DIFFERENTIAL SYSTEMS OF TYPE $(1,1)$ ON HERMITIAN SYMMETRIC SPACES AND THEIR SOLUTIONS

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Abstract. This paper concerns G-invariant systems of second order differential operators on irreducible Hermitian symmetric spaces G/K. The systems of type $(1,1)$ are obtained from K-invariant subspaces of $\mathfrak{p}_+ \otimes \mathfrak{p}_-$. We show that all such systems can be derived from a decomposition $\mathfrak{p}_+ \otimes \mathfrak{p}_- = \mathcal{H}' \oplus \mathcal{L} \oplus \mathcal{H}^c$. Here $\mathcal{L}$ gives the Laplace-Beltrami operator and $\mathcal{H} = \mathcal{H}' \oplus \mathcal{L}$ is the celebrated Hua system, which has been extensively studied elsewhere. Our main result asserts that for G/K of rank at least two, a bounded real-valued function is annihilated by the system $\mathcal{L} \oplus \mathcal{H}^c$ if and only if it is the real part of a holomorphic function. In view of previous work, one obtains a complete characterization of the bounded functions that are solutions for any system of type $(1,1)$ which contains the Laplace-Beltrami operator.

Contents

1. Introduction

Let G/K be a non-compact irreducible Hermitian symmetric space of rank r. The algebra $\mathbf{D}(G/K)$ of left-G-invariant differential operators on G/K has r algebraically

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independent generators. $\mathbf{D}(G/K)$ contains no first degree operators and has only one second degree generator, the Laplace-Beltrami operator. In order to fully exploit the virtues of G-invariance in low degrees, one is lead naturally to consider invariant *systems* of differential operators.

An abstract formalism for such systems (on any homogeneous space) can be found in [BV]. (See also Chapter V, Section 4 in [H3].) Following [BV], a G-invariant system is determined by a representation of K in some vector space V together with a K-equivariant map $d : V^* \to \mathbf{D}(G)$ from V^* to the algebra of left-G-invariant differential operators on G. The associated system maps smooth functions on G/K to smooth sections in the vector bundle $G \times_K V$ over G/K. We loose no generality by assuming that d is injective and can replace the data (V, d) by the image $W = d(V^*)$, a K-invariant subspace in $\mathbf{D}(G)$. The corresponding system becomes a map D_W from $C^\infty(G/K)$ to $\Gamma(G \times_K W^*)$. We will describe this construction in greater detail below in Section 2.

In the present context, there is a natural notion of *type* for systems D_W. Roughly speaking, we say that D_W has type (a, b) if each operator in W has the form

$$\sum_{|\alpha|=a,|\beta|=b} c_{\alpha,\beta}\partial^\alpha\overline{\partial}^\beta$$

at the identity, where α, β are multi-indices and "∂" denotes derivatives with respect to holomorphic coordinates in directions tangent to G/K. This will be made precise below.

The holomorphic (type $(a, 0)$) and anti-holomorphic (type $(0, b)$) systems on irreducible Hermitian symmetric spaces are completely classified in view of results of Johnson [J]. In general, the classification of all systems of specified type (a, b) on a given G/K reduces to a problem in Invariant Theory. Our focus here is the systems of type $(1, 1)$, determined by K-invariant subspaces in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$. To our knowledge, such systems have not been the subject of any systematic study.

Proposition 4.2, formulated below, classifies the systems of type $(1, 1)$ on any noncompact irreducible Hermitian symmetric space. In general, the possibilities are quite limited. Apart from the Laplace-Beltrami operator, these include the so-called Hua system. This system, which we denote by $D_{\mathcal{H}}$, is given by a canonical K-invariant subspace $\mathcal{H}$ in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$. The action of K on $\mathcal{H}$ is equivalent to its complexified adjoint representation on $\mathfrak{k}_{\mathbb{C}}$. The space $\mathcal{H}$ further decomposes under the action of K as

$$\mathcal{H} = \mathcal{L} \oplus \mathcal{H}'$$

where $\mathcal{L}$ corresponds to the (necessarily one dimensional) center in $\mathfrak{k}_{\mathbb{C}}$ and $\mathcal{H}'$ to the (semi-simple) derived algebra $\mathfrak{k}'_{\mathbb{C}}$. If G/K has type **A III** and rank at least two then $\mathfrak{k}'_{\mathbb{C}}$ has two simple factors and $\mathcal{H}' = \mathcal{H}'_1 \oplus \mathcal{H}'_2$ is the sum of two irreducible subspaces. In all other cases $\mathfrak{k}'_{\mathbb{C}}$ is simple and hence $\mathcal{H}'$ is irreducible. The one dimensional space $\mathcal{L}$ consists of all multiples of the Laplace-Beltrami operator, which we also denote by $\mathcal{L}$, the meaning being clear from the context.

One can also consider the subspace $\mathcal{H}^c$ orthogonal to $\mathcal{H}$ in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$. This is necessarily K-invariant and we call $D_{\mathcal{H}^c}$ the *complementary Hua system*. When G/K has rank one, $\mathcal{H}^c = 0$. We will show that for $rank(G/K) \geq 2$, $\mathcal{H}^c$ is always non-zero and K-irreducible. We will provide two additional abstract characterizations of $\mathcal{H}^c$:

- $\mathcal{H}^c$ is the kernel of the linear map $\mathfrak{p}_+ \otimes \mathfrak{p}_- \to \mathfrak{k}_{\mathbb{C}}$ given by the Lie bracket.
- $\mathcal{H}^c$ is the *Cartan component* in the tensor product $\mathfrak{p}_+ \otimes \mathfrak{p}_-$ of the irreducible K-modules $\mathfrak{p}_\pm$.

These facts support the viewpoint that the complementary Hua system is a natural object for study. For the classical families of non-compact irreducible Hermitian symmetric spaces, concrete descriptions are given in Section 3 for the spaces $\mathcal{H}$ and $\mathcal{H}^c$.

The solutions for a G-invariant system D_W are of particular interest. We say that a smooth function f on G/K is *W-harmonic* when $D_W f = 0$. The aim of this paper is to characterize such functions for each invariant system of type $(1,1)$ which contains the Laplace-Beltrami operator $\mathcal{L}$.

In [JK] it is shown that for G/K of tube type, a smooth bounded function is $\mathcal{H}$-harmonic if and only if it is a Poisson-Szegö integral over the Shilov boundary. It was previously known (see [Hua]) that for $G/K = SU(n,n)/S(U(n) \times U(n))$, Poisson-Szegö integrals f necessarily satisfy $D_{\mathcal{H}} f = 0$. Hua harmonicity on tube domains is also the subject of [L]. The Hua system for non-tube domains has been studied in [BBDHPT] and [B]. In this context, the real valued functions f on G/K satisfying $D_{\mathcal{H}} f = 0$ are the pluriharmonic functions [B]. That is, f is the real part of a holomorphic function on G/K. To describe Poisson-Szegö integrals on general non-tube domains a third order system $\mathcal{BV}$ of type $(2,1)$ is needed [BV].

Our principal object of study is the system $D_{\widetilde{\mathcal{H}}^c}$, where

$$\widetilde{\mathcal{H}}^c = \mathcal{L} \oplus \mathcal{H}^c.$$

That is, we augment the complementary Hua system to include the Laplace-Beltrami operator. This ensures that $\widetilde{\mathcal{H}}^c$-harmonic functions are harmonic in the usual sense. Theorem 5.3 below asserts that for $rank(G/K) \geq 2$ a bounded real valued function on G/K is $\widetilde{\mathcal{H}}^c$-harmonic if and only if it is pluriharmonic. This is the main result in the current work.

Our proof of Theorem 5.3 is based essentially on an interplay between the "G/K"-picture and "S" - picture of a Hermitian symmetric space, S being a solvable Lie group acting simply transitively on the corresponding Siegel domain ${}^c\mathcal{D}$. In Section 6 we use classical structure theory for Hermitian symmetric spaces to exhibit some operators on G/K that belong to the system $\mathcal{L} \oplus \mathcal{H}^c$. In Section 7 we leave the "G/K-picture" and express these operators in terms of S. This allows us to apply techniques and results from [BBDHPT] and [B] to the proof of Theorem 5.3 in Section 8. We show, in particular, that a bounded function f annihilated by $\mathcal{L} \oplus \mathcal{H}^c$ is a Poisson-Szegö integral. In the tube case, it follows from [JK] that f is $\mathcal{H}$-harmonic and we immediately conclude that f is pluriharmonic, since $\mathcal{H}' \oplus \mathcal{L} \oplus \mathcal{H}^c = \mathfrak{p}_+ \otimes \mathfrak{p}_-$. In the non-tube case, in view of [BV], f is annihilated by the third order system $\mathcal{BV}$. We use this fact and a technique from [B] to complete the proof for non-tube spaces.

As explained in Section 5, Theorem 5.3 together with previously known results concerning the Hua system yields a complete characterization for the bounded solutions of all possible type $(1,1)$ systems which contain the Laplace-Beltrami operator.

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2. Invariant systems of differential operators

Given a Lie group G, we let $\mathbf{D}(G)$ ($\cong \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$) denote the algebra of left-G-invariant differential operators on G. The group G acts on $\mathbf{D}(G)$ via the adjoint representation:

$$Ad(g)X(f) = X(f \circ r_{g^{-1}}) \circ r_g \quad \text{for } g \in G,\ X \in \mathbf{D}(G),\ f \in C^\infty(G),$$

where $r_g : G \to G$ denotes right multiplication. We recall that the *symmetrization map*

$$\lambda : S(\mathfrak{g}_{\mathbb{C}}) \to \mathbf{D}(G)$$

is a canonical $Ad(G)$-equivariant vector space isomorphism from the symmetric algebra on the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of G to $\mathbf{D}(G)$. We refer the reader to §4 in Chapter II of [H2] for details.

Now suppose K is a closed Lie subgroup of G. For $f \in C^\infty(G/K)$, we let $\widetilde{f} \in C^\infty(G)$ denote the map

$$\widetilde{f}(g) = f(gK).$$

If $X \in \mathbf{D}(G)$ is $Ad(K)$-invariant then X yields a left-G- invariant differential operator on G/K (also denoted X) via the rule

$$(Xf)\widetilde{\ } = X\widetilde{f}.$$

More generally, an $Ad(K)$-invariant *subspace* W of $\mathbf{D}(G)$ yields a left-G-invariant *system* D_W of differential operators on G/K as explained below.

To describe D_W, one forms the (complex) vector bundle $G \times_K W^*$ associated to the principle bundle $G \to G/K$ via the representation $Ad^* : K \to GL(W^*)$ contragredient to $Ad|_K$ on W ($Ad^*(k)\alpha = \alpha \circ Ad(k^{-1})$). The total space is

$$G \times_K W^* = (G \times W^*)/\sim$$

where $\sim$ is the equivalence relation on $G \times W^*$ given by

$$(gk, \alpha) \sim (g, Ad^*(k)\alpha)$$

for all $k \in K$.

The projection map $\pi : G \times_K W^* \to G/K$ is just

$$\pi\big([g,\alpha]\big) = gK,$$

where $[g, \alpha] \in G \times_K W^*$ denotes the equivalence class of $(g, \alpha) \in G \times W^*$. We let $\Gamma(G \times_K W^*)$ denote the space of smooth sections in the bundle $G \times_K W^*$. Smooth sections $s \in \Gamma(G \times_K W^*)$ are in one-to-one correspondence with smooth maps $\widetilde{s} : G \to W^*$ satisfying

$$\widetilde{s}(gk) = Ad^*(k^{-1})\widetilde{s}(g) = \widetilde{s}(g) \circ Ad(k), \tag{2.1}$$

for $g \in G$, $k \in K$. The correspondence $s \leftrightarrow \widetilde{s}$ is given by

$$s(gK) = [g, \widetilde{s}(g)].$$

Definition 2.2. The system D_W of differential operators on G/K determined by an $Ad(K)$-invariant subspace $W \subset \mathbf{D}(G)$ is the map

$$D_W : C^\infty(G/K) \to \Gamma(G \times_K W^*)$$

defined via

$$(D_W f)\widetilde{\ }(g)(X) = X(\widetilde{f})(g),$$

for $f \in C^\infty(G/K)$, $g \in G$, $X \in W$.

To justify Definition 2.2 one verifies (easily) that the map $(D_W f)\widetilde{\ }$satisfies the K-equivariance property (2.1). The $Ad(K)$-invariance of W is needed here. When W is finite dimensional we can write

$$(D_W f)\widetilde{\ }(g) = \sum_{j=1}^{n} X_j(\widetilde{f})(g) X_j^*,$$

where $\{X_j \ : \ j = 1, \ldots, n\}$ is any basis for W and $\{X_j^*\}$ is the dual basis for W^*. There is a natural left action of G on $\Gamma(G \times_K W^*)$:

$$(s \circ L_g)\widetilde{\ }(h) = \widetilde{s}(gh).$$

The system D_W is left-G-invariant in the sense that

$$(D_W f) \circ L_g = D_W(f \circ \ell_g),$$

where ℓ_g is left multiplication by g on G/K. This follows immediately from the fact that each operator $X \in W$ is left-G-invariant.

It is of interest to study zeros for systems of the form D_W. Observe that $f \in C^\infty(G/K)$ satisfies $D_W f = 0$ if and only if $X\widetilde{f} = 0$ for all $X \in W$. In this case, we will say that f is a *W-harmonic* function and often write $W(f) = 0$ in place of $D_W f = 0$.

2.1. **Invariant systems on symmetric spaces.** Now suppose that G/K is a symmetric space of non-compact type and let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

denote the Cartan decomposition for the Lie algebra $\mathfrak{g}$ of G. Since $\mathfrak{p}$ is $Ad(K)$-invariant, we have

$$\mathbf{D}(G) = \mathbf{D}(G)\mathfrak{k} \oplus \lambda(S(\mathfrak{p}_{\mathbb{C}})),$$

where $\lambda : S(\mathfrak{g}_{\mathbb{C}}) \to \mathbf{D}(G)$ is the symmetrization map. (See Lemma 4.7 in Chapter II of [H2].) Both $\mathbf{D}(G)\mathfrak{k}$ and $\lambda(S(\mathfrak{p}_{\mathbb{C}}))$ are $Ad(K)$-invariant subspaces of $\mathbf{D}(G)$, because λ is $Ad(G)$-equivariant. For $f \in C^\infty(G/K)$ one has $X\widetilde{f} = 0$ for all $X \in \mathbf{D}(G)\mathfrak{k}$. Thus we obtain:

Lemma 2.3. *Let W be an $Ad(K)$-invariant subspace of $\mathbf{D}(G)$ and W' be the projection of W onto $\lambda(S(\mathfrak{p}_{\mathbb{C}}))$ with respect to the decomposition $\mathbf{D}(G) = \mathbf{D}(G)\mathfrak{k} \oplus \lambda(S(\mathfrak{p}_{\mathbb{C}}))$. Then W' is $Ad(K)$-invariant, and for any $f \in C^\infty(G/K)$:*

$$f \text{ is } W\text{-harmonic if and only if } f \text{ is } W'\text{-harmonic.}$$

Lemma 2.3 shows that if we wish to study the zeros of invariant systems of differential operators on G/K, we can restrict attention to systems obtained from $Ad(K)$-invariant subspaces W of $\lambda(S(\mathfrak{p}_{\mathbb{C}}))$.

2.2. **Invariant systems on Hermitian symmetric spaces.** Next suppose that G/K is a *Hermitian* symmetric space of non-compact type. The complex structure on G/K yields an almost complex structure $\mathcal{J}$ on $T_{eK}(G/K) \cong \mathfrak{p}$. This extends to a complex linear map $\mathcal{J} : \mathfrak{p}_{\mathbb{C}} \to \mathfrak{p}_{\mathbb{C}}$ and one has

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$$

where $\mathfrak{p}_\pm$ are the $(\pm i)$-eigenspaces for $\mathcal{J}$. The spaces $\mathfrak{p}_\pm$ are $Ad(K)$-invariant abelian subalgebras of $\mathfrak{g}_{\mathbb{C}}$.

In this context, the canonical algebra isomorphism $S(\mathfrak{p}_{\mathbb{C}}) \cong S(\mathfrak{p}_+)\otimes S(\mathfrak{p}_-)$ is also an isomorphism of K-modules. (The group K acts on $S(\mathfrak{p}_{\mathbb{C}})$ and $S(\mathfrak{p}_\pm)$ by symmetric powers of the Adjoint representation.) It will here be convenient to replace the symmetrization map $\lambda : S(\mathfrak{p}_{\mathbb{C}}) \to \mathbf{D}(G)$ by

$$\lambda^\otimes : S(\mathfrak{p}_+) \otimes S(\mathfrak{p}_-) \to \mathbf{D}(G), \quad \lambda^\otimes = \lambda \otimes \lambda.$$

Explicitly,

$$\lambda^\otimes\big((X_1 \cdots X_a) \otimes (\overline{Y}_1 \cdots \overline{Y}_b)\big) = \frac{1}{a!b!} \sum_{\sigma \in S_a, \sigma' \in S_b} X_{\sigma(1)} \cdots X_{\sigma(a)} \overline{Y}_{\sigma'(1)} \cdots \overline{Y}_{\sigma'(b)},$$

for $(X_1 \cdots X_a) \otimes (\overline{Y}_1 \cdots \overline{Y}_b)$ in $S^a(\mathfrak{p}_+) \otimes S^b(\mathfrak{p}_-)$. The map $\lambda^\otimes$ is $Ad(K)$-equivariant and Lemma 2.3 remains true if we replace $\lambda(S(\mathfrak{p}_{\mathbb{C}}))$ by $\lambda^\otimes(S(\mathfrak{p}_+) \otimes S(\mathfrak{p}_-))$.

Definition 2.4. If W is a K-invariant subspace of $\lambda^\otimes(S^a(\mathfrak{p}_+) \otimes S^b(\mathfrak{p}_-))$ then we say that D_W is a *system of type* (a, b) on G/K.

Thus if D_W has type (a, b) then each element of W is a linear combination of terms of the form $X_1 \cdots X_a \overline{Y}_1 \cdots \overline{Y}_b$ where $X_i \in \mathfrak{p}_+$ and $\overline{Y}_j \in \mathfrak{p}_-$.

3. Preliminaries on Hermitian symmetric spaces and Siegel domains

For the remainder of this paper, G/K will denote a Hermitian symmetric space of non-compact type. For our purposes we can assume, moreover, that G/K is *irreducible*, since any Hermitian symmetric space is a direct product of irreducible factors. Thus G is a connected non-compact simple Lie group with trivial center and K is a maximal compact subgroup of G with center analytically isomorphic to $\mathbb{T}$. (See Theorem 6.1 and Proposition 6.2 of Chapter VIII in [H1].)

In this section we collect notation and background material concerning the structure of G/K and its realizations as bounded and unbounded domains. For details we refer the reader to any of the standard references. See for example, [H1] , [KW] and [W].

3.1. **Algebraic preliminaries.** As in the preceding section, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ denotes the Cartan decomposition for the Lie algebra of G and $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ is the eigenspace decomposition determined by the complex structure. We have

$$[\mathfrak{k}_{\mathbb{C}}, \mathfrak{p}_{\mathbb{C}}] \subset \mathfrak{p}_{\mathbb{C}}, \quad [\mathfrak{p}_+, \mathfrak{p}_+] = 0 = [\mathfrak{p}_-, \mathfrak{p}_-], \quad [\mathfrak{p}_+, \mathfrak{p}_-] = \mathfrak{k}_{\mathbb{C}}.$$

The Lie algebra $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ is a compact real form of $\mathfrak{g}_{\mathbb{C}}$. For $Z \in \mathfrak{g}_{\mathbb{C}}$ we let $\overline{Z}$ and $\tau(Z)$ denote the complex conjugates for Z with respect to the real forms $\mathfrak{g}$ and $\mathfrak{u}$. The two conjugation operators are related via

$$\tau(\theta(Z)) = \overline{Z} = \theta(\tau(Z))$$

where θ is the complexified Cartan involution. If B is the Killing form on $\mathfrak{g}_{\mathbb{C}}$, then the bilinear form defined by

$$B_\tau(X, Y) = -B(X, \tau Y)$$

is a positive definite Hermiian inner product on $\mathfrak{g}_{\mathbb{C}}$. Recall that $\mathcal{J}$ denotes the almost complex structure on the tangent space $\mathfrak{p}$ to G/K at eK and its complexification $\mathfrak{p}_{\mathbb{C}} \to \mathfrak{p}_{\mathbb{C}}$. Let $\mathfrak{c}$ denote the (one dimensional) center of $\mathfrak{k}$. It is a key fact that there exists an element $Z_0 \in \mathfrak{c}$ with

$$\mathcal{J} = ad(Z_0)|\mathfrak{p}_{\mathbb{C}}.$$

Choose a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{k}$. Then $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Define Δ to be the system of roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$. Then each root $\alpha \in \Delta$ is real valued on $i\mathfrak{h}$. We specify an ordering on Δ as follows: for two roots α, β we say that α is bigger than β when $-i(\alpha - \beta)(Z_0) > 0$. In this way we obtain the sets Δ^+ and Δ^- of positive and negative roots.

Each root space $\mathfrak{g}^\alpha$ is contained either in $\mathfrak{k}_{\mathbb{C}}$ or in $\mathfrak{p}_{\mathbb{C}}$. In the first case α is called compact and in the second case noncompact. We write

$$\Delta = C \cup Q$$

where C is the set of compact roots and Q is the set of noncompact roots and let $Q^\pm = \Delta^\pm \cap Q$ denote the sets of positive and negative noncompact roots.

For each $\alpha \in \Delta$ we associate elements H_α, E_α and $E_{-\alpha}$ which span a subalgebra of $\mathfrak{g}_{\mathbb{C}}$, isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. We do this in a standard way as follows. The Killing form B is positive definite on $i\mathfrak{h}$. Thus for each $\alpha \in \Delta$ there is a unique element $\widetilde{H}_\alpha \in i\mathfrak{h}$ for which

$$\alpha(H) = B(H, \widetilde{H}_\alpha).$$

For roots $\alpha, \beta \in \Delta$ let

$$(\alpha, \beta) = B(\widetilde{H}_\alpha, \widetilde{H}_\beta)$$

and set

$$H_\alpha = \frac{2}{(\alpha,\alpha)}\widetilde{H}_\alpha,$$

so that $\alpha(H_\alpha) = 2$.

Now for $\alpha \in \Delta^+$ choose $E_\alpha \in \mathfrak{g}^\alpha$ with $B_\tau(E_\alpha, E_\alpha) = 2/(\alpha,\alpha)$ and set $E_{-\alpha} = -\tau(E_\alpha)$. With these conventions, we have

$$[E_\alpha, E_{-\alpha}] = H_\alpha, \quad [H_\alpha, E_a] = 2E_\alpha, \quad [H_\alpha, E_{-\alpha}] = -2E_{-\alpha},$$

$$\overline{E}_\alpha = E_{-\alpha} \quad \text{for } \alpha \in Q, \quad \overline{E}_\alpha = -E_{-\alpha} \quad \text{for } \alpha \in C,$$

and

$$\begin{aligned} \mathfrak{p}_+ &= \sum_{\alpha\in Q^+} \mathfrak{g}^\alpha = \sum_{\alpha \in Q^+} \mathbb{C}E_\alpha, \\ \mathfrak{p}_- &= \sum_{\alpha\in Q^+} \mathfrak{g}^{-\alpha} = \sum_{\alpha \in Q^+} \mathbb{C}E_{-\alpha}. \end{aligned} \tag{3.1}$$

For $\alpha \in Q^+$, let

$$\begin{aligned} X_\alpha &= E_\alpha + E_{-\alpha}, \\ Y_\alpha &= i(E_\alpha - E_{-\alpha}). \end{aligned} \tag{3.2}$$

Then the set $\{X_\alpha, Y_\alpha\}_{\alpha \in Q^+}$ is a basis for the real vector space $\mathfrak{p}$. One has

$$\begin{aligned} \mathcal{J}X_\alpha &= Y_\alpha, \\ \mathcal{J}Y_\alpha &= -X_\alpha, \\ E_\alpha &= \frac{1}{2}(X_\alpha - iY_\alpha), \\ E_{-\alpha} &= \frac{1}{2}(X_\alpha + iY_\alpha). \end{aligned} \tag{3.3}$$

3.2. **Restricted roots.** Two roots $\alpha, \beta \in \Delta$ are called *strongly orthogonal* if neither $\alpha + \beta$ nor $\alpha - \beta$ are roots. This implies orthogonality in the usual sense: $(\alpha, \beta) = 0$. Let

$$\Gamma = \{\gamma_1, \dots, \gamma_r\} \subset Q^+ \tag{3.4}$$

be a maximal set of strongly orthogonal positive noncompact roots. Then

$$\mathfrak{a} = \sum_{\gamma\in\Gamma} \mathbb{R}X_\gamma, \tag{3.5}$$

is a maximal abelian subalgebra of $\mathfrak{p}$ and $r = \dim(\mathfrak{a})$ is the rank of G/K.

Take $\mathfrak{h}^-$ to be the real span of the elements iH_γ ($\gamma \in \Gamma$), and $\mathfrak{h}^+$ to be the orthogonal complement of $\mathfrak{h}^-$ in $\mathfrak{h}$ via the Killing form B:

$$\begin{aligned} \mathfrak{h}^- &= \sum_{\gamma\in\Gamma} \mathbb{R}iH_\gamma, \\ \mathfrak{h} &= \mathfrak{h}^- \oplus_B \mathfrak{h}^+. \end{aligned} \tag{3.6}$$

For $\alpha, \beta \in \Delta$ write $\alpha \sim \beta$ if and only if $\alpha|_{\mathfrak{h}^-} = \beta|_{\mathfrak{h}^-}$ and define:

$$\begin{aligned} C_i &= \{\ \alpha \in C\colon\ \alpha \sim -\tfrac{1}{2}\gamma_i\} && \text{for } i = 1, \ldots, r, \\ C_{ij} &= \{\ \alpha \in C\colon\ \alpha \sim \tfrac{1}{2}(\gamma_j - \gamma_i)\} && \text{for } 1 \le i < j \le r, \\ Q_i &= \{\ \alpha \in Q\colon\ \alpha \sim \tfrac{1}{2}\gamma_i\} && \text{for } i = 1, \ldots, r, \\ Q_{ij} &= \{\ \alpha \in Q\colon\ \alpha \sim \tfrac{1}{2}(\gamma_i + \gamma_j)\} && \text{for } 1 \le i < j \le r. \end{aligned} \tag{3.7}$$

Some important properties of the above sets are contained in the following theorem:

Theorem 3.8 (Restricted Roots Theorem, [H3]). *The map $\alpha \mapsto \gamma_i + \alpha$ is a bijection of C_i onto Q_i (for $1 \le i \le r$) and C_{ij} onto Q_{ij} (for $1 \le i < j \le r$). Q^+ is the disjoint union of the sets Γ, Q_i, Q_{ij}.*

In addition, we will need these facts:

- All elements $\alpha \in Q^+$ have a common length $(\alpha, \alpha)^{\frac{1}{2}}$.
- For α, β in Q^+,

$$\alpha(H_\beta) = \beta(H_\alpha). \tag{3.9}$$

- The sets Q_{ij} have a common cardinality for $1 \le i < j \le r$. Likewise, the sets $Q_1, \ldots, Q_r$ have a common cardinality. We let q_1 denote the common cardinality of the Q_i's and q_2 the common cardinality of the Q_{ij}'s.
- For $\alpha \in Q_{ij}$ set

$$\widetilde{\alpha} = \gamma_i + \gamma_j - \alpha. \tag{3.10}$$

  Proposition 8 in [L] shows that $\widetilde{\alpha}$ is a noncompact positive root. So clearly $\widetilde{\alpha} \in Q_{ij}$.

3.3. **Harish-Chandra realization.** Let $G_{\mathbb{C}}$ denote the adjoint group for $\mathfrak{g}_{\mathbb{C}}$ and $K_{\mathbb{C}}$ be the analytic subgroup corresponding to $\mathfrak{k}_{\mathbb{C}}$. The analytic subgroups of $G_{\mathbb{C}}$ corresponding to subalgebras $\mathfrak{p}_+$, $\mathfrak{p}_-$ will be denoted by P_+ and P_- respectively. They are abelian. The exponential map from $\mathfrak{p}_\pm$ to $P_\pm$ is biholomorphic, so $P_\pm$ is biholomorphically equivalent with $\mathbb{C}^n$ for some n.

The mapping $(p_1, k, p_2) \mapsto p_1 k p_2$ is a diffeomorphism of $P_+ \times K_{\mathbb{C}} \times P_-$ onto an open submanifold of $G_{\mathbb{C}}$ containing G. For $g \in G$ let $p_+(g)$ denote the unique element in $\mathfrak{p}_+$ such that $g \in \exp(p_+(g))K_{\mathbb{C}}P_-$. Then $p_+(g) = p_+(gk)$ and p_+ is a diffeomorphism of G/K onto a bounded domain $\mathcal{D} \subset \mathfrak{p}_+$. G acts biholomorphically on $\mathcal{D}$ by $g \cdot p_+(\widetilde{g}) = p_+(g\widetilde{g})$. Let $o = p_+(e)$, then $\mathcal{D}$ is the G-orbit of o and the group K is the stabilizer of the point o. This is the Harish–Chandra embedding and in fact realizes G/K as a bounded symmetric domain.

3.4. **Siegel domains.** Let $\mathcal{D}$ denote the bounded realization of G/K described above. Set $X_\Gamma = \sum_{\gamma\in\Gamma} X_\gamma = \sum_{\gamma\in\Gamma}(E\gamma + E_{-\gamma})$, $E_\Gamma = \sum_{\gamma\in\Gamma} E_\gamma$ and define an element of $G_{\mathbb{C}}$ called the Cayley transform:

$$c = \exp(\frac{\pi}{4} i X_\Gamma). \tag{3.11}$$

Let

$$\begin{aligned} {}^cG &= Ad(c)G, \\ {}^cK &= Ad(c)K, \\ {}^c\mathfrak{g} &= Ad(c)\mathfrak{g}. \end{aligned}$$

For $g \in G$, $c\exp(p_+(g)) \in P_+{}^cK_{\mathbb{C}}P_-$ and by [KW], the mapping $x \mapsto c \cdot x$, where

$$c \cdot p_+(g) = p_+(c \exp p_+(g)),$$

defines a biholomorphism of $\mathcal{D}$ onto a domain ${}^c\mathcal{D} \subset \mathfrak{p}^+$. Clearly, ${}^c\mathcal{D}$ is the orbit of the point $c \cdot o = iE_\Gamma$ under the action of the group ${}^cG$ and ${}^cK$ is the isotropy group of iE_Γ.

It was proved in [KW] that ${}^c\mathcal{D}$ is a Siegel domain. We briefly recall the definition and notation of Siegel domains. The reader is referred to the book of J. Faraut and A. Koranyi [FK] for more details.

Let V be a Euclidean Jordan algebra and Ω be an irreducible symmetric cone contained in V. We denote by $L(x)$ the self-adjoint endomorphism of V given by left multiplication by x, i.e. $L(x)y = xy$. We fix a Jordan frame $\{c_1, \dots, c_r\}$ in V. The Peirce decomposition of V related to the Jordan frame $\{c_1, \dots, c_r\}$ ([FK], Theorem IV.2.1) may be written as

$$V = \bigoplus_{1\le i\le j\le r} V_{ij}\,. \tag{3.12}$$

It is given by the common diagonalization of the self-adjoint endomorphisms $L(c_j)$ with respect to their only eigenvalues 0, $\frac{1}{2}$, 1. In particular $V_{jj} = \mathbb{R}c_j$ is the eigenspace of $L(c_j)$ related to 1, and, for $i < j$, V_{ij} is the intersection of the eigenspaces of $L(c_i)$ and $L(c_j)$ related to $\frac{1}{2}$. All V_{ij}, for $i < j$, have the same dimension d.

Suppose that we are given a complex vector space $\mathcal{Z}$ and a Hermitian symmetric bilinear mapping

$$\Phi : \ \mathcal{Z} \times \mathcal{Z} \mapsto V^{\mathbb{C}}.$$

We assume that

$$\Phi(\zeta,\zeta) \in \overline{\Omega}, \quad \zeta \in \mathcal{Z}\,,$$
$$\text{and } \Phi(\zeta,\zeta) = 0 \ \text{ implies } \zeta = 0.$$

The associated Siegel domain is then

$$\widetilde{\mathcal{D}} = \{(\zeta, z) \in \mathcal{Z} \times V^{\mathbb{C}} : \ \Im z - \Phi(\zeta,\zeta) \in \Omega\}. \tag{3.13}$$

One says that $\widetilde{\mathcal{D}}$ is of type I (or has tube type) when $\mathcal{Z} = \{0\}$. Otherwise, $\widetilde{\mathcal{D}}$ is said to be of type II (non-tube type).

The data V, $\mathcal{Z}$ and Φ can be defined in terms of some subspaces of $\mathfrak{g}_{\mathbb{C}}$ so that

$$\mathfrak{p}_+ = \mathcal{Z} \times V^{\mathbb{C}} \quad \text{and} \quad \widetilde{\mathcal{D}} = {}^c\mathcal{D}.$$

(For details we refer to [KW], [B]). Moreover it is known that ${}^c\mathcal{D}$ has tube type if and only if the sets Q_i in (3.7) are all empty.

3.5. **Iwasawa decomposition of ${}^cG$.** Consider the Iwasawa decomposition of ${}^cG = N\widetilde{A}\,{}^cK$ and denote by S its solvable part: $S = N\widetilde{A}$. Let $\mathfrak{n}$, $\widetilde{\mathfrak{a}}$, $\mathfrak{s}$ be the corresponding Lie algebras. Then $\widetilde{\mathfrak{a}}$ can be chosen as a subalgebra of ${}^c\mathfrak{g}$ consisting of elements $H = L(a)$, where

$$a = \sum_{j=1}^{r} a_j c_j.$$

We let λ_j denote the linear form on $\widetilde{\mathfrak{a}}$ given by $\lambda_j(H) = a_j$. All endomorphisms of $\mathfrak{s}$ having the form: adH for $H \in \widetilde{\mathfrak{a}}$ admit joint diagonalization. Therefore $\mathfrak{s}$ can be decomposed as a direct sum of corresponding root spaces. The forms of all roots are well-known:

$$\mathfrak{s} = \left(\bigoplus_j \mathfrak{n}_{\frac{\lambda_j}{2}}\right) \oplus \left(\bigoplus_{1\le i\le j\le r} \mathfrak{n}_{\frac{\lambda_i+\lambda_j}{2}}\right) \oplus \left(\bigoplus_{1\le i<j\le r} \mathfrak{n}_{\frac{\lambda_j-\lambda_i}{2}}\right) \oplus \widetilde{\mathfrak{a}} \tag{3.14}$$

To simplify our notation put

$$\begin{aligned} \mathcal{Z}_j &= \mathfrak{n}_{\frac{\lambda_j}{2}}\,, \\ \mathfrak{n}_{ij} &= \mathfrak{n}_{\frac{\lambda_j-\lambda_i}{2}}\,, \\ V_{ij} &= \mathfrak{n}_{\frac{\lambda_i+\lambda_j}{2}}\,. \end{aligned}$$

Then it is known that

$$\mathcal{Z} = \bigoplus_j \mathcal{Z}_j, \quad V = \bigoplus_{i,j} V_{ij},$$

and we set

$$\mathfrak{n}_0 = \bigoplus_{1\leq i<j\leq r} \mathfrak{n}_{ij}.$$

Denote by $N(\Phi)$ and N_0 the subgroups of S corresponding to the subalgebras $\mathcal{Z}\oplus V$ and $\mathfrak{n}_0$ of $\mathfrak{s}$. Then $S = N(\Phi)N_0\widetilde{A}$, $N(\Phi)$ is two-step nilpotent with center V and each of $N(\Phi)$, N_0 and $N = N(\Phi)N_0$ is normal in S. Since S acts simply transitively on the domain ${}^c\mathcal{D}$, we may identify S and ${}^c\mathcal{D}$:

$$S \ni s \sim s\cdot(c\cdot o) \in {}^c\mathcal{D}. \tag{3.15}$$

Now we describe an orthonormal basis of $\mathfrak{s}$ corresponding to the decomposition (3.14). This will be the same basis as in [DHMP], [BDH] and [B]. By $\overline{Q}_{ij}$ we shall denote a subset of Q_{ij} that contains exactly one from each pair of roots $(\alpha, \widetilde{\alpha} = \gamma_i+\gamma_j-\alpha)$ when $\widetilde{\alpha}\neq\alpha$. Let us define

$$\begin{aligned}
X_i &= E_{\gamma_i} && \text{for } \gamma_i\in\Gamma,\\
X^1_\alpha &= \tfrac{1}{\sqrt{2}}(E_\alpha - \varepsilon_\alpha E_{\widetilde{\alpha}}) && \text{for } \alpha\in\overline{Q}_{ij} \text{ and } \alpha\neq\widetilde{\alpha},\\
X^2_\alpha &= \tfrac{i}{\sqrt{2}}(E_\alpha + \varepsilon_\alpha E_{\widetilde{\alpha}}) && \text{for } \alpha\in\overline{Q}_{ij} \text{ and } \alpha\neq\widetilde{\alpha},\\
\mathcal{X}_\alpha &= \tfrac{1}{\sqrt{2}}\psi(E_\alpha) && \text{for } \alpha\in Q_i,
\end{aligned} \tag{3.16}$$

where $\psi = I + Ad(c^2)\tau$, and $\varepsilon_\alpha = \pm 1$ (the precise value was determined in [B]). If $\alpha = \widetilde{\alpha}$ then instead of X^1_α, X^2_α we define $X^1_\alpha = E_\alpha$.

In the rest of the paper we will use the notation X^k_α without specifying the set of indices k's. In particular, the summation over $\alpha\in\overline{Q}_{ij}$ will always denote the summation over α and k together.

Then by [B]

$$\begin{aligned}
V_{ii} &= span\{X_i\},\\
V_{ij} &= span\{X^k_\alpha\}_{\alpha\in\overline{Q}_{ij}},\\
\mathcal{Z}_j &= (span\{\mathcal{X}_\alpha\}_{\alpha\in Q_j})^{\mathbb{C}}.
\end{aligned}$$

Next we transport the complex structure $\mathcal{J}$ from ${}^c\mathcal{D}$ to $\mathfrak{s}$ and define:

$$\begin{aligned}
H_j &= \mathcal{J}(X_j),\\
Y^k_\alpha &= \mathcal{J}(X^k_\alpha),\\
\mathcal{Y}_\alpha &= \mathcal{J}(\mathcal{X}_\alpha).
\end{aligned} \tag{3.17}$$

It was proved in [B] that

$$X_j, X^k_\alpha, \mathcal{X}_\alpha, H_j, Y^k_\alpha, \mathcal{Y}_\alpha$$

form an orthonormal basis of $\mathfrak{s}$ with respect to the Hermitian product B_τ.

Finally,

$$
\begin{aligned}
Z_j &= X_j - iH_j, \quad (1 \le j \le r,) \\
Z_\alpha^k &= X_\alpha^k - iY_\alpha^k, \quad (\alpha \in \bigcup \overline{Q}_{ij}, k = 1, 2), \\
\mathcal{Z}_\alpha &= \mathcal{X}_\alpha - i\mathcal{Y}_\alpha \quad (\alpha \in \bigcup_j Q_j)
\end{aligned}
\tag{3.18}
$$

is a basis of S-invariant holomorphic vector fields.

4. Systems of type $(1,1)$

We suppose here, as in the previous section, that G/K is an irreducible Hermitian symmetric space of non-compact type. To determine all systems D_W of a given type (a,b) on G/K, one needs to find all K-invariant subspaces W of $S^a(\mathfrak{p}_+) \otimes S^b(\mathfrak{p}_-)$ (and apply the modified symmetrization map $\lambda^\otimes$). Recall that K acts on $S(\mathfrak{p}_\pm)$ by symmetric powers of the adjoint representation. Since G/K is irreducible, $\mathfrak{p}_\pm$ are irreducible K-modules. Moreover, it is known that the representations of K on $S(\mathfrak{p}_\pm)$ are multiplicity free. The decompositions for $S(\mathfrak{p}_\pm)$ under the action of K are described in [J] on a case-by-case basis using the classification of irreducible Hermitian symmetric spaces. Thus, in principle, all systems of types $(a,0)$ and $(0,b)$ are known.

The current work concerns systems of type $(1,1)$. Thus, we must describe the K-invariant subspaces of $\mathfrak{p}_+ \otimes \mathfrak{p}_-$. We begin with some observations concerning the adjoint representations of K on $\mathfrak{p}_\pm$, which we now denote by $\sigma_\pm$:

$$\sigma_+(k) = Ad(k)|\mathfrak{p}_+, \quad \sigma_-(k) = Ad(k)|\mathfrak{p}_-.$$

These representations are unitary with respect to the positive definite Hermitian inner product B_τ on $\mathfrak{p}_\pm$. The conjugation map $Z \mapsto \overline{Z}$ (with respect to the real form $\mathfrak{g} \subset \mathfrak{g}_\mathbb{C}$) interchanges $\mathfrak{p}_+$ with $\mathfrak{p}_-$, as shown by Equations 3.1. Since

$$\sigma_-(k)(\overline{Z}) = \overline{\sigma_+(k)(Z)}$$

for $k \in K$, $Z \in \mathfrak{p}_+$, we see that σ_- is (unitarily) equivalent to $\overline{\sigma}_+$, the conjugate representation for σ_+. (Recall that the conjugate for a complex representation is obtained by replacing the complex structure on the representation space by its conjugate. The conjugate for a matrix representation is obtained by conjugation of matrix entries.) The Hermitian inner product B_τ on $\mathfrak{p}_+$ yields a further isomorphism of complex vector spaces:

$$\overline{\mathfrak{p}}_+ \to \mathfrak{p}_+^*, \quad Z \mapsto B_\tau(\cdot, Z),$$

establishing a unitary equivalence of $\overline{\sigma}_+$ with σ_+^*, the contragredient representation for σ_+. In summary, we have canonical unitary equivalences

$$\sigma_- \simeq \overline{\sigma}_+ \simeq \sigma_+^*.$$

We will denote elements of $\mathfrak{p}_-$ as "$\overline{Z}$" for $Z \in \mathfrak{p}_+$. One can choose to interpret this literally, as the result of applying the conjugation map on $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$ to Z, or as simply a notation for Z itself, but viewed as an element of $\mathfrak{p}_+$ with the conjugate complex structure.

The representation $\sigma_+ \otimes \sigma_-$ always contains a copy of

$$Ad_K : K \to U(\mathfrak{k}_{\mathbb{C}}),$$

the complexified adjoint representation of K on $\mathfrak{k}_{\mathbb{C}}$. Indeed, the linear map $\varphi : \mathfrak{p}_+ \otimes \mathfrak{p}_- \to \mathfrak{k}_{\mathbb{C}}$ determined by

$$\varphi(X \otimes \overline{Y}) = [X, \overline{Y}]$$

intertwines $\sigma_+ \otimes \sigma_-$ with Ad_K and is surjective. Thus, the dual map

$$\varphi^* : \mathfrak{k}_{\mathbb{C}}^* \to (\mathfrak{p}_+ \otimes \mathfrak{p}_-)^* \approxeq \mathfrak{p}_+^* \otimes \mathfrak{p}_-^* \approxeq \mathfrak{p}_- \otimes \mathfrak{p}_+ \approxeq \mathfrak{p}_+ \otimes \mathfrak{p}_-$$

is injective and intertwines the contragredient representation Ad_K^* for Ad_K with $\sigma_+ \otimes \sigma_-$. In fact, Ad_K^* and Ad_K are equivalent representations, since K is compact. So

$$\mathcal{H} = \varphi^*(\mathfrak{k}_{\mathbb{C}}^*)$$

is a K-invariant subspace of $\mathfrak{p}_+ \otimes \mathfrak{p}_-$ and $(\sigma_+ \otimes \sigma_-)|\mathcal{H}$ is equivalent to Ad_K.

As explained in the Introduction, $D_{\mathcal{H}}$ is the Hua system. It is the subject of works including [Hua], [JK], [BV], [L] and [BBDHPT]. The Hua system is not irreducible. Indeed, the Lie algebra $\mathfrak{k}_{\mathbb{C}}$ decomposes as

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{k}'_{\mathbb{C}} \oplus \mathfrak{c}_{\mathbb{C}}$$

where $\mathfrak{c}$ is the (one-dimensional) center of $\mathfrak{k}$ and the derived algebra $\mathfrak{k}'_{\mathbb{C}}$ is semi-simple. Thus, the K-invariant subspace $\mathcal{H} = \varphi^*(\mathfrak{k}_{\mathbb{C}}^*)$ in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$ further decomposes under $\sigma_+ \otimes \sigma_-$ as

$$\mathcal{H} = \varphi^*((\mathfrak{k}'_{\mathbb{C}})^*) \oplus \varphi^*(\mathfrak{c}_{\mathbb{C}}^*) = \mathcal{H}' \oplus \mathcal{L}$$

The group K acts trivially on the one-dimensional space $\mathcal{L} = \varphi^*(\mathfrak{c}_{\mathbb{C}}^*)$ and $\mathcal{H}' = \varphi^*((\mathfrak{k}'_{\mathbb{C}})^*)$ contains one K-irreducible subspace for each simple factor in $\mathfrak{k}'_{\mathbb{C}}$. The classification of irreducible Hermitian symmetric spaces of non-compact type, discussed below, shows that $\mathfrak{k}'_{\mathbb{C}}$ has at most two simple factors. The elements of $\mathcal{L}$ yield scalar multiples of the Laplace-Beltrami operator on G/K. These are the only second order left-G-invariant differential operators on G/K, so $\mathfrak{p}_+ \otimes \mathfrak{p}_-$ contains no further copies of the trivial representation.

We now have $\mathfrak{p}_+ \otimes \mathfrak{p}_- = \mathcal{H} \oplus \mathcal{H}^c$, where

$$\mathcal{H}^c = \mathcal{H}^\perp = Ker(\varphi) \tag{4.1}$$

is the orthogonal complement to $\mathcal{H}$ in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$ and also the kernel of $\varphi : \mathfrak{p}_+ \otimes \mathfrak{p}_- \to \mathfrak{k}_{\mathbb{C}}$. The space $\mathcal{H}^c$ is K-invariant and, as will be shown below, is non-zero except when G/K has rank one. (The rank one cases are the complex hyperbolic spaces, up to isomorphism.) $D_{\mathcal{H}^c}$ is the *complementary Hua system*. We will show that $\mathcal{H}^c$ is K-irreducible and inequivalent to any of the irreducible constituents of $\mathcal{H}$.

Moreover, we will show that $\mathcal{H}^c$ is the *Cartan component* in $\sigma_+ \otimes \sigma_-$. That is, a highest weight vector in $\mathcal{H}^c$ is obtained as the tensor product of highest weight vectors in $\mathfrak{p}_+$ and $\mathfrak{p}_-$. If $X, Y \in \mathfrak{p}_+$ are highest and lowest weight vectors for σ_+ then $\mathcal{H}^c = Span\{(\sigma_+ \otimes \sigma_-)(K)(X \otimes \overline{Y})\}$ and $X \otimes \overline{Y}$ is a highest weight vector for $(\sigma_+ \otimes \sigma_-)|\mathcal{H}^c$. This provides another characterization of $\mathcal{H}^c$. We summarize the preceding discussion as follows:

Proposition 4.2. *Let G/K be an irreducible Hermitian symmetric space of non-compact type. Then:*

- *The representation $\sigma_+ \otimes \sigma_-$ of K on $\mathfrak{p}_+ \otimes \mathfrak{p}_-$ is multiplicity free, with a canonical decomposition*
$$\mathfrak{p}_+ \otimes \mathfrak{p}_- = \mathcal{H} \oplus \mathcal{H}^c$$
as an orthogonal direct sum of K-invariant subspaces. (The space $\mathcal{H}$ yields the Hua system and $\mathcal{H}^c$ the complementary Hua system.)
- *The representation $(\sigma_+ \otimes \sigma_-)|\mathcal{H}$ is equivalent to Ad_K, the complexified adjoint representation of K on $\mathfrak{k}_{\mathbb{C}}$ and decomposes as*
$$\mathcal{H} = \mathcal{H}' \oplus \mathcal{L},$$
where $\mathcal{L}$ is a copy of the trivial representation of K on $\mathbb{C}$ and $\mathcal{H}'$ contains an irreducible subspace for each simple factor in $\mathfrak{k}'_{\mathbb{C}}$;
- *The space $\mathcal{H}^c$ is $\mathcal{H}^c = Ker(\varphi)$ where $\varphi : \mathfrak{p}_+ \otimes \mathfrak{p}_- \to \mathfrak{k}_{\mathbb{C}}$ is the linear map with $\varphi(X \otimes \overline{Y}) = [X, \overline{Y}]$.*
- *If G/K has rank at least two then $\mathcal{H}^c$ is non-zero, $(\sigma_+ \otimes \sigma_-)|\mathcal{H}^c$ is irreducible and is the Cartan component in $\sigma_+ \otimes \sigma_-$.*

Note that since $\sigma_+ \otimes \sigma_-$ is multiplicity free, any K-invariant subspace of $\mathfrak{p}_+ \otimes \mathfrak{p}_-$ is a sum of the irreducibles described above. Thus, Proposition 4.2 determines *all* systems of type $(1,1)$ on G/K.

To complete the proof of Proposition 4.2 we employ case-by-case analysis using the classification for irreducible Hermitian symmetric spaces of non-compact type. We refer the reader to [H1], Chapter X, for details concerning this classification and for definitions of the Lie groups that arise. The spaces in question fall into 4 classical families (types **A III**, **C I**, **D III**, **BD I**) and two exceptional cases (types **E III**, **E VII**). For the classical families, we will exhibit the spaces $\mathcal{H}$ and $\mathcal{H}^c$ in Proposition 4.2 explicitly.

4.1. **Type A III.** Here $G/K = SU(n,m)/S(U(n) \times U(m))$ with $1 \le n \le m$. The rank of G/K is n. One can realize $\mathfrak{p}_+$ as $\mathfrak{p}_+ = M_{n,m}(\mathbb{C})$ with Hermitian inner product $\langle X, Y \rangle = tr(XY^*)$, where Y^* denotes conjugate-transpose. The bounded realization for G/K as a domain in $\mathfrak{p}_+$ is
$$\mathcal{D} = \{Z \in \mathfrak{p}_+ \ : \ Z^t\overline{Z} < I_m\} = \{Z \in \mathfrak{p}_+ \ : \ ZZ^* < I_n\}.$$

(Here the inequality $ZZ^* < I_n$, for example, means that $I_n - ZZ^*$ is positive definite.) $\mathcal{D}$ is a tube domain if and only if $n = m$

The group $K = S(U(n) \times U(m))$ acts on $\mathfrak{p}_+$ via

$$\sigma_+(k_1, k_2)Z = k_1 Z k_2^*,$$

$(k_1 \in U(n), k_2 \in U(m), det(k_1)det(k_2) = 1)$. Writing elements of $\mathfrak{p}_- = \overline{\mathfrak{p}}_+$ as $\overline{Z}$ for $Z \in \mathfrak{p}_+$, one has

$$(\sigma_+ \otimes \sigma_-)(k_1, k_2)(X \otimes \overline{Y}) = (k_1 X k_2^*) \otimes (\overline{k_1 Y k_2^t}).$$

Let $E_{a,b}$ $(1 \le a \le n,\ 1 \le b \le m)$ denote the $(n \times m)$-matrix with a one in position (a, b) and set

$$L_{i,j} = \sum_{\ell=1}^{m} E_{i,\ell} \otimes \overline{E}_{j,\ell} \quad (1 \le i, j \le n)$$

$$R_{i',j'} = \sum_{\ell=1}^{n} E_{\ell,i'} \otimes \overline{E}_{\ell,j'} \quad (1 \le i', j' \le m).$$

One verifies that

$$(\sigma_+ \otimes \sigma_-)(k_1, k_2)L_{i,j} = (k_1 L k_1^*)_{i,j}, \quad (\sigma_+ \otimes \sigma_-)(k_1, k_2)R_{i',j'} = (k_2 R k_2^*)_{i',j'},$$

where "$k_1 L k_1^*$" and "$k_2 L k_2^*$" mean formal matrix multiplication. Hence

$$\mathcal{H} = Span\{L_{i,j}, R_{i',j'} \ : \ 1 \le i, j \le n, 1 \le i', j' \le m\}$$

is K-invariant and K acts on $\mathcal{H}$ by a copy of Ad_K.

The space $\mathcal{H}$ decomposes under K as $\mathcal{H} = \mathcal{L} \oplus \mathcal{H}_1' \oplus \mathcal{H}_2'$. Here K acts trivially on $\mathcal{L}$ of dimension 1 and $\mathcal{H}_1'$, $\mathcal{H}_2'$ correspond to the simple factors $sl(n, \mathbb{C})$, $sl(m, \mathbb{C})$ in $\mathfrak{k}_{\mathbb{C}}' = sl(n, \mathbb{C}) \oplus sl(m, \mathbb{C})$. Explicitly:

$$\mathcal{L} = \mathbb{C}\left(\sum_{i=1}^{n} L_{i,i}\right) = \mathbb{C}\left(\sum_{i'=1}^{m} R_{i',i'}\right),$$

$$\mathcal{H}_1' = \left\{\sum_{i,j=1}^{n} c_{i,j} L_{i,j} \ : \ \sum_i c_{i,i} = 0\right\},$$

$$\mathcal{H}_2' = \left\{\sum_{i',j'=1}^{m} c_{i',j'} R_{i',j'} \ : \ \sum_{i'} c_{i',i'} = 0\right\}.$$

The space $\mathcal{H}_1' = 0$ whenever $n = 1$ and $\mathcal{H}_2' = 0$ when $n = m = 1$. Note that for $n = m = 1$, $\mathcal{D} \cong SU(1,1)/S(U(1) \times U(1))$ is the unit ball in $\mathbb{C}$. When $n = 1$ and $m \ge 2$, $SU(1,m)/S(U(1) \times U(m))$, is a complex hyperbolic space of dimension m. These are the rank one cases.

One can easily exhibit bases for the spaces $\mathcal{H}_1'$, $\mathcal{H}_2'$. For example,

$$\{L_{i,j} \ : \ i \ne j\} \cup \{L_{i,i} - L_{i+1,i+1} \ : \ 1 \le i \le n - 1\}$$

is a basis for $\mathcal{H}'_1$.

From above, we see that $\mathcal{H}^c$ can be written as

$$\mathcal{H}^c = \left\{ \sum_{i,j=1}^{n} \sum_{i',j'=1}^{m} c_{i,i',j,j'} E_{i,i'} \otimes \overline{E}_{j,j'} \ : \ \sum_{i=1}^{n} c_{i,i',i,j'} = 0 = \sum_{i'=1}^{m} c_{i,i',j,i'} \right\}.$$

Note that $\mathcal{H}^c = 0$ when $n = 1$. That is, when G/K has rank one. The action of $K' = SU(n) \times SU(m)$ on $\mathcal{H}^c$ is equivalent to the (exterior) tensor product of the complexified adjoint representations for $SU(n)$, $SU(m)$ on $sl(n, \mathbb{C})$, $sl(m, \mathbb{C})$. In particular, $\mathcal{H}^c$ is K-irreducible. Highest weight vectors in $\mathfrak{p}_\pm$ are given by $E_{1,1}$ and $\overline{E}_{n,m}$. As $E_{1,1} \otimes \overline{E}_{n,m}$ belongs to $\mathcal{H}^c$, we see that $\mathcal{H}^c$ is the Cartan component in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$. The space $\mathcal{H}^c$ has dimension $(n^2-1)(m^2-1)$. The following elements form a basis:

- $E_{i,i'} \otimes \overline{E}_{j,j'}$ with $i \neq j$; $i' \neq j'$,
- $E_{i,i'} \otimes \overline{E}_{j,i'} - E_{i,i'+1} \otimes \overline{E}_{j,i'+1}$ with $i \neq j$; $1 \leq i' \leq m-1$,
- $E_{i,i'} \otimes \overline{E}_{i,j'} - E_{i+1,i'} \otimes \overline{E}_{i+1,j'}$ with $i' \neq j'$; $1 \leq i \leq n-1$,
- $E_{1,1} \otimes \overline{E}_{1,1} - E_{i,1} \otimes \overline{E}_{i,1} - E_{1,j'} \otimes \overline{E}_{1,j'} + E_{i,j'} \otimes \overline{E}_{i,j'}$ with $2 \leq i \leq n$; $2 \leq j' \leq m$.

4.2. **Type C I.** These are the spaces $G/K = Sp(n, \mathbb{R})/U(n)$. Since $Sp(1, \mathbb{R})/U(1) \approxeq U(1,1)/S(U(1) \times U(1))$, we can assume that $n \geq 2$. The rank of G/K is n. The space $\mathfrak{p}_+$ is realized as the space of $n \times n$ symmetric matrices:

$$\mathfrak{p}_+ = \{ Z \in M_{n,n}(\mathbb{C}) \ : \ Z^t = Z \}$$

with Hermitian inner product $\langle X, Y \rangle = tr(XY^*) = tr(X\overline{Y})$. The bounded realization for G/K as a domain in $\mathfrak{p}_+$ is

$$\mathcal{D} = \{ Z \in \mathfrak{p}_+ \ : \ Z\overline{Z} < I_n \}.$$

This is a tube domain.

The group K acts on $\mathfrak{p}_+$ via

$$\sigma_+(k)Z = kZk^t,$$

and on $\mathfrak{p}_+ \otimes \mathfrak{p}_- = \mathfrak{p}_+ \otimes \overline{\mathfrak{p}}_+$ by

$$(\sigma_+ \otimes \sigma_-)(k)(X \otimes \overline{Y}) = (kXk^t) \otimes (\overline{k}\overline{Y}k^*).$$

Define elements $F_{i,j} \in \mathfrak{p}_+$ $(1 \leq i, j \leq n)$ as

$$F_{i,j} = E_{i,j} + E_{j,i}.$$

Note that $F_{i,j} = F_{j,i}$ and that the $F_{i,j}$'s are pair-wise orthogonal with

$$\langle F_{i,j}, F_{i,j} \rangle = \begin{cases} 4 & \text{for } i = j \\ 2 & \text{for } i \neq j \end{cases}.$$

Define

$$T_{i,j} = \sum_{\ell=1}^{n} F_{i,\ell} \otimes \overline{F}_{\ell,j}, \ (1 \le i,j \le n).$$

Then one has

$$(\sigma_+ \otimes \sigma_-)(k) T_{i,j} = (kTk^*)_{i,j}.$$

Thus

$$\mathcal{H} = Span\{T_{i,j} \ : \ 1 \le i,j \le n\}$$

is K-invariant and K acts on $\mathcal{H}$ by a copy of Ad_K. The space $\mathcal{H}$ contains two irreducible components, corresponding to the decomposition $\mathfrak{k}_{\mathbb{C}} = \mathbb{C} \oplus sl(n,\mathbb{C})$. This can be written as $\mathcal{H} = \mathcal{L} \oplus \mathcal{H}'$ where

$$\mathcal{L} = \mathbb{C}\left(\sum_{i=1}^{n} T_{i,i}\right) = \mathbb{C}\left(\sum_{i,\ell} F_{i,\ell} \otimes \overline{F}_{\ell,i}\right),$$

$$\mathcal{H}' = \left\{\sum_{i,j=1}^{n} c_{i,j} T_{i,j} \ : \ \sum_{i=1}^{n} c_{i,i} = 0\right\}.$$

A basis for $\mathcal{H}'$ is given by

$$\{T_{i,j} \ : \ i \ne j\} \cup \{T_{i,i} - T_{i+1,i+1} \ : \ 1 \le i \le n-1\}.$$

Let $\mathcal{H}^c$ denote the orthogonal complement to $\mathcal{H}$ in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$. As representations of $SU(n)$, $\sigma_\pm$ have highest weights $(2,0,\dots,0)$ and $(2,2,\dots,2,0)$, and $F_{1,1}$, $\overline{F}_{n,n}$ are highest weight vectors. An easy application of the Littlewood-Richardson rules (see [FH], Appendix A) shows that $\sigma_+ \otimes \sigma_-$ has exactly three irreducible components. Thus $\mathcal{H}^c$ is necessarily irreducible. The highest weight for $(\sigma_+ \otimes \sigma_-)|\mathcal{H}^c$ is $(4,2,\dots,2,0)$. Since $F_{1,1} \otimes \overline{F}_{n,n}$ belongs to $\mathcal{H}^c$ (it is orthogonal to $\mathcal{H}$), $\mathcal{H}^c$ is the Cartan component in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$.

The dimension of $\mathcal{H}^c$ is $[n(n+1)/2]^2 - n^2 = n^2(n-1)(n+3)/4$. Working from the above description of $\mathcal{H}$ and using the fact that the $F_{i,j} \otimes \overline{F}_{k,\ell}$'s are pair-wise orthogonal in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$, we see that the following vectors form a basis for $\mathcal{H}^c$:

- $F_{i,i} \otimes \overline{F}_{j,j}$ with $i \ne j$,
- $F_{i,i} \otimes \overline{F}_{j,k}$ with $i \ne j,k$; $j < k$,
- $F_{i,j} \otimes \overline{F}_{k,k}$ with $k \ne i,j$; $i < j$,
- $F_{i,j} \otimes \overline{F}_{k,\ell}$ with $\{i,j\} \cap \{k,\ell\} = \emptyset$; $i < j$; $k < \ell$,
- $||F_{i,k}||^2 ||F_{k,j}||^2 F_{i,1} \otimes \overline{F}_{1,j} - ||F_{i,1}||^2 ||\overline{F}_{1,j}||^2 F_{i,k} \otimes \overline{F}_{k,j}$ with $i \ne j$; $2 \le k \le n$,
- $F_{i,i} \otimes \overline{F}_{i,i} + F_{j,j} \otimes \overline{F}_{j,j} - 4F_{i,j} \otimes \overline{F}_{j,i}$ with $i < j$.

4.3. **Type D III.** Here $G/K = SO^*(2n)/U(n)$. Since $SO^*(4)/U(2)$ is non-irreducible and $SO^*(6)/U(3) \approxeq SU(1,3)/S(U(1)\times U(3))$, we can take $n \ge 4$ here. Thus G/K has rank $\lfloor n/2 \rfloor \ge 2$. The space $\mathfrak{p}_+$ can be realized as the space of skew-symmetric $n \times n$-matrices:

$$\mathfrak{p}_+ = \{Z \in M_{n,n}(\mathbb{C}) \ : \ Z^t = -Z\}$$

with Hermitian inner product $\langle X, Y\rangle = tr(XY^*) = -tr(X\overline{Y})$. The bounded realization of G/K is

$$\mathcal{D} = \{Z \in \mathfrak{p}_+ \ : \ Z^t\overline{Z} < I_n\}.$$

$\mathcal{D}$ is a tube domain if and only if n is even.

As in the preceding case, $K = U(n)$ acts on $\mathfrak{p}_+$ via $\sigma_+(k)Z = kZk^t$. The description of $\mathcal{H} \subset \mathfrak{p}_+ \otimes \mathfrak{p}_- = \mathfrak{p}_+ \otimes \overline{\mathfrak{p}}_+$ parallels that for type **C I**. We set

$$F'_{i,j} = E_{i,j} - E_{j,i}$$

and note that $F'_{i,j} = -F'_{j,i}$, $F'_{i,i} = 0$. The $F'_{i,j}$'s are pair-wise orthogonal with $\langle F'_{i,j}, F'_{i,j}\rangle = 2$ for $i \neq j$. Letting

$$T'_{i,j} = \sum_{\ell=1}^{n} F'_{i\ell} \otimes \overline{F}'_{\ell,j},$$

and

$$\mathcal{H} = Span\{T'_{i,j} \ : \ 1 \leq i, j \leq n\},$$

we see that $(\sigma_+ \otimes \sigma_-)|\mathcal{H}$ is a copy of Ad_K. The space $\mathcal{H}$ decomposes as $\mathcal{H} = \mathcal{L} \oplus \mathcal{H}'$, were $\mathcal{L}$, $\mathcal{H}'$ are defined as for type **C I**.

An application of the Littlewood-Richardson rules shows that the orthogonal complement $\mathcal{H}^c$ to $\mathcal{H}$ in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$ must be irreducible with highest weight $(2, 2, 1, \ldots, 1, 0)$ and dimension $[n(n-1)/2]^2 - n^2 = n^2(n+1)(n-3)/4$. The space $\mathcal{H}^c$ contains the tensor product $F'_{1,2} \otimes \overline{F}'_{n-1,n}$ of highest weight vectors in $\mathfrak{p}_\pm$. Thus $\mathcal{H}^c$ is the Cartan component in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$. Working from the description of $\mathcal{H}$ one obtains the following spanning set for $\mathcal{H}^c$:

- $F'_{i,j} \otimes \overline{F}'_{k,\ell}$ with $\{i, j\} \cap \{k, \ell\} = \emptyset$; $i < j$; $k < \ell$,
- $F'_{i,1} \otimes \overline{F}'_{1,j} - F'_{i,k} \otimes \overline{F}'_{k,j}$ with $k, i, j > 1$; $i \neq j$; $k \neq i, j$,
- $F'_{i,j} \otimes \overline{F}'_{j,i} + \frac{1}{(n-1)(n-2)} \sum_{\ell=1}^{n} T'_{\ell,\ell} - \frac{1}{n-2}(T'_{i,i} + T'_{j,j})$ with $i < j$.

This spanning set for $\mathcal{H}^c$ is not, however, linearly independent. For example, summing the elements of the third kind over $j \neq i$ for fixed i gives zero.

4.4. **Type BD I.** Here $G/K = SO_\circ(2, n)/(SO(2) \times SO(n))$. We can take $n \geq 5$, in view of isomorphisms in low dimensions. ($SO_\circ(2, 2)/(SO(2) \times SO(2))$ is not irreducible, $SO_\circ(2, 3)/(SO(2) \times SO(3)) \cong Sp(2, \mathbb{R})/U(2)$, and $SO_\circ(2, 4)/(SO(2) \times SO(4)) \cong SU(2, 2)/S(U(2) \times U(2))$.) The rank of G/K is 2. In this case, $\mathfrak{p}_+$ is realized as $\mathfrak{p}_+ = M_{2,n}(\mathbb{R})$ with complex structure

$$J \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix},$$

for row vectors $x, y \in \mathbb{R}^n$. The map $T : M_{2,n}(\mathbb{R}) \to \mathbb{C}^n$ defined as

$$T(Z) = x^t + iy^t, \text{ for } Z = \begin{bmatrix} y \\ x \end{bmatrix}$$

is an isomorphism from the complex vector space $(\mathfrak{p}_+, J)$ to $\mathbb{C}^n$ (column vectors). The bounded realization for G/K is

$$\mathcal{D} = \{Z \in \mathfrak{p}_+ \ : \ ZZ^t < I_2\}.$$

This is a tube domain.

The group $K = SO(2) \times SO(n)$ acts on $\mathfrak{p}_+$ via

$$\sigma_+(k_1, k)Z = k_1 Z k^t, \ (k_1 \in SO(2), k \in SO(n)).$$

For $k_1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ one computes

$$T\big((\sigma_+ \otimes \sigma_-)(k_1, k)Z\big) = e^{i\theta} k T(Z).$$

That is, the action of K on $\mathfrak{p}_+$ coincides, via T, with the standard action of $\mathbb{T} \times SO(n)$ on $\mathbb{C}^n$. In particular, $SO(2)$ acts on $\mathfrak{p}_+$ by (complex) scalars, so the irreducible subspaces in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$ under the actions of K and $K' = SO(n)$ agree. Thus, we need to decompose $\mathbb{C}^n \otimes (\mathbb{C}^n)^*$, or equivalently $\mathbb{C}^n \otimes \mathbb{C}^n$, under the diagonal action of $SO(n)$. Identifying $\mathbb{C}^n \otimes \mathbb{C}^n$ with $M_{n,n}(\mathbb{C})$, $SO(n)$ acts via

$$k \cdot A = kAk^t, \ (k \in SO(n), A \in M_{n,n}(\mathbb{C})).$$

From this viewpoint, the decomposition into $SO(n)$-irreducible subspaces is transparent:

$$M_{n,n}(\mathbb{C}) = \mathbb{C}I_n \oplus \{A \ : \ A^t = -A\} \oplus \{A \ : \ A^t = A, tr(A) = 0\}.$$

One can use the above isomorphism $\mathfrak{p}_+ \otimes \mathfrak{p}_- \approxeq M_{n,n}(\mathbb{C})$ to obtain the corresponding decomposition for our original model. Letting $X_j = E_{2,j} \in (\mathfrak{p}_+ = M_{2,n}(\mathbb{R}))$,

$$\mathfrak{p}_+ \otimes \mathfrak{p}_- = \mathcal{L} \oplus \mathcal{H}' \oplus \mathcal{H}^c$$

where

$$\begin{aligned}
\mathcal{L} &= \mathbb{C}\left(\sum_{j=1}^n X_j \otimes \overline{X}_j\right), \\
\mathcal{H}' &= \left\{\sum_{i,j=1}^n c_{i,j} X_i \otimes \overline{X}_j \ : \ c_{i,j} = -c_{j,i}\right\}, \\
\mathcal{H}^c &= \left\{\sum_{i,j=1}^n c_{i,j} X_i \otimes \overline{X}_j \ : \ c_{i,j} = c_{j,i}, \sum_i c_{i,i} = 0\right\}.
\end{aligned}$$

$\mathcal{H} = \mathcal{L} \oplus \mathcal{H}'$ yields the Hua system. A highest weight vector in $\mathfrak{p}_+$ is given by

$$X_1 + iX_2 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Since $(X_1 + iX_2) \otimes (\overline{X}_1 + i\overline{X}_2) = X_1 \otimes \overline{X}_1 - X_2 \otimes \overline{X}_2 + iX_1 \otimes \overline{X}_2 + iX_2 \otimes \overline{X}_1$ belongs to $\mathcal{H}^c$, we see that $\mathcal{H}^c$ is the Cartan component in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$.

4.5. **Type E III.** In this case $K = Spin(10) \times \mathbb{T}$ and G has Lie algebra $\varepsilon_{6(-14)}$, a certain real form for the complex Lie algebra ε_6. The space G/K has rank 2 and is of non-tube type.

One can identify $\mathfrak{p}_+$ with $\Lambda^{\text{even}}(\mathbb{C}^5) \approxeq \mathbb{C}^{16}$ and σ_+ is the positive half-spin representation. The contragredient representation $\sigma_+^* \simeq \sigma_-$ is equivalent to the negative half-spin representation on $\Lambda^{\text{odd}}(\mathbb{C}^5) \approxeq \mathbb{C}^{16}$.

As $\mathfrak{k}_{\mathbb{C}} = so(10,\mathbb{C}) \oplus \mathbb{C}$, the subspace $\mathcal{H} = \varphi^*(\mathfrak{k}_{\mathbb{C}}^*)$ of $\mathfrak{p}_+ \otimes \mathfrak{p}_-$ has two irreducible components:

$$\mathcal{H} = \mathcal{L} \oplus \mathcal{H}' = \varphi^*(\mathbb{C}^*) \oplus \varphi^*(so(10,\mathbb{C})^*).$$

The highest weights for $\sigma_\pm$ are $(1/2)(L_1 + L_2 + L_3 + L_4 \pm L_5)$ in the notation of [FH]. (See [FH], Proposition 20.15.). Thus, the Cartan component W in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$ has highest weight $L_1 + L_2 + L_3 + L_4$. A application of the Weyl dimension formula shows that $dim(W) = 210$. (The Weyl dimension formula for $SO(2m)$ is Equation (24.41) in [FH].) Since

$$dim(\mathcal{H}) + dim(W) = \dim(\mathfrak{k}_{\mathbb{C}}) + 210 = 46 + 210 = 256 = 16^2 = dim(\mathfrak{p}_+ \otimes \mathfrak{p}_-),$$

we conclude that $W = \mathcal{H}^c$, the orthogonal complement to $\mathcal{H}$ in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$. This proves that $\mathcal{H}^c$ is irreducible and is the Cartan component in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$.

4.6. **Type E VII.** Finally, we consider the exceptional case where $K = E_6 \times \mathbb{T}$ and G has Lie algebra $\varepsilon_{7(-25)}$, a real form of ε_7. The space G/K has rank 3 and is a tube domain.

In this case $\mathfrak{p}_+$ can be identified with an exceptional Jordan algebra $\mathcal{J}$ of dimension 27. The representation of E_6 on $\mathcal{J}$ is described in [CS]. We have the decomposition

$$\mathfrak{p}_+ \otimes \mathfrak{p}_- = \mathcal{L} \oplus \mathcal{H}' \oplus \mathcal{H}^c = \varphi^*(\mathbb{C}^*) \oplus \varphi^*(\varepsilon_6^*) \oplus Ker(\varphi)$$

as usual. It remains to show that $\mathcal{H}^c$ is irreducible and is the Cartan component in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$.

The fundamental weights for the complex simple Lie algebra ε_6 are usually denoted $\omega_1 \ldots, \omega_6$. (See, for example, the tables in [Bou].) The representation σ_+ has highest weight ω_1 and its contragredient $\sigma_+^* \simeq \sigma_-$ has highest weight ω_6. Hence $\omega_1 + \omega_6$ is the highest weight for the Cartan component W in $\mathfrak{p}_+ \otimes \mathfrak{p}_-$. One can use the Weyl dimension formula to show that $dim(W) = 650$. In fact, this dimension can be obtained from a table in [GS]. We now see that

$$dim(\mathcal{H}) + dim(W) = \dim(\mathfrak{k}_{\mathbb{C}}) + 650 = 79 + 650 = 729 = 27^2 = dim(\mathfrak{p}_+ \otimes \mathfrak{p}_-).$$

Hence $W = \mathcal{H}^c$, completing the proof for Proposition 4.2.

5. The main theorem

The aim of this paper is to characterize functions on G/K which are annihilated by a system of type (1,1) containing the Laplace-Beltrami operator. Proposition 4.2 shows that, apart from the systems $\mathcal{L}$ and $\mathfrak{p}_+ \otimes \mathfrak{p}_-$, we have the Hua system $\mathcal{H} = \mathcal{L} \oplus \mathcal{H}'$

and the complementary system $\mathcal{L}\oplus\mathcal{H}^c$. For $G/K = SU(n,m)/S(U(n)\times U(m))$, the Hua system can be decomposed: $\mathcal{H} = \mathcal{L}\oplus\mathcal{H}'_1\oplus\mathcal{H}'_2$ and one can consider two further systems of type (1,1): $\mathcal{L}\oplus\mathcal{H}'_1$ and $\mathcal{L}\oplus\mathcal{H}'_2$. These exhaust the possibilities.

Theorems 5.1 through 5.5 below characterize the functions annihilated by each of these systems. The first two results concern zeros of the Hua system.

Theorem 5.1 ([JK])**.** *A function f on a Hermitian symmetric space of tube type satisfies $\mathcal{H}(f) = 0$ if and only if it is the Poisson-Szegö integral of a hyperfunction supported on the Shilov boundary of G/K.*

Theorem 5.2 ([BBDHPT],[B])**.** *Let G/K be an irreducible Hermitian symmetric space of non-tube type and let f be a real-valued function on G/K. Then $\mathcal{H}(f) = 0$ if and only if f is pluriharmonic.*

Let us recall that f defined on $\mathcal{D}\subset\mathbb{C}^n$ is pluriharmonic if it is the real part of a holomorphic function. Equivalently, f is annihilated by all operators $\partial_{z_j}\partial_{\overline{z}_k}$ $(1\leq j,k\leq n)$. That is, the pluriharmonic functions are those annihilated by the system $\mathfrak{p}_+\otimes\mathfrak{p}_-$. The main result of this paper is the following.

Main Theorem 5.3. *Let G/K be an irreducible Hermitian symmetric space of rank $r\geq 2$ and let f be a bounded real-valued function on G/K. Then f is $\mathcal{L}\oplus\mathcal{H}^c$-harmonic if and only if f is pluriharmonic.*

In the statement of Theorem 5.3, we require $r\geq 2$, because $\mathcal{H}^c = 0$ for the rank one Hermitian symmetric spaces. It seems likely that the boundedness hypothesis on f can be removed from Theorem 5.3, as in the statement of Theorem 5.2, but we are not able to show this using the methods of this paper. This is one problem for future research.

For $G/K = S(n,m)/S(U(n)\times U(m))$ we must also consider the systems $\mathcal{L}\oplus\mathcal{H}'_1$ and $\mathcal{L}\oplus\mathcal{H}'_2$. As in Section 4.1 we assume that $1\leq n\leq m$ and have $\mathcal{H}'_1\cong sl(n,\mathbb{C})$, $\mathcal{H}'_2\cong sl(m,\mathbb{C})$. The following result is due to N. Berline and M. Vergne.

Theorem 5.4 ([BV])**.** *Let $G/K = SU(n,m)/S(U(n)\times U(m))$ with $1\leq n\leq m$. A function f on G/K satisfies $\mathcal{L}\oplus\mathcal{H}'_1(f) = 0$ if and only if it is the Poisson-Szegö integral of a hyperfunction supported on the Shilov boundary.*

By symmetry, when $n = m$ the $\mathcal{L}\oplus\mathcal{H}'_2$ system also characterizes Poisson-Szegö integrals. These are tube domains. On the other hand, when $n < m$ we have a non-tube domain and the following result.

Theorem 5.5. *Let $G/K = SU(n,m)/S(U(n)\times U(m))$ with $1\leq n < m$. A real-valued function f on G/K satisfies $\mathcal{L}\oplus\mathcal{H}'_2(f) = 0$ if and only if f is pluriharmonic.*

Proof. Let f be $\mathcal{L}\oplus\mathcal{H}'_2$-harmonic. Note that the system $\mathcal{H}'_2$ contains a copy of $\mathcal{H}'_1$ via the obvious inclusion $sl(n,\mathbb{C})\hookrightarrow sl(m,\mathbb{C})$. Thus Theorem 5.4 implies that f is the Poisson-Szegö integral of a hyperfunction. But now, Theorem 5.4 also shows that f

is $\mathcal{L} \oplus \mathcal{H}'_1$-harmonic. Thus in fact f is annihilated by all of $\mathcal{H}'_1 \oplus \mathcal{L} \oplus \mathcal{H}'_2 = \mathcal{H}$. Finally as $n < m$, G/K is of non-tube type, so Theorem 5.2 shows f is pluriharmonic. $\square$

The rest of this paper concerns the proof of Theorem 5.3. Our strategy can be outlined as follows. Let G/K have rank at least two and f be a bounded real-valued function on G/K. First note that if f is pluriharmonic then $\mathcal{L} \oplus \mathcal{H}^c(f) = 0$ because pluriharmonic functions are annihilated by all of $\mathfrak{p}_+ \otimes \mathfrak{p}_-$. Hence we need only show that $\mathcal{L} \oplus \mathcal{H}^c(f) = 0$ implies f is pluriharmonic. Having the boundedness assumption (which is invariant under biholomorphic mappings), we transfer our problem to the group S acting simply transitively on the Siegel domain ${}^c\mathcal{D}$. We show that f is annihilated by some S-invariant operators derived from the $\mathcal{L} \oplus \mathcal{H}^c$ system. Using two of these operators and a technique from [BBDHPT], we show that $f \circ c^{-1}$ is a Poisson-Szegö integral. In the tube case, Theorem 5.1 now implies that f is Hua-harmonic. So f is in fact annihilated by all of $\mathfrak{p}_+ \otimes \mathfrak{p}_- = \mathcal{H}' \oplus \mathcal{L} \oplus \mathcal{H}^c$, hence pluriharmonic. In the non-tube case, according to [BV], Poisson-Szegö integrals are annihilated by a certain system of third order operators. We will use this fact and methods described in [BDH] and [B] to obtain pluriharmonicity of f in the non-tube case.

6. The $\mathcal{L} \oplus \mathcal{H}^c$ system and Berline-Vergne operators

6.1. **The $\mathcal{L} \oplus \mathcal{H}^c$ system.** Below we introduce some operators which belong to the $\mathcal{L} \oplus \mathcal{H}^c$ system. We make extensive use of the notation and results from Section 3.2. As before, r always denotes the rank of G/K. We assume throughout that $r \geq 2$, since $\mathcal{H}^c = 0$ when $r = 1$. (Most formulas to be derived below are vacuous when $r = 1$.)

For $\alpha \in Q_{ij}$, put

$$\Delta_\alpha = E_\alpha E_{-\alpha} + E_{\widetilde{\alpha}} E_{-\widetilde{\alpha}} - E_{\gamma_i} E_{-\gamma_i} - E_{\gamma_j} E_{-\gamma_j}, \tag{6.1}$$

regarded as an operator arising from $\mathfrak{p}_+ \otimes \mathfrak{p}_-$.

Recall that the system $\mathcal{H}^c$ is the kernel of the map $\varphi : \mathfrak{p}_+ \otimes \mathfrak{p}_- \to \mathfrak{k}_{\mathbb{C}}$ given by $\varphi(X, Y) = [X, Y]$.

Lemma 6.2. *For $\alpha \in Q_{ij}$ the operator Δ_α belongs to the system $\mathcal{H}^c$.*

Proof. Recall from (3.10) that we have $\widetilde{\alpha} = \gamma_i + \gamma_j - \alpha$, also in Q_{ij}. Thus

$$\widetilde{H}_\alpha + \widetilde{H}_{\widetilde{\alpha}} - \widetilde{H}_{\gamma_i} - \widetilde{H}_{\gamma_j} = 0, \tag{6.3}$$

and hence

$$\frac{(\alpha, \alpha)}{2} H_\alpha + \frac{(\widetilde{\alpha}, \widetilde{\alpha})}{2} H_{\widetilde{\alpha}} - \frac{(\gamma_i, \gamma_i)}{2} H_{\gamma_i} - \frac{(\gamma_j, \gamma_j)}{2} H_{\gamma_j} = 0. \tag{6.4}$$

But all elements of Q^+ have a common length, so $(\alpha, \alpha) = (\widetilde{\alpha}, \widetilde{\alpha}) = (\gamma_i, \gamma_i) = (\gamma_j, \gamma_j)$ and hence $H_\alpha + H_{\widetilde{\alpha}} - H_{\gamma_i} - H_{\gamma_j} = 0$.

Now we have

$$\begin{aligned}\varphi(\Delta_\alpha) &= [E_\alpha, E_{-\alpha}] + [E_{\widetilde{\alpha}}, E_{-\widetilde{\alpha}}] - [E_{\gamma_i}, E_{-\gamma_i}] - [E_{\gamma_j}, E_{-\gamma_j}] \\ &= H_\alpha + H_{\widetilde{\alpha}} - H_{\gamma_i} - H_{\gamma_j} \\ &= 0,\end{aligned}$$

which yields the lemma. □

Proposition 6.5. *The operator*

$$\Delta_1 = \sum_{\alpha\in\bigcup_i Q_i} E_\alpha E_{-\alpha} + c\sum E_{\gamma_i}E_{-\gamma_i}$$

is an element of $\mathcal{L}\oplus\mathcal{H}^c$ for some positive constant c.

Proof. As $\mathcal{L}$ is the unique K-invariant operator in $\mathfrak{p}_+\otimes\mathfrak{p}_-\cong\mathfrak{p}_+\otimes(\mathfrak{p}_+)^*$, we have

$$\mathcal{L} = \sum_{\alpha\in Q^+} E_\alpha E_{-\alpha} = \Big(\sum_{\alpha\in\bigcup_i Q_i} + \sum_{\alpha\in\bigcup_{i<j} Q_{ij}} + \sum_{\alpha\in\Gamma}\Big) E_\alpha E_{-\alpha}.$$

The operator $\mathcal{L} - \frac{1}{2}\sum_{\alpha\in\bigcup_{i<j}Q_{ij}}\Delta_\alpha$ belongs to $\mathcal{L}\oplus\mathcal{H}^c$ in view of Lemma 6.2. Let $q_2 = |Q_{ij}|$ denote the common cardinality of the sets Q_{ij}. Then we can write

$$\begin{aligned}\mathcal{L} - \frac{1}{2}\sum_{\alpha\in\bigcup_{i<j}Q_{ij}}\Delta_\alpha &= \sum_{\alpha\in\bigcup_i Q_i} E_\alpha E_{-\alpha} + \sum_{k=1}^{r} E_{\gamma_k}E_{-\gamma_k} + \frac{q_2}{2}\sum_{i<j}(E_{\gamma_i}E_{-\gamma_i} + E_{\gamma_j}E_{-\gamma_j}) \\ &= \sum_{\alpha\in\bigcup_i Q_i} E_\alpha E_{-\alpha} + \left(1 + \frac{(r-1)q_2}{2}\right)\sum_{k=1}^{r} E_{\gamma_k}E_{-\gamma_k}.\end{aligned}$$

So $\Delta_1 = \sum_{\alpha\in\bigcup_i Q_i} E_\alpha E_{-\alpha} + c\sum E_{\gamma_i}E_{-\gamma_i}$ belongs to $\mathcal{L}\oplus\mathcal{H}^c$ for $c = 1+(r-1)q_2/2$. □

Definition 6.6. $\widetilde{Q_r} = Q_r \cup \bigcup_i Q_{ir}$.

Lemma 6.7. *If $\alpha\in Q_{ij}\cup Q_i$ and $i,j<r$ then there exist $\beta,\delta\in\widetilde{Q_r}$ such that*

$$\Delta'_\alpha = E_\alpha E_{-\alpha} + E_{\gamma_r}E_{-\gamma_r} - E_\beta E_{-\beta} - E_\delta E_{-\delta} \in \mathcal{H}^c.$$

Proof. Since $\alpha\notin\widetilde{Q_r}$, we see that $\alpha\pm\gamma_r$ is not a root, and so α and γ_r are strongly orthogonal. Let Γ' be a maximal set of strongly orthogonal roots containing α and γ_r. Take any root $\beta\sim_{\Gamma'}\frac{\alpha+\gamma_r}{2}$ and put $\delta = \alpha+\gamma_r-\beta$. Then by (3.10) δ is a root, and $\delta\sim_{\Gamma'}\frac{\alpha+\gamma_r}{2}$. Since $(\beta,\gamma_r)\neq 0$, $(\delta,\gamma_r)\neq 0$ and neither is equal to γ_r, we have $\beta,\delta\in Q_r\cup\bigcup_i Q_{ir}$. Now Lemma 6.2 concludes the proof. □

Proposition 6.8. *The operator*

$$\Delta_2 = \sum_{\alpha\in Q_r\cup\bigcup Q_{ir}} c_\alpha E_\alpha E_{-\alpha} - c_{\gamma_r}E_{\gamma_r}E_{-\gamma_r}$$

is an element of $\mathcal{L}\oplus\mathcal{H}^c$ for some positive constants c_α.

Proof. Take

$$\mathcal{L} - \frac{1}{c}\Delta_1 - (1 - \frac{1}{c}) \sum_{\alpha \in \bigcup_{i<r} Q_i} \Delta'_\alpha - \sum_{\alpha \in \bigcup_{i<j<r} Q_{ij}} \Delta'_\alpha$$

with c as in Proposition 6.5. □

6.2. **Berline-Vergne operators.** In this subsection we restrict our attention to non-tube spaces. In Section 8 we will prove that any bounded function annihilated by the $\mathcal{L} \oplus \mathcal{H}^c$ system is a Poisson-Szegö integral from the Shilov boundary. This class of functions is characterized by operators due to N. Berline and M. Vergne.

Theorem 6.9 ([BV])**.** *Let G/K be a Hermitian symmetric non-tube space and let $f \in C^\infty(G/K)$. Then f is a Poisson-Szegö integral if and only if f is G-harmonic and*

$$\mathcal{BV}(f) = \sum_{\alpha,\beta,\delta \in Q^+} E_\alpha E_{-\beta} E_\delta \widetilde{f}[E_{-\alpha}, [E_\beta, E_{-\delta}]] = 0. \tag{6.10}$$

The phrase "f is G-harmonic" means $Df = 0$ for all operators $D \in \mathbf{D}(G/K)$ with no constant term. (In [BV] and other works, G-harmonic functions are simply referred to as "harmonic".)

Note that $[E_{-\alpha}, [E_\beta, E_{-\delta}]]$ in the expression above is an element of $\mathfrak{p}_-$. We will compute the projection of $\mathcal{BV}(f)$ onto the one-dimensional space spanned by $E_{-\gamma_r}$. So we need to understand the effect of the operators $E_\alpha E_{-\beta} E_\delta$ on $\widetilde{f}$ when $\alpha - \beta + \delta = \gamma_r$. We get a non-zero contribution to the sum only when $\beta - \delta = \alpha - \gamma_r$. We consider the possibilities, assuming throughout that $f \in C^\infty(G/K)$ is annihilated by the $\mathcal{L} \oplus \mathcal{H}^c$ system.

To simplify our notation put

$$U_\alpha = E_\alpha E_{-\alpha} E_{\gamma_r}. \tag{6.11}$$

Since $[\mathfrak{p}_+, \mathfrak{p}_-] \subset \mathfrak{k}_\mathbb{C}$ and $\mathfrak{p}_+$ is abelian , we have

$$E_\alpha E_{-\delta} E_\gamma \widetilde{f} = E_\alpha E_\gamma E_{-\delta} \widetilde{f} = E_\gamma E_\alpha E_{-\delta} \widetilde{f} = E_\gamma E_{-\delta} E_\alpha \widetilde{f}$$

for all α, β, γ in Q^+. In particular, $U_\alpha = E_\alpha E_{-\alpha} E_{\gamma_r} = E_{\gamma_r} E_{-\alpha} E_\alpha$.

Case 1. $\alpha = \beta = \gamma_r$, then $\delta = -\alpha + \beta + \gamma_r = \gamma_r$ and

$$[E_{-\gamma_r}, [E_{\gamma_r}, E_{-\gamma_r}]] = [E_{-\gamma_r}, H_{\gamma_r}] = 2E_{-\gamma_r}.$$

The coefficient of $E_{-\gamma_r}$ will be $2E_{\gamma_r} E_{-\gamma_r} E_{\gamma_r} \widetilde{f} = 2U_{\gamma_r} \widetilde{f}$.

Case 2. $\alpha = \beta \neq \gamma_r$, then again $\delta = \gamma_r$. E_β does not commute with $E_{-\gamma_r}$ if and only if $\alpha = \beta \in \widetilde{Q}_r$. Hence by the Jacobi identity

$$\begin{aligned} [E_{-\alpha}, [E_\alpha, E_{-\gamma_r}]] &= -[E_\alpha, [E_{-\gamma_r}, E_{-\alpha}]] - [E_{-\gamma_r}, [E_{-\alpha}, E_\alpha]] = -[H_\alpha, E_{-\gamma_r}] \\ &= E_{-\gamma_r}. \end{aligned}$$

The coefficient will be $E_\alpha E_{-\alpha} E_{\gamma_r} \widetilde{f} = U_\alpha \widetilde{f}$.

Case 3. $\alpha \neq \beta = \delta$, then $\alpha = \gamma_r$ and as before

$$[E_{-\gamma_r}, [E_\beta, E_{-\beta}]] = E_{-\gamma_r}$$

for $\beta \in \widetilde{Q}_r$. The coefficient will be $E_{\gamma_r} E_{-\beta} E_\beta \widetilde{f} = U_\beta \widetilde{f}$.

Case 4. Assume that $\alpha \neq \beta$ and $\beta \neq \delta$ and $\beta - \delta = \alpha - \gamma_r$ is a root.

Then $(\alpha - \gamma_r)(H_{\gamma_r}) = \pm 1$, so α must belong to $\widetilde{Q_r}$. Thus $\beta - \delta \sim (\gamma_i + \gamma_r)/2$ or $-\gamma_r/2$. Since $\beta \in Q^+$, we must have $\delta \in \widetilde{Q_r}$ or $\delta = \gamma_r$. But $\alpha \neq \beta$, so we conclude that both α, δ are in $\widetilde{Q_r}$.

We further analyze this case in a series of lemmas:

Lemma 6.12. *Suppose that $\alpha, \delta \in \widetilde{Q_r}$, $\beta \in Q^+$ such that $\beta - \delta = \alpha - \gamma_r$ is a root. Then there is some $d = \pm 1$ such that $dE_{-\beta} E_\delta - E_{-\alpha} E_{\gamma_r}$ and $dE_{-\beta} E_\alpha - E_{-\delta} E_{\gamma_r}$ are in $\mathcal{H}^c$.*

Proof. Since $\beta - \delta = \alpha - \gamma_r$ is a root, there is some $d = \pm 1$ such that

$$[E_\beta, E_{-\delta}] = d[E_\alpha, E_{-\gamma_r}]. \tag{6.13}$$

By applying the conjugation τ, we also have

$$[E_{-\beta}, E_\delta] = d[E_{-\alpha}, E_{\gamma_r}]. \tag{6.14}$$

From (6.13), and the fact that $\mathfrak{p}_-$ is abelian, we obtain:

$$\begin{aligned}
[E_{-\alpha}, [E_\beta, E_{-\delta}]] &= d[E_{-\alpha}, [E_\alpha, E_{-\gamma_r}]] \\
&= d[E_{-\gamma_r}, [E_\alpha, E_{-\alpha}]] + d[E_\alpha, [E_{-\alpha}, E_{-\gamma_r}]] \\
&= d[E_{-\gamma_r}, H_\alpha] \\
&= d\gamma_r(H_\alpha) E_{-\gamma_r} = dE_{-\gamma_r}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[E_{-\alpha}, [E_\beta, E_{-\delta}]] &= [E_\beta, [E_{-\alpha}, E_{-\delta}] + [E_{-\delta}, [E_\beta, E_{-\alpha}]] \\
&= [E_{-\delta}, [E_\beta, E_{-\alpha}]]
\end{aligned}$$

Thus we obtain:

$$\begin{aligned}
dH_{\gamma_r} &= d[E_{\gamma_r}, E_{-\gamma_r}] \\
&= [E_{\gamma_r}, [E_{-\delta}, [E_\beta, E_{-\alpha}]]] \\
&= [E_{-\delta}, [E_{\gamma_r}, [E_\beta, E_{-\alpha}]]] + [[E_\beta, E_{-\alpha}], [E_{-\delta}, E_{\gamma_r}]] \\
&= \epsilon_1 H_\delta + \epsilon_2 H_{\beta-\alpha}.
\end{aligned}$$

We compare this last equation to

$$\widetilde{H}_{\gamma_r} = \widetilde{H}_\delta - \widetilde{H}_{\beta-\alpha}.$$

If $\beta - \alpha$ and δ are not linearly independent, then $\delta - \gamma_r = \beta - \alpha = c\delta$ for some $c \neq 0$, so δ is a multiple of γ_r, namely $\delta = \gamma_r/2$. Then our last calculation implies that $\delta H_{\gamma_r} = (\epsilon_1 + \epsilon_2)H_{-\gamma_r}$, which is impossible. Thus we conclude that $\epsilon_2 \neq 0$, and therefore $\beta - \alpha = \delta - \gamma_r$ is a root.

So in fact we have $H_{\gamma_r} = H_\delta - H_{\beta-\alpha}$, and therefore $\epsilon_1 = d$, $\epsilon_2 = -d$. Now

$$[[E_\beta, E_{-\alpha}], [E_{-\delta}, E_{\gamma_r}]] = -dH_{\beta-\alpha},$$

and

$$[E_{-\delta}, E_{\gamma_r}] = \rho E_{\gamma_r-\delta} = \rho E_{\alpha-\beta}$$

with $\rho = \pm 1$. Thus

$$[E_\beta, E_{-\alpha}] = -\rho d E_{\beta-\alpha}$$

and hence, by applying the conjugation τ,

$$[E_{-\beta}, E_\alpha] = \rho d E_{\alpha-\beta} = d[E_{-\delta}, E_{\gamma_r}]. \tag{6.15}$$

Combining (6.14) and (6.15), we see that both $dE_{-\beta}E_\delta - E_{-\alpha}E_{-\gamma_r}$ and $dE_{-\beta}E_\alpha - E_{-\delta}E_{\gamma_r}$ are in $Ker(\varphi) = \mathcal{H}^c$. □

Lemma 6.16. *Suppose that $\alpha, \delta \in \widetilde{Q_r}$, $\beta \in Q^+$ and $\beta - \delta = \alpha - \gamma_r$ is a root. Then $U_\alpha \widetilde{f} = U_\delta \widetilde{f}$.*

Proof. By Lemma 6.12,

$$U_\alpha \widetilde{f} = E_\alpha E_{-\alpha} E_{\gamma_r} \widetilde{f} = dE_\alpha E_{-\beta} E_\delta \widetilde{f} = dE_\delta E_{-\beta} E_\alpha \widetilde{f} = E_\delta E_{-\delta} E_{\gamma_r} \widetilde{f} = U_\delta \widetilde{f}.$$

□

Lemma 6.17. *For any $\alpha \in Q_{ir}$, $U_\alpha \widetilde{f} = U_{\widetilde{\alpha}} \widetilde{f}$.*

Proof. Take $\beta = \gamma_i$, $\delta = \widetilde{\alpha}$ in Lemma 6.16. □

Lemma 6.18. *For any $\alpha \in Q_{ir}$ and $\delta \in Q_r$, $U_\alpha \widetilde{f} = U_\delta \widetilde{f}$.*

Proof. In light of lemmas 6.16 and 6.17 we need to show that, given $\alpha \in Q_{ir}$ and $\delta \in Q_r$, there is a $\beta \in Q_i$ such that

$$\beta - \delta = \alpha - \gamma_r \in \Delta \text{ or } \beta - \delta = \widetilde{\alpha} - \gamma_r \in \Delta.$$

If $\delta - \alpha \in \Delta$, then $\delta - \alpha \sim -\gamma_i/2$, and hence it is an element of C_i. Then $\beta = \delta - \alpha + \gamma_i \in Q_i$, so $\beta - \delta = \widetilde{\alpha} - \gamma_r \in \Delta$.

If $\delta - \alpha \notin \Delta$, then α and δ are strongly orthogonal, and we can find a maximal set Γ' of strongly orthogonal roots containing this pair. Since $\gamma_r(H_\alpha) = \alpha(H_{\gamma_r}) = 1$ and $\gamma_r(H_\delta) = \delta(H_{\gamma_r}) = 1$, we must have $\gamma_r \sim_{\Gamma'} (\alpha + \delta)/2$. Then $\beta = \alpha + \gamma - \gamma_r$ is a root, and $\beta - \delta = \alpha - \gamma_r \in \Delta$. □

Thus we have shown that $U_\alpha \widetilde{f}$ has a common value in the sets Q_{ir}, Q_r, and hence:

Proposition 6.19. *$U_\alpha \widetilde{f}$ has the same value for every α in $\widetilde{Q_r}$.*

Combining all four cases, we now have the following consequence of Theorem 6.9:

Theorem 6.20. *Let G/K be a Hermitian symmetric non-tube space and let f be a Poisson-Szegö integral annihilated by the $\mathcal{L}\oplus\mathcal{H}^c$ system. Then*

$$BV(\widetilde{f}) = \big(E_{\gamma_r}E_{-\gamma_r}E_{\gamma_r} + c\sum_{\alpha\in\widetilde{Q}_r} E_\alpha E_{-\alpha}E_{\gamma_r}\big)\widetilde{f} = 0,$$

with c greater than 1.

7. The $\mathcal{L}\oplus\mathcal{H}^c$ system on Siegel domains

Our proof of Theorem 5.3 in Section 8 uses the operators $\mathcal{L}$, Δ_1, Δ_2 and BV together with some Fourier analysis on the group S. Therefore, we are going to transfer these operators to S and make them act on corresponding functions there.

The scheme is as follows. Given a function f on $\mathcal{D}$ and a function $\widetilde{f}(g) = f(g\cdot o)$ on G, we choose coordinates $z = \sum z_\alpha E_\alpha$ in $\mathfrak{p}_+$ to express $\mathcal{L}$, Δ_1, Δ_2, BV in partial derivatives (see (7.10)).

Let $\widetilde{U}$ be one of operators $\mathcal{L}$, Δ_1, Δ_2, BV and U the corresponding right-hand side operator in (7.10). Since $\widetilde{U}$ is left-invariant on G we have

$$U(f\circ g)(o) = \widetilde{U}\widetilde{f}(g).$$

Next we need to have (7.10), but on ${}^c\mathcal{D}$ for the function

$$ {}^cf(x) = f(c^{-1}\cdot x). \tag{7.1}$$

For that we compute the differential of the Cayley transform c, we choose convenient coordinates (7.11) in $\mathfrak{p}_+$ and we write $\widetilde{U}\widetilde{f}(e)$ as in (7.12) i.e.

$$\widetilde{U}\widetilde{f}(e) = ({}^cU{}^cf)(c\cdot o),$$

there ${}^cU$ denotes the corresponding operator on the right-hand side of (7.12).

Finally, we extend ${}^cU$ by S-invariance

$$({}^cU{}^c\widetilde{f})(s) ={}^c U({}^cf\cdot s)(c\cdot o),$$

where

$$ {}^c\widetilde{f}(s) ={}^c f(s\cdot(c\cdot o)). \tag{7.2}$$

Then, by invariance, $\widetilde{U}\widetilde{f}=0$ implies ${}^cU{}^c\widetilde{f}=0$. Then $\mathcal{L}$, Δ_1, Δ_2, BV are written in Theorem 7.24 in terms of some basic building blocks $\Delta(W,Z)$, as explained in (7.13).

7.1. **Operators on the domain $\mathcal{D}$.** All our operators are sums of elements having the form $E_\alpha\overline{E_\alpha}E_{\gamma_r}$and $E_\alpha\overline{E_\alpha}$. In $\mathcal{D}\subset\mathfrak{p}_+$ let us introduce complex coordinates $\{z_\alpha\}_{\alpha\in Q^+}$ (we shall denote $z_\alpha = x_\alpha + iy_\alpha$) corresponding to the root space decomposition $z=\sum z_\alpha E_\alpha$. Then we prove:

Theorem 7.3. *Let f be a function on $\mathcal{D}$, then*

$$E_\alpha \overline{E_\alpha} \widetilde{f}(e) = \partial_{z_\alpha} \partial_{\overline{z}_\alpha} f(o), \tag{7.4}$$

$$E_\alpha \overline{E_\alpha} E_{\gamma_r} \widetilde{f}(e) = \partial_{z_\alpha} \partial_{\overline{z}_\alpha} \partial_{z_{\gamma_r}} f(o). \tag{7.5}$$

Notice first that for right K-invariant functions $\widetilde{f} \in C^\infty(G)$:

$$\begin{aligned} E_\alpha \overline{E_\alpha} \widetilde{f} =& (X_\alpha^2 + Y_\alpha^2) \widetilde{f}, \\ E_\alpha \overline{E_\alpha} E_{\gamma_r} \widetilde{f} =& E_{\gamma_r} E_\alpha \overline{E_\alpha} \widetilde{f} = (X_{\gamma_r} X_\alpha^2 + X_{\gamma_r} Y_\alpha^2 - i(Y_{\gamma_r} X_\alpha^2 + Y_{\gamma_r} Y_\alpha^2)) \widetilde{f}. \end{aligned} \tag{7.6}$$

To prove the above theorem we need to understand better the action of the group G on the domain $\mathcal{D}$. It is known ([K]) that for the case of $\mathcal{D} \cong SU(n,m)/S(U(n) \times U(m))$ and $\mathfrak{p}_+ = M_{n,m}$:

$$g \cdot 0 = BD^{-1} \tag{7.7}$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(n,m)$. Using this formula we shall be able to compute the action of some elements of G on $\mathcal{D} \cong G/K$ in the general case.

Lemma 7.8 ([H1],[KW])**.** *Let $\alpha \in Q^+$, then*

$$\exp tX_\alpha \cdot o = \tanh(t) E_\alpha, \quad \exp rX_\alpha \cdot tE_\alpha = \tfrac{t\cosh(r) + \sinh(r)}{\cosh(r) + t\sinh(r)} E_\alpha,$$
$$\exp tY_\alpha \cdot o = i\tanh(t) E_\alpha, \quad \exp rY_\alpha \cdot tE_\alpha = \tfrac{t\cosh(r) + i\sinh(r)}{\cosh(r) - it\sinh(r)} E_\alpha.$$

Proof. The algebra spanned by $E_\alpha, \overline{E_\alpha}, H_\alpha$ is isomorphic with $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(1,1)^\mathbb{C}$. By j^{-1} we shall denote the isomorphism:

$$j^{-1}(E_\alpha) = E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; j^{-1}(E_{-\alpha}) = \overline{E} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; j^{-1}(H_\alpha) = H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which extends to an isomorphism j of $\exp Lie(E_\alpha, \overline{E_\alpha}, H_\alpha)$ and $SL(2,\mathbb{C})$. Moreover, j commutes with taking the P^+ component in both $G^\mathbb{C}$ and $SL(2,\mathbb{C})$:

$$p_+ \circ j = j \circ \widetilde{p}_+.$$

Therefore, by (7.7):

$$\begin{aligned} \exp tX_\alpha \cdot o &= p_+(\exp tX_\alpha) = p_+ \circ j(\exp tX) = j \circ \widetilde{p}_+(\begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}) \\ &= j \circ \log \begin{pmatrix} 1 & \tanh(t) \\ 0 & 1 \end{pmatrix} = \tanh(t) E_\alpha, \\ \exp rX_\alpha \cdot tE_\alpha &= p_+(\exp rX_\alpha \exp tE) = j \circ \widetilde{p}_+(\begin{pmatrix} \cosh(r) & t\cosh(r) + \sinh(r) \\ \sinh(r) & \cosh(r) + t\sinh(r) \end{pmatrix}) \\ &= j \circ \log \begin{pmatrix} 1 & \frac{t\cosh(r)+\sinh(r)}{\cosh(r)+t\sinh(r)} \\ 0 & 1 \end{pmatrix} = \frac{t\cosh(r) + \sinh(r)}{\cosh(r) + t\sinh(r)} E_\alpha. \end{aligned}$$

In the same way we prove the two remaining formulas. $\square$

Using Lemma 7.8 we conclude:

$$
\begin{aligned}
X_\alpha^2\widetilde{f}(e) &= \partial_{x_\alpha}^2 f(o), & Y_\alpha^2\widetilde{f}(e) &= \partial_{y_\alpha}^2 f(o),\\
X_\alpha^3\widetilde{f}(e) &= \partial_{x_\alpha}^3 f(o) - 2\partial_{x_\alpha} f(o), & Y_\alpha^3\widetilde{f}(e) &= \partial_{y_\alpha}^3 f(o) - 2\partial_{y_\alpha} f(o),\\
X_\alpha Y_\alpha^2\widetilde{f}(e) &= \partial_{x_\alpha}\partial_{y_\alpha}^2 f(o) + 2\partial_{x_\alpha} f(o), & Y_\alpha X_\alpha^2\widetilde{f}(e) &= \partial_{y_\alpha}\partial_{x_\alpha}^2 f(o) + 2\partial_{y_\alpha} f(o),
\end{aligned}
$$

which by (7.6) proves (7.4) and (7.5) for $\alpha = \gamma_r$. To cope with the general version of equation (7.5) we need a stronger version of Lemma 7.8:

Lemma 7.9. *If $\alpha \in Q_r \cup \bigcup_i Q_{ir}$ then*

$$
\begin{aligned}
\exp sX_{\gamma_r}\cdot tE_\alpha &= \frac{t}{cosh(s)}E_\alpha + \tanh(s)E_{\gamma_r},\\
\exp sY_{\gamma_r}\cdot tE_\alpha &= \frac{t}{\cosh(s)}E_\alpha + \tanh(s)(iE_{\gamma_r}).
\end{aligned}
$$

Proof. Put $\beta = \alpha - \gamma_r$, then by the Restricted Roots Theorem β is a root and we may assume

$$
\begin{aligned}
E_\beta &= [E_\alpha, E_{-\gamma_r}],\\
E_{-\beta} &= [E_{\gamma_r}, E_{-\alpha}],\\
H_\beta &= [E_\beta, E_{-\beta}] = H_\alpha - H_{\gamma_r}.
\end{aligned}
$$

The vectors $E_\alpha, E_\beta, E_{\gamma_r}, E_{-\alpha}, E_{-\beta}, E_{-\gamma_r}, H_\alpha, H_\beta$ span a subalgebra of $\mathfrak{g}_{\mathbb{C}}$ isomorphic with $\mathfrak{sl}(3,\mathbb{C}) = \mathfrak{su}(2,1)^{\mathbb{C}}$:

$$
j^{-1}(E_\beta) = E_1 = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\ j^{-1}(E_\alpha) = E_2 = \begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},
$$

$$
j^{-1}(E_\gamma) = E_3 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}.
$$

Applying (7.7) we obtain

$$
\begin{aligned}
\exp sX_{\gamma_r}\cdot tE_\alpha &= p_+ \circ j(\exp sX_3 \exp tE_2) = j\circ \widetilde{p}_+ \begin{pmatrix} 1 & 0 & t\\ 0 & \cosh(s) & \sinh(s)\\ 0 & \sinh(s) & \cosh(s)\end{pmatrix}\\
&= j\circ \log \begin{pmatrix} 1 & 0 & \frac{t}{cosh(s)}\\ 0 & 1 & \tanh(s)\\ 0 & 0 & 1\end{pmatrix} = \frac{t}{cosh(s)}E_\alpha + \tanh(s)E_{\gamma_r},\\
\exp sY_{\gamma_r}\cdot tE_\alpha &= j\circ \widetilde{p}_+ \begin{pmatrix} 1 & 0 & t\\ 0 & \cosh(s) & i\sinh(s)\\ 0 & -i\sinh(s) & \cosh(s)\end{pmatrix}
\end{aligned}
$$

$$= \frac{t}{\cosh(s)}E_\alpha + \tanh(s)(iE_{\gamma_r}).$$

□

By Lemma 7.9:

$$X_{\gamma_r}X_\alpha^2\widetilde{f}(e) = \partial_{x_{\gamma_r}}\partial_{x_\alpha}^2 f(o), \quad X_{\gamma_r}Y_\alpha^2\widetilde{f}(e) = \partial_{x_{\gamma_r}}\partial_{y_\alpha}^2 f(o),$$
$$Y_{\gamma_r}X_\alpha^2\widetilde{f}(e) = \partial_{y_{\gamma_r}}\partial_{x_\alpha}^2 f(o), \quad Y_{\gamma_r}Y_\alpha^2\widetilde{f}(e) = \partial_{y_{\gamma_r}}\partial_{y_\alpha}^2 f(o),$$

which finishes the proof of Theorem 7.3. □

Having this result it is easy to write our operators on the domain at the point o:

$$\begin{aligned}
\mathcal{L}\widetilde{f}(e) &= \Big(\sum_{\alpha\in Q^+}\partial_{z_\alpha}\partial_{\overline{z}_\alpha}\Big)f(o),\\
\Delta_1\widetilde{f}(e) &= \Big(\sum_{\alpha\in\bigcup_i Q_i}\partial_{z_\alpha}\partial_{\overline{z}_\alpha} + c_0\sum_{\gamma\in\Gamma}\partial_{z_\gamma}\partial_{\overline{z}_\gamma}\Big)f(o),\\
\Delta_2\widetilde{f}(e) &= \Big(\sum_{\alpha\in Q_r\cup\bigcup_i Q_{ir}} c_\alpha\partial_{z_\alpha}\partial_{\overline{z}_\alpha} - c_{\gamma_r}\partial_{z_\gamma}\partial_{\overline{z}_\gamma}\Big)f(o),\\
BV\widetilde{f}(e) &= \Big(c_1\sum_{\alpha\in Q_r\cup\bigcup_i Q_{ir}}\partial_{z_\alpha}\partial_{\overline{z}_\alpha}\partial_{z_{\gamma_r}} + \partial_{z_{\gamma_r}}\partial_{\overline{z}_{\gamma_r}}\partial_{z_{\gamma_r}}\Big)f(o).
\end{aligned}\tag{7.10}$$

7.2. **The Cayley transform.** In order to write down the above operators on S, we compute the differential of c. We shall use the formula given by [S], Lemma II.5.3, which says that the Jacobian of the mapping $z \mapsto c\cdot z$ at the point o is given by

$$Jac(o \mapsto c\cdot o) = Ad\ c_K|_{\mathfrak{p}^+},$$

where c_K denotes the $K^{\mathbb{C}}$ component of c in the decomposition $P^+K^{\mathbb{C}}P^-$. By [KW], Lemma 3.5:

$$c_K = \exp\left(-\sum_{1\le i\le r}\log\cosh(\frac{\pi}{4}i)\cdot H_{\gamma_i}\right) = \prod_{1\le i\le r}\exp\left(-\log\frac{\sqrt{2}}{2}\cdot H_{\gamma_i}\right).$$

Hence the vectors E_α are eigenvectors of $Ad(c_K)$:

$$Adc_K(E_\alpha) = \begin{cases} 2E_\alpha, & \text{for } \alpha\in\Gamma,\\ 2E_\alpha, & \text{for } \alpha\in Q_{ij},\\ \sqrt{2}E_\alpha, & \text{for } \alpha\in Q_j.\end{cases}$$

Therefore, to have the operators on ${}^c\mathcal{D}$ it is enough to multiply all terms by appropriate constants. But it will be more convenient for us to write the result using the basis (3.16):

$$v = u + it = \sum v_jX_j + \sum_{\alpha\in\bigcup Q_i} v_\alpha\mathcal{X}_\alpha + \sum_{\alpha\in\bigcup\overline{Q}_{ij}} v_\alpha^kX_\alpha^k.$$

We assume below that G/K is of non-tube type (for tube spaces using similar methods we get the same values of ${}^c\mathcal{L}$ and ${}^c\Delta_1$). Then it is known that $\alpha \neq \widetilde{\alpha}$ for $\alpha \in \overline{Q}_{ij}$. By (3.16):

$$\begin{aligned} x_\alpha &= \frac{1}{\sqrt{2}}(u^1_\alpha - t^2_\alpha), \quad y_\alpha = \frac{1}{\sqrt{2}}(u^2_\alpha + t^1_\alpha), \\ x_{\widetilde{\alpha}} &= -\frac{\varepsilon_\alpha}{\sqrt{2}}(u^1_\alpha + t^2_\alpha), \quad y_{\widetilde{\alpha}} = \frac{\varepsilon_\alpha}{\sqrt{2}}(u^2_\alpha - t^1_\alpha), \end{aligned} \tag{7.11}$$

therefore

$$\begin{aligned} \partial_{z_\alpha}\partial_{\overline{z}_\alpha}\ {}^cf(c\cdot o) &= (\partial^2_{x_\alpha} + \partial^2_{y_\alpha})\ {}^cf(c\cdot o) \\ &= \Big(\frac{1}{2}(\partial_{u^1_\alpha} - \partial_{t^2_\alpha})^2 + \frac{1}{2}(\partial_{u^2_\alpha} + \partial_{t^1_\alpha})^2\Big)\ {}^cf(c\cdot o), \\ \partial_{z_{\widetilde{\alpha}}}\partial_{\overline{z}_{\widetilde{\alpha}}}\ {}^cf(c\cdot o) &= \Big(\frac{1}{2}(\partial_{u^1_\alpha} + \partial_{t^2_\alpha})^2 + \frac{1}{2}(\partial_{u^2_\alpha} - \partial_{t^1_\alpha})^2\Big)\ {}^cf(c\cdot o), \\ (\partial_{z_\alpha}\partial_{\overline{z}_\alpha} + \partial_{z_{\widetilde{\alpha}}}\partial_{\overline{z}_{\widetilde{\alpha}}})\ {}^cf(c\cdot o) &= (\partial_{v^1_\alpha}\partial_{\overline{v}^1_\alpha} + \partial_{v^2_\alpha}\partial_{\overline{v}^2_\alpha})\ {}^cf(c\cdot o). \end{aligned}$$

Finally for any Hermitian symmetric space G/K:

$$\begin{aligned} \mathcal{L}\widetilde{f}(e) &= 4\Big(\sum_{\alpha\in\bigcup Q_i} \partial_{v_\alpha}\partial_{\overline{v}_\alpha} + \sum_{\alpha\in\bigcup \overline{Q}_{ij}} \partial_{v^k_\alpha}\partial_{\overline{v}^k_\alpha} + \sum_j \partial_{v_j}\partial_{\overline{v}_j}\Big)\ {}^cf(c\cdot o), \\ \Delta_1\widetilde{f}(e) &= 4\Big(\sum_{\alpha\in\bigcup Q_i} \partial_{v_\alpha}\partial_{\overline{v}_\alpha} + c_0\sum_j \partial_{v_j}\partial_{\overline{v}_j}\Big)\ {}^cf(c\cdot o), \end{aligned} \tag{7.12}$$

and for G/K being of non-tube type:

$$\begin{aligned} BV\widetilde{f}(e) &= 8\Big(c_1\big(\sum_{\alpha\in Q_r} \partial_{v_\alpha}\partial_{\overline{v}_\alpha}\partial_{v_r} + \sum_{\alpha\in\bigcup_i \overline{Q}_{ir}} \partial_{v^k_\alpha}\partial_{\overline{v}^k_\alpha}\partial_{v_r}\big) + \partial_{v_r}\partial_{\overline{v}_r}\partial_{v_r}\Big)\ {}^cf(c\cdot o), \\ \Delta_2\widetilde{f}(e) &= 4\Big(\sum_{\alpha\in Q_r} c_\alpha\partial_{v_\alpha}\partial_{\overline{v}_\alpha} + \sum_{\alpha\in\bigcup \overline{Q}_{ir}} (c_\alpha\partial_{z_\alpha}\partial_{\overline{z}_\alpha} + c_{\widetilde{\alpha}}\partial_{z_{\widetilde{\alpha}}}\partial_{\overline{z}_{\widetilde{\alpha}}}) - c_{\gamma_r}\partial_{v_r}\partial_{\overline{v}_r}\Big)\ {}^cf(c\cdot o), \end{aligned}$$

where in the last operator it is more convenient not to change coordinates for $\alpha \in Q_{ir}$.

7.3. *S*-**invariant operators on Siegel domains.** Now we extend the above operators by S-invariance. Let W and Z be S-invariant holomorphic vector fields which agree with ∂_w and ∂_z at o. We have to find S-invariant operators on ${}^c\mathcal{D}$, $\Delta(W,Z)$ and $D(W,Z)$, such that:

$$\begin{aligned} \Delta(W,Z)\ {}^c\widetilde{f}(e) &= \partial_w\partial_{\overline{z}}\ {}^cf(c\cdot o), \\ D(W,Z)\ {}^c\widetilde{f}(e) &= \partial_w\partial_{\overline{w}}\partial_z\ {}^cf(c\cdot o) \end{aligned}$$

for any function ${}^cf$ on ${}^c\mathcal{D}$.

The operators $\Delta(W,Z)$ are explicitly computed in [DHMP]:

$$\Delta(W,Z) = W\overline{Z} - \nabla_W\overline{Z} = \overline{Z}W - \nabla_{\overline{Z}}W \tag{7.13}$$

where ∇ denotes the Riemannian connection on S. $\Delta(Z,Z)$ is *a real, second order, elliptic degenerate operator which annihilates holomorphic (consequently pluriharmonic) functions.* Moreover any left-invariant operator with the above properties is a linear combination of $\Delta(W,Z)$'s. To simplify our notation we shall denote $\Delta(Z,Z)$ by Δ_Z. In view of (3.18)

$$\begin{aligned}
\Delta_{Z_j}\ {}^c\widetilde{f}(e) &= \partial_{v_j}\partial_{\overline{v}_j}\ {}^cf(c\cdot o)\\
\Delta_{Z^k_\alpha}\ {}^c\widetilde{f}(e) &= \partial_{v^k_\alpha}\partial_{\overline{v}^k_\alpha}\ {}^cf(c\cdot o)\\
\Delta_{\mathcal{Z}_\alpha}\ {}^c\widetilde{f}(e) &= \partial_{v_\alpha}\partial_{\overline{v}_\alpha}\ {}^cf(c\cdot o),
\end{aligned}$$

and for $\alpha \in \overline{Q}_{ir}$

$$\begin{aligned}
\Delta_{W_\alpha}\ {}^c\widetilde{f}(e) &= \partial_{z_\alpha}\partial_{\overline{z}_\alpha}\ {}^cf(c\cdot o)\\
\Delta_{W_{\tilde{\alpha}}}\ {}^c\widetilde{f}(e) &= \partial_{z_{\tilde{\alpha}}}\partial_{\overline{z}_{\tilde{\alpha}}}\ {}^cf(c\cdot o),
\end{aligned}$$

where

$$\begin{aligned}
W_\alpha =& \frac{1}{\sqrt{2}}\Big[(X^1_\alpha - Y^2_\alpha) - i(X^2_\alpha + Y^1_\alpha)\Big],\\
W_{\tilde{\alpha}} =& -\frac{\varepsilon_\alpha}{\sqrt{2}}\Big[(X^1_\alpha + Y^2_\alpha) + i(X^2_\alpha - Y^1_\alpha)\Big].
\end{aligned} \tag{7.14}$$

It remains to calculate the operators $D(W, Z_r)$.

Theorem 7.15. *Let W be one of the holomorphic vector fields $\mathcal{Z}_\alpha, Z^k_\beta, Z_r$, for $\alpha \in Q_r, \beta \in \bigcup_i \overline{Q}_{ir}$ then*

$$D(W, Z_r) = W\Delta(W, Z_r) - ic_W\Delta_{Z_r}$$

with a positive constant c_W.

Proof. We begin the proof with some general observations. Given any coordinates $\{w_i\}$ in ${}^c\mathcal{D}$, take W_n to be the S-invariant holomorphic vector field such that:

$$W_n g(c\cdot o) = \partial_{w_n} g(c\cdot o), \tag{7.16}$$

then W_n can be written in the form

$$W_n g(z) = \sum_i h^n_i(z)\partial_{w_i} g(z)$$

and of course by (7.16)

$$h^n_i(c\cdot o) = \delta_{in}.$$

Let us write the building blocks $\Delta(W_p, W_q)$ in these coordinates.

$$\begin{aligned}
\Delta(W_p, W_q) &= \overline{W}_p W_q - \nabla_{\overline{W}_p} W_q \\
&= \big(\sum_i \overline{h_i^p(z)}\partial_{\overline{w}_i}\big)\big(\sum_j h_j^q(z)\partial_{w_j}\big) - \sum_i \overline{h_i^p(z)}\nabla_{\partial\overline{w}_i}\big(\sum_j h_j^q(z)\partial_{w_j}\big) \\
&= \sum_{i,j} \overline{h_i^p(z)} h_j^q(z)\partial_{\overline{w}_i}\partial_{w_j} + \sum_{i,j} \overline{h_i^p(z)}\partial_{\overline{w}_i} h_j^q(z)\partial_{w_j} \\
&\quad - \sum_{i,j} \overline{h_i^p(z)}\partial_{\overline{w}_i} h_j^q(z)\partial_{w_j} - \sum_{i,j} \overline{h_i^p(z)} h_j^q(z)\nabla_{\partial\overline{w}_i}\partial_{w_j} \\
&= \sum_{i,j} \overline{h_i^p(z)} h_j^q(z)\partial_{\overline{w}_i}\partial_{w_j}.
\end{aligned}$$

Since

$$\begin{aligned}
W_n\Delta(W_n, W_k)g(c\cdot o) &= W_n\big(\sum_{i,j} \overline{h_i^n(z)} h_j^k(z)\partial_{\overline{w}_i}\partial_{w_j}\big)g(c\cdot o) \\
&= \big(\sum_{i,j} \overline{h_i^n(z)} h_j^k(z)\partial_{w_n}\partial_{\overline{w}_i}\partial_{w_j} + \sum_{i,j} \partial_{w_n}(\overline{h_i^n} h_j^k)(z)\partial_{\overline{w}_i}\partial_{w_j}\big)g(c\cdot o) \\
&= \partial_{w_n}\partial_{\overline{w}_n}\partial_{w_k} g(c\cdot o) + \sum_{i,j} \partial_{w_n}\big(\overline{h_i^n} h_j^k\big)(c\cdot o)\partial_{\overline{w}_i}\partial_{w_j} g(c\cdot o),
\end{aligned}$$

we see that

$$D(W_n, W_k) = W_n\Delta(\overline{W}_n, W_k) - \sum_{i,j} \partial_{w_n}\big(\overline{h_i^n} h_j^k\big)(c\cdot o)\Delta(\overline{W}_i, W_j), \tag{7.17}$$

Thus we need to compute:

$$\partial_{w_n}(\overline{h_i^n} h_j^k)(c\cdot o) = (\partial_{w_n}\overline{h_i^n})(c\cdot o)h_j^k(c\cdot o) + \overline{h_i^n}(c\cdot o)(\partial_{w_n} h_j^k)(c\cdot o) \tag{7.18}$$

and to do it, we are going to use the action of the W_i's on ${}^c\mathcal{D}$.

Taking $g(z) = z_j$ we get

$$W_k g(z) = \sum_i h_i^k(z)\partial_{w_i} g(z) = h_j^k(z)$$

For $z = s\cdot(c\cdot o)$:

$$\begin{aligned}
h_j^k(z) &= W_k g(s) = \frac{1}{2}(X_k - iY_k)g(s) \\
&= \frac{1}{2}\Big(\partial_t g(s\exp tX_k)|_{t=0} - i\partial_t g(s\exp tY_k)|_{t=0}\Big)
\end{aligned}$$

(To simplify the notation we write $g(s)$ instead of $g(s \cdot o)$ identifying the function on S with that on ${}^c\mathcal{D}$.) Furthermore,

$$\begin{aligned}\partial_{w_n} h_j^k(c \cdot o) =& W_n h_j^k(c \cdot o) = \frac{1}{2}(X_n - iY_n) h_j^k(c \cdot o) \\ =& \frac{1}{2}\Big(\partial_s h_j^k(\exp sX_n)|_{s=0} - i\partial_s h_j^k(\exp sY_n)|_{s=0}\Big) \\ =& \frac{1}{4}\partial_s\partial_t \Bigg(\big(g(\exp sX_n \exp tX_k) - g(\exp sY_n \exp tY_k)\big) \\ & - i\big(g(\exp sX_n \exp tY_k) + g(\exp sY_n \exp tX_k)\big)\Bigg)\Big|_{\substack{t=0\\ s=0}}\end{aligned} \tag{7.19}$$

and

$$\begin{aligned}\partial_{w_n} \overline{h_i^n(c \cdot o)} =& \overline{\partial_{\overline{w}_n} h_i^n(c \cdot o)} \\ =& \frac{1}{4}\partial_s\partial_t \Bigg(\big(\overline{g}(\exp sX_n \exp tX_n) + \overline{g}(\exp sY_n \exp tY_n)\big) \\ & + i\big(\overline{g}(\exp sX_n \exp tY_n) - \overline{g}(\exp sY_n \exp tX_n)\big)\Bigg)\Big|_{\substack{t=0\\ s=0}}\end{aligned} \tag{7.20}$$

From now we shall use the standard notation (3.13) of Siegel domains. To prove the theorem we need to consider $W_n = X - iY \in \{\mathcal{Z}_\alpha, Z_\alpha^k, Z_r\}$ and $W_k = Z_r = X_r - iY_r$. Let us also recall the action of the group S on the domain ${}^c\mathcal{D}$:

$$((\zeta, x)s) \cdot (w, z) = (\zeta + \sigma(s)w, x + sz + 2i\Phi(\sigma(s)w, \zeta) + i\Phi(\zeta, \zeta)), \tag{7.21}$$

where σ is a representation of $N_0\widetilde{A}$:

$$\sigma : N_0\widetilde{A} \ni s \mapsto \sigma(s) \in GL(\mathcal{Z}).$$

Observe that (7.21) implies:

$$\begin{aligned}\partial_s\partial_t g(\exp sX \exp tX_r)|_{\substack{t=0\\ s=0}} &= 0, \\ \partial_s\partial_t g(\exp sX \exp tH_r)|_{\substack{t=0\\ s=0}} &= 0,\end{aligned}$$

which simplifies (7.19) and (7.20). Furthermore

$$\partial_s\partial_t g(\exp sY \exp tX_r)|_{\substack{t=0\\ s=0}} \quad \text{and} \quad \partial_s\partial_t g(\exp sY \exp tH_r)|_{\substack{t=0\\ s=0}} \tag{7.22}$$

are nonzero only for $Y = H_r$. Thus for $W_n \in \{\mathcal{Z}_\alpha, Z_\alpha^k\}$ the formula (7.18) reduces to:

$$\partial_{w_n}(\overline{h_i^n(z)} h_r^r(z))(c \cdot o) = (\partial_{w_n}\overline{h_i^n})(c \cdot o). \tag{7.18'}$$

while for $W_n = Z_r$ (8.12) becomes

$$\partial_{z_r}(\overline{h_i^r(z)} h_r^r(z))(c \cdot o) = (\partial_{z_r}\overline{h_i^r} + \partial_{z_r} h_i^r)(c \cdot o). \tag{7.18''}$$

Now we are going to calculate (7.18') and (7.18").
CASE I. $W_n = Z_r$. Then $\exp tX_r \in V_{rr}$, $\exp tH_r \in \mathfrak{a}_r$. Notice that in (7.19) the values $\partial_s\partial_t g$ are nonzero only if $g((\zeta, z)) = z_r$. Then

$$\begin{aligned}
\partial_s\partial_t g(\exp sH_r \exp tX_r \cdot (c \cdot o))|_{\substack{t=0\\s=0}} &= \partial_s\partial_t g(\exp sH_r \cdot \exp tX_r + i \exp sH_r)|_{\substack{t=0\\s=0}} = 1,\\
\partial_s\partial_t g(\exp sH_r \exp tH_r \cdot (c \cdot o))|_{\substack{t=0\\s=0}} &= \partial_s\partial_t g(i \exp(s+t)H_r)|_{\substack{t=0\\s=0}} = i,
\end{aligned}$$

and we get

$$\partial_{z_r} h_r^r(o) = -2i.$$

Similarly using (7.20) we obtain:

$$\partial_{z_r} \overline{h_r^r(o)} = -2i.$$

Hence by (7.18")

$$\partial_{z_r}(\overline{h_r^r}(z) h_r^r(z))(c \cdot o) = -4i.$$

CASE II. $W_n = Z_\alpha^k$ for $\alpha \in \overline{Q}_{jr}$. Let $x_t = \exp tX_\alpha^k$ then $x_t \in V_{jr}$. To describe the action of $y_t = \exp tY_\alpha^k = \exp(2x_t \Box c_j)$ we need a Jordan algebra lemma ([FK], Lemma VI.3.1):

Lemma 7.23. *Take $z \in V$ and decompose $z = z_1 + z_{\frac{1}{2}} + z_0$ into its Peirce decomposition with respect to c_j. Then for $a = \exp(2x_t \Box c_j)z$, we have:*

$$\begin{aligned}
a_1 &= z_1,\\
a_{\frac{1}{2}} &= 2L(x_t)z_1 + z_{\frac{1}{2}},\\
a_0 &= 2L(e - c_j)L(x_t)^2 z_1 + 2L(e - c_j)L(x_t)z_{\frac{1}{2}} + z_0. \qquad \Box
\end{aligned}$$

We shall also use a key fact: $V_{ij} \cdot V_{ij} \subset V_{ii} + V_{jj}$. We have:

$$y_s(x_t) = x_t + 2L(e - c_j)L(x_s)(x_t) = x_t + CstX_r,\ C \neq 0.$$

Therefore,

$$\partial_s\partial_t \overline{g}(\exp sY \exp tX)|_{\substack{t=0\\s=0}} = C$$

for $g(\zeta, z) = z_r$ and it is zero for other variables. We have

$$\begin{aligned}
y_t \cdot e &= e + 2L(x_t)c_j + 2L(e - c_j)L(x_t)^2 c_j\\
&= e + x_t + L(e - c_j)L(x_t)x_t
\end{aligned}$$

To calculate

$$\partial_s\partial_t \overline{g}(\exp sY \exp tY)|_{\substack{t=0\\s=0}}$$

we need to know only the term with st in $y_s \cdot (y_t \cdot e)$. Therefore it is enough to calculate

$$y_s \cdot (e + x_t) = e + x_s + L(e - c_j)L(x_s)x_s + x_t + 2L(e - c_j)L(x_s)x_t.$$

Only the last term matters, it is equal $iCstX_r$ and so

$$\partial_s\partial_t \overline{g}(\exp sY \exp tY)|_{\substack{t=0\\s=0}} = -iC.$$

Finally the only nonzero term in (7.18') is

$$\partial_{z^k_\alpha}\overline{h^{\alpha,k}_r(o)} = -2iC.$$

Case III. $W_n = \mathcal{Z}_\alpha$ for $\alpha \in Q_r$. Then $\exp tX = \exp t\mathcal{X}_\alpha = t\mathcal{X}_\alpha$, $\Phi(\mathcal{X}_\alpha, \mathcal{X}_\alpha) = CX_r$ and for $g((\zeta, z)) = z_r$

$$\partial_s\partial_t\overline{g}(\exp sX \exp tX)|_{\substack{t=0\\s=0}} = \partial_s\partial_t\overline{2sti\Phi(x,x)} = -2iC.$$

By analogy, for $\exp tY = \exp t\mathcal{Y}_\alpha = it\mathcal{X}_\alpha$:

$$\begin{aligned}
\partial_s\partial_t\overline{g}(\exp sY \exp tY)|_{\substack{t=0\\s=0}} &= -2iC,\\
\partial_s\partial_t\overline{g}(\exp sX \exp tY)|_{\substack{t=0\\s=0}} &= -2C,\\
\partial_s\partial_t\overline{g}(\exp sY \exp tX)|_{\substack{t=0\\s=0}} &= 2C.
\end{aligned}$$

Thus

$$\partial_{z_\alpha}\overline{h^\alpha_r(o)} = -8iC.$$

□

Now we are ready to transfer operators $\mathcal{L}, \Delta_1, \Delta_2, BV$ to S:

Theorem 7.24. *The images of operators $\mathcal{L}, \Delta_1, \Delta_2$ and BV are respectively:*

$$\begin{aligned}
{}^c\mathcal{L} &= \sum_{\alpha\in\bigcup Q_i} \Delta_{\mathcal{Z}_\alpha} + \sum_i \Delta_{Z_i} + \sum_{\alpha\in\bigcup \overline{Q}_{ij}} \Delta_{Z^k_\alpha},\\
{}^c\Delta_1 &= \sum_{\alpha\in\bigcup Q_i} \Delta_{\mathcal{Z}_\alpha} + c_0\sum_i \Delta_{Z_i},\\
{}^c\Delta_2 &= \sum_{\alpha\in Q_r} c_\alpha\Delta_{\mathcal{Z}_\alpha} + \sum_{\alpha\in\bigcup_i \overline{Q}_{ir}} (c_\alpha\Delta_{W_\alpha} + c_{\widetilde{\alpha}}\Delta_{W_{\widetilde{\alpha}}}) - c_r\Delta_{Z_r},\\
{}^cBV &= A\Big(\sum_{\alpha\in Q_r} \mathcal{Z}_\alpha\Delta(\mathcal{Z}_\alpha, Z_r) + \sum_{\alpha\in\bigcup_j \overline{Q}_{jr}} Z^k_\alpha\Delta(Z^k_\alpha, Z_r)\Big) + Z_r\Delta_{Z_r} - iB\Delta_{Z_r},
\end{aligned}$$

where W_α, $W_{\widetilde{\alpha}}$ are as in (7.14), c_0, c_α, c_r, B are strictly positive constants and $A > 1$.

One can simplify the operator ${}^cBV$. For that we use the following formula for the Bergmann connection

$$\nabla_{\overline{W}}Z = \pi_Q([\overline{W}, Z]),$$

(see e.g. [BBDHPT]) where W, Z are holomorphic vectors fields and π_Q denotes the projection onto the space of holomorphic vectors fields. Then for $Z \in \{\mathcal{Z}_\alpha, Z^k_\alpha\}$ we have

$$\nabla_{\overline{Z}}Z_r = \pi_Q[X + iY, X_r - iH_r] = \frac{1}{2}\pi_Q(iX - Y) = \frac{1}{2}\pi_Q(i(X - iY)) = 0$$

which by (7.13) implies

$$\Delta(\overline{Z}, Z_r) = \overline{Z} Z_r.$$

Furthermore applying

$$Z\overline{Z} = (X - iY)(X + iY) = X^2 + Y^2 + i[X, Y] = X^2 + Y^2 - iX_r,$$

we obtain

$$\begin{aligned} {}^cBV = &A\Big(\sum_{\alpha\in Q_r}\big((\mathcal{X}_\alpha)^2 + (\mathcal{Y}_\alpha)^2\big) + \sum_{\alpha\in\bigcup_i \overline{Q}_{ir}}\big((X_\alpha^k)^2 + (Y_\alpha^k)^2\big)\Big)Z_r \\ &- kAiX_rZ_r + Z_r\Delta_{Z_r} - iB\Delta_{Z_r} \end{aligned} \tag{7.25}$$

for $k = q_1 + (r-1)q_2$. (Recall that q_1 denotes the cardinality of the sets Q_i and q_2 the cardinality of the Q_{ij}'s.)

Similarly, we rewrite ${}^c\Delta_2$. If $\alpha \in Q_{jr}$ then

$$[Y_\alpha^{\varepsilon_1}, X_\alpha^{\varepsilon_2}] = X_r \tag{7.26}$$

when $\varepsilon_1 = \varepsilon_2$ and $[Y_\alpha^{\varepsilon_1}, X_\alpha^{\varepsilon_2}] = 0$ when $\varepsilon_1 \neq \varepsilon_2$. Now using (7.13) and (7.26) we calculate

$$\begin{aligned} \Delta_{Z_r} &= X_r^2 + H_r^2 - H_r \\ \Delta_{W_\alpha} &= \frac{1}{2}(X_\alpha^1 - Y_\alpha^2)^2 + \frac{1}{2}(X_\alpha^2 + Y_\alpha^1)^2 - H_r \\ \Delta_{W_{\widetilde{\alpha}}} &= \frac{1}{2}(X_\alpha^1 + Y_\alpha^2)^2 + \frac{1}{2}(X_\alpha^2 - Y_\alpha^1)^2 - H_r \\ \Delta_{\mathcal{Z}_\alpha} &= \mathcal{X}_\alpha^2 + \mathcal{Y}_\alpha^2 - H_r \end{aligned}$$

and so

$$\begin{aligned} {}^c\Delta_2 = 4\Big(&- c_{\gamma_r}(X_r^2 + H_r^2 - H_r) + \sum_{\alpha\in\bigcup_i\overline{Q}_{ir}} c_\alpha\big(\frac{1}{2}(X_\alpha^1 - Y_\alpha^2)^2 + \frac{1}{2}(X_\alpha^2 + Y_\alpha^1)^2 - H_r\big) \\ &+ \sum_{\alpha\in\bigcup_i\overline{Q}_{ir}} c_{\widetilde{\alpha}}\big(\frac{1}{2}(X_\alpha^1 + Y_\alpha^2)^2 + \frac{1}{2}(X_\alpha^2 - Y_\alpha^1)^2 - H_r\big) + \sum_{\alpha\in Q_r} c_\alpha(\mathcal{X}_\alpha^2 + \mathcal{Y}_\alpha^2 - H_r)\Big). \end{aligned} \tag{7.27}$$

8. Proof of the main theorem

We turn now to the proof of Theorem 5.3. Our proof will depend on whether or not G/K is of tube type. The first step, given by Proposition 8.1, applies to both cases.

8.1. **Poisson-Szegö integrals.** We will show that bounded $\mathcal{L} \oplus \mathcal{H}^c$-harmonic functions are Poisson-Szegö integrals. In fact, we have the following.

Proposition 8.1. *Let F be a bounded function on S annihilated by ${}^c\mathcal{L}$ and ${}^c\Delta_1$. Then F is the Poisson-Szegö integral of a function $f \in L^\infty(N(\Phi))$.*

Proof. The first part of this proof closely follows that for Proposition 3.3 in [BBDHPT]. Since ${}^c\mathcal{L}F = 0$, F can be written as the Poisson integral of a function $\widetilde{f}$ over $N = N(\Phi)N_0$. As in [BBDHPT] we can assume that $\widetilde{f}$ is continuous. We will prove below that $y \mapsto \widetilde{f}(y)$ is constant on N_0. It then follows that F is the Poisson-Szegö integral of $f = \widetilde{f}|_{N(\Phi)}$ over $N(\Phi)$, completing the proof.

Let

$$H = \sum_j H_j$$

and set

$$F_t(g) = F(exp(tH)g)$$

for $g \in S$. By Formula (37) in [BBDHPT], $F_H = \lim_{t \to -\infty} F_t$ exists and is given by the formula

$$F_H(wya) = \int_{N_0} \widetilde{f}(yaua^{-1}) \left(\int_{N(\Phi)} \widetilde{P}_{\mathcal{L}}(vu)dv \right) du \tag{8.2}$$

where $w, v \in N(\Phi)$, $y, u \in N_0$, $a \in \widetilde{A}$, and $\widetilde{P}_{\mathcal{L}}$ denotes the Poisson kernel for ${}^c\mathcal{L}$ on $N = N(\Phi)N_0$.

The operators ${}^c\mathcal{L}$ and ${}^c\Delta_1$ act from the right and annihilate F. So ${}^c\mathcal{L}F_t = 0 = {}^c\Delta_1 F_t$ and hence

$$\begin{aligned} {}^c\mathcal{L}F_H &= 0, && (8.3)\\ {}^c\Delta_1 F_H &= 0. && (8.4) \end{aligned}$$

Moreover, as in Lemma 3.4 in [BBDHPT], one can use (8.2) to show that $t \mapsto F_H(wyaexp(tH))$ is constant and hence

$$HF_H = 0. \tag{8.5}$$

Equation (8.2) shows that $F_H(wya) = F_H(ya)$, independent of $w \in N(\Phi)$. Substituting the expressions

$$\Delta_{Z_i} = X_i^2 + H_i^2 - H_i, \quad \Delta_{\mathcal{Z}_\alpha} = \mathcal{X}_\alpha^2 + \mathcal{Y}_\alpha^2 - H_i \ (\alpha \in Q_i)$$

into the formula for ${}^c\Delta_1$ in Proposition 7.24 and noting that X_i, $\mathcal{X}_\alpha$, $\mathcal{Y}_\alpha$ belong to the Lie algebra $\mathcal{Z} \oplus V$ for $N(\Phi)$, we see that

$${}^c\Delta_1 F_H(ya) = \left(c_0 \sum H_i^2 - (c_0 + q_1) \sum H_i \right) F_H(ya)$$

where q_1 is the common cardinality of the sets Q_i. Now using (8.4) and (8.5), we conclude that $F_H(ya)$ is annihilated by the Laplace-Beltrami operator $\sum H_i^2$ for the abelian group $\widetilde{A}$. As F_H is bounded, $F_H(ya)$ can not depend on $a \in \widetilde{A}$. We have shown $F_H(wya) = F_H(y)$ and (8.3) now yields

$$0 = (\sum H_i^2 + \sum (Y_\alpha^k)^2 - \sum a_i H_i) F_H(y) = (\sum (Y_\alpha^k)^2) F_H(y).$$

So $F_H(y)$ is annihilated by a left-invariant second order elliptic differential operator on the nilpotent group N_0. Thus $F_H(y)$ is a constant function. Now from (8.2) we conclude that $\widetilde{f}(y)$ is also constant on N_0. □

8.2. **Proof of Theorem 5.3 for tube domains.** Let G/K be of tube type and F be a real-valued bounded $\mathcal{L} \oplus \mathcal{H}^c$-harmonic function on G/K. Proposition 8.1 shows that F is a Poisson-Szegö integral. The Johnson-Koranyi Theorem 5.1 implies that F is $\mathcal{H}$-harmonic. So F is in fact annihilated by all of $\mathcal{H}' \oplus \mathcal{L} \oplus \mathcal{H}^c = \mathfrak{p}_+ \otimes \mathfrak{p}_-$. This means F is pluriharmonic.

8.3. **Proof of Theorem 5.3 in the non-tube case.** The Hua system does not annihilate the Poisson-Szegö kernel on a non-tube domain, so Poisson-Szegö integrals need not be Hua harmonic. Thus we require a different proof for Theorem 5.3 in the non-tube case.

For the rest of this section we assume G/K to be of non-tube type. As above, F denotes a real-valued bounded $\mathcal{L} \oplus \mathcal{H}^c$-harmonic function on G/K and ${}^cF$, ${}^c\widetilde{F}$ are as in (7.1) and (7.2) . We note the following consequence of Theorem 6.20 and Proposition 8.1.

Corollary 8.6. ${}^cBV({}^c\widetilde{F}) = 0$.

Our proof will make use of the operators ${}^cBV$, ${}^c\Delta_2$ and a characterization of bounded pluriharmonic functions on Siegel domains from [BDH]. It is shown in [BDH] that F is pluriharmonic if and only if ${}^c\widetilde{F}$ is annihilated by three operators. Each of these is a linear combination of building-blocks $\Delta_{\mathcal{Z}_\alpha}$, Δ_{Z_i} and $\Delta_{Z_\alpha^k}$. Thus we have:

Theorem 8.7 ([BDH])**.** *If a real bounded function ${}^c\widetilde{F}$ is annihilated by the operators*

$$\Delta_{\mathcal{Z}_\alpha}\ (\alpha \in \cup_i Q_i), \quad \Delta_{Z_i}\ (1 \le i \le r) \quad \textit{and} \quad \Delta_{Z_\alpha^k}\ (\alpha \in \cup \overline{Q}_{ij})$$

then it is pluriharmonic.

Consider the decomposition

$$\mathfrak{s} = \mathfrak{s}^{r-1} \oplus \mathfrak{s}_r, \tag{8.8}$$

where

$$\begin{aligned}\mathfrak{s}^{r-1} &= \left(\bigoplus_{j<r} \mathcal{Z}_j\right) \oplus \left(\bigoplus_{1\le i\le j<r} V_{ij}\right) \oplus \left(\bigoplus_{1\le i<j<r} \mathfrak{n}_{ij}\right) \oplus \left(\bigoplus_{1\le j<r} \mathbb{R}H_j\right),\\ \mathfrak{s}_r &= \mathcal{Z}_r \oplus \left(\bigoplus_{1\le j<r} V_{jr}\right) \oplus \left(\bigoplus_{1\le j<r} \mathfrak{n}_{jr}\right) \oplus V_{rr} \oplus \mathbb{R}H_r.\end{aligned} \tag{8.9}$$

$\mathfrak{s}^{r-1}$ is a subalgebra of $\mathfrak{s}$ and $\mathfrak{s}_r$ is an ideal. Therefore we can decompose the group as a semidirect product of a subgroup S^{r-1} and normal subgroup S_r,

$$S = S^{r-1}S_r. \tag{8.10}$$

Let

$$\mathfrak{h}_r = \mathcal{Z}_r \oplus \left(\bigoplus_{j<r} V_{jr}\right) \oplus \left(\bigoplus_{j<r} \mathfrak{n}_{jr}\right) \oplus V_{rr}, \quad \widetilde{\mathfrak{a}}_r = \mathbb{R}H_r, \tag{8.11}$$

so that $\mathfrak{s}_r = \mathfrak{h}_r \oplus \widetilde{\mathfrak{a}}_r$. Using the bracket relations, one sees that $\mathfrak{h}_r$ is a Heisenberg Lie algebra of dimension $2k+1$, where, as in formula (7.25), $k = q_1 + (r-1)q_2$. Here

$$\begin{aligned} \mathcal{Z}_r &\quad \text{has basis} \quad \{\mathcal{X}_a, \mathcal{Y}_a \ : \ \alpha \in Q_r\},\\ \oplus_{j<r}V_{jr} &\quad \text{has basis} \quad \{X^1_\alpha, X^2_\alpha \ : \ \alpha \in \cup_{j<r}Q_{jr}\},\\ \oplus_{j<r}\mathfrak{n}_{jr} &\quad \text{has basis} \quad \{Y^1_\alpha, Y^2_\alpha \ : \ \alpha \in \cup_{j<r}Q_{jr}\}\end{aligned}$$

and $V_{rr} = \mathbb{R}X_r$ is the center of $\mathfrak{h}_r$. The basis

$$\mathcal{X}_\alpha, \mathcal{Y}_\alpha, X^k_\alpha, Y^k_\alpha, X_r, H_r, \quad \alpha \in Q_r \cup \bigcup Q_{jr}$$

for $\mathfrak{s}_r$ is orthonormal with respect to the Riemannian form g_r on S_r. The complex structure $\mathcal{J}$ on $\mathfrak{s}$ restricts to yield a complex structure on $\mathfrak{s}_r$:

$$\begin{aligned}\mathcal{J}(X_r) = H_r, &\quad \mathcal{J}(H_r) = -X_r,\\ \mathcal{J}(X^k_\alpha) = Y^k_\alpha, &\quad \mathcal{J}(Y^k_\alpha) = -X^k_\alpha,\\ \mathcal{J}(\mathcal{X}_\alpha) = \mathcal{Y}_\alpha, &\quad \mathcal{J}(\mathcal{Y}_\alpha) = -\mathcal{X}_\alpha.\end{aligned}$$

The operators $\widetilde{\Delta}_{Z_r}, \widetilde{\Delta}_{\mathcal{Z}_\alpha}, \widetilde{\Delta}_{Z^k_\alpha}$ defined on S_r as in (7.13), act from the right and make perfect sense on both S_r and S. Thus for any smooth function g on S we may write:

$$\begin{aligned}\widetilde{\Delta}_{Z_r}(g_{s'})(s_r) &= (\Delta_{Z_r}g)(s's_r),\\ \widetilde{\Delta}_{\mathcal{Z}_\alpha}(g_{s'})(s_r) &= (\Delta_{\mathcal{Z}_\alpha}g)(s's_r),\\ \widetilde{\Delta}_{Z^k_\alpha}(g_{s'})(s_r) &= (\Delta_{Z^k_\alpha}g)(s's_r).\end{aligned} \tag{8.12}$$

where $s' \in S^{r-1}$, $s_r \in S_r$ and $g_{s'}(s_r) = g(s's_r)$. Likewise, Equations (7.25) and (7.27) show that ${}^cBV$ and ${}^c\Delta_2$ can be applied to functions on S_r. As ${}^cBV$ and ${}^c\Delta_2$ annihilate ${}^c\widetilde{F}$ we have

$${}^cBV^c(\widetilde{F}_{s'}) = {}^c\Delta_2{}^c(\widetilde{F}_{s'}) = 0. \tag{8.13}$$

Theorem 8.18, proved below, shows that (8.13) implies ${}^c\widetilde{F}_{s'}$ is a pluriharmonic function on S_r. Hence

$$\widetilde{\Delta}_{Z_r}({}^c\widetilde{F}_{s'}) = \widetilde{\Delta}_{\mathcal{Z}_\alpha}({}^c\widetilde{F}_{s'}) = \widetilde{\Delta}_{Z^k_\beta}({}^c\widetilde{F}_{s'}) = 0 \text{ for all } \alpha \in Q_r,\ \beta \in \bigcup_i \overline{Q}_{ir}$$

and by (8.12)

$$\Delta_{Z_r}({}^c\widetilde{F}) = \Delta_{\mathcal{Z}_\alpha}({}^c\widetilde{F}) = \Delta_{Z^k_\beta}({}^c\widetilde{F}) = 0 \text{ for all } \alpha \in Q_r,\ \beta \in \bigcup_i \overline{Q}_{ir}. \tag{8.14}$$

Thus the function ${}^cF$ on ${}^c\mathcal{D}$ satisfies

$$\partial_{v_\alpha}\partial_{\overline{v}_\alpha}{}^cF(c\cdot o) = \partial_{v^k_\beta}\partial_{\overline{v}^k_\beta}{}^cF(c\cdot o) = 0, \text{ for } \alpha \in \{\gamma_r\}\cup Q_r, \beta \in \bigcup_i \overline{Q}_{ir}, \tag{8.15}$$

where, as before, v_α, v^k_β denote coordinates on ${}^c\mathcal{D}$ with respect to basis (3.18). Notice that our considerations do not depend on the order of the roots in Γ, hence the role of all γ_j's is equivalent as far as the conclusion is made on the domain ${}^c\mathcal{D}$ not on the group S. In particular, we may interchange γ_r with γ_j and so obtain (8.15) for $\alpha \in \{\gamma_j\}\cup Q_j, \beta \in \bigcup_i \overline{Q}_{ij}$. But now we may apply S-invariance (with respect to the group S defined by the original ordering) and conclude that all building blocks Δ_{Z_j}, $\Delta_{\mathcal{Z}_\alpha}$, $\Delta_{Z^k_\alpha}$ annihilate ${}^c\widetilde{F}$. This shows that F is pluriharmonic, in view of Theorem 8.7.

8.4. **Rank one analysis.** The group S_r in (8.10) can be identified with the semidirect product

$$S_0 = \mathbb{H}^k A_0$$

of the Heisenberg group $\mathbb{H}^k = \mathbb{C}^k \times \mathbb{R}$ with $A_0 = \mathbb{R}^+$ acting via dilations. The solvable group S_0 arises in connection with the rank one Hermitian symmetric space $G/K = SU(1,k)/S(U(1)\times U(k))$, whose bounded realization $\mathcal{D}_0$ is biholomorphically equivalent to the unit ball in $\mathbb{C}^{k+1}$. S_0 is identified with the classical Siegel domain

$${}^c\mathcal{D}_0 = \{(z, z_{k+1}) \in \mathbb{C}^k \times \mathbb{C} :\ \Im z_{k+1} > \frac{1}{4}|z|^2\}.$$

We write points $s \in S_0$ as

$$s = \big((\zeta, u), a\big) = \big(((\zeta_1\ \ldots, \zeta_k), u), a\big), \quad (\zeta_j = x_j + iy_j \in \mathbb{C}, u \in \mathbb{R}, a \in \mathbb{R}^+)$$

and denote by $\mathcal{X}_j$, $\mathcal{Y}_j$, T the left-invariant fields on $\mathbb{H}^k$, which at $\mathbf{e}$ agree respectively with ∂_{x_j}, ∂_{y_j}, ∂_u. Then the operators $\widetilde{\mathcal{X}}_j$, $\widetilde{\mathcal{Y}}_j$, $\widetilde{T}$, H given by

$$\begin{aligned} \widetilde{\mathcal{X}}_j &= \sqrt{a}\mathcal{X}_j, \\ \widetilde{\mathcal{Y}}_j &= \sqrt{a}\mathcal{Y}_j, \\ \widetilde{T} &= aT, \\ H &= a\partial_a, \end{aligned} \tag{8.16}$$

are left-invariant on the group S_0, and form a basis of the Lie algebra $\mathfrak{s}_0$ of S_0. The elements X_r and H_r of $\mathfrak{s}_r$ correspond to $\widetilde{T}$ and H respectively. X^k_α's and $\mathcal{X}_\alpha$'s in $\mathfrak{s}_r$ correspond to the $\widetilde{\mathcal{X}}_j$'s in $\mathfrak{s}_0$ and the Y^k_α's, $\mathcal{Y}_\alpha$'s correspond to the $\mathcal{Y}_j$'s. Let $\ell = (r-1)q_2$ and order the bases for $\mathfrak{s}_r$ and $\mathfrak{s}_0$ so that the X^k_α's and Y^k_α's ($\alpha \in \bigcup_j \overline{Q}_{jr}$) correspond with $\widetilde{\mathcal{X}}_1, \dots, \widetilde{\mathcal{X}}_\ell$ and $\widetilde{\mathcal{Y}}_1, \dots, \widetilde{\mathcal{Y}}_\ell$ respectively. The elements $\widetilde{\mathcal{X}}_\alpha$, $\widetilde{\mathcal{Y}}_a$ ($\alpha \in Q_r$) correspond to $\widetilde{\mathcal{X}}_j$, $\widetilde{\mathcal{Y}}_j$ with $\ell < j \leq k$.

Recall that the operators ${}^cBV$ and ${}^c\Delta_2$ on S are now regarded as living on the subgroup S_r. Identifying S_r with S_0 as described above and working with equations (7.25) and (7.27) we obtain the following expressions in the notation of (8.16).

$$
\begin{aligned}
{}^cBV =& AaL_BZ - ikA\widetilde{T}Z + Z\Delta_Z - iB\Delta_Z, \\
{}^c\Delta_2 =& a \sum_{0\leq j<l} \Big(b_{2j+1}((\mathcal{X}_{2j+1} - \mathcal{Y}_{2j+2})^2 + (\mathcal{X}_{2j+2} + \mathcal{Y}_{2j+1})^2) \\
&+ \ b_{2j+2}((\mathcal{X}_{2j+1} + \mathcal{Y}_{2j+2})^2 + (\mathcal{X}_{2j+2} - \mathcal{Y}_{2j+1})^2) \Big) \\
&+ \ a \sum_{2l<j\leq k} b_j((\mathcal{X}_j)^2 + (\mathcal{Y}_j)^2) - CH - c_r(\widetilde{T}^2 + H^2),
\end{aligned} \tag{8.17}
$$

where

$$
Z = \widetilde{T} - iH, \quad \Delta_Z = \widetilde{T} + H^2 - H, \quad L_B = \sum_{1\leq j\leq n} \left((\mathcal{X}_j)^2 + (\mathcal{Y}_j)^2 \right).
$$

To complete the proof of Theorem 5.3 in the non-tube case, it remains to establish the following result.

Theorem 8.18. *Let F be a real bounded function on S_0 annihilated by two operators: ${}^cBV$ and ${}^c\Delta_2$. Then F is pluriharmonic.*

Observe that F is a real function while ${}^cBV$ is a complex operator. Therefore F is annihilated by two real operators:

$$
\begin{aligned}
U =& AaL_B\widetilde{T} - kA\widetilde{T}H + \widetilde{T}(\widetilde{T}^2 + H^2 - H), \\
V =& AaL_BH + kA\widetilde{T}^2 + H(\widetilde{T}^2 + H^2 - H) + B(\widetilde{T}^2 + H^2 - H),
\end{aligned} \tag{8.19}
$$

$A > 1, B > 0$.

Let ϕ be a Schwartz function on $\mathbb{R}$ whose Fourier transform satisfies

$$
\hat{\phi}(\lambda) = \begin{cases} 1 & \text{for } |\lambda| \leq 1, \\ 0 & \text{for } |\lambda| \geq 2. \end{cases}
$$

By [BDH] (Lemma 4.4) there exists a sequence $\{k_n\}_{n=1,2,\dots}$ of natural numbers tending to infinity such that for

$$
\phi_n(x) = \frac{1}{k_n}\phi\left(\frac{x}{k_n}\right),
$$

convolving in the central direction in $\mathbb{H}^k$ the limit

$$\lim_{n\to\infty} \phi_n *_{\mathbb{R}} F((\zeta, x)a)$$

exists for every $\zeta \in \mathbb{C}^k$, $a \in A_0$ and does not depend on the central variable x. Denote this limit by $G(\zeta, a)$, then G is annihilated by the same operators as F: $U, V, {}^c\Delta_2$.

Lemma 8.20. *The function $G(\zeta, a)$ is constant.*

Proof. Let $\widetilde{G}(\zeta, a) = HG(\zeta, a) = (a\partial_a)G(\zeta, a)$, then by Harnack's inequality the function $\widetilde{G}$ is bounded and by (8.19)

$$VG = (AaL_B + H^2 + (B-1)H - B)\widetilde{G}(\zeta, a) = 0.$$

This equation implies that $\widetilde{G} = 0$. Indeed, let μ_t be the semigroup of measures with infinitesimal generator $AaL_B + H^2 + (B-1)H$. Then $AaL_B + H^2 + (B-1)H - B$ generates the semigroup $e^{-Bt}\mu_t$ and so $\widetilde{G} * e^{-Bt}\mu_t = \widetilde{G}$. Now letting $t \to \infty$ we see that $\widetilde{G}$ vanishes, G does not depend on a and so it is a function on $\mathbb{R}^{2k} = \mathbb{C}^k$ annihilated by the elliptic operator (compare (8.17)):

$$\Bigg(\sum_{0\le j<l} \Big(b_{2j+1}((\partial_{x_{2j+1}} - \partial_{y_{2j+2}})^2 + (\partial_{x_{2j+2}} + \partial_{y_{2j+1}})^2)$$

$$+ b_{2j+2}((\partial_{x_{2j+1}} + \partial_{y_{2j+2}})^2 + (\partial_{x_{2j+2}} - \partial_{y_{2j+1}})^2)\Big) + \sum_{2l<j\le k} b_j(\partial^2_{x_j} + \partial^2_{y_j})\Bigg) G(x, y) = 0,$$

Taking the Fourier transform of both sides (in the distribution sense) we get

$$\Bigg(\sum_{0\le j<l} \Big(b_{2j+1}((\eta_{2j+1} - \xi_{2j+2})^2 + (\eta_{2j+2} + \xi_{2j+1})^2)$$

$$+ b_{2j+2}((\eta_{2j+1} + \xi_{2j+2})^2 + (\eta_{2j+2} - \xi_{2j+1})^2)\Big) + \sum_{2l<j\le k} b_j(\eta_j^2 + \xi_j^2)\Bigg) \widehat{G}(\eta, \xi) = 0,$$

which implies

$$supp\widehat{G} \subset \{0\}.$$

Boundedness of G forces it to be constant. □

We remark that the only use of the operator Δ_2 appears above in the proof of Lemma 8.20. In summary, we have proved the following.

Theorem 8.21. *Assume that F satisfies the assumptions of Theorem 8.18. Denote by $F(\zeta, \hat{\lambda}, a)$ the distributional partial Fourier transform of F along the center of $\mathbb{H}^k$.*

Let

$$\begin{aligned}\eta_n(x) &= k_n\phi(k_nx) - k_n\phi(k_n^{-1}x),\\ F_n(\zeta,x,a) &= \eta_n *_{\mathbb{R}} F(\zeta,x,a).\end{aligned}$$

Then

$$suppF_n(\cdot,\hat{\cdot},\cdot) \subset \mathbb{C}^n \times \{\lambda\in\mathbb{R}:\ k_n^{-1}\le|\lambda|\le 2k_n\}\times\mathbb{R}^+,$$

F_n is annihilated by U and V and there is a constant c such that the sequence F_n tends to $F+c$.

As a corollary, we see that to justify Theorem 8.18 it is enough to justify it for functions satisfying

$$\mathrm{supp}F(\cdot,\hat{\cdot},\cdot) \subset \mathbb{C}^n \times \{\lambda\in\mathbb{R}:\ \varepsilon\le|\lambda|\le \varepsilon^{-1}\}\times\mathbb{R}^+, \tag{8.22}$$

Writing the operator U on the partial Fourier transform side and using the above assumptions one can easily prove that such F is annihilated by

$$U_1 = AaL_B - kAH + \widetilde{T}^2 + H^2 - H, \tag{8.23}$$

which is a second order elliptic operator. Therefore by [DH], [R] the function F may be written as

$$F((\zeta,x)a) = f *_{\mathbb{H}^k} P_a(\zeta,x)$$

for the Poisson kernel P_a determined by U and $f\in L^\infty(\mathbb{H}^k)$. Furthermore

$$\lim_{a\to0} F((\zeta,x)a) = f(\zeta,x)$$

in the *weak sense.

Proposition 8.24. *The boundary value f of F satisfies the following differential equation:*

$$(L_B^2 + k^2\partial_x^2)f(\zeta,x) = 0$$

Proof. In the proof we shall use the following simple lemma

Lemma 8.25. *Let $h\in C^1(0,\infty)$. Assume that*

$$ah'(a) - \gamma h(a) = v(a)$$

for some $\gamma>0$ and

$$\lim_{a\to0} v(a) = v_0.$$

Then

$$\lim_{a\to0} h(a) = -\frac{v_0}{\gamma}.$$

Let $F(a)(\zeta,x) = F((\zeta,x)a)$ and let ϕ be a test function. The lemma is applied to $h(a) = \langle \partial_a^p F(a), \phi\rangle$, for $p = 1,2,3$. Observe that by (8.23)

$$a\partial_a^2 F(a) - kA\partial_a F(a) = -(AL_B + a\partial_x^2)F(a). \tag{8.26}$$

Applying Lemma 8.25 we get

$$\lim_{a\to 0} \partial_a F(a) = \frac{1}{k} L_B f.$$

Differentiating (8.26) along a we obtain:

$$a\partial_a^3 F(a) - (kA-1)\partial_a^2 F(a) = -(AL_B\partial_a + \partial_x^2 + a\partial_x^2\partial_a)F(a) \tag{8.27}$$

and using again Lemma 8.25 we prove

$$\lim_{a\to 0} \partial_a^2 F(a) = \frac{1}{kA-1}\Big(\frac{A}{k}L_B^2 + \partial_x^2\Big) f.$$

Repeating this procedure one can also show that:

$$\lim_{a\to 0} \partial_a^3 F(a) \text{ exists.}$$

Applying the operator V we have:

$$\begin{aligned}
0 &= \lim_{a\to 0}\Big(AL_B\partial_a + kA\partial_x^2 + a\partial_a(\partial_x^2+\partial_a^2) + B'(\partial_x^2+\partial_a^2)\Big)F(a) \qquad (B' = B+2)\\
&= \Big(\frac{A}{k}L_B^2 + kA\partial_x^2 + B'\partial_x^2 + \frac{B'}{kA-1}\big(\frac{A}{k}L_B^2 + \partial_x^2\big)\Big) f\\
&= \frac{1}{k(kA-1)}\Big((A(kA-1) + B'A)L_B^2\\
&\quad + (k^2A(kA-1) + B'k(kA-1) + B'k)\partial_x^2\Big) f\\
&= \frac{1}{k(kA-1)}\Big(A(kA-1+B')L_B^2 + k^2A(kA-1+B')\partial_x^2\Big) f\\
&= C(L_B^2 + k^2\partial_x^2) f.
\end{aligned}$$

□

Lemma 8.28. *There is an ε' such that for every $\zeta \in \mathbb{C}^n$*

$$supp\widehat{f_\zeta}(\lambda) \subset \{\varepsilon' < |\lambda| < \varepsilon'^{-1}\}.$$

Proof. Let ε be as in 8.22. Taking $\phi \in \mathcal{S}(\mathbb{R})$ such that:

$$\hat{\phi}(\lambda) = \begin{cases} 1 & \text{for } \varepsilon < |\lambda| < \varepsilon^{-1}, \\ 0 & \text{for } |\lambda| \in (0, \frac{\varepsilon}{2}) \cup (\frac{2}{\varepsilon}, \infty), \end{cases}$$

with ε as in (8.22), we obtain $F = \phi *_{\mathbb{R}} F$. Thus

$$(f - \phi *_{\mathbb{R}} f) *_{\mathbb{H}^k} P_s = F(s) - \phi *_{\mathbb{R}} F(s) = 0,$$

but the mapping $L^\infty(\mathbb{H}^k) \ni g \mapsto g *_{\mathbb{H}^k} P_s \in L^\infty(S_0)$ is injective ([DH]), therefore $f = \phi *_{\mathbb{R}} f$, establishing the lemma with $\varepsilon' = \varepsilon/2$. □

Now we are in the situation described in [B] (see also [BBDHJ]), where the following was proved:

Theorem 8.29 ([B]). *Let $f \in L^\infty(\mathbb{H}^k)$ be a boundary value of a real bounded function F on S_0 i.e.:*

$$F((\zeta, x)a) = f *_{\mathbb{H}^k} P_a(\zeta, x).$$

Assume:

- $(L_B^2 + k^2\partial_x^2)f(\zeta, x) = 0,$
- *supp*$\widehat{f_\zeta}(\lambda) \subset \{\varepsilon' < |\lambda| < \varepsilon'^{-1}\}$ *for every* $\zeta \in \mathbb{C}^k$.

Then F is pluriharmonic.

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