Spherical Functions for the Action of a Finite Unitary Group on a Finite Heisenberg Group

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ABSTRACT. The action of the unitary group on the real Heisenberg group yields a Gelfand pair. The associated spherical functions are well known and have been computed independently by many authors. In this paper we develop a discrete counterpart to this story by replacing the real numbers by a finite field of odd characteristic. This produces a finite Gelfand pair whose spherical functions are computed explicitly. Our formulae resemble classical Gauss sums.

1. Introduction and overview of results

Given any field $\mathbb{F}$ of characteristic not equal to 2 one can form the $(2n)$-dimensional symplectic vector space $\big(\mathcal{W} = \mathbb{F}^n \times \mathbb{F}^n,\ [\cdot,\cdot]\big)$ where

$$[\mathbf{z},\mathbf{z}'] = \big[(\mathbf{x},\mathbf{y}),(\mathbf{x}',\mathbf{y}')\big] = \mathbf{x}\cdot\mathbf{y}' - \mathbf{y}\cdot\mathbf{x}'.$$

The associated polarized Heisenberg group is $H_n(\mathbb{F}) = \mathcal{W}\times\mathbb{F}$ with product

$$(\mathbf{z},\,t)(\mathbf{z}',\,t') = \big(\mathbf{z}+\mathbf{z}',\ t+t'+2^{-1}[\mathbf{z},\mathbf{z}']\big).$$

This is a two-step nilpotent group with center $\mathbb{F}$. The symplectic group $Sp(n,\mathbb{F})$ acts by automorphisms on $H_n(\mathbb{F})$ via

$$k\cdot(\mathbf{z},t) = (k\mathbf{z},t).$$

In the classical situation one has $\mathbb{F}=\mathbb{R}$. Identifying $\mathcal{W} = \mathbb{R}^n\times\mathbb{R}^n$ with $\mathbb{C}^n$ realizes the unitary group $U(n) = U(n,\mathbb{C})$ as a maximal compact connected subgroup of $Sp(n,\mathbb{R})$. The action of $U(n)$ on $H_n = H_n(\mathbb{R})$ yields a Gelfand pair $(U(n)\ltimes H_n,\ U(n))$ whose bounded spherical functions are well known [**1, 4, 5, 10, 12, 20, 21**]. They amount to certain $U(n)$-invariant functions on H_n which restrict to an additive character on the center. One has two distinct types of behavior according to whether or not this central character is trivial:

Type 1: (Non-trivial on the center.) For each pair $(\lambda,m)\in\mathbb{R}^\times\times(\mathbb{Z}_{\geq 0})$ one has a spherical function $\phi_{\lambda,m}(\mathbf{z},t) = L_m^{(n-1)}\left(|\lambda||\mathbf{z}|^2/2\right)e^{-|\lambda||\mathbf{z}|^2/4}e^{i\lambda t}$. Here $L_m^{(n-1)}$ is the order $(n-1)$ Laguerre polynomial of degree m normalized so that $L_m^{(n-1)}(0)=1$. The polynomials $\{L_m^{(n-1)}\mid m\in\mathbb{Z}_{\geq 0}\}$ are orthogonal on $(0,\infty)$ with respect to the measure $x^{n-1}e^{-x}\,dx$. Explicitly $L_m^{(n-1)}(x) = (n-1)!\sum_{j=0}^m\binom{m}{j}(-x)^j/(j+n-1)!$.

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Type 2: (Trivial on the center.) These are obtained by $U(n)$-averaging additive unitary characters on $\mathcal{W}$. One has $\eta_r(\mathbf{z},t) = 2^{n-1}(n-1)!J_{n-1}(r|\mathbf{z}|)/(r|\mathbf{z}|)^{n-1}$ for each $r > 0$ together with $\eta_0 \equiv 1$.

In this paper we derive a counterpart to these formulae in the context of finite fields. Throughout we let $\mathbb{F} = \mathbb{F}_q$ denote the field with q elements where

$$q = p^m$$

for some *odd* prime p and positive integer m. Choose any non-square

$$\varepsilon \in \mathbb{F}^\times - (\mathbb{F}^\times)^2$$

in $\mathbb{F}$ and form the quadratic extension field

$$\widetilde{\mathbb{F}} = \mathbb{F}(\sqrt{\varepsilon}).$$

We adapt "complex notation" to $\widetilde{\mathbb{F}}$, writing, for $(z = x + y\sqrt{\varepsilon}) \in \widetilde{\mathbb{F}}$,

$$Re(z) = x, \qquad Im(z) = y, \qquad \overline{z} = x - y\sqrt{\varepsilon}.$$

Now

$$\langle \mathbf{z}, \mathbf{z}' \rangle = z_1\overline{z_1'} + \cdots + z_n\overline{z_n'}$$

is a Hermitian inner product on $\widetilde{\mathbb{F}}^n$. (That is, non-degenerate $\mathbb{F}$-bilinear and $\widetilde{\mathbb{F}}$-linear in the first variable with $\overline{\langle z, z' \rangle} = \langle z', z \rangle$.) The unitary group $U(n,\widetilde{\mathbb{F}})$ is the set of $\widetilde{\mathbb{F}}$-linear transformations on $\widetilde{\mathbb{F}}^n$ preserving $\langle \cdot, \cdot \rangle$. This is a finite group of order

$$|U(n,\widetilde{\mathbb{F}})| = \frac{\prod_{j=1}^n((q^2-q-1)^j - 1)}{(q-2)^n}.$$

Identifying $\mathcal{W}$ with $\widetilde{\mathbb{F}}^n$ embeds $U(n,\widetilde{\mathbb{F}})$ in $Sp(n,\mathbb{F})$ because $[\mathbf{z},\mathbf{z}'] = -Im(\langle \mathbf{z}, \mathbf{z}' \rangle)$. It is known that $(U(n,\widetilde{\mathbb{F}}) \ltimes H_n(\mathbb{F}),\ U(n,\widetilde{\mathbb{F}}))$ is a finite Gelfand pair [**2**].

As in the classical situation we can regard spherical functions for the pair $(U(n,\widetilde{\mathbb{F}}) \ltimes H_n(\mathbb{F}),\ U(n,\widetilde{\mathbb{F}}))$ as $U(n,\widetilde{\mathbb{F}})$-invariant functions on $H_n(\mathbb{F})$ given on the center by additive characters of $\mathbb{F}$. The type 2 spherical functions, having trivial central character, are obtained by $U(n,\widetilde{\mathbb{F}})$-averaging additive characters $\widetilde{\psi} \in \widehat{\mathcal{W}}$:

$$\eta_{\widetilde{\psi}}(\mathbf{z},t) = \frac{1}{|U(n,\widetilde{\mathbb{F}})|} \sum_{k \in U(n,\widetilde{\mathbb{F}})} \widetilde{\psi}(k\mathbf{z}).$$

One obtains a distinct type 2 spherical function for each $U(n,\widetilde{\mathbb{F}})$-orbit in $\mathcal{W}$. When $n = 1$ this gives q such spherical functions and for $n \geq 2$ there are $q+1$ in all. See Lemma 4.1 below.

The type 2 spherical functions for $(U(n,\widetilde{\mathbb{F}}) \ltimes H_n(\mathbb{F}),\ U(n,\widetilde{\mathbb{F}}))$ are obvious analogues for their classical counterparts. The type 1 spherical functions (non-trivial on the center) are less transparent. We will describe them first for the case $n = 1$. Note that $U(1,\widetilde{\mathbb{F}}) = \{z \in \widetilde{\mathbb{F}} \ :\ z\overline{z} = 1\}$ is the kernel of the norm mapping

$$N : \widetilde{\mathbb{F}}^\times \to \mathbb{F}^\times, \qquad N(z) = z\overline{z} = x^2 - y^2\varepsilon$$

for the field extension $\widetilde{\mathbb{F}}/\mathbb{F}$. As N is surjective $U(1,\widetilde{\mathbb{F}})$ is a cyclic subgroup of $\widetilde{\mathbb{F}}^\times$ with order $q+1$.

THEOREM 1.1. *The $U(1,\widetilde{\mathbb{F}})$-spherical functions of type 1 on $H_1(\mathbb{F})$ are given as follows. Let $\psi \in \widehat{\mathbb{F}}$ be a non-trivial additive character on $\mathbb{F}$ and $\widetilde{\chi} \in (\widetilde{\mathbb{F}}^\times/\mathbb{F}^\times)^\frown$ a non-trivial multiplicative character on $\widetilde{\mathbb{F}}$ whose restriction to $\mathbb{F}^\times$ is trivial. One has a spherical function $\phi_{\psi,\widetilde{\chi}}(z,t) = \phi_{\psi,\widetilde{\chi}}(z)\psi(t)$ where $\phi_{\psi,\widetilde{\chi}}(0) = 1$ and*

$$\phi_{\psi,\widetilde{\chi}}(z) = \frac{-1}{q^2-1} \sum_{w \in \widetilde{\mathbb{F}}-\mathbb{F}} \widetilde{\chi}(w)\psi\left(-\frac{1}{4\varepsilon}\frac{Re(w)}{Im(w)}N(z)\right) \tag{1.1}$$

for $z \neq 0$. Distinct pairs $(\psi,\widetilde{\chi})$ yield distinct spherical functions $\phi_{\psi,\widetilde{\chi}}$. So there are $(q-1)q = q^2-q$ spherical functions of type 1.

Our description of the type 1 spherical functions when $n \geq 2$ is less concise. We will show, however, that they are computable in terms of the corresponding functions for the case $n = 1$:

THEOREM 1.2. *For $n \geq 2$ the $U(n,\widetilde{\mathbb{F}})$-spherical functions of type 1 on $H_n(\mathbb{F})$ may be indexed by pairs $\big(\psi \in \widehat{F} - \{1\},\ \widetilde{\chi} \in (\widetilde{\mathbb{F}}^\times/\mathbb{F}^\times)^\frown\big)$. One has $\phi_{\psi,\widetilde{\chi}}(\mathbf{z},t) = \phi_{\psi,\widetilde{\chi}}(\mathbf{z})\psi(t)$ where*

$$\phi_{\psi,\widetilde{\chi}}(\mathbf{z}) = \frac{1}{d_n(\widetilde{\chi})} \sum \phi^1_{\psi,\widetilde{\chi}_1}(z_1)\cdots\phi^1_{\psi,\widetilde{\chi}_n}(z_n).$$

Here

- *$\phi^1_{\psi,\widetilde{\chi}_j}(z_j)$ is a $U(1,\widetilde{\mathbb{F}})$-spherical function (as in Theorem 1.1),*
- *the sum is over all $(\widetilde{\chi}_1,\ldots,\widetilde{\chi}_n)$ with $\widetilde{\chi}_j \not\equiv 1$ and $\widetilde{\chi}_1\cdots\widetilde{\chi}_n = \widetilde{\chi}$,*
- *$d_n(\widetilde{\chi}) = \begin{cases} \frac{q^n+(-1)^{n-1}}{q+1} & \text{for } \widetilde{\chi} \not\equiv 1 \\ \frac{q^n+(-1)^n q}{q+1} & \text{for } \widetilde{\chi} \equiv 1 \end{cases}$.*

Distinct pairs $(\psi,\widetilde{\chi})$ yield distinct spherical functions $\phi_{\psi,\widetilde{\chi}}$. So there are $(q-1)(q+1) = q^2-1$ spherical functions of type 1.

Recall that a *Gauss sum*, in the context of finite fields $\mathbb{F}$, is a sum of the sort

$$G(\chi,\psi) = \sum_{\alpha \in \mathbb{F}^\times} \chi(\alpha)\psi(\alpha), \tag{1.2}$$

for some multiplicative character $\chi \in (\mathbb{F}^\times)^\frown$ and additive character $\psi \in \widehat{\mathbb{F}}$. Equation 1.1 presents $\phi_{\widetilde{\chi},\psi}$ as a "modified Gauss sum" over the extension field $\widetilde{\mathbb{F}}$. The sum contains an honest multiplicative character but the additive character has been altered. Working from (1.1) we will derive another expression for $\phi_{\widetilde{\chi},\psi}$, as a "modified Gauss sum" over $\mathbb{F}$. In this form we have an honest additive character but the multiplicative character has been altered:

COROLLARY 1.3. *The $U(1,\widetilde{\mathbb{F}})$-spherical functions of type 1 on $H_1(\mathbb{F})$ may be written as $\phi_{\psi,\chi'}(z,t) = \phi_{\psi,\chi'}(z)\psi(t)$ where $\phi_{\psi,\chi'}(0) = 1$ and*

$$\phi_{\psi,\chi'}(z) = \frac{-1}{q+1}\sum_{a \in \mathbb{F}} \chi'(a)\psi\left(-\frac{a}{4\varepsilon}N(z)\right) \tag{1.3}$$

for $z \neq 0$. Here $\psi \in \widehat{\mathbb{F}}$ is a non-trivial additive character on $\mathbb{F}$ and $\chi' : \mathbb{F} \to \mathbb{C}$ satisfies $\chi' \not\equiv 1$ and

$$\chi'(a)\chi'(b) = \begin{cases} 1 & \text{if } b = -a \\ \chi'\left(\frac{ab+\varepsilon}{a+b}\right) & \text{if } b \neq -a \end{cases}. \tag{1.4}$$

Gauss sums and their variants arise frequently in connection with representation theory and analysis on finite groups ([**16**], [**22**]). So the general flavor of our formulae is not surprising. They also bear some resemblance to Soto-Andrade's spherical functions on the finite Poincaré upper half plane [**18, 19**].

The remainder of this paper is organized as follows. The next section summarizes background material concerning finite Gelfand pairs, spherical functions and the representation theory of Heisenberg groups and related semidirect products. Section 3 contains the proofs for Theorem 1.1 and Corollary 1.3. The proof for Theorem 1.2 is given in Section 4.

2. Preliminaries

For a finite set S, the symbol $\mathbb{C}[S]$ will denote the set of all $\mathbb{C}$-valued functions on S. This is a complex vector space of dimension $|S|$ which carries a Hermitian inner product

$$\langle f, g\rangle_S = \frac{1}{|S|}\sum_{x\in S} f(x)\overline{g(x)}.$$

When $S = G$, a finite group, $\mathbb{C}[G]$ is the group algebra with convolution product

$$f \star g(x) = \sum_{y\in G} f(xy^{-1})g(y).$$

The symbol $\widehat{G}$ denotes the set of irreducible unitary representations of a group G, modulo unitary equivalence. For $\rho \in \widehat{G}$ we write $\mathcal{H}_\rho$ for the representation space of ρ and let $d_\rho = dim(\mathcal{H}_\rho)$.

2.1. Generalities on finite Gelfand pairs and spherical functions.

We recall briefly the definition and basic properties of Gelfand pairs and associated spherical functions in the context of finite groups. Proofs may be found in [**14**, Chapter 7], [**22**, Chapter 20].

Let G be a finite group and K a subgroup of G. One says that (G,K) is a *Gelfand pair* when the set $\mathbb{C}[K\backslash G/K]$ of K-bi-invariant complex valued functions on G is a *commutative* subalgebra of the group algebra $\mathbb{C}[G]$. Equivalently $Ind_K^G(1_K)$ is a multiplicity free representation of G and

$$dim(\mathcal{H}_\rho^K) \leq 1 \text{ for each } \rho \in \widehat{G}.$$

Suppose that (G,K) is a finite Gelfand pair. For each representation $\rho \in \widehat{G}$ one obtains a $\mathbb{C}$-valued function on G via

$$\phi_\rho(x) = \frac{1}{|K|}\sum_{k\in K} Tr(\rho(kx)) = \frac{1}{|K|}\sum_{k\in K} Tr(\rho(xk)), \tag{2.1}$$

the K-average of the trace character for ρ. The K-*spherical representations* are

$$\widehat{G}_K = \{\rho \in \widehat{G} \;:\; dim(\mathcal{H}_\rho^K) = 1\} = \{\rho \in \widehat{G} \;:\; \phi_\rho(e) = 1\},$$

the spectrum of $Ind_K^G(1_K)$, and $\{\phi_\rho \;:\; \rho \in \widehat{G}_K\}$ are the *spherical functions* for (G,K). These are precisely the functions $\phi : G \to \mathbb{C}$ for which $f \mapsto (f \star \phi)(e)$ is a non-zero algebra mapping from $\mathbb{C}[K\backslash G/K]$ to $\mathbb{C}$. They form an orthogonal basis for $\mathbb{C}[K\backslash G/K]$ with $\langle \phi_\rho, \phi_\rho\rangle_G = 1/d_\rho$. An alternate formula for ϕ_ρ reads

$$\phi_\rho(x) = \langle \rho(x)v_\rho,\; v_\rho\rangle \tag{2.2}$$

where v_ρ is a unit K-fixed vector in $\mathcal{H}_\rho$.

2.2. Semidirect products. Now suppose K and H are finite groups and that K acts on H via automorphisms. We call (K, H) a Gelfand pair when (G, K) is a Gelfand pair with $G = K \ltimes H$, the semidirect product of K with H. The restriction mapping $\mathbb{C}[G] \to \mathbb{C}[H]$ gives an isometry from $\mathbb{C}[K\backslash G/K]$ onto $\mathbb{C}[H]^K$, the K-invariant functions on H, compatible with convolution. So (K, H) is a Gelfand pair if and only if $\mathbb{C}[H]^K$ is a commutative subalgebra of $\mathbb{C}[H]$. In this case we regard the spherical functions as elements of $\mathbb{C}[H]^K$, via restriction.

2.3. Schrödinger and oscillator representations. Now let $\mathbb{F} = \mathbb{F}_q$ as in Section 1. The unitary dual of $H = H_n(\mathbb{F})$ is well understood. Each irreducible representation is of dimension 1 or q^n. The q^n-dimensional representations are those with non-trivial central characters. Let $\psi \in \widehat{\mathbb{F}}$ be a non-trivial additive character on $\mathbb{F}$. The *Schrödinger representation* π_ψ is realized on the space $\mathbb{C}[\mathbb{F}^n]$ as

$$\pi_\psi(\mathbf{x}, \mathbf{y}, t) f(\mathbf{u}) = \psi\left(t + \mathbf{y}\cdot\mathbf{u} + 2^{-1}\mathbf{x}\cdot\mathbf{y}\right) f(\mathbf{u}+\mathbf{x}). \tag{2.3}$$

Now if $k \in Sp(n, \mathbb{F})$ then $\pi_\psi \circ k$ has central character ψ and hence is unitarily equivalent to π_ψ. The *oscillator representation* ω_ψ of $Sp(n, \mathbb{F})$ (also called the metaplectic or Weil representation) intertwines $\pi_\psi \circ k$ with π_ψ:

$$\pi_\psi(k\mathbf{z}, t) = \omega_\psi(k)\pi_\psi(\mathbf{z}, t)\omega_\psi(k)^{-1} \qquad (k \in Sp(n, \mathbb{F}),\ (\mathbf{z}, t) \in H). \tag{2.4}$$

This completely characterizes the representation ω_ψ except when $n = 1$ and $q = 3$ [**8**]. The oscillator representation is given explicitly in [**15**] and [**3**].

2.4. Type 1 representations of $G = K \ltimes H$. Now let $H = H_n(\mathbb{F})$, let K be a subgroup of $Sp(n, \mathbb{F})$ and set $G = K \ltimes H$. An application of the *Mackey machine* gives a description of $\widehat{G}$. The type 1 representations $\rho \in \widehat{G}$ are those non-trivial on the center of H. These have the form

$$\rho(k, \mathbf{z}, t) = \rho_{\psi,\sigma}(k, \mathbf{z}, t) = \sigma(k) \otimes \pi_\psi(\mathbf{z}, t)\omega_\psi(k), \tag{2.5}$$

where $\psi \in \widehat{F} - \{1\}$, $\sigma \in \widehat{K}$.

2.5. Gelfand pairs (K, H). Let $\rho = \rho_{\psi,\sigma}$ be as in (2.5). Now $dim(\mathcal{H}_\rho^K)$ is the multiplicity of the contragredient representation $\sigma^* \in \widehat{K}$ in $\omega_\psi|_K$. So if (K, H) is a Gelfand pair we must have that $\omega_\psi|_K$ is multiplicity free. On the other hand one can check that $dim(\mathcal{H}_\rho^K) \leq 1$ holds for all type 2 representations $\rho \in \widehat{G}$. So one has:

Proposition 2.1. [**2**] *Let $H = H_n(\mathbb{F})$ and K be a subgroup of $Sp(n, \mathbb{F})$. Then (K, H) is a Gelfand pair if and only if $\omega_\psi|_K$ is a multiplicity free representation of K for every $\psi \in \widehat{F} - \{1\}$.*

In practice it can be difficult to obtain the decomposition of $\omega_\psi|K$ into irreducible components. When (K, H) is a Gelfand pair the total number of irreducibles that occur is, however, determined by the following:

Lemma 2.2. *Let K be a subgroup of $Sp(n, \mathbb{F})$ for which (K, H) is a Gelfand pair. The number of irreducible components in $\omega_\psi|_K$ $(\psi \in \widehat{\mathbb{F}} - \{1\})$ coincides with $|\mathcal{W}/K|$, the number of K-orbits in $\mathcal{W}$.*

This follows easily from the fact that the tensor product $\omega_\psi \otimes \omega_\psi^*$ of the oscillator representation for $Sp(n, \mathbb{F})$ with its contragredient can be identified with its permutation representation in $\mathbb{C}[\mathcal{W}]$ [**8, 9**].

2.6. Type 1 spherical functions for (K, H). The preceding discussion shows that the type 1 spherical functions for a Gelfand pair of the sort (K, H), $(H = H_n(\mathbb{F})$, $K \subset Sp(n, \mathbb{F}))$ are

$$\left\{\phi_{\psi,\sigma} \ : \ \psi \in \widehat{\mathbb{F}} - \{1\}, \ \sigma \in \widehat{K}, \ \sigma^* \leq \omega_\psi|_K\right\} \tag{2.6}$$

where $\phi_{\psi,\sigma} = \phi_{\rho_{\psi,\sigma}}$ (see (2.1)) and "$\sigma^* \leq \omega_\psi|_K$" means σ^* occurs in $\omega_\psi|_K$.

For $\sigma \in \widehat{K}$ with $\sigma^* \leq \omega_\psi|_K$ let $P_\sigma \subset \mathbb{C}[\mathbb{F}^n]$ denote the (unique) subspace on which $\omega_\psi|_K$ acts by a copy of σ^*. Working from Equation 2.2 one can show that

$$\phi_{\psi,\sigma}(\mathbf{z}, t) = \frac{1}{d_\sigma}\sum_{j=1}^{d_\sigma}\langle \pi_\psi(\mathbf{z}, t)u_j, \ u_j\rangle \tag{2.7}$$

where $\{u_j\}$ is any orthonormal basis for P_σ [**1**, Corollary 2.3].

2.7. Restricting to subgroups of K. Suppose that K and K' are subgroups of $Sp(n, \mathbb{F})$ with $K' \subset K$ and that (K', H) is a Gelfand pair. Clearly (K, H) is also a Gelfand pair. The type 1 K-spherical functions $\phi_{\psi,\sigma}$ are related to the K'-spherical functions $\phi'_{\psi,\sigma'}$ via

$$\phi_{\psi,\sigma} = \frac{1}{d_\sigma}\sum_{\sigma' \leq \sigma|_{K'}} d_{\sigma'}\phi'_{\psi,\sigma'}. \tag{2.8}$$

This follows easily from (2.7) by using an orthonormal basis for P_σ compatible with its decomposition into K'-irreducible subspaces $P_{\sigma'}$ [**1**, Proposition 2.4].

3. Type 1 spherical functions for $(U(1, \widetilde{\mathbb{F}}), \ H_1(\mathbb{F}))$

In this section we will prove Theorem 1.1 and Corollary 1.3. We adopt the notation

$$H_1 = H_1(\mathbb{F}), \qquad U_1 = U(1, \widetilde{\mathbb{F}}) = \{k \in \widetilde{\mathbb{F}} \ : \ N(k) = k\overline{k} = 1\}$$

and take $G = U_1 \ltimes H_1$. It is shown in [**2**] that (U_1, H_1) is a Gelfand pair. Another proof of this fact will be given below, in Remark 3.3, as a byproduct of our spherical function calculation.

As U_1 is abelian (in fact cyclic of order $q + 1$) the type 1 representations of G (see (2.5)) become

$$\rho_{\psi,\chi}(k, z, t) = \chi(k)\pi_\psi(z, t)\omega_\psi(k) \tag{3.1}$$

for characters $\psi \in \widehat{F} - \{1\}$, $\chi \in \widehat{U_1}$. To apply Equation 2.1 we must compute $Tr(\rho_{\psi,\chi}(k, z, t))$. Letting $\pi_\psi(z) = \pi_\psi(z, 0)$, one has $\pi_\psi(z, t) = \pi_\psi(z)\psi(t)$ and hence

$$Tr(\rho_{\psi,\chi}(k, z, t)) = \chi(k)Tr\big(\pi_\psi(z)\omega_\psi(k)\big)\psi(t). \tag{3.2}$$

Calculation of $Tr\big(\pi_\psi(z)\omega_\psi(k)\big)$ is the key technical step in this paper. We write $Sgn_{\mathbb{F}^\times}$ for the *sign character* on the multiplicative group $\mathbb{F}^\times$,

$$Sgn_{\mathbb{F}^\times}(t) = \left\{\begin{array}{ll} +1 & \text{if } t \text{ is a square in the group } \mathbb{F}^\times \\ -1 & \text{otherwise} \end{array}\right. .$$

LEMMA 3.1.

$$Tr(\pi_\psi(z)\omega_\psi(k)) = \left\{\begin{array}{ll} q\delta_{z,0} & \textit{if } k = 1 \\ Sgn_{\mathbb{F}^\times}(-1) & \textit{if } k = -1 \\ Sgn_{\mathbb{F}^\times}\left(\frac{a-1}{2}\right)\psi\left(-\frac{1+a}{4b\varepsilon}N(z)\right) & \textit{if } (k = a + b\sqrt{\varepsilon}) \neq \pm 1 \end{array}\right\}.$$

PROOF. The additive characters ψ on $\mathbb{F}$ can be written explicitly as

$$\psi_\lambda(x) = exp\left(\frac{2\pi i}{p} Tr_{\mathbb{F}/\mathbb{F}_p}(\lambda x)\right) \quad \text{for } \lambda \in \mathbb{F} \tag{3.3}$$

where $Tr_{\mathbb{F}/\mathbb{F}_p}$ is the trace mapping for the field extension $\mathbb{F}/\mathbb{F}_p$ ($\mathbb{F} = \mathbb{F}_q = \mathbb{F}_{p^m}$). Throughout this proof we take $\psi = \psi_\lambda$ with $\lambda \in \mathbb{F}^\times$ and write π_λ, ω_λ in place of π_{ψ_λ}, ω_{ψ_λ}. Also let $z = x + y\sqrt{\varepsilon}$ here and $\delta_u \in \mathbb{C}[\mathbb{F}]$ denote the function

$$\delta_u(s) = \delta_{u,s} = \begin{cases} 1 & \text{if } s = u \\ -1 & \text{if } s \neq u \end{cases}.$$

Note that $\{\delta_u \ : \ u \in \mathbb{F}\}$ is an (orthogonal) basis for $\mathbb{C}[\mathbb{F}]$. Formula 2.3 gives

$$\pi_\lambda(z)\delta_u = \psi_\lambda(yu - xy/2)\delta_{u-x}. \tag{3.4}$$

The calculations for $k = 1$ and $k = -1$ are straightforward and left to the reader. Below we consider $k = a + b\sqrt{\varepsilon}$ where $k \neq \pm 1$ and $N(k) = a^2 - b^2\varepsilon = 1$, so $b \neq 0$. Also assume, for the moment, that $a \neq 0$. (Pure imaginary elements $k = b\sqrt{\varepsilon}$ in U_1 will be treated separately.)

Viewed as an element of $Sp(1,\mathbb{F}) = SL(2,\mathbb{F})$, k factors as

$$\begin{aligned}\begin{bmatrix} a & b\varepsilon \\ b & a \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ a^{-1}b & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -a^{-1}b\varepsilon & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= L_{a^{-1}b} D_a J L_{-a^{-1}b\varepsilon} J^{-1}.\end{aligned}$$

where $L_c = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$. We now apply the following formulae from [**15**]:

$$\begin{gathered} \omega_\lambda(D_a)f(s) = Sgn_{\mathbb{F}^\times}(a) f(a^{-1}s) \\ \omega_\lambda(L_c)f(s) = \psi_\lambda(-cs^2/2)f(s) \\ \omega_\lambda(J)f(s) = (-1)^{m+1}(-i)^{m(p-1)/2} Sgn_{\mathbb{F}^\times}(\lambda)\tfrac{1}{\sqrt{q}}\textstyle\sum_{\alpha\in\mathbb{F}} f(\alpha)\psi_\lambda(\alpha s) \end{gathered}. \tag{3.5}$$

These give

$$\begin{aligned}\omega_\lambda(k)f(s) &= \psi_\lambda(-a^{-1}bs^2/2) Sgn_{\mathbb{F}^\times}(a)\big(\omega_\lambda(JL_{-a^{-1}b\varepsilon}J^{-1})f\big)(a^{-1}s) \\ &= \frac{Sgn_{\mathbb{F}^\times}(a)}{q}\psi_\lambda\left(-\frac{b}{2a}s^2\right)\sum_{\alpha,\beta\in\mathbb{F}}\psi_\lambda\left(\frac{b\varepsilon}{2a}\alpha^2 + \alpha\left(\frac{s}{a} - \beta\right)\right) f(\beta).\end{aligned}$$

Setting $f = \delta_u$ and completing the square in α yields

$$\begin{aligned}\omega_\lambda(k)\delta_u(s) &= \frac{Sgn_{\mathbb{F}^\times}(a)}{q}\psi_\lambda\left(-\frac{b}{2a}s^2\right)\sum_{\alpha\in\mathbb{F}}\psi_\lambda\left(\frac{b\varepsilon}{2a}\alpha^2 + \left(\frac{s}{a} - u\right)\alpha\right) \\ &= \frac{Sgn_{\mathbb{F}^\times}(a)}{q}\psi_\lambda\left(-\frac{a}{2b\varepsilon}u^2 + \frac{1}{b\varepsilon}us - \frac{a}{2b\varepsilon}s^2\right) G_\lambda\left(\frac{b\varepsilon}{2a}\right)\end{aligned}$$

where $G_\lambda(c)$ denotes the *quadratic Gauss sum*[1]

$$G_\lambda(c) = \sum_{\alpha\in\mathbb{F}}\psi_\lambda(c\alpha^2) \tag{3.6}$$

[1]Quadratic Gauss sums are special cases of Equation 1.2. Indeed $G_\lambda(c) = G(Sgn_{\mathbb{F}^\times}, \psi_{c\lambda})$.

for $c \in \mathbb{F}^\times$. So now, applying (3.4),

$$\begin{aligned}\pi_\lambda(z)\omega_\lambda(k)\delta_u &= \frac{Sgn_{\mathbb{F}^\times}(a)}{q} G_\lambda\left(\frac{b\varepsilon}{2a}\right)\sum_{v\in\mathbb{F}}\psi_\lambda\left(-\frac{a}{2b\varepsilon}u^2+\frac{1}{b\varepsilon}uv-\frac{a}{2b\varepsilon}v^2\right)\pi_\lambda(z)\delta_v \\ &= \frac{Sgn_{\mathbb{F}^\times}(a)}{q} G_\lambda\left(\frac{b\varepsilon}{2a}\right)\sum_{v\in\mathbb{F}}\psi_\lambda\left(-\frac{a}{2b\varepsilon}u^2+\frac{1}{b\varepsilon}uv-\frac{a}{2b\varepsilon}v^2+vy-\frac{1}{2}xy\right)\delta_{v-x}\end{aligned}$$

and hence

$$\begin{aligned}Tr(\pi_\lambda(z)\omega_\lambda(k)) = \frac{Sgn_{\mathbb{F}^\times}(a)}{q} G_\lambda\left(\frac{b\varepsilon}{2a}\right)\psi_\lambda\left(\frac{1}{2}xy\right)\times \\ \sum_{u\in\mathbb{F}}\psi_\lambda\left(-\frac{a}{2b\varepsilon}u^2+\frac{1}{b\varepsilon}u(u+x)-\frac{a}{2b\varepsilon}(u+x)^2+uy\right)\end{aligned}$$

Upon completing the square in u the last summation becomes

$$\begin{aligned}&\sum_{u\in\mathbb{F}}\psi_\lambda\left(\left(\frac{1-a}{b\varepsilon}\right)\left(u+\frac{x}{2}+\frac{by\varepsilon}{2(1-a)}\right)^2-\frac{1}{4}\left[\left(\frac{1-a}{b\varepsilon}\right)x^2+2xy+\left(\frac{b\varepsilon}{1-a}\right)y^2\right]-\frac{ax^2}{2b\varepsilon}\right) \\ &\quad = G_\lambda\left(\frac{1-a}{b\varepsilon}\right)\psi_\lambda\left(-\frac{1}{2}xy\right)\psi_\lambda\left(-\frac{1}{4}\left[\left(\frac{1+a}{b\varepsilon}\right)x^2+\left(\frac{b\varepsilon}{1-a}\right)y^2\right]\right).\end{aligned}$$

But $b\varepsilon/(1-a) = -(1+a)/b$ since $a^2-b^2\varepsilon = 1$ and so

$$\left(\frac{1+a}{b\varepsilon}\right)x^2+\left(\frac{b\varepsilon}{1-a}\right)y^2 = \frac{1+a}{b\varepsilon}(x^2-y^2\varepsilon) = \frac{1+a}{b\varepsilon}N(z).$$

We obtain

$$Tr(\pi_\lambda(z)\omega_\lambda(k)) = \frac{Sgn_{\mathbb{F}^\times}(a)}{q} G_\lambda\left(\frac{b\varepsilon}{2a}\right)G_\lambda\left(\frac{1-a}{b\varepsilon}\right)\psi_\lambda\left(-\frac{1+a}{4b\varepsilon}N(z)\right).$$

The quadratic Gauss sum $G_\lambda(c)$ has been famously evaluated. In fact (see [**13**, Section 5.2])

$$G_\lambda(c) = G_1(\lambda c) = Sgn_{\mathbb{F}^\times}(\lambda c)G_1(1) \tag{3.7}$$

where

$$G_1(1) = \begin{cases} (-1)^{m-1}\sqrt{q} & \text{if } p\equiv 1 \ (\bmod 4) \\ (-1)^{m-1}i^m\sqrt{q} & \text{if } p\equiv 3 \ (\bmod 4)\end{cases}. \tag{3.8}$$

Applying these facts and multiplicativity of $Sgn_{\mathbb{F}^\times}$ we obtain

$$Sgn_{\mathbb{F}^\times}(a)G_\lambda\left(\frac{b\varepsilon}{2a}\right)G_\lambda\left(\frac{1-a}{b\varepsilon}\right) = Sgn_{\mathbb{F}^\times}\left(\frac{1-a}{2}\right)(G_1(1))^2$$

where

$$(G_1(1))^2 = \left\{\begin{array}{ll} q & \text{if } p\equiv 1 \ (\bmod 4) \\ (-1)^m q & \text{if } p\equiv 3 \ (\bmod 4)\end{array}\right\} = Sgn_{\mathbb{F}^\times}(-1)q.$$

So finally

$$Tr(\pi_\lambda(z)\omega_\lambda(k)) = Sgn_{\mathbb{F}^\times}\left(\frac{a-1}{2}\right)\psi_\lambda\left(-\frac{1+a}{4b\varepsilon}N(z)\right)$$

as in the statement of our Lemma.

To complete the proof it remains to consider pure imaginary elements $k = b\sqrt{\varepsilon}$ in U_1. As $-b^2\varepsilon = N(k) = 1$ the element -1 must fail to be a square in $\mathbb{F}^\times$. That is $Sgn_{\mathbb{F}^\times}(-1) = -1$ and $q \equiv 3 \pmod 4$. Equivalently

$$p \equiv 3 \pmod 4 \text{ and } m \text{ is odd.}$$

In this case U_1 contains exactly two such pure imaginary elements.

As $b\varepsilon = -b^{-1}$, one can factor k in $Sp(1,\mathbb{F})$ as

$$\begin{bmatrix} 0 & b\varepsilon \\ b & 0 \end{bmatrix} = \begin{bmatrix} b^{-1} & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = D_{b^{-1}} J^{-1}.$$

Equations 3.4 and 3.5 yield

$$\begin{aligned} \pi_\lambda(z)\omega_\lambda(k)\delta_u &= Sgn_{\mathbb{F}^\times}(b) Sgn_{\mathbb{F}^\times}(\lambda)(+i)^{m(p-1)/2} \frac{1}{\sqrt{q}} \sum_{v\in\mathbb{F}} \psi_\lambda(-buv)\pi_\lambda(z)\delta_v \\ &= \frac{i^{m(p-1)/2} Sgn_{\mathbb{F}^\times}(b\lambda)}{\sqrt{q}} \sum_{v\in\mathbb{F}} \psi_\lambda\left(-buv + yv - \frac{1}{2}xy\right)\delta_{v-x} \end{aligned}$$

and hence

$$\begin{aligned} Tr(\pi_\lambda(z)\omega_\lambda(k)) &= \frac{i^{m(p-1)/2} Sgn_{\mathbb{F}^\times}(b\lambda)}{\sqrt{q}} \sum_{u\in\mathbb{F}} \psi_\lambda\left(-bu(x+u) + y(x+u) - \frac{1}{2}xy\right) \\ &= \frac{i^{m(p-1)/2} Sgn_{\mathbb{F}^\times}(b\lambda)}{\sqrt{q}} \psi_\lambda\left(\frac{1}{2}xy + \frac{1}{4b}(bx-y)^2\right) \sum_{u\in\mathbb{F}} \psi_\lambda\left(-b\left[u + \frac{bx-y}{2b}\right]^2\right) \\ &= \frac{i^{m(p-1)/2} Sgn_{\mathbb{F}^\times}(b\lambda)}{\sqrt{q}} G_\lambda(-b)\psi_\lambda\left(\frac{b}{4}\left[x^2 + \frac{y^2}{b^2}\right]\right). \end{aligned}$$

But $1/b^2 = -\varepsilon$, so $x^2 + y^2/b^2 = x^2 - y^2\varepsilon = N(z)$ and hence

$$Tr(\pi_\lambda(z)\omega_\lambda(k)) = \frac{i^{m(p-1)/2} Sgn_{\mathbb{F}^\times}(b\lambda)}{\sqrt{q}} G_\lambda(-b)\psi_\lambda\left(\frac{b}{4}N(z)\right). \tag{3.9}$$

Now applying (3.7) and (3.8) gives

$$G_\lambda(-b) = Sgn_{\mathbb{F}^\times}(-\lambda b)G_1(1) = -Sgn_{\mathbb{F}^\times}(b\lambda) i^m \sqrt{q}$$

as $p \equiv 3 \pmod 4$ and m is odd. The coefficient in (3.9) is thus $-i^{m(p+1)/2} = -i^{(p+1)/2}$, again using $p \equiv 3 \pmod 4$, m odd. So finally

$$Tr(\pi_\lambda(z)\omega_\lambda(k)) = -i^{(p+1)/2}\psi_\lambda\left(\frac{b}{4}N(z)\right). \tag{3.10}$$

On the other hand setting $a = 0$ in the formula given for $Tr(\pi_\lambda(z)\omega_\lambda(k))$ with $k \neq \pm 1$ in the statement of Lemma 3.1 produces

$$Sgn_{\mathbb{F}^\times}\left(-\frac{1}{2}\right)\psi_\lambda\left(-\frac{1}{4b\varepsilon}N(z)\right) = -Sgn_{\mathbb{F}^\times}(2)\psi_\lambda\left(\frac{b}{4}N(z)\right),$$

as $Sgn_{\mathbb{F}^\times}(-1) = -1$ and $-1/b\varepsilon = b$ here. This coincides with (3.10) because, in fact,

$$Sgn_{\mathbb{F}^\times}(2) = i^{(p+1)/2}$$

when $p \equiv 3 \pmod 4$ and m odd. Indeed 2 is a square in $\mathbb{F}^\times_{(q=p^m)}$ if and only if 2 is a square in $\mathbb{F}_p^\times$. For if 2 were a square in $\mathbb{F}_q^\times$ but not a square in $\mathbb{F}_p^\times$ then a square root for 2 in $\mathbb{F}_q$ would be an element of degree 2 over $\mathbb{F}_p$. But this is not possible as the field extension $\mathbb{F}_q/\mathbb{F}_p$ has odd degree m. So now $Sgn_{\mathbb{F}^\times}(2)$ coincides with the Legendre symbol, which for $p \equiv 3 \pmod 4$ is (see [**7**, Theorem 95])

$$\left(\frac{2}{p}\right) = \left\{ \begin{array}{rl} 1 & \text{if } p \equiv 7 \pmod 8 \\ -1 & \text{if } p \equiv 3 \pmod 8 \end{array} \right\} = i^{(p+1)/2}.$$

This completes the proof. □

REMARK 3.2. Setting $z = 0$ in the formula from Lemma 3.1 gives the following expression for the trace character of $\omega_\psi|_{U_1}$.

$$Tr(\omega_\psi(k)) = \left\{ \begin{array}{cl} q & \text{if } k = 1 \\ Sgn_{\mathbb{F}^\times}\left(\frac{a-1}{2}\right) & \text{for } k = a + b\sqrt{\varepsilon} \neq 1 \end{array} \right. .$$

The character for the oscillator representation of $Sp(n, \mathbb{F})$ is the subject of [**9, 6, 17**].

Let $\rho_{\psi,\chi}$ be a type 1 representation of G (see (3.1)) and write

$$\phi_{\psi,\chi}(z,t) = \frac{1}{q+1} \sum_{k \in U_1} Tr(\rho_{\psi,\chi}(k,z,t)).$$

In view of (3.2) one has $\phi_{\psi,\chi}(z,t) = \phi_{\psi,\chi}(z)\psi(t)$ where

$$\phi_{\psi,\chi}(z) = \frac{1}{q+1} \sum_{k \in U_1} \chi(k) Tr(\pi_\psi(z)\omega_\psi(k)). \tag{3.11}$$

The result is a spherical function when $\rho_{\psi,\chi} \in \widehat{G}_{U_1}$. We will now complete the proof for Theorem 1.1 by deriving Formula 1.1 for the type 1 spherical functions. The parameter $\widetilde{\chi}$ in (1.1) is related to χ in a simple way, Equation 3.14 below. We will show that the condition that $\rho_{\psi,\chi}$ be U_1-spherical requires $\widetilde{\chi}$ to be non-trivial.

Proof of Theorem 1.1. The group homomorphism

$$\eta : \widetilde{\mathbb{F}}^\times \to U_1, \quad \eta(w) = \frac{w}{\overline{w}} = \frac{w^2}{N(w)} \tag{3.12}$$

has kernel $\mathbb{F}^\times$ and hence image of order $|\widetilde{\mathbb{F}}^\times|/|\mathbb{F}^\times| = q + 1 = |U_1|$. So η is a group epimorphism.[2] We use the map η to lift the summation over U_1 in (3.11) to a summation over $\widetilde{\mathbb{F}}^\times$. Lemma 3.1 now yields

$$\begin{aligned} \phi_{\psi,\chi}(z) &= \frac{1}{q^2-1} \sum_{w \in \widetilde{\mathbb{F}}^\times} \chi(\eta(w)) Tr[\pi_\psi(z)\omega_\psi(\eta(w))] \\ &= \frac{1}{q^2-1} \Bigg[(q-1)q\delta_{z,0} + (q-1)\chi(-1)Sgn_{\mathbb{F}^\times}(-1) \\ &\quad + \sum_{w \in \widetilde{\mathbb{F}}^\times - (\mathbb{F}^\times \cup \mathbb{F}^\times\sqrt{\varepsilon})} \chi(\eta(w)) Sgn_{\mathbb{F}^\times}\left(\frac{Re(\eta(w)) - 1}{2}\right) \psi\left(-\frac{1 + Re(\eta(w))}{4Im(\eta(w))\varepsilon} N(z)\right)\Bigg]. \end{aligned}$$

Taking $(w = x + y\sqrt{\varepsilon}) \in \widetilde{\mathbb{F}}^\times$ one computes

$$Sgn_{\mathbb{F}^\times}\left(\frac{Re(\eta(w)) - 1}{2}\right) = Sgn_{\mathbb{F}^\times}\left(\frac{y^2\varepsilon}{x^2 - y^2\varepsilon}\right) = -Sgn_{\mathbb{F}^\times}(N(w)),$$

[2]This observation is a special case of *Hilbert's Satz 90* concerning cyclic field extensions. See [**11**, Theorem 4.28] for the general result.

$$\frac{1+Re(\eta(w))}{Im(\eta(w))} = \frac{x}{y} = \frac{Re(w)}{Im(w)} \qquad \text{for } Im(w) \neq 0.$$

So now

$$\phi_{\psi,\chi}(z) = \frac{1}{q^2-1}\Bigg[(q-1)q\delta_{z,0} + (q-1)Sgn_{\mathbb{F}^\times}(-1)\chi(-1) - \sum_{w\in\widetilde{\mathbb{F}}^\times-(\mathbb{F}^\times\cup\mathbb{F}^\times\sqrt{\varepsilon})} Sgn_{\mathbb{F}^\times}(N(w))\chi(\eta(w))\psi\left(-\frac{1}{4\varepsilon}\frac{Re(w)}{Im(w)}N(z)\right)\Bigg].$$

But for the $(q-1)$ pure imaginary elements $w = y\sqrt{\varepsilon}$ in $\widetilde{\mathbb{F}}^\times$ one has

$$\begin{aligned} Sgn_{\mathbb{F}^\times}(N(w))\chi(\eta(w))\psi\left(-\frac{1}{4\varepsilon}\frac{Re(w)}{Im(w)}N(z)\right) &= Sgn_{\mathbb{F}^\times}(-y^2\varepsilon)\chi(-1)\psi(0) \\ &= -Sgn_{\mathbb{F}^\times}(-1)\chi(-1). \end{aligned}$$

So we can write $\phi_{\psi,\chi} = \phi_{\psi,\widetilde{\chi}}$ where

$$\phi_{\psi,\widetilde{\chi}}(z) = \frac{1}{q^2-1}\left[(q-1)q\delta_{z,0} - \sum_{w\in\widetilde{\mathbb{F}}-\mathbb{F}} \widetilde{\chi}(w)\psi\left(-\frac{1}{4\varepsilon}\frac{Re(w)}{Im(w)}N(z)\right)\right] \tag{3.13}$$

and

$$\widetilde{\chi}(w) = Sgn_{\mathbb{F}^\times}(N(w))\chi(\eta(w)) = Sgn_{\widetilde{\mathbb{F}}^\times}(w)\chi(\eta(w)). \tag{3.14}$$

Each $\widetilde{\chi}$ is a multiplicative character on $\widetilde{\mathbb{F}}$ whose restriction to $\mathbb{F}^\times$ is trivial. Distinct characters $\chi \in \widehat{U_1}$ yield distinct characters $\widetilde{\chi} \in (\widetilde{\mathbb{F}}^\times)\widehat{}$ because η is surjective. As $|\widetilde{\mathbb{F}}^\times/\mathbb{F}^\times| = q+1 = |U_1|$ we conclude that

$$\{\widetilde{\chi} \;:\; \chi \in \widehat{U_1}\} = (\widetilde{\mathbb{F}}^\times/\mathbb{F}^\times)\widehat{},$$

the set of all multiplicative characters on $\widetilde{\mathbb{F}}$ whose restrictions to $\mathbb{F}^\times$ are trivial. Thus (3.13) is precisely the formula given for type 1 spherical functions in the statement of Theorem 1.1. In fact, however, (3.13) is a spherical function only when $\rho_{\psi,\chi}$ is a spherical representation. It remains to verify that this corresponds to the condition that $\widetilde{\chi}$ be non-trivial. For this it suffices to evaluate $\phi_{\psi,\widetilde{\chi}}$ at the identity in H_1. The result must be 1 or 0, depending on whether or not $\rho_{\psi,\chi}$ is a spherical representation.

Equation 3.13 yields

$$\phi_{\psi,\widetilde{\chi}}(0) = \frac{q}{q+1} - \frac{1}{q^2-1}\sum_{w\in\widetilde{\mathbb{F}}-\mathbb{F}} \widetilde{\chi}(w).$$

When $\widetilde{\chi} \equiv 1$ this gives $\phi_{\psi,\widetilde{\chi}}(0) = 0$. When $\widetilde{\chi}$ is non-trivial one has

$$0 = \sum_{w\in\widetilde{\mathbb{F}}^\times} \widetilde{\chi}(w) = \sum_{w\in\widetilde{\mathbb{F}}-\mathbb{F}} \widetilde{\chi}(w) + \sum_{w\in\mathbb{F}^\times} \widetilde{\chi}(w) = \sum_{w\in\widetilde{\mathbb{F}}-\mathbb{F}} \widetilde{\chi}(w) + (q-1),$$

so $\sum_{w\in\widetilde{\mathbb{F}}-\mathbb{F}} \widetilde{\chi}(w) = -(q-1)$ and $\phi_{\psi,\widetilde{\chi}}(0) = 1$. Hence the type 1 spherical functions are obtained by taking $\widetilde{\chi} \in (\widetilde{\mathbb{F}}^\times/\mathbb{F}^\times)\widehat{}$ non-trivial as claimed. This completes the proof for Theorem 1.1. $\square$

REMARK 3.3. As $\phi_{\psi,\widetilde{\chi}}(0) = dim(\mathcal{H}^{U_1}_{\rho_{\psi,\chi}})$ the proof shows $dim(\mathcal{H}^{U_1}_{\rho}) \leq 1$ for all type 1 representations $\rho \in \widehat{G}$. This gives another proof of the fact that (U_1, H_1) is a Gelfand pair. Now in view of Proposition 2.1 the restriction $\omega_\psi|_{U_1}$ of the oscillator representation to U_1 is multiplicity free for each non-trivial $\psi \in \widehat{\mathbb{F}}$. Hence the q-dimensional representation $\omega_\psi|_{U_1}$ decomposes as the sum of all but one of the $q+1$ distinct characters $\chi \in \widehat{U_1}$. But the sign characters for U_1 and $\widetilde{\mathbb{F}}^\times$ are related via $Sgn_{U_1} \circ \eta = Sgn_{\widetilde{\mathbb{F}}^\times}$ and hence $\widetilde{Sgn_{U_1}} = \left(Sgn_{\widetilde{\mathbb{F}}^\times}\right)^2 \equiv 1$. The proof shows that $\rho_{\psi,\chi}$ is U_1-spherical if and only if $\widetilde{\chi} \neq 1$. Equivalently χ occurs in $\omega_\psi|_{U_1}$ if and only if $\chi \neq Sgn_{U_1}$. This proves the following result of Gérardin.[3]

LEMMA 3.4. [**6**, Theorem 3.3(c)] *The representation $\omega_\psi|_{U_1}$ decomposes as the sum of all characters $\chi \neq Sgn_{U_1}$ in $\widehat{U_1}$.*

Proof of Corollary 1.3. Working from (1.1) one has, for $z \neq 0$,

$$\begin{aligned}\phi_{\psi,\widetilde{\chi}}(z) &= -\frac{1}{q^2-1}\sum_{w\in\widetilde{\mathbb{F}}-\mathbb{F}} \widetilde{\chi}(w)\psi\left(-\frac{1}{4\varepsilon}\frac{Re(w)}{Im(w)}N(z)\right)\\ &= -\frac{1}{q^2-1}\sum_{a\in\mathbb{F}}\sum_{b\in\mathbb{F}^\times}\widetilde{\chi}\big(b(a+\sqrt{\varepsilon})\big)\psi\left(-\frac{1}{4\varepsilon}\frac{ba}{b}N(z)\right)\\ &= -\frac{q-1}{q^2-1}\sum_{a\in\mathbb{F}}\widetilde{\chi}(a+\sqrt{\varepsilon})\psi\left(-\frac{a}{4\varepsilon}N(z)\right)\\ &= -\frac{1}{q+1}\sum_{a\in\mathbb{F}}\chi'(a)\psi\left(-\frac{a}{4\varepsilon}N(z)\right),\end{aligned}$$

where now

$$\chi'(a) = \widetilde{\chi}(a+\sqrt{\varepsilon}). \tag{3.15}$$

As $\widetilde{\chi}$ is non-trivial one has $\chi' \not\equiv 1$ and one can check that χ' satisfies condition (1.4) in the statement of Corollary 1.3.

To complete the proof we will show that every map $\chi' : \mathbb{F} \to \mathbb{C}$ satisfying (1.4) has the form (3.15) for some character $\widetilde{\chi} \in (\widetilde{\mathbb{F}}^\times/\mathbb{F}^\times)^\wedge$. Let χ' satisfy (1.4) and define $\widetilde{\chi} : \widetilde{\mathbb{F}}^\times \to \mathbb{C}$ as

$$\widetilde{\chi}(x+y\sqrt{\varepsilon}) = \begin{cases} 1 & \text{if } y = 0 \\ \chi'(x/y) & \text{if } y \neq 0 \end{cases}.$$

By construction $\chi'(a) = \widetilde{\chi}(a+\sqrt{\varepsilon})$ for $a \in \mathbb{F}$ and $\widetilde{\chi}|_{\mathbb{F}^\times} \equiv 1$. Finally a routine computation shows that $\widetilde{\chi}$ is a multiplicative character on $\widetilde{\mathbb{F}}$. □

4. Type 1 spherical functions for $(U(n,\widetilde{\mathbb{F}}),\ H_n(\mathbb{F}))$

We now write $H_n = H_n(\mathbb{F})$, $U_n = U(n,\mathbb{F})$ and let $\omega^{(n)}_\psi$ denote the restriction of the oscillator representation from $Sp(n,\mathbb{F})$ to U_n. Lemma 3.4 asserts that $\omega^{(1)}_\psi$

[3] A different convention for the oscillator representation on $U(1,\widetilde{\mathbb{F}})$ is employed in [**6**]. In our context this amounts to replacing $\omega_\psi|_{U_1}$ by $Sgn_{U_1} \otimes \omega_\psi|_{U_1}$. This twisted variant of $\omega_\psi|_{U_1}$ then decomposes as the sum of all non-trivial characters for U_1.

decomposes with q irreducible components,

$$\omega_\psi^{(1)} \simeq \bigoplus_{\chi\in\mathcal{S}} \chi \quad \text{where} \quad \mathcal{S} = \widehat{U_1} - \{Sgn_{U_1}\}. \tag{4.1}$$

When $n \geq 2$, however, $\omega_\psi^{(n)}$ has $q+1$ irreducible constituents. This is a consequence of Lemma 2.2 together with the following:

Lemma 4.1. *For $n \geq 2$ the group $U_n = U(n, \mathbb{F}_q)$ has $q+1$ orbits on $\mathcal{W} = \widetilde{\mathbb{F}}^n$.*

Proof. Equivalently the non-zero vectors in $\mathcal{W}$ form q orbits. We will outline an argument, omitting details. To begin it can be shown that each U_n-orbit in $\mathcal{W} = \widetilde{\mathbb{F}}^n$ meets $\widetilde{\mathbb{F}}^2 \subset \widetilde{\mathbb{F}}^n$. So it suffices to consider the case $n = 2$.

Suppose that $\mathbf{z} \neq 0$ is not an *isotropic vector* in $\mathcal{W} = \widetilde{\mathbb{F}}^2$. That is, $\langle \mathbf{z}, \mathbf{z}\rangle \neq 0$. One shows that $U_2 \cdot \mathbf{z}$ contains a vector of the form $(\lambda, 0)$ with $\lambda \in \widetilde{\mathbb{F}}^\times$. Moreover $U_2 \cdot (\lambda, 0) = U_n \cdot (\lambda', 0)$ if and only if $N(\lambda) = N(\lambda')$. As the norm mapping is surjective it follows that the non-isotropic vectors form $|\mathbb{F}^\times| = q-1$ orbits.

Next let

$$\mathcal{N} = \{\mathbf{z} \in \mathcal{W} \;:\; \mathbf{z} \neq 0,\ \langle \mathbf{z}, \mathbf{z}\rangle = 0\}$$

denote the set of non-zero isotropic vectors. One has $\mathcal{N} \neq \emptyset$ since, for example, $(1, \lambda) \in \mathcal{N}$ for any $\lambda \in \widetilde{\mathbb{F}}^\times$ with $N(\lambda) = -1$. It can be shown that $U_2 \cdot (1, \lambda) = \mathcal{N}$. Thus $\mathcal{N}$ is a single U_2-orbit, completing the proof. $\square$

Now take $n \geq 2$, let T^n denote the torus

$$T^n = \underbrace{U_1 \times \cdots \times U_1}_{n \text{ times}},$$

and identify U_1 with the scalar matrices kI_n in U_n $(k \in U_1)$ so that

$$U_1 \subset T^n \subset U_n.$$

As U_1 is a central subgroup in U_n each irreducible representation $\sigma \in \widehat{U_n}$ is given on U_1 by some character $\chi^\sigma \in \widehat{U_1}$:

$$\sigma(kI_n) = \chi^\sigma(k) I_{\mathcal{H}_\sigma} \quad (k \in U_1).$$

We will refer to χ^σ as the *central character* for σ.

Lemma 4.2. *Let $\sigma \in \widehat{U_n}$ and suppose that σ occurs in $\omega_\psi^{(n)}$. Then*

$$\sigma|_{T^n} \simeq \bigoplus \big(\chi_1 \otimes \chi_2 \otimes \cdots \otimes \chi_n\big)$$

where the sum is over all $(\chi_1, \ldots, \chi_n) \in \mathcal{S}^n$ $(\mathcal{S} = \widehat{U_1} - \{Sgn_{U_1}\})$ with product $\chi_1\chi_2\cdots\chi_n = \chi^\sigma$. Moreover each character $\chi \in \widehat{U_1}$ arises as the central character $\chi = \chi^\sigma$ for precisely one irreducible representation $\sigma \leq \omega_\psi^{(n)}$.

Proof. The restriction of $\omega_\psi^{(n)}$ to T^n can be identified with the tensor product representation $\omega_\psi^{(1)} \otimes \cdots \otimes \omega_\psi^{(1)}$. So, by (4.1),

$$\omega_\psi^{(n)}|_{T^n} \simeq \bigoplus_{\chi_j\in\mathcal{S}} \big(\chi_1 \otimes \cdots \otimes \chi_n\big). \tag{4.2}$$

Each representation $\chi_1 \otimes \cdots \otimes \chi_n$ in $\omega_\psi^{(n)}|_{T^n}$ occurs in the restriction of exactly one $\sigma \leq \omega_\psi^{(n)}$. As the restriction of $\chi_1 \otimes \cdots \otimes \chi_n$ to the scalar matrices is given by $kI_n \mapsto \chi_1(k) \cdots \chi_n(k)$ we have that

$$\chi_1 \otimes \cdots \otimes \chi_n \leq \sigma|_{T^n} \implies \chi_1\chi_2 \cdots \chi_n = \chi^\sigma.$$

On the other hand it is easy to see that each of the $q+1$ characters χ for U_1 can be obtained as a product $\chi = \chi_1 \cdots \chi_n$ with each factor $\chi_j \neq Sgn_{U_1}$. So all characters $\chi \in \widehat{U_1}$ appear as central characters χ^σ for irreducible representations $\sigma \leq \omega_\psi^{(n)}$. As we know $\omega_\psi^{(n)}$ has exactly $q+1$ irreducible constituents it now follows that the mapping

$$Spec(\omega_\psi^{(n)}) \to \widehat{U_1}, \quad \sigma \mapsto \chi^\sigma$$

is a bijection. So now

- each $\sigma \leq \omega_\psi^{(n)}$ is determined by its central character χ^σ,
- every $\chi \in \widehat{U_1}$ has the form $\chi = \chi^\sigma$ for just one $\sigma \leq \omega_\psi^{(n)}$, and
- for $\chi_1, \ldots, \chi_n \in \mathcal{S}$ we have $\chi_1 \otimes \cdots \otimes \chi_n \leq \sigma|_{T^n}$ if and only if $\chi_1 \cdots \chi_n = \chi^\sigma$.

This completes the proof. □

In view of Lemma 4.2 we now let $\sigma_\chi^{(n)} \in \widehat{U_n}$ denote the unique irreducible representation occurring in $\omega_\psi^{(n)}$ with central character $\chi^\sigma = \chi$.

LEMMA 4.3. *For $n \geq 2$ the dimension $d_n(\chi)$ of $\sigma_\chi^{(n)}$ is given by*

$$d_n(\chi) = \begin{cases} \frac{q^n+(-1)^{n-1}}{q+1} & \text{for } \chi \neq (Sgn_{U_1})^n \\ \frac{q^n+(-1)^n q}{q+1} & \text{for } \chi = (Sgn_{U_1})^n \end{cases}.$$

Here $(Sgn_{U_1})^n = 1$ or Sgn_{U_1} according to the parity of n. In any case $\omega_\psi^{(n)}$ decomposes a a sum of q irreducibles of dimension $(q^n + (-1)^{n-1})/(q+1)$ and one of dimension $(q^n + (-1)^n q)/(q+1)$. This is consistent with the fact that $dim(\mathbb{C}[\mathbb{F}^n]) = q^n$.

PROOF. For $n = 2$ we have

$$\begin{aligned} d_2(\chi) &= \left|\{(\chi_1, \chi_2) \in \mathcal{S}^2 \ : \ \chi_1\chi_2 = \chi\}\right| = \left|\{\chi_1 \in \widehat{U_1} \ : \ \chi_1 \neq Sgn_{U_1}, \, \chi \cdot Sgn_{U_1}\}\right| \\ &= \begin{cases} q-1 & \text{for } \chi \neq 1 \\ q & \text{for } \chi = 1 \end{cases}, \end{aligned}$$

which agrees with the formula in the statement of the lemma. For $n \geq 3$ one has the recurrence

$$\begin{aligned} d_n(\chi) &= \left|\{(\chi_1, \ldots, \chi_n) \in \mathcal{S}^n \ : \ \chi_1 \cdots \chi_n = \chi\}\right| \\ &= \left|\bigcup_{\chi' \in \mathcal{S}} \{(\chi_1, \ldots, \chi_{n-1}) \in \mathcal{S}^{n-1} \ : \ \chi_1 \cdots \chi_{n-1} = \chi\chi'\}\right| \\ &= \sum_{\chi' \in \mathcal{S}} d_{n-1}(\chi\chi'), \end{aligned}$$

which can be used to complete the proof using induction on n. When $\chi = (Sgn_{U_1})^n$ we have $\chi\chi' \neq (Sgn_{U_1})^{n-1}$ for each $\chi' \in \mathcal{S}$. In this case the sum yields

$$d_n(\chi) = q\left[\frac{q^{n-1} + (-1)^{n-2}}{q+1}\right] = \frac{q^n + (-1)^n q}{q+1}.$$

When $\chi \neq (Sgn_{U_1})^n$ we have $\chi\chi' = (Sgn_{U_1})^{n-1}$ for exactly one $\chi' \in \mathcal{S}$, namely $\chi' = \chi^{-1}(Sgn_{U_1})^{n-1}$. In this case the sum yields

$$d_n(\chi) = (q-1)\left[\frac{q^{n-1} + (-1)^{n-2}}{q+1}\right] + \frac{q^{n-1} + (-1)^{n-1}q}{q+1} = \frac{q^n + (-1)^{n-1}}{q+1}$$

as claimed. □

Now (T^n, H_n) is a Gelfand pair since (4.2) shows $\omega_\psi|_{T^n}$ to be multiplicity free. As T^n is a subgroup of U_n it follows that (U_n, H_n) is also a Gelfand pair.

Proof of Theorem 1.2. We will apply Equation 2.8 with $K = U_n$, $K' = T^n$. The type 1 spherical functions for (T^n, H_n) are indexed by $(\widehat{\mathbb{F}} - \{1\}) \times \mathcal{S}^n$ and can be written as

$$\phi_{\psi;\chi_1,\ldots,\chi_n}(\mathbf{z}, t) = \phi^1_{\psi,\chi_1}(z_1)\cdots\phi^1_{\psi,\chi_n}(z_n)\psi(t)$$

where ϕ^1_{ψ,χ_j} denotes the type 1 spherical function for (U_1, H_1) with parameters (ψ, χ_j). On the other hand the type 1 spherical functions for (U_n, H_n) are indexed by

$$(\widehat{\mathbb{F}} - \{1\}) \times Spec(\omega_\psi^{(n)}) = (\widehat{\mathbb{F}} - \{1\}) \times \{\sigma_\chi^{(n)} \ : \ \chi \in \widehat{U_1}\}.$$

Lemma 4.2 in combination with (2.8) shows that $\phi_{\psi,\chi} = \phi_{\psi,\sigma_\chi^{(n)}}$ satisfies

$$\phi_{\psi,\chi}(\mathbf{z}) = \frac{1}{d_n(\chi)}\sum \phi^1_{\psi,\chi_1}(z_1)\cdots\phi^1_{\psi,\chi_n}(z_n)$$

where the sum is over all $(\chi_1, \ldots, \chi_n) \in \mathcal{S}^n$ with $\chi_1\cdots\chi_n = \chi$ and $d_n(\chi) = dim(\sigma_\chi^{(n)})$ is as in Lemma 4.3. Finally lifting each $\chi \in \widehat{U_1}$ to a character $\widetilde{\chi}$ on $\widetilde{\mathbb{F}}^\times$, via Formula 3.14, puts the type 1 spherical functions for (U_n, H_n) in the form given in the statement of Theorem 1.2. □

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