Gelfand pairs associated with finite Heisenberg groups

Chal Benson and Gail Ratcliff

Dept of Mathematics, East Carolina University, Greenville, NC 27858
bensonf@ecu.edu, ratcliffg@ecu.edu

A topological group G together with a compact subgroup K are said to form a *Gelfand pair* if the set $L^1(K\backslash G/K)$ of K-bi-invariant integrable functions on G is a commutative algebra under convolution. The situation where G and K are Lie groups has been the focus of extensive and ongoing investigation. Riemannian symmetric spaces G/K furnish the most widely studied and best understood examples. ([Hel84] is a standard reference.) Apart from these, key examples arise as semi-direct products $G = K \ltimes N$, of compact Lie groups K with two-step nilpotent Lie groups N. Such pairs are the focus of [BJR90], [Vin03] and [Yak06], among other works. There are many examples where $N = H_n(\mathbb{R})$, a (real) Heisenberg group, and K is a subgroup of the unitary group $U(n)$.

Gelfand pairs also arise in connection with analysis on *finite* groups, but, to our knowledge, have been studied less extensively. Known examples include the symmetric group modulo the hyperoctahedral group [Mac] and finite analogues of the hyperbolic plane [SA87, Ter99]. In this paper we introduce a family of Gelfand pairs associated with finite Heisenberg groups. They provide finite analogues for the Gelfand pairs associated with $H_n(\mathbb{R})$. Our examples appear elsewhere, in the literature on the *oscillator representation*, but their relevance to the study of Gelfand pairs has not, however, been previously emphasized.

1 Preliminaries

To begin we must establish notation and recall some ideas concerning the representation theory for Heisenberg groups over finite fields. For a finite set S, the symbol $\mathbb{C}[S]$ will denote the set of all $\mathbb{C}$-valued functions on S. This is a complex vector space of dimension $|S|$ which carries a positive-definite hermitian inner product

$$\langle f, g\rangle_S = \frac{1}{|S|}\sum_{x\in S} f(x)\overline{g(x)}.$$

1.1 Heisenberg groups

Let $\mathbb{F}$ be a field of odd characteristic. The polarized *Heisenberg group* $H_n(\mathbb{F})$ is the set
$$H_n(\mathbb{F}) = \mathbb{F}^n \times \mathbb{F}^n \times \mathbb{F}$$
with product
$$(\mathbf{x}, \mathbf{y}, t)(\mathbf{x}', \mathbf{y}', t') = \left(\mathbf{x} + \mathbf{x}', \mathbf{y} + \mathbf{y}', t + t' + \frac{1}{2}(\mathbf{x} \cdot \mathbf{y}' - \mathbf{y} \cdot \mathbf{x}')\right). \tag{1}$$
Inclusion of the factor $1/2$ is motivated by the use of exponential coordinates in connection with the real Heisenberg group $H_n(\mathbb{R})$. (See [Fol89].) Some authors omit this factor but the resulting group is isomorphic with that defined here via the mapping $(\mathbf{x}, \mathbf{y}, t) \mapsto (\mathbf{x}, \mathbf{y}, 2t)$.

An alternate notation is useful in connection with certain examples and constructions. We write
$$\mathcal{W} = \mathcal{W}_n = \mathbb{F}^n \times \mathbb{F}^n,$$
so that $H_n(\mathbb{F}) = \mathcal{W} \times \mathbb{F}$ and Equation (1) becomes
$$(\mathbf{z}, t)(\mathbf{z}', t') = (\mathbf{z} + \mathbf{z}', t + t' + 2^{-1}[\mathbf{z}, \mathbf{z}']) \tag{2}$$
where $[\mathbf{z}, \mathbf{z}']$ denotes the usual symplectic form on $\mathcal{W}$, namely
$$[\mathbf{z}, \mathbf{z}'] = [(\mathbf{x}, \mathbf{y}), (\mathbf{x}', \mathbf{y}')] = \mathbf{x} \cdot \mathbf{y}' - \mathbf{y} \cdot \mathbf{x}'. \tag{3}$$
More generally, for *any* finite dimensional symplectic vector space $(\mathcal{W}, [\cdot, \cdot])$ over $\mathbb{F}$ we let
$$H_{\mathcal{W}} = \mathcal{W} \times \mathbb{F} \text{ with product given by (2).} \tag{4}$$

1.2 Unitary dual of $H_n(\mathbb{F}_q)$

Throughout this paper we take
$$\mathbb{F} = \mathbb{F}_q,$$
the finite field with q elements where $q = p^m$ for some odd prime p. The field $\mathbb{F}$ is an extension of its prime field $\mathbb{Z}_p = \mathbb{Z}/(p\mathbb{Z})$. The characters on $\mathbb{F}$ are
$$\widehat{\mathbb{F}} = \{\psi_a \ : \ a \in \mathbb{F}\}$$
where
$$\psi_a(t) = \exp\left(\frac{2\pi i}{p} \mathrm{Tr}_{\mathbb{F}/\mathbb{Z}_p}(at)\right) \tag{5}$$
and
$$\mathrm{Tr}_{\mathbb{F}/\mathbb{Z}_p} : \mathbb{F} \to \mathbb{Z}_p$$

is the trace map for the field extension $\mathbb{F}/\mathbb{Z}_p$. Explicitly one can write

$$\mathrm{Tr}_{\mathbb{F}/\mathbb{Z}_p}(t) = t + t^p + t^{p^2} + \cdots + t^{p^{m-1}}.$$

(See Chapter 2 in [LN].) The basic identity

$$\sum_{t\in\mathbb{F}} \psi_a(t) = \left\{ \begin{matrix} q \text{ if } a = 0 \\ 0 \text{ if } a \neq 0 \end{matrix} \right\} = q\delta_{a,0}. \tag{6}$$

shows that the characters are pair-wise orthogonal unit vectors in $\mathbb{C}[\mathbb{F}]$.

The mappings

$$\Psi_{\mathbf{a},\mathbf{b}} : H_n(\mathbb{F}) \to \mathbb{T}, \qquad \Psi_{\mathbf{a},\mathbf{b}}(\mathbf{x},\mathbf{y},t) = \prod_{i=1}^{n} \psi_{a_i}(x_i) \prod_{j=1}^{n} \psi_{b_j}(y_j) \tag{7}$$

for $\mathbf{a},\mathbf{b} \in \mathbb{F}^n$ give q^{2n} distinct 1-dimensional representations of $H_n(\mathbb{F})$. One can verify, moreover, that for $\lambda \in \mathbb{F}^\times$, the formula

$$\pi_\lambda(\mathbf{x},\mathbf{y},t)f(\mathbf{u}) = \psi_\lambda\left(t + \mathbf{y}\cdot\mathbf{u} + 2^{-1}\mathbf{x}\cdot\mathbf{y}\right) f(\mathbf{u}+\mathbf{x}) \tag{8}$$

defines a unitary representation (analogous to the Schrödinger model in the real case) of $H_n(\mathbb{F})$ in the the inner product space $\mathbb{C}[\mathbb{F}^n]$. The trace character $\chi_\lambda(\mathbf{x},\mathbf{y},t) = tr(\pi_\lambda(\mathbf{x},\mathbf{y},t))$ for π_λ is

$$\chi_\lambda(\mathbf{x},\mathbf{y},t) = q^n\delta_{\mathbf{x},0}\delta_{\mathbf{y},0}\psi_\lambda(t),$$

which yields

$$\langle \chi_\lambda, \chi_{\lambda'}\rangle_{H_n(\mathbb{F})} = \langle \psi_\lambda, \psi_{\lambda'}\rangle_{\mathbb{F}} = \delta_{\lambda,\lambda'}$$

in view of orthogonality for the characters of $\mathbb{F}$. It follows that the representations $\{\pi_\lambda \;:\; \lambda \in \mathbb{F}^\times\}$ are inequivalent and irreducible.

Summing the squares of the dimensions for the representations (7) and (8) gives

$$q^{2n} \times 1^2 + (q-1) \times (q^n)^2 = q^{2n+1} = |H_n(\mathbb{F})|.$$

Thus (7) and (8) exhaust the unitary dual of $H_n(\mathbb{F})$:

$$\widehat{H_n(\mathbb{F})} = \{\Psi_{\mathbf{a},\mathbf{b}} \;:\; \mathbf{a},\mathbf{b} \in \mathbb{F}^n\} \uplus \{\pi_\lambda \;:\; \lambda \in \mathbb{F}^\times\}. \tag{9}$$

On the center of $H_n(\mathbb{F})$ we have $\pi_\lambda(0,0,t) = \psi_\lambda(t)I_{\mathbb{C}[\mathbb{F}^n]}$. So the q^n-dimensional irreducible representations are determined by their central characters. This proves:

Theorem 1.1 (Stone-von Neumann Theorem) *Let $\lambda \in \mathbb{F}^\times$ and $\beta : H_n(F) \to U(V)$ be an irreducible unitary representation with central character ψ_λ. (That is, $\beta(0,0,t) = \psi_\lambda(t)I_V$.) Then β is unitarily equivalent to the Schrödinger representation π_λ defined by Equation 8.*

1.3 Oscillator representation

The symplectic group for $\big(\mathcal{W} = \mathbb{F}^n \times \mathbb{F}^n, [\cdot,\cdot]\big)$,

$$Sp(n,\mathbb{F}) = \{g \in GL(2n,\mathbb{F}) \ : \ [g\mathbf{z}, g\mathbf{z}'] = [\mathbf{z},\mathbf{z}']\},$$

acts by automorphisms on $H_n(\mathbb{F})$ via

$$g \cdot (\mathbf{z}, t) = (g\mathbf{z}, t).$$

Fix $\lambda \in \mathbb{F}^\times$. For given $g \in Sp(n,\mathbb{F})$,

$$(\mathbf{z}, t) \mapsto \pi_\lambda \circ g(\mathbf{z}, t) = \pi_\lambda(g\mathbf{z}, t)$$

is an irreducible representation with central character ψ_λ. The Stone-von Neumann Theorem ensures that $\pi_\lambda \circ g$ is unitarily equivalent to π_λ. Thus there is a unitary operator $\omega_\lambda(g)$ on $\mathbb{C}[\mathbb{F}^n]$ satisfying

$$\pi_\lambda(g\mathbf{z}, t) = \omega_\lambda(g)\pi_\lambda(\mathbf{z}, t)\omega_\lambda(g)^{-1}. \tag{10}$$

Schur's Lemma shows that (10) defines $\omega_\lambda(g)$ up to a multiplicative scalar of modulus one. In the context of finite fields, there is a systematic choice of scalars for which

$$\omega_\lambda : Sp(n,\mathbb{F}) \to U(\mathbb{C}[\mathbb{F}^n])$$

is a representation of the group $Sp(n,\mathbb{F})$. In the literature, ω_λ is variously called the *oscillator*, *metaplectic*, or *Weil-Segal-Shale representation*. It is known that $Sp(n,\mathbb{F})$ coincides with its commutator subgroup provided $n > 1$ or $q > 3$. Thus (10) completely determines the representation ω_λ, except when $n = 1$ and $q = 3$. (See [How].)

The contragredient representation π_λ^* for π_λ has central character $\psi_{-\lambda}$. Thus π_λ^* is unitarily equivalent to $\pi_{-\lambda}$, by the Stone-von Neumann Theorem. Moreover the contragredient ω_λ^* of the oscillator representation satisfies $\pi_\lambda^*(g\mathbf{z}, t) = \omega_\lambda^*(g)\pi_\lambda^*(\mathbf{z}, t)\omega_\lambda^*(g)^{-1}$. It follows that

$$\omega_\lambda^* \text{ is unitarily equivalent to } \omega_{-\lambda}. \tag{11}$$

There are, in fact, just two distinct oscillator representations ω_λ, up to unitary equivalence. Indeed

Proposition 1.2 (See [How], [Neu02]) *For $\lambda, \lambda' \in \mathbb{F}^\times$ one has $\omega_\lambda' \simeq \omega_\lambda$ if and only if λ'/λ is a square in $\mathbb{F}^\times$.*

The oscillator representation can be rendered explicitly, at least on a set of generators for $Sp(n,\mathbb{F})$. The formulas are given below in Theorem 1.3. Writing $(2n) \times (2n)$-matrices in block form,

$$g = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \qquad (A, B, C, D \text{ of size } n \times n),$$

one has

$$g \in Sp(n,\mathbb{F}) \iff \{A^tC = C^tA, \quad B^tD = D^tB, \quad A^tD - C^tB = I\}.$$

The group $Sp(n,\mathbb{F})$ is generated by the subset

$$\{A_{diag} \; : \; A \in GL(n,\mathbb{F})\} \cup \{C_{lower} \; : \; C^t = C\} \cup \{J\}$$

where

$$A_{diag} = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & (A^t)^{-1} \end{array}\right], \quad C_{lower} = \left[\begin{array}{c|c} I & 0 \\ \hline C & I \end{array}\right], \quad J = \left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right]. \tag{12}$$

Theorem 1.3 (See [Neu02]) *For $\lambda \in \mathbb{F}_q^\times$ ($q = p^m$) the oscillator representation*

$$\omega_\lambda : Sp(n,\mathbb{F}) \to U(\mathbb{C}[\mathbb{F}^n])$$

is given on the generators (12) for $Sp(n,\mathbb{F})$ as follows.

- $\omega_\lambda(A_{diag})f(\mathbf{u}) = sgn(\det A)f(A^{-1}\mathbf{u})$ *where*

$$sgn(t) = \begin{cases} +1 \text{ if } t \text{ is a square in } \mathbb{F} \\ -1 \text{ otherwise} \end{cases}.$$

- $\omega_\lambda(C_{lower})f(\mathbf{u}) = \psi_\lambda\left(-\frac{1}{2}\mathbf{u}^tC\mathbf{u}\right)f(\mathbf{u})$.
- $\omega_\lambda(J)f(\mathbf{u}) = (-1)^{n(m+1)}(-i)^{nm(p-1)/2}sgn(\lambda)\overline{\mathcal{F}}_\lambda f(\mathbf{u})$ *where*

$$\overline{\mathcal{F}}_\lambda f(\mathbf{u}) = \frac{1}{\sqrt{q^n}} \sum_{\mathbf{x}\in\mathbb{F}^n} f(\mathbf{x})\psi_\lambda(\mathbf{x}\cdot\mathbf{u}).$$

Note that $\overline{\mathcal{F}}_\lambda f$ is a λ-weighted variant of the (n-dimensional) inverse discrete Fourier transform (DFT).

2 The group algebra $\mathbb{C}[H_n(\mathbb{F}_q)]$ and Gelfand pairs

Let H denote the Heisenberg group $H = H_n(\mathbb{F}) = \mathcal{W} \times \mathbb{F}$ where, as before,

$$\mathcal{W} = \mathbb{F}^n \times \mathbb{F}^n, \quad \mathbb{F} = \mathbb{F}_q, \quad q = p^m, \quad p \text{ an odd prime.}$$

The convolution product on $\mathbb{C}[H]$ is

$$\begin{aligned}(f \star g)(\mathbf{z},t) &= \sum_{(\mathbf{z}',t')\in H} f\big((\mathbf{z},t)(\mathbf{z}',t')^{-1}\big)g(\mathbf{z}',t') \\ &= \sum_{(\mathbf{z}',t')\in H} f\left(\mathbf{z}-\mathbf{z}', t-t'-2^{-1}[\mathbf{z},\mathbf{z}']\right)g(\mathbf{z}',t').\end{aligned}$$

2.1 Twisted convolution on $\mathbb{C}[\mathcal{W}]$

Twisted convolution is well-known in connection with analysis on the real Heisenberg group $H_n(\mathbb{R})$. (See [Fol89].) Here we require its discrete analog.

Definition 2.1 For $f \in \mathbb{C}[H]$ and $a \in \mathbb{F}$ define $f_a \in \mathbb{C}[\mathcal{W}]$ via

$$f_a(\mathbf{z}) = \frac{1}{\sqrt{q}} \sum_{t \in \mathbb{F}} f(\mathbf{z}, t)\psi_a(t).$$

For fixed $\mathbf{z} \in \mathcal{W}$, $f_a(\mathbf{z})$ is the (one-dimensional) inverse discrete Fourier transform of $t \mapsto f(\mathbf{z}, t)$ evaluated at a. The Fourier inversion formula yields

$$f(\mathbf{z}, t) = \frac{1}{\sqrt{q}} \sum_{a \in \mathbb{F}} f_a(\mathbf{z})\psi_a(-t). \tag{13}$$

In particular, a function $f \in \mathbb{C}[H]$ is completely determined by $\{f_a \,:\, a \in \mathbb{F}\}$. So for given $f, f' \in \mathbb{C}[H]$,

$$f = f' \iff f_a = f'_a \text{ for all } a \in \mathbb{F}. \tag{14}$$

Definition 2.2 For functions $f, g \in \mathbb{C}[\mathcal{W}]$ and given $a \in \mathbb{F}$ we define the *twisted convolution* $f \natural_a g \in \mathbb{C}[\mathcal{W}]$ via

$$(f \natural_a g)(\mathbf{z}) = \sum_{\mathbf{w} \in \mathcal{W}} f(\mathbf{z} - \mathbf{w})g(\mathbf{w})\psi_a\left(\frac{1}{2}[\mathbf{z}, \mathbf{w}]\right).$$

A straightforward calculation using (13) and (6) yields the following.

Lemma 2.3 *For $f, g \in \mathbb{C}[H]$ and $a \in \mathbb{F}$ one has*

$$(f \star g)_a = \sqrt{q} f_a \natural_a g_a.$$

2.2 *K*-invariant functions on *H* and $\mathcal{W}$

The symplectic group $Sp(n, \mathbb{F})$ acts on $\mathbb{C}[\mathcal{W}]$ and $\mathbb{C}[H]$ via

$$k \cdot f(\mathbf{z}) = f(k^{-1}\mathbf{z}) \quad \text{and} \quad k \cdot f(\mathbf{z}, t) = f(k^{-1} \cdot (\mathbf{z}, t)) = f(k^{-1}z, t).$$

For subgroups K of $Sp(n, \mathbb{F})$ we let $\mathbb{C}[\mathcal{W}]^K$ and $\mathbb{C}[H]^K$ denote the sets of K-fixed elements in $\mathbb{C}[\mathcal{W}]$ and $\mathbb{C}[H]$ respectively. These are easily seen to be subalgebras of $\mathbb{C}[\mathcal{W}]$ and $\mathbb{C}[H]$ with respect to the convolutions $\natural_a$ and $\star$.

Definition 2.4 Given a subgroup K of $Sp(n, \mathbb{F})$, we say that (K, H) is a *Gelfand pair* when $\mathbb{C}[H]^K$ is a *commutative* algebra under convolution.

Remark 2.5 One can identify $\mathbb{C}[H]^K$ with the algebra $\mathbb{C}[K \backslash G / K]$ of K-bi-invariant functions on the semidirect product $G = K \ltimes H$. So (G, K) is a Gelfand pair in the traditional sense when Definition 2.4 applies.

Proposition 2.6 *Let K be a subgroup of $Sp(n,\mathbb{F})$. Then (K,H) is a Gelfand pair if and only if $hh' \in (Kh')(Kh)$ for all $h,h' \in H$.*

Proof. This result is the discrete analog of Theorem 1.12 in [BJR90]. □

Proposition 2.7 *Let K be a subgroup of $Sp(n,\mathbb{F})$. Then (K,H) is a Gelfand pair if and only if $(\mathbb{C}[\mathcal{W}]^K, \natural_\lambda)$ is commutative for all $\lambda \in \mathbb{F}^\times$.*

Proof. First note that $(\mathbb{C}[\mathcal{W}]^K, \natural_0)$ is, in any case, commutative since $\natural_0$ is the standard (untwisted) convolution on $\mathbb{C}[\mathcal{W}]$. To complete the proof use (14) together with Lemma 2.3 and the the obvious identity

$$(k \cdot f)_a = k \cdot f_a,$$

$(k \in Sp(n,\mathbb{F}), \quad f \in \mathbb{C}[H], \quad a \in \mathbb{F})$. □

Two immediate but useful properties of Gelfand pairs are noted in the following lemma.

Lemma 2.8 *Let K_1 and K_2 be a pair of subgroups of $Sp(n,\mathbb{F})$ and suppose that (K_1,H) a Gelfand pair.*

(a) If $K_1 \subset K_2$ then (K_2,H) is a Gelfand pair.
(b) If K_1, K_2 are conjugate in $Sp(n,\mathbb{F})$ then (K_2,H) is a Gelfand pair.

3 Gelfand pairs and the oscillator representation

3.1 Operator valued Fourier transform on $\mathbb{C}[\mathcal{W}]$

For $f \in \mathbb{C}[\mathcal{W}]$ and $\lambda \in \mathbb{F}^\times$ let $\pi_\lambda(f)$ denote the operator

$$\pi_\lambda(f) = \sum_{\mathbf{z}\in\mathcal{W}} f(\mathbf{z})\pi_\lambda(\mathbf{z}) \tag{15}$$

on $\mathbb{C}[\mathcal{W}]$. Here $\pi_\lambda(\mathbf{z}) = \pi_\lambda(\mathbf{z},0)$ and π_λ is the Schrödinger representation (8). The following standard result is easily verified.

Lemma 3.1 *$\pi_\lambda(f\natural_\lambda g) = \pi_\lambda(f)\pi_\lambda(g)$ for $f,g \in \mathbb{C}[\mathcal{W}]$ and $\lambda \in \mathbb{F}^\times$.*

Lemma 3.2 *The map $\pi_\lambda : \mathbb{C}[\mathcal{W}] \to End(\mathbb{C}[\mathbb{F}^n])$ is a vector space isomorphism for each $\lambda \in \mathbb{F}^\times$. In fact $q^{-3n/2}\pi_\lambda$ is a* unitary *isomorphism of $\mathbb{C}[\mathcal{W}]$ onto $End(\mathbb{C}[\mathbb{F}^n])$ equipped with the Hilbert-Schmidt inner product*

$$\langle T,S\rangle_{HS} = tr(TS^*).$$

Proof. The set $\{q^{n/2}\delta_{\mathbf{u}} \ : \ \mathbf{u} \in \mathbb{F}^n\}$ is an orthonormal basis for $\mathbb{C}[\mathbb{F}^n]$ with

$$\pi_\lambda(\mathbf{x}, \mathbf{y})\delta_{\mathbf{u}} = \psi_\lambda(\mathbf{u} \cdot \mathbf{y} - 2^{-1}\mathbf{x} \cdot \mathbf{y})\delta_{\mathbf{u}-\mathbf{x}}.$$

So for $f, f' \in \mathbb{C}[\mathcal{W}]$ we compute

$$\begin{aligned}\langle \pi_\lambda(f), \pi_\lambda(f')\rangle_{HS} &= \sum_{\mathbf{u}\in\mathbb{F}^n} q^n \langle \pi_\lambda(f)\delta_{\mathbf{u}}, \pi_\lambda(f')\delta_{\mathbf{u}}\rangle_{\mathbb{F}^n} \\ &= \sum_{\mathbf{u}\in\mathbb{F}^n;\mathbf{z},\mathbf{z}'\in\mathcal{W}} f(\mathbf{z}), \overline{f'(\mathbf{z}')} q^n \langle \pi_\lambda(\mathbf{z})\delta_{\mathbf{u}}, \pi_\lambda(\mathbf{z}')\delta_{\mathbf{u}}\rangle_{\mathbb{F}^n}.\end{aligned}$$

A calculation using (8) shows

$$\sum_{\mathbf{u}\in\mathbb{F}^n} \langle \pi_\lambda(\mathbf{z})\delta_{\mathbf{u}}, \pi_\lambda(\mathbf{z}')\delta_{\mathbf{u}}\rangle_{\mathbb{F}^n} = \delta_{\mathbf{z},\mathbf{z}'}.$$

Thus

$$\langle \pi_\lambda(f), \pi_\lambda(f')\rangle_{HS} = q^n \sum_{\mathbf{z}\in\mathcal{W}} f(\mathbf{z})\overline{f'(\mathbf{z})} = q^{3n}\langle f, f'\rangle_{\mathcal{W}}.$$

This shows $q^{-3n/2}\pi_\lambda : \mathbb{C}[\mathcal{W}] \to End(\mathbb{C}[\mathbb{F}^n])$ is unitary, hence injective. As the spaces $\mathbb{C}[\mathcal{W}]$ and $End(\mathbb{C}[\mathbb{F}^n])$ have equal dimension it follows that π_λ is an isomorphism of $\mathbb{C}[\mathcal{W}]$ onto $End(\mathbb{C}[\mathbb{F}^n])$. □

3.2 Oscillator representation

Recall that $\omega_\lambda : Sp(n, \mathbb{F}) \to U(\mathbb{C}[\mathbb{F}^n])$ denotes the oscillator representation, characterized by Equation 10. Now for $k \in Sp(n, \mathbb{F})$ let $\widetilde{\omega}_\lambda(k)$ be the operator on $End(\mathbb{C}[\mathbb{F}^n])$ defined as

$$\widetilde{\omega}_\lambda(k)T = \omega_\lambda(k)T\omega_\lambda(k)^{-1} \tag{16}$$

One checks that $\widetilde{\omega}_\lambda$ defines a unitary representation of $Sp(n, \mathbb{F})$ on the Hermitian vector space $(End(\mathbb{C}[\mathbb{F}^n], \langle \cdot, \cdot\rangle_{HS})$. Moreover for $k \in Sp(n, \mathbb{F})$ and $f \in \mathbb{C}[\mathcal{W}]$ one has

$$\pi_\lambda(k \cdot f) = \sum_{\mathbf{z}\in\mathcal{W}} f(k^{-1}\mathbf{z})\pi_\lambda(\mathbf{z}) = \sum_{\mathbf{w}\in\mathcal{W}} f(\mathbf{w})\pi_\lambda(k\mathbf{w}) = \widetilde{\omega}_\lambda(k)\pi_\lambda(f), \tag{17}$$

since $\pi_\lambda(k\mathbf{w}) = \omega_\lambda(k)\pi_\lambda(\mathbf{w})\omega_\lambda(k)^{-1}$. So the isomorphism $\pi_\lambda : \mathbb{C}[\mathcal{W}] \to End(\mathbb{C}[\mathbb{F}^n])$ intertwines the natural representation of $Sp(n, \mathbb{F})$ on $\mathbb{C}[\mathcal{W}]$ with $\widetilde{\omega}_\lambda$.

Definition 3.3 Let K be a subgroup of $Sp(n, \mathbb{F})$. For $\lambda \in \mathbb{F}^\times$ we define the *commutant* $\mathcal{C}_{\lambda,K}$ of $\widetilde{\omega}_\lambda(K)$ in $End(\mathbb{C}[\mathbb{F}^n])$ as

$$\mathcal{C}_{\lambda,K} = End(\mathbb{C}[\mathbb{F}^n])^{\widetilde{\omega}_\lambda(K)} = \{T \in End(\mathbb{C}[\mathbb{F}^n]) \ : \ \omega_\lambda(k)T = T\omega_\lambda(k) \ \forall k \in K\}.$$

Note that $\mathcal{C}_{\lambda,K}$ is a subalgebra of $End(\mathbb{C}[\mathbb{F}^n])$.

Proposition 3.4 π_λ *yields an algebra isomorphism of* $(\mathbb{C}[\mathcal{W}]^K, \natural_\lambda)$ *onto* $\mathcal{C}_{\lambda,K}$.

Proof. Taken together, Lemmas 3.1 and 3.2 show that $\pi_\lambda : \mathbb{C}[\mathcal{W}] \to End(\mathbb{C}[\mathbb{F}^n])$ is an algebra isomorphism of $(\mathbb{C}[\mathcal{W}], \natural_\lambda)$ onto $End(\mathbb{C}[\mathbb{F}^n])$. Equation 17 shows that π_λ maps $\mathbb{C}[\mathcal{W}]^K$ onto $\mathcal{C}_{\lambda,K}$. □

Proposition 3.5 *Let* K *be a subgroup of* $Sp(n, \mathbb{F})$ *and* $\lambda \in \mathbb{F}^\times$. *Then* $\omega_\lambda|_K$ *is multiplicity free if and only if* $\mathcal{C}_{\lambda,K}$ *is commutative.*

Proof. Suppose that $\omega_\lambda|_K$ is multiplicity free. So $\mathbb{C}[\mathbb{F}^n]$ has a canonical decomposition into pair-wise inequivalent $\omega_\lambda(K)$-irreducible subspaces:

$$\mathbb{C}[\mathbb{F}^n] = P_1 \oplus \cdots \oplus P_m$$

say. Schur's Lemma shows that each operator $T \in \mathcal{C}_{\lambda,K}$ must preserve the P_j's and act by a scalar on each. Any two such operators commute with one another.

Next suppose that $\omega_\lambda|_K$ is *not* multiplicity free. Hence $\mathbb{C}[\mathbb{F}^n]$ has a decomposition of the sort

$$\mathbb{C}[\mathbb{F}^n] = W_1 \oplus W_2 \oplus V,$$

where W_1, W_2, V are $\omega_\lambda(K)$-invariant and W_1, W_2 are $\omega_\lambda(K)$-irreducible and equivalent. Thus $\mathcal{C}_{\lambda,K}$ contains a copy of $GL(2, \mathbb{F})$, and it fails to be commutative. □

Definition 3.6 We say that a subgroup K of $Sp(n, \mathbb{F})$ is *ω-multiplicity free* if the restriction $\omega_\lambda|_K$ of the oscillator representation to K is multiplicity free for all $\lambda \in \mathbb{F}^\times$.

Together Propositions 2.7, 3.4 and 3.5 imply the following.

Theorem 3.7 *Let* K *be a subgroup of* $Sp(n, \mathbb{F})$. *Then* (K, H) *is a Gelfand pair if and only if* K *is* ω*-multiplicity free.*

When applying Definition 3.6 it suffices to check that $\omega_\lambda|_K$ is multiplicity free for at most two values of λ.

Proposition 3.8 *A subgroup* K *of* $Sp(n, \mathbb{F})$ *is* ω*-multiplicity free if and only if* $\omega_1|_K$ *and* $\omega_\varepsilon|_K$ *are multiplicity free for any fixed choice of* $\varepsilon \in \mathbb{F}^\times$ *which is not a square. Moreover when* $q \equiv 3 \mod 4$ *it suffices that* $\omega_1|_K$ *be multiplicity free.*

Proof. Proposition 1.2 implies that each oscillator representation ω_λ is unitarily equivalent to one of ω_1 or ω_ε. If -1 is not a square in $\mathbb{F}$, equivalently when $q \equiv 3 \mod 4$, we can take $\varepsilon = -1$. But ω_{-1} is contragredient to ω_1 by (11). So when $\omega_1|_K$ is multiplicity free so is $\omega_{-1}|_K$. □

Remark 3.9 We do not know of an example where $\omega_1|_K$ is multiplicity free but K fails to be ω-multiplicity free.

4 Counting and convolving K-orbits in $\mathcal{W}$

Let ρ denote the natural (unitary) representation of $Sp(n,\mathbb{F})$ on $\mathbb{C}[\mathcal{W}]$:

$$\rho(k)f(\mathbf{z}) = k\cdot f(\mathbf{z}) = f(k^{-1}\mathbf{z}),$$

and $\widetilde{\omega}_\lambda : Sp(n,\mathbb{F}) \to U(End(\mathbb{C}[\mathbb{F}^n])$ be as in (16). We have seen that ρ and $\widetilde{\omega}_\lambda$ are unitarily equivalent via $q^{-3n/2}\pi_\lambda$, a multiple of the operator valued Fourier transform. There is another viewpoint on this equivalence. Consider the standard isomorphism

$$\Phi : \mathbb{C}[\mathbb{F}^n] \otimes \mathbb{C}[\mathbb{F}^n]^* \to End(\mathbb{C}[F^n]), \qquad \Phi(f\otimes\varphi)(g) = \varphi(g)f.$$

One checks easily that

- Φ is unitary. (As before $End(\mathbb{C}[F^n])$ carries the Hilbert-Schmidt inner product and we give $\mathbb{C}[\mathbb{F}^n]\otimes\mathbb{C}[\mathbb{F}^n]^*$ the tensor product of $\langle\cdot,\cdot\rangle_{\mathbb{F}^n}$ with its dual inner product.)
- $\Phi\big(\omega_\lambda(k)f\otimes\omega_\lambda^*(k)\varphi\big) = \widetilde{\omega}_\lambda(k)\Phi(f\otimes\varphi)$.

So Φ establishes a unitary equivalence

$$\omega_\lambda\otimes\omega_\lambda^* \simeq \widetilde{\omega}_\lambda,$$

and the composite $q^{3n/2}\pi_\lambda^{-1}\circ\Phi$ yields

$$\omega_\lambda\otimes\omega_\lambda^* \simeq \rho. \tag{18}$$

This basic fact plays a central role in [How].

4.1 Counting orbits

Now let K be a subgroup of $Sp(n,\mathbb{F})$ and for $\lambda\in\mathbb{F}^\times$ decompose $\omega_\lambda|_K$:

$$\omega_\lambda|_K \simeq \sum_{\sigma\in\widehat{K}} m_{\sigma,\lambda}\sigma, \qquad m_{\sigma,\lambda} = mult(\sigma,\omega_\lambda|_K).$$

Writing $d_\sigma = \dim(\sigma)$ one has

$$\sum_{\sigma\in\widehat{K}} m_{\sigma,\lambda}d_\sigma = \dim(\mathbb{C}[\mathbb{F}^n]) = q^n. \tag{19}$$

Also, applying (18):

$$\rho|_K \simeq (\omega_\lambda\otimes\omega_\lambda^*)|_K \simeq \sum_{\sigma,\sigma'\in\widehat{K}} m_{\sigma,\lambda}m_{\sigma',\lambda}\sigma\otimes(\sigma')^*.$$

We know that $\sigma\otimes(\sigma)^*$ has a one-dimensional space of K-fixed vectors and that $(\sigma\otimes(\sigma')^*)^K = 0$ when $\sigma'\not\simeq\sigma$. So

$$\dim(\mathbb{C}[\mathcal{W}]^K) = \sum_{\sigma\in\widehat{K}} m_{\sigma,\lambda}^2.$$

But $\dim(\mathbb{C}[\mathcal{W}]^K) = |\mathcal{W}/K|$, the number of K-orbits in $\mathcal{W}$. (Indeed, if $K\mathbf{z}_1,\ldots,K\mathbf{z}_r$ are the distinct K-orbits in $\mathcal{W}$ then the characteristic functions $\{\delta_{K\mathbf{z}_1},\ldots,\delta_{K\mathbf{z}_r}\}$ form a basis for $\mathbb{C}[\mathcal{W}]^K$.) So now

$$\sum_{\sigma\in\widehat{K}} m_{\sigma,\lambda}^2 = |\mathcal{W}/K|. \tag{20}$$

Proposition 4.1 *If K is ω-multiplicity free (equivalently (K,H) is a Gelfand pair) then we must have*

- $|\mathcal{W}/K| \le q^n$ *and*
- $|K| \ge q^n+1$.

Proof. Suppose that $|\mathcal{W}/K| > q^n$. Using Equations 19 and 20 one obtains

$$\sum_{\sigma\in\widehat{K}} m_{\sigma,\lambda}^2 > \sum_{\sigma\in\widehat{K}} m_{\sigma,\lambda}d_\sigma \ge \sum_{\sigma\in\widehat{K}} m_{\sigma,\lambda}.$$

So we must have $m_{\sigma,\lambda}\ge 2$ for some $\sigma\in\widehat{K}$. Hence K fails to be ω-multiplicity free unless $|\mathcal{W}/K|\le q^n$.

Next observe that

$$|\mathcal{W}/K| \ge 1 + \frac{q^{2n}-1}{|K|},$$

since $\{0\}$ is a K-orbit in $\mathcal{W}$ and $\mathcal{W}\backslash\{0\}$ contains at least $\frac{q^{2n}-1}{|K|}$ K-orbits. So in view of the inequality $|\mathcal{W}/K|\le q^n$ we must have

$$1+\frac{q^{2n}-1}{|K|} \le q^n \implies \frac{q^{2n}-1}{q^n-1} \le |K| \implies q^n+1 \le |K|,$$

as claimed. □

Proposition 4.2 *Let K be an* abelian *subgroup of $Sp(n,\mathbb{F})$. Then $|\mathcal{W}/K|\ge q^n$ and K is ω-multiplicity free if and only if $|\mathcal{W}/K| = q^n$. Moreover, this is possible only when $|K|\ge q^n+1$.*

Proof. When K is abelian we have $d_\sigma = 1$ for all $\sigma\in\widehat{K}$. So Equations 19 and 20 become

$$\sum_{\sigma\in\widehat{K}} m_{\sigma,\lambda} = q^n, \qquad \sum_{\sigma\in\widehat{K}} m_{\sigma,\lambda}^2 = |\mathcal{W}/K|.$$

As $\sum_{\sigma\in\widehat{K}} m_{\sigma,\lambda}^2 \ge \sum_{\sigma\in\widehat{K}} m_{\sigma,\lambda}$ we conclude that $|\mathcal{W}/K|\ge q^n$ must hold. Also K is ω-multiplicity free if and only if $\sum_{\sigma\in\widehat{K}} m_{\sigma,\lambda}^2 = \sum_{\sigma\in\widehat{K}} m_{\sigma,\lambda}$. Equivalently, $|\mathcal{W}/K| = q^n$ must hold. Proposition 4.1 shows, moreover, that this implies $|K|\ge q^n+1$. □

4.2 Convolving orbits

The characteristic functions

$$\{\delta_{K\mathbf{z}} \ : \ K\mathbf{z} \in \mathcal{W}/K\}$$

for the K-orbits in $\mathcal{W}$ yield an (orthogonal) basis for $\mathbb{C}[\mathcal{W}]^K$. We compute

$$\begin{aligned}\delta_{K\mathbf{z}} \natural_\lambda \delta_{K\mathbf{z}'}(\mathbf{w}) &= \sum_{\mathbf{v}\in\mathcal{W}} \delta_{K\mathbf{z}}(\mathbf{w}-\mathbf{v})\delta_{K\mathbf{z}'}(\mathbf{v})\psi_\lambda(2^{-1}[\mathbf{w},\mathbf{v}]) \\ &= \sum_{\mathbf{v}\in(\mathbf{w}-K\mathbf{z})\cap K\mathbf{z}'} \psi_\lambda(2^{-1}[\mathbf{w},\mathbf{v}]). \end{aligned} \tag{21}$$

In view of Proposition 2.7, (K,H) will be a Gelfand pair if and only if $\delta_{K\mathbf{z}}\natural_\lambda\delta_{K\mathbf{z}'}(\mathbf{w}) = \delta_{K\mathbf{z}'}\natural_\lambda\delta_{K\mathbf{z}}(\mathbf{w})$ for all $\mathbf{z},\mathbf{z}',\mathbf{w}\in\mathcal{W}$ and all $\lambda\in\mathbb{F}^\times$. It is enough to consider $\mathbf{z}\neq 0\neq\mathbf{z}'$ since $\delta_{K0}=\delta_0$ is a two-sided identity in $(\mathbb{C}[\mathcal{W}],\natural_\lambda)$. Moreover we can take $\mathbf{w}\neq 0$ because, in any case, $f\natural_\lambda g(0) = f\natural_\lambda g(0)$ for functions $f,g\in\mathbb{C}[\mathcal{W}]$. This discussion yields the following.

Lemma 4.3 *(K,H) is a Gelfand pair if and only if*

$$\sum_{\mathbf{v}\in(\mathbf{w}-K\mathbf{z})\cap K\mathbf{z}'} \psi_\lambda([\mathbf{w},\mathbf{v}]) = \sum_{\mathbf{v}\in(\mathbf{w}-K\mathbf{z}')\cap K\mathbf{z}} \psi_\lambda([\mathbf{w},\mathbf{v}])$$

for all $\mathbf{z},\mathbf{z}',\mathbf{w}\in\mathcal{W}\backslash\{0\}$ and $\lambda\in\mathbb{F}^\times$.

Remark 4.4 By Proposition 3.8 it suffices to check the condition in Lemma 4.3 for at most two values of the parameter λ.

Lemma 4.5 *Suppose that there are K-invariant subspaces $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{W}$ with*

$$\mathcal{X}\cap\mathcal{Y}=0, \qquad [\mathcal{X},\mathcal{Y}]\neq 0.$$

Then (K,H) is not *a Gelfand pair.*

Proof. Choose points $\mathbf{z}\in\mathcal{X}$ and $\mathbf{z}'\in\mathcal{Y}$ with $[\mathbf{z},\mathbf{z}']=1$ and let $\mathbf{w}=\mathbf{z}+\mathbf{z}'$. We have

$$(\mathbf{w}-K\mathbf{z})\cap K\mathbf{z}' = \{\mathbf{z}'\}\,.$$

Indeed, suppose that $\mathbf{v}\in(\mathbf{w}-K\mathbf{z})\cap K\mathbf{z}'$. Thus for some $k,k'\in K$,

$$\mathbf{w}-k\mathbf{z}=\mathbf{v}=k'\mathbf{z}' \quad \text{and hence} \quad \mathbf{z}-k\mathbf{z}=k'\mathbf{z}'-\mathbf{z}'.$$

But $\mathbf{z}-k\mathbf{z}\in\mathcal{X}$ and $k'\mathbf{z}'-\mathbf{z}'\in\mathcal{Y}$. So $\mathbf{v}=k'\mathbf{z}'=\mathbf{z}'$ since $\mathcal{X}\cap\mathcal{Y}=0$.

Thus for these choices of $\mathbf{z}$, $\mathbf{z}'$ and $\mathbf{w}$ we have

$$\sum_{\mathbf{v}\in(\mathbf{w}-K\mathbf{z})\cap K\mathbf{z}'} \psi_1([\mathbf{w},\mathbf{v}]) = \psi_1([\mathbf{z}+\mathbf{z}',\mathbf{z}']) = \psi_1(1) = e^{2\pi i/p}.$$

Likewise

$$\sum_{\mathbf{v}\in(\mathbf{w}-K\mathbf{z}')\cap K\mathbf{z}} \psi_1([\mathbf{w},\mathbf{v}]) = \psi_1([\mathbf{z}+\mathbf{z}',\mathbf{z}]) = \psi_1(-1) = e^{-2\pi i/p}.$$

As p is odd, these values are necessarily different. So (K,H) fails to be a Gelfand pair in view of Lemma 4.3. □

Corollary 4.6 *Let $(K,H_{\mathcal{W}})$ be a Gelfand pair where K acts reductively but non-irreducibly on $\mathcal{W}$. So $\mathcal{W}$ decomposes as a direct sum of K-invariant subspaces,*

$$\mathcal{W} = \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_\ell$$

say. Let $K_j \subset GL(\mathcal{W}_j)$ denote the restriction of K to $\mathcal{W}_j$. Then:

(a) $[\mathcal{W}_i,\mathcal{W}_j]=0$ for all $i\neq j$.
(b) Each $\mathcal{W}_j$ is a symplectic subspace of $\mathcal{W}$ and $K_j \subset Sp(\mathcal{W}_j)$.
(c) Each $(K_j,H_{\mathcal{W}_j})$ is a Gelfand pair.

Corollary 4.7 *Suppose we are given symplectic vector spaces $(\mathcal{W}_j,[\cdot,\cdot]_j)$ and subgroups $K_j\subset Sp(\mathcal{W}_j)$ for $j=1,\dots\ell$. We form the symplectic direct sum*

$$W = \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_\ell, \qquad [(\mathbf{w}_1,\dots,\mathbf{w}_\ell),(\mathbf{w}'_1,\dots,\mathbf{w}'_\ell)] = \sum_{j=1}^{\ell}[\mathbf{w}_j,\mathbf{w}'_j]_j$$

and let $K\subset Sp(\mathcal{W})$ denote the product

$$K = K_1\times\cdots\times K_\ell.$$

Then $(K,H_{\mathcal{W}})$ is a Gelfand pair if and only if $(K_j,H_{\mathcal{W}_j})$ is a Gelfand pair for $j=1,\dots,\ell$.

Recall that a subspace $\mathcal{X}$ of the symplectic vector space $\mathcal{W}$ is said to be *isotropic* if $[\mathcal{X},\mathcal{X}]=0$.

Corollary 4.8 *Suppose that K acts reductively on $\mathcal{W}$ and that the action preserves a non-zero isotropic subspace $\mathcal{X}$. Then (K,H) is not a Gelfand pair.*

Proof. As K acts reductively on $\mathcal{W}$ we have $\mathcal{W}=\mathcal{X}\oplus\mathcal{Y}$ for some K-invariant subspace $\mathcal{Y}$. As $\mathcal{X}$ is isotropic we necessarily have $[\mathcal{X},\mathcal{Y}]\neq 0$. □

5 Examples

5.1 Symplectic groups

A trivial application of Lemma 4.3 shows that $(Sp(n,\mathbb{F}),H_n(\mathbb{F}))$ is a Gelfand pair. Indeed, for $K=Sp(n,\mathbb{F})$ we have only one non-zero K-orbit. That is

$K\mathbf{z} = \mathcal{W}\backslash\{0\}$ for all $\mathbf{z} \neq 0$ in $\mathcal{W}$. Equation 20 now shows that the oscillator representation ω_λ must decompose into exactly two inequivalent irreducible constituents. This fact is well known. (See [How], [Neu02].) The $\omega_\lambda(Sp(n,\mathbb{F}))$-irreducible subspaces of $\mathbb{C}[\mathbb{F}^n]$ can be identified as follows.

The matrix $-I$ belongs to the center of $Sp(n,\mathbb{F})$ and the first formula from Theorem 1.3 shows that $\omega_\lambda(-I) = sgn((-1)^n)T$ where

$$Tf(\mathbf{u}) = f(-\mathbf{u}).$$

So the eigenspaces for T must be $\omega_\lambda(Sp(n,\mathbb{F}))$-invariant. These are the spaces of even and odd functions. As ω_λ has exactly two irreducible components these spaces are necessarily irreducible.

To obtain more interesting examples we must consider smaller subgroups of $Sp(n,\mathbb{F})$.

5.2 General linear groups

The group $GL(n,\mathbb{F})$ embeds diagonally in $Sp(n,\mathbb{F})$ via $\{A_{diag} : A \in GL(n,\mathbb{F})\}$. (See (12).) The action of $GL(n,\mathbb{F})$ on $\mathcal{W} = \mathbb{F}^n \times \mathbb{F}^n$ preserves the isotropic subspaces

$$\mathcal{X} = \mathbb{F}^n \times \{0\}, \quad \mathcal{Y} = \{0\} \times \mathbb{F}^n.$$

So $(GL(n,\mathbb{F}), H_n(\mathbb{F}))$ is not a Gelfand pair, in view of Corollary 4.8. More generally, if $K \subset Sp(n,\mathbb{F})$ is conjugate in $Sp(n,\mathbb{F})$ to a subgroup of $GL(n,\mathbb{F})$ then (K,H) is not a Gelfand pair.

For $n \geq 2$ there are $q+3$ distinct $GL(n,\mathbb{F})$-orbits in $\mathcal{W}$, namely

$$\{(0,0)\}, \qquad (\mathbb{F}^n\backslash\{0\}) \times \{0\}, \qquad \{0\} \times (\mathbb{F}^n\backslash\{0\}),$$

$$\{(\mathbf{x},\mathbf{y}) \;:\; \mathbf{x} \neq 0 \neq \mathbf{y},\ \mathbf{x}\cdot\mathbf{y} = 0\} \quad \text{and} \quad \{(\mathbf{x},\mathbf{y}) \;:\; \mathbf{x}\cdot\mathbf{y} = a\}$$

for each $a \in \mathbb{F}^\times$. So here $|\mathcal{W}/GL(n,\mathbb{F})| < q^n$, but $(GL(n,\mathbb{F}), H_n(\mathbb{F}))$ is not a Gelfand pair. This shows that, for K non-abelian, there can be no converse for Proposition 4.1.

5.3 Borel subgroups

A subgroup of $Sp(n,\mathbb{F})$ conjugate to

$$B = \left\{ \left[\begin{array}{c|c} A & 0 \\ \hline C & (A^t)^{-1} \end{array}\right] \;:\; A \in GL(n,\mathbb{F}) \text{ lower triangular, } A^tC \text{ symmetric} \right\}$$

is called a Borel subgroup.

Proposition 5.1 *If K is a subgroup of $Sp(n,\mathbb{F})$ that contains a Borel subgroup then (K,H) is a Gelfand pair.*

Proof. It suffices to show that (B, H) is a Gelfand pair. This can be done using Proposition 2.6. There are exactly $2n$ non-zero B-orbits in $\mathcal{W} = \mathbb{F}^n \times \mathbb{F}^n$:

$$\mathcal{W}\backslash\{0\} = \biguplus_{j=1}^{n} B\mathbf{u}_j \uplus \biguplus_{j=1}^{n} B\mathbf{v}_j \quad \text{where} \quad \mathbf{u}_j = (\mathbf{e}_j, 0),\ \mathbf{v}_j = (0, \mathbf{e}_j)$$

and $\{\mathbf{e}_j\}$ is the standard basis for $\mathbb{F}^n$.

Suppose that $\mathbf{z} \in B\mathbf{u}_i$ and $\mathbf{w} \in B\mathbf{u}_j$. One has $\mathbf{z} \pm \mathbf{w} \in B\mathbf{u}_{\min(i,j)}$, and hence $\mathbf{z} + \mathbf{w} = k(\mathbf{z} - \mathbf{w})$ for some $k \in B$. Let $k' = -k$ and note that $k' \in B$, since $-I \in B$. Now

$$\begin{aligned}(\mathbf{z}, s)(\mathbf{w}, t) &= (\mathbf{z} + \mathbf{w},\ s + t + 2^{-1}[\mathbf{z}, \mathbf{w}]) \\ &= (k\mathbf{z} - k\mathbf{w},\ s + t + 2^{-1}[k\mathbf{z}, k\mathbf{w}]) \\ &= (k'\mathbf{w} + k\mathbf{z},\ t + s + 2^{-1}[k'\mathbf{w}, k\mathbf{z}]) \\ &= (k'\mathbf{w}, t)(k\mathbf{z}, s).\end{aligned}$$

So $(\mathbf{z}, s)(\mathbf{w}, t) \in (B(\mathbf{w}, t))(B(\mathbf{z}, s))$. Similar calculations apply for other combinations of orbits. □

One can also give an explicit description of the algebra $(\mathbb{C}[\mathcal{W}]^B, \natural_\lambda)$ by determining the twisted convolution of pairs of characteristic functions for B-orbits in $\mathcal{W}$. We adopt the notation

$$E_\lambda(\mathbf{z}) = \sum_{\mathbf{w} \in B\mathbf{z}} \psi_\lambda(2^{-1}[\mathbf{z}, \mathbf{w}]) \qquad (\mathbf{z} \in \mathcal{W},\ \lambda \in \mathbb{F}^\times).$$

Brute force calculation yields the following.

$$\begin{aligned}\delta_{B\mathbf{u}_i} \natural_\lambda \delta_{B\mathbf{u}_j} &= \begin{cases} E_\lambda(\mathbf{u}_{\max(i,j)})\delta_{B\mathbf{u}_{\min(i,j)}} & \text{for } i \neq j \\ E_\lambda(\mathbf{u}_i) \sum_{\ell \geq i} \delta_{B\mathbf{u}_\ell} - \delta_{B\mathbf{u}_i} & \text{for } i = j \end{cases} \\ \delta_{B\mathbf{v}_i} \natural_\lambda \delta_{B\mathbf{v}_j} &= \begin{cases} E_\lambda(\mathbf{v}_{\min(i,j)})\delta_{B\mathbf{v}_{\max(i,j)}} & \text{for } i \neq j \\ E_\lambda(\mathbf{v}_i) \sum_{\ell \leq i} \delta_{B\mathbf{v}_\ell} - \delta_{B\mathbf{v}_i} & \text{for } i = j \end{cases} \\ \delta_{B\mathbf{u}_j} \natural_\lambda \delta_{B\mathbf{v}_i} &= \delta_{B\mathbf{v}_i} \natural_\lambda \delta_{B\mathbf{u}_j} = E_\lambda(\mathbf{v}_i)\delta_{B\mathbf{u}_j}\end{aligned}$$

These formulas show, in particular, that $(\mathbb{C}[\mathcal{W}]^B, \natural_\lambda)$ is commutative, as guaranteed by Proposition 5.1.

5.4 Unitary groups

Let $\widetilde{\mathbb{F}}$ denote a quadratic extension of the field $\mathbb{F} = \mathbb{F}_q$. Up to isomorphism $\widetilde{\mathbb{F}}$ is a copy of $\mathbb{F}_{q^2}$. More concretely we choose any non-square

$$\varepsilon \in \mathbb{F}^\times \backslash (\mathbb{F}^\times)^2$$

in $\mathbb{F}$ and take

$$\widetilde{\mathbb{F}} = \mathbb{F}(\sqrt{\varepsilon}).$$

The Galois involution $\widetilde{\mathbb{F}} \to \widetilde{\mathbb{F}}$ will be written as $z \mapsto \overline{z}$. One has $\overline{z} = z^q$ and

$$\overline{a + b\sqrt{\varepsilon}} = a - b\sqrt{\varepsilon} \quad (a, b \in \mathbb{F}).$$

Let $(\widetilde{\mathcal{W}}, \langle\cdot,\cdot\rangle)$ be a (finite dimensional) Hermitian vector space over $\widetilde{\mathbb{F}}$. That is $\langle\cdot,\cdot\rangle : \widetilde{\mathcal{W}} \times \widetilde{\mathcal{W}} \to \widetilde{\mathbb{F}}$ is

- $\mathbb{F}$-bilinear and non-degenerate with
- $\langle \lambda\mathbf{z}, \mathbf{z}'\rangle = \lambda\langle \mathbf{z}, \mathbf{z}'\rangle$ and $\overline{\langle \mathbf{z}, \mathbf{z}'\rangle} = \langle \mathbf{z}', \mathbf{z}\rangle \qquad (\mathbf{z}, \mathbf{z}' \in \widetilde{\mathcal{W}},\ \lambda \in \widetilde{\mathbb{F}})$.

The unitary group $U(\widetilde{\mathcal{W}})$ is the set of $\widetilde{\mathbb{F}}$-linear operators preserving $\langle\cdot,\cdot\rangle$.

Let $\mathcal{W}$ denote the underlying $\mathbb{F}$-vector space for $\widetilde{\mathcal{W}}$. One obtains a symplectic form on $\mathcal{W}$ via

$$[\mathbf{z}, \mathbf{z}'] = \frac{1}{2\sqrt{\varepsilon}}\Big(\overline{\langle \mathbf{z}, \mathbf{z}'\rangle} - \langle \mathbf{z}, \mathbf{z}'\rangle\Big) = \frac{1}{2\sqrt{\varepsilon}}\Big(\langle \mathbf{z}', \mathbf{z}\rangle - \langle \mathbf{z}, \mathbf{z}'\rangle\Big).$$

(Writing $\langle \mathbf{z}, \mathbf{z}'\rangle \in \mathbb{F}(\sqrt{\varepsilon})$ as $\langle \mathbf{z}, \mathbf{z}'\rangle = \langle \mathbf{z}, \mathbf{z}'\rangle_r + \langle \mathbf{z}, \mathbf{z}'\rangle_i\sqrt{\varepsilon}$, one has $[\mathbf{z}, \mathbf{z}'] = -\langle \mathbf{z}, \mathbf{z}'\rangle_i$.) Clearly $U(\widetilde{\mathcal{W}})$ is a subgroup of $Sp(\mathcal{W})$, the symplectic group for $(\mathcal{W}, [\cdot,\cdot])$. We will prove that:

Proposition 5.2 *$(U(\widetilde{\mathcal{W}}), H_{\mathcal{W}})$ is a Gelfand pair.*

It is well known that a given finite dimensional $\widetilde{\mathbb{F}}$-vector space admits *exactly one* hermitian inner product, up to equivalence.[1] In fact one can find an orthonormal basis

$$\mathcal{B} = \{e_1, \ldots, e_n\}$$

for $(\widetilde{\mathcal{W}}, \langle\cdot,\cdot\rangle)$ with $\langle e_i, e_j\rangle = \delta_{i,j}$. Using $\mathcal{B}$ to identify $\widetilde{\mathcal{W}}$ with $\widetilde{\mathbb{F}}^n$ we have the usual formula

$$\langle \mathbf{z}, \mathbf{z}'\rangle = z_1\overline{z_1'} + \cdots + z_n\overline{z_n'}.$$

Let

$$\widetilde{\mathcal{W}}_j = \widetilde{\mathbb{F}}e_j, \quad \mathcal{W}_j \text{ denote } \widetilde{\mathcal{W}}_j \text{ viewed as an } \mathbb{F}\text{-vector space}, \quad f_j = \sqrt{\varepsilon}e_j.$$

Now

- $\{e_j, f_j\}$ is a basis for $\mathcal{W}_j$,
- $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ is a symplectic basis for $\mathcal{W}$ (i.e. $[e_i, e_j] = 0 = [f_i, f_j]$, $[e_i, f_j] = \delta_{i,j}$), and
- $\mathcal{W} = \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_n$ is a symplectic direct sum.

[1] In contrast, the complex vector space $\mathbb{C}^n$ admits $\lfloor (n+2)/2 \rfloor$ inequivalent Hermitian inner products. These yield distinct unitary groups $U(r,s)$ with $r+s=n$. The analogs for these Hermitian inner products in the finite fields context are, however, mutually equivalent.

The restriction of $\langle\cdot,\cdot\rangle$ to $\widetilde{\mathcal{W}}_j$ is a hermitian inner product on $\widetilde{\mathcal{W}}_j$. We consider the subgroups $U(\widetilde{\mathcal{W}}_i) \subset Sp(\mathcal{W}_j)$ and their direct product

$$U(\widetilde{\mathcal{W}}_1) \times \cdots \times U(\widetilde{\mathcal{W}}_n) \subset U(\widetilde{\mathcal{W}}) \subset Sp(\mathcal{W}),$$

the subgroup of $U(\widetilde{\mathcal{W}})$ preserving the decomposition $\mathcal{W} = \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_n$.

The following result evidentally implies Proposition 5.2.

Proposition 5.3 $\big(U(\widetilde{\mathcal{W}}_1) \times \cdots \times U(\widetilde{\mathcal{W}}_n), H_{\mathcal{W}}\big)$ *is a Gelfand pair.*

Proof. In view of Corollary 4.7 it suffices to show that each $\big(U(\widetilde{\mathcal{W}}_j), H_{\mathcal{W}_j}\big)$ is a Gelfand pair. This amounts to showing that $(U(\widetilde{\mathbb{F}}), H_1(\mathbb{F}))$ is a Gelfand pair, where $\widetilde{\mathbb{F}}$ carries the Hermitian inner product

$$\langle z, z'\rangle = z\overline{z'}.$$

Now

$$U(\widetilde{\mathbb{F}}) = \{\lambda \in \widetilde{\mathbb{F}}^\times \;:\; \lambda\overline{\lambda} = 1\},$$

is the kernel of the norm mapping

$$N : \widetilde{\mathbb{F}} \to \mathbb{F}, \qquad N(\lambda) = \lambda\overline{\lambda}$$

restricted to the multiplicative group $\widetilde{\mathbb{F}}^\times$ for the field $\widetilde{\mathbb{F}}$. So $U(\widetilde{\mathbb{F}})$ is, in particular, abelian. Moreover a pair of points $z, z' \in \widetilde{\mathbb{F}}$ belong to a common $U(\widetilde{\mathbb{F}})$-orbit in $\widetilde{\mathbb{F}}$ if and only if $N(z) = N(z')$. So

$$|\widetilde{\mathbb{F}}/U(\widetilde{\mathbb{F}})| = |N(\widetilde{\mathbb{F}})| = |\mathbb{F}| = q,$$

as it is well known that N is surjective. Proposition 4.2 now implies that $(U(\widetilde{\mathbb{F}}), H_1(\mathbb{F}))$ is a Gelfand pair as desired. $\square$

Remark 5.4 As $U(\widetilde{\mathbb{F}})$ is the kernel of the epimorphism $N : \widetilde{\mathbb{F}}^\times \to \mathbb{F}^\times$, it is cyclic of order $q+1$.

Our final result asserts that the Gelfand pair in Proposition 5.3 is minimal. The analogous theorem for real Heisenberg groups $H_n(\mathbb{R})$ is due to Leptin [Lep85].

Proposition 5.5 $(K, H_{\mathcal{W}})$ *fails to be a Gelfand pair for all proper subgroups K of the torus $U(\widetilde{\mathcal{W}}_1) \times \cdots \times U(\widetilde{\mathcal{W}}_n)$.*

Proof. We can take $\widetilde{\mathcal{W}}_j = \widetilde{\mathbb{F}}$ and $\widetilde{\mathcal{W}} = \widetilde{\mathbb{F}}^n$ with the usual hermitian inner product. Now the torus

$$T = U(\widetilde{\mathbb{F}}) \times \cdots \times U(\widetilde{\mathbb{F}})$$

coincides with a T-orbit in $\widetilde{\mathcal{W}}$. Namely one has

$$T \cdot \mathbf{z}_\circ = T \text{ for } \mathbf{z}_\circ = (1, 1, \ldots, 1).$$

Let K denote a proper subgroup of T. The T-orbit $T \cdot \mathbf{z}_\circ$ is a disjoint union of $|T/K|$ orbits for the subgroup K. These correspond to the cosets of K in T. As $(T, H_{\mathcal{W}})$ is a Gelfand pair we have $|\mathcal{W}/T| = q^n$ and

$$|\mathcal{W}/K| \geq q^n - 1 + |T/K| > q^n.$$

So $(K, H_{\mathcal{W}})$ fails to be a Gelfand pair by Proposition 4.2. □

References

[BJR90] Chal Benson, Joe Jenkins, and Gail Ratcliff. On Gel′fand pairs associated with solvable Lie groups. *Trans. Amer. Math. Soc.*, 321(1):85–116, 1990.

[Fol89] Gerald B. Folland. *Harmonic analysis in phase space*, volume 122 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1989.

[Gér77] Paul Gérardin. Weil representations associated to finite fields. *J. Algebra*, 46(1):54–101, 1977.

[Hel84] Sigurdur Helgason. *Groups and geometric analysis*, volume 113 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1984. Integral geometry, invariant differential operators, and spherical functions.

[How] Roger E. Howe. Invariant theory and duality for classical groups over finite fields, with applications to their singular representation theory. Unpublished manuscript.

[How73] Roger E. Howe. On the character of Weil's representation. *Trans. Amer. Math. Soc.*, 177:287–298, 1973.

[Lep85] Horst Leptin. A new kind of eigenfunction expansions on groups. *Pacific J. Math.*, 116(1):45–67, 1985.

[LN] Rudolf Lidl and Harald Niederreiter. *Finite fields*, volume 20 of *Encyclopedia of Mathematics and its Applications*. Second edition.

[Mac] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. Second edition.

[Neu02] Markus Neuhauser. An explicit construction of the metaplectic representation over a finite field. *J. Lie Theory*, 12(1):15–30, 2002.

[SA87] Jorge Soto-Andrade. Geometrical Gel′fand models, tensor quotients, and Weil representations. In *The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986)*, volume 47 of *Proc. Sympos. Pure Math.*, pages 305–316. Amer. Math. Soc., Providence, RI, 1987.

[Shi80] Ken-ichi Shinoda. The characters of Weil representations associated to finite fields. *J. Algebra*, 66(1):251–280, 1980.

[Ter99] Audrey Terras. *Fourier analysis on finite groups and applications*, volume 43 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1999.

[Vin03] È. B. Vinberg. Commutative homogeneous spaces of Heisenberg type. *Tr. Mosk. Mat. Obs.*, 64:54–89, 2003.

[Yak06] Oksana Yakimova. Principal Gelfand pairs. *Transform. Groups*, 11(2):305–335, 2006.