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Combinatorial Properties of Generalized Binomial Coefficients

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Abstract. The generalized binomial coefficients discussed in this paper were first studied in the context of spherical functions for Gelfand pairs associated with the Heisenberg group. We now define generalized binomial coefficients in a more general context, and show that they satisfy most of the combinatorial properties obtained for Gelfand pairs. The results depend on an isometric involution defined on polynomials on $\mathbb{C}^n$.

1. Introduction

Our initial study of generalized binomial coefficients involved the Heisenberg group $H_n = V \times \mathbb{R}$, $V \cong \mathbb{C}^n$ [**BR98**]. Any subgroup $K \subseteq U(V)$ acts on V, giving automorphisms of H_n. We say that $(K \ltimes H_n, K)$ is a **Gelfand pair** if the convolution algebra of K-bi-invariant functions on $K \ltimes H_n$ is commutative. Equivalently, the convolution algebra of K-invariant functions on H_n is commutative.

Generic representations of H_n act on Fock space $\mathcal{F}$, the completion of $\mathbb{C}[V]$ with respect to the inner product

$$\langle p, q \rangle = \int p(z)\overline{q(z)}e^{-|z|^2/2}\, d\tilde{z},$$

where the measure $d\tilde{z}$ on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ is normalized so that $\int e^{-|z|^2/2}\, d\tilde{z} = 1$. The unitary group acts on $\mathcal{F}$ by intertwining operators, namely $k \cdot p(z) = p(k^{-1}z)$. By a theorem of Carcano [**Car87**], $(K \ltimes H_n, K)$ is a Gelfand pair if and only if the action of K on $\mathbb{C}[V]$ is **multiplicity free**.

Spherical functions for the Gelfand pair $(K \ltimes H_n, K)$ are the K-invariant eigenfunctions for the K-invariant, left-H_n-invariant differential operators on H_n. We have several algorithms for computing the spherical functions. Let π be the standard representation of H_n on $\mathcal{F}$. Given $u, v \in \mathcal{F}$, define the matrix coefficient

$$\Phi(u,v)(z,t) = \langle \pi(z,t)u, v\rangle. \tag{1.1}$$

Let

$$\mathbb{C}[V] = \sum_{\alpha\in\Lambda} V_\alpha \tag{1.2}$$

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be the multiplicity free decomposition with respect to the action of K. Given a unit vector $v_\alpha \in V_\alpha$, define

$$\phi_\alpha(z,t) = \int_K \Phi(v_\alpha, v_\alpha)(kz,t)dk.$$

Then ϕ_α is a spherical function. Alternatively, let $d_\alpha = \dim V_\alpha$ and $\{v_i : i = 1, \ldots, d_\alpha\}$ be an orthonormal basis of V_α. Then

$$\phi_\alpha = \frac{1}{d_\alpha} \sum_{i=1}^{d_\alpha} \Phi(v_i, v_i).$$

Let $\mathcal{P}(V) = \mathbb{C}[V] \otimes \overline{\mathbb{C}[V]}$. Define two inner products on the space $\mathcal{P}(V)$:

$$\langle p, q\rangle_* = (p(2\partial, 2\overline{\partial})q)(0,0);$$

$$\langle p, q\rangle_{\mathcal{F}} = \int p(z,\overline{z})\overline{q(z,\overline{z})}e^{-|z|^2/2}d\tilde{z}.$$

We define a canonical basis for the K-invariant polynomials $\mathcal{P}(V)^K$ by

$$p_\alpha(z,\overline{z}) = \frac{1}{d_\alpha}\sum v_i(z)\overline{v_i(z)},$$

for each $\alpha \in \Lambda$. These invariant polynomials are related to the K-spherical functions by

$$\phi_\alpha(z,t) = e^{it}q_\alpha(z,\overline{z})e^{-|z|^2/4},$$

where $q_\alpha \in \mathcal{P}(V)^K$, $q_\alpha =$ const. p_α+L.O.T.

Thus the sets $\{p_\alpha : \alpha \in \Lambda\}$ and $\{q_\alpha : \alpha \in \Lambda\}$ are both bases for the space of K-invariant polynomials. The p_α's are orthogonal with respect to $\langle,\rangle_*$, while the q_α's are orthogonal with respect to $\langle,\rangle_{\mathcal{F}}$.

In the current work, we replace the space $\mathcal{P}(V)^K$ with a subspace $\mathcal{V} \subseteq \mathcal{P}(V)$ which is invariant under the Laplacian, and under multiplication by $|z|^2$. We define an involution $\mathcal{T} : \mathcal{V} \to \mathcal{V}$, which can be described in four ways. Given a $\langle,\rangle_*$-orthogonal basis of homogeneous polynomials $\{p_\alpha\}$, we define another family $\{q_\alpha = \mathcal{T}p_\alpha\}$. We write

$$q_\alpha = \sum_\beta \left[\begin{array}{c}\alpha\\ \beta\end{array}\right](-1)^{|\beta|}p_\beta,$$

where $|\beta|$ is the degree of homogeneity of p_β in $\overline{z}$. The coefficients $\left[\begin{array}{c}\alpha\\ \beta\end{array}\right]$ are called **generalized binomial coefficients**.

In Section 2, we discuss the motivating example of the Gelfand pair $(U(n) \ltimes H_n, U(n))$, where the p_α's are monomials, the q_α's are Laguerre polynomials, and the generalized binomial coefficients are the usual binomial coefficients.

The transformation $\mathcal{T}$ is introduced in Section 3, and we show that it can be defined in four very different ways. In Section 4, we derive the combinatorial properties of generalized binomial coefficients, which were previously known only for Gelfand pairs. [**BR98**].

We calculate generalized binomial coefficients for several examples in Section 5, namely the Gelfand pair $(U(n) \ltimes H_n, U(n))$, monomials, monomial symmetric functions, and Schur polynomials.

2. Motivating example

Let $V = \mathbb{C}^n$, $K = U(n)$. Then $\mathbb{C}[V] = \sum_{m\geq 0} V_m$, where V_m is the space of homogeneous polynomials of degree m. Throughout this paper, we will use multinomial notation, so that for $a, b \in \mathbb{N}^n$, $z^a = z_1^{a_1} \ldots z_n^{a_n}$, $|a| = a_1 + \ldots + a_n$, $a! = a_1! \ldots a_n!$, and $\binom{a}{b} = a!/b!(a-b)!$ provided $b_j \leq a_j$ for all j.

The monomials $\{z^a/\sqrt{2^{|a|}a!} : |a| = m\}$ form an orthonormal basis of V_m, yielding the canonical invariants $p_m = (n-1)!\gamma^m/2^m(m+n-1)!$, where $\gamma(z) = |z|^2$. The spherical functions are then given by $q_m = (n-1)!L_m^{n-1}(\gamma/2)$, where L_m^{n-1} is the Laguerre polynomial of degree m and order $n-1$. Explicitly $L_m^{n-1} = \sum_{j=0}^m \binom{m}{j}(-x)^j/(j+n-1)!$. (See (5.1) for a proof in the current context.) The two families of invariant polynomials are related by

$$q_m = \sum_{k=0}^{m} \binom{m}{k} (-1)^k p_k.$$

For general Gelfand pairs $(K \ltimes H_n, K)$, we have the decomposition (1.2). Writing $|\alpha| = m$ if $V_\alpha \subseteq V_m$, we have

$$q_\alpha = \sum_{|\beta|\leq|\alpha|} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (-1)^{|\beta|} p_\beta.$$

The coefficients $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ are called "Generalized Binomial Coefficients."

There are many interesting combinatorial properties of Generalized Binomial Coefficients which motivate this work. These were first derived in [**Yan**] [**BR98**], with a different treatment in [**BR04**].

3. General Setting

We replace the space $\mathcal{P}(V)^K$ with a subspace $\mathcal{V} \subseteq \mathcal{P}(V)$ satisfying:

(1) $\mathcal{V}$ has a $\langle,\rangle_*$-orthogonal basis of homogeneous polynomials $\{p_\alpha : \alpha \in \Lambda\}$. We write $|\alpha| = m$ if p_α is homogeneous of degree m in $\overline{z}$. The set Λ is given the partial order $\alpha < \beta$ if and only if $|\alpha| < |\beta|$.
(2) $\Delta : \mathcal{V} \to \mathcal{V}$, where $\Delta = \sum_{j=1}^n \partial_j \overline{\partial}_j$ is the real Laplacian.
(3) If $p \in \mathcal{V}$, then $\gamma p \in \mathcal{V}$, where $\gamma(z) = |z|^2$.

Now define the operator $\mathcal{T} : P(V) \to P(V)$ by

$$\mathcal{T}(p) = e^{-2\Delta} p(z, -\overline{z}).$$

Clearly, we have $\mathcal{T} : \mathcal{V} \to \mathcal{V}$. The operator $\mathcal{T}$ was defined in [**BR04**] in the context of multiplicity free actions. We have since realized that the results are applicable in this more general setting. We will show that there are three additional ways to define the operator $\mathcal{T}$. First, define the symplectic Fourier transform

$$\widehat{f}(w) = \int f(z) e^{-iIm(z\cdot\overline{w})} d\tilde{z}.$$

PROPOSITION 3.1. $(pe^{-\gamma/2})\widehat{\ } = \mathcal{T}(p)e^{-\gamma/2}$

PROOF. We calculate $\mathcal{T}(p)$ for monomials $p(z) = z^a\overline{z}^b$:

$$\begin{aligned}\mathcal{T}p(z,\overline{z}) &= (-1)^{|b|}e^{-2\Delta}z^a\overline{z}^b \\ &= (-1)^{|b|}\sum_c \frac{(-2)^{|c|}}{c!}\partial^c\overline{\partial}^c z^a\overline{z}^b \\ &= (-1)^{|b|}\sum_c \frac{(-2)^{|c|}a!b!}{c!(a-b)!(b-c)!}z^{a-c}\overline{z}^{b-c}\end{aligned}$$

On the other hand,

$$\begin{aligned}(pe^{-\gamma/2})\widehat{\ } &= (-2\overline{\partial})^a(2\partial)^b e^{-\gamma/2} \\ &= (2\partial)^b z^a e^{-\gamma/2} \\ &= \sum_c \binom{b}{c}((2\partial)^c z^a)\left((2\partial)^{b-c}e^{-\gamma/2}\right) \\ &= \sum_c \binom{b}{c}(2)^{|c|}\frac{a!}{(a-c)!}z^{a-c}(-\overline{z})^{b-c}e^{-\gamma/2} \\ &= \sum_c (-1)^{|b|}\frac{(-2)^{|c|}a!b!}{c!(b-c)!(a-c)!}z^{a-c}\overline{z}^{b-c}e^{-\gamma/2}\end{aligned}$$

□

Since the symplectic Fourier transform is its own inverse, we immediately see:

PROPOSITION 3.2. *$\mathcal{T}$ is an involution on $\mathcal{P}(V)$.*

We can also define $\mathcal{T}$ in terms of the matrix coefficients (1.1) for the Fock space representation of the Heisenberg group.

PROPOSITION 3.3.

$$\Phi(u,v) = \mathcal{T}(u\overline{v})e^{-\gamma/4}$$

PROOF. Note that our normalization of the measure results in $\int z^a\overline{z}^a e^{-|z|2/2}d\tilde{z} = 2^{|a|}a!$, and distinct monomials are orthogonal. Let $u(z) = z^a$, $v(z) = z^b$. Then

$$\begin{aligned}\Phi(u,v)(z) &= \langle \pi(z,0)u, v\rangle \\ &= \int \pi(z,0)u(w)\overline{v(w)}e^{-|w|^2/2}d\tilde{w} \\ &= \int u(w+z)e^{-|z|^2/4}e^{-\overline{z}\cdot w/2}\overline{v(w)}e^{-|w|^2/2}d\tilde{w} \\ &= e^{-|z|^2/4}\int (w+z)^a\overline{w}^b e^{-\overline{z}\cdot w/2}e^{-|w|^2/2}d\tilde{w} \\ &= e^{-|z|^2/4}\int \sum_c \binom{a}{c} w^c z^{a-c}\overline{w}^b e^{-\overline{z}\cdot w/2}e^{-|w|^2/2}d\tilde{w} \\ &= e^{-|z|^2/4}\int \sum_c \binom{a}{c} w^c z^{a-c}\overline{w}^b \sum_l \frac{(-\overline{z})^l(w/2)^l}{l!}e^{-|w|^2/2}d\tilde{w} \\ &= e^{-|z|^2/4}\sum_{c,l}\binom{a}{c}\frac{(-1)^{|l|}}{2^{|l|}l!}\int w^c z^{a-c}\overline{w}^b\overline{z}^l w^l e^{-|w|^2/2}d\tilde{w}\end{aligned}$$

$$
\begin{aligned}
&= e^{-|z|^2/4} \sum_{c,l} \binom{a}{c} \frac{(-1)^{|l|}}{2^{|l|} l!} z^{a-c} \overline{z}^{l} \int w^{c+l} \overline{w}^{b} e^{-|w|^2/2} d\tilde{w} \\
&= \sum_{c} \binom{a}{c} \frac{(-1)^{|b-c|}}{2^{|b-c|}(b-c)!} z^{a-c} \overline{z}^{b-c} 2^{|b|} b! e^{-|z|^2/4} \\
&= (-1)^{|b|} \sum_{c} \frac{(-2)^{|c|} a! b!}{c!(a-c)!(b-c)!} z^{a-c} \overline{z}^{b-c} e^{-|z|^2/4} \\
&= \mathcal{T}(u\overline{v}) e^{-|z|^2/4},
\end{aligned}
$$

as shown in the proof of Proposition 3.1. □

PROPOSITION 3.4. *$\mathcal{T} : (\mathcal{P}(V), \langle, \rangle_*) \to (\mathcal{P}(V), \langle, \rangle_{\mathcal{F}})$ is an isometry.*

PROOF.

$$
\begin{aligned}
\langle \mathcal{T}(u\overline{v}), \mathcal{T}(u'\overline{v'}) \rangle_{\mathcal{F}} &= \langle \Phi(u,v) e^{\gamma/4}, \Phi(u,v) e^{\gamma/4} \rangle_{\mathcal{F}} \\
&= \langle \Phi(u,v), \Phi(u',v') \rangle_{L^2} \\
&= \langle u, u' \rangle \langle v, v' \rangle \\
&= \langle u\overline{v}, u'\overline{v'} \rangle_*
\end{aligned}
$$

□

PROPOSITION 3.5. *The family $\{\mathcal{T} p_\alpha : \alpha \in \Lambda\}$ is obtained, up to constants, from $\{p_\alpha : \alpha \in \Lambda\}$ by Gram-Schmidt orthogonalization with respect to the inner product $\langle, \rangle_{\mathcal{F}}$ and the partial order.*

PROOF. Note that $\mathcal{T} p_\alpha = (-1)^{|\alpha|} e^{-2\Delta} p_\alpha = (-1)^{|\alpha|} p_\alpha$+L.O.T. Thus span$\{p_\beta : |\beta| \leq |\alpha|\} =$ span$\{\mathcal{T} p_\beta : |\beta| \leq |\alpha|\}$. Since the p_α's are orthogonal with respect to $\langle, \rangle_*$, the $\mathcal{T} p_\alpha$'s will be orthogonal with respect to $\langle, \rangle_{\mathcal{F}}$. □

4. Combinatorial results

Since p_α is homogeneous, we have

$$\mathcal{T}(p_\alpha) = (-1)^{|\alpha|} e^{-2\Delta} p_\alpha.$$

Define a new family of polynomials $\{q_\alpha\} \subseteq \mathcal{V}$ and new generalized binomial coefficients by

$$q_\alpha = \mathcal{T}(p_\alpha) = \sum_{|\beta| \leq |\alpha|} \left[\begin{array}{c} \alpha \\ \beta \end{array} \right] (-1)^{|\beta|} p_\beta.$$

These new generalized binomial coefficients satisfy most of the combinatorial properties of the original generalized binomial coefficients.

Since T is an involution, we immediately obtain:

PROPOSITION 4.1.

$$p_\alpha = \sum_{|\beta| \leq |\alpha|} \left[\begin{array}{c} \alpha \\ \beta \end{array} \right] (-1)^{|\beta|} q_\beta$$

PROPOSITION 4.2.

$$\frac{(2\Delta)^k}{k!} p_\alpha = \sum_{|\beta| = |\alpha| - k} \left[\begin{array}{c} \alpha \\ \beta \end{array} \right] p_\beta$$

Proof. Since the p_α's are a homogeneous basis for $\mathcal{V}$, we know that

$$\frac{(2\Delta)^k}{k!}p_\alpha = \sum_{|\beta|=|\alpha|-k} c_{\alpha,\beta}p_\beta$$

for some coefficients $c_{\alpha,\beta}$.

Thus

$$\begin{aligned}\mathcal{T}(p_\alpha) &= (-1)^{|\alpha|}\sum_k \frac{(-2\Delta)^k}{k!}p_\alpha \\ &= \sum_k (-1)^{|\alpha|+k}\sum_{|\beta|=|\alpha|-k} c_{\alpha,\beta}p_\beta \\ &= \sum_\beta (-1)^{|\beta|}c_{\alpha,\beta}p_\beta,\end{aligned}$$

and hence $c_{\alpha,\beta} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$. □

Proposition 4.3. *"Pieri Formula"*

$$\frac{(2\gamma)^k}{k!}d_\beta p_\beta = \sum_{|\alpha|=|\beta|+k} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} d_\alpha p_\alpha,$$

where $1/d_\alpha = \langle p_\alpha, p_\alpha\rangle_*$.

Proof. Let

$$\frac{(2\gamma)^k}{k!}d_\beta p_\beta = \sum_{|\alpha|=|\beta|+k} c_{\alpha,\beta}d_\alpha p_\alpha.$$

Then by orthogonality, we obtain

$$\begin{aligned}c_{\alpha,\beta} &= \left\langle \frac{(2\gamma)^k}{k!}d_\beta p_\beta, p_\alpha\right\rangle_* \\ &= \left\langle d_\beta p_\beta, \frac{(2\Delta)^k}{k!}p_\alpha\right\rangle_* \\ &= \left\langle d_\beta p_\beta, \sum_{|\delta|=|\alpha|-k}\begin{bmatrix} \alpha \\ \delta \end{bmatrix} p_\delta\right\rangle_* \\ &= \begin{bmatrix} \alpha \\ \beta \end{bmatrix}\end{aligned}$$

□

Writing $(2\gamma)^m/m!$ as $(2\gamma)^k/k!\cdot(2\gamma)^{m-k}/(m-k)!\cdot\binom{m}{k}^{-1}$, Proposition 4.3 gives us:

Proposition 4.4. *For any* l *with* $|\delta| < l < |\alpha|$,

$$\sum_{|\beta|=\ l}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\begin{bmatrix} \beta \\ \delta \end{bmatrix} = \binom{|\alpha|-|\delta|}{|\alpha|-l}\begin{bmatrix} \alpha \\ \delta \end{bmatrix}.$$

By repeatedly multiplying p_β by 2γ, we also obtain:

Proposition 4.5.

$$\left[\begin{array}{c}\alpha\\ \beta\end{array}\right]=\sum\left[\begin{array}{c}\alpha\\ \delta_1\end{array}\right]\left[\begin{array}{c}\delta_1\\ \delta_2\end{array}\right]\cdots\left[\begin{array}{c}\delta_{|\beta|-1}\\ \beta\end{array}\right],$$

where the sum is taken over all $(\delta_1,\ldots,\delta_{|\beta|-1})$ with $|\delta_k|=|\alpha|-k$.

5. Examples

5.1. Motivating Example Revisited. The motivating example for much of this work is the Gelfand pair $(U(n)\ltimes H_n, U(n))$. We take the canonical basis for $\mathcal{P}(V)^{U(n)}$, namely $\{(n-1)!|z|^{2m}/2^m(m+n-1)!\}$. The corresponding set of invariants is $q_m=(n-1)!L_m^{n-1}(|z|^2/2)$. For completeness, we provide the proof. Let $\gamma=|z|^2$, so that $p_m=(n-1)!\gamma^m/2^m(m+n-1)!$ Then

$$\begin{aligned}\Delta(\gamma^m)&=\sum_j\partial_j\overline{\partial}_j(\gamma^m)\\&=\sum_j\partial_j(mz_j\gamma^{m-1})\\&=\sum_j(m\gamma^{m-1}+m(m-1)z_j\overline{z}_j\gamma^{m-2})\\&=m(n+m-1)\gamma^{m-1}.\end{aligned}$$

$$\begin{aligned}\mathcal{T}((n-1)!\gamma^m/2^m(m+n-1)!)&=(-1)^m(n-1)!e^{-2\Delta}\gamma^m/2^m(m+n-1)!\\&=\frac{(-1)^m(n-1)!}{2^m(m+n-1)!}\sum_k\frac{(-2)^k}{k!}\Delta^k\gamma^m\\&=\frac{(-1)^m(n-1)!}{2^m(m+n-1)!}\sum_k\frac{(-2)^k}{k!}\frac{m!(n+m-1)!}{(m-k)!(n+m-1-k)!}\gamma^{m-k}\\&=\frac{(-1)^m(n-1)!}{2^m}\sum_k\frac{(-2)^{m-k}m!}{(m-k)!k!(k+n-1)!}\gamma^k\\&=(n-1)!\sum_k\binom{m}{k}\frac{(-\gamma/2)^k}{(k+n-1)!}\\&=(n-1)!L_m^{n-1}(\gamma/2)\qquad(5.1)\end{aligned}$$

5.2. Monomials. Let $\mathcal{V}=\mathcal{P}(V)$, with basis $p_{a,b}=z^a\overline{z}^b$. In the proof of Proposition 3.1, we calculated $\mathcal{T}p_{a,b}$, obtaining

$$q_{a,b}=\sum_c\frac{(-1)^{|b|-|c|}2^{|c|}a!b!}{c!(a-c)!(b-c)!}z^{a-c}\overline{z}^{b-c},$$

and thus

$$\left[\begin{array}{c}(a,b)\\ (a-c,b-c)\end{array}\right]=\frac{2^{|c|}a!b!}{c!(a-c)!(b-c)!}$$

5.3. Monomial Symmetric Functions. Let $K = S_n \ltimes T_n$ act on $V = \mathbb{C}^n$ by

$$(\sigma, \gamma) \cdot (z_1, \dots z_n) = (\gamma_1 z_{\sigma(1)}, \dots, \gamma_n z_{\sigma(n)}),$$

and let $\mathcal{V} = \mathcal{P}(V)^K$. (Note that $(K \ltimes H_n, K)$ is also a Gelfand pair.) Given a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, let $m_\lambda = z_1^{\lambda_1} \dots z_n^{\lambda_n}$.

Proposition 5.1. *$V_\lambda = span\{m_{\sigma(\lambda)} : \sigma \in S_n\}$ is irreducible under the action of $S_n \ltimes T_n$.*

Proof. Suppose that $\lambda = (1^{a_1} 2^{a_2} \dots)$, with $a_1 + a_2 + \dots = |\lambda|$. Then stab $m_\lambda \cong S_{a_1} \times S_{a_2} \times \dots$, so $\dim V_\lambda = n!/|\text{stab } m_\lambda| = n!/a_1! a_2! \dots$. The character for the action of K on V_λ is given by

$$\chi(\sigma, \gamma) = \sum_{\mu \in (S_n \cdot \lambda) : \sigma(\mu) = \mu} \gamma^\mu,$$

and hence $\int_{T^n} |\chi(\sigma, \gamma)|^2 d\gamma = |\{\mu \in (S_n \cdot \lambda) : \sigma(\mu) = \mu\}|$. Thus we get

$$\begin{aligned} \langle \chi, \chi \rangle &= \frac{1}{n!} \sum_{\sigma \in S_n} |\{\mu \in (S_n \cdot \lambda) : \sigma(\mu) = \mu\}| \\ &= \frac{1}{n!} \sum_{\mu \in (S_n \cdot \lambda)} |\{\sigma : \sigma(\mu) = \mu\}| \\ &= \frac{1}{n!} \dim V_\lambda |\text{stab } \lambda| \\ &= 1. \end{aligned}$$

□

Our canonical family of K-invariant polynomials would be

$$\frac{1}{\dim V_\lambda} \sum_{\mu \in S_n \cdot \lambda} \frac{m_\mu \overline{m_\mu}}{2^{|\mu|} \mu!}.$$

For convenience, we take a multiple of the canonical invariants and define

$$p_\lambda = \sum_{\sigma \in S_n} \frac{|m_{\sigma(\lambda)}|^2}{2^{|\lambda|} \lambda!} = |\text{stab } \lambda| \sum_{\mu \in S_n \cdot \lambda} \frac{m_\mu \overline{m_\mu}}{2^{|\mu|} \mu!}.$$

These are the monomial symmetric functions in the variables $(|z_1|^2, \dots, |z_n|^2)$. Define the Laguerre-type polynomial

$$\mathcal{L}_\lambda(z) = L_{\lambda_1}(|z_1|^2/2) \dots L_{\lambda_n}(|z_n|^2/2),$$

where $L_{\lambda_j} := L_{\lambda_j}^{(0)}$. Using the fact that $\mathcal{T}(|m_\lambda|^2) = \mathcal{T}(|z_1|^{2\lambda_1}) \dots \mathcal{T}(|z_n|^{2\lambda_n})$, we apply (5.1) to obtain

$$q_\lambda = \sum_{\sigma \in S_n} \mathcal{L}_{\sigma(\lambda)}.$$

We can now derive the formula for the generalized binomial coefficients:

$$\begin{aligned} q_\lambda(z,\overline{z}) &= \sum_{\sigma\in S_n} L_{\sigma(\lambda_1)}(|z_1|^2/2)\dots L_{\sigma(\lambda_n)}(|z_n|^2/2) \\ &= \sum_{\sigma\in S_n}\sum_{\mu_1}\binom{\sigma(\lambda)_1}{\mu_1}\frac{(-|z_1|^2/2)^{\mu_1}}{\mu_1!}\dots\sum_{\mu_n}\binom{\sigma(\lambda)_n}{\mu_n}\frac{(-|z_n|^2/2)^{\mu_n}}{\mu_n!} \\ &= \sum_{\sigma\in S_n}\sum_{\mu}\binom{\sigma(\lambda)}{\mu}\frac{(-1)^{|\mu|}|m_\mu|^2}{2^{|\mu|}\mu!} \end{aligned}$$

The coefficient of p_μ will agree with the coefficient of $|m_\mu|^2/2^|\mu|\mu!$, so we obtain

$$q_\lambda = \sum_\mu \sum_{\sigma\in S_n}(-1)^{|\mu|}\binom{\sigma(\lambda)}{\mu}p_\mu,$$

where the first sum is taken over μ in distinct S_n-orbits. Since $\left[\begin{array}{c}\sigma(\lambda)\\ \mu\end{array}\right] = \left[\begin{array}{c}\lambda\\ \sigma^{-1}(\mu)\end{array}\right]$, we obtain the generalized binomial coefficients

$$\left[\begin{array}{c}\lambda\\ \mu\end{array}\right] = \sum_{\sigma\in S_n}\binom{\lambda}{\sigma(\mu)}.$$

Note that the summands are non-zero only when $\sigma(\mu)_j \le \lambda_j$ for each j.

5.4. Schur Polynomials. There are other choices for bases of symmetric polynomials in $(|z_1|^2,\dots,|z_n^2|)$. Let $p_\lambda(z,\overline{z}) = S_\lambda(|z_1|^2,\dots,|z_n|^2)$, where S_λ is the Schur polynomial associated with the partition λ. (See [**Sta99**] [**Mac95**] for a thorough treatment.)

Explicitly,

$$S_\lambda(x_1,\dots,x_n) = \frac{\det[x_i^{\lambda_j+n-j}]}{\det[x_i^{n-j}]}.$$

The classical Pieri formula says that

$$(x_1+\dots+x_n)S_\lambda = \sum S_\mu,$$

where the sum is taken over all partitions μ with Young's diagram obtained by adding one box to λ. The Pieri formula for the family $\{p_\lambda\}$ becomes

$$\gamma^m p_\lambda = \sum K_{\lambda,\mu}p_\mu,$$

where $K_{\lambda,\mu}$ is a Kostka number [**Sta99**]. That is, $K_{\lambda,\mu}$ is the number of ways to build the Young's diagram for μ from λ, adding one box at a time. From this we conclude that the generalized binomial coefficients are given by

$$\left[\begin{array}{c}\mu\\ \lambda\end{array}\right] = \frac{2^{|\mu|-|\lambda|}}{(|\mu|-|\lambda|)!}\frac{d_\lambda}{d_\mu}K_{\lambda,\mu}.$$

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