COMBINATORICS AND SPHERICAL FUNCTIONS ON THE HEISENBERG GROUP

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ABSTRACT. Let V be a finite dimensional Hermitian vector space and K be a compact Lie subgroup of $U(V)$ for which the representation of K on $C[V]$ is multiplicity free. One obtains a canonical basis $\{p_\alpha\}$ for the space $C[V_R]^K$ of K-invariant polynomials on V_R and also a basis $\{q_\alpha\}$ via orthogonalization of the p_α's. The polynomial p_α yields the homogeneous component of highest degree in q_α. The coefficients that express the q_α's in terms of the p_β's are the *generalized binomial coefficients* of Z. Yan. We present some new combinatorial identities that involve these coefficients. These have applications to analysis on Heisenberg groups. Indeed, the polynomials q_α completely determine the generic bounded spherical functions for a Gelfand pair obtained from the action of K on a Heisenberg group $H = V \times R$.

CONTENTS

1. INTRODUCTION

Throughout this paper, V will denote a complex vector space of dimension n with a Hermitian inner product $\langle \cdot, \cdot \rangle$ and K will denote a compact Lie subgroup of the group $U(V)$ of unitary operators on V. The group $U(V)$ acts on the ring $\mathbb{C}[V]$ of holomorphic polynomials on V via

$$(k \cdot p)(z) = p(k^{-1}z)$$

for $k \in U(V)$, $p \in \mathbb{C}[V]$, $z \in V$. One decomposes $\mathbb{C}[V]$ as an algebraic direct sum

$$\mathbb{C}[V] = \sum_{\alpha \in \Lambda} P_\alpha \tag{1.1}$$

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of (finite-dimensional) K-irreducible subspaces. Here Λ is a countably infinite index set that parameterizes the decomposition. The action of K on V is said to *multiplicity free* when the representations of K on P_α and P_β are inequivalent for $\alpha \neq \beta$. In this case, the above decomposition of $\mathbb{C}[V]$ is canonical. We will assume throughout this paper that the action of K on V is multiplicity free. There is a rich theory for such actions (see for example [How95]) and a complete classification (see [Kac80], [BR96], [Lea]).

The representation of $U(V)$ on $\mathbb{C}[V]$ preserves the spaces $\mathcal{P}_m(V)$ of homogeneous polynomials of degree m. Hence each P_α is a space of homogeneous polynomials. We write $|\alpha|$ for the degree of homogeneity of the polynomials in P_α, so that $P_\alpha \subset \mathcal{P}_{|\alpha|}(V)$. We will also let $d_\alpha = \dim(P_\alpha)$ and write $0 \in \Lambda$ for the index with $P_0 = \mathcal{P}_0(V) = \mathbb{C}$.

There are no non-constant K-invariants in $\mathbb{C}[V]$ since the action of K on V is multiplicity free. The algebra $\mathbb{C}[V_\mathbb{R}]^K$ of K-invariant polynomials on the underlying real space $V_\mathbb{R}$ of V is, however, of interest. One always has, for example, that

$$\gamma(z) = |z|^2/2$$

(where $|z|^2 = \langle z, z\rangle$) belongs to $\mathbb{C}[V_\mathbb{R}]^K$. One can describe a canonical basis for $\mathbb{C}[V_\mathbb{R}]^K$ as follows. We equip $\mathbb{C}[V]$ (and also $\mathbb{C}[V_\mathbb{R}]^K$) with the *Fock inner product* defined by

$$\langle f, g\rangle_{\mathcal{F}} = \left(\frac{1}{2\pi}\right)^n \int_V f(z)\overline{g(z)}e^{-\gamma(z)}dz. \tag{1.2}$$

Here "dz" denotes Lebesgue measure on $V_\mathbb{R} \cong \mathbb{R}^{2n}$. Given $\alpha \in \Lambda$ and any orthonormal basis $\{v_1, \ldots, v_{d_\alpha}\}$ for P_α, we let

$$p_\alpha(z) := \frac{1}{d_\alpha}\sum_{j=1}^{d_\alpha} v_j(z)\overline{v_j(z)}. \tag{1.3}$$

The polynomial p_α is an $\mathbb{R}^+$-valued, K-invariant polynomial on $V_\mathbb{R}$ homogeneous of degree $2|\alpha|$. The definition of p_α does not depend on the choice of basis for P_α and $\{p_\alpha \mid \alpha \in \Lambda\}$ is a vector space basis for $\mathbb{C}[V_\mathbb{R}]^K$ (see [BJR92]). We remark that a result in [HU91] ensures that $\mathbb{C}[V_\mathbb{R}]^K$ is itself a polynomial algebra $\mathbb{C}[V_\mathbb{R}]^K = \mathbb{C}[\gamma_1, \ldots, \gamma_r]$, where $\{\gamma_1, \ldots, \gamma_r\}$ is a canonical subset of $\{p_\alpha \mid \alpha \in \Lambda\}$.

We let $\{q_\alpha \mid \alpha \in \Lambda\}$ be the polynomials obtained from $\{p_\alpha \mid \alpha \in \Lambda\}$ via Gram-Schmidt orthogonalization with respect to the Fock inner product. Here we:

1. Order the set index set Λ so that α precedes β if $|\alpha| < |\beta|$. The indices $\{\alpha \mid |\alpha| = m\}$ can be ordered arbitrarily.
2. Normalize the polynomials q_α so that $q_\alpha(0) = 1$.

It is shown in [BJR92] that the polynomials obtained in this way are independent of the ordering chosen for the indices $\{\alpha \mid |\alpha| = m\}$. Thus $\{q_\alpha \mid \alpha \in \Lambda\}$ is another canonical basis for the space $\mathbb{C}[V_\mathbb{R}]^K$. One has that $(-1)^{|\alpha|}p_\alpha$ is the homogeneous component of highest degree in q_α:

$$q_\alpha = (-1)^{|\alpha|}p_\alpha + \text{ lower order terms.}$$

The set $\{q_\alpha \mid \alpha \in \Lambda\}$ is a complete orthogonal system in the space $L^2_K(V_{\mathbb{R}}, e^{-\gamma(z)}dz)$ of K-invariant functions that are square integrable with respect to the weight appearing in the Fock inner product. (See [Yan], [BJR].) Moreover, the norms of the q_α's are given by

$$\langle q_\alpha, q_\alpha \rangle_{\mathcal{F}} = \frac{1}{d_\alpha}.$$

In this paper, we will study various "combinatorial" problems that concern the polynomials $\{p_\alpha\}$ and $\{q_\alpha\}$ for a multiplicity free action. These objects are natural and the problems seem of interest in their own right. Our motivation for this work arises, however, from it's applications to analysis on Heisenberg groups. Each multiplicity free action yields a Gelfand pair associated with a Heisenberg group. The polynomials $\{q_\alpha\}$ determine a dense set of full Plancherel measure in the space of bounded spherical functions for this Gelfand pair. This connection, which accounts for the title of this paper, is discussed in more detail below in Section 2. For an analytical application of some of the combinatorial identities proved here, we refer the reader to [BJR].

For $\alpha \in \Lambda$, the functions $\{p_\beta \mid |\beta| \le |\alpha|\}$ form a basis for the space of K-invariant polynomials on $V_{\mathbb{R}}$ of degree at most $2|\alpha|$. Since q_α belongs to this space, we can write:

$$q_\alpha = \sum_{|\beta| \le |\alpha|} (-1)^{|\beta|} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} p_\beta \tag{1.4}$$

for some well-defined numbers $\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right]$. We call these values *generalized binomial coefficients* for the action of K on V. Since the functions q_α and p_β are all real valued, the generalized binomial coefficients are real numbers. The factor of $(-1)^{|\beta|}$ in the definition ensures that the $\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right]$'s are non-negative. This will be shown below in Lemma 3.9. Since $(-1)^{|\alpha|}p_\alpha$ is the homogeneous component of highest degree in q_α, we see that $\left[\begin{smallmatrix} \alpha \\ \alpha \end{smallmatrix}\right] = 1$ and that $\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] = 0$ for $\beta \neq \alpha$ with $|\beta| = |\alpha|$. We extend the definition of the generalized binomial coefficients by setting $\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] = 0$ for $|\beta| > |\alpha|$. Also note that $\left[\begin{smallmatrix} \alpha \\ 0 \end{smallmatrix}\right] = 1$ since $p_0 = 1 = q_\alpha(0)$.

The generalized binomial coefficients were introduced by Z. Yan in an unpublished manuscript [Yan]. The terminology is motivated by the case where $K = U(n)$ acts in the standard fashion on $V = \mathbb{C}^n$. Here the generalized binomial coefficients coincide with the usual binomial coefficients. (See Example 5.1 below.) Moreover, it is shown in [Yan] that for multiplicity free actions that arise from Hermitian symmetric spaces, the generalized binomial coefficients agree with those introduced by Herz, Dib and Faraut-Koranyi (see [Dib90, FK94, Yan92]).

Our main results appear in Sections 3 and 4. These establish properties of the generalized binomial coefficients, including the following identities:

$$p_\alpha = \sum_{|\beta|\le|\alpha|} (-1)^{|\beta|} \begin{bmatrix}\alpha\\ \beta\end{bmatrix} q_\beta, \tag{1.5}$$

$$\frac{\gamma^k}{k!} d_\beta p_\beta = \sum_{|\alpha|=|\beta|+k} \begin{bmatrix}\alpha\\ \beta\end{bmatrix} d_\alpha p_\alpha, \tag{1.6}$$

$$\gamma q_\alpha = -\sum_{|\beta|=|\alpha|+1} \frac{d_\beta}{d_\alpha} \begin{bmatrix}\beta\\ \alpha\end{bmatrix} q_\beta + (2|\alpha|+n) q_\alpha - \sum_{|\beta|=|\alpha|-1} \begin{bmatrix}\alpha\\ \beta\end{bmatrix} q_\beta, \tag{1.7}$$

$$\frac{\Delta^k}{k!} p_\alpha = \frac{1}{2^k} \sum_{|\beta|=|\alpha|-k} \begin{bmatrix}\alpha\\ \beta\end{bmatrix} p_\beta, \tag{1.8}$$

$$\frac{\Delta^k}{k!} q_\alpha = \left(-\frac{1}{2}\right)^k \sum_{|\beta|=|\alpha|-k} \begin{bmatrix}\alpha\\ \beta\end{bmatrix} q_\beta. \tag{1.9}$$

A comparison of Equations 1.4 and 1.5 shows the generalized binomial coefficients are "self-inverting" and thus a remarkable symmetry exists between the p_α,s and the q_α's. Equations 1.6 and 1.7 provide product formulae for the invariant polynomials p_α and $q_\alpha \in \mathbb{C}[V_\mathbb{R}]^K$ with powers of the fundamental invariant $\gamma(z) = |z|^2/2$. In Equations 1.8 and 1.9, $\Delta := \sum_{j=1}^n \frac{\partial}{\partial z_j}\frac{\partial}{\partial \bar{z}_j}$ is (a multiple of) the usual Laplace operator. These equations are in some sense dual to Equations 1.6 and 1.7. Equation 1.6 is a central result in [Yan]. We provide a new proof below in Section 3 and subsequently make frequent use of this fundamental identity.

We see that the solutions to a variety of combinatorial problems that arise in connection with multiplicity free actions can all be expressed in terms of generalized binomial coefficients. Moreover, we show in Section 4 that the generalized binomial coefficients determine the eigenvalues for K-invariant polynomial coefficient differential operators on $\mathbb{C}[V]$ as well as the eigenvalues for K-invariant and left-invariant differential operators applied to spherical functions on the Heisenberg group. These eigenvalues have recently been studied in [OO97], [Ols] and [Sah94].

Section 5 concerns the computation of the generalized binomial coefficients for several examples. We show how Equation 1.6 can be used to obtain the generalized binomial coefficients for the standard actions of $K = U(n)$ and $K = SO(n,\mathbb{R}) \times \mathbb{T}$ on $V = \mathbb{C}^n$. Example 5.3 concerns the action of $U(n) \times U(n)$ on $\mathbb{C}^n \otimes \mathbb{C}^n$. Here we reduce the computation of the generalized binomial coefficients to standard combinatorial values together with evaluation of the p_α polynomials at a single base point. In this case, the p_α's can be expressed in terms of Schur polynomials and Equation 1.6 is related to the classical Pieri formula for certain products of Schur polynomials. We wish to thank Jacques Faraut for drawing our attention to these connections.

We conclude this section with some remarks regarding the normalization conventions adopted in our definition of the inner product 1.2. If one uses an orthonormal basis for $(V, \langle\cdot,\cdot\rangle)$ to identify V with $\mathbb{C}^n$ then the monomials $z^I := z_1^{i_1}\cdots z_n^{i_n}$ in $\mathbb{C}[V] \cong \mathbb{C}[z_1,\ldots,z_n]$ are orthogonal with respect to $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ and satisfy $\langle z^I, z^I\rangle_{\mathcal{F}} = 2^I I! := 2^{i_1+\cdots+i_n} i_1!\cdots i_n!$. It is natural, however, to ask whether the choice of scaling in the Gaussian factor of $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ plays any role in the definition and properties of the generalized binomial coefficients. The answer is "no". Indeed, suppose that we were to replace $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ by an inner product $\langle\cdot,\cdot\rangle_\lambda$ of the form

$$\langle f, g\rangle_\lambda = c_\lambda \int_V f(z)\overline{g(z)}e^{-\lambda\gamma(z)}dz$$

where λ and c_λ are positive constants. We would obtain bases $\{p_\alpha^\lambda \mid \alpha\in\Lambda\}$ and $\{q_\alpha^\lambda \mid \alpha\in\Lambda\}$ for $\mathbb{C}[V_{\mathbb{R}}]^K$ using $\langle\cdot,\cdot\rangle_\lambda$ in place of $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ in the definition of the p_α's and q_α's. It is then easy to show that $p_\alpha^\lambda(z) = \frac{\lambda^n}{c_\lambda}p_\alpha\left(\sqrt{\lambda}z\right)$ and $q_\alpha^\lambda(z) = q_\alpha\left(\sqrt{\lambda}z\right)$. Thus if we define coefficients $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]_\lambda$ via $q_\alpha^\lambda = \sum_{|\beta|\le|\alpha|}(-1)^{|\beta|}\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]_\lambda p_\alpha^\lambda$ then one simply obtains $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]_\lambda = \frac{c_\lambda}{\lambda^n}\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$. We see that our choice of normalization conventions has no significant effect on the theory developed in this paper.

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2. Spherical functions on the Heisenberg group

A connection is established in [BJR92] between the polynomials $\{q_\alpha \mid \alpha\in\Lambda\}$ for a multiplicity free action and analysis on the Heisenberg group. We recall this connection briefly in this section and refer the reader to [BJR92] and [BJRW96] for further details.

One forms the Heisenberg group $H_n = V\times\mathbb{R}$ with group law

$$(z,t)(z',t') = \left(z+z', t+t' - \frac{1}{2}Im\langle z, z'\rangle\right).$$

The unitary group acts by automorphisms on H_n via

$$k\cdot(z,t) := (kz,t) \quad \text{for } k\in U(n) \text{ and } (z,t)\in H_n.$$

Given a compact Lie subgroup K of $U(V)$, we say that (K, H_n) is a *Gelfand pair* when the algebra $L^1_K(H_n)$ of K-invariant L^1-functions on H_n is commutative under convolution. It is known that (K,H_n) is a Gelfand pair if and only if the action of K on V is multiplicity free. (See [Car87], [BJR92].)

There is a well developed theory of spherical functions associated to Gelfand pairs (K, H_n). These are the smooth K-invariant functions ψ on H_n with $\psi(0,0) = 1$ which are joint eigenfunctions for the differential operators on H_n that are invariant under the action of K and under the left action of H_n. Integration against the bounded

K-spherical functions on H_n yields the Gelfand space $\Delta(K, H_n)$ for the commutative Banach $\star$-algebra $L^1_K(H_n)$.

The functions $\phi_{\alpha,\lambda}$ defined for $\alpha \in \Lambda$ and $\lambda \in \mathbb{R}^\times$ by

$$\phi_{\alpha,\lambda} = q_\alpha\left(|\lambda|^{1/2}z\right)e^{-\gamma\left(|\lambda|^{1/2}z\right)/2}e^{i\lambda t}$$

are pair-wise distinct bounded K-spherical functions on H_n. The set $\Delta_1(K, H_n) = \{\phi_{\alpha,\lambda} \mid \alpha \in \Lambda, \lambda \in \mathbb{R}^\times\}$ is dense and of full Godement-Plancherel measure in $\Delta(K, H_n)$. We see that the determination of these functions completely reduces to the computation of the polynomials q_α. The bounded K-spherical functions that do not belong to $\Delta_1(K, H_n)$ are also of interest but will not play a role in this paper.

Recall that the polynomials q_α can be obtained from the p_α's by Gram-Schmidt orthogonalization. The paper [BJR92] contains a "Rodrigues' type formula" which provides another procedure to compute the polynomial q_α's from the p_α's. We present this here as we will need it later in Sections 3 and 4. We use an orthonormal basis to identify V with $\mathbb{C}^n$ so that $\langle z, z'\rangle = z \cdot \overline{z'}$ for $z, z' \in \mathbb{C}^n$. Proposition 4.7 in [BJR92] asserts that

$$p_\alpha\left(2\frac{\partial}{\partial\overline{z}}, -2\frac{\partial}{\partial z}\right)\left(e^{-\gamma}\right) = q_\alpha e^{-\gamma}. \tag{2.1}$$

Here $p_\alpha\left(2\frac{\partial}{\partial\overline{z}}, -2\frac{\partial}{\partial z}\right)$ denotes the differential operator on $V_\mathbb{R}$ obtained by replacing each occurrence of z_j in p_α by $2\frac{\partial}{\partial\overline{z}_j}$ and each occurrence of $\overline{z}_j$ by $-2\frac{\partial}{\partial z_j}$. One can also state this result in terms of the symplectic Fourier transform on $V_\mathbb{R}$ given for $f \in L^1(V_\mathbb{R})$ by

$$\mathcal{F}^\omega_V(f)(\zeta) = \int_V f(z)e^{-iIm\langle z,\zeta\rangle}dz \tag{2.2}$$

where "dz" denotes Euclidean measure on $V_\mathbb{R}$. One has (see Corollary 4.17 in [BJR92])

$$\mathcal{F}^\omega_V\left(p_\alpha e^{-\gamma}\right) = (2\pi)^n q_\alpha e^{-\gamma} \tag{2.3}$$

and by Fourier inversion

$$\int_V q_\alpha(\zeta)e^{-\gamma(\zeta)}e^{iIm\langle z,\zeta\rangle}d\zeta = (2\pi)^n p_\alpha(z)e^{-\gamma(z)}. \tag{2.4}$$

3. Generalized binomial coefficients

In this section, we develop some properties of the generalized binomial coefficients, beginning with a fundamental result due to Z. Yan. We view this as a "Pieri formula". (In Example 5.3 it will be shown how Theorem 3.1 is related to the classical Pieri formula.)

Theorem 3.1 (cf. [Yan]).

$$\frac{\gamma^k}{k!} d_\beta p_\beta = \sum_{|\alpha|=|\beta|+k} \begin{bmatrix}\alpha\\ \beta\end{bmatrix} d_\alpha p_\alpha.$$

Theorem 3.1 shows how the generalized binomial coefficients can be computed using only the p_α's. The proof below uses some techniques from [Yan]. First, however, we require the following lemma.

Lemma 3.2. $\sum_{|\alpha|=m} d_\alpha p_\alpha = \gamma^m/m!$.

Proof. Recall that $p_\alpha = \frac{1}{d_\alpha}\sum v_j \overline{v}_j$ where $\{v_1,\ldots,v_{d_\alpha}\}$ is a orthonormal basis for P_α. Thus $\sum_{|\alpha|=m} d_\alpha p_\alpha$ can be expressed as $\sum_{|I|=m} z^I\overline{z}^I/2^{|I|}I!$, using the fact that $\{z^I/\sqrt{2^{|I|}I!}\}$ is an orhonormal basis for $\mathcal{P}_m(V)$. This yields

$$\sum_{|\alpha|=m} d_\alpha p_\alpha = \frac{1}{2^m m!}\sum_{|I|=m}\frac{m!}{I!} z^I\overline{z}^I = \frac{1}{2^m m!}\left(|z|^2\right)^m = \frac{\gamma(z)^m}{m!}.$$

□

Proof of Theorem 3.1. Let the values $A_{\alpha,\beta}$ be defined by

$$\frac{\gamma^k}{k!} d_\beta p_\beta = \sum_{|\alpha|=|\beta|+k} A_{\alpha,\beta} d_\alpha p_\alpha$$

and consider the function $F : V\times V \to \mathbb{C}$ defined by

$$F(z,\zeta) = \int_K e^{iIm\langle kz,\zeta\rangle} dk.$$

$F(z,\zeta)$ is real analytic and invariant under the action of $K\times K$ on $V\times V$. Fixing z, we expand $\zeta\mapsto F(z,\zeta)$ in a Taylor series as

$$F(z,\zeta) = \sum_{\alpha\in\Lambda} c_\alpha(z) d_\alpha p_\alpha(\zeta). \tag{3.1}$$

This convergesin the usual tolology on $C^\infty(V\times V)$ and thus we can differentiate termwise. Applying the operator $p_\alpha\left(\frac{\partial}{\partial\overline{\zeta}},\frac{\partial}{\partial\zeta}\right)\Big|_{\zeta=0}$ to each side of Equation 3.1 and noting that

$$p_\alpha\left(\frac{\partial}{\partial\overline{\zeta}},\frac{\partial}{\partial\zeta}\right)\Big|_{\zeta=0} p_\beta(z) = \left\{\begin{array}{ll} 1/d_\alpha 4^{|\alpha|} & \text{for } \alpha=\beta \\ 0 & \text{for } \alpha\neq\beta\end{array}\right\}, \tag{3.2}$$

we obtain the identity $p_\alpha(z/2,-\overline{z}/2) = c_\alpha(z)/4^{|\alpha|}$. Thus $c_\alpha(z) = (-1)^{|\alpha|}p_\alpha(z)$ and Equation 3.1 becomes

$$F(z,\zeta) = \sum_{\alpha\in\Lambda}(-1)^{|\alpha|} d_\alpha p_\alpha(z) p_\alpha(\zeta). \tag{3.3}$$

Next recall that $\{q_\alpha(\zeta) \mid \alpha \in \Lambda\}$ is a complete orthogonal system in $L^2(V_{\mathbb{R}}, e^{-|\zeta|^2/2}d\zeta)$ with $||q_\alpha||^2 = (2\pi)^n/d_\alpha$. Since the function $\zeta \mapsto F(z,\zeta)$ belongs to $L^2(V_{\mathbb{R}}, e^{-|\zeta|^2/2}d\zeta)$, we also obtain an expansion of the form

$$F(z,\zeta) = \sum_{\alpha\in\Lambda} a_\alpha(z) q_\alpha(\zeta),$$

where

$$\begin{aligned} a_\alpha(z) &= \frac{1}{||q_\alpha||^2}\left(\int_V F(z,\zeta) q_\alpha(\zeta) e^{-|\zeta|/2} d\zeta\right) \\ &= \frac{d_\alpha}{(2\pi)^n}\int_V F(z,\zeta) q_\alpha(\zeta) e^{-|\zeta|/2} d\zeta \\ &= \frac{d_\alpha}{(2\pi)^n}\int_V \int_K e^{iIm\langle kz,\zeta\rangle} q_\alpha(\zeta) e^{-|\zeta|/2} dk d\zeta \\ &= \frac{d_\alpha}{(2\pi)^n}\int_V \int_K e^{iIm\langle z,\zeta\rangle} q_\alpha(k\zeta) e^{-|k\zeta|/2} dk d\zeta \\ &= \frac{d_\alpha}{(2\pi)^n}\int_V q_\alpha(\zeta) e^{-|\zeta|/2} e^{iIm\langle z,\zeta\rangle} d\zeta \\ &= \frac{d_\alpha}{(2\pi)^n}(2\pi)^n p_\alpha(z) e^{-|z|^2/2} = d_\alpha p_\alpha(z) e^{-\gamma(z)}. \end{aligned}$$

Here we have applied Equation 2.4 in the last step above. We obtain

$$F(z,\zeta) = \sum_{\alpha\in\Lambda} d_\alpha p_\alpha(z) e^{-\gamma(z)} q_\alpha(\zeta) \tag{3.4}$$

in $L^2(V_{\mathbb{R}}, e^{-|\zeta|^2/2}d\zeta)$. In fact, this series also converges absolutely and uniformly on compact subsets of $V \times V$. Indeed, since $\phi_{\alpha,1}(\zeta) = q_\alpha(\zeta)e^{-\gamma(\zeta)/2}e^{it}$ is bounded by 1, we have that $|q_\alpha(\zeta)| \le e^{\gamma(\zeta)/2}$ for all $\zeta \in V$. This fact together with Lemma 3.2 gives

$$\begin{aligned} \sum_{\alpha\in\Lambda} |d_\alpha p_\alpha(z) e^{-\gamma(z)} q_\alpha(\zeta)| &= \sum_{\alpha\in\Lambda} d_\alpha p_\alpha(z) e^{-\gamma(z)} |q_\alpha(\zeta)| \\ &\le e^{-\gamma(z)} e^{-|\zeta|^2/2} \sum_{m=0}^{\infty}\left(\sum_{|\alpha|=m} d_\alpha p_\alpha(z)\right) \\ &= e^{-\gamma(z)} e^{-|\zeta|^2/2} \sum_{m=0}^{\infty} \frac{\gamma(z)^m}{m!} \\ &= e^{-|\zeta|^2/2}. \end{aligned}$$

The function

$$G(z,\zeta) = f(z,\zeta) e^{\gamma(z)}$$

is also analytic and $K\times K$-invariant on $V\times V$. We thus have a Taylor series expansion of the form

$$G(z,\zeta)=\sum_{m=0}^{\infty}\sum_{|\alpha|+|\beta|=m} c_{\alpha,\beta}p_\alpha(z)p_\beta(\zeta)$$

with

$$p_\alpha\left(\frac{\partial}{\partial\overline{z}},\frac{\partial}{\partial z}\right)p_\beta\left(\frac{\partial}{\partial\overline{\zeta}},\frac{\partial}{\partial\zeta}\right)G(z,\zeta)\bigg|_{(z,\zeta)=(0,0)}=\frac{c_{\alpha,\beta}}{4^{|\alpha|+|\beta|}d_\alpha d_\beta}.$$

Here we have used Equation 3.2. This last quantity is equal to

$$p_\alpha\left(\frac{\partial}{\partial\overline{z}},\frac{\partial}{\partial z}\right)\left(p_\beta\left(\frac{\partial}{\partial\overline{\zeta}},\frac{\partial}{\partial\zeta}\right)G(z,\zeta)\bigg|_{\zeta=0}\right)\bigg|_{z=0}$$

and to

$$p_\beta\left(\frac{\partial}{\partial\overline{\zeta}},\frac{\partial}{\partial\zeta}\right)\left(p_\alpha\left(\frac{\partial}{\partial\overline{z}},\frac{\partial}{\partial z}\right)G(z,\zeta)\bigg|_{z=0}\right)\bigg|_{\zeta=0}.$$

In the following, we assume that $|\beta|\leq|\alpha|$.

Our series expansions for $F(z,\zeta)$ yield two expressions for $G(z,\zeta)$:

$$\sum_{\alpha\in\Lambda} d_\alpha p_\alpha(z)q_\alpha(\zeta)\text{ and }\sum_{\alpha\in\Lambda} d_\alpha(-1)^{|\alpha|}p_\alpha(z)e^{\gamma(z)}p_\alpha(\zeta).$$

Both series converge absolutely and uniformly on compact subsets of $V\times V$. The first series is a Taylor series for $z\mapsto G(z,\zeta)$ for each ζ and the second is a Taylor series for $\zeta\mapsto G(z,\zeta)$ for each z. Thus the first expression gives

$$p_\alpha\left(\frac{\partial}{\partial\overline{z}},\frac{\partial}{\partial z}\right)G(z,\zeta)\bigg|_{z=0}=\frac{q_\alpha(\zeta)}{4^{|\alpha|}},$$

and hence

$$\begin{aligned}
p_\beta&\left(\frac{\partial}{\partial\overline{\zeta}},\frac{\partial}{\partial\zeta}\right)\left(p_\alpha\left(\frac{\partial}{\partial\overline{z}},\frac{\partial}{\partial z}\right)G(z,\zeta)\bigg|_{z=0}\right)\bigg|_{\zeta=0}\\
&=p_\beta\left(\frac{\partial}{\partial\overline{\zeta}},\frac{\partial}{\partial\zeta}\right)\bigg|_{\zeta=0}\frac{q_\alpha(\zeta)}{4^{|\alpha|}}\\
&=\frac{1}{4^{|\alpha|}}\sum_{|\delta|\leq|\alpha|}\begin{bmatrix}\alpha\\ \delta\end{bmatrix}(-1)^{|\delta|}\,p_\beta\left(\frac{\partial}{\partial\overline{\zeta}},\frac{\partial}{\partial\zeta}\right)\bigg|_{\zeta=0}p_\delta(z)\\
&=\frac{1}{4^{|\alpha|}}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\frac{(-1)^{|\beta|}}{4^{|\beta|}d_\beta}.
\end{aligned}$$

On the other hand, the second expression yields:

$$p_\beta\left(\frac{\partial}{\partial\overline{\zeta}},\frac{\partial}{\partial\zeta}\right)G(z,\zeta)\bigg|_{\zeta=0} = (-1)^{|\beta|}d_\beta p_\beta(z)e^{\gamma(z)}\; p_\beta\left(\frac{\partial}{\partial\overline{\zeta}},\frac{\partial}{\partial\zeta}\right)\bigg|_{\zeta=0} p_\beta(z)$$
$$= (-1)^{|\beta|}d_\beta p_\beta(z)e^{\gamma(z)}\frac{1}{d_\beta 4^{|\beta|}}$$

and hence

$$p_\alpha\left(\frac{\partial}{\partial\overline{z}},\frac{\partial}{\partial z}\right)\left(p_\beta\left(\frac{\partial}{\partial\overline{\zeta}},\frac{\partial}{\partial\zeta}\right)G(z,\zeta)\bigg|_{\zeta=0}\right)\Bigg|_{z=0}$$
$$= p_\alpha\left(\frac{\partial}{\partial\overline{z}},\frac{\partial}{\partial z}\right)\bigg|_{z=0}\left(\left(-\frac{1}{4}\right)^{|\beta|}p_\beta(z)e^{\gamma(z)}\right)$$
$$= \left(-\frac{1}{4}\right)^{|\beta|}p_\alpha\left(\frac{\partial}{\partial\overline{z}},\frac{\partial}{\partial z}\right)\bigg|_{z=0}\left[\sum_{N=0}^{\infty}\frac{\gamma(z)^N}{N!}p_\beta(z)\right]$$
$$= \left(-\frac{1}{4}\right)^{|\beta|}\sum_{N=0}^{\infty}p_\alpha\left(\frac{\partial}{\partial\overline{z}},\frac{\partial}{\partial z}\right)\bigg|_{z=0}\left[\frac{\gamma(z)^N}{N!}p_\beta(z)\right]$$
$$= \left(-\frac{1}{4}\right)^{|\beta|}\sum_{N=0}^{\infty}p_\alpha\left(\frac{\partial}{\partial\overline{z}},\frac{\partial}{\partial z}\right)\bigg|_{z=0}\left[\sum_{|\delta|=|\beta|+N}A_{\delta,\beta}\frac{d_\delta}{d_\beta}p_\delta(z)\right]$$
$$= \left(-\frac{1}{4}\right)^{|\beta|}A_{\alpha,\beta}\frac{d_\alpha}{d_\beta}\frac{1}{4^{|\alpha|}d_\alpha}.$$

In these calculations we have applied Equation 3.2 several times. Comparing the two expressions for $p_\alpha\left(\frac{\partial}{\partial\overline{z}},\frac{\partial}{\partial z}\right)p_\beta\left(\frac{\partial}{\partial\overline{\zeta}},\frac{\partial}{\partial\zeta}\right)G(z,\zeta)\Big|_{(z,\zeta)=(0,0)}$ finally yields $A_{\alpha,\beta}=\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ as desired. ∎

Since $\{q_\alpha \mid \alpha\in\Lambda\}$ is a basis for $\mathbb{C}[V_\mathbb{R}]^K$, we can express each p_α in terms of the q_α's. Interestingly, the coefficients that arise in this way are identical to the the coefficients for q_α in terms of the p_α's. The generalized binomial coefficients are thus self-inverting.

Proposition 3.3.

$$p_\alpha = \sum_{|\beta|\le|\alpha|}(-1)^{|\beta|}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}q_\beta.$$

Proof. Let the values $b_{\alpha,\beta}$ be defined via

$$p_\alpha = \sum_{|\beta|\le|\alpha|}(-1)^{|\beta|}b_{\alpha,\beta}q_\beta \tag{3.5}$$

and rewrite Equation **??** as

$$\sum_{\alpha\in\Lambda}(-1)^{|\alpha|}d_\alpha p_\alpha(z)p_\alpha(\zeta)=\sum_{\alpha\in\Lambda}d_\alpha p_\alpha(z)e^{-\gamma(z)}q_\alpha(\zeta). \tag{3.6}$$

Using Theorem 3.1, we have

$$\begin{aligned} d_\alpha p_\alpha(z)e^{-\gamma(z)} &= \sum_{N=0}^{\infty}d_\alpha p_\alpha(z)(-\gamma(z))^N/N! = \sum_{N=0}^{\infty}(-1)^N\sum_{|\beta|=|\alpha|+N}d_\beta\begin{bmatrix}\beta\\ \alpha\end{bmatrix}p_\beta(z)\\ &= \sum_{|\beta|\ge|\alpha|}(-1)^{|\beta|-|\alpha|}d_\beta\begin{bmatrix}\beta\\ \alpha\end{bmatrix}p_\beta(z). \end{aligned}$$

Substituting this identity and Equation 3.5 into Equation 3.6 gives

$$\sum_{|\beta|\le|\alpha|}(-1)^{|\alpha|+|\beta|}d_\alpha b_{\alpha,\beta}q_\beta(z)p_\alpha(\zeta)=\sum_{|\beta|\ge|\alpha|}(-1)^{|\beta|-|\alpha|}d_\beta\begin{bmatrix}\beta\\ \alpha\end{bmatrix}p_\beta(z)q_\alpha(\zeta).$$

It follows that $(-1)^{|\alpha|+|\beta|}d_\alpha b_{\alpha,\beta}=(-1)^{|\alpha|-|\beta|}d_\alpha\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ and hence that $b_{\alpha,\beta}=\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ as claimed. □

Corollary 3.4. *Suppose $|\alpha|=m$, $|\delta|=k$, and $k\le\ell\le m$. Then*

$$\sum_{|\beta|=\ell}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\begin{bmatrix}\beta\\ \delta\end{bmatrix}=\frac{(m-k)!}{(m-\ell)!(\ell-k)!}\begin{bmatrix}\alpha\\ \delta\end{bmatrix}, \quad \textit{and} \tag{3.7}$$

$$\sum_{\beta}(-1)^{|\beta|}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\begin{bmatrix}\beta\\ \delta\end{bmatrix}=\begin{cases}(-1)^{|\alpha|} & \textit{if } \alpha=\delta\\ 0 & \textit{if } \alpha\ne\delta\end{cases}. \tag{3.8}$$

Proof. Theorem 3.1 shows that $\gamma^{m-k}d_\delta p_\delta=(m-k)!\sum_{|\alpha|=m}\left[\begin{smallmatrix}\alpha\\ \delta\end{smallmatrix}\right]d_\alpha p_\alpha$. On the other hand, two applications of Theorem 3.1 yield

$$\begin{aligned}\gamma^{m-k}d_\delta p_\delta=\gamma^{m-\ell}\gamma^{\ell-k}d_\delta p_\delta &= \gamma^{m-\ell}(\ell-k)!\sum_{|\beta|=\ell}\begin{bmatrix}\beta\\ \delta\end{bmatrix}d_\beta p_\beta\\ &=(\ell-k)!(m-\ell)!\sum_{\substack{|\alpha|=m\\ |\beta|=\ell}}\begin{bmatrix}\beta\\ \delta\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}d_\alpha p_\alpha.\end{aligned}$$

This establishes the first identity. For the second equation, we use Proposition 3.3 together with the definition of the generalized binomial coefficients to write

$$q_\alpha=\sum_\beta(-1)^{|\beta|}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}p_\beta=\sum_{\beta,\delta}(-1)^{|\beta|}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(-1)^{|\delta|}\begin{bmatrix}\beta\\ \delta\end{bmatrix}q_\delta.$$

□

We write $\Delta = \sum_{j=1}^n \frac{\partial}{\partial z_j}\frac{\partial}{\partial \overline{z}_j}$, so that 4Δ is the Laplace operator on $V_{\mathbb{R}}$. Proposition 3.5 provides a formula for $\Delta(p_\alpha)$ in terms of the generalized binomial coefficients. First, we compute that for $a \in \mathbb{Z}^+$,

$$\begin{aligned}
\Delta\left(p_\alpha e^{-\gamma/a}\right) &= \sum_{j=1}^n \frac{\partial}{\partial z_j}\frac{\partial}{\partial \overline{z}_j}(p_\alpha)\, e^{-\gamma/a} + \sum_{j=1}^n \frac{\partial}{\partial z_j}(p_\alpha)\frac{\partial}{\partial \overline{z}_j}\left(e^{-\gamma/a}\right) \\
&\quad + \sum_{j=1}^n \frac{\partial}{\partial \overline{z}_j}(p_\alpha)\frac{\partial}{\partial z_j}\left(e^{-\gamma/a}\right) + p_\alpha \sum_{j=1}^n \frac{\partial}{\partial z_j}\frac{\partial}{\partial \overline{z}_j}\left(e^{-\gamma/a}\right) \\
&= (\Delta p_\alpha)\, e^{-\gamma/a} - \sum_{j=1}^n \frac{z_j}{2a}\frac{\partial}{\partial z_j}(p_\alpha)\, e^{-\gamma/a} - \sum_{j=1}^n \frac{\overline{z}_j}{2a}\frac{\partial}{\partial \overline{z}_j}(p_\alpha)\, e^{-\gamma/a} \\
&\quad + p_\alpha \sum_{j=1}^n \left[-\frac{1}{2a}e^{-\gamma/a} + \frac{z_j\overline{z}_j}{4a^2}e^{-\gamma/a}\right] \\
&= \left(\Delta p_\alpha - \frac{2|\alpha|+n}{2a}p_\alpha + \frac{\gamma}{2a^2}p_\alpha\right) e^{-\gamma/a}.
\end{aligned} \tag{3.9}$$

Proposition 3.5.

$$\frac{\Delta^k}{k!}p_\alpha = \frac{1}{2^k}\sum_{|\beta|=|\alpha|-k}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}p_\beta.$$

Proof. We first prove the result for $k=1$. Equation 3.9 with $a=2$ shows that

$$(4\Delta - \gamma/2)\left(p_\alpha e^{-\gamma/2}\right) = (4\Delta(p_\alpha) - (2|\alpha|+n)p_\alpha)e^{-\gamma/2}.$$

On the other hand, it is known that

$$(4\Delta - \gamma/2)\left(q_\alpha e^{-\gamma/2}\right) = -(2|\alpha|+n)q_\alpha e^{-\gamma/2}.$$

(This can be found for example in [BJRW96].) Using this together with Proposition 3.3, we can write

$$\begin{aligned}
(4\Delta - \gamma/2)\left(p_\alpha e^{-\gamma/2}\right) &= (4\Delta - \gamma/2)\left(\sum_\beta (-1)^{|\beta|}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}q_\beta e^{-\gamma/2}\right) \\
&= \sum_\beta (-1)^{|\beta|}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(-2|\beta|-n)q_\beta e^{-\gamma/2} \\
&= \left((-1)^{|\alpha|+1}(2|\alpha|+n)q_\alpha + \sum_{|\beta|=|\alpha|-1}(-1)^{|\alpha|}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(2|\alpha|-2+n)q_\beta + \text{L.O.T.}\right) e^{-\gamma/2}
\end{aligned}$$

$$= \left((-1)^{|\alpha|+1}(2|\alpha|+n) \left[(-1)^{|\alpha|} p_\alpha + (-1)^{|\alpha|-1} \sum_{|\beta|=|\alpha|-1} \begin{bmatrix}\alpha\\ \beta\end{bmatrix} p_\beta \right] + \sum_{|\beta|=|\alpha|-1} (-1)^{|\alpha|} \begin{bmatrix}\alpha\\ \beta\end{bmatrix} (-1)^{|\alpha|-1}(2|\alpha|-2-n)p_\beta + \text{L.O.T.} \right) e^{-\gamma/2}$$

where "L.O.T." denotes the homogeneous summands of lower degree.

Equating the polynomial terms of homogeneous degree $2(|\alpha|-1)$, in the two expressions for $(4\Delta - \gamma/2)\left(p_\alpha e^{-\gamma/2}\right)$, we conclude that

$$4\Delta(p_\alpha) = (2|\alpha|+n) \sum_{|\beta|=|\alpha|-1} \begin{bmatrix}\alpha\\ \beta\end{bmatrix} p_\beta - (2|\alpha|-2+n) \sum_{|\beta|=|\alpha|-1} \begin{bmatrix}\alpha\\ \beta\end{bmatrix} p_\beta$$

and hence $\Delta(p_\alpha) = \frac{1}{2}\sum_{|\beta|=|\alpha|-1} \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] p_\beta$ as claimed.

The general result follows by repeated application of the Contraction Formula 3.7. ∎

Corollary 3.6.

$$e^{-2\Delta} p_\alpha = (-1)^{|\alpha|} q_\alpha.$$

Proposition 3.7.

$$\gamma q_\alpha = -\sum_{|\beta|=|\alpha|+1} \frac{d_\beta}{d_\alpha} \begin{bmatrix}\beta\\ \alpha\end{bmatrix} q_\beta + (2|\alpha|+n) q_\alpha - \sum_{|\beta|=|\alpha|-1} \begin{bmatrix}\alpha\\ \beta\end{bmatrix} q_\beta.$$

Proof. Equation 3.9 with $a = 1$ reads

$$\Delta(p_\alpha e^{-\gamma}) = \left(\Delta(p_\alpha) - \frac{1}{2}(2|\alpha|+n)p_\alpha + \frac{\gamma}{2} p_\alpha \right) e^{-\gamma}.$$

The symplectic Fourier transform given by Equation 2.2 satisfies

$$\mathcal{F}_V^\omega\left(\frac{\partial f}{\partial z_j}\right)(\zeta) = \frac{\overline{\zeta}_j}{2}\mathcal{F}_V^\omega(f)(\zeta), \qquad \mathcal{F}_V^\omega\left(\frac{\partial f}{\partial \overline{z}_j}\right)(\zeta) = -\frac{\zeta_j}{2}\mathcal{F}_V^\omega(f)(\zeta)$$

and hence

$$\mathcal{F}_V^\omega(\Delta f)(\zeta) = -\frac{\gamma(\zeta)}{2}\mathcal{F}_V^\omega(f)(\zeta).$$

Thus we can apply Equation 2.3 together with Theorem 3.1 and Proposition 3.5 to obtain

$$\begin{aligned}(2\pi)^n\gamma q_\alpha e^{-\gamma} &= \gamma\mathcal{F}_V^\omega\left(p_\alpha e^{-\gamma}\right) = -2\mathcal{F}_V^\omega\left(\Delta\left(p_\alpha e^{-\gamma}\right)\right)\\ &= \mathcal{F}_V^\omega\left(-2\Delta(p_\alpha)e^{-\gamma} + (2|\alpha|+n)p_\alpha e^{-\gamma} - \gamma p_\alpha e^{-\gamma}\right)\\ &= \mathcal{F}_V^\omega\left(-\sum_{|\beta|=|\alpha|-1}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}p_\beta e^{-\gamma} + (2|\alpha|+n)p_\alpha e^{-\gamma} - \sum_{|\beta|=|\alpha|+1}\frac{d_\beta}{d_\alpha}\begin{bmatrix}\beta\\ \alpha\end{bmatrix}p_\beta e^{-\gamma}\right)\\ &= (2\pi)^n\left[-\sum_{|\beta|=|\alpha|-1}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}q_\beta + (2|\alpha|+n)q_\alpha - \sum_{|\beta|=|\alpha|+1}\frac{d_\beta}{d_\alpha}\begin{bmatrix}\beta\\ \alpha\end{bmatrix}q_\beta\right]e^{-\gamma}.\end{aligned}$$

□

Proposition 3.8.

$$\frac{\Delta^k}{k!}q_\alpha = \left(-\frac{1}{2}\right)^k\sum_{|\beta|=|\alpha|-k}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}q_\beta.$$

Proof. It suffices to consider the case $k=1$. The general case follows by induction as in the proof of Proposition 3.5. Let $\Delta(q_\alpha) = \sum_{|\beta|<|\alpha|}c_{\alpha,\beta}q_\beta$. Since the polynomials q_α form a complete orthogonal system in $L^2(V_{\mathbb{R}}, e^{-|\zeta|^2/2}d\zeta)$ with $||q_\alpha||^2 = (2\pi)^n/d_\alpha$ and the q_α's are real valued, we have

$$\int\Delta q_\alpha(z)q_\beta(z)e^{-\gamma(z)}dz = (2\pi)^n\frac{c_{\alpha,\beta}}{d_\beta}.$$

On the other hand, we can use Equation 2.1 and integration by parts to write:

$$\begin{aligned}\int\Delta q_\alpha(z)q_\beta(z)e^{-\gamma(z)}dz &= \int\Delta q_\alpha(z)p_\beta\left(2\frac{\partial}{\partial\overline{z}}, -2\frac{\partial}{\partial z}\right)e^{-\gamma(z)}dz\\ &= (-4)^{|\beta|}\int q_\alpha(z)(2\gamma p_\beta)\left(\frac{\partial}{\partial\overline{z}}, \frac{\partial}{\partial z}\right)\left(e^{-\gamma(z)}\right)dz.\end{aligned}$$

An application of Theorem 3.1 yields

$$(\gamma p_\beta)\left(\frac{\partial}{\partial\overline{z}}, \frac{\partial}{\partial z}\right) = \sum_{|\delta|=|\beta|+1}\frac{d_\delta}{d_\beta}\begin{bmatrix}\delta\\ \beta\end{bmatrix}p_\delta\left(\frac{\partial}{\partial\overline{z}}, \frac{\partial}{\partial z}\right).$$

Thus we have

$$\begin{aligned}\int\Delta q_\alpha(z)q_\beta(z)e^{-\gamma(z)}dz &= \frac{(-4)^{|\beta|+1}}{(-4)}\int 2q_\alpha(z)\sum_{|\delta|=|\beta|+1}\frac{d_\delta}{d_\beta}\begin{bmatrix}\delta\\ \beta\end{bmatrix}p_\delta\left(\frac{\partial}{\partial\overline{z}}, \frac{\partial}{\partial z}\right)\left(e^{-\gamma(z)}\right)dz\\ &= -\frac{1}{2}\int q_\alpha(z)\sum_{|\delta|=|\beta|+1}\frac{d_\delta}{d_\beta}\begin{bmatrix}\delta\\ \beta\end{bmatrix}q_\delta(z)e^{-\gamma(z)}dz,\end{aligned}$$

where we have again used Equation 2.1 in the last step. Using orthogonality of the q_α's, we obtain:

$$\int \Delta q_\alpha(z) q_\beta(z) e^{-\gamma(z)} dz = \begin{cases} 0 & \text{if } |\beta| \neq |\alpha| - 1 \\ -\frac{1}{2}\frac{d_\alpha}{d_\beta}\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]\frac{(2\pi)^n}{d_\alpha} & \text{if } |\beta| = |\alpha| - 1 \end{cases}.$$

Thus, for $|\beta| = |\alpha| - 1$, we have

$$(2\pi)^n \frac{c_{\alpha,\beta}}{d_\beta} = -\frac{1}{2}\frac{1}{d_\beta}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(2\pi)^n$$

and thus $c_{\alpha,\beta} = -\frac{1}{2}\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ as claimed. □

We conclude this section by proving nonnegativity of the generalized binomial coefficients as promised earlier.

Lemma 3.9. $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ *is a nonnegative number for all* $\alpha, \beta \in \Lambda$.

Proof. Suppose that $|\alpha| = |\beta| + k$. Repeated application of Equation 3.7 yields

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{k!}\sum \begin{bmatrix}\varepsilon_1\\ \beta\end{bmatrix}\begin{bmatrix}\varepsilon_2\\ \varepsilon_1\end{bmatrix}\cdots\begin{bmatrix}\varepsilon_{k-1}\\ \varepsilon_{k-2}\end{bmatrix}\begin{bmatrix}\alpha\\ \varepsilon_{k-1}\end{bmatrix} \tag{3.10}$$

where the sum is over all $(\varepsilon_1, \ldots, \varepsilon_{k-1})$ with $|\varepsilon_j| = |\beta| + j$. This identity shows that it suffices to prove Lemma 3.9 for the case where $|\alpha| = |\beta| + 1$. We consider this case below.

Let $\beta \in \Lambda$ and $\{v_1, \ldots, v_m\}$ be an orthonormal basis for P_β ($m = d_\beta$). Since $z_i v_j \in \mathcal{P}_{|\beta|+1}(V) = \sum_{|\alpha|=|\beta|+1} P_\alpha$ we can write

$$z_i v_j = \sum_{|\alpha|=|\beta|+1} v_\alpha(i,j)$$

where $v_\alpha(i,j) \in P_\alpha$. Thus also

$$2m\gamma p_\beta = \sum_{i=1}^n \sum_{j=1}^m z_i v_j \overline{z_i v_j} = \sum_{|\alpha|=|\beta|+1=|\alpha'|} \sum_{i=1}^n \sum_{j=1}^m v_\alpha(i,j)\overline{v_{\alpha'}(i,j)}.$$

Note that the sum $\sum_{|\alpha|=|\beta|+1=|\alpha'|} P_\alpha \otimes \overline{P}_{\alpha'}$ is direct in $\mathbb{C}[V_\mathbb{R}] = \mathbb{C}[V] \otimes \overline{\mathbb{C}[V]}$ and that each $P_\alpha \otimes \overline{P}_{\alpha'}$ is a K-invariant subspace. Since $2m\gamma p_\beta$ is a K-invariant polynomial, it follows that

$$\sum_{i=1}^n \sum_{j=1}^m v_\alpha(i,j)\overline{v_{\alpha'}(i,j)} \in P_\alpha \otimes \overline{P}_{\alpha'}$$

is K-invariant for each $|\alpha| = |\beta| + 1 = |\alpha'|$. Schur's Lemma implies, however, that

$$\left(P_\alpha \otimes \overline{P}_{\alpha'}\right)^K = \begin{cases} \{0\} & \text{for } \alpha' \neq \alpha \\ \mathbb{C}p_\alpha & \text{for } \alpha' = \alpha \end{cases}$$

and hence we conclude that $\sum_{i=1}^n \sum_{j=1}^m v_\alpha(i,j)\overline{v_{\alpha'}(i,j)} = 0$ when $\alpha' \neq \alpha$ and that $\sum_{i=1}^n \sum_{j=1}^m |v_\alpha(i,j)|^2 = a_\alpha p_\alpha$ for some value $a_\alpha \in \mathbb{C}$. As p_α and $\sum_{i=1}^n \sum_{j=1}^m |v_\alpha(i,j)|^2$ are both non-negative real valued polynomials, we must have that $a_\alpha \geq 0$. Thus

$$2m\gamma p_\beta = \sum_{|\alpha|=|\beta|+1} a_\alpha p_\alpha$$

for some values $a_\alpha \geq 0$. Theorem 3.1 now implies that $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] = a_\alpha/2d_\alpha \geq 0$. □

Remark 3.1. Note that the subspace $\mathrm{Span}(\mathcal{P}_k(V)P_\beta)$ of $\mathbb{C}[V]$ spanned by $\mathcal{P}_k(V)P_\beta$ is K-invariant. The proof of Lemma 3.9 shows that for $|\alpha| = |\beta|+1$, we can have $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] \neq 0$ only when $v_\alpha(i,j) \neq 0$ for some i,j. As $z_i v_j \in \mathcal{P}_k(V)P_\beta$, this implies that $P_\alpha \subset \mathrm{Span}(\mathcal{P}_1(V)P_\beta)$. Using Equation 3.10, we obtain that for $|\alpha| = |\beta| + k$

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \neq 0 \Rightarrow P_\alpha \subset \mathrm{Span}(\mathcal{P}_k(V)P_\beta).$$

4. Eigenvalues for K-invariant differential operators

One basis for the Lie algebra of left invariant vector fields on H_n is written as $\{Z_1, Z_2, \ldots, Z_n, \overline{Z}_1, \overline{Z}_2, \ldots, \overline{Z}_n, T\}$ where

$$Z_j = 2\frac{\partial}{\partial \overline{z}_j} + i\frac{z_j}{2}\frac{\partial}{\partial t}, \quad \overline{Z}_j = 2\frac{\partial}{\partial z_j} - i\frac{\overline{z}_j}{2}\frac{\partial}{\partial t}, \tag{4.1}$$

and

$$T := \frac{\partial}{\partial t}.$$

With these conventions one has $[Z_j, \overline{Z}_j] = -2iT$.

The first order operators $Z_1, \ldots, Z_n, \overline{Z}_1, \ldots, \overline{Z}_n, T$ generate the algebra $\mathbb{D}(H_n)$ of left-invariant differential operators on H_n. We denote the subalgebra of K-invariant differential operators by

$$\mathbb{D}_K(H_n) := \{D \in \mathbb{D}(H_n) \mid D(f \circ k) = D(f) \circ k \text{ for } k \in K,\ f \in C^\infty(H_n)\}.$$

Recall that the K-spherical functions for the Gelfand pair (K, H_n), discussed in Section 2, are eigenfunctions for the operators $D \in \mathbb{D}_K(H_n)$. We denote the eigenvalue for $D \in \mathbb{D}_K(H_n)$ on a spherical function ψ by $\widehat{D}(\psi)$:

$$D\psi = \widehat{D}(\psi)\psi.$$

Given a polynomial $p \in \mathbb{C}[V_\mathbb{R}]^K$, we construct operators $p(Z, \overline{Z})$, $p(\overline{Z}, Z)$ and L_p in $\mathbb{D}_K(H_n)$. Here $p(Z, \overline{Z}) = p(Z_1, \ldots, Z_n, \overline{Z}_1, \ldots, \overline{Z}_n)$ is obtained from $p(z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n)$ by replacing each occurrence of the variable z_j by the operator Z_j and each $\overline{z}_j$ by $\overline{Z}_j$. Within each monomial $z_1^{a_1} \cdots z_n^{a_n} \overline{z}_1^{b_1} \cdots \overline{z}_n^{b_n}$, we ensure that the holomorphic variables z_j appear before any anti-holomorphic variables $\overline{z}_j$. The operator $p(\overline{Z}, Z) = p(\overline{Z}_1, \ldots, \overline{Z}_n, Z_1, \ldots, Z_n)$ is also obtained from $p(z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n)$ by replacing the variable z_j by Z_j and the variable $\overline{z}_j$ by $\overline{Z}_j$. Here, however, we place the anti-holomorphic variables $\overline{z}_j$ before the the holomorphic variables z_j within monomials. The operator L_p is obtained in a similar fashion but by averaging over all possible orderings of the variables. One can give basis free descriptions of the operators $p(Z,\overline{Z})$, $p(\overline{Z},Z)$ and L_p (this is done for L_p in [BJR92]), but these will not needed here. The operators $\{p_\alpha(Z,\overline{Z}) \mid \alpha \in \Lambda\}$, or alternatively $\{p_\alpha(\overline{Z},Z) \mid \alpha \in \Lambda\}$ or $\{L_{p_\alpha} \mid \alpha \in \Lambda\}$, together with T, generate the algebra $\mathbb{D}_K(H_n)$. Using the fact that $\{p_\alpha \mid \alpha \in \Lambda\}$ is a basis for $\mathbb{C}[V_\mathbb{R}]^K$ together with the commutation relations $[Z_j,\overline{Z}_j] = -2iT$, we see that

$$L_{p_\beta} = \sum_{|\delta|\le|\beta|} c_{\beta,\delta}\, p_\delta(Z,\overline{Z}) T^{|\beta|-|\delta|}$$

for some numbers $c_{\beta,\delta}$. We will see that the coefficients $c_{\beta,\delta}$ can be written in terms of generalized binomial coefficients.

Theorem 4.1 (cf. [Yan]). *For $\beta,\alpha \in \Lambda$, $\lambda \in \mathbb{R}^\times$ one has*

$$\widehat{L}_{p_\beta}(\phi_{\alpha,\lambda}) = \left(-\frac{|\lambda|}{2}\right)^{|\beta|} \sum_\delta \begin{bmatrix}\alpha\\ \delta\end{bmatrix}\begin{bmatrix}\beta\\ \delta\end{bmatrix}\frac{2^{|\delta|}}{d_\delta}.$$

Proof. Lemma 3.4 in [BJRW96] shows that $\widehat{L}_{p_\beta}(\phi_{\alpha,\lambda}) = |\lambda|^{|\beta|}\widehat{L}_{p_\beta}(\phi_{\alpha,1})$. Hence we need only prove the result for $\phi_\alpha = \phi_{\alpha,1}$. Proposition 3.9 in [BJRW96] shows that we have a series expansion for the (real analytic) function

$$\phi_\alpha(z,0) = \sum_\beta d_\beta \widehat{L}_{p_\beta}(\phi_\alpha)\, p_\beta(z).$$

On the other hand, we compute

$$\begin{aligned}
\phi_\alpha(z,0) = q_\alpha(z)e^{-\gamma(z)/2} &= \sum_{m=0}^\infty \left(-\frac{1}{2}\right)^m \frac{\gamma^m}{m!} q_\alpha \\
&= \sum_{m=0}^\infty \left(-\frac{1}{2}\right)^m \sum_\delta (-1)^{|\delta|}\begin{bmatrix}\alpha\\ \delta\end{bmatrix}\frac{\gamma^m}{m!}p_\delta \\
&= \sum_{m=0}^\infty \left(-\frac{1}{2}\right)^m \sum_\delta (-1)^{|\delta|}\begin{bmatrix}\alpha\\ \delta\end{bmatrix}\sum_{|\beta|=|\delta|+m}\frac{d_\beta}{d_\delta}\begin{bmatrix}\beta\\ \delta\end{bmatrix}p_\beta \\
&= \sum_{\beta,\delta}\frac{(-1)^{|\beta|}}{2^{|\beta|-|\delta|}}\frac{d_\beta}{d_\delta}\begin{bmatrix}\alpha\\ \delta\end{bmatrix}\begin{bmatrix}\beta\\ \delta\end{bmatrix}p_\beta
\end{aligned}$$

Equating coefficients in these two series expansions for $\phi_\alpha(z,0)$ in terms of p_β yields

$$\widehat{L}_{p_\beta}(\phi_\alpha) = \left(-\frac{1}{2}\right)^{|\beta|}\sum_\delta \begin{bmatrix}\alpha\\ \delta\end{bmatrix}\begin{bmatrix}\beta\\ \delta\end{bmatrix}\frac{2^{|\delta|}}{d_\delta}$$

as claimed. □

Theorem 4.2.

$$p_\beta(Z,\overline{Z})^\wedge\left(\phi_{\alpha,\lambda}\right)=\frac{(-|\lambda|)^{|\beta|}}{d_\beta}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}.$$

Proof. First we note that

$$\langle p_\alpha, q_\beta\rangle_{\mathcal{F}}=\frac{(-1)^{|\beta|}}{d_\beta}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}. \tag{4.2}$$

Indeed, we have

$$\langle p_\alpha, q_\beta\rangle_{\mathcal{F}}=\sum_\delta(-1)^{|\delta|}\begin{bmatrix}\alpha\\ \delta\end{bmatrix}\langle q_\delta, q_\beta\rangle_{\mathcal{F}}=\frac{(-1)^{|\beta|}}{d_\beta}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}.$$

Now let $\{u_1,\ldots,u_{d_\beta}\}$ and $\{v_1,\ldots,v_{d_\alpha}\}$ be orthonormal bases for the subspaces P_β and P_α in $\mathbb{C}[V]$. We can use Equation 2.1 and integration by parts to write

$$\begin{aligned}
\langle p_\alpha, q_\beta\rangle_{\mathcal{F}} &= \frac{1}{(2\pi)^n}\int p_\alpha(z)\left[p_\beta\left(2\frac{\partial}{\partial\overline{z}},-2\frac{\partial}{\partial z}\right)e^{-\gamma(z)}\right]dz\\
&=\frac{1}{d_\beta(2\pi)^n}\int p_\alpha(z)\sum_i\overline{u}_i\left(-2\frac{\partial}{\partial z}\right)u_i\left(2\frac{\partial}{\partial\overline{z}}\right)e^{-\gamma(z)}dz\\
&=\frac{1}{d_\beta(2\pi)^n}\sum_i\int\left[\overline{u}_i\left(2\frac{\partial}{\partial z}\right)p_\alpha(z)\right]u_i(-z)e^{-\gamma(z)}dz\\
&=\frac{1}{d_\alpha d_\beta(2\pi)^n}\sum_{i,j}\int u_i(-z)\left[\overline{u}_i\left(2\frac{\partial}{\partial z}\right)v_j(z)\overline{v}_j(\overline{z})\right]e^{-\gamma(z)}dz\\
&=\frac{1}{d_\alpha(2\pi)^n}\sum_j\int\left[p_\beta\left(-z,2\frac{\partial}{\partial z}\right)v_j(z)\right]\overline{v}_j(\overline{z})e^{-\gamma(z)}dz\\
&=\frac{1}{d_\alpha}\sum_j\langle\pi\left(p_\beta\left(Z,\overline{Z}\right)\right)v_j,v_j\rangle_{\mathcal{F}}.
\end{aligned}$$

In the last line, π denotes the irreducible representation of H_n on Fock space (the Hilbert space obtained by completing $\mathbb{C}[V]$ with respect to the Fock inner product) with $\pi(0,1)=I$. One has that

$$\pi(Z_j)=-z_j\quad\text{and}\quad\pi(\overline{Z}_j)=2\frac{\partial}{\partial z}.$$

Proposition 3.20 in [BJR92] asserts that the operator $\pi(D)$ for $D\in\mathbb{D}_K(H_n)$ acts on P_α as the scalar $\widehat{D}(\phi_\alpha)$. Thus we obtain:

$$\langle p_\alpha, q_\beta\rangle_{\mathcal{F}}=p_\beta(Z,\overline{Z})^\wedge(\phi_\alpha).$$

Comparing this with Equation 4.2 yields

$$p_\beta(Z,\overline{Z})^\wedge(\phi_\alpha) = \frac{(-1)^{|\beta|}}{d_\beta}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}.$$

This completes the proof, since $p_\beta(Z,\overline{Z})^\wedge\left(\phi_{\alpha,\lambda}\right) = |\lambda|^{|\beta|}p_\beta(Z,\overline{Z})^\wedge(\phi_\alpha)$. $\square$

Theorem 4.3.

$$L_{p_\beta} = \sum_\delta \begin{bmatrix}\beta\\ \delta\end{bmatrix} p_\delta(Z,\overline{Z})\left(\frac{iT}{2}\right)^{|\beta|-|\delta|}.$$

Proof. Let $D := \sum_\delta \left[\begin{smallmatrix}\beta\\ \delta\end{smallmatrix}\right] p_\delta(Z,\overline{Z})\left(\frac{iT}{2}\right)^{|\beta|-|\delta|}$. Since $D \in \mathbb{D}_K(H_n)$, D is completely determined by its eigenvalues $\{\widehat{D}(\psi) \mid \psi \in \Delta(K,H_n)\}$. Moreover, since $\psi \mapsto \widehat{D}(\psi)$ is continuous on $\Delta(K,H_n)$ and the set $\Delta_1(K,H_n) = \{\phi_{\alpha,\lambda} \mid \alpha\in\Lambda, \lambda\in\mathbb{R}^\times\}$ is dense in $\Delta(K,H_n)$ (see [BJRW96]), we need only show that $\widehat{D}(\phi_{\alpha,\lambda}) = \widehat{L}_{p_\beta}(\phi_{\alpha,\lambda})$ for all $\alpha\in\Lambda$, $\lambda\in\mathbb{R}^\times$. As $\widehat{D}(\phi_{\alpha,\lambda}) = |\lambda|^{|\beta|}\widehat{D}(\phi_\alpha)$ and $\widehat{L}_{p_\beta}(\phi_{\alpha,\lambda}) = |\lambda|^{|\beta|}\widehat{L}_{p_\beta}(\phi_\alpha)$, we need only check that

$$\widehat{D}(\phi_\alpha) = \widehat{L}_{p_\beta}(\phi_\alpha)$$

for all $\alpha\in\Lambda$. As $T(\phi_\alpha) = i\phi_\alpha$ we see that

$$\begin{aligned}
\widehat{D}(\phi_\alpha) &= \sum_\delta \begin{bmatrix}\beta\\ \delta\end{bmatrix}\left(-\frac{1}{2}\right)^{|\beta|-|\delta|} p_\delta(Z,\overline{Z})^\wedge(\phi_\alpha)\\
&= \sum_\delta \begin{bmatrix}\beta\\ \delta\end{bmatrix}\left(-\frac{1}{2}\right)^{|\beta|-|\delta|} \frac{(-1)^{|\delta|}}{d_\delta}\begin{bmatrix}\alpha\\ \delta\end{bmatrix} \quad \text{by Theorem 4.2}\\
&= \left(-\frac{1}{2}\right)^{|\beta|}\sum_\delta \begin{bmatrix}\beta\\ \delta\end{bmatrix}\begin{bmatrix}\alpha\\ \delta\end{bmatrix}\frac{2^{|\delta|}}{d_\delta}\\
&= \widehat{L}_{p_\beta}(\phi_\alpha) \quad \text{by Theorem 4.1.}
\end{aligned}$$

$\square$

Theorem 4.4.

$$p_\beta(\overline{Z},Z)^\wedge(\phi_{\alpha,\lambda}) = (-|\lambda|)^{|\beta|}\sum_\delta \frac{1}{d_\delta}\begin{bmatrix}\alpha\\ \delta\end{bmatrix}\begin{bmatrix}\beta\\ \delta\end{bmatrix}.$$

Proof. As in the proofs of Theorems 4.1 and 4.2, it suffices to prove the result for $\phi_\alpha = \phi_{\alpha,1}$. Let $\{v_1,\dots,v_{d_\alpha}\}$ and $\{u_1,\dots,u_{d_\beta}\}$ be orthonormal bases for P_α and P_β. Note that $p_\beta(\overline{Z},Z) = \frac{1}{d_\beta}\sum_i \overline{u}_i(\overline{Z})u_i(Z)$. As in the proof on Theorem 4.2, a straightforward calculation shows that

$$\begin{aligned}
p_\beta(\overline{Z},Z)^\wedge(\phi_\alpha) &= \frac{1}{d_\alpha}\sum_j \langle \pi(p_\beta(\overline{Z},Z))v_j, v_j\rangle_{\mathcal{F}} \\
&= \frac{1}{d_\alpha(2\pi)^n}\sum_j \int \left[p_\beta\left(2\frac{\partial}{\partial z}, -z\right) v_j(z)\right]\overline{v_j(z)}e^{-\gamma(z)}dz \\
&= \frac{1}{d_\alpha d_\beta(2\pi)^n}\sum_{i,j} \int \left[\overline{u}_i\left(2\frac{\partial}{\partial z}\right) u_i(-z)v_j(z)\right]\overline{v_j(z)}e^{-\gamma(z)}dz \\
&= \frac{1}{d_\alpha d_\beta(2\pi)^n}\sum_{i,j} \int u_i(-z)v_j(z)\overline{v_j(z)}\overline{u}_i\left(-2\frac{\partial}{\partial z}\right) e^{-\gamma(z)}dz \\
&= \frac{1}{d_\beta(2\pi)^n}\sum_{i} \int p_\alpha(z)u_i(-z)\overline{u}_i(\overline{z})e^{-\gamma(z)}dz \\
&= (-1)^{|\beta|}\langle p_\alpha, p_\beta\rangle_{\mathcal{F}} \\
&= (-1)^{|\beta|}\sum_{\delta,\varepsilon}(-1)^{|\delta|}\begin{bmatrix}\alpha\\ \delta\end{bmatrix}(-1)^{|\varepsilon|}\begin{bmatrix}\beta\\ \varepsilon\end{bmatrix}\langle q_\delta, q_\varepsilon\rangle_{\mathcal{F}} \\
&= (-1)^{|\beta|}\sum_{\delta}\frac{1}{d_\delta}\begin{bmatrix}\alpha\\ \delta\end{bmatrix}\begin{bmatrix}\beta\\ \delta\end{bmatrix}.
\end{aligned}$$

□

Corollary 4.5.

$$p_\beta(\overline{Z},Z) = \sum_\delta \begin{bmatrix}\beta\\ \delta\end{bmatrix} p_\delta(Z,\overline{Z})(iT)^{|\beta|-|\delta|}.$$

Proof. Let D be the operator on the right hand side of this equation. We have

$$\widehat{D}(\phi_{\alpha,\lambda}) = \sum_\delta \begin{bmatrix}\beta\\ \delta\end{bmatrix}\frac{(-|\lambda|)^{|\delta|}}{d_\delta}\begin{bmatrix}\alpha\\ \delta\end{bmatrix}(-|\lambda|)^{|\beta|-|\delta|} = (-|\lambda|)^{|\beta|}\sum_\delta \frac{1}{d_\delta}\begin{bmatrix}\alpha\\ \delta\end{bmatrix}\begin{bmatrix}\beta\\ \delta\end{bmatrix}.$$

□

We conclude this section by describing some related results for differential operators on V. We let $\mathcal{PD}(V)^K$ denote the space of K-invariant polynomial coefficient differential operators on V. The action of $D \in \mathcal{PD}(V)^K$ on $\mathbb{C}[V]$ is diagonalized by decomposition 1.1. That is, $D|_{P_\alpha}$ is given by a scalar for each $\alpha \in \Lambda$. Given $\beta \in \Lambda$, one can form the Wick ordered operators $p_\beta\left(z, \frac{\partial}{\partial z}\right)$ and $q_\beta\left(z, \frac{\partial}{\partial z}\right)$, the anti-Wick ordered operators $p_\beta\left(\frac{\partial}{\partial z}, z\right)$ and $q_\beta\left(\frac{\partial}{\partial z}, z\right)$ and the symmetrized operators $Sym\left(p_\beta\left(z, \frac{\partial}{\partial z}\right)\right)$ and $Sym\left(q_\beta\left(z, \frac{\partial}{\partial z}\right)\right)$. These all belong to $\mathcal{PD}(V)^K$. We will see that the eigenvalues for each of these operators on P_α can be expressed in terms of generalized binomial coefficients.

Proposition 4.6. *Define a continuous family of operators* $\{D_\alpha(t) \mid t \in \mathbb{C}\} \subset \mathcal{P}D(V)^K$ *by*

$$D_\alpha(t) = \sum_\beta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \left(\frac{t}{2}\right)^{|\alpha|-|\beta|} p_\beta\left(z, \frac{\partial}{\partial z}\right).$$

Then:

$$\begin{aligned} D_\alpha(0) &= p_\alpha\left(z, \frac{\partial}{\partial z}\right) \\ D_\alpha(1/2) &= Sym\left(p_\alpha\left(z, \frac{\partial}{\partial z}\right)\right) \\ D_\alpha(1) &= p_\alpha\left(\frac{\partial}{\partial z}, z\right) \\ D_\alpha(-1) &= (-1)^{|\alpha|} q_\alpha\left(\frac{\partial}{\partial z}, z\right) \\ D_\alpha(-3/2) &= (-1)^{|\alpha|} Sym\left(q_\alpha\left(z, \frac{\partial}{\partial z}\right)\right) \\ D_\alpha(-2) &= (-1)^{|\alpha|} q_\alpha\left(z, \frac{\partial}{\partial z}\right). \end{aligned}$$

Proof. The proof of Theorem 4.2 shows that $\pi(p_\beta(Z, \overline{Z})) = p_\beta\left(-z, 2\frac{\partial}{\partial z}\right) = (-2)^{|\beta|} p_\beta\left(z, \frac{\partial}{\partial z}\right)$ acts on P_δ as the scalar $p_\beta(Z,\overline{Z})^\wedge(\phi_\delta) = \frac{(-1)^{|\beta|}}{d_\beta}\left[\begin{smallmatrix}\delta\\ \beta\end{smallmatrix}\right]$. Thus, $D_\alpha(t)$ acts on P_δ as the scalar

$$\sum_\beta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \left(\frac{t}{2}\right)^{|\alpha|-|\beta|} \left(\frac{1}{2}\right)^{|\beta|} \frac{1}{d_\beta} \begin{bmatrix} \delta \\ \beta \end{bmatrix} = \left(\frac{1}{2}\right)^{|\alpha|} \sum_\beta \frac{1}{d_\beta} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \begin{bmatrix} \delta \\ \beta \end{bmatrix} t^{|\alpha|-|\beta|}.$$

Theorem 4.4 shows that $\pi(p_\alpha(\overline{Z}, Z)) = p_\alpha\left(2\frac{\partial}{\partial z}, -z\right) = (-2)^{|\alpha|} p_\alpha\left(\frac{\partial}{\partial z}, z\right)$ acts on P_δ as the scalar $p_\alpha(\overline{Z}, Z)^\wedge(\phi_\delta) = (-1)^{|\alpha|}\sum_\beta \frac{1}{d_\beta}\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]\left[\begin{smallmatrix}\delta\\ \beta\end{smallmatrix}\right]$ and hence $p_\alpha\left(\frac{\partial}{\partial z}, z\right) = D_\alpha(1)$. Theorem 4.1 shows that $\pi(L_{p_\alpha}) = (-2)^{|\alpha|} Sym\left(p_\alpha\left(z, \frac{\partial}{\partial z}\right)\right)$ acts on P_δ by the scalar $\widehat{L}_{p_\alpha}(\phi_\delta) = \left(-\frac{1}{2}\right)^{|\alpha|}\sum_\beta \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]\left[\begin{smallmatrix}\delta\\ \beta\end{smallmatrix}\right]\frac{2^{|\beta|}}{d_\beta}$, and hence $Sym\left(p_\alpha\left(z, \frac{\partial}{\partial z}\right)\right) = D_\alpha(1/2)$.

Since $q_\alpha = \sum_\beta (-1)^{|\beta|}\left[{\alpha \atop \beta}\right] p_\beta$, we see that $D_\alpha(-2) = (-1)^{|\alpha|} q_\alpha\left(z, \frac{\partial}{\partial z}\right)$. We also have that $q_\alpha\left(\frac{\partial}{\partial z}, z\right)$ acts on P_δ by the scalar

$$\begin{aligned}
\sum_\beta (-1)^{|\beta|} \begin{bmatrix}\alpha\\ \beta\end{bmatrix} \frac{1}{2^{|\beta|}} \sum_\varepsilon \frac{1}{d_\varepsilon} \begin{bmatrix}\delta\\ \varepsilon\end{bmatrix} \begin{bmatrix}\beta\\ \varepsilon\end{bmatrix} &= \sum_\varepsilon \frac{1}{d_\varepsilon} \begin{bmatrix}\delta\\ \varepsilon\end{bmatrix} \sum_{m=|\varepsilon|}^{|\alpha|} \left(-\frac{1}{2}\right)^m \sum_{|\beta|=m} \begin{bmatrix}\alpha\\ \beta\end{bmatrix} \begin{bmatrix}\beta\\ \varepsilon\end{bmatrix} \\
&= \sum_\varepsilon \frac{1}{d_\varepsilon} \begin{bmatrix}\delta\\ \varepsilon\end{bmatrix} \sum_{m=|\varepsilon|}^{|\alpha|} \left(-\frac{1}{2}\right)^m \frac{(|\alpha|-|\varepsilon|)!}{(|\alpha|-m)!(m-|\varepsilon|)!} \begin{bmatrix}\alpha\\ \varepsilon\end{bmatrix} \quad \text{by Equation 3.7} \\
&= \sum_\varepsilon \frac{1}{d_\varepsilon} \begin{bmatrix}\delta\\ \varepsilon\end{bmatrix} \left(-\frac{1}{2}\right)^{|\varepsilon|} \left(1-\frac{1}{2}\right)^{|\alpha|-|\varepsilon|} \begin{bmatrix}\alpha\\ \varepsilon\end{bmatrix} \\
&= \left(-\frac{1}{2}\right)^{|\alpha|} \sum_\varepsilon \frac{1}{d_\varepsilon} \begin{bmatrix}\delta\\ \varepsilon\end{bmatrix} \begin{bmatrix}\alpha\\ \varepsilon\end{bmatrix} (-1)^{|\alpha|-|\varepsilon|}.
\end{aligned}$$

Thus $q_\alpha\left(\frac{\partial}{\partial z}, z\right) = (-1)^{|\alpha|} D_\alpha(-1)$. A similar calculation shows that $Sym\left(q_\alpha\left(z, \frac{\partial}{\partial z}\right)\right)$ acts on P_δ by the scalar

$$\sum_\beta \begin{bmatrix}\alpha\\ \beta\end{bmatrix} \left(-\frac{1}{4}\right)^{|\beta|} \sum_\varepsilon \begin{bmatrix}\delta\\ \varepsilon\end{bmatrix} \begin{bmatrix}\beta\\ \varepsilon\end{bmatrix} \frac{2^{|\varepsilon|}}{d_\varepsilon} = \left(-\frac{1}{2}\right)^{|\alpha|} \sum_\varepsilon \frac{1}{d_\varepsilon} \begin{bmatrix}\delta\\ \varepsilon\end{bmatrix} \begin{bmatrix}\alpha\\ \varepsilon\end{bmatrix} \left(-\frac{3}{2}\right)^{|\alpha|-|\varepsilon|},$$

which gives $Sym\left(q_\alpha\left(z, \frac{\partial}{\partial z}\right)\right) = (-1)^{|\alpha|} D_\alpha(-3/2)$. □

The proof of Proposition 4.6 determines the spectrum of the operators $p_\alpha\left(z, \frac{\partial}{\partial z}\right)$, $q_\alpha\left(z, \frac{\partial}{\partial z}\right)$, $p_\alpha\left(\frac{\partial}{\partial z}, z\right)$, $q_\alpha\left(\frac{\partial}{\partial z}, z\right)$, $Sym\left(p_\alpha\left(z, \frac{\partial}{\partial z}\right)\right)$ and $Sym\left(q_\alpha\left(z, \frac{\partial}{\partial z}\right)\right)$ on $\mathbb{C}[V]$. We have:

$$\begin{aligned}
p_\alpha\left(z, \tfrac{\partial}{\partial z}\right)\Big|_{P_\delta} &= \left(\tfrac{1}{2}\right)^{|\alpha|} \tfrac{1}{d_\alpha}\left[{\delta \atop \alpha}\right] \\
q_\alpha\left(z, \tfrac{\partial}{\partial z}\right)\Big|_{P_\delta} &= \textstyle\sum_\beta \left(-\tfrac{1}{2}\right)^{|\beta|} \tfrac{1}{d_\beta}\left[{\alpha \atop \beta}\right]\left[{\delta \atop \beta}\right] \\
p_\alpha\left(\tfrac{\partial}{\partial z}, z\right)\Big|_{P_\delta} &= \left(\tfrac{1}{2}\right)^{|\alpha|} \textstyle\sum_\beta \tfrac{1}{d_\beta}\left[{\alpha \atop \beta}\right]\left[{\delta \atop \beta}\right] \\
q_\alpha\left(\tfrac{\partial}{\partial z}, z\right)\Big|_{P_\delta} &= \left(\tfrac{1}{2}\right)^{|\alpha|} \textstyle\sum_\beta \tfrac{(-1)^{|\beta|}}{d_\beta}\left[{\alpha \atop \beta}\right]\left[{\delta \atop \beta}\right] \\
Sym\left(p_\alpha\left(z, \tfrac{\partial}{\partial z}\right)\right)\Big|_{P_\delta} &= \left(\tfrac{1}{4}\right)^{|\alpha|} \textstyle\sum_\beta \tfrac{2^{|\beta|}}{d_\beta}\left[{\alpha \atop \beta}\right]\left[{\delta \atop \beta}\right] \\
Sym\left(q_\alpha\left(z, \tfrac{\partial}{\partial z}\right)\right)\Big|_{P_\delta} &= \left(\tfrac{3}{4}\right)^{|\alpha|} \textstyle\sum_\beta \left(-\tfrac{2}{3}\right)^{|\beta|} \tfrac{1}{d_\beta}\left[{\alpha \atop \beta}\right]\left[{\delta \atop \beta}\right].
\end{aligned}$$

5. Examples

In this paper we have studied various combinatorial problems that arise in connection with a multiplicity free action and have related these to analysis with an associated Gelfand pair. We have seen that these problems are related to each other in the sense that their solutions can all be expressed in terms of the generalized binomial coefficients for the action. In this section we consider the explicit determination of generalized binomial coefficients for some specific examples of multiplicity free actions. We begin with the usual action of $K = U(n)$ on $V = \mathbb{C}^n$. This is the simplest and most classical example. Here the generalized binomial coefficients coincide with the usual binomial coefficients, motivating the terminology.

5.1. $\mathbf{U(n)}$. Consider the standard action of $K = U(n)$ on $V = \mathbb{C}^n$. Here decomposition 1.1 reads $\mathbb{C}[V] = \sum_{m=0}^{\infty} \mathcal{P}_m(V)$ and the $U(n)$-invariant polynomial p_m associated with $\mathcal{P}_m(V)$ is $p_m = \frac{(n-1)!}{(m+n-1)!}\gamma^m$ (see Proposition 6.2 in [BJR92]). Theorem 3.1 shows that in this case we have

$$\begin{bmatrix} m+k \\ m \end{bmatrix} = \begin{pmatrix} m+k \\ m \end{pmatrix}, \tag{5.1}$$

so that the generalized binomial coefficients are the standard binomial coefficients.

Using Equation 1.4 and our formulas for p_m and $\left[{m \atop j}\right]$ gives

$$q_m(z) = \sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix} p_j(z) = (n-1)! \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} \frac{(-\gamma(z))^j}{(j+n-1)!} = L_m^{(n-1)}(\gamma(z)).$$

Here $L_m^{(n-1)}$ is the *generalized Laguerre polynomial* of degree m and order $n-1$, normalized so that $L_m^{(n-1)}(0) = 1$. This result is well known. As a consequence of Proposition 3.3, we also see that the $(m+1) \times (m+1)$-matrix $\left[(-1)^j \binom{i}{j}\right]$ is self-inverting.

5.2. $\mathbf{SO(n, \mathbb{R}) \times \mathbb{T}}$. Next we consider the usual action of the group $K = SO(n, \mathbb{R}) \times \mathbb{T}$ on $V = \mathbb{C}^n$. A detailed treatment of this example appeared in [BJR93]. The multiplicity free decomposition of $\mathbb{C}[V]$ under the action of K is related to the theory of spherical harmonics. Let $\varepsilon(z) := z_1^2 + \cdots + z_n^2$ and $\mathcal{H}_k := \{p \in \mathcal{P}_k(V) \mid \frac{\partial^2 p}{\partial z_1^2} + \cdots \frac{\partial^2 p}{\partial z_n^2} = 0\}$ be the homogeneous "ε-harmonic" polynomials of degree k. Decomposition 1.1 can be written as

$$\mathbb{C}[V] = \sum_{(k,\ell) \in \Lambda} P_{k,\ell}$$

where $\Lambda = \mathbb{N}^2$ and $P_{k,\ell} = \mathcal{H}_k \varepsilon^\ell$. Note that $|(k,\ell)| = k + 2\ell$ and that $d_{k,\ell} = \dim(P_{k,\ell})$ is given by

$$d_{k,\ell} = d_k := \dim(\mathcal{H}_k) = \begin{pmatrix} n+k-1 \\ k \end{pmatrix} - \begin{pmatrix} n+k-3 \\ k-2 \end{pmatrix}$$

for all ℓ. We adopt the notation $\gamma_1(z) = \gamma(z) = |z|^2/2$ and $\gamma_2(z) = |\varepsilon(z)|^2/4$, so that $\gamma_1, \gamma_2 \in \mathbb{C}[V_{\mathbb{R}}]^K$ have degrees 2 and 4 respectively. It is shown in [BJR93] that

$$p_{k,\ell} = \frac{1}{c_{k,\ell}} p_k \gamma_2^\ell \tag{5.2}$$

where $p_k := p_{k,0}$ and

$$c_{k,\ell} := 4^\ell (\ell!)^2 \binom{k+\frac{n}{2}+\ell-1}{\ell}. \tag{5.3}$$

Note that the $c_{k,\ell}$'s also depend on the dimension n of V.

We wish to compute the generalized binomial coefficients $\left[{K,L \atop k,\ell}\right]$ for indices (K,L), $(k,\ell) \in \Lambda$. First note that $\left[{K,L \atop k,\ell}\right] = 0$ when $K+2L < k+2\ell$. Thus we suppose below that $K+2L \geq k+2\ell$. Theorem 3.1 shows that $\left[{K,L \atop k,\ell}\right]$ is the coefficient of $d_K p_{K,L}$ in $\left(\frac{\gamma_1^m}{m!}\right) d_k p_{k,\ell}$ for $m = (K+2L)-(k+2\ell)$. Since $p_{K,L}$ is divisible by γ_2^L and $p_{k,\ell}$ is divisible by γ_2^ℓ, we conclude that

$$\begin{bmatrix} K,L \\ k,\ell \end{bmatrix} = 0 \quad \text{for } L < \ell.$$

Theorem 3.1 thus gives

$$\begin{aligned} \frac{\gamma_1^m}{m!} d_k p_{k,\ell} &= \sum_{j=0}^{\lfloor \frac{k+m}{2} \rfloor} \begin{bmatrix} k+m-2j, \ell+j \\ k,\ell \end{bmatrix} d_{k+m-2j} p_{k+m-2j,\ell+j} \\ &= \sum_{j=0}^{\lfloor \frac{k+m}{2} \rfloor} \begin{bmatrix} k+m-2j, \ell+j \\ k,\ell \end{bmatrix} \frac{d_{k+m-2j}}{c_{k+m-2j,\ell+j}} p_{k+m-2j} \gamma_2^{\ell+j}. \end{aligned}$$

We can however also write

$$\begin{aligned} \frac{\gamma_1^m}{m!} d_k p_{k,\ell} = \frac{\gamma_2^\ell}{c_{k,\ell}} \frac{\gamma_1^m}{m!} d_k p_k &= \frac{\gamma_2^\ell}{c_{k,\ell}} \sum_{j=0}^{\lfloor \frac{k+m}{2} \rfloor} \begin{bmatrix} k+m-2j, j \\ k,0 \end{bmatrix} d_{k+m-2j} p_{k+m-2j,j} \\ &= \sum_{j=0}^{\lfloor \frac{k+m}{2} \rfloor} \begin{bmatrix} k+m-2j, j \\ k,0 \end{bmatrix} \frac{d_{k+m-2j}}{c_{k,\ell} c_{k+m-2j,j}} p_{k+m-2j} \gamma_2^{\ell+j}. \end{aligned}$$

Comparing these expressions for $\gamma_1^m d_k p_{k,\ell}/m!$ yields

$$\begin{bmatrix} K,L \\ k,\ell \end{bmatrix} = \frac{c_{K,L}}{c_{k,\ell} c_{K,L-\ell}} \begin{bmatrix} K,L-\ell \\ k,0 \end{bmatrix}$$

for $L \geq \ell$. Thus it now suffices to compute the generalized binomial coefficients of the form $\left[{K,L \atop k,0}\right]$ with $k \leq K+2L$.

Since $\gamma_1^k/k! = \sum_{j=1}^{\lfloor k/2\rfloor} d_{k-2j}p_{k-2j,j}$, we can use Theorem 3.1 to write

$$\begin{aligned}\frac{\gamma_1^{k+m}}{k!m!} &= \sum_{j=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{\ell=0}^{\lfloor\frac{k+m}{2}-j\rfloor}\begin{bmatrix} m+k-2j-2\ell, j+\ell \\ k-2j, j\end{bmatrix} d_{m+k-2j-2\ell}p_{m+k-2j-2\ell,j+\ell} \\ &= \sum_{L=0}^{\lfloor\frac{k+m}{2}\rfloor}\sum_{j=0}^{\min(\lfloor\frac{k}{2}\rfloor,L)}\begin{bmatrix} m+k-2L, L \\ k-2j, j\end{bmatrix} d_{m+k-2L}p_{m+k-2L,L}.\end{aligned}$$

We also have

$$\frac{\gamma_1^{k+m}}{k!m!} = \binom{k+m}{k}\frac{\gamma_1^{k+m}}{(k+m)!} = \binom{k+m}{k}\sum_{L=0}^{\lfloor\frac{k+m}{2}\rfloor} d_{k+m-2L}p_{k+m-2L,L}.$$

Comparing these expressions and replacing $k+m$ by K gives

$$\binom{K}{k} = \sum_{j=0}^{\min(\lfloor\frac{k}{2}\rfloor,L)}\begin{bmatrix} K-2L, L \\ k-2j, j\end{bmatrix} = \sum_{j=0}^{\min(\lfloor\frac{k}{2}\rfloor,L)}\frac{c_{K-2L,L}}{c_{k-2j,j}c_{K-2L,L-j}}\begin{bmatrix} K-2L, L-j \\ k-2j, 0\end{bmatrix}$$

for $k \le K$ and each $L = 0,\ldots,\lfloor K/2\rfloor$. In particular, letting $L=0$ gives

$$\begin{bmatrix} K,0 \\ k,0\end{bmatrix} = \binom{K}{k}$$

for all $k \le K$. Moreover, for $L = 1,\ldots,\lfloor K/2\rfloor$ we have

$$\begin{bmatrix} K-2L, L \\ k,0\end{bmatrix} = \binom{K}{k} - \sum_{j=1}^{\min(\lfloor\frac{k}{2}\rfloor,L)}\frac{c_{K-2L,L}}{c_{k-2j,j}c_{K-2L,L-j}}\begin{bmatrix} K-2L, L-j \\ k-2j, 0\end{bmatrix}$$

or equivalently

$$\begin{bmatrix} K, L \\ k,0\end{bmatrix} = \binom{K+2L}{k} - \sum_{j=1}^{\min(\lfloor\frac{k}{2}\rfloor,L)}\frac{c_{K,L}}{c_{k-2j,j}c_{K,L-j}}\begin{bmatrix} K, L-j \\ k-2j, 0\end{bmatrix}$$

for $k \le K+2L$ and $L \ge 1$. This provides a recurrence relation that reduces the calculation of $\left[\begin{smallmatrix} K,L \\ k,0\end{smallmatrix}\right]$ to that for coefficients $\left[\begin{smallmatrix} K,L' \\ k',0\end{smallmatrix}\right]$ with values $L' < L$ and $k' < k$. Repeated application of the recurrence terminates with initial conditions of the form $\left[\begin{smallmatrix} K,0 \\ k,0\end{smallmatrix}\right] = \binom{K}{k}$ and $\left[\begin{smallmatrix} K,L \\ 0,0\end{smallmatrix}\right] = 1$. We summarize our results below in Theorem 5.1. This provides an algorithm to compute all generalized binomial coefficients for this example.

Theorem 5.1. *The generalized binomial coefficients* $\left[\begin{smallmatrix} K,k \\ L,\ell\end{smallmatrix}\right]$ *for the action of* $SO(n,\mathbb{R})\times\mathbb{T}$ *on* $\mathbb{C}^n$ *are determined as follows.*

- $\left[\begin{smallmatrix} K,L \\ k,\ell\end{smallmatrix}\right] = 0$ *if either* $K+2L < k+2\ell$ *or* $L < \ell$.
- $\left[\begin{smallmatrix} K,L \\ 0,0\end{smallmatrix}\right] = 1$ *and* $\left[\begin{smallmatrix} K,0 \\ k,0\end{smallmatrix}\right] = \binom{K}{k}$ *for* $K \ge k$.

- $\left[{K,L \atop k,\ell}\right] = \frac{c_{K,L}}{c_{k,\ell}c_{K,L-\ell}}\left[{K,L-\ell \atop k,0}\right]$ *for* $L \geq \ell$.
- $\left[{K,L \atop k,0}\right] = \binom{K+2L}{k} - \sum_{j=1}^{\min(\lfloor\frac{k}{2}\rfloor,L)} \frac{c_{K,L}}{c_{k-2j,j}c_{K,L-j}}\left[{K,L-j \atop k-2j,0}\right]$ *for* $k \leq K+2L$ *and* $L \geq 1$.

Here the values $c_{k,\ell}$ *are given by Equation 5.3.*

5.3. $\mathbf{U(n)} \otimes \mathbf{U(n)}$. The group $K = U(n) \times U(n)$ and its complexification $K_C = GL(n,\mathbb{C}) \times GL(n,\mathbb{C})$ act on the space $V = M_{n,n}(\mathbb{C})$ of $n \times n$ complex matrices via

$$(k_1,k_2) \cdot Z = k_1 Z k_2^t.$$

The decomposition of $\mathbb{C}[V]$ under the action of K (and of K_C) is classical and elegant. (See for example [How95].) Let Λ denote the set of Young's diagrams with at most n rows and write σ_α for the irreducible representation of $U(n)$ (or $GL(n,\mathbb{C})$) given by the Young's diagram $\alpha \in \Lambda$. One has

$$\mathbb{C}[V] = \sum_{\alpha\in\Lambda} P_\alpha$$

where the representation of $U(n) \times U(n)$ (or $(GL(n,\mathbb{C}) \times GL(n,\mathbb{C}))$ on P_α is equivalent to $\sigma_\alpha^* \otimes \sigma_\alpha^*$, the outer tensor product of two copies of the representation σ_α^* contragredient to σ_α. A highest weight vector in P_α is given by $w_1^{j_1} w_2^{j_2} \ldots w_m^{j_m}$ where j_ℓ is the number of columns in D having length ℓ and

$$w_\ell(Z) := \begin{vmatrix} z_{11} & \cdots & z_{1\ell} \\ \vdots & & \vdots \\ z_{\ell 1} & \cdots & z_{\ell\ell} \end{vmatrix} \quad \text{for } \ell = 1,\ldots,m.$$

The degree $|\alpha|$ of the polynomials in P_α is given by the size (total number of boxes) of the Young's diagram α. Let $H \cong (\mathbb{R}^+)^n$ denote the set of $n \times n$ diagonal matrices with (strictly) positive real entries. Note that $H \subset GL(n,\mathbb{C})$ and that $H \subset V$. The character $tr(\sigma_\alpha(h))$ for the representation σ_α at a matrix $h = diag(d_1,\ldots,d_n) \in H$ is given by

$$tr(\sigma_\alpha(h)) = s_\alpha(d_1,\ldots,d_n)$$

where s_α is the *Schur polynomial* in n variables obtained from the Young's diagram α. (See for example [Mac95].) The following result asserts that the polynomials $p_\alpha \in \mathbb{C}[V_\mathbb{R}]^K$ are also "essentially" Schur polynomials.

Theorem 5.2. *The polynomial* $p_\alpha \in \mathbb{C}[V_\mathbb{R}]^K$ *is completely determined by its restriction to the set* $H \cong (\mathbb{R}^+)^n$ *in* $V = M_{n,n}(\mathbb{C})$*. Moreover, for* $h = diag(d_1,\ldots,d_n) \in H$ *we have*

$$p_\alpha(h) = c_\alpha s_\alpha(d_1^2,\ldots,d_n^2)$$

where s_α *is the Schur polynomial for the Young's diagram* $\alpha \in \Lambda$ *and* c_α *is the positive constant*

$$c_\alpha = \frac{p_\alpha(I)}{\dim(\sigma_\alpha)}.$$

Proof. The polynomial p_α is determined by its restriction to the open dense set of non-singular matrices Z in V. Given a non-singular $Z \in V$, the matrix ZZ^* is Hermitian with positive real eigenvalues, $a_1, \dots, a_n \in \mathbb{R}^+$ say. One has

$$p_\alpha(Z) = p_\alpha\left(diag\left(\sqrt{a_1}, \dots, \sqrt{a_n}\right)\right).$$

Indeed, the spectral theorem ensures that one can find a matrix $k_1 \in U(n)$ with $k_1 Z Z^* k_1^{-1} = (k_1 Z)(k_1 Z)^* = diag(a_1, \dots, a_n)$. Thus $k_1 Z$ has pair-wise orthogonal rows with norms $\sqrt{a_1}, \dots, \sqrt{a_n}$. We see that $k_2 := diag\left(1/\sqrt{a_1}, \dots, 1/\sqrt{a_n}\right) k_1 Z$ is a unitary matrix. Thus $k_1 Z k_2^{-1} = diag\left(\sqrt{a_1}, \dots, \sqrt{a_n}\right)$ and one obtains $p_\alpha(Z) = p_\alpha\left(diag\left(\sqrt{a_1}, \dots, \sqrt{a_n}\right)\right)$ by K-invariance of p_α.

Let σ_α be realized in a Hermitian vector space V_α with $\dim(V_\alpha) = m$. The operators $\{\sigma_\alpha(h) \mid h \in H\}$ are simultaneously diagonalizable with positive real eigenvalues. Let $\{v_1, \dots, v_m\}$ be an orthonormal basis for V_α consisting of eigenvectors for the operators $\sigma_\alpha(h)$ $(h \in H)$. We thus have homomorphisms $\sigma_{\alpha,i} : H \to \mathbb{R}^+$ with

$$\sigma_\alpha(h) v_i = \sigma_{\alpha,i}(h) v_i$$

for $i = 1, \dots, n$. The associated dual basis $\{v_1^*, \dots, v_n^*\}$ for V_α^* consists of eigenvectors for the operators $\{\sigma_\alpha^*(h) \mid h \in H\}$ obtained from the contragredient representation for σ_α. Indeed, we have

$$\sigma_\alpha^*(h) v_i^* = \sigma_{\alpha,i}(h^{-1}) v_i^*.$$

Let $w_{i,j} \in P_\alpha$ be the polynomial that corresponds to $v_i^* \otimes v_j^* \in V_\alpha^* \otimes V_\alpha^*$ under a unitary isomorphism $P_\alpha \cong V_\alpha^* \otimes V_\alpha^*$ that intertwines the representation of K_C on P_α with $\sigma_\alpha^* \otimes \sigma_\alpha^*$. $\{w_{i,j} \mid i,j = 1, \dots, m\}$ is a basis for P_α and for $h_1, h_2 \in H, Z \in V$ one has

$$w_{i,j}(h_1 Z h_2) = w_{i,j}(h_1 Z h_2^t) = \left(\left(h_1^{-1}, h_2^{-1}\right) \cdot w_{i,j}\right)(Z) = \sigma_{\alpha,i}(h_1)\sigma_{\alpha,j}(h_2) w_{i,j}(Z).$$

Since $p_\alpha = \frac{1}{m^2} \sum_{i,j} |w_{i,j}|^2$, we see that

$$p_\alpha(h_1 Z h_2) = \frac{1}{m^2} \sum_{i,j} \sigma_{\alpha,i}(h_1)^2 \sigma_{\alpha,j}(h_2)^2 |w_{i,j}(Z)|^2 = \frac{1}{m^2} \sum_{i,j} \sigma_{\alpha,i}(h_1^2) \sigma_{\alpha,j}(h_2^2) |w_{i,j}(Z)|^2$$

for $h_1, h_2 \in H, Z \in V$.

There is an alternate formula for p_α. One can always replace the average in the definition of p_α by an integral over the compact group K (see [BJR92]). In the present case this gives

$$\int_{U(n)} \int_{U(n)} \left|w_{i,j}(k_1 Z k_2^t)\right| dk_1 dk_2 = p_\alpha(Z)$$

for each i, j. Here we use normalized Haar measure on $U(n)$. Thus we can write

$$\begin{aligned}\int_{U(n)}\int_{U(n)} & p_\alpha(h_1k_1Zk_2^th_2)dk_1dk_2 \\ &= \frac{1}{m^2}\sum_{i,j}\sigma_{\alpha,i}(h_1^2)\sigma_{\alpha,j}(h_2^2)\int_{U(n)}\int_{U(n)}|w_{i,j}(k_1Zk_2^t)|^2dk_1dk_2 \\ &= \frac{p_\alpha(Z)}{m^2}\sum_{i,j}\sigma_{\alpha,i}(h_1^2)\sigma_{\alpha,j}(h_2^2)\end{aligned}$$

for $h_1, h_2 \in H, Z \in V$. Setting $h_1 = I = Z$ and $h_2 = h = diag(d_1, \ldots, d_n) \in H$ in this equation and using K-invariance of p_α gives

$$p_\alpha(h) = \frac{p_\alpha(I)}{m^2}\sum_{i,j}\sigma_{\alpha,j}(h^2) = \frac{p_\alpha(I)}{m}\sum_j\sigma_{\alpha,j}(h^2) = \frac{p_\alpha(I)}{\dim(\sigma_\alpha)}s_\alpha(d_1^2, \ldots, d_n^2).$$

Since $p_\alpha \neq 0$, the nonnegative constant $c_\alpha := p_\alpha/\dim(\sigma_\alpha)$ is necessarily positive. This completes the proof. □

Given two Young's diagrams $\alpha, \beta \in \Lambda$, we write $\beta \subset \alpha$ when β is a sub-diagram of α. That is, α has at least as many boxes as does β on each row. If $|\alpha| = |\beta| + k$ then $\mathcal{C}_{\alpha\beta}$ will denote the number of sequences $(\varepsilon_0, \ldots, \varepsilon_k)$ of Young's diagrams such that

- $\beta = \varepsilon_0 \subset \varepsilon_1 \subset \cdots \subset \varepsilon_{k-1} \subset \varepsilon_k = \alpha$, and
- $|\varepsilon_j| = |\beta| + j$ for $j = 1, \ldots, k$.

That is, ε_j is obtained from ε_{j-1} by adding a single box $\square$ to some row. Note that $\mathcal{C}_{\alpha\beta} = 0$ if $\beta \not\subset \alpha$. When $\beta \subset \alpha$, $\mathcal{C}_{\alpha\beta}$ is the number of *standard tableaux* of shape $\alpha - \beta$. That is, the number of ways to assign the values $1, 2, \ldots, k$ to the boxes of the skew-diagram $\alpha - \beta$ so that values increase as we move along rows from left to right and as we move down columns [Mac95].

Theorem 5.3. *For $\alpha, \beta \in \Lambda$ with $|\alpha| = |\beta| + k$ we have*

$$\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \frac{p_\beta(I)\dim(\sigma_\beta)}{2^kk!p_\alpha(I)\dim(\sigma_\alpha)}\mathcal{C}_{\alpha\beta}.$$

In particular, $\left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right] \neq 0$ if and only if $\beta \subset \alpha$.

We note that $\dim(\sigma_\alpha)$ is given by the formula

$$\dim(\sigma_\alpha) = \prod_{i<j}\frac{\alpha_i - \alpha_j + j - i}{j - i}$$

where α_j denotes the length of the j'th row in α. Thus Theorem 5.3 reduces the computation of the generalized binomial coefficients for this example to the classical problem of counting standard tableaux and evaluation of the p_α polynomials at a single point $I \in V$. For other viewpoints on Theorem 5.3, we refer the reader to [KS96] and [OO97].

Proof of Theorem 5.3. First suppose that $|\alpha| = |\beta| + 1$. Theorem 3.1 asserts that

$$d_\beta \gamma p_\beta = \sum_{|\alpha|=|\beta|+1} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} d_\alpha p_\alpha \tag{5.4}$$

where $d_\alpha = \dim(P_\alpha) = \dim(\sigma_\alpha)^2$ and $d_\beta = \dim(P_\beta) = \dim(\sigma_\beta)^2$. Theorem 5.2 shows that

$$p_\beta(h) = \frac{p_\beta(I)}{\dim(\sigma_\beta)} s_\beta(d_1^2, \ldots, d_n^2) \quad \text{and} \quad p_\alpha(h) = \frac{p_\alpha(I)}{\dim(\sigma_\alpha)} s_\alpha(d_1^2, \ldots, d_n^2)$$

for $h = diag(d_1, \ldots, d_n) \in H$. Moreover,

$$\gamma(h) = \frac{d_1^2 + \cdots + d_n^2}{2} = \frac{1}{2} s_{\square}(d_1^2, \ldots, d_n^2)$$

where $\square$ denotes the Young's diagram consisting of a single box. Thus Equation 5.4 yields

$$s_{\square}(d_1^2, \ldots, d_n^2) s_\beta(d_1^2, \ldots, d_n^2) = \sum_{|\alpha|=|\beta|+1} \frac{2\dim(\sigma_\alpha) p_\alpha(I)}{\dim(\sigma_\beta) p_\beta(I)} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} s_\alpha(d_1^2, \ldots, d_n^2).$$

On the other hand, a special case of the *Pieri formula* for Schur polynomials (see [Mac95], page 331) gives

$$s_{\square} s_\beta = \sum_{\substack{|\alpha|=|\beta|+1 \\ \beta \subset \alpha}} s_\alpha.$$

Thus we conclude that

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \left\{ \begin{array}{ll} \frac{p_\beta(I)\dim(\sigma_\beta)}{2p_\alpha(I)\dim(\sigma_\alpha)} & \text{if } \beta \subset \alpha \\ 0 & \text{if } \beta \not\subset \alpha \end{array} \right\} = \frac{p_\beta(I)\dim(\sigma_\beta)}{2p_\alpha(I)\dim(\sigma_\alpha)} \mathcal{C}_{\alpha\beta}.$$

The general case where $|\beta| = |\alpha| + k$ now follows by an easy application of Equation 3.10. $\square$

We conclude by noting that Theorems 5.2 and 5.3 can be generalized as follows. One considers the action of $K = U(n) \times U(m)$ on the space $V = M_{n,m}(\mathbb{C})$ of $n \times m$ matrices via $(k_1, k_2) \cdot Z = k_1 Z k_2^t$. We can assume without loss of generality that $m \leq n$. The decomposition of $\mathbb{C}[V]$ is parameterized by the set Λ of Young's diagrams with at most m rows. Let $H \subset V$ denote the set of matrices of the form $h = \left[\frac{D}{0}\right]$ where $D = diag(d_1, \ldots, d_m)$ with $d_j \in \mathbb{R}^+$. For $\alpha \in \Lambda$, p_α is determined by its restriction to H via the formula $p_\alpha(h) = p_\alpha(Z_\circ) s_\alpha(d_1^2, \ldots, d_m^2)/\dim(\sigma_\alpha^m)$. Here s_α is a Schur polynomial in m variables, $Z_\circ := \left[\frac{I}{0}\right]$ and σ_α^m is the irreducible representation of $U(m)$ given by $\alpha \in \Lambda$. The generalized binomial coefficients can be expressed as

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{p_\beta(Z_\circ)\dim(\sigma^n_\beta)}{2^k k! p_\alpha(Z_\circ)\dim(\sigma^n_\alpha)} \mathcal{C}_{\alpha\beta}$$

where σ^n_α is the irreducible representation of $U(n)$ given by $\alpha \in \Lambda$.

References

[BJR] C. Benson, J. Jenkins, and G. Ratcliff. The spherical transform of a Schwartz function on the Heisenberg group. (*to appear in J. Functional Analysis*).

[BJR92] C. Benson, J. Jenkins, and G. Ratcliff. Bounded K-spherical functions on Heisenberg groups. *J. Functional Analysis*, 105:409–443, 1992.

[BJR93] C. Benson, J. Jenkins, and G. Ratcliff. $O(n)$-spherical functions on Heisenberg groups. *Contemporary Math.*, 145:181–197, 1993.

[BJRW96] C. Benson, J. Jenkins, G. Ratcliff, and T. Worku. Spectra for Gelfand pairs associated with the Heisenberg group. *Colloq. Math.*, 71:305–328, 1996.

[BR96] C. Benson and G. Ratcliff. A classification for multiplicity free actions. *Journal of Algebra*, 181:152–186, 1996.

[Car87] G. Carcano. A commutativity condition for algebras of invariant functions. *Boll. Un. Mat. Italiano*, 7:1091–1105, 1987.

[Dib90] H. Dib. Fonctions de Bessel sur une algèbre de Jordan. *J. Math Pures et Appl.*, 69:403–448, 1990.

[FK94] J. Faraut and A. Koranyi. *Analysis on Symmetric Cones.* Oxford University Press, New York, 1994.

[How95] R. Howe. *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond*, volume 8 of *Israel Math. Conf. Proc.* Bar-Ilan Univ., Ramat Gan, 1995.

[HU91] R. Howe and T. Umeda. The Capelli identity, the double commutant theorem and multiplicity-free actions. *Math. Annalen*, 290:565–619, 1991.

[Kac80] V. Kac. Some remarks on nilpotent orbits. *Journal of Algebra*, 64:190–213, 1980.

[KS96] F. Knop and S. Sahi. Difference equations and symmetric polynomials defined by their zeroes. *International Math. Research Notes*, 10:473–486, 1996.

[Lea] A. Leahy. Ph.D. Thesis, Rutgers University.

[Mac95] I. G. Macdonald. *Symmetric Functions and Hall Polynomials, Second Edition.* Clarendon Press, Oxford, 1995.

[Ols] G. Olshanski. Quasi-symmetric functions and factorial Schur functions. (*preprint*).

[OO97] A. Okounkov and G. Olshanski. Shifted Jack polynomials, binomial formula, and applications. *Math. Res. Letters*, 4:69–78, 1997.

[Sah94] S. Sahi. The spectrum of certain invariant differential operators associated to Hermitian symmetric spaces. In J. L. Brylinski, editor, *Lie Theory and Geometry*, volume 123 of *Progress in Math.*, pages 569–576. Birkhäuser, Boston, 1994.

[Yan] Z. Yan. Special functions associated with multiplicity-free representations. (*preprint*).

[Yan92] Z. Yan. Generalized hypergeometric functions and Laguerre polynomials in two variables. *Contemporary Math.*, 138:239–259, 1992.

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