THE SPHERICAL TRANSFORM OF A SCHWARTZ FUNCTION ON THE HEISENBERG GROUP

CHAL BENSON, JOE JENKINS, AND GAIL RATCLIFF

ABSTRACT. Suppose that $K \subset U(n)$ is a compact Lie group acting on the $(2n+1)$-dimensional Heisenberg group H_n. We say that (K, H_n) is a Gelfand pair if the convolution algebra $L^1_K(H_n)$ of integrable K-invariant functions on H_n is commutative. In this case, the Gelfand space $\Delta(K, H_n)$ is equipped with the Godement-Plancherel measure, and the spherical transform $^\wedge : L^2_K(H_n) \to L^2(\Delta(K, H_n))$ is an isometry. The main result in this paper provides a complete characterization of the set $\mathcal{S}_K(H_n)^\wedge = \{\widehat{f} \mid f \in \mathcal{S}_K(H_n)\}$ of spherical transforms of K-invariant Schwartz functions on H_n. We show that a function F on $\Delta(K, H_n)$ belongs to $\mathcal{S}_K(H_n)^\wedge$ if and only if the functions obtained from F via application of certain derivatives and difference operators satisfy decay conditions. We also consider spherical series expansions for K-invariant Schwartz functions on H_n modulo its center.

CONTENTS

1. INTRODUCTION

Given a complex vector space V of dimension n with Hermitian inner product $\langle \cdot, \cdot \rangle$, one forms the Heisenberg group $H_n = V \times \mathbb{R}$ with group law

$$(z,t)(z',t') = \left(z + z', t + t' - \frac{1}{2} Im\langle z, z'\rangle\right).$$

In order to present explicit formulae, one can use an orthonormal basis to identify V with $\mathbb{C}^n$ so that $\langle z, z'\rangle = z \cdot \overline{z'}$ for $z, z' \in \mathbb{C}^n$. The group $U(n)$ of unitary transformation

1991 *Mathematics Subject Classification.* Primary 22E30, 43A55.

of $(V, \langle\cdot,\cdot\rangle)$ acts by automorphisms on H_n via

$$k \cdot (z,t) := (kz,t) \quad \text{for } k \in U(n) \text{ and } (z,t) \in H_n$$

and on the space of polynomials $\mathbb{C}[V]$ via

$$k \cdot p(z) := p(k^{-1}z) \quad \text{for } k \in U(n),\ p \in \mathbb{C}[V] \text{ and } z \in V.$$

If K is a compact Lie subgroup of $U(n)$ then we say that (K, H_n) is a *Gelfand pair* when the algebra $L^1_K(H_n)$ of K-invariant L^1-functions on H_n is commutative under convolution. One has the following result:

Theorem 1.1 (cf. [Car87], [BJR90]). *(K, H_n) is a Gelfand pair if and only if the representation of K on $\mathbb{C}[V]$ is multiplicity free.*

Using Theorem 1.1, one sees that the group $U(n)$ and many of its proper subgroups K yield Gelfand pairs. Working from Theorem 1.1, one obtains a complete classification of all such subgroups [Kac80, BR96]. This classification shows that the theory of Gelfand pairs associated with Heisenberg groups is quite rich. There are, for example, twenty distinct families of Gelfand pairs (K, H_n) where K is connected and acts irreducibly on V. The current paper concerns analysis in this setting.

There is a well developed theory of spherical functions associated with Gelfand pairs of the form (K, H_n). We denote the space of bounded K-spherical functions, equipped with the compact-open topology, by $\Delta(K, H_n)$. There are two types of bounded K-spherical functions:

- Type 1: $\Delta_1(K, H_n) = \{\phi_{\alpha,\lambda} \mid \alpha \in \Lambda, \lambda \in \mathbb{R}^\times\}$. Here Λ is a countable index set that parameterizes the decomposition
$$\mathbb{C}[V] = \sum_{\alpha\in\Lambda} P_\alpha$$
of $\mathbb{C}[V]$ into K-irreducible subspaces. $\Delta_1(K, H_n)$ is dense and of full Godement-Plancherel measure in $\Delta(K, H_n)$.
- Type 2: $\Delta_2(K, H_n) = \{\eta_{Kw} \mid w \in V\}$, where Kw is a K-orbit in V.

The reader will find the relevant definitions below in Section 2 along with a summary of some results from earlier work. The K-spherical transform $\widehat{f} : \Delta(K, H_n) \to \mathbb{C}$ for a function $f \in L^1_K(H_n)$ is defined by

$$\widehat{f}(\psi) := \int_{H_n} f(z,t)\overline{\psi(z,t)}\, dz dt,$$

were "$dzdt$" denotes Haar measure for the group H_n. $\widehat{f}$ belongs to the space $C_\circ(\Delta(K, H_n))$ of continuous functions on $\Delta(K, H_n)$ that vanish at infinity.

What can one say about the spherical transform of a K-invariant Schwartz function on H_n? More precisely, letting $\mathcal{S}_K(H_n)$ denote the space of K-invariant Schwartz functions on H_n, we seek to characterize the subspace

$$\mathcal{S}_K(H_n)^\wedge = \left\{\widehat{f} \mid f \in \mathcal{S}_K(H_n)\right\}$$

of $C_\circ(\Delta(K,H_n))$. The main result in this paper is Theorem 6.1 below, which provides a complete solution to this problem. In order for a function F on $\Delta(K,H_n)$ to belong to the space $\mathcal{S}_K(H_n)^\wedge$, the following conditions are both necessary and sufficient:

1. F is continuous on $\Delta(K,H_n)$.
2. $w \mapsto F(\eta_{Kw})$ is a Schwartz function on V.
3. $\lambda \mapsto F(\phi_{\alpha,\lambda})$ is smooth on $\mathbb{R}^\times$ and the functions $\partial_\lambda^m F(\phi_{\alpha,\lambda})$ satisfy certain decay conditions. In particular, $\partial_\lambda^m F(\phi_{\alpha,\lambda})$ is a rapidly decreasing sequence in α for each fixed $\lambda \in \mathbb{R}^\times$.
4. Certain "derivatives" of F also satisfy the three conditions above. These are defined on $\Delta_1(K,H_n)$ as specific combinations of ∂_λ and "difference operators" which play the role of differentiation in the discrete parameter $\alpha \in \Lambda$.

The precise formulation of these conditions can be found in Section 6. The "derivatives" of functions in $\mathcal{S}_K(H_n)^\wedge$ referred to above are operators corresponding to multiplication of functions in $\mathcal{S}_K(H_n)$ by certain polynomials. The difference operators in the discrete parameter $\alpha \in \Lambda$ are linear operators whose coefficients are "*generalized binomial coefficients*". These coefficients were introduced by Z. Yan in [Yan] and appear in many formulas concerning the type 1 spherical functions. A summary of their properties is given below in Section 4. A more complete discussion of generalized binomial coefficients will appear in [BR].

One consequence of the estimates involved in our characterization of $\mathcal{S}_K(H_n)^\wedge$ is that $f \in \mathcal{S}_K(H_n)$ can be recovered from $F = \widehat{f}$ via inversion of the spherical transform. One has

$$f(z,t) = \left(\frac{1}{2\pi}\right)^{n+1} \int_{\mathbb{R}^\times} \sum_{\alpha\in\Lambda} \dim(P_\alpha) F\left(\phi_{\alpha,\lambda}\right) \phi_{\alpha,\lambda}(z,t)|\lambda|^n d\lambda. \tag{1.1}$$

The first two conditions in our characterization of $\mathcal{S}_K(H_n)^\wedge$ play a rather subtle role in the sufficiency proof. Indeed, the bounded K-spherical functions η_{Kw} of type 2 do not appear in Formula 1.1. In order to control the growth of $f(z,t)$ in the central direction, we introuce a λ-derivative into the right hand side of Formula 1.1. An application of integration by parts produces two boundary terms at $\lambda = 0$ and we need to show that these cancel. This is done in Section 7 where we prove that

$$\lim_{\lambda\to 0^\pm} \sum_{\alpha\in\Lambda} \dim(P_\alpha)|\lambda|^n F\left(\phi_{\alpha,\lambda}\right) \phi_{\alpha,\lambda}(z,0) = \left(\frac{1}{2\pi}\right)^n \int_V F\left(\eta_{Kw}\right) \eta_{Kw}(z) dw.$$

(See Proposition 7.1.) It is here that the continuity and behavior of F on $\Delta_2(K,H_n)$ comes into play.

We also consider the related problem of characterizing the the space $\mathcal{S}_K(V)$ of K-invariant Schwartz functions on V via the spherical transform. The functions ϕ_α° on V defined by $\phi_\alpha^\circ(z) = \phi_\alpha(z,0)$ form a complete orthogonal system in the space $L^2_K(V)$ of square integrable K-invariant functions on V (see Proposition 3.1). The

ϕ_α°'s are eigenfunctions for differential operators that arise from K-invariant and left-invariant differential operators on the Heisenberg group and (modulo dilations) for the Fourier transform. The coefficients in the L^2-expansion of $f \in L^2_K(V)$ as a series in $\{\phi_\alpha^\circ \mid \alpha \in \Lambda\}$ are (up to normalization) $\widehat{f}(\alpha) := \langle f, \phi_\alpha^\circ \rangle_2$. Theorem 5.1 asserts that F belongs to

$$\mathcal{S}_K(V)^\wedge = \{\widehat{f} \mid f \in \mathcal{S}_K(V)\}$$

if and only if $\{F(\alpha) \mid \alpha \in \Lambda\}$ is a rapidly decreasing sequence.

The remainder of this paper is structured as follows. Section 2 contains preliminary material and summarizes results from previous work concerning spherical functions. Section 3 concerns the spherical series expansions that arise by suppressing the central direction in the Heisenberg group. The material in this section appeared in [Yan] but we have provided new and complete proofs here. Section 4 concerns combinatorial properties of the generalized binomial coefficients. Our characterizations of $\mathcal{S}_K(V)^\wedge$ and $\mathcal{S}_K(H_n)^\wedge$ appear in Section 5 and Section 6 respectively. As a corollary to Theorem 6.1, we show that for $K = U(n)$ or $\mathbb{T}^n$, one can construct a function $f \in \mathcal{S}_K(H_n)$ whose spherical transform $\widehat{f}$ has a pre-determined compact support in $\Delta(K, H_n)$. The sufficiency proof for Theorem 6.1 is completed in Section 7, which concerns the analysis of boundary terms.

The first and third authors wish to thank the Université de Metz for their support during the completion of this research.

2. Notation and Preliminaries

We need to establish notation and recall some results concerning spherical functions associated with Gelfand pairs (K, H_n). We refer the reader to [BJR90], [BJR92] and [BJRW] for complete details on this material.

Throughout this paper, K will always denote a closed Lie subgroup of $U(n)$ for which (K, H_n) is a Gelfand pair.

2.1. **Decomposition of $\mathbb{C}[V]$.** We decompose $\mathbb{C}[V]$ into K-irreducible subspaces P_α,

$$\mathbb{C}[V] = \sum_{\alpha \in \Lambda} P_\alpha \tag{2.1}$$

where Λ is some countably infinite index set. Theorem 1.1 ensures that this decomposition is canonical. Since the representation of K on $\mathbb{C}[V]$ preserves the space $\mathcal{P}_m(V)$ of homogeneous polynomials of degree m, each P_α is a subspace of some $\mathcal{P}_m(V)$. We write $|\alpha|$ for the degree of homogeneity of the polynomials in P_α, so that $P_\alpha \subset \mathcal{P}_{|\alpha|}(V)$. We will write d_α for the dimension of P_α and denote by $0 \in \Lambda$ the index for the scalar polynomials $P_0 = \mathcal{P}_0(V) = \mathbb{C}$.

2.2. **Invariant polynomials.** Since the representation of K on $\mathbb{C}[V]$ is multiplicity free, there can be no non-constant K-invariant holomorphic polynomials. One does, however, have invariant polynomials on the underlying real vector space $V_{\mathbb{R}}$ for V. We denote the set of these by $\mathbb{C}[V_{\mathbb{R}}]^K$. One obtains a canonical basis for $\mathbb{C}[V_{\mathbb{R}}]^K$ as follows. Given $\alpha \in \Lambda$ and any orthonormal basis $\{v_1, \ldots, v_{d_\alpha}\}$ for P_α, we let

$$p_\alpha(z) := \frac{1}{d_\alpha} \sum_{j=1}^{d_\alpha} v_j(z)\overline{v_j(z)}. \tag{2.2}$$

The polynomial p_α on $V_{\mathbb{R}}$ is $\mathbb{R}^+$-valued, K-invariant, and homogeneous of degree $2|\alpha|$. The definition of p_α does not depend on the choice of basis for P_α and $\{p_\alpha \mid \alpha \in \Lambda\}$ is a vector space basis for $\mathbb{C}[V_{\mathbb{R}}]^K$. One computes that

$$\sum_{|\alpha|=m} d_\alpha p_\alpha(z) = \frac{1}{m!}\gamma(z)^m \tag{2.3}$$

where $\gamma(z)$ is defined by

$$\gamma(z) := |z|^2/2.$$

A result in [HU91] ensures that the algebra $\mathbb{C}[V_{\mathbb{R}}]^K$ is freely generated by a canonical finite subset $\{\gamma_1, \gamma_2, \ldots, \gamma_d\} \subset \{p_\alpha \mid \alpha \in \Lambda\}$,

$$\mathbb{C}[V_{\mathbb{R}}]^K = \mathbb{C}[\gamma_1, \gamma_2, \ldots, \gamma_d]. \tag{2.4}$$

We call the generators $\{\gamma_1, \ldots, \gamma_d\}$ the *fundamental invariants* for the action of K on V. When K acts irreducibly on V, the polynomial γ is one of the fundamental invariants and we can suppose that $\gamma_1 = \gamma$.

2.3. **Invariant differential operators.** One basis for the Lie algebra of left invariant vector fields on H_n is written as $\{Z_1, Z_2, \ldots, Z_n, \overline{Z}_1, \overline{Z}_2, \ldots, \overline{Z}_n, T\}$ where

$$Z_j = 2\frac{\partial}{\partial \overline{z}_j} + i\frac{z_j}{2}\frac{\partial}{\partial t}, \quad \overline{Z}_j = 2\frac{\partial}{\partial z_j} - i\frac{\overline{z}_j}{2}\frac{\partial}{\partial t}, \tag{2.5}$$

and

$$T := \frac{\partial}{\partial t}.$$

With these conventions one has $[Z_j, \overline{Z}_j] = -2iT$. The first order operators $Z_1, \ldots, Z_n, \overline{Z}_1, \ldots, \overline{Z}_n, T$ generate the algebra $\mathbb{D}(H_n)$ of left-invariant differential operators on H_n. We denote the subalgebra of K-invariant differential operators by

$$\mathbb{D}_K(H_n) := \{D \in \mathbb{D}(H_n) \mid D(f \circ k) = D(f) \circ k \text{ for } k \in K,\ f \in C^\infty(H_n)\}.$$

Since (K, H_n) is a Gelfand pair, $\mathbb{D}_K(H_n)$ is an abelian algebra. A result due to Thomas [Tho82] shows that the converse is also true, at least when K is connected.

One differential operator will play a key role in this paper. This is the *Heisenberg sub-Laplacian* defined by

$$\mathcal{L} := \frac{1}{2}\sum_{j=1}^{n}\left(Z_j\overline{Z}_j + \overline{Z}_j Z_j\right). \tag{2.6}$$

$\mathcal{L}$ is $U(n)$-invariant and hence belongs to $\mathbb{D}_K(H_n)$ for all Gelfand pairs (K, H_n). Since the formal adjoints of Z_j and $\overline{Z}_j$ as operators on $L^2(H_n)$ are $Z_j^* = -\overline{Z}_j$ and $\overline{Z}_j^* = -Z_j$, we see that $\mathcal{L}$ is essentially self-adjoint on $L^2(H_n)$.

The algebra $\mathbb{D}_K(H_n)$ is generated by $\{L_{\gamma_1}, \ldots, L_{\gamma_d}, T\}$, where L_{γ_j} is the operator obtained from $\gamma_j(Z, \overline{Z})$ by symmetrization.

2.4. **Spherical functions.** A smooth function $\psi : H_n \to \mathbb{C}$ is called *K-spherical* if

1. ψ is K-invariant,
2. ψ is an eigenfunction for every $D \in \mathbb{D}_K(H_n)$, and
3. $\psi(0, 0) = 1$.

We write $\widehat{D}(\psi)$ for the eigenvalue of $D \in \mathbb{D}_K(H_n)$ on a K-spherical function ψ, that is $D(\psi) = \widehat{D}(\psi)\psi$.

We denote the set of positive definite K-spherical functions on H_n by $\Delta(K, H_n)$. In [BJR90], it is shown that every bounded K-spherical function is positive definite, so $\Delta(K, H_n)$ is also the set of bounded K-spherical functions. We remark that this result contrasts with the situation for symmetric spaces. (See eg. [GV88].) As shown in [BJR92], the bounded K-spherical functions can be derived from the representation theory for H_n together with the action of K on V.

The infinite dimensional irreducible unitary representations of H_n can be realized in *Fock space*. This is the space $\mathcal{F}$ consisting of entire functions $f : V \to \mathbb{C}$ which are square integrable with respect to $e^{-|z|^2/2}dz$ with Hilbert space structure

$$\langle f, g\rangle_{\mathcal{F}} = \left(\frac{1}{2\pi}\right)^n \int_V f(z)\overline{g(z)}e^{-|z|^2/2}dz.$$

Here "dz" denotes Lebesgue measure on $V_{\mathbb{R}} \cong \mathbb{R}^{2n}$. The holomorphic polynomials $\mathbb{C}[V]$ form a dense subspace in $\mathcal{F}$. One has an irreducible unitary representation π of H_n on $\mathcal{F}$ defined as

$$(\pi(z, t)f)(w) = e^{it - \frac{1}{2}<w,z> - \frac{1}{4}|z|^2} f(w + z).$$

For $\alpha \in \Lambda$ let

$$\phi_\alpha(z, t) := \frac{1}{d_\alpha}\sum_{j=1}^{d_\alpha}\langle \pi(z, t)v_j, v_j\rangle_{\mathcal{F}}, \tag{2.7}$$

where $\{v_1, \ldots, v_{d_\alpha}\}$ is an orthonormal basis for P_α. This description of ϕ_α does not depend on our choice of basis $\{v_j\}$. Define $\phi_{\alpha,\lambda}$ for $\lambda \in \mathbb{R}^\times$ and $\alpha \in \Lambda$ by

$$\phi_{\alpha,\lambda}(z,t) := \phi_\alpha\left(\sqrt{|\lambda|}z, \lambda t\right), \tag{2.8}$$

so that $\phi_\alpha = \phi_{\alpha,1}$. The $\phi_{\alpha,\lambda}$'s are distinct bounded K-spherical functions. We refer to these elements of $\Delta(K, H_n)$ as the spherical functions of *type 1.* From Equation 2.7 one can show that ϕ_α has the general form

$$\phi_\alpha(z,t) = e^{it} q_\alpha(z) e^{-\frac{1}{4}|z|^2} = e^{it} q_\alpha(z) e^{-\gamma(z)/2}$$

where q_α is a K-invariant polynomial on $V_{\mathbb{R}}$ with homogeneous component of highest degree given by $(-1)^{|\alpha|} p_\alpha$. As for the p_α's, the set $\{q_\alpha \mid \alpha \in \Lambda\}$ is a basis for the vector space $\mathbb{C}[V_{\mathbb{R}}]^K$.

The eigenvalues of the Heisenberg sub-Laplacian $\mathcal{L}$ on the type 1 K-spherical functions are given by

$$\widehat{\mathcal{L}}\left(\phi_{\alpha,\lambda}\right) = -|\lambda|(2|\alpha| + n). \tag{2.9}$$

This follows from Proposition 3.20 in [BJR92] together with Lemma 3.4 in [BJRW]. The key point here is that the *quantum harmonic oscillator* $\pi(\mathcal{L})$ on Fock space $\mathcal{F}$ acts via the scalar $-(2m + n)$ on $\mathcal{P}_m(V)$.

In addition to the K-spherical functions of type 1, there are K-spherical functions which arise from the one dimensional representations of H_n. For $w \in V$, let

$$\eta_w(z,t) := \int_K e^{iRe\langle w,kz\rangle} dk = \int_K e^{iRe\langle z,kw\rangle} dk \tag{2.10}$$

where "dk" denotes normalized Haar measure on K. These are the bounded K-spherical functions of *type 2.* Note that η_0 is the constant function 1 and $\eta_w = \eta_{w'}$ if and only if $Kw = Kw'$. Thus we have one K-spherical function for each K-orbit in V and sometimes write "η_{Kw}" in place of "η_w".

It is shown in [BJR92] that every bounded K-spherical function is of type 1 or type 2. Thus we have:

Theorem 2.1. *The bounded K-spherical functions on H_n are parameterized by the set $\left(\Lambda \times \mathbb{R}^\times\right) \cup (V/K)$ via $\Delta(K, H_n) = \Delta_1(K, H_n) \cup \Delta_2(K, H_n)$ where*

$$\Delta_1(K, H_n) = \left\{\phi_{\alpha,\lambda} \mid \alpha \in \Lambda, \lambda \in \mathbb{R}^\times\right\} \quad \textit{and} \quad \Delta_2(K, H_n) = \left\{\eta_{Kw} \mid w \in V\right\}.$$

2.5. **Topology on $\Delta(K, H_n)$.** We give $\Delta(K, H_n)$ the (compact-open) topology of uniform convergence on compact sets. A detailed discussion of this topology can be found in [BJRW]. A key fact is that $\Delta_1(K, H_n)$ is dense in $\Delta(K, H_n)$. Moreover, the space $\Delta(K, H_n)$ can be embedded in $(\mathbb{R}^+)^d \times \mathbb{R}$. Indeed, let $L_{\gamma_1}, \ldots, L_{\gamma_d} \in \mathbb{D}_K(H_n)$ be differential operators obtained from the fundamental invariants via symmetrization and let $T = \frac{\partial}{\partial t}$.

Theorem 2.2 (cf. [BJRW]). *The map* $E : \Delta(K, H_n) \to (\mathbb{R}^+)^d \times \mathbb{R}$ *defined by*

$$E(\psi) = \left(\left|\widehat{L_{\gamma_1}}(\psi)\right|, \ldots, \left|\widehat{L_{\gamma_d}}(\psi)\right|, -i\widehat{T}(\psi)\right)$$

is a homeomorphism onto its image.

The eigenvalues $\widehat{L_{\gamma_j}}(\psi)$ for the L_{γ_j}'s on $\Delta(K, H_n)$ are real numbers with constant sign and one can describe the map E more explicitly as follows. Let $e_j(\alpha) = \left|\widehat{L_{\gamma_j}}(\phi_\alpha)\right|$ and $2m_j$ be the degree of γ_j. Then

$$E(\phi_{\alpha,\lambda}) = \left(e_1(\alpha)|\lambda|^{m_1}, \ldots, e_d(\alpha)|\lambda|^{m_d}, \lambda\right) \quad \text{and} \quad E(\eta_w) = \left(\gamma_1(w), \ldots, \gamma_d(w), 0\right).$$

We can use the map E and a metric on $(\mathbb{R}^+)^d \times \mathbb{R}$ to produce a metric on $\Delta(K, H_n)$ which induces the compact-open topology. For example,

$$d(\psi_1, \psi_2) = \sum_{j=1}^{d} \left(\left|\widehat{L_{\gamma_1}}(\psi_j) - \widehat{L_{\gamma_1}}(\psi_j)\right| + \left|\widehat{T}(\psi_1) - \widehat{T}(\psi_2)\right|\right. \tag{2.11}$$

is one such metric on $\Delta(K, H_n)$.

2.6. **The K-spherical transform.** The K-spherical transform for $f \in L^1_K(H_n)$ is the function $\widehat{f}$ on $\Delta(K, H_n)$ defined by

$$\widehat{f}(\psi) := \int_{H_n} f(z,t)\overline{\psi(z,t)}\, dzdt.$$

Here "$dzdt$" denotes Haar measure for the group H_n, which is simply Euclidean measure on $V_{\mathbb{R}} \times \mathbb{R}$. $\widehat{f}$ is a bounded function with

$$||\widehat{f}||_\infty \le ||f||_1$$

for $f \in L^1_K(H_n)$. This follows immediately from the fact that for $\psi \in \Delta(K, H_n)$ one has $|\psi(z,t)| \le \psi(0,0) = 1$, since ψ is positive definite.

The compact-open topology on $\Delta(K, H_n)$ is the smallest topology which makes all of the maps $\{\widehat{f} \mid f \in L^1_K(H_n)\}$ continuous. Since $L^1_K(H_n)$ is a Banach $\star$-algebra (with involution defined via $f^*(z,t) := \overline{f(-z,-t)}$), $\widehat{f}$ belongs to the space $C_\circ(\Delta(K, H_n))$ of continuous functions on $\Delta(K, H_n)$ that vanish at infinity.

2.7. **Godement-Plancherel measure.** Godement's Plancherel Theory for Gelfand pairs (G, K) (cf. [God62], or section 1.6 in [GV88]) ensures that there exists a unique positive Borel measure $d\mu$ on the space $\Delta(K, H_n)$ for which

$$\int_{H_n} |f(z,t)|^2 dzdt = \int_{\Delta(K,H_n)} |\widehat{f}(\psi)|^2 d\mu(\psi) \tag{2.12}$$

for all continuous functions $f \in L^1_K(H_n) \cap L^2_K(H_n)$. If $f \in L^1_K(H_n) \cap L^2_K(H_n)$ is continuous and $\widehat{f}$ is integrable with respect to $d\mu$ then one has the *Inversion Formula*:

$$f(z,t) = \int_{\Delta(K,H_n)} \widehat{f}(\psi)\psi(z,t)\, d\mu(\psi). \tag{2.13}$$

The spherical transform $f \mapsto \widehat{f}$ extends uniquely to an isomorphism between $L^2_K(H_n)$ and $L^2(\Delta(K, H_n), d\mu)$.

The following result makes the Godement-Plancherel measure on $\Delta(K, H_n)$ explicit. Given $F : \Delta(K, H_n) \to \mathbb{C}$, we write $F(\alpha, \lambda)$ in place of $F(\phi_{\alpha,\lambda})$. The reader can find proofs of Theorem 2.3 in [BJRW] and [Yan]. The result for $K = U(n)$ is also discussed in [Far87] and [Str91].

Theorem 2.3. *The Godement-Plancherel measure $d\mu$ on $\Delta(K, H_n)$ is given by*

$$\int_{\Delta(K,H_n)} F(\psi)\, d\mu(\psi) = \left(\frac{1}{2\pi}\right)^{n+1} \int_{\mathbb{R}^\times} \sum_{\alpha\in\Lambda} d_\alpha F(\alpha, \lambda)\, |\lambda|^n d\lambda.$$

Note that $\Delta_2(K, H_n)$ is a set of measure zero in $\Delta(K, H_n)$.

2.8. **Fourier transforms.** We define the Fourier transform $\mathcal{F}_V(f) : V \to \mathbb{C}$ of $f \in L^1(V)$ by

$$\mathcal{F}_V(f)(w) := \int_V f(z) e^{-iRe\langle z,w\rangle} dz$$

where "dz" denotes Euclidean measure on $V_{\mathbb{R}}$. With this normalization one has

$$||\mathcal{F}_V(f)||_2 = (2\pi)^n ||f||_2$$

and the inversion formula reads

$$\int_V \mathcal{F}_V(f)(w) e^{iRe\langle z,w\rangle} dw = (2\pi)^{2n} f(z)$$

for suitable $f : V \to \mathbb{C}$. Similarly, the Euclidean Fourier transform $\mathcal{F}_H(f)$ of $f \in L^1(H_n)$ is defined by

$$\mathcal{F}_H(f)(w, s) = \int_{H_n} f(z, t) e^{-iRe\langle z,w\rangle} e^{-its} dz dt. \tag{2.14}$$

If $f \in L^1_K(H_n)$ then the Fourier and spherical transforms are related via

$$\widehat{f}(\eta_w) = \mathcal{F}_H(f)(w, 0). \tag{2.15}$$

Indeed, for $w \in V$, one has

$$\begin{aligned}
\widehat{f}(\eta_w) &= \int_{H_n} \int_K f(z, t) e^{-iRe\langle kz,w\rangle} dk dz dt \\
&= \int_{H_n} \int_K f(kz, t) e^{-iRe\langle z,w\rangle} dk dz dt \\
&= \int_{H_n} f(z, t) e^{-iRe\langle z,w\rangle} dz dt \\
&= \mathcal{F}_H(f)(w, 0).
\end{aligned}$$

3. K-SPHERICAL SERIES EXPANSIONS ON V

The dependence of a K-spherical function $\psi(z,t)$ on the central variable t is rather trivial. Thus, it is natural to suppress this direction and consider functions on V. The Godement-Plancherel Theory in section 2.6 leads to orthogonal eigenfunction expansions for K-invariant functions on V. The ideas here are quite standard and well known. For $f : H_n \to \mathbb{C}$, we define $f^\circ : V \to \mathbb{C}$ by

$$f^\circ(z) := f(z,0).$$

Given $g : V \to \mathbb{C}$ we define $g_1 : H_n \to \mathbb{C}$ by

$$g_1(z,t) := g(z)e^{it}.$$

If f is a K-invariant function on H_n then f° is K-invariant on V. If g is K-invariant on V then g_1 is K-invariant on H_n and $(g_1)^\circ = g$. For $D \in \mathbb{D}(H_n)$ define a differential operator on D' on $V_{\mathbb{R}}$ by

$$D'(g) := (D\,(g_1))^\circ \quad \text{for } g \in C^\infty(V_{\mathbb{R}}).$$

It is easy to verify that if $D \in \mathbb{D}_K(H_n)$ then D' is a K-invariant differential operator on $V_{\mathbb{R}}$. Moreover

$$D'\left(\phi_\alpha^\circ\right) = \left(D\left((\phi_\alpha^\circ)_1\right)\right)^\circ = \left(D\left(\phi_\alpha\right)\right)^\circ = \widehat{D}\left(\phi_\alpha\right)\phi_\alpha^\circ.$$

Thus $\phi_\alpha^\circ(z) = q_\alpha(z)e^{-\gamma(z)/2}$ is an eigenfunction for D' whenever $D \in \mathbb{D}_K(H_n)$.

As an example, consider the Heisenberg sub-Laplacian $\mathcal{L}$ given by Equation 2.6. Using Equations 2.5 one computes that

$$\mathcal{L}' = 4\Delta - \gamma/2 + E \tag{3.1}$$

where $\Delta := \sum_{j=1}^n \frac{\partial}{\partial z_j}\frac{\partial}{\partial \overline{z}_j}$ (so that 4Δ is the usual Laplace operator on $V_{\mathbb{R}}$) and $E := \sum_{j=1}^n \left(\overline{z}_j \frac{\partial}{\partial \overline{z}_j} - z_j \frac{\partial}{\partial z_j}\right)$. A homogeneity argument shows that $E(\phi_\alpha^\circ) = 0$ for $\alpha \in \Lambda$. Equation 2.9 thus yields

$$(4\Delta - \gamma/2)\left(\phi_\alpha^\circ\right) = \mathcal{L}'(\phi_\alpha^\circ) = -(2|\alpha| + n)\phi_\alpha^\circ. \tag{3.2}$$

Proposition 3.1 (cf. [Yan]). *$\{\phi_\alpha^\circ \mid \alpha \in \Lambda\}$ is a complete orthogonal system in $L^2_K(V)$ with $||\phi_\alpha^\circ||_2^2 = (2\pi)^n/d_\alpha$.*

Proof. The functional form for $\phi_{\alpha,\lambda}$ shows that $\phi_\alpha^\circ = q_\alpha(z)e^{-\frac{1}{4}|z|^2}$ for some $q_\alpha \in \mathbb{C}[V_{\mathbb{R}}]^K$. Thus ϕ_α° is a K-invariant Schwartz function and hence belongs to $L^2_K(V_{\mathbb{R}})$. For $v \in \mathcal{F}$, define $M_v(z) := \langle \pi(z,0)v, v\rangle_{\mathcal{F}}$. Since π is a square-integrable representation, M_v is square-integrable. Let $\{v_1, \ldots, v_{d_\alpha}\}$, $\{u_1, \ldots, u_{d_\beta}\}$ be orthonormal bases for P_α, P_β. One has $\phi_\alpha^\circ = \frac{1}{d_\alpha}\sum_i M_{v_i}$, $\phi_\beta^\circ = \frac{1}{d_\beta}\sum_j M_{u_j}$ so that

$$\langle \phi_\alpha^\circ, \phi_\beta^\circ \rangle_2 = \frac{1}{d_\alpha d_\beta}\sum_{i,j}\langle M_{v_i}, M_{u_j}\rangle_2.$$

Standard facts concerning square-integrable representations ensure that

$$\langle M_{v_i}, M_{u_j}\rangle_2 = c\left|\langle v_i, u_j\rangle_{\mathcal{F}}\right|^2$$

for some constant c. Thus we have

$$\langle \phi^\circ_\alpha, \phi^\circ_\beta\rangle_2 = \frac{c\delta_{\alpha,\beta}}{d_\alpha},$$

and hence the ϕ°_α's are pair-wise orthogonal.

To compute the L^2-norm of ϕ°_α, one must determine c. As 1 is a unit vector in $\mathcal{F}$, one has

$$\begin{aligned} c &= \langle M_1, M_1\rangle_2 = \|e^{-\frac{1}{4}|z|^2}\|_2^2 \\ &= \int_V e^{-\frac{1}{2}|z|^2} dz \\ &= \frac{2\pi^n}{(n-1)!}\int_0^\infty e^{-r^2/2} r^{2n-1} dr \\ &= (2\pi)^n. \end{aligned}$$

It remains to show that $\{\phi^\circ_\alpha \mid \alpha \in \Lambda\}$ is complete in $L^2_K(V)$. Suppose that $f \in L^2_K(V)$ and that $\langle f, \phi^\circ_\alpha\rangle_2 = 0$ for all $\alpha \in \Lambda$. Since $\phi^\circ_\alpha(z) = q_\alpha(z)e^{-|z|^2/4}$ where $\{q_\alpha \mid \alpha \in \Lambda\}$ is a basis for $\mathbb{C}[V_\mathbb{R}]^K$, we see that

$$\int_V f(z)p(z)e^{-|z|^2/4} dz = 0$$

for all polynomials $p \in \mathbb{C}[V_\mathbb{R}]$. It follows that $f = 0$ almost everywhere and hence that $\{\phi^\circ_\alpha \mid \alpha \in \Lambda\}$ is complete. $\square$

Note that $|\phi^\circ_\alpha(z)|$ is bounded by $\phi^\circ_\alpha(0) = 1$. For $f \in L^p_K(V)$ and $\alpha \in \Lambda$ let $\widehat{f}(\alpha)$ be defined by

$$\widehat{f}(\alpha) = \langle f, \phi^\circ_\alpha\rangle_2 = \int_V f(z)\phi^\circ_\alpha(z) dz.$$

Corollary 3.2. *If $f \in L^2_K(V)$ then*

$$f = \sum_{\alpha\in\Lambda} \widehat{f}(\alpha)\frac{\phi^\circ_\alpha}{\|\phi^\circ_\alpha\|_2^2} = \left(\frac{1}{2\pi}\right)^n \sum_{\alpha\in\Lambda} d_\alpha \widehat{f}(\alpha)\phi^\circ_\alpha \quad \textit{in } L^2(V) \textit{ and}$$

$$\|f\|_2^2 = \sum_{\alpha\in\Lambda} \frac{|\widehat{f}(\alpha)|^2}{\|\phi^\circ_\alpha\|_2^2} = \left(\frac{1}{2\pi}\right)^n \sum_{\alpha\in\Lambda} d_\alpha |\widehat{f}(\alpha)|^2.$$

It is shown in [Yan] that the ϕ°_α's are essentially eigenfunctions for $\mathcal{F}_V$. More precisely,

Proposition 3.3 (cf. [Yan]). *The function* $\widetilde{\phi_\alpha}(z) := \phi^\circ_\alpha(\sqrt{2}z)$ *satisfies*

$$\mathcal{F}_V\left(\widetilde{\phi_\alpha}\right) = (2\pi)^n(-1)^{|\alpha|}\widetilde{\phi_\alpha}.$$

The factor of $\sqrt{2}$ appears above but not in [Yan] because different coordinates on H_n are used there. The key fact used in Yan's proof of Proposition 3.3 is that ϕ°_α can be expressed in terms of the Hermite-Weber transform of p_α. One can also prove Proposition 3.3 by using the fact that a suitable multiple of the Hermite operator $-4\Delta + 2\gamma$ is an infinitesimal generator for the Fourier transform. (See for example, pg 122 in [HT92].) With the notational conventions here, this fact can be written as

$$\mathcal{F}_V = (2\pi i)^n \exp(i\pi(\Delta - \gamma/2)).$$

In view of Equation 3.2, we have

$$(\Delta - \gamma/2)\widetilde{\phi_\alpha} = -\frac{2|\alpha| + n}{2}\widetilde{\phi_\alpha}$$

and hence

$$\mathcal{F}_V\left(\widetilde{\phi_\alpha}\right) = (2\pi i)^n e^{-i\pi(2|\alpha|+n)/2}\widetilde{\phi_\alpha} = (2\pi)^n(-1)^{|\alpha|}\widetilde{\phi_\alpha}$$

as stated.

For $r > 0$, we let $\delta_r : V \to V$ be the dilation

$$\delta_r(z) := rz.$$

Since one has $\mathcal{F}_V(f \circ \delta_r) = \left(\frac{1}{r}\right)^{2n}\mathcal{F}_V(f) \circ \delta_{1/r}$, we obtain the formulas

$$\mathcal{F}_V(\phi^\circ_\alpha) = (4\pi)^n(-1)^{|\alpha|}\phi^\circ_\alpha \circ \delta_2 \quad \text{and more generally} \tag{3.3}$$

$$\mathcal{F}_V(\phi^\circ_\alpha \circ \delta_r) = \left(\frac{4\pi}{r^2}\right)^n(-1)^{|\alpha|}\phi^\circ_\alpha \circ \delta_{2/r}. \tag{3.4}$$

In summary, the set $\{\phi^\circ_\alpha \mid \alpha \in \Lambda\}$ forms a complete orthogonal system in $L^2_K(V)$. Each ϕ°_α is a simultaneous eigenfunction for the differential operators $\{D' \mid D \in \mathbb{D}_K(H_n)\}$ and (modulo dilation) for the Fourier transform $\mathcal{F}_V$.

4. Generalized binomial coefficients

For $\alpha \in \Lambda$, the functions $\{p_\beta \mid |\beta| \le |\alpha|\}$ form a basis for the space of K-invariant polynomials on $V_\mathbb{R}$ of degree at most $2|\alpha|$. Since q_α belongs to this space, we can write:

$$q_\alpha = \sum_{|\beta|\le|\alpha|}(-1)^{|\beta|}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}p_\beta \tag{4.1}$$

for some well defined numbers $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$. We call these values *generalized binomial coefficients* for the action of K on V. They will play a crucial role in the formulation and proof of our characterization of K-invarint Schwartz functions via the spherical transform. The results that we will need concerning generalized binomial coefficients are contained in this section.

Since the functions q_α and p_β are all real valued, the generalized binomial coefficients are real numbers. It is shown in [BR], moreover, that the $\left[\begin{array}{c}\alpha\\ \beta\end{array}\right]$'s are non-negative. Since $(-1)^{|\alpha|}p_\alpha$ is the homogeneous component of highest degree in q_α, we see that $\left[\begin{array}{c}\alpha\\ \alpha\end{array}\right] = 1$ and that $\left[\begin{array}{c}\alpha\\ \beta\end{array}\right] = 0$ for $\beta \neq \alpha$ with $|\beta| = |\alpha|$. We extend the definition of generalized binomial coefficients by setting $\left[\begin{array}{c}\alpha\\ \beta\end{array}\right] = 0$ when $|\beta| > |\alpha|$. Since $q_\alpha(0) = 1 = p_0$ and $p_\beta(0) = 0$ for $|\beta| > 0$, we see that $\left[\begin{array}{c}\alpha\\ 0\end{array}\right] = 1$. Here recall that $0 \in \Lambda$ denotes the index with $P_0 = \mathbb{C}$.

The generalized binomial coefficients were introduced by Z. Yan in [Yan], where one finds the important identity

$$\frac{\gamma^k}{k!}p_\beta = \sum_{|\alpha|=|\beta|+k} \frac{d_\alpha}{d_\beta}\left[\begin{array}{c}\alpha\\ \beta\end{array}\right]p_\alpha. \tag{4.2}$$

For the case where $K = U(n)$, decomposition 2.1 reads $\mathbb{C}[V] = \sum_{m=0}^{\infty}\mathcal{P}_m(V)$ and the $U(n)$-invariant polynomial p_m associated with $\mathcal{P}_m(V)$ is $p_m = \frac{(n-1)!}{(m+n-1)!}\gamma^m$ (see Proposition 6.2 in [BJR92]). Equation 4.2 shows that in this case we have

$$\left[\begin{array}{c}m+k\\ m\end{array}\right] = \binom{m+k}{k}. \tag{4.3}$$

The generalized binomial coefficients are studied further in [BR], where the reader will find proofs of the following identities:

$$\gamma q_\alpha = -\sum_{|\beta|=|\alpha|+1}\frac{d_\beta}{d_\alpha}\left[\begin{array}{c}\beta\\ \alpha\end{array}\right]q_\beta + (2|\alpha|+n)q_\alpha - \sum_{|\beta|=|\alpha|-1}\left[\begin{array}{c}\alpha\\ \beta\end{array}\right]q_\beta, \tag{4.4}$$

$$\sum_{|\beta|=\ell}\left[\begin{array}{c}\alpha\\ \beta\end{array}\right]\left[\begin{array}{c}\beta\\ \delta\end{array}\right] = \frac{(|\alpha|-|\delta|)!}{(|\alpha|-\ell)!(\ell-|\delta|)!}\left[\begin{array}{c}\alpha\\ \delta\end{array}\right]. \tag{4.5}$$

Letting $\delta = 0$ in Equation 4.5, we see that

$$\sum_{|\beta|=\ell}\left[\begin{array}{c}\alpha\\ \beta\end{array}\right] = \binom{|\alpha|}{\ell}, \tag{4.6}$$

since $\left[\begin{array}{c}\beta\\0\end{array}\right]=\left[\begin{array}{c}\alpha\\0\end{array}\right]=1$. It follows that for fixed β, $\left[\begin{array}{c}\alpha\\\beta\end{array}\right]$ has polynomial growth in $|\alpha|$. Equation 4.6 gives, in particular,

$$\sum_{|\beta|=|\alpha|-1}\left[\begin{array}{c}\alpha\\\beta\end{array}\right]=|\alpha|. \tag{4.7}$$

Evaluating Equation 4.4 at $z=0$ and using Equation 4.7 yields also

$$\sum_{|\beta|=|\alpha|+1}\frac{d_\beta}{d_\alpha}\left[\begin{array}{c}\beta\\\alpha\end{array}\right]=|\alpha|+n. \tag{4.8}$$

Our characterization of $\mathcal{S}_K(H_n)^\wedge$, presented below in Section 6, involves the application of *difference operators* $\mathcal{D}^+$, $\mathcal{D}^-$ defined as follows.

Definition 4.1. Given a function g on Λ, $\mathcal{D}^+g$ and $\mathcal{D}^-g$ are the functions on Λ defined by

$$\mathcal{D}^+g(\alpha)=\sum_{|\beta|=|\alpha|+1}\frac{d_\beta}{d_\alpha}\left[\begin{array}{c}\beta\\\alpha\end{array}\right]g(\beta)-(|\alpha|+n)g(\alpha)=\sum_{|\beta|=|\alpha|+1}\frac{d_\beta}{d_\alpha}\left[\begin{array}{c}\beta\\\alpha\end{array}\right](g(\beta)-g(\alpha))$$

$$\mathcal{D}^-g(\alpha)=|\alpha|g(\alpha)-\sum_{|\beta|=|\alpha|-1}\left[\begin{array}{c}\alpha\\\beta\end{array}\right]g(\beta)=\sum_{|\beta|=|\alpha|-1}\left[\begin{array}{c}\alpha\\\beta\end{array}\right](g(\alpha)-g(\beta))$$

for $|\alpha|>0$ and $\mathcal{D}^-g(0)=0$.

The two formulae presented for $\mathcal{D}^\pm g$ agree in view of Equations 4.8 and 4.7.

Using this notation, Equation 4.4 can be rewritten as

$$\gamma q_\alpha=-(\mathcal{D}^+-\mathcal{D}^-)q_\alpha. \tag{4.9}$$

Since $\phi_\alpha^\circ(z)=q_\alpha(z)e^{-\gamma(z)/2}$ and $\phi_{\alpha,\lambda}(z,t)=\phi_\alpha^\circ(\sqrt{|\lambda|}z)e^{i\lambda t}$, we obtain also

$$\gamma\phi_\alpha^\circ=-(\mathcal{D}^+-\mathcal{D}^-)\phi_\alpha^\circ,\quad \gamma\phi_{\alpha,\lambda}=-\frac{1}{|\lambda|}(\mathcal{D}^+-\mathcal{D}^-)\phi_{\alpha,\lambda}. \tag{4.10}$$

The right sides in these equations denote the functions on V and H_n obtained by applying the difference operators to the α-index. Thus, for example, $\mathcal{D}^-\phi_{\alpha,\lambda}$ denotes the function on H_n defined by $\mathcal{D}^-\phi_{\alpha,\lambda}(z,t)=|\alpha|\phi_{\alpha,\lambda}(z,t)-\sum_{|\beta|=|\alpha|-1}\left[\begin{array}{c}\alpha\\\beta\end{array}\right]\phi_{\beta,\lambda}(z,t)$.

Lemma 4.1. *$\partial_\lambda q_\alpha\left(\sqrt{|\lambda|}z\right)=\frac{1}{\lambda}\mathcal{D}^-q_\alpha\left(\sqrt{|\lambda|}z\right)$ for $\lambda\neq 0$.*

Proof. Equation 4.1 shows that

$$q_\alpha\left(\sqrt{|\lambda|}z\right)=\sum_{|\beta|\le|\alpha|}\left[\begin{array}{c}\alpha\\\beta\end{array}\right](-1)^{|\beta|}p_\beta\left(\sqrt{|\lambda|}z\right)=\sum_{|\beta|\le|\alpha|}\left[\begin{array}{c}\alpha\\\beta\end{array}\right](-|\lambda|)^{|\beta|}p_\beta(z).$$

Suppose here that $\lambda > 0$. We compute that

$$\begin{aligned}
\partial_\lambda q_\alpha\left(\sqrt{|\lambda|}z\right) &= \sum_{|\beta|\le|\alpha|}\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](-1)^{|\beta|}|\beta|\lambda^{|\beta|-1}p_\beta(z)\\
&= \frac{1}{\lambda}\sum_{|\beta|\le|\alpha|}\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](-1)^{|\beta|}|\beta|\lambda^{|\beta|}p_\beta(z)\\
&= \frac{|\alpha|}{\lambda}\sum_{|\beta|\le|\alpha|}\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](-\lambda)^{|\beta|}p_\beta(z) - \frac{1}{\lambda}\sum_{|\beta|\le|\alpha|}\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](|\alpha|-|\beta|)(-\lambda)^{|\beta|}p_\beta(z)\\
&= \frac{|\alpha|}{\lambda}q_\alpha\left(\sqrt{|\lambda|}z\right) - \frac{1}{\lambda}\sum_{|\beta|\le|\alpha|}\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](|\alpha|-|\beta|)(-1)^{|\beta|}p_\beta\left(\sqrt{|\lambda|}z\right)
\end{aligned}$$

Equation 4.5 shows that $\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](|\alpha|-|\beta|) = \sum_{|\delta|=|\alpha|-1}\left[\begin{array}{c}\alpha\\ \delta\end{array}\right]\left[\begin{array}{c}\delta\\ \beta\end{array}\right]$. Hence

$$\begin{aligned}
\partial_\lambda q_\alpha\left(\sqrt{|\lambda|}z\right) &= \frac{|\alpha|}{\lambda}q_\alpha\left(\sqrt{|\lambda|}z\right) - \frac{1}{\lambda}\sum_{|\beta|\le|\alpha|}\sum_{|\delta|=|\alpha|-1}\left[\begin{array}{c}\alpha\\ \delta\end{array}\right]\left[\begin{array}{c}\delta\\ \beta\end{array}\right](-1)^{|\beta|}p_\beta\left(\sqrt{|\lambda|}z\right)\\
&= \frac{|\alpha|}{\lambda}q_\alpha\left(\sqrt{|\lambda|}z\right) - \frac{1}{\lambda}\sum_{|\delta|=|\alpha|-1}\left[\begin{array}{c}\alpha\\ \delta\end{array}\right]\left(\sum_{|\beta|\le|\alpha|}\left[\begin{array}{c}\delta\\ \beta\end{array}\right](-1)^{|\beta|}p_\beta\left(\sqrt{|\lambda|}z\right)\right)\\
&= \frac{|\alpha|}{\lambda}q_\alpha\left(\sqrt{|\lambda|}z\right) - \frac{1}{\lambda}\sum_{|\delta|=|\alpha|-1}\left[\begin{array}{c}\alpha\\ \delta\end{array}\right]q_\delta\left(\sqrt{|\lambda|}z\right)\\
&= \frac{1}{\lambda}\mathcal{D}^- q_\alpha\left(\sqrt{|\lambda|}z\right).
\end{aligned}$$

A similar analysis applies when $\lambda < 0$. $\square$

Using Lemma 4.1 we obtain that for $\lambda > 0$,

$$\partial_\lambda\phi_{\alpha,\lambda}(z,t) = \partial_\lambda\left[q_\alpha\left(\sqrt{|\lambda|}z\right)e^{-\lambda\gamma(z)/2}e^{i\lambda t}\right] = \frac{1}{\lambda}\mathcal{D}^-\phi_{\alpha,\lambda}(z,t) - \frac{\gamma(z)}{2}\phi_{\alpha,\lambda}(z,t) + it\phi_{\alpha,\lambda}(z,t).$$

In view of Equation 4.10 we can also write

$$\partial_\lambda\phi_{\alpha,\lambda}(z,t) = \frac{1}{\lambda}\mathcal{D}^+\phi_{\alpha,\lambda}(z,t) + \frac{\gamma(z)}{2}\phi_{\alpha,\lambda}(z,t) + it\phi_{\alpha,\lambda}(z,t).$$

Thus we have

$$\partial_\lambda\phi_{\alpha,\lambda} = \left\{\begin{array}{c}\frac{1}{\lambda}\mathcal{D}^-\phi_{\alpha,\lambda} - \frac{\gamma}{2}\phi_{\alpha,\lambda} + it\phi_{\alpha,\lambda}\\ \frac{1}{\lambda}\mathcal{D}^+\phi_{\alpha,\lambda} + \frac{\gamma}{2}\phi_{\alpha,\lambda} + it\phi_{\alpha,\lambda}\end{array}\right\} \quad \text{for } \lambda > 0, \tag{4.11}$$

and similarly

$$\partial_\lambda\phi_{\alpha,\lambda} = \left\{\begin{array}{c}\frac{1}{\lambda}\mathcal{D}^-\phi_{\alpha,\lambda} + \frac{\gamma}{2}\phi_{\alpha,\lambda} + it\phi_{\alpha,\lambda}\\ \frac{1}{\lambda}\mathcal{D}^+\phi_{\alpha,\lambda} - \frac{\gamma}{2}\phi_{\alpha,\lambda} + it\phi_{\alpha,\lambda}\end{array}\right\} \quad \text{for } \lambda < 0. \tag{4.12}$$

Equivalently

$$\left(\frac{\gamma}{2}+it\right)\phi_{\alpha,\lambda}=\left\{\begin{array}{ll}(\partial_\lambda-\frac{1}{\lambda}\mathcal{D}^+)\,\phi_{\alpha,\lambda} & \text{for } \lambda>0\\ (\partial_\lambda-\frac{1}{\lambda}\mathcal{D}^-)\,\phi_{\alpha,\lambda} & \text{for } \lambda<0\end{array}\right\}\quad\text{and}\tag{4.13}$$

$$\left(\frac{\gamma}{2}-it\right)\phi_{\alpha,\lambda}=\left\{\begin{array}{ll}-(\partial_\lambda-\frac{1}{\lambda}\mathcal{D}^-)\,\phi_{\alpha,\lambda} & \text{for } \lambda>0\\ -(\partial_\lambda-\frac{1}{\lambda}\mathcal{D}^+)\,\phi_{\alpha,\lambda} & \text{for } \lambda<0\end{array}\right\}\quad.\tag{4.14}$$

5. K-Invariant Schwartz functions on V

We let $\mathcal{S}(V)$ denote the space of Schwartz functions on V and $\mathcal{S}(H_n)$ the Schwartz functions on $H_n = V\times\mathbb{R}$. As usual, $\mathcal{S}_K(V)$ and $\mathcal{S}_K(H_n)$ will denote the K-invariant elements in $\mathcal{S}(V)$ and $\mathcal{S}(H_n)$ respectively. Note that the ϕ_α°'s belong to $\mathcal{S}_K(V)$. We would like to characterize the spaces $\mathcal{S}_K(V)$ and $\mathcal{S}_K(H_n)$ in terms of the spherical transform. The problem for $\mathcal{S}_K(V)$ is related to that for $\mathcal{S}_K(H_n)$ as follows. Since $U(n)$ acts trivially on the center of H_n, we have that $\mathcal{S}_K(H_n)\cong\mathcal{S}_K(V)\otimes\mathcal{S}(\mathbb{R})$. Moreover, for $g\in\mathcal{S}_K(V)$, $h\in\mathcal{S}(\mathbb{R})$ and $(\alpha,\lambda)\in\Lambda\times\mathbb{R}^\times$ one computes easily that

$$(g\otimes h)^\wedge(\alpha,\lambda)=\left(\frac{1}{|\lambda|}\right)^n\left(g\circ\delta_{\frac{1}{\sqrt{|\lambda|}}}\right)^\wedge(\alpha)\mathcal{F}(h)(\lambda)\tag{5.1}$$

where $\mathcal{F}(h)(\lambda)=\int_{-\infty}^{\infty}h(t)e^{-i\lambda t}dt$ is the one dimensional Fourier transform in the central direction. Thus the problem of characterizing $\mathcal{S}_K(H_n)$ via the spherical transform leads us to seek conditions on maps $\widehat{f}:\Lambda\to\mathbb{C}$ that are necessary and sufficient to ensure that $f\in L^1_K(V)$ is in fact a Schwartz function. This problem, which is of interest in its own right, is answered cleanly by Theorem 5.1 below. In Section 6 we will return to the problem of characterizing $\mathcal{S}_K(H_n)$ via the spherical transform.

Definition 5.1. We say that a function $F:\Lambda\to\mathbb{C}$ is *rapidly decreasing* if for every sequence $(\alpha_m)_{m=1}^\infty$ in Λ and every $N\in\mathbb{Z}^+$ one has

$$\lim_{m\to\infty}F(\alpha_m)|\alpha_m|^N=0.$$

Equivalently, for each $N\in\mathbb{Z}^+$, there is a constant C_N for which

$$|F(\alpha)|\leq\frac{C_N}{(2|\alpha|+n)^N}.$$

Theorem 5.1. *If $f\in\mathcal{S}_K(V)$ then $\widehat{f}$ is rapidly decreasing on Λ. Conversely, if F is rapidly decreasing on Λ then $F=\widehat{f}$ for some $f\in\mathcal{S}_K(V)$. Moreover, the map*

$$^\wedge:\mathcal{S}_K(V)\to\{F\mid F\ \textit{is rapidly decreasing on}\ \Lambda\}$$

is a bijection.

Lemma 5.2. *If F is rapidly decreasing on Λ then so are $\mathcal{D}^+F$ and $\mathcal{D}^-F$.*

Proof. Let $c_m := \max_{|\alpha|=m} |F(\alpha)|$. The function $m \mapsto c_m$ is rapidly decreasing on $\mathbb{Z}^+$. Using Equation 4.8 and the fact that the generalized binomial coefficients are non-negative (proved in [BR]) we obtain

$$\begin{aligned} |\mathcal{D}^+ F(\alpha)| &\leq \sum_{|\beta|=|\alpha|+1} \frac{d_\beta}{d_\alpha} \left[\begin{array}{c} \beta \\ \alpha \end{array} \right] |F(\beta)| + (|\alpha| + n)|F(\alpha)| \\ &\leq c_{|\alpha|+1} \sum_{|\beta|=|\alpha|+1} \frac{d_\beta}{d_\alpha} \left[\begin{array}{c} \beta \\ \alpha \end{array} \right] + c_{|\alpha|}(|\alpha| + n) \\ &= (|\alpha| + n) \left(c_{|\alpha|} + c_{|\alpha|+1} \right), \end{aligned}$$

which shows that $\mathcal{D}^+ F$ is rapidly decreasing. Using Equation 4.7 in a similar fashion yields

$$|\mathcal{D}^- F(\alpha)| \leq |\alpha| \left(c_{|\alpha|-1} + c_{|\alpha|} \right),$$

showing that $\mathcal{D}^- F$ is rapidly decreasing. □

Lemma 5.3. *Let F be a rapidly decreasing function on Λ and G be a bounded function on Λ. Then*

$$\sum_{\alpha\in\Lambda} d_\alpha F(\alpha) \mathcal{D}^+ G(\alpha) = -\sum_{\alpha\in\Lambda} d_\alpha \left(\mathcal{D}^- + n \right) F(\alpha) G(\alpha),$$
$$\sum_{\alpha\in\Lambda} d_\alpha F(\alpha) \mathcal{D}^- G(\alpha) = -\sum_{\alpha\in\Lambda} d_\alpha \left(\mathcal{D}^+ + n \right) F(\alpha) G(\alpha).$$

Here all four series converge absolutely.

Proof. We compute formally that

$$\begin{aligned} \sum_{\alpha\in\Lambda} d_\alpha F(\alpha) \mathcal{D}^+ G(\alpha) &= \sum_{\alpha\in\Lambda} d_\alpha F(\alpha) \left(\sum_{|\beta|=|\alpha|+1} \frac{d_\beta}{d_\alpha} \left[\begin{array}{c} \beta \\ \alpha \end{array} \right] G(\beta) - (|\alpha| + n) G(\alpha) \right) \\ &= \sum_{\alpha\in\Lambda} \sum_{|\beta|=|\alpha|+1} d_\beta \left[\begin{array}{c} \beta \\ \alpha \end{array} \right] F(\alpha) G(\beta) - \sum_{\alpha\in\Lambda} d_\alpha (|\alpha| + n) F(\alpha) G(\alpha) \\ &= \sum_{\beta\in\Lambda} d_\beta \left(\sum_{|\alpha|=|\beta|-1} \left[\begin{array}{c} \beta \\ \alpha \end{array} \right] F(\alpha) \right) G(\beta) - \sum_{\alpha\in\Lambda} d_\alpha (|\alpha| + n) F(\alpha) G(\alpha) \\ &= \sum_{\alpha\in\Lambda} d_\alpha \left(-\mathcal{D}^- - n \right) F(\alpha) G(\alpha). \end{aligned}$$

The hypotheses on F and G ensure that the terms in each series above are products of rapidly decreasing functions with functions of polynomial growth in $|\alpha|$. Thus all of these series converge absolutely and the above rearrangements are justified. Indeed, suppose that $|G(\alpha)| \leq M$ for all $\alpha \in \Lambda$. Equations 4.7 and 4.8 show that

$$\sum_{|\beta|=|\alpha|+1} \frac{d_\beta}{d_\alpha} \left[\begin{array}{c} \beta \\ \alpha \end{array} \right] G(\beta) - (|\alpha|+n)G(\alpha) \leq 2M(|\alpha|+n).$$

Moreover, since $P_\alpha \subset \mathcal{P}_{|\alpha|}(V)$, we have that

$$d_\alpha \leq \dim\left(\mathcal{P}_{|\alpha|}(V)\right) = \binom{|\alpha|+n-1}{|\alpha|} \leq \frac{(|\alpha|+n-1)^{n-1}}{(n-1)!}, \tag{5.2}$$

a polynomial bound on d_α. Hence $\alpha \mapsto d_\alpha(|\alpha|+n)G(\alpha)$ is bounded by $M(|\alpha|+n)^n$ and $\beta \mapsto d_\beta G(\beta)$ by $M(|\beta|+n-1)^{n-1}$. Finally note that $\beta \mapsto \sum_{|\alpha|=|\beta|-1} \left[\begin{array}{c} \beta \\ \alpha \end{array} \right] F(\alpha)$ is bounded by

$$\dim\left(\mathcal{P}_{|\beta|-1}(V)\right)|\beta|H(\beta) \leq (|\beta|+n-2)^{n-1}|\beta|H(\beta)$$

where $H(\beta) = \max\{|F(\alpha)| \ : \ |\alpha| = |\beta|-1\}$ is rapidly decreasing in β.

The second identity follows from the first by interchanging the roles of F and G. $\square$

Proof of Theorem 5.1. First suppose that $f \in \mathcal{S}_K(V)$ and let $\mathcal{L}$ denote the Heisenberg sub-Laplacian given by Equation 2.6. The associated differential operator $\mathcal{L}'$ on V, given in Equation 3.1, is self-adjoint and preserves $\mathcal{S}_K(V)$. In view of Equation 3.2, we have that for $N \in \mathbb{Z}^+$,

$$\begin{aligned} (2|\alpha|+n)^N \left|\widehat{f}(\alpha)\right| &= \left|(-(2|\alpha|+n))^N \langle f, \phi_\alpha^\circ \rangle_2\right| \\ &= \left|\langle f, (\mathcal{L}')^N \phi_\alpha^\circ \rangle_2\right| = \left|\langle (\mathcal{L}')^N f, \phi_\alpha^\circ \rangle_2\right| \\ &\leq ||(\mathcal{L}')^N f||_1 \end{aligned}$$

since $|\phi_\alpha^\circ(z)| \leq 1$ for all $z \in V$. Hence we obtain a bound of the form

$$|\widehat{f}(\alpha)| \leq \frac{C_N}{(2|\alpha|+n)^N}$$

for each $N \in \mathbb{Z}^+$ where $C_N = ||(\mathcal{L}')^N f||_1$. It follows immediately that $\widehat{f}$ is a rapidly decreasing function on Λ.

Conversely, suppose that F is a rapidly decreasing function on Λ. The estimate given above in Equation 5.2 shows that d_α is bounded by a polynomial function of $|\alpha|$. It follows that the series $\sum_{\alpha \in \Lambda} d_\alpha (F(\alpha))^p$ converges absolutely for all $p \in \mathbb{Z}^+$. As $|\phi_\alpha^\circ(z)| \leq 1$, we conclude that the series $\sum_{\alpha \in \Lambda} d_\alpha F(\alpha) \phi_\alpha^\circ(z)$ converges absolutely and uniformly in z. Define a function f on V via

$$f(z) = \left(\frac{1}{2\pi}\right)^n \sum_{\alpha \in \Lambda} d_\alpha F(\alpha) \phi_\alpha^\circ(z). \tag{5.3}$$

Since $\sum_{\alpha\in\Lambda} d_\alpha |F(\alpha)|^2$ converges, Corollary 3.2 shows that $f \in L^2_K(V)$ with $\widehat{f} = F$. Moreover, as

- $\sum_{\alpha\in\Lambda} d_\alpha |F(\alpha)|$ converges, and
- each ϕ°_α is a Schwartz function, and
- the ϕ°_α's are uniformly bounded by 1,

it follows easily that $\lim_{|z|\to\infty} f(z) = 0$. To prove that f is a Schwartz function, it suffices to show that f is smooth and that $\lim_{|z|\to\infty} \gamma(z)^a(\Delta^b f)(z) = 0$ for all non-negative integers a, b. In view of the preceding analysis, this follows by induction from Lemma 5.4 below.

To complete the proof, note that if $f \in \mathcal{S}_K(V)$ and $F(\alpha) = \widehat{f}(\alpha) = 0$ for all $\alpha \in \Lambda$ then Equation 5.3 implies that $f = 0$ in $L^2_K(V)$. As f is continuous, it follows that $f(z) = 0$ for all $z \in V$. This shows that $f \mapsto \widehat{f}$ is injective on $\mathcal{S}_K(V)$. $\square$

Lemma 5.4. *Let $f = (\frac{1}{2\pi})^n \sum_{\alpha\in\Lambda} d_\alpha F(\alpha)\phi^\circ_\alpha$ where F is rapidly decreasing. Then*

1. *γf can be written in the form $(\frac{1}{2\pi})^n \sum_{\alpha\in\Lambda} d_\alpha G(\alpha)\phi^\circ_\alpha$ where G is rapidly decreasing, and*
2. *f is twice differentiable and Δf can be written in the form $(\frac{1}{2\pi})^n \sum_{\alpha\in\Lambda} d_\alpha H(\alpha)\phi^\circ_\alpha$ where H is rapidly decreasing.*

Proof. We use Equation 4.10 together with Lemma 5.3 to write

$$\begin{aligned}
\gamma(z)f(z) &= \left(\frac{1}{2\pi}\right)^n \sum_{\alpha\in\Lambda} d_\alpha F(\alpha)\gamma(z)\phi^\circ_\alpha(z) \\
&= -\left(\frac{1}{2\pi}\right)^n \sum_{\alpha\in\Lambda} d_\alpha F(\alpha)\left(\mathcal{D}^+ - \mathcal{D}^-\right)\phi^\circ_\alpha(z) \\
&= \left(\frac{1}{2\pi}\right)^n \sum_{\alpha\in\Lambda} d_\alpha \left(\mathcal{D}^- - \mathcal{D}^+\right) F(\alpha)\phi^\circ_\alpha(z).
\end{aligned}$$

In view of Lemma 5.2, $G = \left(\mathcal{D}^- - \mathcal{D}^+\right) F$ is rapidly decreasing. This establishes (1) in the statement of Lemma 5.4.

Equation 3.2 shows that $(4\Delta - \gamma/2)\phi^\circ_\alpha = -(2|\alpha| + n)\phi^\circ_\alpha$ and hence

$$\Delta\phi^\circ_\alpha = \frac{\gamma}{8}\phi^\circ_\alpha - \frac{2|\alpha| + n}{4}\phi^\circ_\alpha.$$

Thus formal application of Δ term-wise to the series $(\frac{1}{2\pi})^n \sum_{\alpha\in\Lambda} d_\alpha F(\alpha)\phi^\circ_\alpha$ for f yields the series $(\frac{1}{2\pi})^n \sum_{\alpha\in\Lambda} d_\alpha H(\alpha)\phi^\circ_\alpha$ where

$$H(\alpha) = \frac{G(\alpha)}{8} - \frac{2|\alpha| + n}{4}F(\alpha)$$

and $\gamma f = (\frac{1}{2\pi})^n \sum_{\alpha\in\Lambda} d_\alpha G(\alpha)\phi^\circ_\alpha$ as above. As both F and G are rapidly decreasing, so is H. As both $\sum_{\alpha\in\Lambda} d_\alpha F(\alpha)\phi^\circ_\alpha(z)$ and $\sum_{\alpha\in\Lambda} d_\alpha H(\alpha)\phi^\circ_\alpha(z)$ converge uniformly in

z, we conclude that f is twice differentiable with $\Delta f = (\frac{1}{2\pi})^n \sum_{\alpha\in\Lambda} d_\alpha H(\alpha)\phi_\alpha^\circ$. This establishes (2) in the statement of Lemma 5.4. $\square$

6. K-Invariant Schwartz functions on H_n

In this section we return to the problem of characterizing the space $\mathcal{S}_K(H_n)$ via the spherical transform. Theorem 6.1 solves this problem and is our main result. Several definitions are required in the formulation of this theorem.

Definition 6.1. Let F be a function on $\Delta(K, H_n)$. We say that F is *rapidly decreasing* on $\Delta(K, H_n)$ if

- F is continuous on $\Delta(K, H_n)$,
- the function $F_\circ$ on V defined by $F_\circ(w) = F(\eta_w)$ belongs to $\mathcal{S}_K(V)$,
- the map $\lambda \mapsto F(\alpha, \lambda)$ is smooth on $\mathbb{R}^\times = (-\infty, 0) \cup (0, \infty)$ for each fixed $\alpha \in \Lambda$,
- for each $m, N \geq 0$ there exists a constant $C_{m,N}$ for which
$$|\partial_\lambda^m F(\alpha, \lambda)| \leq \frac{C_{m,N}}{|\lambda|^{m+N}(2|\alpha| + n)^N}$$
for all $(\alpha, \lambda) \in \Lambda \times \mathbb{R}^\times$.

We say that a continuous function on $\Delta_1(K, H_n)$ is rapidly decreasing if it extends to a rapidly decreasing function on $\Delta(K, H_n) = \Delta_1(K, H_n) \cup \Delta_2(K, H_n)$. Since $\Delta_1(K, H_n)$ is dense in $\Delta(K, H_n)$, such an extension is necessarily unique.

Note that if F is rapidly decreasing on $\Delta(K, H_n)$ then $\alpha \mapsto F(\alpha, \lambda)$ is rapidly decreasing on Λ, in the sense of Definition 5.1, for each $\lambda \neq 0$. We see that F is bounded by letting $m = N = 0$ and one can show, moreover, that F vanishes at infinity by letting $m = 0$ and $N = 1$. We remark that the functions $\partial_\lambda^m F(\alpha, \lambda)$ defined on $\Delta_1(K, H_n)$ need not extend continuously across $\Delta_2(K, H_n)$. Example 6.1 below illustrates this behavior.

Definition 6.2. Let F be a function on $\Delta_1(K, H_n)$ which is smooth in λ. M^+F and M^-F are the functions on $\Delta_1(K, H_n)$ defined by
$$M^+F(\alpha,\lambda) = \left\{ \begin{array}{ll} \left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^+\right) F(\alpha,\lambda) & \text{for } \lambda > 0 \\ \left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^-\right) F(\alpha,\lambda) & \text{for } \lambda < 0 \end{array} \right\} \quad \text{and}$$
$$M^-F(\alpha,\lambda) = \left\{ \begin{array}{ll} \left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^-\right) F(\alpha,\lambda) & \text{for } \lambda > 0 \\ \left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^+\right) F(\alpha,\lambda) & \text{for } \lambda < 0 \end{array} \right\}.$$

We remind the reader that the difference operators $\mathcal{D}^\pm$ are defined by
$$\mathcal{D}^+F(\alpha,\lambda) = \sum_{|\beta|=|\alpha|+1} \frac{d_\beta}{d_\alpha}\left[\begin{array}{c}\beta\\ \alpha\end{array}\right] F(\beta,\lambda) - (|\alpha|+n)F(\alpha,\lambda),$$
$$\mathcal{D}^-F(\alpha,\lambda) = |\alpha|F(\alpha,\lambda) - \sum_{|\beta|=|\alpha|-1}\left[\begin{array}{c}\alpha\\ \beta\end{array}\right] F(\beta,\lambda).$$

Definition 6.3. $\widehat{\mathcal{S}}(K,H_n)$ is the set of all functions $F:\Delta(K,H_n)\to\mathbb{C}$ for which $(M^+)^\ell(M^-)^m F$ is rapidly decreasing for all $\ell,m\geq 0$.

If F is rapidly decreasing on $\Delta(K,H_n)$ then $\lambda\mapsto F(\alpha,\lambda)$ is smooth on $\mathbb{R}^\times$ and we have well defined functions $(M^+)^\ell(M^-)^m F$ on $\Delta_1(K,H_n)$. F belongs to $\widehat{\mathcal{S}}(K,H_n)$ if and only if these functions extend continuously to rapidly decreasing functions on $\Delta(K,H_n)$.

Theorem 6.1. *If $f\in\mathcal{S}_K(H_n)$ then $\widehat{f}\in\widehat{\mathcal{S}}(K,H_n)$. Conversely, if $F\in\widehat{\mathcal{S}}(K,H_n)$ then $F=\widehat{f}$ for some $f\in\mathcal{S}_K(H_n)$. Moreover, the map $^\wedge:\mathcal{S}_K(H_n)\to\widehat{\mathcal{S}}(K,H_n)$ is a bijection.*

If $f\in\mathcal{S}_K(H_n)$ and $\widehat{f}=0$ then the inversion formula for the spherical transform (Equation 2.13) yields that $f=0$. Thus the spherical transform is injective on $\mathcal{S}_K(H_n)$. To prove Theorem 6.1, it remains to show that $\mathcal{S}_K(H_n)^\wedge\subset\widehat{\mathcal{S}}(K,H_n)$ and that $\widehat{\mathcal{S}}(K,H_n)\subset\mathcal{S}_K(H_n)^\wedge$. This will require most of the remainder of this section. First, however, we will present an example.

Example 6.1. Consider the case where $K=U(n)$. Decomposition 2.1 reads $\mathbb{C}[V]=\sum_{m=0}^\infty\mathcal{P}_m(V)$ and one has $\Delta(U(n),H_n)\cong\left(\mathbb{R}^\times\times\mathbb{Z}^+\right)\cup\mathbb{R}^+$. Using Equation 4.3, one computes that the difference operators $\mathcal{D}^\pm$ appearing in the definition of the set $\widehat{\mathcal{S}}(U(n),H_n)$ are given by

$$\mathcal{D}^+g(m)=(m+n)(g(m+1)-g(m)),\qquad \mathcal{D}^-g(m)=m(g(m)-g(m-1))$$

for functions $g:\mathbb{Z}^+\to\mathbb{C}$.

Let $f\in\mathcal{S}_K(H_n)$ be defined by $f(z,t)=g(z)h(t)$ where $g(z)=e^{-|z|^2}$ and $h\in\mathcal{S}(\mathbb{R})$. We will compute the spherical transform $\widehat{f}:\Delta(U(n),H_n)\to\mathbb{C}$.

The polynomial q_m associated with $\mathcal{P}_m(V)$ is a suitably normalized generalized Laguerre polynomial which can be written explicitly as (see eg. [BJR92])

$$q_m(z)=(n-1)!\sum_{j=0}^m\binom{m}{j}\frac{1}{(n+j-1)!}\left(-\frac{|z|^2}{2}\right)^j.$$

Thus, for $\lambda\neq 0$ we have

$$\begin{aligned}\left(g\circ\delta_{1/\sqrt{|\lambda|}}\right)^\wedge(m)&=\int_V e^{-|z|^2/|\lambda|}q_m(z)e^{-|z|^2/4}dz\\&=(n-1)!\sum_{j=0}^m\binom{m}{j}\frac{(-1)^j}{2^j(n+j-1)!}\int_V|z|^{2j}e^{-K|z|^2}dz\end{aligned}$$

where $K=\frac{1}{|\lambda|}+\frac{1}{4}=\frac{4+|\lambda|}{4|\lambda|}$. One has

$$\int_V|z|^{2j}e^{-K|z|^2}dz=\frac{2\pi^n}{(n-1)!}\int_0^\infty e^{-Kr^2}r^{2(n+j)-1}dr=\frac{\pi^n(n+j-1)!}{(n-1)!K^{n+j}}$$

and hence

$$\begin{aligned}\left(g \circ \delta_{1/\sqrt{|\lambda|}}\right)^{\wedge}(m) &= \left(\frac{\pi}{K}\right)^n \sum_{j=0}^{m} \binom{m}{j} \left(-\frac{1}{2K}\right)^j \\ &= \left(\frac{\pi}{K}\right)^n \left(1-\frac{1}{2K}\right)^m = \left(\frac{4\pi|\lambda|}{4+|\lambda|}\right)^n \left(\frac{4-|\lambda|}{4+|\lambda|}\right)^m.\end{aligned}$$

Thus we have (see Equation 5.1)

$$\begin{aligned}\widehat{f}(m,\lambda) &= \left(\frac{1}{|\lambda|}\right)^n \left(g \circ \delta_{\frac{1}{\sqrt{|\lambda|}}}\right)^{\wedge}(m)\mathcal{F}(h)(\lambda) \\ &= \left(\frac{4\pi}{4+|\lambda|}\right)^n \left(\frac{4-|\lambda|}{4+|\lambda|}\right)^m \mathcal{F}(h)(\lambda).\end{aligned}$$

For $w \in V$ one has (see Equation 2.15)

$$\begin{aligned}\widehat{f}(w) = \mathcal{F}_H(f)(w,0) &= \mathcal{F}_V(g)(w)\mathcal{F}(h)(0) \\ &= \left(\pi^n \int_{-\infty}^{\infty} h(t)dt\right) e^{-|w|^2/4}.\end{aligned}$$

One can verify directly that $\widehat{f} \in \widehat{\mathcal{S}}(U(n), H_n)$, but we will not do this here. In particular, the functions $\lambda \mapsto \widehat{f}(\lambda, m)$ are differentiable on $\mathbb{R}^{\times}$ for each $m \in \mathbb{Z}^+$, and the derivatives $\partial_\lambda \widehat{f}(\lambda, m)$ satisfy estimates as in Definition 6.1. Note, however, that these derivatives do not agree as $\lambda \to 0$. Specifically, $\lim_{\lambda\to 0} \partial_\lambda \widehat{f}(\lambda, m)$ depends on m. Hence there is no function $g \in L^1_{U(n)}(H_n)$ whose $U(n)$-spherical transform satisfies $\widehat{g}(\lambda, m) = \partial_\lambda \widehat{f}(\lambda, m)$ on $\mathbb{R}^{\times} \times \mathbb{Z}^+$.

This example illustrates why the characterization of $\mathcal{S}_K(H_n)^{\wedge}$ is somewhat complicated. Although the behavior of the derivatives of the functions $\lambda \mapsto \widehat{f}(\lambda, m)$ does come into play, as expected, the space $\mathcal{S}_K(H_n)^{\wedge}$ is not "closed under λ-derivatives". Indeed, one must "replace" ∂_λ by the operators $M^{\pm}$ which involve both ∂_λ and the difference operators $\mathcal{D}^{\pm}$.

Proof that $\mathcal{S}_K(H_n)^{\wedge} \subset \widehat{\mathcal{S}}(K, H_n)$. Suppose that $f \in \mathcal{S}_K(H_n)$ and let $F := \widehat{f}$. We begin by showing that F is rapidly decreasing. F is continuous on $\Delta(K, H_n)$, as is the spherical transform of any integrable K-invariant function. Moreover, Equation 2.15 shows that $F_\circ(w) = \mathcal{F}_H(f)(w, 0)$. Since f is a Schwartz function, so is $F_\circ(w)$. Thus F satisfies the first two conditions in Definition 6.1.

Next we will show that F satisfies the estimates in Definition 6.1 for $m = 0$. The argument is similar to that in the first part of the proof for Theorem 5.1. Recall that the Heisenberg sub-Laplacian $\mathcal{L}$ is a self-adjoint operator on $L^2(H_n)$ with $\mathcal{L}(\phi_{\alpha,\lambda}) =$

$-|\lambda|(2|\alpha|+n)\phi_{\alpha,\lambda}$. Thus we have

$$\begin{aligned}|\lambda|^N(2|\alpha|+n)^N\,|F(\alpha,\lambda)| &= \left|(-|\lambda|(2|\alpha|+n))^N\langle f,\phi_{\alpha,\lambda}\rangle_2\right| \\ &= \left|\langle f,\mathcal{L}^N\phi_{\alpha,\lambda}\rangle_2\right| = \left|\langle \mathcal{L}^N f,\phi_{\alpha,\lambda}\rangle_2\right| \\ &\leq ||\mathcal{L}^N f||_1\end{aligned}$$

since $|\phi_{\alpha,\lambda}(z,t)|\leq 1$ for all $(z,t)\in H_n$. Letting $C_{0,N}:=||\mathcal{L}^N f||_1$, we see that the inequalities in Definition 6.1 hold for $m=0$.

Since $\phi_{\alpha,\lambda}(z,t)$ is smooth in $\lambda\in\mathbb{R}^\times$ for fixed (z,t) and $f\overline{\phi_{\alpha,\lambda}}$ is a Schwartz function, $F(\alpha,\lambda)=\widehat{f}(\alpha,\lambda)$ is smooth in $\lambda\in\mathbb{R}^\times$ with

$$\partial_\lambda F(\alpha,\lambda)=\int f(z,t)\partial_\lambda\overline{\phi_{\alpha,\lambda}(z,t)}dzdt. \tag{6.1}$$

Equations 4.11 and 4.12 provide formulae for $\partial_\lambda\overline{\phi_{\alpha,\lambda}}=\partial_\lambda\phi_{\alpha,-\lambda}$ but we require a different approach here. We write $\phi_\alpha(z,t)=\phi_\alpha^\circ(z)e^{it}$ as $\phi_\alpha^\circ(z,\overline{z})e^{it}$ so that

$$\overline{\phi_{\alpha,\lambda}(z,t)}=\phi_\alpha^\circ(|\lambda|z,\overline{z})e^{-i\lambda t}.$$

We see that

$$\partial_\lambda\overline{\phi_{\alpha,\lambda}(z,t)}=\frac{1}{\lambda}\left[\left(\sum_{j=1}^n z_j\frac{\partial}{\partial z_j}\right)\phi_\alpha^\circ\right](|\lambda|z,\overline{z})-it\overline{\phi_{\alpha,\lambda}(z,t)}.$$

Substituting this expression into Equation 6.1 and integrating by parts gives

$$\partial_\lambda F(\alpha,\lambda)=\frac{1}{\lambda}(Df)^\wedge(\alpha,\lambda)-i(tf)^\wedge(\alpha,\lambda)$$

where $Df:=-\sum_{j=1}^n\frac{\partial}{\partial z_j}(z_jf)$. Since tf and Df are both Schwartz functions on H_n, they satisfy estimates as above. Thus, given $N\geq 0$, one can find constants A and B with

$$|\partial_\lambda F(\alpha,\lambda)|\leq\frac{A}{|\lambda|^{N+1}(2|\alpha|+n)^N}+\frac{B}{|\lambda|^{N+1}(2|\alpha|+n)^{N+1}}\leq\frac{C_{1,N}}{|\lambda|^{N+1}(2|\alpha|+n)^N}$$

where $C_{1,N}=A+B$. By induction on m we see that $|\partial_\lambda^m F(\alpha,\lambda)|$ satisfies an estimate as in Definition 6.1. This completes the proof that F is rapidly decreasing.

Equations 4.13 and 4.13 show that

$$M^+F=\left(\left(\frac{\gamma}{2}+it\right)f\right)^\wedge\Big|_{\Delta_1(K,H_n)}\quad\text{and}\quad M^-F=-\left(\left(\frac{\gamma}{2}-it\right)f\right)^\wedge\Big|_{\Delta_1(K,H_n)}$$

Thus $(M^+)^\ell(M^-)^mF$ is the restriction of $\widehat{g}$ to $\Delta_1(K,H_n)$ where

$$g=(-1)^m(\gamma/2+it)^\ell(\gamma/2-it)^mf.$$

Since $g\in\mathcal{S}_K(H_n)$, it now follows that $(M^+)^\ell(M^-)^mF$ is rapidly decreasing. Thus $F\in\widehat{\mathcal{S}}(K,H_n)$ as desired. □

The following proposition is required to complete the proof of Theorem 6.1.

Proposition 6.2. *Let F be a bounded measurable function on $\Delta(K, H_n)$ with*

$$|F(\alpha,\lambda)| \leq \frac{C}{|\lambda|^N(2|\alpha|+n)^N}$$

for some $N \geq n+2$ and some constant C. Then

1. *$F \in L^p(\Delta(K, H_n))$ for all $p \geq 1$, and*
2. *$F = \widehat{f}$ for some bounded continuous function $f \in L^2_K(H_n)$.*

Suppose, for example, that f is a continuous function in $L^1_K(H_n)$ with $F = \widehat{f}$ rapidly decreasing. Proposition 6.2 shows that f is square integrable and that F is integrable. Thus, the inversion formula 2.13 applies and we can recover f from F via

$$f(z,t) = \left(\frac{1}{2\pi}\right)^{n+1} \int_{\mathbb{R}^\times} \sum_{\alpha\in\Lambda} d_\alpha F(\alpha,\lambda)\phi_{\alpha,\lambda}(z,t)|\lambda|^n d\lambda. \tag{6.2}$$

In particular, we see that this formula certainly holds for any function $f \in \mathcal{S}_K(H_n)$.

Proof of Proposition 6.2. To establish the first assertion, it suffices to prove that $F \in L^1(\Delta(K, H_n))$. Indeed, $|F(\alpha, \lambda)|^p$ satisfies an inequality as in the statement of the proposition with N replaced by pN.

Let $A_1 := \{\phi_{\alpha,\lambda} \mid |\lambda|(2|\alpha| + n) \leq 1\}$ and $A_2 := \{\phi_{\alpha,\lambda} \mid |\lambda|(2|\alpha| + n) > 1\}$. Let M be a constant for which $|F(\psi)| \leq M$ for all $\psi \in \Delta(K, H_n)$ and let

$$d_m := \dim\left(\mathcal{P}_m(V)\right) = \binom{m+n-1}{m}. \tag{6.3}$$

Note that

$$\sum_{|\alpha|=m} d_\alpha = d_m \leq (m+n-1)^{n-1}.$$

We compute

$$\begin{aligned}
\int_{A_1} |F(\psi)|d\mu(\psi) &= \left(\frac{1}{2\pi}\right) \sum_{\alpha\in\Lambda} d_\alpha \int_{0<|\lambda|<\frac{1}{2|\alpha|+n}} |F(\alpha,\lambda)||\lambda|^n d\lambda \\
&\leq \frac{2M}{(2\pi)^{n+1}} \sum_{m=0}^{\infty} d_m \int_0^{\frac{1}{2m+n}} \lambda^n d\lambda \\
&= \frac{2M}{(n+1)(2\pi)^{n+1}} \sum_{m=0}^{\infty} d_m \left(\frac{1}{2m+n}\right)^{n+1}.
\end{aligned}$$

Since $d_m = O(m^{n-1})$, we see that the last series converges. Hence $|F|$ is integrable over A_1. Next we compute

$$\begin{aligned}\int_{A_2} |F(\psi)| d\mu(\psi) &= \left(\frac{1}{2\pi}\right) \sum_{\alpha\in\Lambda} d_\alpha \int_{|\lambda| > \frac{1}{2|\alpha|+n}} |F(\alpha,\lambda)||\lambda|^n d\lambda \\ &\leq \frac{2C}{(2\pi)^{n+1}} \sum_{m=0}^{\infty} d_m \int_{\frac{1}{2m+n}}^{\infty} \frac{\lambda^n}{\lambda^N (2m+n)^N} d\lambda \\ &\leq \frac{2C}{(N-n-1)(2\pi)^{n+1}} \sum_{m=0}^{\infty} \frac{d_m}{(2m+n)^{n+1}}.\end{aligned}$$

The hypothesis that $N-n \geq 2$ was used above to evaluate the integral of $1/\lambda^{N-n}$ over $\frac{1}{2m+n} < \lambda < \infty$. Since $d_m = O(m^{n-1})$, we see that the series in the last expression converges. Hence $|F|$ is integrable over A_1. As $A_1 \cup A_2 = \Delta_1(K, H_n)$ is a set of full measure in $\Delta(K, H_n)$, it follows that $F \in L^1(\Delta(K, H_n))$.

Next let f be the function on H_n defined by

$$f(z,t) = \int_{\Delta(K,H_n)} F(\psi)\psi(z,t) d\mu(\psi).$$

Since $F \in L^1(\Delta(K, H_n))$ and the spherical functions are continuous and bounded by 1, we see that f is well defined, continuous and bounded by $||F||_{L^1(\Delta(K,H_n))}$. Moreover, since $F \in L^2(\Delta(K, H_n))$ and $\wedge : L^2(\Delta(K, H_n)) \to L^2_K(H_n)$ is an isometry, we have that $f \in L^2_K(H_n)$ with $||f||_2^2 = \int_{\Delta(K,H_n)} |F(\psi)|^2 d\mu(\psi)$ and $\widehat{f} = F$. This establishes the second assertion in Proposition 6.2. $\square$

Remark 6.1. One can show that the set A_1 used in the proof of Proposition 6.2 is compact in $\Delta(K, H_n)$. Thus any bounded measurable function is integrable over A_1. This observation motivates the decomposition used in the proof.

Proof that $\widehat{\mathcal{S}}(K, H_n) \subset \mathcal{S}_K(H_n)^\wedge$. Suppose now that $F \in \widehat{\mathcal{S}}(K, H_n)$. Proposition 6.2 shows that $F = \widehat{f}$ where

$$f(z,t) = \left(\frac{1}{2\pi}\right)^{n+1} \int_{\mathbb{R}^\times} \sum_{\alpha\in\Lambda} d_\alpha F(\alpha,\lambda)\phi_{\alpha,\lambda}(z,t)|\lambda|^n d\lambda$$

is K-invariant, continuous, bounded and square integrable. To show that $f \in \mathcal{S}_K(H_n)$, we will show that f is smooth and that

$$\left(\frac{\gamma^2}{4} + t^2\right)^a \left(\frac{\partial}{\partial t}\right)^b \Delta^c f \in L^2_K(H_n)$$

for all $a, b, c \geq 0$. This will follow from the facts

1. $\Delta f \in L^2_K(H_n)$ with $(\Delta f)^\wedge \in \widehat{\mathcal{S}}(K, H_n)$,
2. $\frac{\partial f}{\partial t} \in L^2_K(H_n)$ with $\left(\frac{\partial f}{\partial t}\right)^\wedge \in \widehat{\mathcal{S}}(K, H_n)$, and

3. $(\frac{\gamma}{2} \pm it)\, f \in L^2_K(H_n)$ with $((\frac{\gamma}{2} \pm it)\, f)^\wedge \in \widehat{\mathcal{S}}(K, H_n)$,

which we prove below. Here $\Delta = \sum_{j=1}^n \frac{\partial}{\partial z_j}\frac{\partial}{\partial \overline{z_j}}$ as before.

Using Equation 2.9 for the eigenvalues of the Heisenberg sub-Laplacian, one obtains

$$\begin{aligned} -|\lambda|(2|\alpha|+n)\phi_{\alpha,\lambda} = \mathcal{L}\phi_{\alpha,\lambda} &= \left[4\Delta - \frac{\gamma}{2}\left(\frac{\partial}{\partial t}\right)^2\right]\phi_{\alpha,\lambda} \\ &= 4\Delta\phi_{\alpha,\lambda} - \frac{\lambda^2}{2}\gamma\phi_{\alpha,\lambda}. \end{aligned}$$

Using Equation 4.10 for $\gamma\phi_{\alpha,\lambda}$ gives

$$\begin{aligned} 4\Delta\phi_{\alpha,\lambda} &= -\frac{|\lambda|}{2}(\mathcal{D}^+ - \mathcal{D}^-)\phi_{\alpha,\lambda} - |\lambda|(2|\alpha|+n)\phi_{\alpha,\lambda} \\ &= -\frac{|\lambda|}{2}\left[\sum_{|\beta|=|\alpha|+1}\frac{d_\beta}{d_\alpha}\left[\begin{array}{c}\beta\\ \alpha\end{array}\right]\phi_{\beta,\lambda} + (2|\alpha|+n)\phi_{\alpha,\lambda} + \sum_{|\beta|=|\alpha|-1}\left[\begin{array}{c}\alpha\\ \beta\end{array}\right]\phi_{\beta,\lambda}\right]. \end{aligned}$$

Define a function F_Δ on $\Delta_1(K, H_n)$ by

$$\begin{aligned} F_\Delta(\alpha,\lambda) &= -\frac{|\lambda|}{2}(\mathcal{D}^+ - \mathcal{D}^-)F(\alpha,\lambda) - |\lambda|(2|\alpha|+n)F(\alpha,\lambda) \\ &= -\frac{|\lambda|}{2}\left[\sum_{|\beta|=|\alpha|+1}\frac{d_\beta}{d_\alpha}\left[\begin{array}{c}\beta\\ \alpha\end{array}\right]F(\beta,\lambda) + (2|\alpha|+n)F(\alpha,\lambda) + \sum_{|\beta|=|\alpha|-1}\left[\begin{array}{c}\alpha\\ \beta\end{array}\right]F(\beta,\lambda)\right]. \end{aligned}$$

It is not hard to show that $F_\Delta \in \widehat{\mathcal{S}}(K, H_n)$. In particular, note that Equations 4.7 and 4.8 give

$$|F_\Delta(\alpha,\lambda)| \le \frac{|\lambda|}{2}\left[(|\alpha|+n)\sum_{|\beta|=|\alpha|+1}|F(\beta,\lambda)| + (2|\alpha|+n)|F(\alpha,\lambda)| + |\alpha|\sum_{|\beta|=|\alpha|-1}|F(\beta,\lambda)|\right].$$

One uses this to show that F_Δ satisfies estimates as in Definition 6.1. Moreover, Lemma 5.3 shows that for each $\lambda \ne 0$

$$\sum_{\alpha\in\Lambda} d_\alpha F(\alpha,\lambda)(\mathcal{D}^+ - \mathcal{D}^-)\phi_{\alpha,\lambda} = \sum_{\alpha\in\Lambda} d_\alpha(\mathcal{D}^+ - \mathcal{D}^-)F(\alpha,\lambda)\phi_{\alpha,\lambda}$$

and hence also

$$\int_{\mathbb{R}^\times}\sum_{\alpha\in\Lambda} d_\alpha F_\Delta(\alpha,\lambda)|\lambda|^n d\lambda = 4\int_{\mathbb{R}^\times}\sum_{\alpha\in\Lambda} d_\alpha F(\alpha,\lambda)\left(\Delta\phi_{\alpha,\lambda}\right)|\lambda|^n d\lambda.$$

We conclude that $\Delta f \in L^2_K(H_n)$ with $4(\Delta f)^\wedge = F_\Delta \in \widehat{\mathcal{S}}(K, H_n)$ This proves item (1) above.

Next note that the function defined on $\Delta_1(K,H_n)$ by $\lambda F(\alpha,\lambda)$ belongs to $\widehat{\mathcal{S}}(K,H_n)$. Since $\frac{\partial\phi_{\alpha,\lambda}}{\partial t} = i\lambda\phi_{\alpha,\lambda}$, we see that $\frac{\partial f}{\partial t} \in L^2_K(H_n)$ with $\left(\frac{\partial f}{\partial t}\right)^\wedge = i\lambda F \in \widehat{\mathcal{S}}(K,H_n)$. This establishes item (2) above.

We begin the proof of item (3) by setting

$$\widetilde{F}(z,\lambda) = \sum_{\alpha\in\Lambda} d_\alpha |\lambda|^n F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z) \tag{6.4}$$

for each $\lambda \neq 0$, so that

$$f(z,t) = \left(\frac{1}{2\pi}\right)^{n+1} \int_{\mathbb{R}^\times} \widetilde{F}(z,\lambda)e^{i\lambda t}d\lambda.$$

Note that we can compute $\partial_\lambda \widetilde{F}$ by taking derivatives term-wise in Equation 6.4. For $\lambda > 0$ we have

$$\begin{aligned}\partial_\lambda\widetilde{F}(z,\lambda) = \sum_{\alpha\in\Lambda} d_\alpha n\lambda^{n-1}F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z) + \sum_{\alpha\in\Lambda} d_\alpha\lambda^n\partial_\lambda F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z) \\ + \sum_{\alpha\in\Lambda} d_\alpha\lambda^n F(\alpha,\lambda)\partial_\lambda\phi^\circ_{\alpha,\lambda}(z).\end{aligned}$$

Since $F \in \widehat{\mathcal{S}}(K,H_n)$, the estimates in Definition 6.1 can be applied to show that the first two sums converge absolutely for each $\lambda > 0$. For the third sum, we use Equations 4.11 for $\partial_\lambda\phi^\circ_{\alpha,\lambda}(z) = \partial_\lambda\phi_{\alpha,\lambda}(z,0)$ together with Lemma 5.3 to derive two identities:

$$\begin{aligned}&\sum_{\alpha\in\Lambda} d_\alpha\lambda^n F(\alpha,\lambda)\partial_\lambda\phi^\circ_{\alpha,\lambda}(z)\\ &\quad= \left\{\begin{array}{l} -\frac{\gamma}{2}\sum_{\alpha\in\Lambda} d_\alpha\lambda^n F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z) + \sum_{\alpha\in\Lambda} d_\alpha\lambda^n F(\alpha,\lambda)\frac{1}{\lambda}\mathcal{D}^-\phi^\circ_{\alpha,\lambda}(z) \\ \frac{\gamma}{2}\sum_{\alpha\in\Lambda} d_\alpha\lambda^n F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z) + \sum_{\alpha\in\Lambda} d_\alpha\lambda^n F(\alpha,\lambda)\frac{1}{\lambda}\mathcal{D}^+\phi^\circ_{\alpha,\lambda}(z)\end{array}\right\}\\ &\quad= \left\{\begin{array}{l} -\frac{\gamma}{2}\widetilde{F}(z,\lambda) - \sum_{\alpha\in\Lambda} d_\alpha\lambda^{n-1}(\mathcal{D}^+ + n)F(\alpha,\lambda) \\ \frac{\gamma}{2}\widetilde{F}(z,\lambda) - \sum_{\alpha\in\Lambda} d_\alpha\lambda^{n-1}(\mathcal{D}^- + n)F(\alpha,\lambda)\end{array}\right\}\end{aligned}$$

Substituting these identities in the expression for $\partial_\lambda\widetilde{F}(z,\lambda)$ gives

$$\partial_\lambda\widetilde{F}(z,\lambda) = \left\{\begin{array}{l} -\frac{\gamma}{2}\widetilde{F}(z,\lambda) + \sum_{\alpha\in\Lambda} d_\alpha\lambda^n\left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^+\right)F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z) \\ \frac{\gamma}{2}\widetilde{F}(z,\lambda) + \sum_{\alpha\in\Lambda} d_\alpha\lambda^n\left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^-\right)F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z)\end{array}\right\} \tag{6.5}$$

both valid for $\lambda > 0$. We have similar identities for $\lambda < 0$:

$$\partial_\lambda\widetilde{F}(z,\lambda) = \left\{\begin{array}{l} -\frac{\gamma}{2}\widetilde{F}(z,\lambda) + \sum_{\alpha\in\Lambda} d_\alpha\lambda^n\left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^-\right)F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z) \\ \frac{\gamma}{2}\widetilde{F}(z,\lambda) + \sum_{\alpha\in\Lambda} d_\alpha\lambda^n\left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^+\right)F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z)\end{array}\right\}. \tag{6.6}$$

Note that $(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^\pm)\, F$ is the restriction of $M^\pm F$ to $\Delta_1^+(K, H_n) = \{\phi_{\alpha,\lambda} \mid \alpha \in \Lambda, \lambda > 0\}$ and also of $M^\mp F$ to $\Delta_1^-(K, H_n) = \{\phi_{\alpha,\lambda} \mid \alpha \in \Lambda, \lambda < 0\}$. Since $M^\pm F \in \widehat{\mathcal{S}}(K, H_n)$, $M^\pm F$ is integrable on $\Delta(K, H_n)$ and Equations 6.5 and 6.6 show that $\partial_\lambda \widetilde{F}(z, \lambda)$ is integrable on $\mathbb{R}^\times = \{\lambda \mid \lambda \neq 0\}$. We have,

$$\begin{aligned}
(2\pi)^{n+1} it f(z,t) &= \int_{\mathbb{R}^\times} \widetilde{F}(z,\lambda)\partial_\lambda \left(e^{i\lambda t}\right) d\lambda \\
&= \int_0^\infty \widetilde{F}(z,\lambda)\partial_\lambda \left(e^{i\lambda t}\right) d\lambda + \int_{-\infty}^0 \widetilde{F}(z,\lambda)\partial_\lambda \left(e^{i\lambda t}\right) d\lambda \\
&= -\int_0^\infty \partial_\lambda \widetilde{F}(z,\lambda) e^{i\lambda t} d\lambda - \int_{-\infty}^0 \partial_\lambda \widetilde{F}(z,\lambda) e^{i\lambda t} d\lambda \\
&\qquad - \lim_{\lambda \to 0^+} \widetilde{F}(z,\lambda) + \lim_{\lambda \to 0^-} \widetilde{F}(z,\lambda).
\end{aligned}$$

It can be shown that the limits $\lim_{\lambda \to 0^\pm} \widetilde{F}(z, \lambda)$ exist and are equal. Here one needs to use the hypotheses that F is continuous across $\Delta_2(K, H_n)$ and that $F_\circ$ is a Schwartz function. The proof, which is rather involved, is presented below in Section 7. Using Equations 6.5 and 6.6 for $\partial_\lambda \widetilde{F}(z, \lambda)$ we obtain

$$\begin{aligned}
(2\pi)^{n+1} \left(\pm\frac{\gamma}{2} + it\right) f(z,t) &= -\int_0^\infty \sum_{\alpha \in \Lambda} d_\alpha |\lambda|^n \left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^\pm\right) F(\alpha,\lambda)\phi_{\alpha,\lambda}(z,t) d\lambda \\
&\qquad - \int_{-\infty}^0 \sum_{\alpha \in \Lambda} d_\alpha |\lambda|^n \left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^\mp\right) F(\alpha,\lambda)\phi_{\alpha,\lambda}(z,t) d\lambda \\
&= \int_{\mathbb{R}^\times} \sum_{\alpha \in \Lambda} d_\alpha M^\pm F(\alpha,\lambda)\phi_{\alpha,\lambda}(z,t)|\lambda|^n d\lambda.
\end{aligned}$$

We conclude that $(\pm\frac{\gamma}{2} + it)\, f \in L_K(H_n)$ with $(2\pi)^{n+1}\left((\pm\frac{\gamma}{2} + it)\, f\right)^\wedge = M^\pm F \in \widehat{\mathcal{S}}(K, H_n)$. This completes the proof for item (3). $\square$

The characterization of the spherical transform of a Schwartz function given in Theorem 6.1 can be used to construct functions on H_n whose transform has a predetermined support. Recall that Theorem 2.2 asserts the existence of a map $E : \Delta(K, H_n) \to (\mathbb{R}^+)^d \times \mathbb{R}$ which is a homeomorphism onto its image. In Corollary 6.3 below, $K = U(n)$ or $\mathbb{T}^n$, so that $d = 1$ or $d = n$. In the first case, $\Delta(U(n), H_n) \cong \{(m, \lambda) \mid m \in \mathbb{Z}^+, \lambda \in \mathbb{R}\} \coprod \{s \mid s > 0\}$ with

$$E(m,\lambda) = \left(\frac{|\lambda|}{2}(2m+n), \lambda\right) \quad \text{and} \quad E(s) = \left(\frac{s}{2}, 0\right).$$

In the second case, we have $\Delta(\mathbb{T}^n, H_n) \cong \{(m,\lambda) \mid m \in (\mathbb{Z}^+)^n, \lambda \in \mathbb{R}\} \coprod \{s \mid s \in (\mathbb{R}^+)^n\}$ with

$$E(m,\lambda) = \left(\frac{|\lambda|}{2}(2m_1+n), \ldots, \frac{|\lambda|}{2}(2m_n+n)\lambda\right) \quad \text{and} \quad E(s) = \left(\frac{s_1}{2}, \ldots, \frac{s_n}{2}, 0\right).$$

Corollary 6.3. *Let F be a smooth function of compact support on $\mathbb{R}^2$ (or $\mathbb{R}^{n+1}$). Then $\widetilde{F} = F \circ E$ is the spherical transform of a $U(n)$- (or $\mathbb{T}^n$-) invariant Schwartz function on H_n.*

Proof. Since F has compact support, we see that $\widetilde{F}(m,\lambda) = 0$ if $|\lambda|$ and $|\lambda|(2|m|+n)$ are sufficiently large. Hence our growth conditions are automatically satisfied. The only property left to verify is that $M^\pm \widetilde{F}$ extends continuously across $\Delta_2(K, H_n)$.

For $K = U(n)$, we first consider $M^+\widetilde{F}(m,\lambda)$ with $\lambda \to 0^+$, $|\lambda|(2m+n) \to s$. We have

$$\begin{aligned}
M^+\widetilde{F}(m,\lambda) &= \partial_\lambda \widetilde{F}(m,\lambda) - \frac{1}{\lambda}\mathcal{D}^+\widetilde{F}(m,\lambda) \\
&= \partial_\lambda \left[F\left(\frac{\lambda}{2}(2m+n), \lambda\right)\right] - \frac{m+n}{\lambda}\left[\widetilde{F}(m+1,\lambda) - \widetilde{F}(m,\lambda)\right] \\
&= \frac{2m+n}{2}\partial_1 F\left(\frac{\lambda}{2}(2m+n),\lambda\right) + \partial_2 F\left(\frac{\lambda}{2}(2m+n),\lambda\right) \\
&\qquad - \frac{m+n}{\lambda}\left[F\left(\frac{\lambda}{2}(2m+n+2),\lambda\right) - F\left(\frac{\lambda}{2}(2m+n),\lambda\right)\right].
\end{aligned}$$

Now substitute $\lambda(2m+n)/2 \approx s/2$, to obtain

$$\begin{aligned}
M^+\widetilde{F}(m,\lambda) &\approx \frac{s}{2\lambda}\partial_1 F\left(\frac{s}{2},\lambda\right) + \partial_2 F\left(\frac{s}{2},\lambda\right) - \frac{s}{2\lambda^2}\left[F\left(\frac{s}{2}+\lambda,\lambda\right) - F\left(\frac{s}{2},\lambda\right)\right] \\
&\quad - \frac{n}{2\lambda}\left[F\left(\frac{s}{2}+\lambda,\lambda\right) - F\left(\frac{s}{2},\lambda\right)\right] \\
&= \partial_2 F\left(\frac{s}{2},\lambda\right) - \frac{s}{2\lambda^2}\left[F\left(\frac{s}{2}+\lambda,\lambda\right) - F\left(\frac{s}{2},\lambda\right) - \lambda\partial_1 F\left(\frac{s}{2},\lambda\right)\right] \\
&\quad - \frac{n}{2\lambda}\left[F\left(\frac{s}{2}+\lambda,\lambda\right) - F\left(\frac{s}{2},\lambda\right)\right],
\end{aligned}$$

which converges to

$$\partial_2 F\left(\frac{s}{2},0\right) - \frac{s}{4}\partial_1^2 F\left(\frac{s}{2},0\right) - \frac{n}{2}\partial_1 F\left(\frac{s}{2},0\right)$$

as $\lambda \to 0^+$. Now for $\lambda \to 0^-$, $\lambda(2m+n)/2 \to -s/2$ and we have:

$$\begin{aligned}
M^+\widetilde{F}(m,\lambda) &= \partial_\lambda \widetilde{F}(m,\lambda) - \frac{1}{\lambda}\mathcal{D}^-\widetilde{F}(m,\lambda) \\
&= \partial_\lambda \left[F\left(-\frac{\lambda}{2}(2m+n),\lambda\right)\right] - \frac{m}{\lambda}\left[\widetilde{F}(m,\lambda) - \widetilde{F}(m-1,\lambda)\right] \\
&\approx -\frac{2m+n}{2}\partial_1 F\left(\frac{s}{2},\lambda\right) + \partial_2 F\left(\frac{s}{2},\lambda\right) - \frac{m}{\lambda}\left[F\left(\frac{s}{2},\lambda\right) - F\left(\frac{s}{2}+\lambda,\lambda\right)\right] \\
&\approx \frac{s}{2\lambda}\partial_1 F\left(\frac{s}{2},\lambda\right) + \partial_2 F\left(\frac{s}{2},\lambda\right) + \frac{s}{2\lambda^2}\left[F\left(\frac{s}{2},\lambda\right) - F\left(\frac{s}{2}+\lambda,\lambda\right)\right] \\
&\quad + \frac{n}{2\lambda}\left[F\left(\frac{s}{2},\lambda\right) - F\left(\frac{s}{2}+\lambda,\lambda\right)\right],
\end{aligned}$$

which converges to

$$\partial_2 F\left(\frac{s}{2},0\right) - \frac{s}{4}\partial_1^2 F\left(\frac{s}{2},0\right) - \frac{n}{2}\partial_1 F\left(\frac{s}{2},0\right)$$

as $\lambda \to 0^-$. A similar calculation shows that, as $\lambda \to 0^\pm$ and $|\lambda|(2m+n)/2 \to s/2$, we get

$$M^-\widetilde{F}(m,\lambda) = \left(\partial_\lambda - \frac{1}{\lambda}\mathcal{D}^\mp\right)\widetilde{F}(m,\lambda) \to \partial_2 F\left(\frac{s}{2},0\right) + \frac{s}{4}\partial_1^2 F\left(\frac{s}{2},0\right) + \frac{n}{2}\partial_1 F\left(\frac{s}{2},0\right).$$

Therefore $M^\pm\widetilde{F}$ extends continuously to a smooth function on $\Delta_2(U(n),H_n)$, and by induction, we see that $(M^+)^\ell(M^-)^m\widetilde{F}$ will also be smooth on $\Delta_2(U(n),H_n)$.

For the case $K = \mathbb{T}^n$, we have:

$$\begin{aligned}
\widetilde{F}(m,\lambda) &= F\left(\frac{|\lambda|}{2}(2m_1+1),\ldots,\frac{|\lambda|}{2}(2m_n+1),\lambda\right) \\
\mathcal{D}^+\widetilde{F}(m,\lambda) &= \sum_{j=1}^n (m_j+1)[\widetilde{F}(m_1,\ldots,m_j+1,\ldots,m_n,\lambda) - \widetilde{F}(m,\lambda)], \\
\mathcal{D}^-\widetilde{F}(m,\lambda) &= \sum_{j=1}^n m_j[\widetilde{F}(m,\lambda) - \widetilde{F}(m_1,\ldots,m_j-1,\ldots,m_n,\lambda)].
\end{aligned}$$

If we approach $\Delta_2(\mathbb{T}^n,H_n)$ by taking $\lambda \to 0$, $|\lambda|(2m_j+1) \to s_j$, then we obtain, with $s = (s_1,\ldots,s_n)$:

$$M^\pm\widetilde{F}(m,\lambda) \to \partial_{n+1}F(s,0) \mp \left[\frac{1}{4}\sum_{j=1}^n s_j\partial_j^2 F(s,0) + \frac{1}{2}\sum_{j=1}^n s_j\partial_j F(s,0)\right].$$

Hence $\widetilde{F}$ is the spherical transform of a $\mathbb{T}^n$-invariant Schwartz function on H_n. □

Equation 6.2 shows how a K-invariant Schwartz function $f \in \mathcal{S}_K(H_n)$ is determined by its K-spherical transform $\widehat{f}$. We conclude this section with a result that shows how the Fourier transform $\mathcal{F}_H(f)$ of f is determined by $\widehat{f}$. The formula in Proposition 6.4

differs from Equation 6.2 in that no integration is involved. Proposition 6.4 shows, in particular, that the Fourier transform $w \mapsto \mathcal{F}_H f(w,s)$ along a fixed central "level" $s \neq 0$ is completely determined by the values of the spherical transform $\{\widehat{f}(\alpha, s) \mid \alpha \in \Lambda\}$ associated with this level.

Proposition 6.4. *For $f \in \mathcal{S}_K(H_n)$ one has*

$$\mathcal{F}_H(f)(w,s) = \begin{cases} 2^n \sum_{\alpha\in\Lambda} (-1)^{|\alpha|} d_\alpha \widehat{f}(\alpha,s) \phi_\alpha^\circ \left(2w/\sqrt{|s|}\right) & s \neq 0 \\ \widehat{f}(\eta_w) & s = 0 \end{cases}.$$

Proof. The identity $\mathcal{F}_H(f)(w,0) = \widehat{f}(\eta_w)$ is Equation 2.15. Next we use Equation 6.2 to write

$$\begin{aligned} \mathcal{F}_H(f)(w,s) &= \left(\frac{1}{2\pi}\right)^{n+1} \int_{H_n} \int_{\mathbb{R}^\times} \sum_{\alpha\in\Lambda} d_\alpha \widehat{f}(\alpha,\lambda) \phi_\alpha^\circ \left(\sqrt{|\lambda|} z\right) e^{i(\lambda - s)t} |\lambda|^n e^{-iRe\langle z,w\rangle} d\lambda dz dt \\ &= \left(\frac{1}{2\pi}\right)^{n+1} \int_{\mathbb{R}} \int_{\mathbb{R}^\times} \sum_{\alpha\in\Lambda} d_\alpha \widehat{f}(\alpha,\lambda) \mathcal{F}_V \left(\phi_\alpha^\circ \circ \delta_{\sqrt{|\lambda|}}\right)(w) e^{i(\lambda-s)t} |\lambda|^n d\lambda dt. \end{aligned}$$

Here one can use the estimate given in Definition 6.1 for $m = 0$ to justify rearranging the series and integrals as above. In view of Equation 3.4, we have

$$\mathcal{F}_V \left(\phi_\alpha^\circ \circ \delta_{\sqrt{|\lambda|}}\right)(w) = (-1)^{|\alpha|} \left(\frac{4\pi}{|\lambda|}\right)^n \phi_\alpha^\circ \left(\frac{2w}{\sqrt{|\lambda|}}\right)$$

and hence

$$\begin{aligned} \mathcal{F}_H(f)(w,s) &= \int_{\mathbb{R}} \int_{\mathbb{R}^\times} \frac{2^n}{2\pi} \sum_{\alpha\in\Lambda} (-1)^{|\alpha|} d_\alpha \widehat{f}(\alpha,\lambda) \phi_\alpha^\circ \left(\frac{2w}{\sqrt{|\lambda|}}\right) e^{i(\lambda-s)t} d\lambda dt \\ &= 2^n \sum_{\alpha\in\Lambda} (-1)^{|\alpha|} d_\alpha \widehat{f}(\alpha,s) \phi_\alpha^\circ \left(2w/\sqrt{|s|}\right) \end{aligned}$$

for $s \neq 0$. Again, one can use the estimate in Definition 6.1 to justify the rearrangement of series and integrals employed above. □

7. Analysis of boundary terms

Throughout this section, F will denote a rapidly decreasing function on $\Delta(K, H_n)$ and $F_\circ \in S_K(V)$ is defined by $F_\circ(w) = F(\eta_w)$. (See Definition 6.1.) In the proof of Theorem 6.1, we defined a function $\widetilde{F}(z,\lambda)$ on $V \times \mathbb{R}^\times$ by

$$\widetilde{F}(z,\lambda) = \sum_{\alpha\in\Lambda} d_\alpha |\lambda|^n F(\alpha,\lambda) \phi_{\alpha,\lambda}^\circ(z).$$

In order to complete the proof, we need to show that the limits $\lim_{\lambda\to 0^\pm} \widetilde{F}(z,\lambda)$ exist and are equal. In fact, we will prove the following result.

Proposition 7.1. $\lim_{\lambda\to 0}\sum_{\alpha\in\Lambda} d_\alpha|\lambda|^n F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z) = \left(\frac{1}{2\pi}\right)^n \int_V F_\circ(w)\eta_w(z)dw.$

It is easy to prove Proposition 7.1 for the case where $F=\widehat{f}$ for some $f\in S_K(H_n)$. Indeed, the inversion formula for the spherical transform yields

$$(2\pi)^{n+1}\int f(z,t)dt = \int\int \widetilde{F}(z,\lambda)e^{i\lambda t}d\lambda dt = 2\pi \lim_{\lambda\to 0^\pm}\widetilde{F}(z,\lambda).$$

Thus $\lim_{\lambda\to 0}\widetilde{F}(z,\lambda)$ exists and one has

$$\begin{aligned}\lim_{\lambda\to 0}\widetilde{F}(z,\lambda) &= (2\pi)^n\int f(z,t)dt = \frac{(2\pi)^n}{(2\pi)^{2n}}\int\int_V \mathcal{F}_V(f)(w,t)e^{iRe\langle w,z\rangle}dwdt\\ &= \left(\frac{1}{2\pi}\right)^n\int_V \mathcal{F}_H(f)(w,0)e^{iRe\langle w,z\rangle}dw = \left(\frac{1}{2\pi}\right)^n\int_V F_\circ(w)e^{iRe\langle w,z\rangle}dw\\ &= \left(\frac{1}{2\pi}\right)^n\int_V F_\circ(w)\eta_w(z)dw.\end{aligned}$$

Here we have used Equation 2.15 and K-invariance of $F_\circ$. This proves Proposition 7.1 for functions $F\in S_K(H_n)^\wedge$. Since we use Proposition 7.1 to prove that $\widehat{\mathcal{S}}(K,H_n)\subset S_K(H_n)^\wedge$, we can not, however, assume here that $F\in S_K(H_n)^\wedge$.

The case where $K=U(n)$ plays a special role in our proof of Proposition 7.1. We will denote the bounded $U(n)$-spherical functions on H_n by

$$\Delta_1(U(n),H_n) = \{\phi^U_{m,\lambda}\mid m=0,1,2,\dots\},\quad \Delta_2(U(n),H_n)=\{\eta^U_w\mid w\in V\}.$$

The K-spherical functions $\phi_{\alpha,\lambda}$ are related to the $U(n)$-spherical functions $\phi^U_{m,\lambda}$ via

$$d_m\phi^U_{m,\lambda} = \sum_{|\alpha|=m} d_\alpha\phi_{\alpha,\lambda} \tag{7.1}$$

where as before, $d_m := \dim(\mathcal{P}_m(V))$. This follows from the fact that $d_m p_m = \sum_{|\alpha|=m} d_\alpha p_\alpha$.

Lemma 7.2. *For $w\in V$ one has*

$$\lim_{N\to\infty}\frac{1}{d_N}\sum_{|\alpha|=N} d_\alpha F\left(\alpha,\frac{|w|^2}{2N+n}\right) = \int_{U(n)} F_\circ(kw)dk.$$

Proof. Let $s = |w|^2$. It is shown in [BJRW] that $\phi^U_{N,\frac{s}{2N+n}}$ converges to η^U_w uniformly on compact sets. Using this fact together with Equation 7.1 one computes

$$\begin{aligned}
\int_{U(n)} F_\circ(kw)dk &= \int_{U(n)} \left(\frac{1}{2\pi}\right)^{2n} \int_V \mathcal{F}_V F_\circ(z) e^{i\langle z,kw\rangle} dz dk \\
&= \left(\frac{1}{2\pi}\right)^{2n} \int_V \mathcal{F}_V F_\circ(z) \eta^U_w(z) dz \\
&= \lim_{N\to\infty} \left(\frac{1}{2\pi}\right)^{2n} \int_V \mathcal{F}_V F_\circ(z) \left(\phi^U_{N,\frac{s}{2N+n}}\right)^\circ (z) dz \\
&= \lim_{N\to\infty} \left(\frac{1}{2\pi}\right)^{2n} \frac{1}{d_N} \sum_{|\alpha|=N} d_\alpha \int_V \mathcal{F}_V F_\circ(z) \phi^\circ_{\alpha,\frac{s}{2N+n}}(z) dz.
\end{aligned}$$

Given $\phi_{\alpha,\lambda} \in \Delta_1(K, H_n)$, choose a point $w_{\alpha,\lambda} \in V$ such that

$$d\left(\phi_{\alpha,\lambda}, \eta_{w_{\alpha,\lambda}}\right) \leq d\left(\phi_{\alpha,\lambda}, \eta\right)$$

for all $\eta \in \Delta_2(K, H_n)$. Here $d(\cdot,\cdot)$ denotes the metric on $\Delta(K, H_n)$ defined by Equation 2.11. We can now write

$$\begin{aligned}
\int_{U(n)} F_\circ(kw)dk &= \lim_{N\to\infty} \left(\frac{1}{2\pi}\right)^{2n} \frac{1}{d_N} \sum_{|\alpha|=N} d_\alpha \int_V \mathcal{F}_V F_\circ(z) \eta_{w_{\alpha,\frac{s}{2N+n}}}(z) dz \\
&= \lim_{N\to\infty} \left(\frac{1}{2\pi}\right)^{2n} \frac{1}{d_N} \sum_{|\alpha|=N} d_\alpha \int_V \mathcal{F}_V F_\circ(z) e^{i\langle w_{\alpha,\frac{s}{2N+n}}, z\rangle} dz \\
&= \lim_{N\to\infty} \frac{1}{d_N} \sum_{|\alpha|=N} d_\alpha F_\circ\left(w_{\alpha,\frac{s}{2N+n}}\right) \\
&= \lim_{N\to\infty} \frac{1}{d_N} \sum_{|\alpha|=N} d_\alpha F\left(\alpha, \frac{s}{2N+n}\right).
\end{aligned}$$

Here we have used continuity of F and the fact that $F_\circ$, and hence also $\mathcal{F}_V F_\circ$, are K-invariant. □

Next we define a function ${}^UF$ on $\Delta(U(n), H_n)$ by

$$\begin{aligned}
{}^UF\left(\phi^U_{m,\lambda}\right) &= \frac{1}{d_m} \sum_{|\alpha|=m} d_\alpha F(\alpha,\lambda), \\
{}^UF\left(\eta^U_w\right) &= \int_{U(n)} F_\circ(kw)dk.
\end{aligned}$$

If $F = \widehat{f}$ for some $f \in \mathcal{S}_K(H_n)$, then one can check that ${}^UF$ is the $U(n)$-spherical transform of the function ${}^Uf \in \mathcal{S}_{U(n)}(H_n)$ defined by ${}^Uf(z,t) = \int_{U(n)} f(kz,t)dk$. This observation motivates the definition of ${}^UF$. In general one has the following result.

Lemma 7.3. *${}^UF$ is rapidly decreasing on $\Delta(U(n), H_n)$.*

Proof. It is clear that $({}^UF)_\circ$ is a Schwartz function and that ${}^UF$ satisfies the estimates in Definition 6.1. We need, however, to show that ${}^UF$ is continuous across $\Delta_2(U(n), H_n)$. Suppose that (m_N, λ_N) is a sequence in $\mathbb{Z}^+ \times \mathbb{R}^\times$ with $\lim_{N\to\infty} \phi^U_{m_N,\lambda_N} = \eta^U_w$ in $\Delta(U(n), H_n)$. This occurs if and only if $|\lambda_N|(2m_N + n) \to |w|^2$ (see [BJRW]). Since

$$d\left(\left(m_N, \frac{|w|^2}{2m_N + n}\right), (m_N, \lambda_N)\right) = \left|\frac{|w|^2}{2m_N + n} - \lambda_N\right| \to 0$$

we see that one also has

$$\lim_{N\to\infty} \phi^U_{m_N, \frac{|w|^2}{2m_N+n}} = \eta^U_w$$

in $\Delta(U(n), H_n)$. Since $\lim_{N\to\infty} m_N = \infty$, we have

$$\begin{aligned}
\lim_{N\to\infty} {}^UF(m_N, \lambda_N) &= \lim_{N\to\infty} {}^UF\left(m_N, \frac{|w|^2}{2m_N + n}\right) \\
&= \lim_{N\to\infty} \frac{1}{d_{m_N}} \sum_{|\alpha|=N} d_\alpha F\left(\alpha, \frac{|w|^2}{2N + n}\right) \\
&= \int_{U(n)} F_\circ(kw)dk \qquad \text{(by Lemma 7.2)} \\
&= {}^UF(\eta^U_s).
\end{aligned}$$

□

Lemma 7.4 shows that Proposition 7.1 holds for the case where $K = U(n)$ and $z = 0$.

Lemma 7.4. *Let G be a rapidly decreasing function on $\Delta(U(n), H_n)$ and $w \in V$. Then $\lim_{\lambda\to 0} \sum_{m=0}^\infty d_m |\lambda|^n G(m, \lambda) = \left(\frac{1}{2\pi}\right)^n \int_V G_\circ(w)dw$.*

Proof. We will show that, for λ small, the left hand side of the above equation is close to a Riemann sum for the integral on the right hand side. Define a function g on $\mathbb{R}^+$ via

$$g\left(\frac{|w|^2}{2}\right) = G_\circ(w).$$

g is continuous and rapidly decreasing on $\mathbb{R}^+$. Using spherical coordinates on V, one sees that

$$\left(\frac{1}{2\pi}\right)^n \int_V G_\circ(w)dw = \frac{1}{(n-1)!} \int_0^\infty g(s) s^{n-1} ds. \tag{7.2}$$

Choose points $w_{m,\lambda} \in V$ with $|w_{m,\lambda}|^2 = |\lambda|(2m + n)$, and hence that

$$d\left(\phi^U_{m,\lambda}, \eta^U_{w_{m,\lambda}}\right) = |\lambda|.$$

We have

$$\lim_{\lambda\to 0}\sum_{m=0}^{\infty} d_m|\lambda|^n G(m,\lambda) = \lim_{\lambda\to 0}\sum_{m=0}^{\infty} d_m|\lambda|^n G\left(\eta^U_{w_{m,\lambda}}\right)$$
$$= \lim_{\lambda\to 0}\sum_{m=0}^{\infty} d_m|\lambda|^n g\left(\frac{|\lambda|}{2}(2m+n)\right).$$

Comparing this with Equation 7.2, we see that to complete the proof we must show that

$$\lim_{\lambda\to 0}\sum_{m=0}^{\infty} d_m|\lambda|^n g\left(\frac{|\lambda|}{2}(2m+n)\right) = \frac{1}{(n-1)!}\int_0^\infty g(s)s^{n-1}ds. \tag{7.3}$$

In fact, as shown below, Equation 7.3 is valid for any function $g \in L^1(\mathbb{R}^+, s^{n-1}ds)$.

It suffices to establish Equation 7.3 for a characteristic function. Suppose that $g(s) = 1$ for $s \in [a,b]$ and $g(s) = 0$ elsewhere. Then

$$\sum_{m=0}^{\infty} d_m|\lambda|^n g\left(\frac{|\lambda|}{2}(2m+n)\right) = |\lambda|^n \sum_{m=A_\lambda}^{B_\lambda} d_m,$$

where A_λ is the smallest integer with $a \le |\lambda|(2A_\lambda+n)/2$ and B_λ is the largest integer with $b \ge |\lambda|(2B_\lambda+n)/2$. Now

$$\sum_{m=A_\lambda}^{B_\lambda} d_m = \sum_{m=0}^{B_\lambda}\binom{m+n-1}{n-1} - \sum_{m=0}^{A_\lambda}\binom{m+n-1}{n-1} = \binom{B_\lambda+n}{n} - \binom{A_\lambda+n}{n}$$

and thus

$$\lim_{\lambda\to 0}|\lambda|^n \sum_{m=A_\lambda}^{B_\lambda} d_m = \lim_{\lambda\to 0}\left[\frac{|\lambda|^n}{n!}(B_\lambda+1)\cdots(B_\lambda+n) - \frac{|\lambda|^n}{n!}(A_\lambda+1)\cdots(A_\lambda+n)\right]$$
$$= \frac{1}{n!}(b^n - a^n) = \frac{1}{(n-1)!}\int_a^b s^{n-1}ds.$$
$$= \frac{1}{(n-1)!}\int_0^\infty g(s)s^{n-1}ds$$

as desired. ☐

Lemma 7.5. $\lim_{\lambda\to 0}\sum_{\alpha\in\Lambda} d_\alpha|\lambda|^n F(\alpha,\lambda) = (\frac{1}{2\pi})^n \int_V F_\circ(w)dw$.

Proof. Lemma 7.3 ensures that the function ${}^U F$ is rapidly decreasing on $\Delta(U(n), H_n)$. Thus, by Lemma 7.4 we have

$$\lim_{\lambda\to 0}\sum_{m=0}^{\infty} d_m|\lambda|^n\, {}^U F(m,\lambda) = \left(\frac{1}{2\pi}\right)^n \int_V ({}^U F)_\circ(w)dw.$$

But

$$\sum_{\alpha\in\Lambda} d_\alpha|\lambda|^n F(\alpha,\lambda) = \sum_{m=0}^{\infty} d_m|\lambda|^n \frac{1}{d_m}\sum_{|\alpha|=m} d_\alpha F(\alpha,\lambda)$$
$$= \sum_{m=0}^{\infty} d_m|\lambda|^n\, {}^U F(m,\lambda),$$

and

$$\int_V F_\circ(w)dw = \int_V\int_{U(n)} F_\circ(kw)dkdw = \int_V ({}^U F)_\circ(w)dw.$$

$\square$

Proof of Proposition 7.1. Lemma 7.5 shows that Proposition 7.1 holds for the case where $z=0$. The proposition now follows by replacing $F(\alpha,\lambda)$ by $F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z)$. For $z\in V$ fixed, we see that $F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z)$ is a rapidly decreasing function on $\Delta_1(K,H_n)$ with continuous extension across $\Delta_2(K,H_n)$ given by $\eta_w\mapsto F_\circ(w)\eta_w(z)$. Note that $w\mapsto F_\circ(w)\eta_w(z)$ belongs to $\mathcal{S}_K(V)$ for fixed z. Thus Lemma 7.5 yields $\lim_{\lambda\to 0}\sum_{\alpha\in\Lambda} d_\alpha|\lambda|^n F(\alpha,\lambda)\phi^\circ_{\alpha,\lambda}(z) = \left(\frac{1}{2\pi}\right)^n\int_V F_\circ(w)\eta_w(z)dw$. $\square$

References

[BJR90] C. Benson, J. Jenkins, and G. Ratcliff. On Gelfand pairs associated with solvable Lie groups. *Trans. Amer. Math. Soc.*, 321:85–116, 1990.

[BJR92] C. Benson, J. Jenkins, and G. Ratcliff. Bounded K-spherical functions on Heisenberg groups. *J. Functional Analysis*, 105:409–443, 1992.

[BJRW] C. Benson, J. Jenkins, G. Ratcliff, and T. Worku. Spectra for Gelfand pairs associated with the Heisenberg group. (*to appear in Colloq. Math.*).

[BR] C. Benson and G. Ratcliff. Combinatorics and spherical functions on the Heisenberg group. (*preprint*).

[BR96] C. Benson and G. Ratcliff. A classification for multiplicity free actions. *Journal of Algebra*, 181:152–186, 1996.

[Car87] G. Carcano. A commutativity condition for algebras of invariant functions. *Boll. Un. Mat. Italiano*, 7:1091–1105, 1987.

[Far87] J. Faraut. Analyse harmonique et fonctions speciales. In *Deux Courses d'Analyse Harmonique*. Birkhäuser, Boston, 1987.

[God62] R. Godement. A theory of spherical functions I. *Trans. Amer. Math. Soc.*, 73:496–556, 1962.

[GV88] R. Gangolli and V.S. Varadarajan. *Harmonic Analysis of Spherical Functions on Real Reductive Groups*. Springer-Verlag, New York, 1988.

[HT92] R. Howe and E.C. Tan. *Non-Abelian Harmonic Analysis, Applications of $SL(2,\mathbb{R})$*. Universitext. Springer-Verlag, New York, 1992.

[HU91] R. Howe and T. Umeda. The Capelli identity, the double commutant theorem and multiplicity-free actions. *Math. Annalen*, 290:565–619, 1991.

[Kac80] V. Kac. Some remarks on nilpotent orbits. *Journal of Algebra*, 64:190–213, 1980.

[Str91] R. Strichartz. L^p harmonic analysis and Radon transforms on the Heisenberg group. *J. Functional Analysis*, 96:350–406, 1991.

[Tho82] E.G.F. Thomas. An infinitesimal characterization of Gelfand pairs. *Proceedings of the conference in Modern Analysis and Probability in honor of Shizuo Kakutani, Yale University*, 1982.

[Yan] Z. Yan. Special functions associated with multiplicity-free representations. (*preprint*).

Dept of Math and Comp Sci, Univ of Missouri-St. Louis, St. Louis, MO 63121
E-mail address: `benson@arch.umsl.edu`

Division of Math Sciences, National Science Foundation, 4210 Wilson Blvd, Arlington, VA 22203-9966
E-mail address: `jjenkins@nsf.gov`

Dept of Math and Comp Sci, Univ of Missouri-St. Louis, St. Louis, MO 63121
E-mail address: `ratcliff@arch.umsl.edu`