Unpublished Appendix to "Time Irreversibility and Business Cycle Asymmetry"

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Abstract
Addendum: Some Results on the Relationship Between Transversely and Longitudinally Asymmetric Functions and Time Reversibility

We wish to prove some lemmas about the relationship between longitudinal and transversal asymmetry for differential periodic functions. We begin with some definitions of symmetry and then demonstrate the main result.

In the following discussion all functions are assumed to be at least $C^2$.

Definition: $f(t)$ is periodic of period $T$, iff $f(t) = f(t + T)$, for all $t$ in the domain of $f(\cdot)$.

By an inessential transformation of variables we can define the domain of one complete cycle that begins with a relative maximum as $[0,T]$; we assume hereafter that this has been done.

Hereafter, we will restrict attention to periodic functions.

Definition: $f(t)$ is longitudinally symmetric, iff, for all $t$ such that $f(t)$ is a relative maximum or a relative minimum:

$$f(t + \ast) = f(t - \ast),$$

for any $\ast$ less than the difference between adjacent points in the domain of $f(\cdot)$ yielding relative maxima and minima.

Otherwise, $f(t)$ is longitudinally asymmetric. See Figure 1 which illustrates a longitudinally asymmetric periodic function.

Definition: A time reversible function is any function that satisfies the following condition:

$$\int_0^T f(t)^2 \cdot f(t-k) dt = \int_0^T f(t-k)^2 \cdot f(t) dt \quad (A.1)$$

An alternative and fully equivalent statement for a definition of time reversibility is:

$$\int_0^T f(t)^2 \cdot f(t-k) dt = \int_0^T f(-t)^2 \cdot f(-t+k) dt \quad (A.2)$$

A function that is not time reversible is said to be time irreversible. Time reversible functions are ones such that the replacement of "$t$" by "$-t$" leaves the function invariant; examples are the equations of motion in classical mechanics. Time irreversible functions are not invariant to such transformations and consequently define a unique direction of time, instead of merely providing an ordering as is the case with time reversible functions.

The equivalence of the two definitions for time reversibility is easily demonstrated. Consider:
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(a): \[ \int_{0}^{T} f(t)^2 f(t-k) \, dt = \int_{0}^{T} f(-t)^2 f(-t+k) \, dt \]

(b): \[ \int_{0}^{T} f(-t)^2 f(-t+k) \, dt = \int_{0}^{T} f(T-t)^2 f(T-t+k) \, dt \]  

(c): \[ \int_{0}^{T} f(t') f(t'k) \, dt' = \int_{0}^{T} f(t'-k)^2 f(t') \, dt' \]

(d): \[ = \int_{0}^{T} f(t-k)^2 f(t) \, dt \]

(3.a) is the alternative definition, (3.b) holds by periodicity, the first term in (3.c) is by transformation of variables and the second term follows by the invariance of the integral to a time shift, and (3.d) follows from a simple transformation of variables. A similar argument in reverse shows that the first definition implies the second.

Definition: \( f(t) \) is transversely symmetric:
(a) \( f(t) \) is longitudinally symmetric; and
(b) for all \( t_{\text{max}}, t_{\text{min}} \) points in the domain \([0,T]\) such that \( f(t_{\text{max}}) \) is a relative maximum and \( f(t_{\text{min}}) \) is a relative minimum and \( f'(\cdot) \) is the derivative:
\[ f'(t_{\text{max}} +/- \ast) = f'(t_{\text{min}} +/- \ast), \text{ for some } \ast. \]

If condition (b) does not hold, \( f(t) \) is transversely asymmetric, but longitudinally symmetric. If condition (a) does not hold, the function is transversely and longitudinally asymmetric. See Figure 2 for an example of a longitudinally symmetric, but transversely asymmetric function. The idea behind a transversely asymmetric function is that the function is "symmetric" to some extent, that is, the peaks and troughs, while individually locally symmetric, have different degrees of curvature. Transversely asymmetric functions are not symmetric in a direction that is "transverse" to the direction of the function.

Definition: \( f(t) \) is completely symmetric if \( f(t) \) is both longitudinally and transversely symmetric.

If \( f(t) \) is longitudinally asymmetric, then it is transversely asymmetric. If \( f(t) \) is longitudinally symmetric, it may, or may not, be transversely asymmetric. Simple sinusoidal functions, for example, \( \sin(\pi t) \), or \( \cos(\pi t) \), \( \pi \in [0,2\pi] \), are completely symmetric.

Lemma 1: Longitudinal asymmetry is a necessary and sufficient condition for time irreversibility.
We wish to prove that, if \( f(\cdot) \) is longitudinally asymmetric:
\[ \int_{-T/2}^{T/2} f(t)^2 f(t-k) \, dt \neq \int_{-T/2}^{T/2} f(-t)^2 f(-t+k) \, dt \]  

The proof precedes by splitting each of the two integrals above into two components and comparing terms after re-expressing all integrals over the domain \([0,T/2]\). We assume without loss of generality that \( t = 0 \) is a relative maximum, or a relative minimum. The left hand integral becomes:
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\[ \int_0^{T/2} f(t)^2 \cdot f(t-k) \, dt + \int_0^{T/2} f(-t)^2 \cdot f(-t-k) \, dt, \tag{A.5} \]

and the right hand integral becomes:

\[ \int_0^{T/2} f(t)^2 \cdot f(t+k) \, dt + \int_0^{T/2} f(-t)^2 \cdot f(-t+k) \, dt. \tag{A.6} \]

Recalling that we have assumed longitudinal asymmetry, we see that, comparing the two integrals involving negative \( t \), they will not be equal since we are subtracting \( k \) in one case and adding \( k \) in the other. Similar remarks hold for the remaining two terms involving the positive \( t \). We conclude that the integrals are not equal in pairs so that the function satisfies the condition for time irreversibility.

If \( f(\cdot) \) is longitudinally symmetric, let \( k = 2^* \) in equations (A.5) and (A.6). By adding a time shift of \( * \) to both expressions, we see immediately that the assumption of longitudinal symmetry implies that the two integrals above are equal in that their constituent parts are equal. Thus, longitudinal symmetry implies time reversibility.

**Lemma 2:** Let \( f(t) \) be a function that is transversely asymmetric, but is longitudinally symmetric; that is:

\[ f(t_{\text{max}} +/\ - \text{*}) = f(t_{\text{max}} -/+ \text{*}), \text{ for any * less than the difference between adjacent points in the domain of } f(\cdot) \text{ yielding relative maxima and minima}; \]

\[ f'(t_{\text{max}} +/\ - \text{*}) \neq f'(t_{\text{min}} -/+ \text{*}), \text{ for some sufficiently large *}. \]

We wish to show that such a function \( f(\cdot) \) is time reversible.

Because longitudinally symmetric functions are time reversible, the result follows immediately.

**Lemma 3:** Let \( f(t) \) be a function that is transversely asymmetric, but is longitudinally symmetric; that is:

\[ f(t_{\text{max}} +/\ - \text{*}) = f(t_{\text{max}} -/+ \text{*}), \text{ for any * less than the difference between adjacent points in the domain of } f(\cdot) \text{ yielding relative maxima and minima}; \]

\[ f'(t_{\text{max}} +/\ - \text{*}) \neq f'(t_{\text{min}} -/+ \text{*}), \text{ for some sufficiently large *}. \]

We wish to prove that \( f'(t) \), the derivative derived from \( f(t) \), is longitudinally asymmetric and therefore time irreversible.

This result is illustrated in Figure 3. Let \( t_{\text{infl}} \) denote an inflection point between, say a minimum and the next maximum, then at such a point \( f'(t_{\text{infl}}) \) is a maximum. Consider an, neighborhood of \( t_{\text{infl}} \). By the statement of the problem and the assumption that the inflection point lies between a minimum to the left and a maximum to the right with different degrees of curvature, then we know that \( |f'(t_{\text{infl}} -\text{*})| \) is not equal to \( |f'(t_{\text{infl}} +\text{*})| \) for some * in a sufficiently large neighborhood of \( t_{\text{infl}} \). Consequently, in terms of the function \( f'(t) \), it satisfies the conditions for longitudinal asymmetry. Assuming that \( f'(t_{\text{infl}}) \) lies between a maximum to the left and a minimum to the right provides the basis for an equivalent argument. We conclude that if \( f(t) \) is transversely asymmetric, but is longitudinally symmetric and thus time reversible, its derivative, however, is longitudinally asymmetric and is therefore time irreversible.

Completely symmetric functions are time reversible and so are their derivatives.