TATE TWISTS OF HODGE STRUCTURES ARISING FROM ABELIAN VARIETIES

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Abstract. We consider the category of Hodge substructures of the cohomology of abelian varieties, and ask when a Tate twist of such a Hodge structure belongs to the same category.

1. Introduction

A (rational) Hodge structure $V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$ is said to be *effective* if $V^{p,q} = 0$ unless p, $q \geq 0$, and, it is said to be *geometric* if it is isomorphic to a Hodge substructure of $H^n(X, \mathbb{Q})$ for some smooth, projective variety X over $\mathbb{C}$. For $m \in \mathbb{Z}$, the Tate twist $V(m)$ is the Hodge structure of weight $n - 2m$ defined by $V(m)^{p,q} = V^{p+m,q+m}$.

A geometric Hodge structure must be effective and polarizable, but not conversely (Grothendieck [17, p. 300, 2nd footnote]). Any effective and polarizable Hodge structure of weight 1 is the first cohomology of an abelian variety, and hence geometric. In [7] we have shown that any Hodge structure of CM-type is geometric. These are the only known criteria for an abstract Hodge structure to be geometric [15, p. 305].

The general Hodge conjecture as formulated by Grothendieck [17] implies that any effective Tate twist of a geometric Hodge structure is again geometric. In a series of papers [1–9] we have shown that, for certain abelian varieties A, every effective Tate twist of a Hodge structure in the cohomology of A is isomorphic to a Hodge structure occurring in the cohomology of some abelian variety. Moreover, we have used this to prove the general Hodge conjecture for certain abelian varieties. We have also shown the existence of a Hodge structure which occurs in the cohomology of an abelian variety, but which has an effective Tate twist that does not occur in the cohomology of *any* abelian variety [5, Theorem 5.5, p. 926].

We now outline the contents and organization of this paper. We begin in §2 with a summary of the definitions and basic properties of the Hodge and Lefschetz groups of an abelian variety, and the Kuga fiber varieties associated with them. In §3 we formulate our problem in representation theoretic form. In the following four sections we review our results for various classes of abelian varieties. In §8, we discuss the abelian varieties for which the general Hodge conjecture is known; the full list of such abelian varieties

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is given in Appendix A—in all these cases the general Hodge conjecture follows from known cases of the usual Hodge conjecture and the results included in this article. §9 is a remark on the open cases.

The results mentioned above have appeared in various papers with inconsistent definitions, notations and terminology. Our goal in this article is to give a unified presentation, with simplifications and generalizations where possible. Certain errors crept into the earlier articles; Appendix B alerts the reader to them.

Notations and conventions. All representations are finite-dimensional and algebraic. The derived group of a group G is denoted by G'. All abelian varieties are over $\mathbb{C}$. For an abelian variety A, we let $D(A) = \mathrm{End}(A)\otimes\mathbb{Q}$ be its endomorphism algebra, $L(A)$ its Lefschetz group, $G(A)$ its Hodge group, $L'(A)$ the derived group of $L(A)$, and, $G'(A)$ the derived group of $G(A)$. For a finite field extension E of a field F, we let $\mathrm{Res}_{E/F}$ be the restriction of scalars functor, from varieties over E to varieties over F. For an algebraic or topological group G, we denote by G^0 the connected component of the identity.

2. Algebraic groups and abelian varieties

2.1. **Abelian varieties.** Let A be an abelian variety over $\mathbb{C}$, and let $V = H_1(A,\mathbb{Q})$. As a complex manifold, A is the torus $V_{\mathbb{R}}/V_{\mathbb{Z}}$ with a complex structure J on $V_{\mathbb{R}}$, where $V_{\mathbb{Z}}$ is the lattice $H_1(A,\mathbb{Z})$. We view V as a left vector space over the endomorphism algebra $D = D(A) = \mathrm{End}(A)\otimes\mathbb{Q}$.

Recall that a polarization of A induces an involution $a \mapsto \overline{a}$ on D, and a Riemann form β, i.e., an alternating $\mathbb{Q}$-bilinear form on V such that $\beta(x, Jy)$ is symmetric and positive definite, $\beta(V_{\mathbb{Z}}, V_{\mathbb{Z}}) \subset \mathbb{Z}$, and such that $\beta(ax, y) = \beta(x, \overline{a}y)$ for $a \in D$. We use the nondegenerate form β to identify V with its dual $H^1(A,\mathbb{Q})$, and thus $\bigwedge^n V$ with $H^n(A,\mathbb{Q})$.

2.2. **Hodge group.** The Hodge group (or Special Mumford-Tate group) of A is defined to be the smallest $\mathbb{Q}$-algebraic subgroup $G = G(A)$ of $GL(V)$ such that $G(\mathbb{R})$ contains $\exp(\theta J)$ for all $\theta \in \mathbb{R}$ (Mumford [28]). It is a connected, reductive group characterized by the property that its invariants in $H^\star(A^k,\mathbb{Q})$ are precisely the Hodge classes for any positive integer k.

2.3. **Lefschetz group.** The Lefschetz group $L(A)$ is defined to be the centralizer of $\mathrm{End}(A)$ in $Sp(V,\beta)$; it is characterized by the property that for any positive integer k, the subring of $H^\star(A^k,\mathbb{Q})$ generated by the classes of divisors equals $H^\star(A^k,\mathbb{C})^{L(A)_{\mathbb{C}}} \cap H^\star(A^k,\mathbb{Q})$ (Milne [27, Theorem 3.2, p. 656] and Murty [31, §3.6.2, p. 93]). Since any divisor class is a Hodge class, it follows that $G(A) \subset L(A) \subset Sp(V,\beta)$.

2.4. **Classification.** Assume, now, that A is simple. Then by Albert's classification, D is one of the following [37]:

type I: a totally real number field F;

type II: a totally indefinite quaternion algebra over a totally real number field F;

type III: a totally definite quaternion algebra over a totally real number field F;

type IV: a division algebra over a CM-field E. In this case let F be the maximal totally real subfield of E.

In each case there exists a unique F-bilinear form $T\colon V\times V\to D$ such that $\beta(x,y)=\mathrm{Tr}_{D/\mathbb{Q}}\,T(x,y)$, $T(ax,by)=aT(x,y)\bar{b}$, and, $T(y,x)=-\overline{T(x,y)}$ for all x, $y\in V$, a, $b\in D$ [38, Lemma 1.2, p. 162]. The Lefschetz group is then the restriction of scalars, from F to $\mathbb{Q}$, of the unitary group of T:

$$L(A)=\mathrm{Res}_{F/\mathbb{Q}}\,U(T)=\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{Aut}_D(V,T). \tag{2.1}$$

Let S be the set of embeddings of F into $\mathbb{R}$. We can write

$$L(A)_{\mathbb{R}}=\prod_{\alpha\in S}L_\alpha \qquad \text{and} \qquad V_{\mathbb{R}}=\bigoplus_{\alpha\in S}V_\alpha, \tag{2.2}$$

where L_α acts trivially on $V_{\alpha'}$ unless $\alpha=\alpha'$. Then L_α and its action on V_α are given as follows [30]:

type I: $L_\alpha=Sp(V_\alpha,\beta_\alpha)$ is a symplectic group acting via its standard representation on V_α;

type II: L_α is a symplectic group acting on V_α as two copies of the standard representation;

type III: $L_{\alpha,\mathbb{C}}$ is an orthogonal group acting on $V_{\alpha,\mathbb{C}}$ as two copies of the standard representation;

type IV: $L_\alpha=U(p_\alpha,q_\alpha)$, and $L_{\alpha,\mathbb{C}}\cong GL_m(\mathbb{C})$ acts on $V_{\alpha,\mathbb{C}}$ as the direct sum of the standard representation and its contragredient.

2.5. **Kuga fiber varieties.** We review below the construction of Kuga fiber varieties. These are families of abelian varieties which include the solutions to fine moduli problems for abelian varieties with additional structures such as Hodge cycles (Mumford [28]) or PEL-structures (polarizations, endomorphisms, level structures) (Shimura [39]).

Let β be a nondegenerate alternating form on a finite-dimensional vector space V over $\mathbb{Q}$, and L a lattice in V. The symplectic group $Sp(V,\beta)$ is a reductive group of hermitian type, which acts by conjugation on the Siegel space $\mathfrak{S}$ consisting of all complex structures J on $V_{\mathbb{R}}$ such that $\beta(x,Jy)$ is symmetric and positive definite. If Γ' is a torsion-free arithmetic subgroup of $Sp(V,\beta)$, then $\mathcal{V}'=\Gamma\backslash\mathfrak{S}$ is a quasiprojective algebraic variety, and there exists a universal family of abelian varieties $\mathcal{A}'\to\mathcal{V}'$ such that the fiber over $P\in\mathcal{V}'$ is the torus $V_{\mathbb{R}}/L$ with the complex structure J, where J is a point of $\mathfrak{S}$ lying over P.

Now let G be a semisimple $\mathbb{Q}$-algebraic group of hermitian type, Γ a torsion-free arithmetic subgroup of $G(\mathbb{Q})$, K a maximal compact subgroup of $G(\mathbb{R})^0$, and, $X=G(\mathbb{R})^0/K$ the corresponding bounded symmetric domain. Suppose given a representation $\rho\colon G\to Sp(V,\beta)$ such that $\rho(\Gamma)\subset\Gamma'$, and an equivariant holomorphic map $\tau\colon X \to \mathfrak{S}$, between the symmetric domains of the two groups. The family of abelian varieties $\mathcal{A} \to \mathcal{V} = \Gamma\backslash X$ obtained as the pullback of the universal family $\mathcal{A}' \to \mathcal{V}'$ is called a *Kuga fiber variety*, We refer the reader to [1] or [33] for details.

For A as in §2.1, the semisimple parts of the groups $G(A)$ and $L(A)^0$ are of hermitian type, and their inclusions into $Sp(V, \beta)$ define Kuga fiber varieties which are the Hodge families of Mumford [28] and (generalizations of) the PEL-families of Shimura [39], respectively.

2.6. **Abelian varieties of PEL-type.** We say that an abelian variety A is of PEL-*type* if the semisimple parts of $G(A)$ and $L(A)^0$ are equal. Thus a simple abelian variety is of PEL-type if and only if it is a general member of a PEL-family of abelian varieties (see [1, §1 and §4.6]). This means in essence that for each $k \geq 1$, A^k has no Hodge classes other than those it is required to have by virtue of its endomorphisms. We note that all abelian varieties of CM-type, all abelian varieties of dimension less than 4, and, all abelian varieties for which the general Hodge conjecture is currently known, are of PEL-type.

3. Dominating varieties

We say that a Hodge structure V of weight w is *fully twisted* if V is effective, and $V^{w,0} \neq 0$. Thus V is fully twisted if and only if V is effective, but $V(1)$ is not effective. More generally, for a subfield F of $\mathbb{C}$, we say that an F-subspace U of V_F is *fully twisted* if $U_{\mathbb{C}} \cap V^{w,0} \neq 0$.

Definition 3.1. A smooth, projective, complex algebraic variety A is said to be *dominated* by a class $\mathcal{X}$ of smooth, projective, complex algebraic varieties if, given any irreducible Hodge structure V in the cohomology of A, there exists a fully twisted Hodge structure V' in the cohomology of some $X \in \mathcal{X}$ such that V' is isomorphic to a Tate twist of V.

This definition is motivated by the following observation of Grothendieck (for a proof, see the proof of [1, Proposition 2.1, p. 243]).

Proposition 3.2 (Grothendieck [17, p. 301])**.** *Let A be a smooth projective variety over $\mathbb{C}$ which is dominated by $\mathcal{X}$. If the usual Hodge conjecture holds for $A \times X$ for each $X \in \mathcal{X}$, then the general Hodge conjecture holds for A.*

Hodge structures in the cohomology of an abelian variety A correspond to $G(A)$-submodules of $H^\star(A, \mathbb{Q})$. Thus A is dominated by $\mathcal{X}$ if and only if given any irreducible $G(A)$-submodule, V, of $H^\star(A, \mathbb{Q})$, there exist $B \in \mathcal{X}$, and a fully twisted $G(B)$-submodule V' of $H^n(B, \mathbb{Q})$ for some n, such that V and V' are isomorphic as $G(A \times B)$-modules. (Note that $G(A \times B)$ is a subgroup of $G(A) \times G(B)$, so it makes sense to consider V and V' as $G(A \times B)$-modules.)

Since it is usually easier to work over $\mathbb{C}$ than over $\mathbb{Q}$, it is natural to modify the above by extending scalars. Given a subfield F of $\mathbb{C}$, which will usually be either $\mathbb{Q}$ or $\mathbb{C}$, we say that A is *F-dominated* by a class $\mathcal{X}$ of abelian varieties if, given any irreducible $G(A)_F$-submodule, V, of $H^\star(A,F)$, there exist $B \in \mathcal{X}$, and a fully twisted $G(B)_F$-submodule V' of $H^n(B,F)$ for some n, such that V and V' are isomorphic as $G(A\times B)_F$-modules. In particular, A is dominated by $\mathcal{X}$ if and only if A is $\mathbb{Q}$-dominated by $\mathcal{X}$. We then have the following results:

Lemma 3.3 (Abdulali [8, Lemma 2, p. 1165])**.** *If an abelian variety A is F-dominated by $\mathcal{X}$ for some subfield F of $\mathbb{C}$, then A is dominated by $\mathcal{X}$.*

Theorem 3.4 (Abdulali [8, Propositions 3 and 4, p. 1166])**.** *Let A and B be abelian varieties such that $G(A \times B) = G(A) \times G(B)$.*

(1) *If A is $\mathbb{C}$-dominated by $\mathcal{X}$ and B is $\mathbb{C}$-dominated by $\mathcal{Y}$, then $A \times B$ is $\mathbb{C}$-dominated by $\mathcal{X} \cdot \mathcal{Y}$.*
(2) *If A is $\mathbb{C}$-dominated by $\mathcal{X}$ and B is dominated by $\mathcal{Y}$, then $A \times B$ is dominated by $\mathcal{X} \cdot \mathcal{Y}$.*

Here, $\mathcal{X} \cdot \mathcal{Y} = \{\, X \times Y \mid X \in \mathcal{X},\ Y \in \mathcal{Y}\,\}$.

4. Semisimple Hodge groups

We begin with abelian varieties with semisimple Hodge groups, without factors of type III. Abelian varieties of type III will be dealt with in §7.

Theorem 4.1 (Abdulali [1, Theorem 5.1, p. 348])**.** *Let A be an abelian variety of* PEL*-type. Suppose that the Hodge group of A is semisimple and A has no factors of type* III*. Then A is $\mathbb{C}$-dominated by the set of powers of itself. The usual Hodge conjecture for A implies the general Hodge conjecture for all powers of A.*

Idea of proof. We illustrate the idea of proof of this theorem in the simplest case: a generic abelian variety A of dimension n. Let β be a Riemann form for A. Then the Hodge group of A is the symplectic group $G = Sp(V,\beta)$, acting on $V = H^1(A,\mathbb{Q})$ via the standard representation. By the representation theory of the symplectic group, for $2 \le k \le \dim A$, we have

$$\bigwedge^k V = \pi_k \oplus \bigwedge^{k-2} V,$$

where π_k is an irreducible representation of G. Thus

$$H^k(A,\mathbb{Q}) = V_k \oplus H^{k-2}(A,\mathbb{Q})(-1),$$

where V_k is equivalent to π_k as a G-module. Since $H^{k-2}(A,\mathbb{C})(-1)$ cannot contain any $(k,0)$-forms, it follows that all $(k,0)$-forms are in $V_{k,\mathbb{C}}$. For each irreducible representation π_k of G in the cohomology of A we have found a fully twisted Hodge structure V_k in $H^\star(A,\mathbb{Q})$ with π_k equivalent to V_k, proving that A is dominated by itself. Since the usual Hodge conjecture is known for $A \times A$, this proves the general Hodge conjecture for A (due originally to Mattuck [26]). □

The abelian varieties for which the general Hodge conjecture follows from this theorem are listed in Appendix A (see item (1)).

5. The CM-case

A polarizable Hodge structure is said to be of CM-type if its Hodge group is commutative. As shown by Mumford [29], an abelian variety A is of CM-type if and only if the Hodge structure $H^1(A, \mathbb{Q})$ is of CM-type.

Theorem 5.1 (Abdulali [7, Theorem 3, p. 159])**.** *Every effective Hodge structure of* CM*-type occurs in the cohomology of an abelian variety of* CM*-type.*

Theorem 5.2 (Abdulali [7, Theorem 4, p. 159])**.** *Any abelian variety of* CM*-type is dominated by the class of all* CM *abelian varieties. If A is a simple abelian variety of* CM*-type with* CM *by E, then any power of A is dominated by the set of products of abelian varieties with* CM *by E.*

Theorem 5.3 (Abdulali [7, Proposition 5, p. 160])**.** *Let $E_1, \dots, E_m$ be* CM*-fields whose Galois closures are linearly disjoint over $\mathbb{Q}$. For each $i = 1, \dots, m$, let A_i be an abelian variety with complex multiplication by E_i, and let $\mathcal{X}_i$ be a class of abelian varieties which dominates A_i. Then $A = \prod_{i=1}^m A_i$ is dominated by $\mathcal{X} = \{ \prod_{i=1}^m X_i \mid X_i \in \mathcal{X}_i \}$.*

Using Theorem 5.2, we can prove the general Hodge conjecture for certain abelian varieties (see Appendix A, item (3)). We note that Hazama [19, 20] had independently proved that the usual Hodge conjecture for all CM abelian varieties implies the general Hodge conjecture for all CM abelian varieties.

For further discussion of Theorem 5.1 see [16].

6. Abelian varieties of type IV

We have dealt above with the extreme cases where the Hodge group is either semisimple or commutative. The general situation turns out to be quite subtle. In light of Theorem 5.1, it is tempting to rephrase the definition of $\mathbb{C}$-domination of abelian varieties in terms of just the semisimple part $G'(A)$ of the Hodge group. One might, for example, say that A is "semisimply dominated" by $\mathcal{X}$ if, given any irreducible $G'(A)_{\mathbb{C}}$-submodule V of $H^\star(A, \mathbb{C})$, there exists a fully twisted $G'(B)_{\mathbb{C}}$-submodule V' of $H^\star(B, \mathbb{C})$ for some $B \in \mathcal{X}$, which is equivalent to V as a $G'(A \times B)_{\mathbb{C}}$-module. It is tempting to hope that in this situation A will be dominated by abelian varieties of the form $B \times C$, where $B \in \mathcal{X}$ and C is of CM-type. This is unfortunately not true. It turns out that we require not just V', but all of its Galois conjugates, to be fully twisted. The formal definition follows.

Definition 6.1. An abelian variety A is *semidominated* by a set $\mathcal{X}$ of abelian varieties if, given any nontrivial irreducible representation ρ of $G'(A)_{\mathbb{C}}$ such that ρ occurs in $H^n(A, \mathbb{C})$ for some n, there exist $A_\rho \in \mathcal{X}$, a positive integer c_ρ, and, $V_\rho \subset H^{c_\rho}(A_\rho, \mathbb{C})$, such that

(1) V_ρ is a $G(A_\rho)_{\mathbb{C}}$-submodule of $H^{c_\rho}(A_\rho, \mathbb{C})$,
(2) the action of $G'(A \times A_\rho)_{\mathbb{C}}$ on V_ρ is equivalent to $\rho \circ p_1$, where
$$G'(A) \times G'(A_\rho) \supset G'(A \times A_\rho) \xrightarrow{p_1} G'(A)$$
is the projection to the first factor, and,
(3) for each $\sigma \in \operatorname{Aut}(\mathbb{C})$, the conjugate $(V_\rho)^\sigma$ contains a nonzero $(c_\rho, 0)$-form.

Theorem 6.2 (Abdulali [9, Theorem 7]). *Let A be an abelian variety semidominated by $\mathcal{X}$. Then A is dominated by the set of abelian varieties of the form $B \times C$, where $B \in \mathcal{X}$, and C is of* CM*-type.*

Remark 6.3. The converse of Theorem 6.2 is false. For example, the abelian varieties in Theorem 7.1 are dominated by the set of powers of themselves, but are not semidominated by any class of abelian varieties [9, Remark 5].

Lemma 6.4 (Abdulali [9, Lemma 6]). *If A is semidominated by $\mathcal{X}$ and B is semidominated by $\mathcal{Y}$, and if $G'(A \times B) = G'(A) \times G'(B)$, then $A \times B$ is semidominated by*
$$\mathcal{X} \cdot \mathcal{Y} = \{X \times Y \mid X \in \mathcal{X},\ Y \in \mathcal{Y}\}.$$

In the next two theorems we use these ideas to show that certain abelian varieties of type IV are dominated by abelian varieties. In a few cases we can also deduce the general Hodge conjecture (see Appendix A, item (4)).

Theorem 6.5 (Abdulali [8, Theorem 10, p. 1167]). *Let A be an abelian variety of* PEL*-type such that each simple factor of A is of type* IV. *Then we can write $G'(A)(\mathbb{R}) \cong \prod_{\alpha \in S} SU(p_\alpha, q_\alpha)$. Assume that for each $\alpha \in S$ we have $|p_\alpha - q_\alpha| = 1$. Then A is semidominated by the set of powers of A.*

Theorem 6.6 (Abdulali [9, Theorem 10]). *Let A be an abelian variety such that each simple factor of $G'(A)(\mathbb{R})$ is isomorphic to $SU(p, 1)$ for some $p > 1$. Then A is dominated by abelian varieties.*

Idea of proof. Using Lemma 6.4, we reduce to the case of a power of a simple abelian variety A_0. Then $A_0 = \mathcal{A}_P$ is the fiber at a general point P of a Hodge family $\mathcal{A} \to \mathcal{V}$. For $k = 1, \dots, p$, Satake [32] constructs Kuga fiber varieties $\mathcal{A}_k \to \mathcal{V}$, from symplectic representations $\rho_k \colon G'(A) \to Sp(V_k, \beta_k)$ such that a simple factor of $G'(A)(\mathbb{R})$ acts as the direct sum of $\bigwedge^k$ and its contragredient. Let A_k be the fiber over P of $\mathcal{A}_k$. It follows from Satake's classification [32] that $\mathcal{A}$ is isogenous to $\mathcal{A}_k$ and hence A_0 is isogenous to A_k for some k. Let
$$\mathcal{X} = \left\{ A_1^{n_1} \times \cdots \times A_p^{n_p} \mid n_i \geq 0 \right\}.$$
We then show that every member of $\mathcal{X}$ is semidominated by $\mathcal{X}$, and hence dominated by abelian varieties. □

Remark 6.7. The families of abelian varieties $\mathcal{A}_k \to \mathcal{V}$ in the proof of Theorem 6.6 are not, in general, PEL-families, so the abelian varieties in $\mathcal{X}$ are not, in general, of PEL-type.

7. Abelian varieties of type III

We now consider a simple abelian variety A of PEL-type whose endomorphism algebra is a totally definite quaternion algebra $D = D(A)$ over a totally real number field F. We use the notations of §2, so that $T\colon V \times V \to D$ is now a quaternionic skew-hermitian form. Let $m = \dim_D V$. The Hodge group $G = G(A)$ is the connected component of the identity of $\operatorname{Res}_{F/\mathbb{Q}} \operatorname{Aut}(V, T)$, and $G(\mathbb{R}) = \prod_{\alpha \in S} G_\alpha$, where each G_α is isomorphic to the group of type D_m called $SO^\star(2m)$ by Helgason [21, p. 445], and $SU^-(m, \mathbb{H})$ by Satake [33, Exercise 2, p. 278].

The discriminant, $\operatorname{discr} T$, plays a crucial role here. Theorem 7.1 deals with the cases where $\operatorname{discr} T$ is not a square. When $\operatorname{discr} T$ is a square, we deal with the cases $m = 4$ and $m > 4$ in Theorems 7.2 and 7.3, respectively.

Theorem 7.1 (Abdulali [5, Theorem 4.1, p. 922])**.** *Let A be a simple abelian variety of* PEL*-type such that $D(A)$ is a totally definite quaternion algebra over $\mathbb{Q}$. Let T be the skew-hermitian form determined by a polarization of A. If* $\operatorname{discr} T$ *is not a square, then any power of A is dominated by the set of powers of A, and the usual Hodge conjecture for A implies the general Hodge conjecture for all powers of A.*

Outline of proof. We have $V_{\mathbb{C}} = W \oplus \overline{W}$, with $G_{\mathbb{C}}$ acting as the standard representation on both W and $\overline{W}$, and, $\dim W = 2m$. As explained in [5, p. 923], $W = W^{1,0} \oplus W^{0,1}$, where $W^{1,0}$ is an m-dimensional space of $(1,0)$-forms, and $W^{0,1}$ is an m-dimensional space of $(0,1)$-forms. Let $\{u_1, \dots, u_m\}$ be a basis of $W^{1,0}$.

Let $\mathfrak{g}$ be the Lie algebra of $G_{\mathbb{C}}$. We denote by $\mu_1, \dots, \mu_m$, the fundamental weights of $\mathfrak{g}$ with respect to a Cartan algebra $\mathfrak{h}$ (see [5, §3.3, p. 921] and [44, p. 329]). For $1 \le j \le m-2$, $\bigwedge^j W$ is an irreducible $\mathfrak{g}$-module with highest weight μ_j which contains the $(j,0)$-form $u_1 \wedge \cdots \wedge u_j$. The irreducible $\mathfrak{g}$-module $\bigwedge^{m-1} W$ has highest weight $\mu_{m-1} + \mu_m$, and contains the $(m-1,0)$-form $u_1 \wedge \cdots \wedge u_{m-1}$. However, $\bigwedge^m W$ is the direct sum of two irreducible $\mathfrak{g}$-modules, say U_1 and U_2, having highest weights $2\mu_{m-1}$ and $2\mu_m$ respectively. The $(m,0)$-form $u_1 \wedge \cdots \wedge u_m$ is in U_2, while U_1 contains no $(m,0)$-forms. The assumption that $\operatorname{discr} T$ is not a square implies that the weights $2\mu_{m-1}$ and $2\mu_m$ belong to the same orbit of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Hence if a Hodge structure U is such that $U_{\mathbb{C}}$ contains U_1, then it must also contain U_2, and thus be fully twisted.

Now let M be an irreducible Hodge structure in the cohomology of a power of A, and M_0 an irreducible $\mathfrak{g}$-submodule of $M_{\mathbb{C}}$. Let μ be the highest weight of M_0, and write $\mu = a_1\mu_1 + \cdots + a_m\mu_m$, for some nonnegative integers $a_1, \dots, a_m$. Since the representation occurs in the tensor algebra generated by $V_{\mathbb{C}}$, we must have $a_{m-1} \equiv a_m \pmod 2$.

If $a_m \ge a_{m-1}$, we have

$$\mu = a_1\mu_1 + \cdots + a_{m-2}\mu_{m-2} + a_{m-1}(\mu_{m-1} + \mu_m) + b_m(2\mu_m)$$

with $b_m = a_m - a_{m-1} \geq 0$. We have seen above that for each of the weights $\mu_1, \ldots, \mu_{m-2}, \mu_{m-1} + \mu_m, 2\mu_m$, there exists an irreducible and fully twisted $\mathfrak{g}$-module in the cohomology of A with that highest weight. Hence, taking tensor products, we obtain a fully twisted $\mathfrak{g}$-module M_1 of highest weight μ in the cohomology of some power of A. An irreducible Hodge structure containing M_1 is fully twisted and isomorphic to a Tate twist of M.

If $a_m < a_{m-1}$, we have

$$\mu = a_1\mu_1 + \cdots + a_{m-2}\mu_{m-2} + a_{m-1}(\mu_{m-1} + \mu_m) + c_m(2\mu_{m-1})$$

with $c_m = a_{m-1} - a_m \geq 0$. This is a Galois conjugate of

$$\mu' = a_1\mu_1 + \cdots + a_{m-2}\mu_{m-2} + a_{m-1}(\mu_{m-1} + \mu_m) + c_m(2\mu_m)$$

so we can apply the arguments of the previous paragraph to a conjugate of M_0 contained in $M_{\mathbb{C}}$. □

Theorem 7.1 implies the general Hodge conjecture for some abelian varieties of type III (see Appendix A, item (2)).

Theorem 7.2 (Abdulali [5, Theorem 5.3, p. 925])**.** *Let A be a simple abelian variety of* PEL*-type of type* III*. Let T be the skew-hermitian form determined by a polarization of A. If* $\operatorname{discr} T$ *is a square and* $\dim_{D(A)} H_1(V, \mathbb{Q}) = 4$*, then there exists an abelian variety B such that each power of A is $\mathbb{C}$-dominated by the set of abelian varieties of the form $A^i \times B^j$.*

Idea of proof. Assume, for simplicity, that the center of $D(A)$ is $\mathbb{Q}$. The spin group $\mathrm{Spin}(6, 2)$ is a 2-fold covering of $SO^\star(8)$, so their symmetric domains are isomorphic (cf. [33, Exercise 1, p. 289]). We thus get two Kuga fiber varieties $\mathcal{A} \to \mathcal{V}$ and $\mathcal{B} \to \mathcal{V}$ over the same arithmetic variety $\mathcal{V}$, where $\mathcal{A} \to \mathcal{V}$ is the PEL-family having A as the fiber over a point P, and, $\mathrm{Spin}(6, 2)$ acts on the first cohomology of $B = \mathcal{B}_P$ via the spin representation with highest weight μ_3. Thus $H^2(B \times B, \mathbb{C})$ contains a fully twisted $\mathrm{Spin}(6, 2)$-module which is equivalent to the representation U_1 constructed in the proof of Theorem 7.1. □

Theorem 7.3 (Abdulali [5, Theorem 5.5, p. 926])**.** *Let A be a simple abelian variety of* PEL*-type of type* III*. Let T be the skew-hermitian form determined by a polarization of A. If* $\operatorname{discr} T$ *is a square, and* $\dim_{D(A)} H_1(V, \mathbb{Q}) > 4$*, then A is not dominated by abelian varieties.*

Sketch of proof. Assume, for simplicity, that the center of $D(A)$ is $\mathbb{Q}$. Let A be the fiber at P of the PEL-family $\mathcal{A} \to \mathcal{V}$. Let M be an irreducible Hodge structure in $H^m(A, \mathbb{Q})$ such that $M_{\mathbb{C}}$ contains the representation U_1 constructed in the proof of Theorem 7.1. Then $M(1)$ is effective, but representation theory shows that a representation equivalent to U_1 cannot occur in $H^{m-2}(A^k, \mathbb{C})$ for any positive integer k [5, Lemma 3.3.1, p. 921]. This shows that A is not dominated by the set of powers of A. Next, we prove that if $M(1)$ were to occur in the cohomology of an abelian variety B, then we could take B to be the fiber over P of a Kuga fiber variety $\mathcal{B} \to \mathcal{V}$.

Finally, we invoke Satake's classification of Kuga fiber varieties [32, 33] to rule out the existence of such an abelian variety. □

8. The general Hodge conjecture

8.1. **Some special cases.** In Appendix A we give a list of abelian varieties for which the general Hodge conjecture can be unconditionally proved using the methods of this paper. These include the following interesting special cases:

(1) Combining cases (1a), (3c), and, (5) of Appendix A, we obtain the general Hodge conjecture for any product of three (or fewer) elliptic curves.
(2) An abelian surface is either a product of two elliptic curves, or, of CM-type (Case (3d)), or, its Hodge group is simple and it belongs to Case (1a). Thus the general Hodge conjecture is true for any power of an abelian surface.
(3) Bardelli [10] proves the general Hodge conjecture for various abelian 3-folds. All of these are included in our list.

8.2. **An open problem.** Let E be a quadratic imaginary number field, and A an abelian m-fold of PEL-type with $D(A) = E$. Then the Lefschetz group of A is a $\mathbb{Q}$-form of $U(p, q)$, where (p, q) is the signature of the hermitian form determined by a polarization of A. The Weil Hodge structure $W \subset H^m(A, \mathbb{Q})$ is a 2-dimensional Hodge structure of type $\{(p, q), (q, p)\}$. Schoen [35, §6] has proved the general Hodge conjecture for W in the following cases:

(1) $m = 4$, $(p, q) = (3, 1), E = \mathbb{Q}(\sqrt{-1})$, and,
(2) $m = 6$, $(p, q) = (5, 1), E = \mathbb{Q}(\sqrt{-3})$.

W is of CM-type, and isomorphic to a Hodge substructure of C^w where C is an elliptic curve with CM by E, and, $w = |p - q|$ [6]. The full general Hodge conjecture for (powers of) these abelian varieties is open, and perhaps approachable using known methods.

9. Concluding remark

I have given above large classes of abelian varieties which are dominated by abelian varieties, and one example of an abelian variety not dominated by abelian varieties. I believe that the exception is the rule, and "most" abelian varieties not considered above are *not* dominated by abelian varieties—and expect to provide evidence for this in a future paper.

Appendix A. Known Cases

The following is a list of abelian varieties for which the general Hodge conjecture is currently known. All these cases can be proved using the methods of this paper.

(1) An abelian variety A of PEL-type such that $G(A)$ is semisimple, and, each simple factor of A is one of the following (see Theorem 4.1):
   (a) Any simple abelian variety A_0 of PEL-type of type I or type II. The usual Hodge conjecture for all powers of A_0 is due to Kuga [25, pp. 79–80]. The general Hodge conjecture was first proved by Hazama [18] and Tankeev [41]. Special cases have been proved by Mattuck [26], Knight [22, 23], Gordon [13, 14], and, Tankeev [42, 43].
   (b) A simple $2m$-dimensional abelian variety A_0 of PEL-type such that $D(A_0) = \mathbb{Q}\left(\sqrt{-d}\right)$, and polarization given by a hermitian form H of signature (m, m), in the following cases:
      (i) $m = 2$ and $d \in \{1, 3\}$;
      (ii) $m = 3$, $d \in \{1, 3\}$, and, $\operatorname{discr} H = -1$.

   The usual Hodge conjecture for these abelian varieties is due to Schoen [34, 36], van Geemen [12], and Koike [24].
(2) Any power of one of the following abelian varieties of type III:
   (a) A 4-dimensional abelian variety A of type III such that $D(A)$ contains a square root of -3 or -1 [5, Corollary 4.3].
   (b) A 6-dimensional abelian variety A of PEL-type of type III such that $D(A)$ contains a square root of -3 or -1, and the polarization is given by a skew-hermitian form of discriminant -1 [2, Example 5.1].
(3) The following abelian varieties of CM-type [7, §4, pp. 160-161]:
   (a) Any power of an abelian variety A with CM by a CM-field E such that $[\overline{E} : \overline{F}] = 2^d$, where F is the maximal totally real subfield of E, $d = [F : \mathbb{Q}]$, and, bars denote Galois closure (Tankeev [41, Theorem 2, p. 180]). Dodson [11, Proposition 2.2.2, p. 82] provides examples of such CM-fields.
   (b) Any power of an elliptic curve of CM-type [40].
   (c) Any abelian variety of the form $A_1^j \times A_2^k \times A_3^\ell$, where A_1, A_2, A_3 are elliptic curves of CM-type.
   (d) Any power of an abelian surface of CM-type.
(4) Any abelian variety of PEL-type which is isogenous to a product of abelian varieties of the following types (Abdulali [8, Theorem 14, p. 1169]):
   (a) a simple 3-dimensional abelian variety with endomorphism algebra either $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$, and with a polarization given by a hermitian form of signature $(2, 1)$;
   (b) a simple 5-dimensional abelian variety with endomorphism algebra either $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$, and with a polarization given by a hermitian form of signature $(3, 2)$;
   (c) an elliptic curve with CM by either $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$.
(5) An abelian variety isogenous to a product $A \times B$, where A is as in Case (1), and B is in one of the other cases above (Theorem 3.4).

Appendix B. Errata to earlier papers

I thank Chad Schoen for kindly pointing out to me that Proposition 4.4.1 of [1] is false. This requires several modifications to subsequent papers.

B.1. **Abelian varieties and the general Hodge conjecture** [1]**.**

(1) page 341, line 7 from bottom: replace $V(-r)$ by $V(r)$.
(2) page 345, line 4: replace $(-r)$ by (r).
(3) Proposition 4.4.1 is false. It should be replaced by [8, Propositions 3 and 4, p. 1166] and [7, Proposition 5, p. 160] (see Theorems 3.4 and 5.3 of this paper.) Consequently,
   (a) Theorem 5.1 requires the additional hypothesis that there is at most one simple factor of type III.
   (b) Theorem 6.1 requires the additional hypothesis that the CM-fields corresponding to the factors of CM-type are linearly disjoint.

B.2. **Filtrations on the cohomology of abelian varieties** [3]**.**

(1) page 7, line 10: replace $V(-r)$ by $V(r)$.
(2) §7 of [3] is corrected, updated, and expanded by Appendix A of the current paper. In particular, [3, 7.1 (2), 7.1 (4), and, 7.2.1] are incorrect.
(3) page 10, line 20: replace $U(-1)$ by $U(1)$.

B.3. **Hodge structures on abelian varieties of CM-type** [4]**.** Theorem 3.1 as stated is false. Consequently, some of the examples in Section 4 are incorrect. Correct results and examples may be found in [7], and are summarized in §5 of this paper.

B.4. **Hodge structures on abelian varieties of type III** [5]**.** Add to the hypotheses of Theorem 4.1 that the center of $D(A)$ is $\mathbb{Q}$ (see Theorem 7.1 of this paper).

B.5. **Hodge structures on abelian varieties of type IV** [6]**.** Theorems 3.1 and 3.2 require the additional hypothesis that the signature (p, q) of the hermitian form satisfies $|p - q| = 1$. Hence Remark 3.4 is invalid, but, Corollary 3.3 is correct as stated.

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