D are $m \times m$ upper triangular, C is $(n-m) \times (n-m)$ upper triangular and B is an arbitrary $m \times (n-m)$-matrix. The derived action for (4.1.2) gives

$$X \cdot v_\circ = - \left[\begin{array}{c|c} A & B \\ \hline O & C \end{array} \right]^t \left[\begin{array}{c} I_m \\ \hline O \end{array} \right] - \left[\begin{array}{c} I_m \\ \hline O \end{array} \right] D = - \left[\begin{array}{c} A^t + D \\ \hline B^t \end{array} \right]. \tag{4.1.4}$$

Here A^t, D are arbitrary $m \times m$ lower and upper triangular matrices respectively and B^t is an arbitrary matrix of size $(n-m) \times m$. We conclude that $\mathfrak{b} \cdot v_\circ = M_{n,m}(\mathbb{C})$. Hence $B \cdot v_\circ$ is open in $M_{n,m}(\mathbb{C})$, so (4.1.2) is a multiplicity free action.

The element X in Equation 4.1.3 belongs to $\mathfrak{h}$ when $B = O$ and A, C, D are diagonal,

$$A = diag(a_1, \ldots, a_m), \quad C = diag(c_1, \ldots, c_{n-m}), \quad D = diag(d_1, \ldots, d_m)$$

say. For such X, Equation 4.1.4 shows $X \cdot v_\circ = 0$ if and only if $D = -A$. So the stabilizer $\mathfrak{h}_\circ$ of $v_\circ$ in $\mathfrak{h}$ is

$$\mathfrak{h}_\circ = \left\{ \left(\begin{bmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_m & & & \\ & & & c_1 & & \\ & & & & \ddots & \\ & & & & & c_{n-m} \end{bmatrix}, \begin{bmatrix} -a_1 & & \\ & \ddots & \\ & & -a_m \end{bmatrix} \right) \right\}.$$

If $\lambda = (\mu, \nu)$ is the highest weight for a representation that occurs in $\mathbb{C}[M_{n,m}(\mathbb{C})]$ then we must have $\lambda(\mathfrak{h}_\circ) = \{0\}$. That is,

$$(\mu_1 - \nu_1)a_1 + \cdots + (\mu_m - \nu_m)a_m + \mu_{m+1}c_1 + \cdots + \mu_n c_{n-m} = 0$$

for all a_j, c_j. Thus we must have

$$\mu_j = \nu_j \text{ for } 1 \leq j \leq m \text{ and } \mu_j = 0 \text{ for } j > m.$$

We have now shown that the only candidates for highest weights of representations occurring in $\mathbb{C}[M_{n,m}(\mathbb{C})]$ have the form

$$\lambda = (\mu, \mu) \text{ where } \mu_1 \geq \cdots \geq \mu_m \geq 0.$$

Note that a representation with highest weight $\lambda = (\mu, \mu)$ is equivalent to $\sigma_n^\mu \otimes \sigma_m^\mu$, the outer tensor product of the irreducible representations of $GL(n, \mathbb{C})$ and $GL(m, \mathbb{C})$ with highest weight μ. We will show that all such weights λ *do* occur in $\mathbb{C}[M_{n,m}(\mathbb{C})]$

by exhibiting a λ-highest weight vector in $\mathbb{C}[M_{n,m}]$. For this, let

$$\xi_k(z) = \begin{vmatrix} z_{11} & \cdots & z_{1k} \\ \vdots & & \vdots \\ z_{k1} & \cdots & z_{kk} \end{vmatrix},$$

the leading minor determinant for $z \in M_{n,m}(\mathbb{C})$, $1 \le k \le m$. For $(h, h') = (diag(h_1, \ldots, h_n), diag(h'_1 \ldots, h'_m)) \in H$ one computes

$$(h, h') \cdot \xi_k(z) = \xi_k(hzh') = \xi_k\big([h_i h'_j z_{ij}]\big) = (h_1 \cdots h_k)(h'_1 \cdots h'_k)\xi_k(z).$$

Hence ξ_k is a highest weight vector with weight $(\varepsilon_1 + \cdots + \varepsilon_k, \varepsilon'_1 + \cdots + \varepsilon'_k) = (1^k, 1^k)$. For any $\mu_1 \ge \cdots \ge \mu_m \ge 0$

$$\xi_\mu = \xi_1^{\mu_1 - \mu_2} \xi_2^{\mu_2 - \mu_3} \cdots \xi_{m-1}^{\mu_{m-1} - \mu_m} \xi_m^{\mu_m} \tag{4.1.5}$$

has weight $\lambda = (\mu, \mu)$ and degree

$$|\mu| = \mu_1 + \cdots + \mu_m.$$

In summary, we have proved:

Theorem 4.1.1. *The action of $G = GL(n, \mathbb{C}) \times GL(m, \mathbb{C})$ on $\mathbb{C}^n \otimes \mathbb{C}^m \cong M_{n,m}(\mathbb{C})$ is multiplicity free. For $n \ge m$ one has the decomposition*

$$\mathbb{C}[M_{n,m}(\mathbb{C})] = \bigoplus_\mu P_\mu$$

where the sum is over all $\mu \in \mathbb{N}^m$ with $\mu_1 \ge \cdots \ge \mu_m \ge 0$. The polynomial ξ_μ given by (4.1.5) is a highest weight vector in P_μ with weight (μ, μ) and $P_\mu \simeq \sigma_n^\mu \otimes \sigma_m^\mu$ as a G-module. Moreover

$$\mathcal{P}_k(M_{n,m}(\mathbb{C})) = \bigoplus_{|\mu| = k} P_\mu.$$

We observe the following points regarding this example.

- The action has rank $m = min(n, m)$. Here $\{(1^k, 1^k) : 1 \le k \le m\}$ are the fundamental highest weights and $\xi_1, \ldots, \xi_m$ are the fundamental highest weight vectors. Indeed, the determinant of a matrix is an irreducible polynomial in the matrix entries.
- For the more conventional (untwisted) action (4.1.1), the decomposition of $\mathbb{C}[M_{n,m}(\mathbb{C})]$ is as in Theorem 4.1.1, but now ξ_μ is a highest weight vector for the *opposite* Borel with weight $(-\mu, -\mu)$.
- The group $GL(n, \mathbb{C}) \times GL(m, \mathbb{C})$ can be replaced by $SL(n, \mathbb{C}) \times GL(m, \mathbb{C})$ or $SL(n, \mathbb{C}) \times SL(m, \mathbb{C}) \times \mathbb{C}^\times$.
- If we restrict to $SL(n, \mathbb{C}) \times SL(m, \mathbb{C})$ then the action remains multiplicity free *provided* $n \ne m$. The Borel subalgebra $\mathfrak{b}'$ consists of matrices X as in (4.1.3) where $tr(A) + tr(C) = 0 = tr(D)$. Equation 4.1.4 shows $\mathfrak{b}' \cdot v_\circ = M_{n,m}(\mathbb{C})$ when $n > m$.

- When $n = m$, the action $\mathbf{SL(n)} \otimes \mathbf{SL(n)}$ fails to be multiplicity free. Indeed, $det : M_{n,n}(\mathbb{C}) \to \mathbb{C}$ is a non-constant $(SL(n,\mathbb{C}) \times SL(n,\mathbb{C}))$-invariant polynomial. So $\mathbb{C}[M_{n,n}(\mathbb{C})]$ contains two copies of the trivial representation of $SL(n,\mathbb{C}) \times SL(n,\mathbb{C})$.
- When $m = 1$ this example reduces to the action of $GL(n,\mathbb{C})$ on $\mathbb{C}^n$ by the defining representation (or its twisted form, the contragredient representation on $(\mathbb{C}^n)^*$). In this case the decomposition in Theorem 4.1.1 reduces to

$$\mathbb{C}[z_1,\ldots,z_n] = \bigoplus_{k=0}^{\infty} \mathcal{P}_k(\mathbb{C}^n).$$

The action has rank one with fundamental highest weight (1^1) and z_1^k is a highest weight vector in $\mathcal{P}_k(\mathbb{C}^n)$.

4.2. $\mathbf{S^2(GL(n))}$. Next we consider the action of $GL(n,\mathbb{C})$ on the symmetric 2-tensors $S^2(\mathbb{C}^2)$ via the symmetric square of the defining representation. Identifying $S^2(\mathbb{C}^n)$ with the complex $n \times n$-symmetric matrices

$$Sym(n,\mathbb{C}) = \{A \in M_{n,n}(\mathbb{C}) \ : \ A^t = A\}$$

our action reads

$$g \cdot v = gvg^t. \tag{4.2.1}$$

As in the preceding example we prefer to twist the action by $g \mapsto (g^{-1})^t$. This gives (4.2.2)

$$g \cdot v = (g^{-1})^t v g^{-1}, \quad g \cdot p(v) = p(g^t v g) \quad \text{for } v \in Sym(n,\mathbb{C}),\, p \in \mathbb{C}[Sym(n,\mathbb{C})].$$

As for $\mathbf{GL(n)} \otimes \mathbf{GL(m)}$, twisting ensures that all weights $\lambda = (\lambda_1,\ldots,\lambda_n)$ that occur in $\mathbb{C}[Sym(n,\mathbb{C})]$ are non-negative. Again we let

$$\xi_\lambda = \xi_1^{\lambda_1-\lambda_2} \cdots \xi_{n-1}^{\lambda_{n-1}-\lambda_n} \xi_n^{\lambda_n}$$

for dominant weights $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, where ξ_k is a leading minor determinant restricted to $Sym(n,\mathbb{C}) \subset M_{n,n}(\mathbb{C})$.

Theorem 4.2.1. *The action of $GL(n,\mathbb{C})$ on $Sym(n,\mathbb{C})$ is multiplicity free. We have the decomposition*

$$\mathbb{C}[Sym(n,\mathbb{C})] = \bigoplus_{\lambda} P_\lambda$$

where the sum is over all $\lambda \in \mathbb{N}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. The polynomial ξ_λ is a highest weight vector in P_λ with weight 2λ. Moreover,

$$\mathcal{P}_k(Sym(n,\mathbb{C})) = \bigoplus_{|\lambda|=k} P_\lambda.$$

Proof. The proof parallels that for Theorem 4.1.1. First we note that for the derived action of $gl(n,\mathbb{C})$ one has

$$X \cdot I = -(X^t + X).$$

We obtain all $n \times n$-symmetric matrices as X ranges over all upper triangular matrices. Thus $\mathfrak{b}_n \cdot I = Sym(n,\mathbb{C})$ and hence $v_\circ = I$ has an open Borel orbit. This proves that our action is multiplicity free.

Suppose that $f \in \mathbb{C}[Sym(n,\mathbb{C})]$ is a highest weight vector with weight μ. For $h = diag(h_1, \ldots, h_n) \in H_n$,

$$f(h^2) = (h \cdot f)(I) = h^\mu f(I) = h_1^{\mu_1} \cdots h_n^{\mu_n} f(I).$$

Now for $h = diag(\pm 1, \ldots, \pm 1)$ we have $h^2 = I$ and hence $f(I) = h^\mu f(I)$ for all such h. As $B_n \cdot I$ is open in $Sym(n,\mathbb{C})$, f is determined by the value $f(I)$ and we must have $f(I) \neq 0$. We conclude that $h^\mu = 1$ for all $h = diag(\pm 1, \ldots, \pm 1)$ and thus each μ_j must be even. (Note that $H_\circ = \{diag(\pm 1, \ldots, \pm 1)\}$ is the stabilizer of $v_\circ = I$ in H_n.)

Since all weights that occur in $\mathbb{C}[Sym(n,\mathbb{C})]$ are non-negative we conclude that all dominant weights μ that occur have the form

$$\mu = 2\lambda = (2\lambda_1, \ldots, 2\lambda_n)$$

for some $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. To complete the proof, one checks that the polynomial ξ_λ is a weight vector with weight 2λ. Indeed for $h \in H_n$ and $1 \leq k \leq n$,

$$h \cdot \xi_k(z) = \xi_k\big([h_i h_j z_{ij}]\big) = h_1^2 \cdots h_k^2 \xi(z),$$

so ξ_k is a (2^k)-weight vector. ☐

As regards this example we note:

- The action has rank n with fundamental highest weights (2^k) and fundamental highest weight vectors ξ_k, $1 \leq k \leq n$. It is known that the determinant of a symmetric matrix is irreducible as a polynomial in the entries z_{ij} with $i \leq j$.
- For the untwisted action (4.2.1), the decomposition of $\mathbb{C}[Sym(n,\mathbb{C})]$ is as in Theorem 4.2.1 but now ξ_λ is a highest weight vector for the opposite Borel with weight -2λ.
- $\mathbf{S^2(SL(n))}$ is not a multiplicity free action because $det \in \mathbb{C}[Sym(n,\mathbb{C})]$ is a non-constant $SL(n,\mathbb{C})$-invariant.

4.3. $\mathbf{\Lambda^2(GL(n))}$. We now turn to the action of $GL(n,\mathbb{C})$ on $\Lambda^2(\mathbb{C}^n)$, the skew-symmetric 2-tensors. Identifying $\Lambda^2(\mathbb{C}^n)$ with

$$Skew(n,\mathbb{C}) = \{A \in M_{n,n}(\mathbb{C}) \;:\; A^t = -A\},$$

the action is given by Equation 4.2.1. As in the preceding examples, we will employ the twisted version, as in Equation 4.2.2.

The leading minor determinant ξ_j vanishes on $Skew(n,\mathbb{C})$ when j is odd and is a perfect square when j is even. We let $\nu_j \in \mathbb{C}[Skew(n,\mathbb{C})]$ denote the Pfaffian polynomial:

$$\nu_j(z) = Pf \begin{bmatrix} z_{1,1} & \dots & z_{1,2j} \\ \vdots & & \vdots \\ z_{2j,1} & \dots & z_{2j,2j} \end{bmatrix},$$

the square root of the principal $2j \times 2j$ subdeterminant. So $deg(\nu_j) = j$ and $\nu_j^2 = \xi_{2j}$ for $j = 1, \dots, \lfloor n/2 \rfloor$. The polynomials ν_j are given explicitly by the formula

$$\nu_j(z) = \frac{1}{j!2^j} \sum_{\sigma \in S_{2j}} sign(\sigma) \prod_{\ell=1}^{j} z_{\sigma(2\ell-1)\sigma(2\ell)}.$$

Let $m = \lfloor n/2 \rfloor$ and write

$$\nu_\lambda = \nu_1^{\lambda_1-\lambda_2} \cdots \nu_{m-1}^{\lambda_{m-1}-\lambda_m} \nu_m^{\lambda_m}$$

for $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$.

Theorem 4.3.1. *The action of $GL(n,\mathbb{C})$ on $Skew(n,\mathbb{C})$ is multiplicity free. We have the decomposition*

$$\mathbb{C}[Skew(n,\mathbb{C})] = \bigoplus_\lambda P_\lambda$$

where the sum is over all $\lambda \in \mathbb{N}^m$ with $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$. The polynomial ν_λ is a highest weight vector in P_λ with weight

$$\widetilde{\lambda} = (\lambda_1, \lambda_1, \lambda_2, \lambda_2, \dots, \lambda_m, \lambda_m).$$

Moreover,

$$\mathcal{P}_k(Skew(n,\mathbb{C})) = \bigoplus_{|\lambda|=k} P_\lambda.$$

Proof. Let $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and let $\widetilde{J}$ be the $n \times n$-matrix

$$\widetilde{J} = \underbrace{J \oplus \cdots \oplus J}_{m\text{-times}} \text{ for } n = 2m \text{ even and } \widetilde{J} = \underbrace{J \oplus \cdots \oplus J}_{m\text{-times}} \oplus 0 \text{ for } n = 2m+1 \text{ odd.}$$

We claim that $\widetilde{J}$ has an open B_n-orbit in $Skew(n,\mathbb{C})$. Hence our action is multiplicity free. In fact, for $X \in gl(n,\mathbb{C})$,

$$X \cdot \widetilde{J} = -(X^t \widetilde{J} + \widetilde{J} X) = -(\widetilde{J} X - (\widetilde{J} X)^t).$$

First suppose that $n = 2m$ and let $X \in \mathfrak{b}_n$. Decompose X into 2×2-blocks X_{ij}, $1 \leq i, j \leq m$. Now $\widetilde{J} X$ has 2×2-blocks JX_{ij}, which are arbitrary for $i < j$. Taking $X_{ii} = \begin{bmatrix} 0 & 0 \\ 0 & c_i \end{bmatrix}$ gives $JX_{ii} = \begin{bmatrix} 0 & c_i \\ 0 & 0 \end{bmatrix}$. This shows that an arbitrary strictly upper triangular matrix can be written in the form $\widetilde{J}X$ for some $X \in \mathfrak{b}_n$. So $\mathfrak{b}_n \cdot \widetilde{J} = Skew(n, \mathbb{C})$ when n is even. When $n = 2m+1$ the same argument shows that for suitable entries in the first $n-1$ rows and columns of $X \in \mathfrak{b}_n$, $\widetilde{J}X$ contains any desired strictly upper triangular matrix in its first $n-1$ rows and columns. As the last column of $\widetilde{J}X$ is

$$\begin{bmatrix} x_{2,n} \\ -x_{1,n} \\ \vdots \\ x_{2m,n} \\ -x_{2m-1,n} \\ 0 \end{bmatrix}$$

we conclude that $\mathfrak{b}_n \cdot \widetilde{J} = Skew(n, \mathbb{C})$ when n is odd.

Suppose that μ is a highest weight that occurs in $\mathbb{C}[Skew(n, \mathbb{C})]$ and that f is a μ-weight vector. Since all weights in $\mathbb{C}[Skew(n, \mathbb{C})]$ are non-negative, we have $\mu_1 \geq \cdots \geq \mu_n \geq 0$. Moreover $f(\widetilde{J}) \neq 0$ as $B_n \cdot \widetilde{J}$ is open. For $h \in H_n$ we have

$$h^\mu f(\widetilde{J}) = (h \cdot f)(\widetilde{J}) = f(h\widetilde{J}h).$$

But

$$h\widetilde{J}h = \begin{bmatrix} h_1h_2J & & \\ & h_3h_4J & \\ & & \ddots \end{bmatrix}$$

so that $h\widetilde{J}h = \widetilde{J}$ for elements $h = diag(h_1, \ldots, h_n)$ of the form

$$h = \begin{cases} diag(h_1, h_1^{-1}, h_2, h_2^{-1}, \ldots, h_m, h_m) & \text{for } n = 2m \\ diag(h_1, h_1^{-1}, h_2, h_2^{-1}, \ldots, h_m, h_m, h_{2m+1}) & \text{for } n = 2m+1 \end{cases}.$$

It follows that $h^\mu = 1$ for all such h and thus

$$\mu_1 = \mu_2, \mu_3 = \mu_4, \ldots, \mu_{2m-1} = \mu_{2m} \text{ and } \mu_n = 0 \text{ when } n \text{ is odd.}$$

Thus all highest weights occurring in $\mathbb{C}[Skew(n, \mathbb{C})]$ have the form

$$\widetilde{\lambda} = (\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_m, \lambda_m)$$

for some $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$. To complete the proof one just needs to check that ν_λ has weight $\widetilde{\lambda}$. This reduces to the calculation

$$\begin{aligned}
(h\cdot\nu_j)(z) &= Pf\left(\begin{bmatrix} h_1 & & \\ & \ddots & \\ & & h_{2j}\end{bmatrix}\begin{bmatrix} z_{1,1} & \cdots & z_{1,2j} \\ \vdots & & \vdots \\ z_{2j,1} & \cdots & z_{2j,2j}\end{bmatrix}\begin{bmatrix} h_1 & & \\ & \ddots & \\ & & h_{2j}\end{bmatrix}\right) \\
&= det\begin{bmatrix} h_1 & & \\ & \ddots & \\ & & h_{2j}\end{bmatrix} Pf\begin{bmatrix} z_{1,1} & \cdots & z_{1,2j} \\ \vdots & & \vdots \\ z_{2j,1} & \cdots & z_{2j,2j}\end{bmatrix} \\
&= h_1\cdots h_{2j} Pf\begin{bmatrix} z_{1,1} & \cdots & z_{1,2j} \\ \vdots & & \vdots \\ z_{2j,1} & \cdots & z_{2j,2j}\end{bmatrix},
\end{aligned}$$

which shows that ν_j is a weight vector with weight (1^{2j}). □

Regarding the action $\mathbf{\Lambda^2(GL(n))}$ we note:

- The Pfaffian of a skew symmetric matrix is irreducible as a polynomial in the entries z_{ij} with $i < j$. Thus $\mathbf{\Lambda^2(GL(n))}$ has rank $m = \lfloor n/2 \rfloor$ with fundamental highest weights (1^{2j}) and fundamental highest weight vectors ν_j.
- The untwisted action has the same decomposition but now ν_λ is a lowest weight vector with weight $-\widetilde{\lambda}$.
- For the trace zero upper triangular matrices $\mathfrak{b}'_n$, one sees, as in the proof of Theorem 4.3.1, that $\mathfrak{b}'_n \cdot \widetilde{J} = Skew(n,\mathbb{C})$ when n is odd. So $\mathbf{\Lambda^2(SL(2m+1))}$ is a multiplicity free action. On the other hand, this fails for $n = 2m$ even. Indeed, $det \in \mathbb{C}[Skew(2m,\mathbb{C})]$ is a non-constant $SL(2m,\mathbb{C})$-invariant so $\mathbf{\Lambda^2(SL(2m))}$ is not multiplicity free.

4.4. $\mathbf{SO(n)} \times \mathbb{C}^\times$**.** The group $G = SO(n,\mathbb{C}) \times \mathbb{C}^\times$ acts on $V = \mathbb{C}^n$ in the usual way:

$$(g,t)\cdot v = tgv.$$

The decomposition for $\mathbb{C}[V] = \mathbb{C}[z_1,\ldots,z_n]$ is given by the classical theory of spherical harmonics. Let

$$\varepsilon(z) = z_1^2 + \cdots + z_n^2$$

and

$$\Delta = \varepsilon(\partial) = \partial_1^2 + \cdots + \partial_n^2,$$

the constant coefficient differential operator with Wick symbol $p(z,w) = \varepsilon(w)$. Since ε is an $SO(n,\mathbb{C})$-invariant polynomial, Δ is an $SO(n,\mathbb{C})$-invariant operator. It follows that the space of *harmonic polynomials*

$$\mathcal{H} = Ker(\Delta) = \{p \in \mathbb{C}[V] \ : \ \Delta p = 0\}$$

is $SO(n,\mathbb{C})$-invariant, as is

$$\mathcal{H}_m = \mathcal{H} \cap \mathcal{P}_m(V),$$

the harmonic polynomials homogeneous of degree m. It is well known that $\mathcal{H}_m$ is an irreducible module for $SO(n,\mathbb{C})$ and

$$\mathcal{P}_m(V) = \mathcal{H}_m \oplus \varepsilon\mathcal{P}_{m-2}(V). \tag{4.4.1}$$

This gives, in particular,

$$dim(\mathcal{H}_m) = \dim(\mathcal{P}_m) - dim(\mathcal{P}_{m-2}) = \binom{m+n-1}{m} - \binom{m+n-3}{m-2} \text{ for } m \geq 2.$$

Now Equation 4.4.1 leads inductively to the decomposition

$$\mathcal{P}_m(V) = \bigoplus_{k+2\ell=m} P_{k,\ell} \qquad \left(P_{k,\ell} = \mathcal{H}_k\varepsilon^\ell\right) \tag{4.4.2}$$

of $\mathcal{P}_m(V)$ into irreducible $SO(n,\mathbb{C})$-modules. The modules $\{P_{k,\ell} \;:\; k+2\ell = m\}$ appearing in (4.4.2) are clearly inequivalent because their dimensions are distinct.

Equation 4.4.2 now gives a decomposition

$$\mathbb{C}[V] = \bigoplus_{k,\ell\geq 0} P_{k,\ell} \tag{4.4.3}$$

of $\mathbb{C}[V]$ into irreducibles for $G = SO(n,\mathbb{C})$. This is *not*, however, multiplicity free because $\mathcal{H}_{k,\ell} \simeq \mathcal{H}_{k,\ell'}$ as $SO(n,\mathbb{C})$-modules for $\ell \neq \ell'$. We use the scalars $\mathbb{C}^\times$ to repair this defect.

The action of $\mathbb{C}^\times$ on $\mathbb{C}[V]$ by scalars commutes with Δ and hence preserves the $\mathcal{H}_m$'s and $\mathcal{P}_{k,\ell}$'s. Now $\mathbb{C}^\times$ acts on $P_{k,\ell}$ by the character $t \mapsto t^{-(k+2\ell)}$. Hence $P_{k,\ell}$ and $P_{k,\ell'}$ are inequivalent as $\mathbb{C}^\times$-modules when $\ell \neq \ell'$. Thus (4.4.3) is multiplicity free as a decomposition for the group $G = SO(n,\mathbb{C}) \times \mathbb{C}^\times$.

A highest weight vector in $P_{k,\ell}$ is given by $(z_1 + iz_2)^k\varepsilon(z)^\ell$, for a suitably chosen Borel subgroup in $G = SO(n,\mathbb{C}) \times \mathbb{C}^\times$. The multiplicity free action $G : \mathbb{C}^n$ has rank 2 with fundamental highest weight vectors $z_1 + iz_2$ and $\varepsilon(z)$.

4.5. $\mathbf{GL(n)} \oplus_{\mathbf{GL(n)}} \mathbf{\Lambda^2(GL(n))}$. The group $G = GL(n,\mathbb{C})$ acts diagonally on

$$V = V_1 \oplus V_2 = \mathbb{C}^n \oplus \Lambda^2(\mathbb{C}^n) \cong \mathbb{C}^n \oplus Skew(n,\mathbb{C}).$$

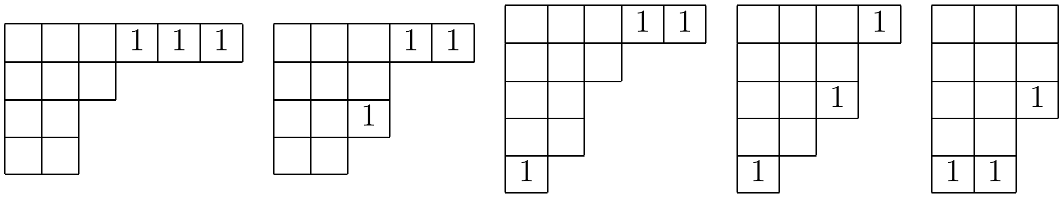
For consistency with Sections 4.1 and 4.3 we twist the action of G on both V_1 and V_2. We have seen how to decompose $\mathbb{C}[V_1]$ and $\mathbb{C}[V_2]$ under the action of G. Writing $\sigma^\mu = \sigma^\mu_n$ for the irreducible representation of G with highest weight $\mu \geq 0$ we have

$$\mathbb{C}[V_1] \simeq \bigoplus_k \sigma^{(k)}, \qquad \mathbb{C}[V_2] = \bigoplus_\lambda \sigma^{\widetilde{\lambda}}$$

as G-modules. Here the second sum is over all $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m \geq 0)$ $(m = \lfloor n/2 \rfloor)$ and $\widetilde{\lambda} = (\lambda_1, \lambda_1, \ldots, \lambda_m, \lambda_m)$. Thus we can write

$$\mathbb{C}[V] = \mathbb{C}[V_1] \otimes \mathbb{C}[V_2] \simeq \sum_{k,\lambda} \sigma^{(k)} \otimes \sigma^{\widetilde{\lambda}}.$$

Tensor product representations of $GL(n, \mathbb{C})$ can be decomposed using the *Littlewood-Richardson rules*. To apply this technique, one identifies each highest weight $\mu \geq 0$ with a Young's diagram consisting of μ_j boxes on row j. The Littlewood-Richardson rules ensure that $\sigma^{(k)} \otimes \sigma^{\widetilde{\lambda}}$ has a multiplicity free decomposition. Moreover, the representations σ^{μ} that occur in the decomposition are given by diagrams that can be obtained from that for $\widetilde{\lambda}$ by adding k boxes, no two of which fall in the same column. For example, when $\lambda = (3,2)$, and $k = 3$, we must add three boxes to the Young's diagram for $\widetilde{\lambda} = (3,3,2,2)$. For $n \geq 5$ this produces five diagrams:



The three boxes added to the diagram for $(3,3,2,2)$ have been marked with 1's in each case. This exercise with the Littlewood-Richardson rules shows

$$\sigma^{(3)} \otimes \sigma^{(3,3,2,2)} \simeq \sigma^{(6,3,2,2)} \oplus \sigma^{(5,3,3,2)} \oplus \sigma^{(5,3,2,2,1)} \oplus \sigma^{(4,3,3,2,1)} \oplus \sigma^{(3,3,3,2,2)}.$$

In general, the Littlewood-Richardson rules yield

$$\sigma^{(k)} \otimes \sigma^{\widetilde{\lambda}} = \sum_{\mu} \sigma^{\mu}$$

where the sum is over all highest weights $\mu \geq 0$ of the form

$$\mu = (\lambda_1 + j_1, \lambda_1, \lambda_2 + j_2, \lambda_2, \dots), \quad j_1 + \cdots + j_m = k, \quad j_i \leq \lambda_{i-1} - \lambda_i.$$

Note that λ can be recovered from μ by extracting every other row. This shows that the representations σ^{μ} that occur in $\sigma^{(k)} \otimes \sigma^{\widetilde{\lambda}}$ and in $\sigma^{(k')} \otimes \sigma^{\widetilde{\lambda}'}$ are distinct for $\lambda' \neq \lambda$. Moreover, the irreducibles σ^{μ} in $\sigma^{(k)} \otimes \sigma^{\widetilde{\lambda}}$ and $\sigma^{(k')} \otimes \sigma^{\widetilde{\lambda}}$ are clearly distinct when $k \neq k'$ as these satisfy $|\mu| = 2|\lambda| + k$ and $|\mu| = 2|\lambda| + k'$ respectively. This shows that $\mathbb{C}[V]$ has a multiplicity free decomposition under the action of G. Note that an arbitrary highest weight $\mu \geq 0$ can be written in the form "$\mu = (\lambda_1 + j_1, \lambda_1, \lambda_2 + j_2, \lambda_2, \dots)$" by letting $j_\ell = \mu_{2\ell-1} - \mu_{2\ell}$. We have proved the following.

Theorem 4.5.1. *The diagonal action of $GL(n, \mathbb{C})$ on $\mathbb{C}^n \oplus \Lambda^2(\mathbb{C}^n)$ is multiplicity free. Moreover*

$$\mathbb{C}[\mathbb{C}^n \oplus \Lambda^2(\mathbb{C}^n)] \simeq \bigoplus_{\mu \geq 0} \sigma^{\mu}$$

as a $GL(n, \mathbb{C})$-module. That is, every non-negative highest weight occurs in $\mathbb{C}[\mathbb{C}^n \oplus \Lambda^2(\mathbb{C}^n)]$ with multiplicity one.

There is another viewpoint on this example. One can identify $V = V_1 \oplus V_2 = \mathbb{C}^n \oplus \Lambda^2(\mathbb{C}^n)$ with $Skew(n+1, \mathbb{C})$ by regarding the first row (or column) of an $(n+1) \times (n+1)$ skew symmetric matrix as an element of $\mathbb{C}^n$, and the remaining entries as an element of $\Lambda^2(\mathbb{C}^n) \cong Skew(n,\mathbb{C})$. For $z \in Skew(n+1,\mathbb{C})$, we write z' for the element of $Skew(n,\mathbb{C})$ obtained by removing the first row and column of z. Under this identification, the diagonal action of $GL(n,\mathbb{C})$ on $V_1 \oplus V_2$ is realized on $Skew(n+1,\mathbb{C})$ by restricting the action of $GL(n+1,\mathbb{C})$ to the subgroup $GL(n,\mathbb{C}) \subset GL(n+1,\mathbb{C})$, embedded as

$$\left\{ \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array} \right] \;:\; A \in GL(n,\mathbb{C}) \right\}.$$

The fundamental highest weight vectors arising from the separate actions of $GL(n,\mathbb{C})$ on V_1 and V_2 are z_{12} and $\nu_k'(z)$ for $k = 1,\ldots,\lfloor n/2 \rfloor$, where $\nu_k'(z)$ is the Pfaffian of the first $2k$ rows and columns of the $n \times n$ matrix z'. There are additional fundamental highest weight vectors, $\nu_k(z)$ for $k = 1,\ldots,\lfloor (n+1)/2 \rfloor$. These are the Pfaffians of the first $2k$ rows and columns of the $(n+1)\times(n+1)$ matrix z. Note that $\nu_1(z) = z_{12}$, so our fundamental highest weight vectors are the ν_k's together with the ν_k''s. For a highest weight $\mu \geq 0$, a μ-highest weight vector in $\mathbb{C}[V]$ is

$$\nu_1^{\mu_1-\mu_2}(\nu_1')^{\mu_2-\mu_3}\nu_2^{\mu_3-\mu_4}\cdots\nu_m^{\mu_{n-1}-\mu_n}(\nu_m')^{\mu_n}$$

when $n = 2m$ is even, and

$$\nu_1^{\mu_1-\mu_2}(\nu_1')^{\mu_2-\mu_3}\nu_2^{\mu_3-\mu_4}\cdots\nu_m^{\mu_{n-2}-\mu_{n-1}}(\nu_m')^{\mu_{n-1}-\mu_n}(\nu_{m+1})^{\mu_n}$$

when $n = 2m+1$ is odd. In particular, this is a rank n multiplicity free action with fundamental highest weights $\{(1^k) \;:\; 1 \leq k \leq n\}$.

4.6. **Section 4 notes.** The decomposition for action $\mathbf{GL(n)} \otimes \mathbf{GL(m)}$ is called $GL(n) - GL(m)$ *duality*. This decomposition and those for the actions $\mathbf{S^2(GL(n))}$ and $\mathbf{\Lambda^2(GL(n))}$ were popularized by Howe in [22]. Reference [23] explains how these results were rediscovered independently by various mathematicians and are implicit in earlier work of Weyl and Schur. $GL(n) - GL(m)$ duality is equivalent to *Schur duality*, which gives the decomposition for $(\mathbb{C}^n)^{\otimes m}$ under the action of $GL(n,\mathbb{C}) \times S_m$. In fact, as explained in [23], both dualities can be derived from the *First Fundamental Theorem* (FFT) of Invariant Theory.

We refer the reader to Section B.2.6 in [17] for the theory of Pfaffian polynomials. The irreducibility of determinants and Pfaffians on $Sym(n,\mathbb{C})$ and $Skew(n,\mathbb{C})$ is proved in Section B.2.7 of [17]. The theory of spherical harmonics can be found in Section 5.2.3 in [17] as well as many other sources. The Littlewood-Richardson rules are discussed in Section I.9 of [39].

Compact forms for the actions $\mathbf{Gl(n)} \otimes \mathbf{GL(m)}$, $\mathbf{S^2(GL(n))}$, $\mathbf{\Lambda^2(GL(n))}$ and $\mathbf{SO(n)} \times \mathbb{C}^\times$ arise in connection with Hermitian symmetric spaces. In fact, let G/K be an irreducible Hermitian symmetric space of non-compact type. Here G is a semi-simple real Lie group and K is a compact Lie subgroup. The symmetric space structure gives a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the Lie algebra. The complex structure on $T_e(G/K)$ is viewed as an $\mathbb{R}$-linear map $J : \mathfrak{p} \to \mathfrak{p}$ with $J^2 = -I$. Now

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$$

where $\mathfrak{p}_\pm$ are the $(\pm i)$-eigenspaces for J on the complexification of $\mathfrak{p}$. The complexified adjoint action of K on $\mathfrak{g}_\mathbb{C}$ preserves $\mathfrak{p}_\pm$. For the classical irreducible Hermitian symmetric spaces this construction leads to the following actions. (See [20].)

- Type A III: $G/K = SU(n,m)/S(U(n)\times U(m))$. The action of $K = S(U(n)\times U(m))$ on $\mathfrak{p}_+$ can be identified with the action of K on $M_{n,m}(\mathbb{C})$ via $(k,k')\cdot v = kv(k')^*$. The group K has a one dimensional center and we essentially have a compact form of the multiplicity free action $\mathbf{SL(n)} \otimes \mathbf{GL(m)}$. More precisely, the action here agrees with action (4.1.1) twisted by $k' \mapsto ((k')^t)^{-1} = \overline{k'}$ on the second factor.
- Type C I: $G/K = Sp(2n,\mathbb{R})/U(n)$. The action of $K = U(n)$ on $\mathfrak{p}_+$ can be identified with that of K on $Sym(n,\mathbb{C})$ via $k\cdot v = kvk^t$. So this is a compact form of $\mathbf{S^2}(\mathbf{GL(n)})$.
- Type D III: $G/K = SO^*(2n)/U(n)$. The action of $K = U(n)$ on $\mathfrak{p}_+$ can be identified with that of K on $Skew(n,\mathbb{C})$ via $k\cdot v = kvk^t$. This is a compact form of the multiplicity free action $\mathbf{\Lambda^2}(\mathbf{GL(n)})$.
- Type BD I: $G/K = SO_\circ(n,2)/SO(n)\times SO(2)$. The action of $K = SO(n,\mathbb{R})\times \mathbb{T}$ on $\mathfrak{p}_+$ can be identified with the action of K on $\mathbb{C}^n$ via $(k,t)\cdot z = tkz$. This is a compact form of $\mathbf{SO(n)} \times \mathbb{C}^\times$.

The classification of irreducible Hermitian symmetric spaces includes two exceptional cases, in addition to the four families described above.

- Type E III: In this case $K = Spin(10)\times\mathbb{T}$ and G has Lie algebra $\mathfrak{e}_{6(-14)}$, a certain real form for the complex Lie algebra $\mathfrak{e}_6$. One can identify $\mathfrak{p}_+$ with $\Lambda^{\text{even}}(\mathbb{C}^5) \approxeq \mathbb{C}^{16}$ and $Spin(10)$ acts by the positive half-spin representation. It is known that (with the scalars included) this action in multiplicity free.
- Type E VII: Here $K = E_6\times\mathbb{T}$ and G has Lie algebra $\mathfrak{e}_{7(-25)}$, a real form of $\mathfrak{e}_7$. In this case $\mathfrak{p}_+$ can be identified with an exceptional Jordan algebra $\mathcal{J}$ of dimension 27. The representation of E_6 on $\mathcal{J}$ is described in [11]. This action (with the scalars included) is multiplicity free.

Thus one has:

Theorem 4.6.1. *(*[25]*) The linear action $K : \mathfrak{p}_+$ associated to any irreducible Hermitian symmetric space of non-compact type is multiplicity free.*

5. A recursive criterion for multiplicity free actions

In this section we present a recursive criterion for multiplicity free actions due to Knop [31]. Given a linear action $G : V$ we let

$$\Psi = \Psi(V) \subset \mathfrak{h}^*$$

be the set of all weights for the representation of H on V, listed with multiplicities. As usual, $B = HN$ is a Borel subgroup and $\Delta = \Delta^+ \cup (-\Delta^+)$ are the roots for G, as in Section 1.4.

For a *highest* weight $\lambda \in \Psi$ let

$$S_\lambda = \{\alpha \in \Delta^+ \ : \ \langle \lambda, \alpha \rangle > 0\}.$$

If $z \in V$ is a λ-highest weight vector then $\mathfrak{g} \cdot z = \mathbb{C}z$ if and only if $S_\lambda = \emptyset$.

If $S_\lambda = \emptyset$ for all highest weights $\lambda \in \Psi$, then G acts by scalars on every weight space in V. In this case, we essentially have a torus action. In particular, $G : V$ is multiplicity free if and only if $H : V$ is multiplicity free. This happens if and only if the set Ψ of weights for $H : V$ is linearly independent.

Suppose that $\lambda \in \Psi$ is a highest weight with $S_\lambda \neq \emptyset$. Let $z_\circ \in V$ be a λ-highest weight vector and let $f_\circ \in V^*$ be the corresponding $(-\lambda)$-lowest weight vector normalized so that $f_\circ(z_\circ) = 1$. Let $P = P_{f_\circ}$ and $\Sigma = \Sigma_{f_\circ}$ be as in Equations 3.2.1 and 3.2.2. Now $P^- = LU^-$ is the opposite parabolic subgroup to $P = LU$ and $P^- = P_{z_\circ}$.

From the definition of Σ one has that $z \in \Sigma$ if and only if $f_\circ(z) \neq 0$ and $(X \cdot f_\circ)(z) = -\lambda(X) f_\circ(z)$ for all $X \in \mathfrak{g} = \mathfrak{p} + \mathfrak{u}^-$. But $X \cdot f_\circ = -\lambda(X) f_\circ$ for $X \in \mathfrak{p}$, so

$$\Sigma = \{z \in V \ : \ f_\circ(z) \neq 0 \text{ and } (\mathfrak{u}^- \cdot f_\circ)(z) = 0\}.$$

Hence Σ is a open set in the subspace

$$W = (\mathfrak{u}^- \cdot f_\circ)^\perp$$

of V. Recall that Σ is invariant under the action of the Levi component L of P. As $U \cdot \Sigma = V_\circ = \{z \in V \ : \ f_\circ(z) \neq 0\}$ by Lemma 3.2.5, we see that there is an open B-orbit in V if and only if there is an open $L \cap B$-orbit in Σ. Equivalently, $G : V$ is multiplicity free if and only if $L : W$ is multiplicity free.

We know, moreover, that $\mathfrak{l} = \mathfrak{h} + \sum_{\alpha \in \Delta(L)} \mathfrak{g}_\alpha$ where

$$\Delta(L) = \{\alpha \in \Delta \ : \ \langle \lambda, \alpha \rangle = 0\} = \Delta \backslash S_\lambda.$$

The positive roots for L are $\Delta^+(L) = \Delta^+ \backslash S_\lambda$. Each of the root vectors $\{X_{-\alpha} \ : \ \alpha \in S_\lambda\} \subset \mathfrak{u}^-$ acts non-trivially on $f_\circ$, so the set of weights in $\mathfrak{u}^- \cdot f_\circ$ is $\{-\lambda + \alpha \ : \ \alpha \in S_\lambda\}$. Thus the set of weights in $W = (\mathfrak{u}^- \cdot f_\circ)^\perp$ is $\Psi \backslash (\lambda - S_\lambda)$.

In summary, we have a recursive algorithm that begins with the pair

$$(\Delta_0^+ = \Delta^+, \Psi_0 = \Psi)$$

where

- Δ^+ is the set of positive roots for G, and
- $\Psi \subset \mathfrak{h}^*$ is the set of all weights for the representation of G on V, listed with multiplicity.

Given a pair (Δ_n^+, Ψ_n) do the following:

- For each highest weight $\lambda \in \Psi_n$ let $S_\lambda = \{\alpha \in \Delta_n^+ \ : \ \langle \lambda, \alpha \rangle > 0\}$;
- If $S_\lambda = \emptyset$ for all highest weights $\lambda \in \Psi_n$, then $G : V$ is a multiplicity free action if and only if Ψ_n is linearly independent;
- Otherwise, choose a highest weight $\lambda \in \Psi_n$ with $S_\lambda \neq \emptyset$ and apply the above steps to the pair $(\Delta_{n+1}^+, \Psi_{n+1}) = (\Delta_n^+ \backslash S_\lambda, \Psi_n \backslash (\lambda - S_\lambda))$.

Eventually all the S_λ's are empty and the algorithm terminates at the second step above.

To illustrate this method, we revisit some of the examples described in Section 4.

5.1. $\mathbf{GL(n)}$. Here $G = GL(n,\mathbb{C})$ acts on $V = \mathbb{C}^n$ as usual. One has

$$\Delta_0^+ = \Delta^+ = \{\varepsilon_i - \varepsilon_j \ : \ i < j\}, \quad \Psi_0 = \Psi = \{\varepsilon_1, \ldots, \varepsilon_n\}.$$

We know $e_1 = (1, 0, \ldots, 0)$ is a highest weight vector in V with weight ε_1, and so

$$S_{\varepsilon_1} = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_1 - \varepsilon_n\}.$$

The coordinate vector $z_1 \in V^*$ dual to e_1 has parabolic subgroup $P = P_{z_1}$ with Lie algebra $\mathfrak{p}$ spanned by $\{E_{11}\} \cup \{E_{ij} \ : \ i > 2\}$. The Levi component is $L = GL(1,\mathbb{C}) \times GL(n-1,\mathbb{C})$. The nilpotent $\mathfrak{u}^- = Span\{E_{1j} \ : \ j \geq 2\}$ gives $\mathfrak{u}^- z_1 = Span\{z_2, \ldots, z_n\}$ and hence $W = (\mathfrak{u}^- z_1)^\perp = \mathbb{C}e_1$. Thus we have

$$\Delta_1^+ = \Delta_0^+ \backslash S_{\varepsilon_1} = \{\varepsilon_i - \varepsilon_j \ : \ 2 \leq i < j\}, \quad \Psi_1 = \Psi_0 \backslash \{\varepsilon_2, \ldots, \varepsilon_n\} = \{\varepsilon_1\}.$$

Since $S_{\varepsilon_1} = \emptyset$, the process terminates. One concludes that $G : V$ is multiplicity free since Ψ_1 is a linearly independent set.

In practice, it is not necessary to identify L and W at each stage in the induction. We illustrate this in the next example.

5.2. $\mathbf{GL(n)} \otimes \mathbf{GL(n)}$. Here $G = GL(n,\mathbb{C}) \times GL(n,\mathbb{C})$ acts on $V = \mathbb{C}^n \otimes \mathbb{C}^n$. Now

$$\Delta_0^+ = \Delta^+ = \{\varepsilon_i - \varepsilon_j \ : \ 1 \leq i < j \leq n\} \cup \{\varepsilon_i' - \varepsilon_j' \ : \ 1 \leq i < j \leq n\}$$

and

$$\Psi_0 = \Psi = \{\varepsilon_i + \varepsilon_j' \ : \ 1 \leq i \leq n, 1 \leq j \leq n\}.$$

$\lambda = \varepsilon_1 + \varepsilon_1'$ is the only highest weight in Ψ_0. One has

$$S_{\varepsilon_1+\varepsilon_1'} = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_1 - \varepsilon_n\} \cup \{\varepsilon_1' - \varepsilon_2', \ldots, \varepsilon_1' - \varepsilon_n'\}.$$

Thus

$$\Delta_1^+ = \{\varepsilon_i - \varepsilon_j \ : \ 2 \leq i < j \leq n\} \cup \{\varepsilon_i' - \varepsilon_j' \ : \ 2 \leq i < j \leq n\}$$

and

$$\Psi_1 = \{\varepsilon_1 + \varepsilon_1'\} \cup \{\varepsilon_i + \varepsilon_j' \ : \ 2 \leq i \leq n, 2 \leq j \leq n\}.$$

From these roots and weights we can see that $L : W$ is equivalent to the action of $\mathbb{C}^\times \times GL(n-1,\mathbb{C}) \times \mathbb{C}^\times \times GL(n-1,\mathbb{C})$ on $\mathbb{C} \oplus (\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1})$.

At the next stage of the algorithm one takes $\lambda = \varepsilon_2 + \varepsilon_2'$ and obtains

$$\Delta_2^+ = \{\varepsilon_i - \varepsilon_j \ : \ 3 \leq i < j\} \cup \{\varepsilon_i' - \varepsilon_j' \ : \ 3 \leq i < j\},$$

$$\Psi_2 = \{\varepsilon_1 + \varepsilon_1', \varepsilon_2 + \varepsilon_2'\} \cup \{\varepsilon_i + \varepsilon_j' \ : \ 3 \leq i, j\}.$$

After n steps, the process terminates with

$$\Delta_n^+ = \emptyset, \quad \Psi_n = \{\varepsilon_1 + \varepsilon_1', \ldots, \varepsilon_n + \varepsilon_n'\}.$$

At this point $L \cong (\mathbb{C}^\times)^n \times (\mathbb{C}^\times)^n$ and $S_\lambda = \emptyset$ for all $\lambda \in \Psi_n$. As Ψ_n is linearly independent, $G : V$ is multiplicity free.

5.3. $\mathbf{GL(n)} \oplus_{\mathbf{GL(n)}} \mathbf{\Lambda^2(GL(n))}$. For the diagonal action of $G = GL(n,\mathbb{C})$ on $V = \mathbb{C}^2 \oplus \Lambda^2(\mathbb{C}^n)$ one has

$$\Delta_0^+ = \Delta^+ = \{\varepsilon_i - \varepsilon_j \ : \ i < j\}, \quad \Psi_0 = \Psi = \{\varepsilon_1, \ldots, \varepsilon_n\} \cup \{\varepsilon_i + \varepsilon_j \ : \ i < j\}.$$

Choose $\lambda = \varepsilon_1$, so that $S_\lambda = \{\varepsilon_1 - \varepsilon_j \ : \ 2 \le j\}$ and

$$\Delta_1^+ = \{\varepsilon_i - \varepsilon_j \ : \ 2 \le i < j\}, \quad \Psi_1 = \{\varepsilon_1\} \cup \{\varepsilon_i + \varepsilon_j \ : \ i < j\}.$$

Next choose $\lambda = \varepsilon_1 + \varepsilon_2$ to obtain $S_\lambda = \{\varepsilon_2 - \varepsilon_j \ : \ 3 \le j\}$ and

$$\Delta_2^+ = \{\varepsilon_i - \varepsilon_j \ : \ 3 \le i < j\}, \quad \Psi_2 = \{\varepsilon_1, \varepsilon_1 + \varepsilon_2\} \cup \{\varepsilon_i + \varepsilon_j \ : \ 2 \le i < j\}.$$

After n steps, the process terminates with $\Delta_n^+ = \emptyset$ and

$$\Psi_n = \{\varepsilon_1, \varepsilon_1 + \varepsilon_2, \varepsilon_2 + \varepsilon_3, \ldots, \varepsilon_{n-1} + \varepsilon_n\}.$$

As this is a linearly independent set, one concludes $G : V$ is a multiplicity free action.

Alternatively, one could begin with $\lambda = \varepsilon_1 + \varepsilon_2$ in the first step. This gives $S_\lambda = \{\varepsilon_1 - \varepsilon_j \ : \ 3 \le j\} \cup \{\varepsilon_2 - \varepsilon_j \ : \ 3 \le j\}$ and

$$\Delta_1^+ = \{\varepsilon_1 - \varepsilon_2\} \cup \{\varepsilon_i - \varepsilon_j \ : \ 3 \le i < j\},$$

$$\Psi_1 = \{\varepsilon_1, \ldots, \varepsilon_n\} \cup \{\varepsilon_1 + \varepsilon_2\} \cup \{\varepsilon_i + \varepsilon_j \ : \ 3 \le i < j\}$$

Next one could choose a highest weight in Ψ_1 that was not available in V, namely $\lambda = \varepsilon_3 + \varepsilon_4$. This gives $S_\lambda = \{\varepsilon_3 - \varepsilon_j, \varepsilon_4 - \varepsilon_j \ : \ 5 \le j\}$ and

$$\Delta_2^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_3 - \varepsilon_4\} \cup \{\varepsilon_i - \varepsilon_j \ : \ 5 \le i < j\},$$

$$\Psi_2 = \{\varepsilon_1 \ \ldots, \varepsilon_n\} \cup \{\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_4\} \cup \{\varepsilon_i + \varepsilon_j \ : \ 5 \le i < j\}.$$

We could then continue in this manner (choosing $\lambda = \varepsilon_5 + \varepsilon_6$) or take $\lambda = \varepsilon_1$. The latter choice gives $S_\lambda = \{\varepsilon_1 - \varepsilon_2\}$,

$$\Delta_3^+ = \{\varepsilon_3 - \varepsilon_4\} \cup \{\varepsilon_i - \varepsilon_j \ : \ 5 \le i < j\}, \text{ and}$$

$$\Psi_3 = \{\varepsilon_1, \varepsilon_3, \ldots, \varepsilon_n\} \cup \{\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_4\} \cup \{\varepsilon_i + \varepsilon_j \ : \ 5 \le i < j\}.$$

For n even, the algorithm could terminate with

$$\Psi_n = \{\varepsilon_1, \varepsilon_3, \varepsilon_5, \ldots, \varepsilon_{n-1}\} \cup \{\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_4, \ldots, \varepsilon_{n-1} + \varepsilon_n\}.$$

This example illustrates how different choices for a highest weight $\lambda \in \Psi_k$ can yield different paths through the algorithm and result in different terminal sets.

6. The classification of linear multiplicity free actions

Let (π, V) be a rational representation of some reductive algebraic group G and consider the question of whether or not this gives a multiplicity free action. The answer depends only on the algebraic subgroup $\pi(G)$ of $GL(V)$. One says that two rational representations (π_1, V_1), (π_2, V_2) of groups G_1, G_2 are *geometrically equivalent* if $\pi_1(G_1)$ coincides with $\pi_2(G_2)$ under some isomorphism $V_1 \to V_2$. For example, (π, V) is geometrically equivalent to $(\pi \circ \varphi, V)$ for any $\varphi \in Aut(G)$. Taking $\varphi(g) = (g^t)^{-1}$ on $G = GL(n, \mathbb{C})$, we see that any rational representation of $GL(n, \mathbb{C})$ is geometrically equivalent to its contragredient.

Linear multiplicity free actions have been completely classified up to geometric equivalence. In this section we present the results of this classification.

6.1. **Irreducible multiplicity free actions.** The multiplicity free actions $G : V$ where G acts irreducibly on V were classified by Victor Kac in [26], building on earlier work including that of Sato and Kimura [47]. The open Borel orbit criterion provided the main technique used to achieve this classification.

First suppose that the image $\mathbf{G}$ of G in $GL(V)$ contains a copy of the scalars. In this case, G is reductive but not semisimple. The group $\mathbf{G}$ coincides with the image of $G' \times \mathbb{C}^\times$, where $G' = (G, G)$ is the commutator subgroup in G, a semisimple group with finite center. The possibilites for $\mathbf{G}'$, the image of G' in $GL(V)$, are listed in Table 3, up to geometric equivalence. That is, for each such $\mathbf{G}'$, the joint action of $\mathbf{G}'$ and $\mathbb{C}^\times$ on V is multiplicity free.

Bold faced type is used in Table 3 to indicate a subgroup of $GL(V)$, the notation indicating the representation involved. The first three entries denote the defining representations of $SL(n, \mathbb{C})$, $SO(n, \mathbb{C})$ and $Sp(2n, \mathbb{C})$ on $\mathbb{C}^n$, $\mathbb{C}^n$ and $\mathbb{C}^{2n}$ respectively. $\mathbf{S^2}(\mathbf{SL}(\mathbf{n}))$ and $\mathbf{\Lambda^2}(\mathbf{SL}(\mathbf{n}))$ denote the images in $GL(S^2(\mathbb{C}^n))$ and $GL(\Lambda^2(\mathbb{C}^n))$ of the symmetric and skew-symmetric squares of the defining representation of $SL(n)$. $\mathbf{SL}(\mathbf{n}) \otimes \mathbf{SL}(\mathbf{m})$ denotes the image of the representation of $SL(n) \times SL(m)$ on $\mathbb{C}^n \otimes \mathbb{C}^m$ (outer tensor product of the two defining representations) and similarly for $\mathbf{SL}(\mathbf{n}) \otimes \mathbf{Sp}(\mathbf{2m})$. $\mathbf{Spin}(\mathbf{7})$, $\mathbf{Spin}(\mathbf{9})$ denote the image of the spin representations of $Spin(7, \mathbb{C})$, $Spin(9, \mathbb{C})$ on $\mathbb{C}^8$ and $\mathbb{C}^{16}$ respectively. $\mathbf{Spin}(\mathbf{10})$ indicates the positive spin representation of $Spin(10, \mathbb{C})$ on $\mathbb{C}^{16}$. $\mathbf{G_2}$ and $\mathbf{E_6}$ denote actions on $\mathbb{C}^7$ and $\mathbb{C}^{27}$ respectively. The conditions on n and m in Table 3 are imposed to eliminate redundancies caused by isomorphisms in low dimensions. Note that the actions $\mathbf{SO}(\mathbf{n})$, $\mathbf{S^2}(\mathbf{SL}(\mathbf{n}))$, $\mathbf{\Lambda^2}(\mathbf{SL}(\mathbf{n}))$ and $\mathbf{SL}(\mathbf{n}) \otimes \mathbf{SL}(\mathbf{m})$ were discussed in Section 4.

The scalars $\mathbb{C}^\times$ act on $\mathcal{P}_m(V)$ by the character $t \mapsto t^{-m}$. Thus $G' \times \mathbb{C}^\times : V$ is a multiplicity free action if and only if the representations of G' on each $\mathcal{P}_m(V)$ are multiplicity free. If we remove the scalars, then multiplicities can arise across different degrees of homogeneity in $\mathbb{C}[V]$. This happens, for example, when $G = SO(n, \mathbb{C}) \times \mathbb{C}^\times$, as discussed in Section 4.4. In fact, most of the actions from Table 3 fail to be multiplicity free when the scalars are removed. Table 4 lists the irreducible multiplicity free actions for semisimple groups up to geometric equivalence.

Irreducible multiplicity free actions $\mathbf{G}' \times \mathbb{C}^\times : V$, $\mathbf{G}'$ semisimple

Semisimple Group $\mathbf{G}'$ ($\mathbf{G}' \times \mathbb{C}^\times : V$ is multiplicity free)	Degrees of fundamental highest weight vectors	Rank
$\mathbf{SL(n)}$ $(n \geq 1)$	1	1
$\mathbf{SO(n)}$ $(n \geq 3)$	$1, 2$	2
$\mathbf{Sp(2n)}$ $(n \geq 2)$	1	1
$\mathbf{S^2(SL(n))}$ $(n \geq 2)$	$1, 2, \ldots, n$	n
$\mathbf{\Lambda^2(SL(n))}$ $(n \geq 4)$	$1, 2, \ldots, \lfloor n/2 \rfloor$	$\lfloor n/2 \rfloor$
$\mathbf{SL(n)} \otimes \mathbf{SL(m)}$ $(n, m \geq 2)$	$1, 2, \ldots, \min(n, m)$	$\min(n, m)$
$\mathbf{Sp(2n)} \otimes \mathbf{SL(2)}$ $(n \geq 2)$	$1, 2, 2$	3
$\mathbf{Sp(2n)} \otimes \mathbf{SL(3)}$ $(n \geq 2)$	$1, 2, 2, 3, 3, 4$	6
$\mathbf{Sp(4)} \otimes \mathbf{SL(n)}$ $(n \geq 4)$	$1, 2, 2, 3, 4, 4$	6
$\mathbf{Spin(7)}$	$1, 2$	2
$\mathbf{Spin(9)}$	$1, 2, 2$	3
$\mathbf{Spin(10)}$	$1, 2$	2
$\mathbf{G_2}$	$1, 2$	2
$\mathbf{E_6}$	$1, 2, 3$	3

Table 3

Irreducible multiplicity free actions $\mathbf{G}' : V$, $\mathbf{G}'$ semisimple

Group
$\mathbf{SL(n)}$ $(n \geq 2)$
$\mathbf{Sp(2n)}$ $(n \geq 2)$
$\mathbf{\Lambda^2(SL(2m+1))}$ $(m \geq 2)$
$\mathbf{SL(n)} \otimes \mathbf{SL(m)}$ $(n, m \geq 2, n \neq m)$
$\mathbf{Sp(4)} \otimes \mathbf{SL(n)}$ $(n \geq 5)$
$\mathbf{Spin(10)}$

Table 4

6.2. **Decomposable actions.** Given linear actions $G_1 : V_1$, $G_2 : V_2$, one can form the product action $G_1 \times G_2 : V_1 \oplus V_2$. If $G_1 : V_1$, $G_2 : V_2$ are both multiplicity free with decompositions $\mathbb{C}[V_j] = \bigoplus_{\lambda \in \Lambda_j} P_\lambda$, then $\mathbb{C}[V_1 \oplus V_2] \cong \mathbb{C}[V_1] \otimes \mathbb{C}[V_2]$ decomposes in a multiplicity free fashion

$$\mathbb{C}[V_1 \oplus V_2] = \bigoplus_{\lambda \in \Lambda_1, \lambda' \in \Lambda_2} P_\lambda \otimes P_{\lambda'}$$

under the action of $G_1 \times G_2$. Thus products of multiplicity free actions are multiplicity free.

One says that an action $G : V$ is *decomposable* if it is geometrically equivalent to a product action $G_1 \times G_2 : V_1 \oplus V_2$ with non-zero V_j. Otherwise, we say $G : V$ is *indecomposable*. Each irreducible multiplicity free action is indecomposable, but the converse is far from true. The action $\mathbf{GL(n)} \oplus_{\mathbf{GL(n)}} \mathbf{\Lambda^2}(\mathbf{GL(n)})$ described in Section 4.5 provides one example of an indecomposable multiplicity free action that is not irreducible.

6.3. **Saturated indecomposable multiplicity free actions.** The classification of linear multiplicity free actions was completed independently by the authors [5] and Andrew Leahy [38]. Given a non-irreducible linear action $G : V$, one decomposes V as a direct sum of G-irreducible subspaces

$$V = V_1 \oplus \cdots \oplus V_m.$$

$G : V$ is said to be *saturated* if the image $\mathbf{G}$ of G in $GL(V)$ contains a full torus $(\mathbb{C}^\times)^m$. That is, the dimension of the center in $\mathbf{G}$ equals the number m of irreducible summands. Given $G : V$ one can always form a saturated action $G' \times (\mathbb{C}^\times)^m : V$. This action is multiplicity free if and only if the representation of G' on

$$\mathcal{P}_{k_1}(V_1) \otimes \cdots \otimes \mathcal{P}_{k_m}(V_m)$$

is multiplicity free for each $(k_1, \ldots, k_m)$.

The saturated indecomposable multiplicity free actions consist of the irreducible actions in Table 3 together with the actions listed in Table 5. Each entry in Table 5 denotes the image $\mathbf{G}'$ of the semisimple part G' of G in $GL(V)$. In each case, V has two irreducible summands, $V = V_1 \oplus V_2$, and one simple factor in $\mathbf{G}'$ is acting diagonally. Thus, for example, $\mathbf{SL(n)} \oplus_{\mathbf{SL(n)}} (\mathbf{SL(n)} \otimes \mathbf{SL(m)})$ denotes the image of $SL(n, \mathbb{C}) \times SL(m, \mathbb{C})$ under the representation on $V = V_1 \oplus V_2 = (\mathbb{C}^n) \oplus (\mathbb{C}^n \otimes \mathbb{C}^m)$ where $SL(n, \mathbb{C})$ acts diagonally on V_1 and V_2. $\mathbf{Spin(8)} \oplus_{\mathbf{Spin(8)}} \mathbf{SO(8)}$ denotes the image of the action of $Spin(8, \mathbb{C})$ on $\mathbb{C}^8 \oplus \mathbb{C}^8$ via the direct sum of the positive spin representation with the natural representation (via $SO(8, \mathbb{C})$). The notation $\mathbf{SL(n)}^*$ denotes the contragredient to the defining representation. For each group $\mathbf{G}'$ in Table 5 the saturated action $\mathbf{G}' \times (\mathbb{C}^\times)^2 : V$ is multiplicity free. Together, Tables 3 and 5 classify all saturated indecomposable multiplicity free actions up to geometric equivalence. In particular, the classification shows that a saturated indecomposable multiplicity free action can have at most two irreducible summands.

6.4. **Non-saturated indecomposable multiplicity free actions.** Only one entry in Table 5 remains multiplicity free when the torus $(\mathbb{C}^\times)^2$ is removed, namely

$$(\mathbf{SL(n)} \otimes \mathbf{SL(2)}) \oplus_{\mathbf{SL(2)}} (\mathbf{SL(2)} \otimes \mathbf{SL(m)}) \quad \text{for } n, m \geq 3$$

In addition, for each group $\mathbf{G}'$ in Table 5 one can consider the joint action of $\mathbf{G}' \times \mathbb{C}^\times$ where $\mathbb{C}^\times$ acts on $V = V_1 \oplus V_2$ via

$$t \cdot (v_1, v_2) = (t^a v_1, t^b v_2)$$

Indecomposable non-irreducible saturated multiplicity free actions

Semisimple Group $\mathbf{G}'$ ($\mathbf{G}' \times (\mathbb{C}^\times)^2 : V$ is multiplicity free)	Degrees of fundamental highest weight vectors (rank)
$\mathbf{SL(n)} \oplus_{\mathbf{SL(n)}} \mathbf{SL(n)}$ $(n \geq 2)$	$1, 1, 2$ (3)
$\mathbf{SL(n)} \oplus_{\mathbf{SL(n)}} \mathbf{SL(n)}^*$ $(n \geq 3)$	$1, 1, 2$ (3)
$\mathbf{SL(n)} \oplus_{\mathbf{SL(n)}} \mathbf{\Lambda^2(SL(n))}$ $(n \geq 4)$	$1, 2, \ldots, \lfloor n/2 \rfloor$, $1, 2, \ldots, \lfloor (n+1)/2 \rfloor$ (n)
$\mathbf{SL(n)}^* \oplus_{\mathbf{SL(n)}} \mathbf{\Lambda^2(SL(n))}$ $(n \geq 4)$	$1, 2, \ldots, \lfloor n/2 \rfloor$, $1, 2, \ldots, \lfloor (n-1)/2 \rfloor$ $(n-1)$
$\mathbf{SL(n)} \oplus_{\mathbf{SL(n)}} (\mathbf{SL(n)} \otimes \mathbf{SL(m)})$ $(n, m \geq 2)$	$1, 2, \ldots, \min(n, m)$, $1, 2, \ldots, \min(n, m+1)$ $(\min(n, m) + \min(n, m+1))$
$\mathbf{SL(n)}^* \oplus_{\mathbf{SL(n)}} (\mathbf{SL(n)} \otimes \mathbf{SL(m)})$ $(n \geq 3, m \geq 2)$	$1, 2, \ldots, \min(n, m)$, $1, 2, \ldots, \min(n, m+1)$ $(\min(n, m) + \min(n, m+1))$
$\mathbf{SL(2)} \oplus_{\mathbf{SL(2)}} (\mathbf{SL(2)} \otimes \mathbf{Sp(2n)})$ $(n \geq 2)$	$1, 1, 2, 2, 2$ (5)
$(\mathbf{SL(n)} \otimes \mathbf{SL(2)}) \oplus_{\mathbf{SL(2)}} (\mathbf{SL(2)} \otimes \mathbf{SL(m)})$ $(n, m \geq 2)$	$1, 1, 2, 2, 2$ (5)
$(\mathbf{SL(n)} \otimes \mathbf{SL(2)}) \oplus_{\mathbf{SL(2)}} (\mathbf{SL(2)} \otimes \mathbf{Sp(2m)})$ $(n, m \geq 2)$	$1, 1, 2, 2, 2, 2$ (6)
$(\mathbf{Sp(2n)} \otimes \mathbf{SL(2)}) \oplus_{\mathbf{SL(2)}} (\mathbf{SL(2)} \otimes \mathbf{Sp(2m)})$ $(n, m \geq 2)$	$1, 1, 2, 2, 2, 2, 2$ (7)
$\mathbf{Sp(2n)} \oplus_{\mathbf{Sp(2n)}} \mathbf{Sp(2n)}$ $(n \geq 2)$	$1, 1, 2, 2$ (4)
$\mathbf{Spin(8)} \oplus_{\mathbf{Spin(8)}} \mathbf{SO(8)}$	$1, 1, 2, 2, 2$ (5)

TABLE 5

for some integers a, b. Such actions are multiplicity free in the cases listed in Table 6.

6.5. **Completing the classification.** Let G be a connected complex algebraic *reductive* group acting on V via some rational representation π. The commutator subgroup G' of G is semi-simple and, by lifting to a finite covering if necessary, we can suppose that $G = G' \times A$ where A is some algebraic torus. Write V as a direct sum

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_r$$

of $\pi(G)$-invariant subspaces W_j which are indecomposable under the action of G'. Letting π_j denote the action of G on W_j, we have $\pi(G') = \mathbf{G_1}' \times \mathbf{G_2}' \times \cdots \times \mathbf{G_r}'$ acting on V via the product action, where $\mathbf{G_j}' = \pi_j(G')$.

Non-saturated multiplicity free actions

$(\mathbf{SL(n)} \otimes \mathbf{SL(2)}) \oplus_{\mathbf{SL(2)}} (\mathbf{SL(2)} \otimes \mathbf{SL(m)})$	$n, m \geq 3$
$\mathbf{SL(n)} \oplus_{\mathbf{SL(n)}} \mathbf{SL(n)}$	$n \geq 3, a \neq b$
$\mathbf{SL(n)}^* \oplus_{\mathbf{SL(n)}} \mathbf{SL(n)}$	$n \geq 3, a \neq -b$
$\mathbf{SL(2m)} \oplus_{\mathbf{SL(2m)}} \mathbf{\Lambda^2(SL(2m))}$	$m \geq 2, b \neq 0$
$\mathbf{SL(2m+1)} \oplus_{\mathbf{SL(2m+1)}} \mathbf{\Lambda^2(SL(2m+1))}$	$m \geq 2, a \neq -mb$
$\mathbf{SL(2m)}^* \oplus_{\mathbf{SL(2m)}} \mathbf{\Lambda^2(SL(2m))}$	$m \geq 2, b \neq 0$
$\mathbf{SL(2m+1)}^* \oplus_{\mathbf{SL(2m+1)}} \mathbf{\Lambda^2(SL(2m+1))}$	$m \geq 2, a \neq mb$
$\mathbf{SL(n)} \oplus_{\mathbf{SL(n)}} (\mathbf{SL(n)} \otimes \mathbf{SL(m)})$	$2 \leq n < m, a \neq 0$
$\mathbf{SL(n)} \oplus_{\mathbf{SL(n)}} (\mathbf{SL(n)} \otimes \mathbf{SL(m)})$	$m \geq 2, n \geq m+2, a \neq b$
$\mathbf{SL(n)}^* \oplus_{\mathbf{SL(n)}} (\mathbf{SL(n)} \otimes \mathbf{SL(m)})$	$2 \leq n < m, a \neq 0$
$\mathbf{SL(n)}^* \oplus_{\mathbf{SL(n)}} (\mathbf{SL(n)} \otimes \mathbf{SL(m)})$	$m \geq 2, n \geq m+2, a \neq -b$
$(\mathbf{SL(2)} \otimes \mathbf{SL(2)}) \oplus_{\mathbf{SL(2)}} (\mathbf{SL(2)} \otimes \mathbf{SL(m)})$	$m \geq 3, a \neq 0$
$(\mathbf{SL(n)} \otimes \mathbf{SL(2)}) \oplus_{\mathbf{SL(2)}} (\mathbf{SL(2)} \otimes \mathbf{Sp(2m)})$	$n \geq 3, m \geq 2, b \neq 0$

TABLE 6

Theorem 6.5.1. *([5]) $G : V$ is a multiplicity free action if and only if A contains a direct product of the form $A_1 \times A_2 \times \cdots \times A_r$ where each A_j is a torus of dimension at most 2 and the actions $\mathbf{G_j}' \times A_j : W_j$ are multiplicity free for $j = 1, \ldots, r$.*

One can choose the A_j's in Theorem 6.5.1 to be minimal, in the sense that the action of $\mathbf{G_j}' \times B$ on W_j fails to be multiplicity free for all proper subgroups B of A_j. Some factors A_j can be trivial and $A_1 \times \cdots \times A_r$ need not act on $W_1 \oplus \cdots \oplus W_r$ via a product action. That is, the A_j's can act diagonally on the indecomposable G'-summands W_j.

Example 6.5.2. As an example, consider $G' = SL(n) \times SL(n)$ for $n \geq 2$ acting on $V = W_1 \oplus W_2 = (\mathbb{C}^n \oplus \mathbb{C}^n) \oplus (\mathbb{C}^n \oplus \mathbb{C}^n)$ via

$$(g, h) \cdot (x_1, y_1, x_2, y_2) = (gx_1, gy_1, hx_2, hy_2) \qquad \text{for } g, h \in SL(n) \text{ and } x_j, y_j \in \mathbb{C}^n.$$

Here $\mathbf{G_1}' = \mathbf{SL(n)} \oplus_{\mathbf{SL(n)}} \mathbf{SL(n)} = \mathbf{G_2}'$ appear in Table 5. Let $A = (\mathbb{C}^\times)^3$ act on V via

$$(t_1, t_2, t_3) \cdot (x_1, y_1, x_2, y_2) = (t_1 t_2 x_1, t_1^2 t_2 y_1, t_1 t_2 x_2, t_1 t_2^2 y_2) \qquad \text{for } t_j \in \mathbb{C}^\times.$$

The joint action of $G' \times A$ on V is multiplicity free. Indeed, if we let $A_1 = \mathbb{C}^\times \times \{1\} \times \{1\}$ and $A_2 = \{1\} \times \mathbb{C}^\times \times \{1\}$ then we see that the actions $\mathbf{G_j}' \times A_j : W_j$ appear in Table 6. Here, however, one can't find subtori A_j as in Theorem 6.5.1 that act independently on W_1 and W_2.

Theorem 6.5.1 completes the classification of multiplicity free actions because we have exhibited all of the possibilities for the groups $\mathbf{G_j}' \subset GL(W_j)$ and for the actions of the A_j's on the W_j's. More precisely, if $G : V$ is multiplicity free then for each $1 \leq j \leq r$ we must have either:

(1) W_j is $\mathbf{G_j}'$ irreducible and
  (a) $\pi_j(A_j) = \mathbb{C}^\times$ and $\mathbf{G_j}'$ appears in Table 3, or
  (b) $A_j = \{1\}$ and $\mathbf{G_j}'$ appears in Table 4.
(2) W_j is a sum of two $\mathbf{G_j}'$-irreducible subspaces and
  (a) $\pi_j(A_j) = (\mathbb{C}^\times)^2$ and $\mathbf{G_j}'$ appears in Table 5, or
  (b) $\mathbf{G_j}'$ appears together with $\pi_j(A_j) \cong \mathbb{C}^\times$ in Table 6, or
  (c) $A_j = \{1\}$ and $\mathbf{G_j}' = (\mathbf{SL(n)} \otimes \mathbf{SL(2)}) \oplus_{\mathbf{SL(2)}} (\mathbf{SL(2)} \otimes \mathbf{SL(m)})$ with $n, m \geq 3$.

6.6. **Proof outline.** The saturated indecomposable multiplicity free actions in Table 5 are obtained from the irreducible actions in Table 3 by extensive case-by-case analysis. Suppose that $G = G'$ is semisimple and that (π, V) is an indecomposable action with $V = V_1 \oplus V_2$, a direct sum of two G-irreducible subspaces. Let $\pi_j = \pi|V_j$ and $\mathbf{G_j} = \pi_j(G) \subset GL(V_j)$ for $j = 1, 2$. If $G \times (\mathbb{C}^\times)^2 : V$ is multiplicity free then it is clear that both $G \times \mathbb{C}^\times : V_1$ and $G \times \mathbb{C}^\times : V_2$ must be multiplicity free. Thus both $\mathbf{G_1}$ and $\mathbf{G_2}$ appear in Table 3.

Define normal subgroups $\mathbf{K_1}$ and $\mathbf{K_2}$ of $\mathbf{G_1}$ and $\mathbf{G_2}$ by

$$\mathbf{K_1} := \pi_1(\mathrm{Ker}(\pi_2)) \quad \text{and} \quad \mathbf{K_2} := \pi_2(\mathrm{Ker}(\pi_1)). \tag{6.6.1}$$

The map

$$F : \mathbf{G_1}/\mathbf{K_1} \to \mathbf{G_2}/\mathbf{K_2} \tag{6.6.2}$$

given by $F(\pi_1(g)\mathbf{K_1}) := \pi_2(g)\mathbf{K_2}$ is a well-defined group isomorphism.

If $\mathbf{K_1} = \mathbf{G_1}$ then it follows that $\mathbf{K_2} = \mathbf{G_2}$ and we have $\pi(G) = \mathbf{G_1} \times \mathbf{G_2}$. In this case, our action decomposes as a direct product of the multiplicity free actions $\mathbf{G_j} \times (\mathbb{C}^\times)$.

Next suppose that the $\mathbf{K_j}$'s are proper subgroups of the $\mathbf{G_j}$'s. Note that $\mathbf{K_j}$ need not be connected. We write $\mathbf{K_j^\circ}$ for the identity component in $\mathbf{K_j}$. Since $\mathbf{K_j}$ is a normal subgroup of $\mathbf{G_j}$, so is $\mathbf{K_j^\circ}$. As $\mathbf{G_j}$ appears in Table 3, $\mathbf{G_j}$ is either a simple group or a product of two simple factors. Thus we can write $\mathbf{G_j} = \mathbf{K_j^\circ H_j}$, where either $\mathbf{H_j} = \mathbf{G_j}$ (when $\mathbf{K_j^\circ} = \{e\}$) or $\mathbf{H_j}$ is one of two simple factors in $\mathbf{G_j}$. As $\mathbf{H_j} \cong \mathbf{G_j}/\mathbf{K_j^\circ}$ covers $\mathbf{G_j}/\mathbf{K_j}$, the derivative of F yields an isomorphism between the Lie algebras of $\mathbf{H_1}$ and $\mathbf{H_2}$. In fact, Table 3 shows that $\mathbf{H_j}$ is simply connected except when $\mathbf{H_j} = \mathbf{SO(n)}$. Thus, we can realize F as a group isomorphism $\mathbf{H_1} \cong \mathbf{H_2}$ or as a covering of one of the $\mathbf{H_j}$'s by the other. We write $F : \mathbf{H_1} \to \mathbf{H_2}$ for this map after interchanging the roles of $\mathbf{G_1}$ and $\mathbf{G_2}$ when $\mathbf{H_2}$ covers $\mathbf{H_1}$. If we define a new group L by $L = \mathbf{K_1^\circ} \times \mathbf{H_1} \times \mathbf{K_2^\circ}$ and a representation σ of L on $V = V_1 \oplus V_2$ by

$$\sigma(k_1, h_1, k_2) := (k_1 h_1, F(h_1) k_2) \in GL(V_1) \times GL(V_2)$$

then we see that $\pi(G) = \sigma(L) = \mathbf{G_1} \oplus_{\mathbf{H_1}} \mathbf{G_2}$. That is, $\mathbf{H_1}$ acts diagonally on V_1 and V_2 via F.

In summary, we have shown that any saturated indecomposable multiplicity free action with two irreducible summands is obtained from two entries in Table 3 via diagonalization along simple factors. Combining pairs of entries in Table 3 which share a common simple factor yields a large number of indecomposable actions. It is necessary to examine each of these in turn to determine which are multiplicity free. Various techniques can be applied in each case. One can look for an open Borel orbit. (Many candidates can be eliminated easily because the underlying vector space has dimension greater than that of a Borel subgroup.) One can apply the Littlewood-Richardson rules (and variants for the classical groups $SO(n,\mathbb{C})$, $Sp(2n,\mathbb{C})$) or use the recursive criterion from Section 5. Table 5 is the end result of this analysis.

If $V = V_1 \oplus \cdots \oplus V_r$ is a sum of r irreducible subspaces, then the above reasoning applies to each pair of groups $\mathbf{G_i} = \pi_i(G)$, $\mathbf{G_j} = \pi_j(G)$. For the action to be indecomposable, at least one simple factor in each $\mathbf{G_i}$ must act diagonally on at least one V_j with $j \neq i$. Diagonalization from Tables 3 and 5 produces indecomposable actions with more than two irreducible summands. The analysis shows, however, that no such actions are multiplicity free.

6.7. **Section 6 notes.** The articles [7] and [31] contain formulas for the fundamental highest weight vectors and further detailed information for each action in the classification. For the irreducible actions, much of this is due to Howe and Umeda [24].

7. Invariant polynomials and differential operators

7.1. **Polynomial coefficient differential operators.** For each $f \in \mathbb{C}[V]$ we have the multiplication operator $M_f \in End(\mathbb{C}[V])$,

$$M_f(h) = fh.$$

Let $\mathcal{P}(V) = \{M_f \ : \ f \in \mathbb{C}[V]\}$, so $\mathcal{P}(V) \cong \mathbb{C}[V]$. For each $v \in V$ we have the directional derivative $\partial_v \in End(\mathbb{C}[V])$,

$$(\partial_v h)(w) = \lim_{t \to 0} \frac{f(w+tv) - f(w)}{t}.$$

The algebra $\mathcal{D}(V)$ generated by $\{\partial_v \ : \ v \in V\}$ is the algebra of constant coefficient differential operators. The embedding $V \hookrightarrow \mathcal{D}(V)$, $v \mapsto \partial_v$ extends to an isomorphism from the symmetric algebra $S(V)$ to $\mathcal{D}(V)$. Thus

$$\mathcal{D}(V) \cong S(V) \cong \mathbb{C}[V^*]$$

as algebras.

We let $\mathcal{PD}(V)$ be the subalgebra of $End(\mathbb{C}[V])$ generated by $\mathcal{P}(V)$ and $\mathcal{D}(V)$. This is the algebra of *polynomial coefficient differential operators.* The product rule for derivatives shows that the map

$$\mathcal{P}(V) \otimes \mathcal{D}(V) \to \mathcal{PD}(V), \quad p \otimes L \mapsto pL$$

given by multiplication is a vector space isomorphism. Composing this with the algebra isomorphisms

$$\mathbb{C}[V \oplus V^*] \cong \mathbb{C}[V] \otimes \mathbb{C}[V^*] \cong \mathcal{P}(V) \otimes \mathcal{D}(V)$$

produces a vector space isomorphism

$$\delta : \mathbb{C}[V \oplus V^*] \to \mathcal{PD}(V), \quad p \mapsto p(z, \partial).$$

We say that $p \in \mathbb{C}[V \oplus V^*]$ is the *Wick symbol* for the operator $p(z, \partial)$. The inverse map

$$\sigma : \mathcal{PD}(V) \to \mathbb{C}[V \oplus V^*]$$

of δ is called the *polarized symbol map.*

The above discussion can be made more concrete by introducing coordinates. Let $(z_1, \ldots, z_n)$ be coordinates on V with respect to some basis $\{e_1, \ldots, e_n\}$ and $(w_1, \ldots, w_n)$ be coordinates on V^* with respect to the dual basis $\{e_1^*, \ldots, e_n^*\}$. We have

$$\mathbb{C}[V] = \mathbb{C}[z_1, \ldots, z_n], \quad \mathbb{C}[V \oplus V^*] = \mathbb{C}[z_1, \ldots, z_n, w_1, \ldots, w_n].$$

The monomials

$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$$

form a basis for $\mathbb{C}[V]$ and the differential operators

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \quad \left(\partial_j = \frac{\partial}{\partial z_j}\right)$$

form a basis for $\mathcal{D}(V)$ as $\alpha = (\alpha_1, \ldots, \alpha_n)$ ranges over all multi-indices $\alpha \in \mathbb{N}^n$. The operator $p(z, \partial)$ with Wick symbol

$$p(z, w) = \sum_{\alpha,\beta} c_{\alpha,\beta} z^\alpha w^\beta.$$

is

$$p(z, \partial) = \sum_{\alpha,\beta} c_{\alpha,\beta} z^\alpha \partial^\beta.$$

The symbol mapping is not an algebra isomorphism because $\mathbb{C}[V \oplus V^*]$ is abelian whereas $\mathcal{PD}(V)$ is not. In fact, one has the Heisenberg commutation relations

$$[\partial_i, z_j] = \delta_{i,j}$$

in $\mathcal{PD}(V)$. To obtain an algebra isomorphism, one can pass to the associated graded algebras. The algebra $\mathbb{C}[V \oplus V^*]$ is filtered by

$$\mathbb{C}^{(k)}[V \oplus V^*] = \sum_{a+b \le k} \mathcal{P}_a(V) \otimes \mathcal{P}_b(V^*).$$

Then

$$\mathcal{PD}^{(k)}(V) = \delta(\mathbb{C}^{(k)}[V \oplus V^*])$$

gives a filtration of the algebra $\mathcal{PD}(V)$. That is

$$\mathbb{C} = \mathcal{PD}^{(0)} \subset \mathcal{PD}^{(1)} \subset \cdots \subset \mathcal{PD}^{(k)}(V) \subset \cdots, \qquad \bigcup_{k=0}^{\infty} \mathcal{PD}^{(k)}(V) = \mathcal{PD}(V),$$

and

$$\mathcal{PD}^{(k)}(V)\mathcal{PD}^{(\ell)}(V) \subset \mathcal{PD}^{(k+\ell)}(V)$$

in view of the commutation relations. In terms of our basis,

$$\mathcal{PD}^{(k)}(V) = Span\{z^\alpha \partial^\beta \ : \ |\alpha| + |\beta| \leq k\}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The map $\delta : \mathbb{C}[V \oplus V^*] \to \mathcal{PD}(V)$ induces an algebra isomorphism from $gr(\mathbb{C}[V \oplus V^*]) \cong \mathbb{C}[V \oplus V^*]$ to

$$gr(\mathcal{PD}(V)) = \sum_{k=0}^{\infty} \mathcal{PD}^{(k+1)}(V)/\mathcal{PD}^{(k)}(V),$$

an abelian algebra that is canonically isomorphic to $\mathcal{PD}(V)$ as a vector space.

Lemma 7.1.1. *$\mathcal{PD}(V)$ is strongly dense in $End(\mathbb{C}[V])$. That is,*

$$\mathcal{PD}(V)|\mathcal{X} = Hom(\mathcal{X}, \mathbb{C}[V])$$

for any finite dimensional subspace $\mathcal{X}$ of $\mathbb{C}[V]$.

Proof. Let $\mathcal{X} \subset \mathbb{C}[V] = \mathbb{C}[z_1, \ldots, z_n]$ be a finite dimensional subspace and $T \in Hom(\mathcal{X}, \mathbb{C}[V])$. Let $\{p_1, \ldots, p_m\}$ be a basis for $\mathcal{X}$, ordered so that $deg(p_1) \leq \cdots \leq deg(p_m)$. By taking suitable linear combinations, one can ensure that p_k contains a monomial m_k which

- has degree $deg(p_k)$ and
- $p_1, \ldots, p_{k-1}$ contain no non-zero multiples of m_k.

Let $f_j = T(p_j)$ for $1 \leq j \leq m$, and let $m_1 = z^\alpha$. Then the operator

$$D_1 = f_1 \frac{1}{\alpha!} \partial^\alpha \quad \big(\alpha! = \alpha_1! \cdots \alpha_n!\big)$$

has $D_1(p_1) = f_1$.

Assume inductively that there is some $D_k \in \mathcal{PD}(V)$ with $D_k(p_j) = f_j$ for $1 \leq j \leq k$. Let $m_{k+1} = z^\beta$. Then $\partial^\beta p_1 = \cdots = \partial^\beta p_k = 0$, and $\partial^\beta p_{k+1} = \beta!$ Now

$$D_{k+1} = \big(f_{k+1} - D_k p_{k+1}\big) \frac{1}{\beta!} \partial^\beta + D_k$$

satisfies $D_{k+1}(p_j) = f_j$ for $1 \leq j \leq k+1$. By induction we conclude that there is an operator $D \in \mathcal{PD}(V)$ with $D|\mathcal{X} = T$. $\square$

7.2. **Invariants in $\mathcal{PD}(V)$.** Now suppose that a reductive algebraic group G is acting linearly on V. For the moment, we do not assume that $G : V$ is a multiplicity free action. The group G acts on $\mathcal{PD}(V)$ via conjugation:

$$(g \cdot D)(f) = g \cdot D(g^{-1} \cdot f).$$

In terms of coordinates one calculates:

$$g \cdot M_{z_j} = \sum_{\ell} (g^{-1})_{j\ell} M_{z_\ell}, \quad g \cdot \partial_j = \sum_{\ell} (g^t)_{j\ell} \partial_\ell.$$

It follows that the polarized symbol mapping intertwines the action of G on $\mathcal{PD}(V)$ with its action on $\mathbb{C}[V \oplus V^*] = \mathbb{C}[z_1, \ldots, z_n, w_1, \ldots, w_n]$ via

$$g \cdot p(z, w) = p(g^{-1} z, g^t w).$$

This formula agrees with that for the representation of G on $\mathbb{C}[V \oplus V^*]$ arising from $G : V$ and its contragredient $G : V^*$,

$$g \cdot (v, \xi) = (g \cdot v, g \cdot \xi) = (g \cdot v, \xi \circ g^{-1}).$$

Thus we have shown:

Lemma 7.2.1. *The G-invariants in $\mathcal{PD}(V)$ are*

$$\mathcal{PD}(V)^G = \mathcal{PD}(V) \cap End_G(\mathbb{C}[V]) = \{p(z, \partial) \ : \ p \in \mathbb{C}[V \oplus V^*]^G\}.$$

Let K be a maximal compact connected subgroup in G. The Unitarian Trick ensures that an operator $D \in \mathcal{PD}(V)$ is G-invariant if and only if it is K-invariant. Now we define the K-average of D via

$$D^\natural = \int_K (k \cdot D) dk$$

where dk denotes normalized Haar measure on K. The operator D belongs to a finite dimensional subspace $\mathcal{PD}^{(m)}(V)$ for some m, and the integral converges in $\mathcal{PD}^{(m)}(V)$. So $D^\natural$ is a polynomial coefficient differential operator. A change of variables and unimodularity of K shows that $k_\circ \cdot D^\natural = D^\natural$ for each $k_\circ \in K$. Thus $D^\natural \in \mathcal{PD}(V)^G$.

Lemma 7.2.2. *$\mathcal{PD}(V)^G$ acts irreducibly on $\mathbb{C}[V]^{B,\lambda}$ for all dominant weights λ.*

Proof. For $D \in \mathcal{PD}(V)^G$, $b \in B$ and h a λ-highest weight vector, we have

$$b \cdot (Dh) = D(b \cdot h) = D(b^\lambda h) = b^\lambda Dh.$$

Thus $\mathcal{PD}(V)^G$ preserves the space $\mathbb{C}[V]^{B,\lambda}$ of λ-highest weight vectors.

Next let h_1, h_2 be two λ-highest weight vectors in $\mathbb{C}[V]$ and consider the finite dimensional G-invariant space

$$\mathcal{X} = Span(G \cdot h_1) + Span(G \cdot h_2).$$

Since $Span(G \cdot h_1)$ and $Span(G \cdot h_2)$ are equivalent as G-modules, there is some $T \in End_G(\mathcal{X})$ with $T(h_1) = h_2$. By Lemma 7.1.1 there is an operator $D \in \mathcal{PD}(V)$ with $D|\mathcal{X} = T$. Now $D^\natural \in \mathcal{PD}(V)^G$ and

$$\begin{aligned} D^\natural h_1 &= \int_K k \cdot D(k^{-1} \cdot h_1) dk \\ &= \int_K k \cdot T(k^{-1} \cdot h_1) dk \quad (\text{as } k^{-1} \cdot h_1 \in \mathcal{X}) \\ &= \int_K T(h_1) dk \quad (\text{as } T \text{ is } G\text{-invariant}) \\ &= T(h_1) \\ &= h_2. \end{aligned}$$

This shows that $\mathcal{PD}(V)^G$ acts irreducibly on $\mathbb{C}[V]^{B,\lambda}$. □

Theorem 7.2.3. *$G : V$ is a multiplicity free action if and only if $\mathcal{PD}(V)^G$ is abelian.*

Proof. Let $G : V$ be a multiplicity free action and

$$\mathbb{C}[V] = \bigoplus_{\lambda \in \Lambda} P_\lambda$$

be the decomposition of $\mathbb{C}[V]$ into pair-wise inequivalent G-irreducibles. Schur's Lemma ensures that any operator $D \in \mathcal{PD}(V)^G$ must preserve each P_λ and acts on P_λ as multiplication by some scalar. It follows that any two operators in $\mathcal{PD}(V)^G$ commute.

Conversely, suppose that $\mathcal{PD}(V)^G$ is abelian. As $\mathcal{PD}(V)^G$ acts irreducibly on $\mathbb{C}[V]^{B,\lambda}$ we must have $dim(\mathbb{C}[V]^{B,\lambda}) \leq 1$ for all dominant weights λ. Hence $G : V$ is multiplicity free. □

Thus when $G : V$ is multiplicity free, both algebras $\mathbb{C}[V \oplus V^*]^G$ and $\mathcal{PD}(V)^G$ are abelian. Although δ and σ are not algebra maps, they induce algebra isomorphisms

$$\mathbb{C}[V \oplus V^*]^G \cong gr\big(\mathbb{C}[V \oplus V^*]^G\big) \cong gr\big(\mathcal{PD}(V)^G\big)$$

between the associated graded algebras. Concretely, this means that although one generally has $p(z, \partial)q(z, \partial) \neq (pq)(z, \partial)$, the operators $p(z, \partial)q(z, \partial)$ and $(pq)(z, \partial)$ have the same top degree terms. Here "degree" in $\mathcal{PD}(V)$ is defined using the filtration $\mathcal{PD}^{(k)}(V)$ from Section 7.1. In particular, $z^\alpha \partial^\beta$ has degree $|\alpha| + |\beta|$.

7.3. **A canonical basis for the invariants.** Suppose that $G : V$ is a (linear) multiplicity free action. The trivial representation of G occurs in $\mathbb{C}[V]$ on $\mathcal{P}_0(V) = \mathbb{C}$, the constant polynomials. As the representation of G on $\mathbb{C}[V]$ is multiplicity free, it follows that $\mathbb{C}[V]^G = \mathbb{C}$. That is, there are no non-constant G-invariants in $\mathbb{C}[V]$. Because of the connection with G-invariant differential operators it is, however, of interest to study the G-invariants in $\mathbb{C}[V \oplus V^*]$.

Let

$$\mathbb{C}[V] = \bigoplus_{\lambda\in\Lambda} P_\lambda$$

denote the multiplicity free decomposition of $\mathbb{C}[V]$ under the action of G. Then

$$\mathbb{C}[V \oplus V^*] = \mathbb{C}[V] \otimes \mathbb{C}[V]^* = \bigoplus_{\lambda,\lambda'\in\Lambda} P_\lambda \otimes P_{\lambda'}^*,$$

where the subspaces $P_\lambda \otimes P_{\lambda'}^*$ are G-invariant in $\mathbb{C}[V \oplus V^*]$. Thus

$$\mathbb{C}[V \oplus V^*]^G = \bigoplus_{\lambda,\lambda'\in\Lambda} \left(P_\lambda \otimes P_{\lambda'}^*\right)^G.$$

But

$$\left(P_\lambda \otimes P_{\lambda'}^*\right)^G \cong \left(Hom(P_{\lambda'}, P_\lambda)\right)^G = Hom_G(P_{\lambda'}, P_\lambda) = \begin{cases} 0 & \text{if } \lambda \neq \lambda' \\ \mathbb{C} & \text{if } \lambda = \lambda' \end{cases}$$

by Schur's Lemma. The first isomporphism is given by $f \otimes \xi \mapsto (p \mapsto \xi(p)f)$. The element in $P_\lambda \otimes P_\lambda^*$ that corresponds to I_{P_λ} under the isomorphism $P_\lambda \otimes P_\lambda^* \cong Hom(P_\lambda, P_\lambda)$ is

$$\widetilde{p}_\lambda = \sum_{j=1}^{d_\lambda} f_j \otimes f_j^* \tag{7.3.1}$$

where $d_\lambda = dim(P_\lambda)$ and $\{f_j \ : \ 1 \leq j \leq d_\lambda\}$ is any basis for P_λ with dual basis $\{f_j^*\}$. Thus we have shown that

$$\mathbb{C}[V \oplus V^*]^G = \bigoplus_{\lambda\in\Lambda} \left(P_\lambda \otimes P_\lambda^*\right)^G = \bigoplus_{\lambda\in\Lambda} \mathbb{C}\widetilde{p}_\lambda.$$

So $\{\widetilde{p}_\lambda \mid \lambda \in \Lambda\}$ is a basis for $\mathbb{C}[V \oplus V^*]^G$. As Equation 7.3.1 does not depend on the choice of basis $\{f_j\}$ for P_λ, the basis $\{\widetilde{p}_\lambda \mid \lambda \in \Lambda\}$ for $\mathbb{C}[V \oplus V^*]^G$ is canonical. Applying Wick quantization we obtain a canonical basis for $\mathcal{PD}(V)^G$.

We call the polynomials $\widetilde{p}_\lambda$ the *unnormalized canonical invariants*. To achieve some simplification in formulae to be derived below, we also introduce the (*normalized*) *canonical invariants*

$$p_\lambda = \frac{1}{d_\lambda}\widetilde{p}_\lambda, \quad (d_\lambda = dim(P_\lambda)).$$

In summary we have proved the following.

Proposition 7.3.1. *$\{p_\lambda \ : \ \lambda \in \Lambda\}$ and $\{p_\lambda(z, \partial) \ : \ \lambda \in \Lambda\}$ are canonical vector space bases for $\mathbb{C}[V \oplus V^*]^G$ and $\mathcal{PD}(V)^G$ respectively.*

7.4. **The fundamental invariants.** Now let r be the rank of the multiplicity free action $G : V$ and

$$\Lambda' = \{\lambda_1, \dots, \lambda_r\}$$

be the set of fundamental highest weights. Recall that $\Lambda = \{m_1\lambda_1 + \cdots + m_r\lambda_r \mid m \in \mathbb{N}^r\}$. (See Proposition 3.3.1.)

Definition 7.4.1. The *fundamental invariants* for $G : V$ are $\{\gamma_1, \dots, \gamma_r\}$ where

$$\gamma_j = p_{\lambda_j}.$$

For $\lambda \in \Lambda$ let $|\lambda| \in \mathbb{N}$ denote the degree of homogeneity of P_λ. That is, $P_\lambda \subset \mathcal{P}_{|\lambda|}(V)$. Then the canonical invariant p_λ is homogeneous of degree $2|\lambda|$.

For any weights $\mu, \nu \in \mathfrak{h}^*$ we will write

$$\mu \prec \nu$$

when $\nu - \mu$ is a sum of positive roots.

Lemma 7.4.2. *For any $\lambda, \mu \in \Lambda$ there are values $c_\nu = c_{\lambda,\mu,\nu}$ for which*

$$p_\lambda p_\mu = \sum_\nu c_\nu p_\nu,$$

where the sum is over all $\nu \in \Lambda$ with $|\nu| = |\lambda| + |\mu|$ and $\nu \preceq \lambda + \mu$. Moreover, $c_{\lambda+\mu} \neq 0$.

Proof. The product $p_\lambda p_\mu$ is G-invariant and belongs to $\mathcal{P}_{|\lambda|+|\mu|}(V) \otimes \mathcal{P}_{|\lambda|+|\mu|}(V^*)$. As the p_ν's form a homogeneous basis for $\mathbb{C}[V \oplus V^*]^G$ we conclude that

$$p_\lambda p_\mu = \sum_{|\nu|=|\lambda|+|\mu|} c_\nu p_\nu$$

for some values c_ν. Let $\{f_j\}_{j=1}^{d_\lambda}$ and $\{h_j\}_{j=1}^{d_\mu}$ be bases of weight vectors for P_λ and P_μ so that f_1, h_1 are highest weight vectors. We know that all other weights in an irreducible representation space precede the highest weight in the partial ordering defined above.

We have

$$p_\lambda p_\mu = \frac{1}{d_\lambda d_\mu} \sum_{i,j} f_i h_j \otimes f_i^* h_j^*.$$

The $\mathbb{C}[V]$-components $f_i h_j$ in this sum are weight vectors with weights $\lambda_i + \mu_j \preceq \lambda + \mu$. It follows that $c_\nu = 0$ unless $\nu \prec \lambda + \mu$. Moreover, the term $f_1 h_1 \otimes f_1^* h_1^*$ contains the $(\lambda + \mu)$-highest weight vector $f_1 h_1$. We conclude that $c_{\lambda+\mu} \neq 0$. $\square$

Theorem 7.4.3. $\mathbb{C}[V \oplus V^*]^G = \mathbb{C}[\gamma_1, \dots, \gamma_r]$. *That is, $\mathbb{C}[V \oplus V^*]^G$ is a polynomial ring freely generated by the fundamental invariants.*

Proof. Given $m \in \mathbb{N}^r$, let $\lambda = m_1\lambda_1 + \cdots + m_r\lambda_r \in \Lambda$. Lemma 7.4.2 shows that

$$\gamma^m = \gamma_1^{m_1} \cdots \gamma_r^{m_r} = p_{\lambda_1}^{m_1} \cdots p_{\lambda_r}^{m_r} = a_\lambda p_\lambda + \sum_{\nu \prec \lambda} a_\nu p_\nu$$

for some coefficients a_ν with $a_\lambda \neq 0$. As $\{p_\lambda\}$ is a basis for $\mathbb{C}[V \oplus V^*]^G$, we conclude that $\{\gamma^m \mid m \in \mathbb{N}^r\}$ is also a basis for $\mathbb{C}[V \oplus V^*]^G$. □

Corollary 7.4.4. *$\mathcal{PD}(V)^G$ is a polynomial ring freely generated by $\{D_j = \gamma_j(z, \partial) \;:\; 1 \leq j \leq r\}$.*

Proof. From Theorem 7.4.3 we see that $\{\gamma^m(z, \partial) \;:\; m \in \mathbb{N}^r\}$ is a basis for the vector space $\mathcal{PD}(V)^G$. Also, given $m = (m_1, \ldots, m_r)$,

$$D^m = D_1^{m_1} \cdots D_r^{m_r} = \gamma_1(z, \partial)^{m_1} \cdots \gamma_r(z, \partial)^{m_r}$$

differs from $\gamma^m(z, \partial)$ by an element of $\mathcal{PD}^{(2|\lambda|-1)}(V)$, where $\lambda = m_1\lambda_1 + \cdots m_r\lambda_r$. By induction on degree in $\mathcal{PD}(V)$ we conclude that $\{D^m \;:\; m \in \mathbb{N}^r\}$ is a vector space basis for $\mathcal{PD}(V)^G$. Thus $\mathcal{PD}(V)^G = \mathbb{C}[D_1, \ldots, D_r]$. □

7.5. **The algebra $\mathbb{C}[V_\mathbb{R}]^K$.** An alternative viewpoint on $\mathbb{C}[V \oplus V^*]^G$ will prove useful. Let K denote a maximal compact connected Lie subgroup of G and $\langle \cdot, \cdot \rangle$ be any K-invariant positive definite Hermitian inner product on V. The conjugate-linear vector space isomorphism

$$V \to V^*, \quad v \mapsto v^* = \langle \cdot, v \rangle$$

is K-equivariant (but not G-equivariant). In view of the Unitarian Trick we obtain an algebra isomorphism

$$\mathbb{C}[V_\mathbb{R}]^K = \mathbb{C}[V \oplus \overline{V}]^K \cong \mathbb{C}[V \oplus V^*]^K = \mathbb{C}[V \oplus V^*]^G. \tag{7.5.1}$$

Here $\overline{V}$ denotes V with the conjugate complex structure and $V_\mathbb{R}$ is the underlying real vector space for V.

Introducing coordinates $(z_1, \ldots, z_n)$ on V with respect to an *orthonormal* basis, one has $\mathbb{C}[V] = \mathbb{C}[z_1, \ldots, z_n]$. This polynomial ring also carries an inner product, namely

$$\langle p, q \rangle_\mathcal{F} = \big(p(\partial)\overline{q}\big)(0) = \big(\overline{q}(\partial)p\big)(0),$$

the so-called *Fischer inner product.* Here $p(\partial) = p(\partial_1, \ldots, \partial_n)$ for $p = p(z_1, \ldots, z_n)$ and for $q(z) = \sum_\alpha c_\alpha z^\alpha$, $\overline{q}(z) = \sum_\alpha \overline{c_\alpha} z^\alpha$.

Thus

$$\langle z^\alpha, z^\beta \rangle_\mathcal{F} = \delta_{\alpha,\beta}\alpha! = \delta_{\alpha,\beta}\alpha_1! \cdots \alpha_n!$$

for multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$.

Lemma 7.5.1. *The subspaces $\{P_\lambda \;:\; \lambda \in \Lambda\}$ in $\mathbb{C}[V]$ are pair-wise orthogonal with respect to the Fischer inner product.*

Proof. This follows from the fact that $K \subset U(V)$ and $U(V)$ preserves $\langle\cdot,\cdot\rangle_{\mathcal{F}}$. This can be seen using an alternative formula for the Fisher inner product:

$$\langle p, q\rangle_{\mathcal{F}} = \frac{1}{\pi^n}\int p(z)\overline{q(z)}e^{-|z|^2}dzd\overline{z}. \tag{7.5.2}$$

In (7.5.2), $n = dim_{\mathbb{C}}(V)$, $|z|^2 = \langle z, z\rangle$ and "$dzd\overline{z}$" denotes Lebesgue measure on $V_{\mathbb{R}}$ normalized using $\langle\cdot,\cdot\rangle$. We see that $\langle k\cdot p, k\cdot q\rangle_{\mathcal{F}} = \langle p, q\rangle_{\mathcal{F}}$ for $k \in U(V)$ via a change of variables in (7.5.2), since both $|z|^2$ and $dzd\overline{z}$ are $U(V)$-invariant.

To establish (7.5.2) it suffices to verify that

$$\int z^\alpha \overline{z}^{\alpha'} e^{-|z|^2}dzd\overline{z} = \pi^n \delta_{\alpha,\alpha'}\alpha!.$$

For this, use polar coordinates $z_j = r_j e^{i\theta_j}$ to write

$$\int z^\alpha \overline{z}^{\alpha'} e^{-|z|^2}dzd\overline{z} = \prod_{j=1}^n \int_0^\infty \int_0^{2\pi} r_j^{\alpha_j+\alpha'_j+1} e^{i(\alpha_j-\alpha'_j)\theta_j} e^{-r_j^2} d\theta_j dr_j.$$

The integral in θ_j is zero unless $\alpha_j = \alpha'_j$, in which case one has

$$2\pi\int_0^\infty r_j^{2\alpha_j+1}e^{-r_j^2}dr_j = 2\pi\int_0^\infty s^{\alpha_j}e^{-s}\frac{ds}{2} = \pi\alpha_j!.$$

□

Using isomorphism (7.5.1) we can regard the canonical invariants p_λ as elements of $\mathbb{C}[V_{\mathbb{R}}]^K = \mathbb{C}[z_1,\ldots,z_n,\overline{z}_1,\ldots,\overline{z}_n]^K$. We have

$$p_\lambda(z,\overline{z}) = \frac{1}{d_\lambda}\sum_{j=1}^{d_\lambda} e_j(z)\overline{e}_j(\overline{z}) = \frac{1}{d_\lambda}\sum_{j=1}^{d_\lambda}|e_j(z)|^2$$

where $\{e_j\}$ is any orthonormal basis for P_λ (with respect to $\langle\cdot,\cdot\rangle_{\mathcal{F}}$). Note that each $p_\lambda \in \mathbb{C}[V_{\mathbb{R}}]^K$ is real valued and non-negative.

We later require the formula

$$\sum_{|\lambda|=k} d_\lambda p_\lambda = \frac{|z|^{2k}}{k!}. \tag{7.5.3}$$

Indeed $\sum_{|\lambda|=k} d_\lambda p_\lambda(z,\overline{z}) = \sum_e |e(z)|^2$ where e ranges over an orthonormal basis for $\mathcal{P}_k(V)$ obtained by concatenation of orthonormal bases for $\{P_\lambda : |\lambda| = k\}$. The sum is, however, independent of the basis and we can use $\{z^\alpha/\sqrt{\alpha!} : |\alpha| = k\}$ to compute

$$\sum_{|\lambda|=k} d_\lambda p_\lambda = \sum_{|\alpha|=k} z^\alpha\overline{z}^\alpha/\alpha! = \frac{1}{k!}\sum_{|\alpha|=k}\frac{k!}{\alpha!}z^\alpha\overline{z}^\alpha = |z|^{2k}/k!$$

as stated.

On $\mathbb{C}[V_{\mathbb{R}}] = \mathbb{C}[z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n]$ we consider the Hermitian inner product

$$\langle p, q\rangle_* = \big(p(\partial, \overline{\partial})\overline{q}\big)(0) = \big(\overline{q}(\partial, \overline{\partial})p\big)(0).$$

This "doubled Fischer inner product" is determined by

$$\langle z^\alpha \overline{z}^\beta, z^{\alpha'}\overline{z}^{\beta'}\rangle_* = \delta_{\alpha,\alpha'}\delta_{\beta,\beta'}\alpha!\beta!.$$

Proposition 7.5.2. *$\{p_\lambda \mid \lambda \in \Lambda\}$ is an orthogonal basis for $\mathbb{C}[V_{\mathbb{R}}]^K$ with respect to the inner product $\langle\cdot,\cdot\rangle_*$. Moreover $\langle p_\lambda, p_\lambda\rangle_* = 1/d_\lambda$.*

Proof. Let $\{e_j\}$, $\{f_j\}$ be $\langle\cdot,\cdot\rangle_{\mathcal{F}}$-orthonormal bases for P_λ, P_μ. Using Lemma 7.5.1 we compute

$$\begin{aligned}
\langle p_\lambda, p_\mu\rangle_* &= \big(p_\lambda(\partial, \overline{\partial})\overline{p}_\mu(z, \overline{z})\big)(0) \\
&= \frac{1}{d_\lambda d_\mu}\sum_{i,j}\big(e_i(\partial)\overline{e}_i(\overline{\partial})\overline{f}_j(z)f_j(\overline{z})\big)(0) \\
&= \frac{1}{d_\lambda d_\mu}\sum_{i,j}\big(e_i(\partial)\overline{f}_j(z)\big)(0)\big(\overline{e}_i(\overline{\partial})f_j(\overline{z})\big)(0) \\
&= \frac{1}{d_\lambda d_\mu}\sum_{i,j}|\langle e_i, f_j\rangle_{\mathcal{F}}|^2 \\
&= \frac{\delta_{\lambda,\mu}d_\lambda}{d_\lambda d_\mu} \\
&= \frac{\delta_{\lambda,\mu}}{d_\lambda}.
\end{aligned}$$

□

7.6. **Section 7 notes.** Theorems 7.2.3, 7.4.3 and Corollary 7.4.4 are from [24]. An action whose invariants form a polynomial ring is said to be *coregular*. The coregular actions for simple groups are classified in [27] and [48].

The Fischer inner product is also called the *Fock inner product*, especially in connection with Equation 7.5.2. Fock space $\mathcal{F}$ is the Hilbert space completion of $\mathbb{C}[V]$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{F}}$. This can be identified as the space of holomorphic functions on V square integrable with respect to the Gaussian measure $e^{-|z|^2}dzd\overline{z}$ [14]. The inner product $\langle\cdot,\cdot\rangle_*$ can be regarded as the restriction of $\langle\cdot,\cdot\rangle_{\mathcal{F}} \otimes \langle\cdot,\cdot\rangle_{\mathcal{F}^*}$ from $\mathcal{F}\otimes\mathcal{F}^*$ to $\mathbb{C}[V \oplus \overline{V}] \cong \mathbb{C}[V]\otimes\mathbb{C}[V]^*$. One can identify $\mathcal{F}\otimes\mathcal{F}^*$ with the space of Hilbert-Schmidt operators on $\mathcal{F}$. Now $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ induces the Hilbert-Schmidt norm. There is also a connection with the Berezin star product, as explained in [1].

8. Generalized binomial coefficients

We continue to assume that $G : V$ is a multiplicity free action. As in the previous section, $\{p_\lambda \mid \lambda \in \Lambda\}$ are the canonical invariants. We view these as living in $\mathbb{C}[V_{\mathbb{R}}]^K$.

8.1. **The polynomials q_λ.** Let $\Delta = \partial \cdot \overline{\partial} = \partial_1\overline{\partial}_1 + \cdots + \partial_n\overline{\partial}_n$ and consider the operator $T : \mathbb{C}[V_\mathbb{R}] \to \mathbb{C}[V_\mathbb{R}]$,

$$(Tp)(z,\overline{z}) = (e^\Delta p)(z,-\overline{z}) = e^{-\Delta}\big(p(z,-\overline{z})\big).$$

Note that T is an involutive automorphism. Indeed, writing $(Mp)(z,\overline{z}) = p(z,-\overline{z})$ one has

$$T = M \circ e^\Delta = e^{-\Delta} \circ M = T^{-1}. \tag{8.1.1}$$

Definition 8.1.1. Let $q_\lambda = T(p_\lambda) = (-1)^{|\lambda|}e^{-\Delta}p_\lambda$ for each $\lambda \in \Lambda$.

The two formulas in the definition for q_λ agree because $p_\lambda(z,-\overline{z}) = (-1)^{|\lambda|}p_\lambda(z,\overline{z})$.

Lemma 8.1.2. *$\{q_\lambda \;:\; \lambda \in \Lambda\}$ is a vector space basis for $\mathbb{C}[V_\mathbb{R}]^K$.*

Proof. First note that $q_\lambda \in \mathbb{C}[V_\mathbb{R}]$ is K-invariant because p_λ is K-invariant and Δ is a $U(V)$-invariant operator. Moreover

$$q_\lambda = (-1)^{|\lambda|}p_\lambda + r_\lambda$$

where $p_\lambda \in \mathcal{P}_{2|\lambda|}(V_\mathbb{R})$ and r_λ is of lower degree. As $\{p_\lambda\}$ is a basis for $\mathbb{C}[V_\mathbb{R}]^K$, so is $\{q_\lambda\}$. $\square$

8.2. **The generalized binomial coefficients.** As q_λ belongs to $\mathbb{C}[V_\mathbb{R}]^K$, it can be written as a finite linear combination of the canonical invariants p_ν.

Definition 8.2.1. The *generalized binomial coefficients* $\begin{bmatrix}\lambda\\\nu\end{bmatrix}$ are defined for $\lambda,\nu \in \Lambda$ via

$$q_\lambda = \sum_{\nu\in\Lambda}(-1)^{|\nu|}\begin{bmatrix}\lambda\\\nu\end{bmatrix}p_\nu.$$

The proof of Lemma 8.1.2 shows that

$$\begin{bmatrix}\lambda\\\lambda\end{bmatrix} = 1, \quad \begin{bmatrix}\lambda\\\nu\end{bmatrix} = 0 \text{ when } |\lambda| = |\nu| \text{ but } \lambda \neq \nu \text{ and } \quad \begin{bmatrix}\lambda\\\nu\end{bmatrix} = 0 \text{ for } |\nu| > |\lambda|.$$

So in fact

$$q_\lambda = \sum_{|\nu|\le|\lambda|}(-1)^{|\nu|}\begin{bmatrix}\lambda\\\nu\end{bmatrix}p_\nu = (-1)^{|\lambda|}p_\lambda + \sum_{|\nu|<|\lambda|}(-1)^{|\nu|}\begin{bmatrix}\lambda\\\nu\end{bmatrix}p_\nu.$$

The terminology "generalized binomial coefficient" is motivated by Example 8.4.1 below. Our immediate goal is to develop some combinatorial properties of these coefficients and to relate them to eigenvalues for operators in $\mathcal{PD}(V)^G$. We begin with the following.

Proposition 8.2.2.

$$\sum_{\mu\in\Lambda}(-1)^{|\mu|}\begin{bmatrix}\lambda\\\mu\end{bmatrix}\begin{bmatrix}\mu\\\nu\end{bmatrix} = \begin{cases} (-1)^{|\lambda|} & \text{for } \lambda = \nu \\ 0 & \text{for } \lambda \neq \nu \end{cases}.$$

Proof. First note that as $q_\lambda = T(p_\lambda)$ and $T^2 = I$ one has

$$p_\lambda = T(q_\lambda) = \sum_\nu (-1)^{|\nu|} \begin{bmatrix}\lambda\\ \nu\end{bmatrix} T(p_\nu) = \sum_\nu (-1)^{|\nu|} \begin{bmatrix}\lambda\\ \nu\end{bmatrix} q_\nu, \tag{8.2.1}$$

an interesting formula in its own right. But now

$$\begin{aligned} q_\lambda &= \sum_\mu (-1)^{|\mu|} \begin{bmatrix}\lambda\\ \mu\end{bmatrix} p_\mu = \sum_\mu (-1)^{|\mu|} \begin{bmatrix}\lambda\\ \mu\end{bmatrix} \left(\sum_\nu (-1)^{|\nu|} \begin{bmatrix}\mu\\ \nu\end{bmatrix} q_\nu \right) \\ &= \sum_\nu (-1)^{|\nu|} \left(\sum_\mu (-1)^{|\mu|} \begin{bmatrix}\lambda\\ \mu\end{bmatrix} \begin{bmatrix}\mu\\ \nu\end{bmatrix} \right) q_\nu \end{aligned}$$

and the result follows from the linear independence of $\{q_\lambda \;:\; \lambda \in \Lambda\}$. □

Proposition 8.2.3. *For $\lambda \in \Lambda$ and $k \in \mathbb{N}$*

$$\frac{\Delta}{k!}^k p_\lambda = \sum_{|\nu| = |\lambda| - k} \begin{bmatrix}\lambda\\ \nu\end{bmatrix} p_\nu \tag{8.2.2}$$

and

$$\frac{\Delta}{k!}^k q_\lambda = (-1)^k \sum_{|\nu| = |\lambda| - k} \begin{bmatrix}\lambda\\ \nu\end{bmatrix} q_\nu \tag{8.2.3}$$

Proof. We have

$$\begin{aligned} \sum_\nu (-1)^{|\nu|} \begin{bmatrix}\lambda\\ \nu\end{bmatrix} p_\nu &= q_\lambda = T(p_\lambda) = (-1)^{|\lambda|} e^{-\Delta} p_\lambda \\ &= (-1)^{|\lambda|} \sum_k (-1)^k \frac{\Delta}{k!}^k p_\lambda. \end{aligned}$$

Equating homogeneous components of degree $2(|\lambda| - k)$ on both sides of this equation yields (8.2.2). Applying the operator T to both sides of (8.2.2) yields (8.2.3) since

$$T(\Delta^k p_\lambda) = (-1)^{|\lambda|-k} e^{-\Delta} \Delta^k p_\lambda = (-1)^k \Delta^k \big((-1)^{|\lambda|} e^{-\Delta} p_\lambda \big) = (-1)^k \Delta^k q_\lambda.$$

□

The next theorem is the key to all subsequent results in this section.

Theorem 8.2.4 (Yan's Pieri Formula)**.** *For $\nu \in \Lambda$, $k \in \mathbb{N}$,*

$$\frac{|z|}{k!}^{2k} d_\nu p_\nu = \sum_{|\lambda| = |\nu| + k} \begin{bmatrix}\lambda\\ \nu\end{bmatrix} d_\lambda p_\lambda.$$

Proof. First note that for polynomials $p, q \in \mathbb{C}[V_{\mathbb{R}}]$ one has

$$\langle z_j p, q\rangle_* = \langle p, \partial_j q\rangle_*, \quad \langle \overline{z}_j p, q\rangle_* = \langle p, \overline{\partial}_j q\rangle_*$$

and hence $\langle |z|^2 p, q\rangle_* = \langle p, \Delta q\rangle_*$. Now using Proposition 7.5.2 and Equation 8.2.2,

$$\begin{aligned}\frac{|z|^{2k}}{k!} p_\nu &= \sum_{|\lambda|=|\nu|+k} \frac{\langle (|z|^{2k}/k!)p_\nu, p_\lambda\rangle_*}{\langle p_\lambda, p_\lambda\rangle_*} p_\lambda \\ &= \sum_{|\lambda|=|\nu|+k} d_\lambda \langle p_\nu, (\Delta^k/k!)p_\lambda\rangle_* p_\lambda \\ &= \sum_{|\lambda|=|\nu|+k} d_\lambda \begin{bmatrix}\lambda\\ \nu\end{bmatrix} \langle p_\nu, p_\nu\rangle_* p_\lambda \\ &= \sum_{|\lambda|=|\nu|+k} \frac{d_\lambda}{d_\nu} \begin{bmatrix}\lambda\\ \nu\end{bmatrix} p_\lambda.\end{aligned}$$

□

Corollary 8.2.5. *For $|\lambda| = |\nu| + k$ one has*

$$\begin{bmatrix}\lambda\\ \nu\end{bmatrix} = \frac{1}{k!} \sum \begin{bmatrix}\varepsilon_1\\ \nu\end{bmatrix} \begin{bmatrix}\varepsilon_2\\ \varepsilon_1\end{bmatrix} \cdots \begin{bmatrix}\varepsilon_{k-1}\\ \varepsilon_{k-2}\end{bmatrix} \begin{bmatrix}\lambda\\ \varepsilon_{k-1}\end{bmatrix}$$

where the sum is over all $(\varepsilon_1, \ldots, \varepsilon_{k-1})$ with $|\varepsilon_j| = |\nu| + j$.

Proposition 8.2.6. *The generalized binomial coefficients are non-negative real numbers.*

Proof. Corollary 8.2.5 shows that it suffices to prove $\left[\begin{smallmatrix}\lambda\\ \nu\end{smallmatrix}\right] \geq 0$ for $|\lambda| = |\nu| + 1$. Let $\{f_1, \ldots, f_{d_\nu}\}$ be an orthonormal basis for P_ν with respect to the Fischer inner product. Since $z_i f_j \in \mathcal{P}_{|\nu|+1}(V) = \bigoplus_{|\lambda|=|\nu|+1} P_\lambda$, we can write

$$z_i f_j = \sum_{|\lambda|=|\nu|+1} f_\lambda(i,j)$$

where $f_\lambda(i,j) \in P_\lambda$. Hence also

$$|z|^2 d_\nu p_\nu = \sum_{i=1}^{n} \sum_{j=1}^{d_\nu} z_i f_j \overline{z_i f_j} = \sum_{|\lambda|=|\lambda'|=|\nu|+1} \sum_{i=1}^{n} \sum_{j=1}^{d_\nu} f_\lambda(i,j)\overline{f_{\lambda'}(i,j)}.$$

The sum $\sum_{|\lambda|=|\lambda'|=|\nu|+1} P_\lambda \otimes \overline{P}_{\lambda'}$ is direct in $\mathbb{C}[V_{\mathbb{R}}] = \mathbb{C}[V] \otimes \mathbb{C}[\overline{V}]$ and each $P_\lambda \otimes \overline{P}_{\lambda'}$ is a K-invariant subspace. Since $|z|^2 d_\nu p_\nu$ is a K-invariant polynomial, it follows that

$$\sum_{i=1}^{n} \sum_{j=1}^{d_\nu} f_\lambda(i,j)\overline{f_{\lambda'}(i,j)} \in P_\lambda \otimes \overline{P}_{\lambda'}$$

is K-invariant for each $|\lambda| = |\nu| + 1 = |\lambda'|$. But

$$(P_\lambda \otimes \overline{P}_{\lambda'})^K = \begin{cases} \{0\} & \text{for } \lambda \neq \lambda' \\ \mathbb{C}p_\lambda & \text{for } \lambda = \lambda' \end{cases}$$

via Schur's Lemma, as in Section 7.3. We conclude that $\sum_{i=1}^n \sum_{j=1}^{d_\nu} f_\lambda(i,j)\overline{f_{\lambda'}(i,j)} = 0$ for $\lambda \neq \lambda'$ and that $\sum_{i=1}^n \sum_{j=1}^{d_\nu} |f_\lambda(i,j)|^2 = c_\lambda p_\lambda$ for some $c_\lambda \in \mathbb{C}$. As p_λ and $\sum_{i=1}^n \sum_{j=1}^{d_\nu} |f_\lambda(i,j)|^2$ are both non-negative real valued polynomials on $V_\mathbb{R}$, we must have $c_\lambda \geq 0$. Hence

$$|z|^2 d_\nu p_\nu = \sum_{|\lambda|=|\nu|+1} c_\lambda p_\lambda$$

for some values $c_\lambda \geq 0$. Theorem 8.2.4 now implies $\left[\begin{smallmatrix}\lambda\\ \nu\end{smallmatrix}\right] = c_\lambda/d_\lambda \geq 0$. □

8.3. **Eigenvalues for operators in $\mathcal{PD}(V)^G$.** The operators $p_\nu(z,\partial)$ in the canonical basis for $\mathcal{PD}(V)^G$ act by scalars on each subspace $P_\lambda \subset \mathbb{C}[V]$. This is a consequence of Schur's Lemma, G-invariance of $p_\nu(z,\partial)$ and the fact that the decomposition $\mathbb{C}[V] = \oplus P_\lambda$ is multiplicity free.

Definition 8.3.1. For $\nu, \lambda \in \Lambda$ let $\widehat{p}_\nu(\lambda) \in \mathbb{C}$ denote the eigenvalue of $p_\nu(z,\partial)$ on P_λ. That is, $p_\nu(z,\partial)|_{P_\lambda} = \widehat{p}_\nu(\lambda) I_{P_\lambda}$.

Proposition 8.3.2. *$d_\nu \widehat{p}_\nu(\lambda) = \left[\begin{smallmatrix}\lambda\\ \nu\end{smallmatrix}\right]$ for all $\lambda, \nu \in \Lambda$.*

Proof. Note that

$$p_\nu(z,\partial) p_\lambda(z,\overline{z}) = \widehat{p}_\nu(\lambda) p_\lambda(z,\overline{z})$$

as $p_\lambda(z,\overline{z}) \in P_\lambda \otimes \overline{P}_\lambda$. Now using Theorem 8.2.4,

$$\begin{aligned} d_\nu p_\nu(z,\partial) e^{|z|^2} &= d_\nu p_\nu(z,\overline{z}) e^{|z|^2} \\ &= d_\nu p_\nu(z,\overline{z}) \sum_k \frac{|z|}{k!}^{2k} \\ &= \sum_k \sum_{|\lambda|=|\nu|+k} \begin{bmatrix}\lambda\\ \nu\end{bmatrix} d_\lambda p_\lambda(z,\overline{z}) \\ &= \sum_\lambda \begin{bmatrix}\lambda\\ \nu\end{bmatrix} d_\lambda p_\lambda(z,\overline{z}). \end{aligned}$$

But we can also use Equation 7.5.3 to write

$$e^{|z|^2} = \sum_k \frac{|z|}{k!}^{2k} = \sum_k \sum_{|\lambda|=k} d_\lambda p_\lambda(z,\overline{z}) = \sum_\lambda d_\lambda p_\lambda(z,\overline{z}).$$

Thus

$$d_\nu p_\nu(z,\partial) e^{|z|^2} = \sum_\lambda d_\nu d_\lambda \widehat{p}_\nu(\lambda) p_\lambda(z,\overline{z}).$$

Comparing these two expressions for $d_\nu p_\nu(z,\partial)e^{|z|^2}$ gives $d_\nu \widehat{p}_\nu(\lambda) = \left[\begin{smallmatrix}\lambda\\ \nu\end{smallmatrix}\right]$ as claimed. $\square$

Together Propositions 8.2.6 and 8.3.2 yield:

Corollary 8.3.3. *The eigenvalues $\widehat{p}_\nu(\lambda)$ are non-negative real numbers.*

8.4. **Examples.** Here are two examples to illustrate the circle of ideas developed above. Even the most basic example is of interest in this context.

Example 8.4.1. GL(n): Consider the usual action of $G = GL(n,\mathbb{C})$ on $V = \mathbb{C}^n$. We have maximal compact subgroup $K = U(n)$ and the standard inner product $\langle z, z'\rangle = z \cdot \overline{z}'$ is K-invariant. $\mathbb{C}[V]$ decomposes as

$$\mathbb{C}[V] = \mathbb{C}[z_1 \ldots, z_n] = \bigoplus_{m\in\mathbb{N}} \mathcal{P}_m(V)$$

under the actions of G and of K. Using the orthonormal basis $\{z^\alpha/\sqrt{\alpha!} \ : \ |\alpha| = m\}$ for $\mathcal{P}_m(V)$ (with respect to $\langle\cdot,\cdot\rangle_{\mathcal{F}}$) we compute the canonical invariant $p_m(z,\overline{z}) \in \mathbb{C}[V_{\mathbb{R}}]^K$:

$$p_m(z,\overline{z}) = \frac{1}{d_m}\sum_{|\alpha|=m}\frac{z^\alpha\overline{z}^\alpha}{\alpha!} = \frac{1}{d_m}\frac{1}{m!}\sum_{|\alpha|=m}\binom{m}{\alpha}|z|^{2\alpha} = \frac{1}{d_m}\frac{1}{m!}|z|^{2m}.$$

Substituting

$$d_m = dim(\mathcal{P}_m(V)) = \binom{m+n-1}{m}$$

one obtains

$$p_m(z,\overline{z}) = \frac{(n-1)!}{(m+n-1)!}|z|^{2m}.$$

The fundamental invariant is $\gamma = p_1 = |z|^2/n$ and the above computation shows that $\mathbb{C}[V_{\mathbb{R}}]^K = \mathbb{C}[\gamma]$, illustrating Theorem 7.4.3.

Next we compute

$$\frac{|z|^{2(m-k)}}{(m-k)!}d_k p_k = \frac{|z|^{2(m-k)}}{(m-k)!}\frac{|z|}{k!}^{2k} = \binom{m}{k}\frac{|z|}{m!}^{2m} = \binom{m}{k}d_m p_m.$$

Now Theorem 8.2.4 implies

$$\begin{bmatrix}m\\ k\end{bmatrix} = \binom{m}{k}$$

for $k, m \in \mathbb{N}$. This fact motivates the terminology "generalized binomial coefficient".

The polynomials $\{q_m \mid m \in \mathbb{N}\}$ are given by

$$q_m(z,\overline{z}) = \sum_k (-1)^k\binom{m}{k}p_k(z,\overline{z}) = \sum_k(-1)^k\binom{m}{k}\frac{(n-1)!}{(k+n-1)!}|z|^{2k} = L_m^{(n-1)}(|z|^2)$$

where

$$L_m^{(r)}(x) = r! \sum_{k=0}^{m} \binom{m}{k} \frac{(-x)^k}{(k+r)!}$$

is the *generalized Laguerre polynomial* of order r and degree m, normalized to have value 1 at $x = 0$.

Now according to Proposition 8.3.2 the operator $d_k p_k(z, \partial) \in \mathcal{PD}(V)^G$ has eigenvalue

$$d_k \widehat{p}_k(m) = \binom{m}{k}$$

on $\mathcal{P}_m(V)$. One can see this directly because

$$d_k p_k(z, \partial)(z_1^m) = \sum_{|\alpha|=k} \frac{z^\alpha \partial^\alpha}{\alpha!}(z_1^m) = \frac{z_1^k \partial_1{}^k}{k!}(z_1^m) = \frac{z_1^k}{k!} \frac{m!}{(m-k)!} z_1^{m-k} = \binom{m}{k} z_1^m$$

for $m \geq k$.

For this example Proposition 8.2.2 asserts that

$$\sum_{k=\ell}^{m} (-1)^k \binom{m}{k} \binom{k}{\ell} = \delta_{m,\ell}(-1)^m. \tag{8.4.1}$$

Equivalently, the lower triangular matrix A with entries $A_{m,k} = (-1)\binom{m}{k}$ $(0 \leq m, k \leq N$ say) is self inverting: $A^2 = A$. One can verify (8.4.1) directly as follows.

$$\begin{aligned} x^m = (1-(1-x))^m &= \sum_k (-1)^k \binom{m}{k} (1-x)^k = \sum_k (-1)^k \binom{m}{k} \sum_\ell (-1)^\ell \binom{k}{\ell} x^\ell \\ &= \sum_\ell (-1)^\ell \left[\sum_k (-1)^k \binom{m}{k} \binom{k}{\ell} \right] x^\ell. \end{aligned}$$

Example 8.4.2. $\mathbf{GL(n)} \otimes \mathbf{GL(n)}$: Recall the (twisted) action (4.1.2) of $G = GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$ on $V = \mathbb{C}^n \otimes \mathbb{C}^n \cong M_{n,n}(\mathbb{C})$. Restricting to the maximal compact subgroup $K = U(n) \times U(n)$ we have

$$(k_1, k_2) \cdot z = \overline{k}_1 v k_2^*$$

where $k^* = \overline{k}^t = k^{-1}$. The inner product $\langle z, w \rangle = tr(zw^*)$ on V is K-invariant. The decomposition

$$\mathbb{C}[M_{n,n}(\mathbb{C})] = \bigoplus_{\lambda \in \Lambda} P_\lambda$$

under the action of K, given in Theorem 4.1.1, is indexed by partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$.

Let $\Sigma \subset V$ denote the set of matrices of the form $h = diag(d_1, \ldots, d_n)$ with $d_j \in \mathbb{R}^+$.

Proposition 8.4.3. *([6]) For $\lambda \in \Lambda$, $p_\lambda \in \mathbb{C}[V_{\mathbb{R}}]^K$ is determined by its restriction to Σ via the formula*

$$p_\lambda(h) = c_\lambda s_\lambda(d_1^2, \dots, d_n^2)$$

where s_λ is a Schur polynomial in n variables and c_λ is a positive constant.

The Schur polynomial s_λ arises as the character of the representation σ_n^λ of $GL(n, \mathbb{C})$ with highest weight λ,

$$s_\lambda(x_1, \dots, x_n) = tr(\sigma_n^\lambda(diag(x_1, \dots, x_n)),$$

and is given explicitly by the determinantal formula

$$s_\lambda(x_1, \dots, x_n) = \frac{det\big[x_i^{\lambda_j+n-j}\big]}{det\big[x_i^{n-j}\big]}. \tag{8.4.2}$$

The Schur function $s_{(1^k)}$ is the k'th elementary symmetric function e_k. Thus the fundamental invariants $\gamma_k = p_{(1^k)} \in \mathbb{C}[V_{\mathbb{R}}]^K$ are determined, up to normalization, by

$$\gamma_k(h) = c_k e_k(d_1^2, \dots, d_n^2)$$

on the cross-section Σ.

It is useful to identify partitions $\lambda \in \Lambda$ with their Young's diagrams. Given two diagrams $\lambda, \nu \in \Lambda$, we write $\nu \subset \lambda$ when ν is a sub-diagram of λ. That is, $\nu_j \leq \lambda_j$ for all j. If $|\lambda| = |\nu| + k$ then $\mathcal{C}_{\lambda\nu}$ will denote the number of sequences $(\varepsilon_0, \dots, \varepsilon_k)$ of Young's diagrams such that

- $\nu = \varepsilon_0 \subset \varepsilon_1 \subset \cdots \subset \varepsilon_{k-1} \subset \varepsilon_k = \lambda$, and
- $|\varepsilon_j| = |\nu| + j$ for $j = 1, \dots, k$.

That is, ε_j is obtained from ε_{j-1} by adding a single box $\square$ to some row. Note that $\mathcal{C}_{\lambda\nu} = 0$ if $\nu \not\subset \lambda$. When $\nu \subset \lambda$, $\mathcal{C}_{\lambda\nu}$ is the number of *standard tableaux* of shape $\lambda - \nu$. That is, the number of ways to assign the values $1, 2, \dots, k$ to the boxes of the skew-diagram $\lambda - \nu$ so that values increase as we move along rows from left to right and as we move down columns.

Proposition 8.4.4. *The generalized binomial coefficients can be expressed as*

$$\begin{bmatrix} \lambda \\ \nu \end{bmatrix} = \frac{1}{k!} \frac{d_\nu c_\nu}{d_\lambda c_\lambda} \mathcal{C}_{\lambda\nu}$$

for $|\lambda| = |\nu| + k$.

Proof. Note that the polynomial $\gamma(z) = |z|^2 = tr(zz^*)$ is given on Σ by

$$\gamma(h) = d_1^2 + \cdots + d_n^2 = s_{(1)}(d_1^2, \dots, d_n^2).$$

Consider the case where $|\lambda| = |\nu| + 1$. Theorem 8.2.4 and Proposition 8.4.3 yield

$$d_\nu c_\nu s_{(1)} s_\nu = \sum_{|\lambda| = |\nu|+1} \begin{bmatrix} \lambda \\ \nu \end{bmatrix} d_\lambda c_\lambda s_\lambda.$$

On the other hand, the classical *Pieri formula* asserts:

$$s_{(1)}s_\nu = \sum_{\substack{|\lambda|=|\nu|+1 \\ \nu\subset\lambda}} s_\lambda.$$

Comparing these expressions gives

$$\begin{bmatrix}\lambda \\ \nu\end{bmatrix} = \left\{ \begin{array}{ll} \frac{d_\nu c_\nu}{d_\lambda c_\lambda} & \text{if } \nu\subset\lambda \\ 0 & \text{if } \nu\not\subset\lambda \end{array} \right\} = \frac{d_\nu c_\nu}{d_\lambda c_\lambda}\mathcal{C}_{\lambda\nu}.$$

An application of Corollary 8.2.5 now yields the result for $|\lambda| = |\nu| + k$. □

The dimensions $d_\mu = dim(P_\mu)$ are $d_\mu = dim(\sigma_n^\mu)^2$, in view of Theorem 4.1.1. A classical formula for these dimensions gives

$$d_\mu = \left[\prod_{i<j} \frac{\mu_i - \mu_j + j - i}{j - i}\right]^2.$$

Remark 8.4.5. This example motivates the name "Yan's Pieri Formula" for Theorem 8.2.4. We will return to this example below in Section 9.4. As a byproduct of this subsequent analysis, one sees that the normalization constants c_λ appearing in Propositions 8.4.3 and 8.4.4 are given by $c_\lambda = 1/H(\lambda)$, where $H(\lambda)$ is as in Proposition 9.4.1.

8.5. **Section 8 notes.** The polynomials q_λ play a role in connection with analysis on the Heisenberg group. Let $H_V = V \times \mathbb{R}$ with product

$$(z,t)(z',t') = \big(z + z', t + t' - \frac{1}{2}Im\langle z, z'\rangle\big).$$

Any compact Lie subgroup K of $U(V)$ act by automorphisms on H_V via

$$k \cdot (z,t) = (kz, t).$$

It is known that $(K \ltimes H_V, K)$ is a Gelfand pair if and only if $K : V$ is a multiplicity free action. A generic set of spherical functions for such a Gelfand pair is completely determined by the q_λ-polynomials. In particular,

$$\phi(z,t) = q_\lambda(z)e^{-|z|^2/2}e^{it}$$

is one such spherical function. We refer the reader to [4] concerning this connection.

One can extend the inner product $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ from $\mathbb{C}[V]$ to $\mathbb{C}[V_\mathbb{R}]$ using Equation 7.5.2. The q_λ's are then orthogonal with respect to $\langle\cdot,\cdot\rangle_{\mathcal{F}}$. In fact, they can be obtained via Gram-Schmidt orthogonalization using $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ from the p_λ's [4].

Generalized binomial coefficients were first introduced in the setting of Hermitian symmetric spaces. See [12], [37] and [13]. Z. Yan subsequently defined generalized binomial coefficients in the more general context of multiplicity free actions in his unpublished manuscript [56]. This contains the first proofs of Theorem 8.2.4 and

Proposition 8.3.2. Further combinatorial identities concerning the generalized binomial coefficients can be found in [6]. Both [56] and [6] use representation theory for the Heisenberg group and exploit the connection with spherical functions outlined above. The treatment given here achieves some simplification.

In [7] it is shown that the generalized binomial coefficients are (non-negative) *rational* numbers. Thus the same holds for the eigenvalues $\widehat{p}_\nu(\lambda)$. The proof involves extensive case-by-case analysis working from the classification for multiplicity free actions presented in Section 6.

For background on Schur functions, including the Pieri formula, we refer the reader to Chapter I in [39].

9. Eigenvalues for operators in $\mathcal{PD}(V)^G$

Recall that for a multiplicity free action $G : V$ each $D \in \mathcal{PD}(V)^G$ acts by scalars on the irreducible subspaces $\{P_\lambda \ : \ \lambda \in \Lambda\}$ in the decomposition of $\mathbb{C}[V]$. For $\nu \in \Lambda$, $\widehat{p}_\nu(\lambda)$ denotes the eigenvalue of $p_\nu(z, \partial)$ on P_λ. This is $d_\nu \widehat{p}_\nu(\lambda) = \begin{bmatrix}\lambda \\ \nu\end{bmatrix}$. We will see that the map $\widehat{p}_\nu : \Lambda \to \mathbb{C}$ extends in a natural way to a polynomial function on the subspace $Span_{\mathbb{C}}(\Lambda)$ of $\mathfrak{h}^*$.

9.1. **Eigenvalue polynomials.**

We say that an operator $D \in \mathcal{PD}(V)$ has *order* m if

$$D \in \delta\big(\mathbb{C}[V] \otimes \sum_{j \leq m} \mathcal{P}_j(V^*)\big).$$

Here $\delta : \mathbb{C}[V \oplus V^*] \to \mathcal{PD}(V)$ is the map given by Wick quantization, $p \mapsto p(z, \partial)$. In terms of coordinates, D has order m when $D \in Span\{z^\alpha \partial^\beta \ : \ |\beta| \leq m\}$.

Proposition 9.1.1. *Let* $D \in \mathcal{PD}(V)$ *and* $f = f_1^{a_1} \cdots f_r^{a_r} \in \mathbb{C}[V]$. *Then*

$$Df(z) = \beta_D(z; a_1, \ldots, a_r) f(z)$$

where $\beta_D \in \mathbb{C}[V][f_1^{-1}, \ldots, f_r^{-1}][a_1, \ldots, a_r]$.

Proof. The proof is by induction on the order of D. For operators of order 1, it is enough to consider $D = \partial_w$, a directional derivative. In this case

$$Df(z) = \sum_j a_j (\partial_w f_j) \left(\frac{f}{f_j}\right),$$

so $\beta_D = \sum_j a_j (\partial_w f_j)/f_j$ in this case.

For D of order greater than 1, write $D = g + \sum_i X_i E_i$ where $g \in \mathbb{C}[V]$ and the X_i's have order 1. Now $E_i f(z) = \beta_i(z; a_1, \ldots, a_r) f(z)$ for suitable β_i by the induction hypothesis. As

$$X_i E_i f(z) = X_i \beta_i f(z) + \beta_i X_i f(z) = \left[X_i \beta_i + \beta_i \sum_j a_j \frac{X_i(f_j)}{f_j}\right] f(z)$$

the result follows. □

Corollary 9.1.2. (1) *As a polynomial in $a = (a_1, \dots, a_r)$, the degree of $\beta_D(z;a)$ is order(D).*

(2) *As polynomials in a, the homogeneous components of highest degree in $\beta_D(z;a)$ and $\sigma_D(z, \frac{df}{f}(z))$ agree. Here $\sigma_D \in \mathbb{C}[V \oplus V^*]$ denotes the Wick symbol of D and*

$$\frac{df}{f} = a_1 \frac{df_1}{f_1} + \cdots + a_r \frac{df_r}{f_r}.$$

Proof. (1) follows by induction from the proof of Proposition 9.1.1. For (2), we first examine $Df(z)$. For $D = \partial_w$ one has

$$\sigma_D\left(z, \frac{df}{f}\right) = a_1 \frac{w^*(f_1)}{f_1} + \cdots + a_r \frac{w^*(f_r)}{f_r} = a_1 \frac{\partial_w f_1}{f_1} + \cdots + a_r \frac{\partial_w f_r}{f_r} = \beta_D(z;a).$$

Given any polynomial $b(a)$ in $a = (a_1 \dots, a_r)$ we will write *top*(b) for the homogeneous component of highest degree. If we look closely at the induction step, with $D = g + \sum_i X_i E_i$, we see that

$$top(\beta_D) = \sum_i top(\beta_i)(z;a) \sum_j a_j \frac{X_i(f_j)}{f_j}.$$

By the induction hypothesis, $top(\beta_i) = top(\sigma_{E_i}(z, df/f))$, and we have just seen that $top(\sigma_{X_i})(z, df/f) = \sum_j a_j X_i(f_j)/f_j$. The result now follows since $top(\sigma_{X_i E_i}) = top(\sigma_{X_i}\sigma_{E_i})$. □

Now suppose that $G : V$ is a multiplicity free action with, as usual,

(1) $\Lambda \subset \mathfrak{h}^*$ the set of highest weights for the representations that occur in $\mathbb{C}[V]$,
(2) $\Lambda' = \{\lambda_1, \dots, \lambda_r\}$ the set of fundamental highest weights, and
(3) $h_j = h_{\lambda_j}$ $(1 \le j \le r)$ the fundamental highest weight vectors.

Then $h = h_1^{a_1} \cdots h_r^{a_r}$ is a highest weight vector in P_λ for $\lambda = a_1\lambda_1 + \cdots + a_r\lambda_r \in \Lambda$. Thus for any $\nu \in \Lambda$,

$$\widehat{p_\nu}(\lambda) h(z) = p_\nu(z, \partial) h(z) = \beta_\nu(z; a_1, \dots, a_r) h(z)$$

with $\beta_\nu = \beta_{p_\nu(z,\partial)}$ as in Proposition 9.1.1. It follows that $\beta_\nu(z;a) = \widehat{p_\nu}(\lambda)$ for all z. Thus $\beta_\nu = \beta_\nu(a)$ is a polynomial in the parameters a, independent of z. Since

$$\widehat{p_\nu}(a_1\lambda_1 + \cdots + a_r\lambda_r) = \beta_\nu(a_1, \dots, a_r)$$

we see that $\widehat{p_\nu}$ extends in a natural way to a polynomial function on $Span_{\mathbb{C}}(\Lambda)$ in $\mathfrak{h}^*$. Moreover, by Corollary 9.1.2 we have

$$top\big(\widehat{p_\nu}(a_1\lambda_1 + \cdots + a_r\lambda_r)\big) = p_\nu\left(z, a_1\frac{dh_1}{h_1} + \cdots + a_r\frac{dh_r}{h_r}\right). \tag{9.1.1}$$

Note that although both arguments on the right hand side of Equation 9.1.1 depend on z, the result is independent of z, and gives a polynomial function of the parameters $a_1, \dots, a_r$.

9.2. **A Harish-Chandra homomorphism for multiplicity free actions.** Let $B = HN$ be a Borel subgroup in G with $\Delta^+ \subset \mathfrak{h}^*$ the associated set of positive roots. Let W denote the Weyl group and

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$$

be half the sum of the positive roots. $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra for $\mathfrak{g}$ with center $\mathcal{Z}\mathcal{U}(\mathfrak{g})$. The Harish-Chandra homomorphism is an algebra isomorphism

$$H : \mathcal{Z}\mathcal{U}(\mathfrak{g}) \to \mathbb{C}[\mathfrak{h}^*]^W$$

such that

$$Z \in \mathcal{Z}\mathcal{U}(\mathfrak{g}) \text{ acts on } V_\lambda \text{ by the scalar } H(Z)(\lambda + \rho) \text{ for all highest weights } \lambda.$$

Let $G : V$ be a multiplicity free action. Knop constructs a map on $\mathcal{P}\mathcal{D}(V)^G$ analogous to the Harish-Chandra homomorphism. Let

$$\mathfrak{a}^* = Span_{\mathbb{C}}(\Lambda) = Span_{\mathbb{C}}\{\lambda_1, \ldots, \lambda_r\} \subset \mathfrak{h}^*, \quad A = \mathfrak{a}^* + \rho$$

and define a map

$$h : \mathcal{P}\mathcal{D}(V)^G \to \mathbb{C}[A], \quad h(D)(\lambda + \rho) = \beta_D(\lambda). \tag{9.2.1}$$

(As before, D acts on P_λ by the scalar $\beta_D(\lambda)$ when $\lambda \in \Lambda$.) Then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{Z}\mathcal{U}(\mathfrak{g}) & \xrightarrow{H} & \mathbb{C}[\mathfrak{h}^*]^W \\ \Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle r} \\ \mathcal{P}\mathcal{D}(V)^G & \xrightarrow{h} & \mathbb{C}[A] \end{array} \tag{9.2.2}$$

The map π is induced by the representation of G on V, and r is the restriction map.

Theorem 9.2.1. *([31]) $h : \mathcal{P}\mathcal{D}(V)^G \cong \mathbb{C}[A]^{W_\circ}$ as algebras where $W_\circ \subset Stab_W(\mathfrak{a}^* + \rho)$ is a finite reflection group.*

The proof of Theorem 9.2.1 is given in Lemmas 9.2.2 through 9.2.6 below:

Lemma 9.2.2. *Let $N = Stab_W(A)$. Then $Im(r) \subset \mathbb{C}[A]^N$ and these two rings have the same field of fractions.*

Proof. From (9.2.2) it is clear that $Im(r) \subset \mathbb{C}[A]^N$ and r factors through $Im(r)$:

$$\mathbb{C}[\mathfrak{h}^*]^W \twoheadrightarrow Im(r) \hookrightarrow \mathbb{C}[A]^N.$$

Letting $\mathfrak{h}^*//W$, $\mathcal{V}$ and $A//N$ denote the varieties with coordinate rings $\mathbb{C}[\mathfrak{h}^*]^W$, $Im(r)$ and $\mathbb{C}[A]^N$ respectively, we have

$$\mathfrak{h}^*//W \hookleftarrow \mathcal{V} \twoheadleftarrow A//N.$$

We see that $r^* : A//N \to \mathfrak{h}^*//W$ fails to be one-to-one whenever $\sigma \in W$, $\sigma \notin N$ but $\sigma(A) \cap A \neq \emptyset$. In this case, $\sigma(A) \neq A$, so $\sigma(A) \cap A$ is a lower-dimensional subvariety in A. Thus r^* fails to be one-to-one on $\cup_{\sigma \in W \setminus N}(\sigma(A) \cap A)$, whose complement is an open set in A. Thus the map $A//N \to \mathcal{V}$ is onto and generically one-to-one, so these varieties have the same field of rational functions. □

Lemma 9.2.3. $\mathbb{C}[A]^N$ *is the integral closure of* $Im(r)$ *in its field of fractions.*

Proof. Since $\mathbb{C}[A]$ is a polynomials ring, it is integrally closed in its quotient field $\mathbb{C}(A)$. Thus if $f \in \mathbb{C}(A)^N$ is integral over $\mathbb{C}[A]^N$ it is integral over $\mathbb{C}[A]$ and hence $f \in \mathbb{C}[A] \cap \mathbb{C}(A)^N = \mathbb{C}[A]^N$. This shows that $\mathbb{C}[A]^N$ is integrally closed in $\mathbb{C}(A)^N$.

It is known that $\mathbb{C}[\mathfrak{h}^*]$ is integral over $\mathbb{C}[\mathfrak{h}^*]^W$. (See Lemma 4.1.2 in [50].) Hence the homomorphic image $\mathbb{C}[A]$ of $\mathbb{C}[\mathfrak{h}^*]$ is integral over $Im(r)$ and in particular $\mathbb{C}[A]^N$ is integral over $Im(r)$. Using Lemma 9.2.2, we conclude that $\mathbb{C}[A]^N$ is the integral closure of $Im(r)$ in its quotient field. □

Lemma 9.2.4. $\mathbb{C}[A]^N \subset Im(h) \subset \mathbb{C}[A]$.

Proof. The map h is injective since each $D \in \mathcal{PD}(V)^G$ is completely determined by its eigenvalues $\{\beta_D(\lambda) \ : \ \lambda \in \Lambda\}$. As $\mathcal{PD}(V)^G$ is a polynomial ring (by Corollary 7.4.4), so is $Im(h)$. We have $Im(r) \subset Im(h) \subset \mathbb{C}[A]$. Given $f \in \mathbb{C}[A]^N$, we know that f is integral over $Im(r)$, hence over $Im(h)$. Thus $f \in Im(h)$. □

Lemma 9.2.5. $Im(h) = \mathbb{C}[A]^{W_\circ}$ *for some subgroup* $W_\circ \subset N$.

Proof. Apply Galois theory to the fraction fields of the rings $\mathbb{C}[A]^N \subset Im(h) \subset \mathbb{C}[A]$. □

Lemma 9.2.6. $W_\circ$ *is a finite reflection group.*

Proof. This follows from the fact that $\mathbb{C}[A]^{W_\circ} = Im(h)$ is a polynomial ring. (See Theorem 4.2.5 in [50].) □

9.3. **Characterizing the eigenvalue polynomials.** Recall that for $\nu \in \Lambda$, $\widehat{p}_\nu$ can be regarded as a polynomial function on $\mathfrak{a}^* = Span_{\mathbb{C}}(\Lambda)$. We now shift $d_\nu \widehat{p}_\nu$ to obtain a polynomial e_ν on $A = \mathfrak{a}^* + \rho$:

$$e_\nu(\lambda + \rho) = d_\nu \widehat{p}_\nu(\lambda)$$

for $\lambda \in \mathfrak{a}^*$. Note that

$$e_\nu = h\big(d_\nu p_\nu(z, \partial)\big)$$

where h is given by (9.2.1). Moreover, $e_\nu(\lambda + \rho) = \left[\begin{smallmatrix}\lambda \\ \nu\end{smallmatrix}\right]$ when $\lambda \in \Lambda$, in view of Proposition 8.3.2.

These polynomials have the following remarkable property.

Theorem 9.3.1. *([31]) For* $\nu \in \Lambda$ *the polynomial* e_ν *is the unique polynomial such that:*

(1) e_ν *is* $W_\circ$*-invariant.*

(2) $deg(e_\nu) \leq |\nu|$.
(3) $e_\nu(\lambda+\rho) = 0$ *for all* $\lambda \in \Lambda$ *with* $|\lambda| \leq |\nu|$, $\lambda \neq \nu$.
(4) $e_\nu(\nu+\rho) = 1$.

Proof. Property (1) holds because e_ν belongs to $Im(h) = \mathbb{C}[\mathfrak{a}^* + \rho]^{W_\circ}$. Property (2) holds by Corollary 9.1.2. Properties (3) and (4) are basic facts concerning the generalized binomial coefficients $\begin{bmatrix}\lambda\\ \nu\end{bmatrix} = e_\nu(\lambda+\rho)$, as explained following Definition 8.2.1.

It remains to show uniqueness of e_ν. Let $d = |\nu|$ and set

$$\Lambda_d = \{\lambda \in \Lambda \ : \ |\lambda| \leq d\}.$$

The map h restricts to yield an isomorphism

$$h : \mathcal{PD}^{(d)}(V)^G \to \mathbb{C}^{(d)}[\mathfrak{a}^* + \rho]^{W_\circ}$$

where $\mathcal{PD}^{(d)}(V)^G = \mathcal{PD}^{(d)}(V) \cap \mathcal{PD}(V)^G$ are the G-invariants in $\mathcal{PD}^{(d)}(V)$ and $\mathbb{C}^{(d)}[\mathfrak{a}^* + \rho]^{W_\circ} = \left(\sum_{m \leq d} \mathcal{P}_m(\mathfrak{a}^* + \rho)\right) \cap \mathbb{C}[\mathfrak{a}^* + \rho]^{W_\circ}$. As $\{p_\lambda(z,\partial) \ : \ \lambda \in \Lambda_d\}$ is a basis for $\mathcal{PD}^{(d)}(V)^G$, we have that $\{e_\lambda \ : \ \lambda \in \Lambda_d\}$ is a basis for $\mathbb{C}^{(d)}[\mathfrak{a}^* + \rho]^{W_\circ}$. Now consider the linear map

$$\varepsilon : \mathbb{C}^{(d)}[\mathfrak{a}^* + \rho]^{W_\circ} \to \mathbb{C}^{\Lambda_d}, \quad \varepsilon(f) = \big(f(\lambda+\rho) \ : \ \lambda \in \Lambda_d\big).$$

In particular we have

$$\varepsilon(e_\nu) = \left(\begin{bmatrix}\lambda\\ \nu\end{bmatrix} \ : \ \lambda \in \Lambda_d\right).$$

As $\begin{bmatrix}\lambda\\ \lambda\end{bmatrix} = 1$ and $\begin{bmatrix}\lambda\\ \nu\end{bmatrix} = 0$ whenever $|\nu| > |\lambda|$ or $|\nu| = |\lambda|$ but $\nu \neq \lambda$, we see that $\{\varepsilon(e_\nu) \ : \ \nu \in \Lambda_d\}$ is a basis for $\mathbb{C}^{\Lambda_d}$. Thus ε is a vector space isomorphism. Now if $f \in \mathbb{C}[\mathfrak{a}^* + \rho]$ satisfies properties (1)-(4) then $f \in \mathbb{C}^{(d)}[\mathfrak{a}^* + \rho]^{W_\circ}$ and $\varepsilon(f) = \varepsilon(e_\nu)$. Hence $f = e_\nu$ as desired. $\square$

9.4. $\mathbf{GL(n) \otimes GL(n)}$ **yet again.** Recall the action of $G = GL(n,\mathbb{C}) \times GL(n,\mathbb{C})$ on $V = \mathbb{C}^n \otimes \mathbb{C}^n \cong M_{n,n}(\mathbb{C})$. Here $\mathfrak{h} = \mathfrak{h}_n \times \mathfrak{h}_n$ and the Weyl group W is isomorphic to $S_n \times S_n$. The highest weights that occur in $\mathbb{C}[V]$ have the form $(\lambda;\lambda)$ for $\lambda \in \mathfrak{h}_n^*$ non-negative and dominant. Thus we identify Λ with the set of all partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ and $\mathfrak{a}^* = Span_{\mathbb{C}}(\Lambda) = Span_{\mathbb{C}}\{(1^k;1^k) \ : \ 1 \leq k \leq n\}$ with $\mathfrak{h}^* \cong \mathbb{C}^n$. We can take

$$\rho = (n-1, n-2, \ldots, 1, 0)$$

under this identification. $W_\circ = Stab_W(\mathfrak{a}^* + \rho)$ is the diagonal subgroup in W which we identify with S_n acting as usual on $\mathbb{C}^n$:

$$\sigma \cdot (z_1, \ldots, z_n) = (z_{\sigma(1)}, \ldots, z_{\sigma(n)}).$$

Thus $\mathbb{C}[\mathfrak{a}^* + \rho]^{W_\circ}$ is identified with

$$\Lambda^*(n) = \{p \in \mathbb{C}[z_1, \ldots, z_n] \ : \ p(z+\rho) = p(\sigma \cdot z + \rho) \text{ for all } \sigma \in S_n\},$$

the algebra of *shifted symmetric polynomials* in n variables.

Now given $\nu \in \Lambda$ the *shifted Schur polynomial* $s^*_\nu(z)$ is defined by

$$s^*_\nu(z_1, \dots, z_n) = \frac{det[(z_i + n - i \downharpoonleft \nu_j + n - j)]}{det[(z_i + n - i \downharpoonleft n - j)]}, \tag{9.4.1}$$

where $(y \downharpoonleft k)$ is the *falling factorial*

$$(y \downharpoonleft k) = y(y-1)\cdots(y-k+1).$$

It is shown in [42] that s^*_ν is a well defined polynomial function. The reader should compare Equation (9.4.1) with the determinantal formula (8.4.2) for Schur polynomials.

Recall that the canonical invariants p_ν for this example are given, up to normalization, by the Schur polynomials s_ν. (See Proposition 8.4.3.) The eigenvalue polynomials e_ν, which interpolate the generalized binomial coefficients, are, up to normalization, the shifted Schur polynomials.

Proposition 9.4.1. *The eigenvalue polynomial e_ν for partition ν is given by*

$$e_\nu(\lambda + \rho) = \frac{1}{H(\nu)} s^*_\nu(\lambda)$$

where

$$H(\nu) = \frac{\prod_i (\nu_i + n - i)!}{\prod_{i<j}(\nu_i - \nu_j - i + j)}$$

is the product of the hook-lengths for ν.

Proof. We apply the characterization Theorem 9.3.1. The definition of s^*_ν shows that s^*_ν is shifted symmetric with degree $|\nu|$.

Suppose that $\lambda \neq \nu$ is a partition with $|\lambda| \leq |\nu|$. We have $s^*_\nu(\lambda) = 0$ as required by property (3) in Theorem 9.3.1. Indeed, $\lambda_\ell < \nu_\ell$ for some ℓ. Thus for all $j \leq \ell \leq i$ one has $\lambda_i \leq \lambda_\ell < \nu_\ell \leq \nu_j$ and hence $(\lambda_i + n - i \downharpoonleft \nu_j + n - j) = 0$. That is, the (i, j)'th entry in the determinant from the numerator in (9.4.1) vanishes for $j \leq \ell \leq i$. It follows that $s^*_\nu(\lambda) = 0$ as claimed.

Finally we check that $s^*_\nu(\nu) = H(\nu)$. First note that $(\nu_i + n - i \downharpoonleft \nu_j + n - j) = 0$ for $i > j$. Hence

$$det[(\nu_i + n - i \downharpoonleft \nu_j + n - j)] = \prod_i (\nu_i + n - i)!$$

The denominator in $s^*_\nu(\nu)$ is the Vandermonde determinant in the variables $\nu + \rho$. This gives

$$det[(\nu_i + n - i \downharpoonleft n - j)] = \prod_{i<j}(\nu_i - \nu_j - i + j),$$

so $s^*_\nu(\nu) = H(\nu)$ as claimed. $\square$

9.5. **Section 9 notes.** The Harish-Chandra isomorphism first appeared in [19]. The reader can find a modern treatment in Secion V.5 of [29]. One reference for facts concerning integrality and integral closures, used in the proof of Theorem 9.2.1, is the text [2] by Atiyah and Macdonald.

The results in this section are due to Knop. A considerably more general version Theorem 9.2.1 was proved in [30]. This asserts that the center of the ring of invariant differential operators for any smooth affine G-variety is a polynomial ring, canonically isomorphic to the ring of invariants for a finite reflection group.

Knop calls $W_\circ$ the *little Weyl group*. This group is given explicitly in [31] for each saturated indecomposable multiplicity free action in the classification from Section 6. In most cases, $W_\circ$ coincides with $N = Stab_W(A)$.

Theorem 9.3.1 was conjectured by Sahi, who proved a special case in [46]. Shifted Schur polynomials are due to Okounkov and Olshanski [42, 43]. The proof of vanishing is taken from [42], which includes many remarkable properties for these functions. Recent work of Knop yields the eigenvalue polynomials e_ν for many other multiplicity free actions. For the actions $GL(n,\mathbb{C}) : S^2(\mathbb{C}^n)$, $GL(n,\mathbb{C}) : \Lambda^2(\mathbb{C}^n)$, $SO(n,\mathbb{C}) \times \mathbb{C}^\times : \mathbb{C}^n$ and $E_6 \times \mathbb{C}^\times : \mathbb{C}^{27}$ this results in *shifted Jack polynomials* with various parameters. See [32, 33].

A multiplicity free action $G : V$ is said to be a *Capelli action* when the map $\pi : \mathcal{ZU}(\mathfrak{g}) \to \mathcal{PD}(V)^G$ in (9.2.2) is surjective. In [24] it is shown that the irreducible multiplicity free actions $\mathbf{G}' \times \mathbb{C}^\times : V$ in Table 3 are all Capelli actions except for

$$\mathbf{G}' = \mathbf{Sp(2n)} \otimes \mathbf{SL(3)}, \mathbf{Spin(9)}, \mathbf{E_6}.$$

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